Appendix A

Partial Derivatives Required for Equation 3.14

This appendix consists of the partial derivatives necessary for calculating the tangent modulus tensor as laid out in Section 3.5 and Equation 3.14. The equations from which these derivatives are determined are all given in Chapter 3.

$$\frac{\partial Q}{\partial \sigma_{ij}} = \omega \frac{\partial F}{d\sigma_{ij}} + (1-\omega)\frac{\partial r}{\partial \sigma_{ij}} \tag{A.1}$$

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial \sigma_{ij}} + \frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial \sigma_{ij}} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial \sigma_{ij}} \tag{A.2}$$

$$\frac{\partial F}{\partial \overline{\epsilon^p}} = \frac{\partial F}{\partial \beta}\frac{\partial \beta}{\partial \psi}\frac{\partial \psi}{\partial \overline{\epsilon^p}} \tag{A.3}$$

$$\frac{\partial F}{\partial r} = 1 \tag{A.4}$$

$$\frac{\partial F}{\partial \xi} = -\begin{cases} \beta\left(\frac{\partial R_{peak}(\xi,\theta)}{\partial \xi} - \frac{\partial R_{yield}(\xi,\theta)}{\partial \xi}\right) + \frac{\partial R_{yield}(\xi,\theta)}{\partial \xi} & \psi \leq \psi_{peak} \\ \beta\left(\frac{\partial R_{peak}(\xi,\theta)}{\partial \xi} - \frac{\partial R_{residual}(\xi,\theta)}{\partial \xi}\right) + \frac{\partial R_{residual}(\xi,\theta)}{\partial \xi} & \psi > \psi_{peak} \end{cases} \tag{A.5}$$

$$\frac{\partial F}{\partial \theta} = -\begin{cases} \beta\left(\frac{\partial R_{peak}(\xi,\theta)}{\partial \theta} - \frac{\partial R_{yield}(\xi,\theta)}{\partial \theta}\right) + \frac{\partial R_{yield}(\xi,\theta)}{\partial \theta} & \psi \leq \psi_{peak} \\ \beta\left(\frac{\partial R_{peak}(\xi,\theta)}{\partial \theta} - \frac{\partial R_{residual}(\xi,\theta)}{\partial \theta}\right) + \frac{\partial R_{residual}(\xi,\theta)}{\partial \theta} & \psi > \psi_{peak} \end{cases} \tag{A.6}$$

$$\frac{\partial F}{\partial \beta} = -\begin{cases} R_{peak}(\xi,\theta) - R_{yield}(\xi,\theta) & \psi \leq \psi_{peak} \\ R_{peak}(\xi,\theta) - R_{residual}(\xi,\theta) & \psi > \psi_{peak} \end{cases} \tag{A.7}$$

$$\frac{\partial \beta}{\partial \psi} = \frac{\kappa e^{1-\left(\frac{\psi}{\psi_{peak}}\right)^{\kappa}}\left[1-\left(\frac{\psi}{\psi_{peak}}\right)^{\kappa}\right]}{\psi_{peak}\left(\frac{\psi}{\psi_{peak}}\right)^{1-\kappa}} \tag{A.8}$$

$$\frac{\partial \psi}{\partial \overline{\epsilon^{p}}} = \frac{1}{\phi + \alpha(|\frac{\xi}{f_c'}|^{\gamma})} \tag{A.9}$$

$$
\begin{aligned}
\frac{\partial R_i(\xi,\theta)}{\partial \xi} = &\frac{1}{4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + (r_{c,i} - 2r_{t,i})^2}\Bigg\{2\frac{dr_{c,i}}{d\xi}(r_{c,i}^2 - r_{t,i}^2)cos(\theta) \\
&+ 2r_{c,i}(2r_{c,i}\frac{dr_{c,i}}{d\xi} - 2r_{t,i}\frac{dr_{t,i}}{d\xi})cos(\theta) \\
&+ \frac{dr_{c,i}}{d\xi}(2r_{t,i} - r_{c,i})\sqrt{4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + 5r_{t,i}^2 - 4r_{t,i}r_{c,i}} \\
&+ r_{c,i}(2\frac{dr_{t,i}}{d\xi} - \frac{dr_{c,i}}{d\xi})\sqrt{4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + 5r_{t,i}^2 - 4r_{t,i}r_{c,i}} \\
&+ \frac{1}{2\sqrt{4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + 5r_{t,i}^2 - 4r_{t,i}r_{c,i}}} \\
&\Big[r_{c,i}(2r_{t,i} - r_{c,i})(4(2r_{c,i}\frac{dr_{c,i}}{d\xi} - 2r_{t,i}\frac{dr_{t,i}}{d\xi})cos^2(\theta) \\
&+ 10r_{t,i}\frac{dr_{t,i}}{d\xi} - 4\frac{dr_{t,i}}{d\xi}r_{c,i} - 4r_{t,i}\frac{dr_{c,i}}{d\xi})\Big]\Bigg\} \\
&- \frac{1}{\Big[4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + (r_{c,i} - 2r_{t,i})^2\Big]^2}\Bigg\{\Big[2r_{c,i}(r_{c,i}^2 - r_{t,i}^2)cos(\theta) \\
&+ r_{c,i}(2r_{t,i} - r_{c,i})\sqrt{4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + 5r_{t,i}^2 - 4r_{t,i}r_{c,i}}\Big] \\
&\Big[4(2r_{c,i}\frac{dr_{c,i}}{d\xi} - 2r_{t,i}\frac{dr_{t,i}}{d\xi})cos^2(\theta) + 2(r_{c,i} - 2r_{t,i})(\frac{dr_{c,i}}{d\xi} - 2\frac{dr_{t,i}}{d\xi})\Big]\Bigg\}
\end{aligned}
\tag{A.10}
$$

$i =$ yield, peak, residual

$$\frac{\partial R_i(\xi,\theta)}{\partial\theta} = \frac{1}{4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + (r_{c,i} - 2r_{t,i})^2}\Bigg\{-2r_{c,i}(r_{c,i}^2 - r_{t,i}^2)sin(\theta)$$
$$- \frac{1}{\sqrt{4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + 5r_{t,i}^2 - 4r_{t,i}r_{c,i}}}$$
$$\Big[4r_{t,i}(2r_{t,i} - r_{c,i})(r_{c,i}^2 - r_{t,i}^2)sin(\theta)cos(\theta)\Big]\Bigg\}$$
$$+ \frac{1}{\Big[4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + (r_{c,i} - 2r_{t,i})^2\Big]^2} \tag{A.11}$$
$$\Bigg\{8\Big[2r_{c,i}(r_{c,i}^2 - r_{t,i}^2)cos(\theta) + r_{c,i}(2r_{t,i} - r_{c,i})$$
$$\sqrt{4(r_{c,i}^2 - r_{t,i}^2)cos^2(\theta) + 5r_{t,i}^2 - 4r_{t,i}r_{c,i}}\Big](r_{c,i}^2 - r_{t,i}^2)sin(\theta)cos(\theta)\Bigg\}$$

i = yield, peak, residual

$$\frac{dr_{t,i}}{d\xi} = \frac{\frac{1}{2}a_{1,i} + a_{2,i}\frac{\xi}{f'_c}}{\sqrt{a_{0,i} + a_{1,i}\frac{\xi}{f'_c} + a_{2,i}\left(\frac{\xi}{f'_c}\right)^2 + \frac{1}{4}a_{3,i}^2}} \qquad i = \text{peak, residual} \tag{A.12}$$

$$\frac{dr_{c,i}}{d\xi} = \frac{\frac{1}{2}b_{1,i} + b_{2,i}\frac{\xi}{f'_c}}{\sqrt{b_{0,i} + b_{1,i}\frac{\xi}{f'_c} + b_{2,i}\left(\frac{\xi}{f'_c}\right)^2 + \frac{1}{4}b_{3,i}^2}} \qquad i = \text{peak, residual} \tag{A.13}$$

$$\frac{dr_{t,yield}}{d\xi} = -\frac{c}{\left(1 - \frac{\xi}{f'_c}\right)^2} \tag{A.14}$$

$$\frac{dr_{c,yield}}{d\xi} = -\frac{d}{\left(1 - \frac{\xi}{f'_c}\right)^2} \tag{A.15}$$

$$\frac{\partial r}{\partial\sigma_{ij}} = \frac{1}{\sqrt{2J_2}}\frac{\partial J_2}{\partial\sigma_{ij}} \tag{A.16}$$

$$\frac{\partial \xi}{\partial \sigma_x} = \frac{\partial \xi}{\partial \sigma_y} = \frac{\partial \xi}{\partial \sigma_z} = \frac{1}{\sqrt{3}} \tag{A.17}$$

$$\frac{\partial \xi}{\partial \tau_{xy}} = \frac{\partial \xi}{\partial \tau_{yz}} = \frac{\partial \xi}{\partial \tau_{xz}} = 0 \tag{A.18}$$

$$\frac{\partial \theta}{\partial \sigma_{ij}} = \frac{\partial \theta}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial \theta}{\partial J_3}\frac{\partial J_3}{\partial \sigma_{ij}} \tag{A.19}$$

$$\frac{\partial \theta}{\partial J_2} = \frac{3\sqrt{3}}{4}\frac{J_3}{J_2^{\frac{5}{2}} sin\left(3\theta\right)} \tag{A.20}$$

$$\frac{\partial \theta}{\partial J_3} = -\frac{\sqrt{3}}{2}\frac{1}{J_2^{\frac{3}{2}} sin\left(3\theta\right)} \tag{A.21}$$

$$\frac{\partial J_2}{\partial \sigma_x} = \frac{1}{3}\left(2\sigma_{xx} - \sigma_{yy} - \sigma_{zz}\right) \tag{A.22}$$

$$\frac{\partial J_2}{\partial \sigma_y} = \frac{1}{3}\left(-\sigma_{xx} + 2\sigma_{yy} - \sigma_{zz}\right) \tag{A.23}$$

$$\frac{\partial J_2}{\partial \sigma_z} = \frac{1}{3}\left(-\sigma_{xx} - \sigma_{yy} + 2\sigma_{zz}\right) \tag{A.24}$$

$$\frac{\partial J_2}{\partial \tau_{xy}} = 2\tau_{xy} \tag{A.25}$$

$$\frac{\partial J_2}{\partial \tau_{yz}} = 2\tau_{yz} \tag{A.26}$$

$$\frac{\partial J_2}{\partial \tau_{xz}} = 2\tau_{xz} \tag{A.27}$$

$$\frac{\partial J_3}{\partial \sigma_x} = \frac{1}{9}\left[4\sigma_y\sigma_z - 2\sigma_x\left(-\sigma_x + \sigma_y + \sigma_z\right) - \left(\sigma_y^2 + \sigma_z^2\right)\right] - \frac{1}{3}\left[2\tau_{yz}^2 - \left(\tau_{xy}^2 + \tau_{xz}^2\right)\right] \tag{A.28}$$

$$\frac{\partial J_3}{\partial \sigma_y} = \frac{1}{9}\left[4\sigma_x\sigma_z - 2\sigma_y\left(\sigma_x - \sigma_y + \sigma_z\right) - \left(\sigma_x^2 + \sigma_z^2\right)\right] - \frac{1}{3}\left[2\tau_{xz}^2 - \left(\tau_{xy}^2 + \tau_{yz}^2\right)\right] \quad \text{(A.29)}$$

$$\frac{\partial J_3}{\partial \sigma_z} = \frac{1}{9}\left[4\sigma_x\sigma_y - 2\sigma_z\left(\sigma_x + \sigma_y - \sigma_z\right) - \left(\sigma_x^2 + \sigma_y^2\right)\right] - \frac{1}{3}\left[2\tau_{xy}^2 - \left(\tau_{xz}^2 + \tau_{yz}^2\right)\right] \quad \text{(A.30)}$$

$$\frac{\partial J_3}{\partial \tau_{xy}} = 2\tau_{xz}\tau_{yz} + \frac{2}{3}\tau_{xy}\left(\sigma_x + \sigma_y - 2\sigma_z\right) \quad \text{(A.31)}$$

$$\frac{\partial J_3}{\partial \tau_{yz}} = 2\tau_{xy}\tau_{xz} + \frac{2}{3}\tau_{yz}\left(-2\sigma_x + \sigma_y + \sigma_z\right) \quad \text{(A.32)}$$

$$\frac{\partial J_3}{\partial \tau_{xz}} = 2\tau_{xy}\tau_{yz} + \frac{2}{3}\tau_{xz}\left(\sigma_x - 2\sigma_y + \sigma_z\right) \quad \text{(A.33)}$$

Note that as $\theta \to 60°$, $\frac{\partial R_i(\xi,\theta)}{\partial \theta} \to 0$ while $\frac{\partial \theta}{\partial J_2} \to \infty$ and $\frac{\partial \theta}{\partial J_3} \to \infty$. Therefore, the product $\frac{\partial R_i(\xi,\theta)}{\partial \theta}\frac{\partial \theta}{\partial J_{2,3}}$ should go to zero. However, due to numerical roundoff, this isn't guaranteed. Therefore, for $\theta \simeq 60°$, a linearization is used for this product.

For $\theta \simeq 60°$:

$$\frac{\partial R_i(\xi,\theta)}{\partial \theta}\frac{\partial \theta}{\partial J_2} = \frac{3\sqrt{3}r_c\left(r_c^2 - r_t^2\right)}{4\left(2r_t - r_c\right)^2}\frac{J_3}{J_2^{\frac{5}{2}}} \quad \text{(A.34)}$$

$$\frac{\partial R_i(\xi,\theta)}{\partial \theta}\frac{\partial \theta}{\partial J_3} = -\frac{\sqrt{3}r_c\left(r_c^2 - r_t^2\right)}{2\left(2r_t - r_c\right)^2 J_2^{\frac{3}{2}}} \quad \text{(A.35)}$$