Chapter 2

Plasticity Review

The creation of the concrete model in this thesis required extensive use of current plasticity theory. In order for the reader to understand the model, a brief review of relevant plasticity theory is presented here. First, Section 2.1 defines the stress invariants used for the definition of this model and discusses their physical significance. Second, the definition and requirements of a failure surface are reviewed in Section 2.2. Third, the definitions for plastic flow and effective plastic strain are discussed in Section 2.3. Finally, the consistency condition, used for determining the tangent material matrix, is outlined in Section 2.4. For a complete discussion of plasticity theory and its application to concrete, see Chen (1982).

2.1 Stress Invariants

For a concrete model to be most useful, the model itself should be defined independent of the coordinate system attached to the material. Thus, it is necessary to define the model in terms of stress invariants which are, by definition, independent of the coordinate system selected. The three-dimensional stress state of the material is traditionally defined by the stress tensor, which can be represented relative to a chosen coordinate system by a matrix:

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \tag{2.1}$$

This stress tensor is often decomposed into two parts: a purely hydrostatic stress, σ_m, defined in Equation 2.2, and the deviatoric stress tensor, s_{ij}, defined in Equation 2.3.

$$\sigma_m = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) \tag{2.2}$$

$$s_{ij} = \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix} \tag{2.3}$$

A common set of stress invariants are the three principal stress invariants. The principal stress coordinate system is the coordinate system in which shear stresses vanish, leaving only normal stresses. This requirement of zero shear stresses leads to the characteristic equation:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \tag{2.4}$$

The first, second, and third invariant of the stress tensor, I_1, I_2, and I_3 are defined in the following equations:

$$I_1 = \sigma_x + \sigma_y + \sigma_z \tag{2.5}$$

$$I_2 = \left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \tag{2.6}$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} \tag{2.7}$$

The three roots of Equation 2.4 are the three principal stress invariants, also called the three principal stresses. They are ordered so that $\sigma_1 > \sigma_2 > \sigma_3$.

The three principal stresses, as well as most other stress invariants, can be rewritten in terms of three core invariants: the first invariant of the stress tensor, I_1, and the second and third invariants of the deviatoric stress tensor, J_2 and J_3. The first invariant of the stress tensor, I_1, was previously defined in Equation 2.5. The second and third invariants of the deviatoric stress tensor are defined as:

$$J_2 = \frac{1}{6}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \tag{2.8}$$

$$J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki} = \begin{vmatrix} \sigma_x - \frac{1}{3}I_1 & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \frac{1}{3}I_1 & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \frac{1}{3}I_1 \end{vmatrix} \tag{2.9}$$

Clearly, a large variety of stress invariants were available to use in defining the model. The three stress invariants ξ, r, and θ were chosen to define the components of the concrete model:

$$\xi = \frac{1}{\sqrt{3}}I_1 = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3) \tag{2.10}$$

$$r = \sqrt{2J_2} = \sqrt{\frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} \tag{2.11}$$

$$cos(3\theta) = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} \tag{2.12}$$

They have a direct physical interpretation which makes it easier to understand the physical implications of the model. To understand the physical significance of each of these invariants, it is helpful to look at them in the principal stress coordinate system $(\sigma_1, \sigma_2, \sigma_3)$. Recall that the principal stress coordinate system corresponds to the orientation in which the material has no shear stresses. A diagram of this coordinate system is shown in Figure 2.1. Consider the case of purely hydrostatic loading with magnitude equal to σ_h. For this load case, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_h$. Thus, the load path travels along the ξ axis. The magnitude of the hydrostatic load, σ_h, is equal to the stress invariant ξ. Therefore, it is clear that the invariant ξ represents the hydrostatic component of the current stress state. Now we consider the planes that lie

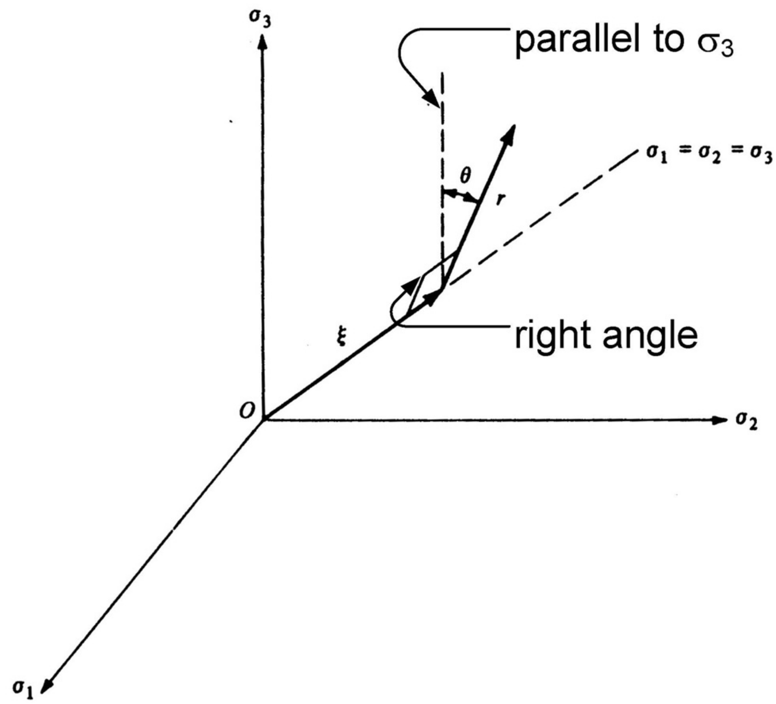

Figure 2.1: Graphical interpretation of stress invariants (ξ, r, θ) in the principal stress space. Modified from Chen (1982).

perpendicular to this hydrostatic axis. For any given stress state lying in one of these planes, the distance between the point representing the stress state in the principal stress coordinate system and the hydrostatic axis is related to the deviatoric stress. The magnitude of this distance is equal to the invariant r. Thus, r represents the stress invariant measure of the deviatoric stress. This leaves only the third invariant, θ, also known as the Lode angle. The invariant θ is controlled by the relationship of the intermediate principal stress to the major and minor principal stresses. When the intermediate principal stress, σ_2, is equal to the minor principal stress, σ_3, the value for θ becomes 60°. When the intermediate principal stress, σ_2, is equal to the major principal stress, σ_1, the value for θ becomes 0°. Thus, θ is an indication of the magnitude of the intermediate principal stress in relation to the minor and major principal stresses.

2.2 Failure Surface

An important component of a concrete plasticity model is the *failure surface*. In traditional plasticity theory, this surface is alternately referred to as the yield or loading surface. In this thesis, to prevent confusion, this surface will be known exclusively as the failure surface, which defines the boundary of elastic behavior. (The concepts of the yield surface and the loading surface will be introduced in Chapter 3 to refer to surfaces which do not follow the rules required of the failure surface.) When the current stress state of the material lies within the failure surface, the material behaves purely elastically. The definition of the failure surface will clearly depend on stress variables, but can also depend upon other variables, such as the plastic strain, or constant parameters, often called hardening parameters. Detailed discussions of the variables on which this surface can depend can be found in other texts [(Chen, 1982), (Lubliner, 1990), (Khan and Huang, 1995)]. The failure surface defined in this thesis is a function of stress variables, through the invariants discussed in Section 2.1, and the effective plastic strain, $\overline{\epsilon^p}$, which will be defined in Section 2.3. Thus, the surface can be expressed as:

$$F = F(\sigma_{ij}, \overline{\epsilon^p}) = 0 \tag{2.13}$$

The failure surface is defined such that, for load states where $F < 0$, the material behaves elastically. Once the load path intersects the failure surface, unloading is defined as returning to a stress state where $F < 0$. While loading continues, the stress state must stay on the failure surface with F = 0, although this surface can move and change shape as $\overline{\epsilon^p}$ varies.

There are well documented behaviors of concrete that affect the definition of the failure surface. Details of the experimental behavior of concrete will be given in Sections 4.2 through 4.5. From these experiments, it is known that the failure surface for concrete should be smooth. Further, consider the failure surface in the (ξ, r) plane, also known as the meridian plane. The shape of the failure surface in this plane describes how the deviatoric stress, r, that can be supported by the concrete will change with the current hydrostatic stress, ξ. Recall that the value for the Lode angle, θ, varies only between 0° and 60°. The two meridian planes corresponding to these two extreme values of the Lode angle are called the tensile and compressive meridians, respectively. The tensile meridian is so named because uniaxial tension is one of the load cases which corresponds to a Lode angle of 0°. The compressive meridian is given that name because uniaxial compression corresponds to a Lode angle of 60°. It is known that, in general, concrete can withstand higher deviatoric stresses when subjected to confinement. There are two forms of confinement: active and passive. Active confinement is a lateral pressure applied to the concrete. Passive confinement consists of wrapping a concrete member in a material such as steel rebar, steel jackets, or fiber reinforced polymer (FRP) jackets, to create a lateral pressure through restraining the expansion of the concrete under axial loading. Confining the concrete results in an increase in apparent strength. In addition, when subjected to loading with θ values near 60°, it will withstand higher deviatoric stresses than when loading occurs near a θ value of 0°. Also, experimental results imply that these meridians should be convex and curved. Thus, in the model's meridian planes: the deviatoric stress increases with ξ; the compressive meridian lies outside the tensile meridian; and the intersection of the surface in all meridian planes is convex. This is exemplified later in Figure 3.2. Experiments have shown that concrete does not fail under purely hydrostatic loading. Therefore, the failure surface should not cross the hydrostatic axis.

It is also useful to consider experimental results in the (r,θ), or deviatoric, plane. The failure surface in this plane exhibits a three-fold symmetry. This is due to the fact that concrete typically behaves as an isotropic material. For small values of ξ, the failure surface is nearly triangular. As ξ increases, the cross section becomes more circular. Physically, this means that the dependence on the intermediate principal stress decreases with increasing confinement.

As previously mentioned, experimental results show that the meridians of the failure surface should be convex. Theory also supports this requirement, since Drucker's stability postulate [(Drucker, 1951), (Drucker, 1960)] requires that the failure surface itself be convex.

2.3 Plastic Flow

To properly model concrete, it is necessary to incorporate the phenomenon called *hardening* into the model. This allows the failure surface to expand and change shape as the concrete is plastically loaded. This requires that the failure surface depend on the plastic strain. Determining the amount of plastic strain that has occurred requires a concept in plasticity known as *plastic flow*.

As previously stated, the failure surface defines the boundary of elastic deformation. When the stress state reaches the failure surface, further loading induces plastic flow. While, by definition of a failure surface, the stress state must stay on the failure surface, due to the presence of hardening in the model, the failure surface can move or change shape due to the plastic flow. However, rules must be established to determine the behavior of this plastic flow. Similar to the failure surface in stress space, a plastic potential function, Q, is defined in strain space. While the potential function is considered to lie in the strain space, the stress and strain variables are commonly thought of as being interchangeable. First, the plastic strain increment is defined:

$$d\epsilon_{ij}^{p} = d\lambda \frac{\partial Q}{\partial \sigma_{ij}} \tag{2.14}$$

The scalar $d\lambda$ represents a proportionality coefficient that can change with loading. The total plastic strain is used to determine the total strain:

$$\epsilon_{ij} = \epsilon_{ij}^{e} + \epsilon_{ij}^{p} \tag{2.15}$$

The elastic strain, ϵ_{ij}^{e}, is determined in the traditional fashion using a generalized Hooke's Law:

$$\sigma_{ij} = D_{ijkl}^{e} \epsilon_{kl}^{e} \tag{2.16}$$

The variable D_{ijkl}^{e} represents the elastic modulus tensor of the material.

A common approach, known as associated plasticity, is to define the plastic potential function to have the same shape as the current failure surface. As will be discussed in Section 3.3, associated plasticity does not correctly predict the behavior of concrete and, therefore, this thesis will use non-associated plasticity with the plastic potential function having a different shape than the current failure surface.

The loading history of the material should affect the location of the failure surface. Within this thesis, this is accomplished through the use of the effective plastic strain increment. The movement of the failure surface will be directly controlled by the effective plastic strain increment:

$$\overline{d\epsilon^p} = \sqrt{\frac{2}{3} d\epsilon^p_{ij} d\epsilon^p_{ij}} \tag{2.17}$$

The introduction of the $\frac{2}{3}$ multiplier comes from requiring a von Mises type material model to satisfy the uniaxial compressive stress test. While the model presented in this thesis shows little resemblance to a von Mises model, this particular definition for effective plastic strain is often used and was, therefore, selected for use here. The effective plastic strain increment is used to control the hardening behavior of the failure surface, as discussed in Section 3.2.

2.4 Consistency Condition

The consistency condition is a mathematical expression of the requirement that the stress state stay on the failure surface as long as loading continues, even though the failure surface itself will be moving and changing shape due to hardening.

$$F + dF = 0 \tag{2.18}$$

However, recall that the failure surface is defined such that $F = 0$ is the onset of plastic flow. Thus, the above equation can be simplified:

$$dF = 0 \tag{2.19}$$

The failure surface in this thesis is written as a function of the stress and the effective plastic strain. Thus, the above condition can be rewritten a final time:

$$dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \overline{\epsilon^p}} d\overline{\epsilon^p} = 0 \tag{2.20}$$

This condition will be used in Section 3.5 to determine the tangent modulus tensor. The tangent modulus tensor is necessary to define the relationship between the stress and strain increments, as will be discussed in Section 3.5.