Nonlinear Optimal Control:
An Enhanced Quasi-LPV Approach

Thesis by

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To My Parents

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Abstract

Realistic models of physical systems are often nonlinear. Our objective is to synthesize controllers for nonlinear systems that not only provide stability, but also deliver good closed-loop performance.

The frozen Riccati equation approach is thoroughly examined. Although it suffers fundamental deficiencies due to its pointwise nature, it is proven that optimality is always possible under a certain assumption on the optimal value of the performance index. This is a consequence of the non-uniqueness of the pointwise linear model of the nonlinear dynamics. However, one cannot assess a priori the guaranteed global performance for a particular model choice.

An alternative to the pointwise design is to treat nonlinear plants as linear parameter varying systems with the underlying parameters being functions of the state variables. By exploiting the variation rate bounds of the parameters, a controller that smoothly schedules on the parameters can be synthesized by solving a convex optimization problem. Depending upon the choice of the variation rate bounds, the resulting controller can range from replicating the pointwise design result, which comes with no guarantee on performance, to providing quadratic stability, in which case it can withstand arbitrarily fast parameter variation.

Under the above quasi-LPV framework, we present a new scheme that incorporates the freedom of choosing the state-dependent linear representation into the control design process. It is shown that the $\mathcal{L}_2$-gain analysis can be reformulated as an infinite dimensional convex optimization problem, and an approximate solution can be obtained by solving a collection of linear matrix inequalities. The synthesis problem is cast as a minimization over an infinite dimensional bilinear matrix inequality constraint. An iterative algorithm, similar to the "$D-K$ iteration" for μ synthesis, is proposed to compute the best achievable performance. It is demonstrated through several examples that this approach can effectively reduce conservatism of the overall design.

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Chapter 1 Introduction

The goal of a control designer is to synthesize controllers for dynamical systems that can achieve certain performance objectives in the presence of disturbances. Modern control theory formulates this task as an optimization problem. Typically, a controller is selected that optimizes a performance index which embodies the desirable objectives.

Optimal control theory for linear time-invariant systems has seen great advances over the last 15 years. Adding to the classical $\mathcal{H}_2$ theory (see [1] for example), elegant formulations of minimizing the $\mathcal{H}_\infty$ [13] and l_1 [11] norms of various closed-loop transfer functions have been developed. The introduction of the structured singular value μ [12] formalized a robust stability index with respect to structured uncertainty. With computationally efficient solutions, these tools have gained wide acceptance among control engineers, and have been successfully applied to solve difficult practical problems.

On the other hand, the nonlinear counterpart has remained mainly at the theoretical level. In practice, control design of nonlinear systems is usually done in an *ad hoc* manner using the linear theory. Unlike linear systems whose behavior can be deduced from the state space matrices, properties of a general nonlinear system cannot be expressed as functions in a finite dimensional space. Therefore, establishing global properties, e.g., obtaining sufficient conditions for performance, involves searching an infinite dimensional space. This hinders the development of computationally tractable algorithms.

Brute-force search of the state space exhibits exponential growth with the number of state variables. Since the dimension of a physical system can easily be more than a dozen, it is prohibitively expensive to search the entire state space. However, the nonlinearity can usually be captured in a lower dimensional space. If the control selection can be conducted over this reduced space, a thorough search may be feasible.

The intrinsic difference between linear and nonlinear systems is that efficient computational algorithms which guarantee near optimal global performance are unlikely to emerge for any general class of nonlinear plants. Nevertheless, some special features that are specific to a certain class of systems may lead to a computationally attractive solution for that class. Therefore, it is very important for us to recognize the strength and weakness of each design methodology.

The usefulness of a well developed linear theory does not stop at linear systems. It has been demonstrated that it can be generalized to handle much broader classes of systems effectively. However, this usually involves a substantial trade off between stability and performance and the computational complexity. We believe that it is advantageous that designers have a systematic way to treat such a trade off as a design parameter.

These considerations of nonlinear control design will be reflected in our development of a new framework. Practical issues like plant uncertainty and controller structural constraints are certainly important, but we will focus on nonlinearity alone as the source of difficulty in this thesis.

1.1 Previous Work

The beginning of optimal control theory, in its modern sense, can be traced back to the 1950s. The performance index initially considered admits a simple interpretation in terms of the control effort and the error. Two different approaches were developed. Dynamic programming, first proposed by Bellman, results in Hamilton-Jacobi partial differential equations [1]. Pontryagin Maximum principle, on the other hand, leads to Euler-Lagrange equations [24, 7]. Since the Hamilton-Jacobi equation (HJE) provides an optimal control law in the feedback form, it is commonly used to characterize the solution to the nonlinear optimal control problem.

Motivated by the breakthrough of $\mathcal{H}_\infty$ (sub)optimal control for linear systems, the nonlinear $\mathcal{H}_\infty$ control has been heavily studied since late 80s. It was shown by several people that sufficient conditions for the existence of global solutions can also be given in terms of Hamilton-Jacobi equations/inequalities [4, 34].

Although the Hamilton-Jacobi equation has been demonstrated to be a powerful tool for nonlinear optimal control theory, its usefulness in practical control design is hampered by the associated computational burden. Different synthesis strategies for nonlinear systems have been developed which aim at achieving the desirable properties while avoiding such computations.

The Lyapunov design techniques rely on finding a control Lyapunov function (CLF) for the nonlinear system, which is an extension of the classical Lyapunov function [25, 26]. Based on the CLF, a control law is then constructed to shape the achievable closed-loop behavior [33, 16]. Although a CLF always exists whenever the system is stabilizable, the task of constructing one may not be feasible except for systems having particular structures, e.g., feedback linearizable, strict feedback and feedforward systems. Even if a CLF is found, it is usually difficult to incorporate the pre-specified performance criterion into the control design, and the performance of the resulting closed-loop system can vary widely. In spite of these difficulties, the Lyapunov design guarantees global stability, which cannot be achieved by many other techniques.

Linear theory has been extended to synthesize controllers for nonlinear systems in a number of ways. Gain scheduling and its variant forms are the most commonly used techniques.

Gain scheduling is an engineering practice used to control nonlinear plants. Traditionally, the design is based on a collection of linear time-invariant approximations to a nonlinear plant at fixed operating points. A linear compensator is constructed for each linearized model, and a global controller is obtained by interpolating the gains of the local compensators. As a pointwise design method, it comes with no guarantee on global performance, or even stability [32].

As a new member to the gain scheduling family, the frozen Riccati equation (FRE) approach was proposed by Cloutier [9]. A similar scheme for nonlinear $\mathcal{H}_\infty$ control was formulated in terms of nonlinear matrix inequalities (NLMI's) by Lu [28]. They deviate from the traditional gain scheduling in that a single state-dependent linear model is used. The implication of this model on the achievable performance will be thoroughly investigated in the thesis.

In contrast to the pointwise design, the quasi linear parameter varying (LPV) approach treats nonlinear plants as linear systems with varying parameters that depend upon state variables. When the varying parameters enter the linear state space structure in an LFT (linear fractional transformation) fashion, a controller smoothly scheduled by the parameters can be derived by considering the parameter variation as uncertainty in the linear model [31, 27, 2]. A recent result [39] illustrates that the explicit information of the variation rate of scheduling parameters can also be incorporated in this framework. Although guaranteeing closed-loop performance, these LPV controllers are apt to be conservative for nonlinear systems. The new concept introduced in this thesis, which attempts to reduce conservatism, is an enhancement to the existing quasi-LPV framework.

1.2 Thesis Outline

How to effectively generalize the well developed linear control theory to nonlinear systems to the maximum degree is the main theme of this thesis. In the area of nonlinear optimal control, the frozen Riccati method has recently gained popularity among some of the control engineers due to its computational advantage over other techniques. This *ad hoc* practice utilizes the state-dependent linear representation of the input-affine nonlinear system, and reduces the Hamilton-Jacobi equation to a pointwise Riccati equation. Though suffering from a lack of theoretical justification, it turns out to be very effective in many cases. Our investigation has ensured the intuition that its failure in providing any guarantee of global stability is caused by ignoring the connection between the parameters used in the linear representation and the state variables. Nevertheless, noticing the non-uniqueness of the linear representation, we have proved that there always exists such choice that the FRE recovers the optimal control under some mild assumption on the solution to the HJE, though finding the right choice is as hard as solving the HJE. One of the contributions of this thesis is the theoretical justification of the existence of the optimal representation of the FRE method. It provides an explanation of the persistently good performance in so many low dimensional examples.

While the FRE is a primitive method that uses linear techniques to tackle nonlinear systems, the problems associated with it also arise in other more sophisticated techniques, such as the linear parameter varying (LPV) scheme. As is revealed in our research, a close connection exists between the FRE and LPV methods. Both can be understood as the search for a Lyapunov function without gridding the entire state space, however the LPV design guarantees stability at a price of heavy computation and potentially severe conservatism. In addition, the LPV method reduces to the FRE when the rate variation of the parameters is set zero. Recognizing that a critical element in both schemes is the choice of linear state-dependent representation of the nonlinear system, a new scheme is proposed in this thesis that incorporates this design freedom into the control selection process in an optimal fashion. This new development constitutes a significant enhancement to the existing LPV methodology.

Chapter 2 reviews the classical nonlinear optimal control problem and the associated Hamilton-Jacobi equation (HJE). A brief summary of the existing practical methods for solving HJE's is presented. The converse HJE methodology proposed by Doyle *et al.* [15] is explained and employed to generate examples in the later chapters.

Chapter 3 investigates the frozen Riccati equation approach to the nonlinear minimax control problem. It is shown that in addition to the existence of a pointwise stabilizing solution to the FRE, a curl type condition is needed to guarantee global optimal performance. Recognizing the non-uniqueness of the linear state-dependent representation of a nonlinear system, it is proven that there always exists such a representation that the FRE recovers the optimal control provided that the gradient of the value function satisfies a certain condition. However, it is demonstrated through several examples that the natural choice does not necessarily lead to good performance. Sometimes, it may fail to provide mere stability.

In Chapter 4, the linear quadratic regulator theory is generalized to a class of nonlinear input-affine systems. Based on the quasi-LPV representation of the nonlinear plant, an upper bound of the performance criterion is derived in terms of affine matrix inequalities (AMI's). A globally stabilizing state feedback law is formed from the solution of the AMI's. A practical computation scheme is proposed to solve the optimization problem under the infinite dimensional AMI constraints. Finally, a discussion of the connection between the quasi-LPV technique and the FRE approach is presented with illustrative examples.

In Chapter 5, after reviewing the induced $\mathcal{L}_2$-norm control of linear parameter varying systems, we propose an enhanced quasi-LPV approach to the nonlinear $\mathcal{H}_\infty$ control problem. Exploiting the non-uniqueness of the state-dependent linear representation of the nonlinear plant, an extra design freedom is introduced to the standard LPV approach. Under the new quasi-LPV framework, the $\mathcal{L}_2$-gain analysis can be formulated as an infinite dimensional convex optimization problem. Unfortunately, the synthesis can only be cast as an infinite dimensional bilinear matrix inequality (BMI) problem which is not jointly convex in both design variables. However, analogous to the "$D - K$ iteration" for μ synthesis, an iterative procedure is proposed to obtain the best achievable performance.

The results of this thesis are summarized in Chapter 6, followed by a discussion of related future research directions.

1.3 Notation

The notation used in the thesis is fairly standard. The set of real numbers is denoted by $\mathbf{R}$, and $\mathbf{R}^+ = \{a : 0 \leq a \in \mathbf{R}\}$ stands for the set of nonnegative reals. A function is said to be of class $\mathcal{C}^k$ if it is continuously differentiable k times. So in particular, $\mathcal{C}^o$ stands for the class of continuous functions, and $\mathcal{C}^1$ represents the class of continuously differentiable functions. For a $\mathcal{C}^1$ function $V : \mathbf{R}^n \rightarrow \mathbf{R}$, we denote by $\frac{\partial V}{\partial x}$ or V_x the row-vector of partial derivatives.

For a matrix $A \in \mathbf{R}^{m \times n}$, the transpose and conjugate transpose are denoted by A^T and A^* respectively. The largest singular value of A is denoted by $\overline{\sigma}(A)$, and

$\|A\|_F$ stands for the Frobenius norm of A, i.e., $\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}$. The kernel of A is ker$(A)$, and the trace of A is tr$A$ if A is square.

The Euclidean norm of a vector is denoted by $\|\cdot\|$, i.e., $\|z\| = \sqrt{z^T z}$ for $z \in \mathbf{R}^k$. The 2-norm of a signal, $z(t)$, is denoted by $\|z\|_2 = \sqrt{\int_0^\infty \|z(t)\|^2 dt}$. The notation $\mathcal{L}_2$ will be used for bounded energy signals, i.e., $\mathcal{L}_2[0,T] = \{w : \int_0^T \|w(t)\|^2 dt < \infty\}$.

Chapter 2 Hamilton-Jacobi Equations

The problem of designing a controller capable of stabilizing a dynamical system while simultaneously providing good performance with respect to some measurement has been thoroughly explored. The optimal solution leads to a Hamilton-Jacobi partial differential equation. The classical nonlinear optimal control problem is reviewed in this chapter, followed by a brief summary of the existing practical methods for solving Hamilton-Jacobi equations. The converse HJE methodology proposed by Doyle *et al.* [15] is presented.

2.1 Nonlinear Optimal Control

The principle of optimality, commonly referred to as dynamic programming, was first proposed by Bellman in 1952. In the context of optimal control, the well-known Hamilton-Jacobi equation (HJE) was derived using the dynamic programming argument. (see, e.g., [1]).

Consider the general nonlinear dynamics described by

$$\dot{x} = f(x, u, t), \qquad x(t_o) \text{ given.} \tag{2.1}$$

Here $x \in \mathbf{R}^n$ represents the state vector, $u \in \mathbf{R}^m$ denotes the input vector, and $f : (\mathbf{R}^n, \mathbf{R}^m, \mathbf{R}) \to \mathbf{R}^n$ is continuously differentiable in all of its arguments. Define the set of admissible control inputs to be the class of piecewise continuous functions. Therefore, for an admissible input $u(t)$ and a given initial condition $x(t_o)$, equation (2.1) has a unique solution over the control interval of interest. The optimal control problem we consider is to determine an admissible control function u to minimize the cost function

$$V(x(t_o), u(\cdot), t_o) = \theta(x(t_f), t_f) + \int_{t_o}^{t_f} \phi(x(\tau), u(\tau), \tau) d\tau \tag{2.2}$$

where both θ and ϕ are continuously differentiable. V is also called the performance index and is a scalar function of the state motion. Define

$$V^*(x(t), t) = \min_{u[t,t_f]} V(x(t), u(\cdot), t).$$

So $V^*(x(t), t)$, commonly referred to as the value function, is the optimal cost accrued over the interval $[t, t_f]$ of all admissible trajectories starting from $x(t)$ at time t.

We assume that V^* exists, and it is twice continuously differentiable. It can be derived from the dynamical programming principle that V^* has to satisfy the following partial differential equation

$$\frac{\partial V^*}{\partial t} = -\min_{u(t)} [\phi(x(t), u(t), t) + \frac{\partial V^*}{\partial x} f(x(t), u(t), t)] \tag{2.3}$$

with the boundary condition

$$V^*(x(t_f), t_f) = \theta(x(t_f)). \tag{2.4}$$

The pair (2.3) and (2.4) are referred to as the Hamilton-Jacobi or Hamilton-Jacobi-Bellman equation. The derivation can be found in the literature (see, e.g., [1]). The HJE in the above format is not precisely a partial differential equation, but a mixture of a functional and partial differential equation. The value of $u(t)$ minimizing the right-hand side of (2.3) depends on $x(t)$, $\frac{\partial V^*}{\partial t}$ and t, thus it is an instantaneous function of these three variables. Assume the optimal control exists and denote it by $\hat{u}(x(t), \frac{\partial V^*}{\partial t}, t)$, then (2.3) becomes a true first-order partial differential equation

$$\frac{\partial V^*}{\partial t} = -[\phi(x(t), \hat{u}(x(t)\frac{\partial V^*}{\partial t}, t), t) + \frac{\partial V^*}{\partial x} f(x(t), \hat{u}(x(t), \frac{\partial V^*}{\partial t}, t), t)] \tag{2.5}$$

with the boundary condition specified by (2.4).

To determine an optimal control, suppose (2.5) and (2.4) have been solved so that the value function V^* is a known function of x and t, thus we only need to substitute the gradient of V^* with respect to x in $\hat{u}$ by the computed value which should result in a state-feedback form denoted by $u^*(x,t)$.

For convenience, we use a shorthand notation to represent the HJE (2.5)

$$\frac{\partial V^*}{\partial t} + \mathcal{H}(\frac{\partial V^*}{\partial x}, x, t) = 0 \tag{2.6}$$

with

$$\mathcal{H}(\frac{\partial V^*}{\partial x}, x, t) = \phi(x(t), \hat{u}(x(t)\frac{\partial V^*}{\partial t}, t), t) + \frac{\partial V^*}{\partial x} f(x(t), \hat{u}(x(t), \frac{\partial V^*}{\partial t}, t), t).$$

In many control applications, the system dynamics are autonomous, i.e., f in (2.1) is not an explicit function of t. If this is the case, the HJB equation defines a steady state value function $V^*(x)$ provided the optimization time horizon is infinite, which simplifies (2.6) as:

$$\mathcal{H}(\frac{\partial V^*}{\partial x}, x) = 0. \tag{2.7}$$

It should be noted that the Hamilton-Jacobi equation only represents a sufficient condition for the optimal value function. If the solution of the HJE satisfies certain differentiability properties, then it is the desired value function and defines the optimal control. But such a solution need not exist, and not every value function satisfies the HJE, e.g., sometimes V^* is not everywhere differentiable. The viscosity solution to the HJE can be used to address this class of value functions [10]. Although the HJE is quite difficult to solve in general, when it can be solved, a candidate for an optimal control function is found as a function of the state trajectory, which is a highly desired feedback form.

2.2 Solving Hamilton-Jacobi Equations

The Hamilton-Jacobi equation is, in general, a nonlinear partial differential equation, and no efficient algorithm is available for problems with more than a few states. The only exception is when the system dynamics are linear and the performance index is quadratic, in which case the HJE is reduced to a Riccati differential equation and only the initial condition problem needs to be solved. So reducing an optimal control problem to the HJE cannot be considered a practical method.

Because of its significance in optimal control and other applications, the Hamilton-Jacobi partial differential equation and its various forms have always been on the studying list of the PDE community. Different approaches have been proposed to solve them. The following is just a sample of some available numerical approximation methods.

Taylor Series Approach

One straightforward approach is to employ Taylor series expansion to approximate the dynamics of the plant, the performance index and the value function. As a result, the Hamilton-Jacobi partial differential equation is reduced to a collection of algebraic matrix equations. As an example, let's consider the nonlinear regulation problem where the performance index is

$$V = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t))dt$$

with $Q \geq 0$ and $R > 0$. Take the Taylor expansion of the dynamics:

$$\dot{x} = Ax + \mathcal{O}(x,x) + Bu + \mathcal{O}(x,u)$$

where $\mathcal{O}(x,x)$ and $\mathcal{O}(x,u)$ are higher order terms. Assume the value function has the following form

$$V^*(x) = x^T Px + \mathcal{O}(x,x,x)$$

where $\mathcal{O}(x,x,x)$ represents the 3rd order terms. By replacing each term in the HJE by its Taylor expansion, one can get the sufficient matching conditions:

$$\begin{cases} A^T P + PA - PBR^{-1}B^T P + Q = 0 \\ \cdots \end{cases} \tag{2.8}$$

The first equation is the algebraic Riccati equation (ARE), which is the solution to the linear quadratic regulator (LQR) problem. The omitted equations denoted by $\cdots$ above should account for the high order terms of the HJE.

The advantage of this approach is obvious: the PDE is reduced to algebraic equations which are more appealing computationally. The disadvantage is that the number of the algebraic equations increases greatly with the order of the Taylor expansion of the dynamics and the value function. If an expansion in x to the N order is used for a system of dimension n, the number of distinct equations in (2.8) is $\sum_{j=1}^{N} \frac{(n-1+j)!}{(n-1)!j!}$. In spite of its limitations, the Taylor series approach is still a very useful tool for obtaining local solutions. A more detailed treatment can be found in [29, 19].

Finite Difference Scheme

Finite difference method is a common tool for solving partial differential equations. For simplicity of illustration, we take a one-dimensional HJE as an example. Consider solving

$$\frac{\partial V}{\partial t}(x,t) + \mathcal{H}(\frac{\partial V}{\partial x}(x,t), x) = 0, \quad V(x,0) = V_0(x),$$

with $t \in \mathbf{R}^+, x \in \mathbf{R}$.

We first grid the $x - t$ plane letting $\Delta x, \Delta t > 0$ be the mesh sizes and V_i^n be the value of the numerical approximation of $V(x,t)$ at $(i\Delta x, n\Delta t)$ $(i, n \in \mathbf{Z})$. The numerical scheme for the solution of the HJE is

$$V_i^{n+1} = V_i^n - \Delta t \mathcal{H}(\frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x}, i\Delta x).$$

Now if $\lim_{t\to\infty} V(t,x)$ exists, then $V(x) = \lim_{t\to\infty} V(t,x)$ is the solution of the original

HJE (2.7). Some simple numerical examples of computing $\mathcal{L}_2$-gains are provided using finite difference schemes in [23]. It should be noted that the finite difference scheme works only for problems with low dimensions because the computational complexity increases exponentially with the dimension of the state space.

Algebraic Technique

Although the HJE characterizes the exact optimal value of the performance criterion, the computational complexity prevents its direct application. Often times, the designer has to sacrifice some performance for a more appealing computational scheme. If the HJE (2.7) is relaxed to the Hamilton-Jacobi inequality (HJI)

$$\mathcal{H}(\frac{\partial V^*}{\partial x}, x) \leq 0, \tag{2.9}$$

any nonnegative solution provides an upper bound on the solution to the HJE. By further assuming the form of the solution, the HJI can be reduced to some algebraic inequality constraint. In the later chapters, we will study the quasi linear parameter varying scheme, which attempts to solve for the "smallest" solution to the algebraic constraint.

2.3 Converse HJE Methodology

As explained in the previous sections, the determination of the optimal feedback law of general nonlinear optimal control problems leads to Hamilton-Jacobi equations. Unfortunately, solving partial differential equations cannot be viewed as a general method for the control design due to the exponential growth of the computation effort involved in terms of the state dimension. Various alternative methods have been developed, which generally give up optimality in favor of reduced computational complexity. With the true optimal solution unknown, it is hard to form a fair comparison of these methods.

The so-called converse HJE methodology was proposed by Doyle *et al.* in [15].

The motivation is to further the understanding of nonlinear control design, and particularly the relationship between the popular nonlinear control design techniques. The different methodologies should be evaluated through benchmark examples that are generated independently from the methodologies themselves. Moreover, the benchmark problems should "appear" complicated enough that they can challenge the design methodologies while their simplicity is hidden from the designer.

The problem is defined as: Given a performance defined as (2.2) and a value function $V : \mathbf{R}^n \to \mathbf{R}^+$, find a class of (nonlinear) dynamics such that the optimal control problem has this as its solution. This converse problem is also characterized by the HJE, but its role is reversed. The converse problem requires only solving the HJE as an algebraic equation in the unknowns f and g with V given.

To illustrate the converse approach, we will present an example of 2-D oscillator.

Example 2.1 *Consider the following dynamics:*

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x) + g(x)u \end{cases}$$

Suppose the performance index to be minimized is

$$\int_0^\infty (x_2^2(t) + u^2(t))dt$$

and the value function is $V^*(x) = x_1^2 + x_2^2$. The corresponding HJE is then

$$\frac{\partial V^*}{\partial x_1}x_2 + \frac{\partial V^*}{\partial x_2}f(x) - \frac{1}{4}(\frac{\partial V^*}{\partial x_2}g(x))^2 + x_2^2 = 0 \tag{2.10}$$

which can be simplified as

$$2x_1x_2 + 2x_2f(x) - \frac{1}{4}(2x_2)^2g^2(x) - x_2^2 = 0.$$

Thus, f and g have to satisfy the equation

$$f(x) = -x_1 - \frac{1}{2}x_2(1 - g^2(x))$$

for this optimal control problem to have V^* as its solution. This leads to the following open loop dynamics

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \frac{1}{2}x_2(1 - g^2(x)) + g(x)u \end{cases}$$

leaving $g(x)$ as a free variable. The optimal control law is given by

$$u^*(x) = -g(x)x_2.$$

Note that we are able to constrain the form of the dynamics for which the assumed value function corresponds to the solution of the optimal control problem. A special case of this type of oscillator is when $g(x) = x_1$, which yields a Van der Pol oscillator with a stable but linearly uncontrollable equilibrium at the origin and an unstable limit cycle. By choosing different functions for $g(x)$, other interesting oscillators can be constructed with known optimal controllers. Another observation of this example is that the choice of the value function $V^*(x)$ and the performance index are constrained by the specified part of the dynamics. To be more specific, the value function has to make $(\frac{\partial V^*}{\partial x_1}x_2 + x_2^2)/\frac{\partial V^*}{\partial x_2}$ well defined for all x.

The converse problem helps to construct an array of examples with possibly high state dimensions, yet for which the optimal controllers are known. Even with quadratic costs and quadratic value functions we are able to make the dynamics carry interesting nonlinearities. This is particularly appealing since this thesis will focus on nonlinearity alone as the source of difficulty. Actually, most of the examples presented in the thesis are generated using this methodology. Although the examples created this way are artificial, it is possible to generate problems of similar characteristics but with explicit physical motivations. However, it is clear that the converse HJE methodology is not useful for actual control design where we have no information about the optimal value of the performance index.

Chapter 3 Frozen Riccati Equation Approach

3.1 Introduction

As reviewed in the previous chapter, the exact solution of the nonlinear optimal control problem leads to the Hamilton-Jacobi equation which cannot be solved efficiently except for some restrictive or low dimensional cases. Most commonly used nonlinear control techniques, on the other hand, aim at providing global (or semi-global) stability, allowing the performance of the resulting closed-loop system to vary widely. The frozen Riccati equation (FRE) method, also called the state-dependent Riccati equation method, has recently been promoted by Cloutier *et al.* [9] intending to synthesize controllers that deliver near optimal performance. As a direct generalization of the linear quadratic theory, the FRE approach relies on a state-dependent linear representation of the input-affine nonlinear system. The control law is obtained by solving the Riccati equation associated with the linear representation along the state trajectory in a pointwise (or "frozen") fashion. It can be viewed as a type of gain scheduled design. Due to its computational simplicity and the potential of on-line implementation, it has been applied to some engineering problems. Despite its success in applications, the performance of FRE controllers can vary from optimal to arbitrarily poor, which makes it remain as an *ad hoc* design methodology.

In this chapter, we will try to build a theoretical foundation for the frozen Riccati approach. The fundamental limitations due to its pointwise nature will be examined. It will be shown that in addition to the existence of a pointwise stabilizing solution to the FRE, a curl type condition is needed to guarantee global optimal performance. The main result is based on the observation of the non-uniqueness of the linear state-dependent representation of a nonlinear system. It will be proven that there always exists such a representation that the FRE recovers the optimal control provided that the gradient of the value function satisfies a certain condition. Although unable to provide a systematic way of spotting the optimal representation in the FRE design paradigm, the theoretical findings point out a direction for improving some other design techniques that employ the same representation of nonlinear systems, which will be presented in Chapter 5.

3.2 The Nonlinear Minimax Problem

In this chapter, the focus will be on the nonlinear infinite horizon optimal control problem. In particular, a class of input-affine nonlinear time-invariant systems are considered

$$\dot{x}(t) = f(x(t)) + g_1(x(t))w(t) + g_2(x(t))u(t), \qquad x(0) = x_o \tag{3.1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^r$ is the control input and $w(t) \in \mathbf{R}^p$ represents the exogenous disturbance entering the system. Here $f(\cdot) \in \mathcal{C}^1$ is the drift vector field with the origin as the equilibrium, i.e, $f(0) = 0$. Both g_1 and g_2 are assumed to be continuous functions of x. The performance objective is to determine a control strategy that minimizes the given performance index under the worst possible disturbance:

$$\inf_{u(t)} \sup_{w(t) \in \mathcal{L}_2[0,\infty)} J(u,w) \tag{3.2}$$

$$J(u,w) = \int_0^\infty (q(x(t)) + u^T(t)u(t) - \gamma^2 w^T(t)w(t))dt \tag{3.3}$$

with $q(x) \geq 0$ for all x and $q(0) = 0$. The allowable disturbance class is assumed to be $\mathcal{L}_2$, i.e., finite power signals. The state x is assumed to be directly available to the control input u. The above problem can be viewed as a zero-sum differential game with minimizing player u and maximizing player w. The detailed treatment of the linear case using game theory can be found in [5].

The Hamilton-Jacobi equation associated with this problem is

$$0 = \min_u \max_w \{V_x(f(x) + g_1(x)w + g_2(x)u) + q(x) + u^T u - \gamma^2 w^T w\}. \tag{3.4}$$

By a simple completion of the square argument, equation (3.4) can be reformulated as

$$0 = \min_u \max_w \{\|u + \frac{1}{2} g_2^T V_x^T\|^2 - \gamma^2 \|w - \frac{1}{2\gamma^2} g_1^T V_x^T\|^2 + V_x f + \frac{1}{4} V_x (\frac{1}{\gamma^2} g_1 g_1^T - g_2 g_2^T) V_x^T + q\}. \tag{3.5}$$

It is clear that the minimizing control should be

$$u^*(x) = -\frac{1}{2} g_2^T(x) V_x^T(x), \tag{3.6}$$

and the worst case disturbance is

$$w^*(x) = \frac{1}{2\gamma^2} g_1^T(x) V_x^T(x). \tag{3.7}$$

Note that u^* and w^* satisfy a saddle point condition for the minimax problem. And w^* is the worst disturbance input in the sense that it maximizes the quantity $J(u, w)$ in (3.3) for $u = u^*$, i.e., w^* solves $\max_{w \in \mathcal{L}_2} J(u^*, w)$.

Substituting u^* and w^* back into (3.5), we obtain the corresponding Hamilton-Jacobi partial differential equation

$$V_x(x) f(x) + \frac{1}{4} V_x(x) (\frac{1}{\gamma^2} g_1(x) g_1^T(x) - g_2(x) g_2^T(x)) V_x^T(x) + q(x) = 0. \tag{3.8}$$

It provides a sufficient condition to the solution of the original minimax control problem.

In order to show the internal stability of the closed-loop system achieved by the optimal controller (3.6), we need to recall a notion of detectability. Assume $h(x)$ is the output of the system (3.1) we are interested in, and $h(0) = 0$.

Definition 3.1 *The pair* $\{f(x), h(x)\}$ *is (zero-state)* **detectable** *if for any integral* *curve* $x(t)$ *of* $\dot{x} = f(x)$, $h(x(t)) = 0$ *for all* $t \geq 0$ *implies* $\lim_{t\to\infty} x(t) = 0$.

The following lemma establishes the asymptotic stability of the closed-loop system with $w = 0$. Similar results can be found in [30, 34, 21].

Lemma 3.2 *Suppose there exists a* C^1 *solution* $V(x) \geq 0$ *to the HJE (3.8) associated with the minimax problem (3.3). Further assume that* $V(x)$ *is proper (i.e., for each* $c > 0$ *the set* $\{x : 0 \leq V(x) \leq c\}$ *is compact). If* $\{f(x), q(x)\}$ *is zero-state detectable, then the closed-loop system under the optimal control law (3.6) is globally asymptotically stable at the origin with* $w = 0$.

Proof We consider $V(x)$ as a candidate Lyapunov function. Along the trajectories of the closed-loop system of (3.1) with $w = 0$ and u is given by (3.6), the rate of change of V is given by

$$\begin{aligned}
\dot{V} &= V_x(x)(f(x) + g_2(x)u^*(x)) \\
&= V_x(x)(f(x) - \frac{1}{2}g_2(x)g_2^T(x)V_x^T(x)) \\
&= -\frac{1}{4}V_x(x)g_2(x)g_2^T(x)V_x^T(x) - \frac{1}{4\gamma^2}V_x(x)g_1(x)g_1^T(x)V_x^T(x) - q(x)
\end{aligned}$$

The last equality comes from the HJE (3.8) that V satisfies. Since $q(x) \geq 0$, V is clearly non-positive along any trajectory. To conclude the asymptotic stability, we will use LaSall's invariance principle [25]. Suppose that $\dot{V}$ is identically zero along a trajectory starting from the initial state x_o. Clearly, $V_x(x)g_2(x)g_2^T(x)V_x^T(x)$ has to be zero along this trajectory by observation. As a result, u^* given by (3.6) has to be zero as well. Therefore, the trajectories of the closed-loop system (3.1) under control u^* with $w = 0$ are the same as those of the free system $\dot{x} = f(x)$. On the other hand, the assumption $\dot{V}$ is identically zero also implies $q(x)$ is identically zero. So following LaSall's invariance principle, we can conclude the global asymptotic stability at zero from the detectability assumption of the pair $\{f(x), q(x)\}$. □

This minimax framework can accommodate a number of various interesting problems. We now proceed with two special cases of the control problem.

3.2.1 The Nonlinear Optimal Regulator Problem

One special case of the minimax problem is when $g_1 = 0$, i.e., there is no disturbance coming into the system. Then the performance index (3.3) becomes

$$V(x) = \min_{u(t)} \int_0^\infty (\, q(x) + u^T u \,) dt. \tag{3.9}$$

The associated Hamilton-Jacobi equation is therefore

$$\frac{\partial V}{\partial x}(x) f(x) - \frac{1}{4}\frac{\partial V}{\partial x}(x) g_2(x) g_2^T(x) \frac{\partial V^T}{\partial x}(x) + q(x) = 0, \tag{3.10}$$

and the optimal control can be constructed from its solution

$$u^* = -\frac{1}{2} g_2^T(x) \frac{\partial V}{\partial x}. \tag{3.11}$$

It is also worth noticing that the solution to the regulator problem corresponds to that of the minimax problem with $\gamma \rightarrow \infty$. By Lemma 3.2, the existence of a proper positive definite solution to the HJE (3.10) implies the global asymptotic stability of the closed-loop system under the state feedback law (3.11) if $\{f(x), q(x)\}$ is zero-state detectable.

3.2.2 Nonlinear $\mathcal{H}_\infty$ Suboptimal Control Problem

If we further assume $q(x)$ in the performance index (3.3) has the following form

$$q(x) = h^T(x) h(x)$$

for some continuous function $h(x) : \mathbf{R}^n \rightarrow \mathbf{R}^q$ with $h(0) = 0$. Take

$$z = [h(x) \;\; u(x)]^T$$

as the regulated output of system (3.1). Then the performance objective (3.3) becomes

$$\min_{u(t)} \max_{w(t)\in\mathcal{L}_2[0,\infty)} \int_0^\infty (\|z\|^2 - \gamma^2\|w\|^2)dt.$$

Let $V(x)$ denote the above minimaxization for the initial condition $x(0) = x$. It is obvious that $V(x) \geq 0$ for all x because taking $w(t) = 0$ sets a lower bound $V(x)$. It can be shown that if there is a well defined positive definite solution to this minimax problem, then the closed-loop system has an $\mathcal{L}_2$-gain less than or equal to γ under the minimizing control law u^*. The function $V(x)$ is called the available storage which measures, for a system with $\mathcal{L}_2$-gain $\leq \gamma$, to what extent the $\mathcal{L}_2$-norm of the output signal can be larger than γ times the $\mathcal{L}_2$-norm of the input signal, depending on the initial state x.

As we shall elaborate in Chapter 5, this minimax problem tries to find a particular $\mathcal{H}_\infty$ suboptimal controller that corresponds to the available storage function. However, just to guarantee the system to have $\mathcal{L}_2$-gain $\leq \gamma$, it is sufficient to find an upper bound on the available storage function, in which case the HJE (3.8) is relaxed to the Hamilton-Jacobi inequality:

$$V_x(x)f(x) + \frac{1}{4}V_x(x)(\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)g_2^T(x))V_x^T(x) + h^T(x)h(x) \leq 0 \quad (3.12)$$

3.3 FRE Approach to the Minimax Problem

When the linear time-invariant system is considered and the performance index is linear quadratic in x (also known as an LQ game problem), the solution to the minimax problem is characterized by an algebraic Riccati equation (ARE). As a direct generalization, Cloutier *et al.* [9] proposed an analogous approach to the nonlinear minimax problem for a class of input-affine systems.

Let us first review the LQ game problem [5]. Consider the linear time-invariant system:

$$\dot{x} = Ax + B_1w + B_2u. \quad (3.13)$$

The performance criterion is

$$\min_{u} \max_{w \in \mathcal{L}_2[0,\infty)} \int_0^\infty (x^T Q x + u^T u - \gamma^2 w^T w) dt \tag{3.14}$$

for some $Q \geq 0$. Under these assumptions, the value function V, if it exists, has to be quadratic, i.e., $V(x) = x^T P x$ for some positive semidefinite matrix P. The corresponding HJE simply reduces to the following ARE:

$$A^T P + PA + P(\frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T) P + Q = 0. \tag{3.15}$$

It was proven in [5] that if there exists a positive semidefinite solution to the above ARE, then the performance index (3.14) admits the optimal value

$$V^*(x) = x^T \bar{P} x$$

where $\bar{P}$ is the minimal such solution. The optimal static state feedback is defined as follows:

$$u^*(x) = -B_2^T \bar{P} x. \tag{3.16}$$

The necessary and sufficient condition for the ARE to admit a stabilizing solution is that $H \in \text{dom(Ric)}$ with the Hamiltonian matrix H defined by

$$H := \begin{bmatrix} A & \frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T \\ -Q & -A^T \end{bmatrix}.$$

Notice that when $B_1 = 0$, i.e., no disturbance is considered, the LQ game becomes the well-known linear quadratic regulator (LQR) problem with unit weight on the control action in the performance index:

$$\min_{u} \int_0^\infty (x^T Q x + u^T u) dt. \tag{3.17}$$

The corresponding ARE becomes

$$A^TP + PA - PB_2B_2^TP + Q = 0. \tag{3.18}$$

The above ARE has a unique positive semidefinite stabilizing solution if $\{A, B_2\}$ is stabilizable and $\{A, Q\}$ is detectable.

Now consider a nonlinear system whose dynamics can be described using the state-dependent linear representation (SDLR)

$$\dot{x} = A(x)x + B_1(x)w + B_2(x)u. \tag{3.19}$$

Clearly, it has the same structure as the linear system (3.13) but with all the state matrices A, B_1 and B_2 being functions of the state variable x instead of being constant. Accordingly, we require the performance index bear the same resemblance to the linear counterpart (3.14):

$$\min_{u} \max_{w \in \mathcal{L}_2[0,\infty)} \int_0^\infty (x^TQ(x)x + u^Tu - \gamma^2 w^Tw)dt. \tag{3.20}$$

Cloutier *et al.* [9] proposed a very simple strategy to deal with this class of problems. It parallels the solution to the LQ game problem: solve the Riccati equation at each point of the state space as though the states are "frozen":

$$A^T(x)P(x) + P(x)A(x) + P(x)(\frac{1}{\gamma^2}B_1(x)B_1^T - B_2(x)B_2^T(x))P(x) + Q(x) = 0. \tag{3.21}$$

A state feedback control law is then constructed from the pointwise stabilizing solution $P(x) \geq 0$ to the above FRE:

$$u(x) = -B_2^T(x)P(x)x. \tag{3.22}$$

Generalizing the results from [13] for linear systems, the following lemma gives the necessary and sufficient condition for the FRE (3.21) to admit a positive semidefinite solution.

Lemma 3.3 *The frozen Riccati equation (3.21) has a positive semidefinite solution* $P(x) \geq 0$ *if and only if the state-dependent Hamiltonian defined by*

$$H(x) := \begin{bmatrix} A(x) & \frac{1}{\gamma^2}B_1(x)B_1^T(x) - B_2(x)B_2^T(x) \\ -Q(x) & -A^T(x) \end{bmatrix} \tag{3.23}$$

is in dom(Ric) for all x, *and* $Ric(H(x)) \geq 0$ *is such a solution. In addition, if we have*

$$\bigcap_{i=0}^{n-1} \ker(Q(x)A^i(x)) = \emptyset \qquad \forall x \tag{3.24}$$

then the stabilizing solution is positive definite, i.e., $P(x) = Ric(H(x)) > 0$.

Note that for $H(x) \in dom(Ric)$, $\{A(x), B_2(x)\}$ must be stabilizable for any x. The condition (3.24) requires that $\{A(x), Q(x)\}$ be observable for all x.

The obvious advantage of the FRE approach is that the characterization of the resulting feedback controller (3.22) has similar structure to the LQ problem, and it is obtained by solving the corresponding FRE (3.21) instead of the HJE. Since the computational complexity of the ARE is only of polynomial growth rate with the state dimension, it provides a possibility to deal with high dimensional nonlinear systems. Moreover, because the FRE depends only on the current state, computation can be carried out online, in which case the FRE (3.21) is defined along the state trajectory.

However, pointwise existence of a positive definite solution to the FRE (3.21) cannot provide any guarantee on the global performance for a nonlinear system. It will be shown that an additional requirement is indeed needed to draw any conclusions. It would be too naive for anyone to expect that the linear theory can be generalized to nonlinear plants in such a straightforward way. But there is evidence that this simple strategy works amazingly well for many examples, including practical ones that arise in the engineering designs. We will conduct a thorough investigation in the sequel, and provide a partial explanation of its "mysterious" success.

3.3.1 State-Dependent Linear Representation

As one can easily see, the SDLR (3.19) of a nonlinear system is not unique when $n > 1$. In fact, there are infinite numbers of them as stated in the following lemma.

Lemma 3.4 *Suppose function $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is continuously differentiable with $f(0) = 0$, then there always exists a continuous matrix-valued function $A_o : \mathbf{R}^n \rightarrow \mathbf{R}^{n\times n}$ such that*

$$f(x) = A_o(x)x.$$

Moreover, all possible matrices $A(x)$ that satisfy $f(x) = A(x)x$ can be parameterized as

$$A(x) = A_o(x) + N(x)$$

where $N : \mathbf{R}^n \rightarrow \mathbf{R}^{n\times n}$ satisfies $N(x)x = 0$.

Proof A candidate for $A_o(x)$ can be obtained from

$$A_o(x) = \int_0^1 \left.\frac{\partial f}{\partial x}\right|_{x=\lambda x} d\lambda. \tag{3.25}$$

This is a standard calculus result whose proof can be found in [36]. The parameterization of all possible $A(x)$'s is trivial. □

There are a few points that need to be emphasized about this parameterization of $f(x)$:

- When $n = 1$, $A(x)$ is uniquely defined by

$$A(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ \frac{df(x)}{dx} & x = 0 \end{cases}$$

  The continuity of $A(x)$ at the origin is preserved because $f(x)$ is assumed to be C^1. When $n > 1$, $A(x)$ can be parameterized by the $n-1$ dimensional kernel space $N(x)$ of x.

- This parameterization is different from taking the gradient of $f(x)$ at x. However, $A(x)$ is equivalent to the linear approximation of $f(x)$ at the origin, i.e.,

$$A(0) = \left.\frac{\partial f}{\partial x}\right|_{x=0}.$$

  Note that this is true for any choice of $A(x)$.

Since the dynamics of a nonlinear system are usually not given in this SDLR form, does it make any difference on the performance if we choose a particular one for the control design? If so, then how should one choose a good representation? These questions will be the focus of our investigation.

3.3.2 Optimality Condition of the FRE Solution

Suppose a particular SDLR of a nonlinear input-affine system is used, and the frozen Riccati equation (3.21) yields the pointwise stablizing solution $P(x) \geq 0$. Since $A(0)$ is the gradient of $f(x)$ at zero, $\bar{V} = x^T P(0)x$ solves the minimax problem locally. Thus $\bar{V}$ is the quadratic approximation of the global value function at the origin. We will reveal next under what condition we can conclude that the FRE controller (3.22) is the global optimal solution to the minimax problem (3.20).

A standard result in nonlinear analysis is needed to establish the condition (see, e.g., [6]).

Lemma 3.5 *Suppose a vector-valued function $\phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is of class $\mathcal{C}^k$ for some integer $k \geq 0$. Let*

$$\phi(x) = [\phi_1(x) \ \cdots \ \phi_n(x)]^T.$$

Then there exists a $\mathcal{C}^{k+1}$ function $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$\frac{\partial \Phi}{\partial x}(x) = \phi^T(x)$$

if and only if

$$\frac{\partial \phi_i}{\partial x_j}(x) = \frac{\partial \phi_j}{\partial x_i}(x)$$

for all x and $i, j = 1, \cdots, n$. Moreover, one such function Φ with $\Phi(0) = 0$ is given by

$$\Phi(x) = x^T \int_0^1 \phi(tx)dt.$$

As a direct consequence of the previous lemma, a curl type condition on the solution to the FRE will be shown to be sufficient to guarantee a positive semidefinite solution of the associated HJE.

Theorem 3.6 *Suppose $P(x) \geq 0$ is the stabilizing solution to the frozen Riccati equation (3.21), and $p(x) = P(x)x$. If $P(x)$ is continuous and $p(x)$ satisfies*

$$\frac{\partial p_i}{\partial x_j}(x) = \frac{\partial p_j}{\partial x_i}(x) \tag{3.26}$$

for $i, j = 1, \cdots, n$, then

$$V^*(x) = 2x^T \int_0^1 p(tx)dt \tag{3.27}$$

is the nonnegative solution to the corresponding Hamilton-Jacobi equation (3.8). Moreover, the feedback control law (3.22) minimizes the performance index (3.20).

Proof By Lemma 3.5, $V(x)$ given by (3.27) is well defined. Since $p(x) = P(x)x$, we have

$$V^*(x) = 2x^T \int_0^1 p(tx)dt = 2\int_0^1 tx^T P(tx)xdt.$$

So $V^*(x)$ is nonnegative for all x because $P(x)$ is positive semidefinite. If we premultiply x^T and postmultiply x to the FRE (3.21), and recognize that $\frac{\partial V^*}{\partial x} = 2x^T P(x)$, we obtain the Hamilton-Jacobi equation (3.8) whose solution is $V^*(x)$. Therefore, the FRE controller (3.22) is optimal since $V^*(x)$ is the value function of the performance index (3.20). □

This theorem says that the existence of the pointwise stabilizing solution is not sufficient to guarantee global performance. In addition, we need to confirm that there indeed exists a $\mathcal{C}^1$ function $V(x)$ with $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$. Jacobson pointed out in [22] a similar condition for the nonlinear regulator problem (3.9). As in most cases we cannot obtain the analytic solution to the frozen Riccati equation, and it is difficult to check the curl condition (3.26) for the numerical solution. Theorem 3.6 is not considered as a practical means to verify optimality of the FRE controller.

The following example from [14] demonstrates that the pointwise stabilizing solution may even fail to yield global stability.

Example 3.7 *The following cost function is to be minimized:*

$$J = \int_0^\infty (q(x) + u^2)dt$$

subject to the differential constraints:

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2, x_3) \\ \dot{x}_2 &= e^{-x_1}x_3 + \epsilon e^{-x_1/2}u \\ \dot{x}_3 &= -e^{x_1}x_2 + \epsilon e^{x_1/2}u \end{cases} \tag{3.28}$$

where

$$f_1 = \begin{cases} \tanh(10\,(e^{x_1}x_2^2 - e^{-x_1}x_3^2)) & |x_1| < 3 \\ 0 & |x_1| \geq 3 \end{cases}$$

$$q(x) = \left\| C(x_1) \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \right\|^2$$

$$C = \epsilon \begin{bmatrix} e^{-x_1/2} & e^{x_1/2} \end{bmatrix}$$

$$\epsilon = 0.1$$

For this example, the dynamics are well suited for the SDLR structure. Consider the natural choice of an SDLR:

$$A(x) = \begin{bmatrix} 0 & e^{-x_1} \\ -e^{x_1} & 0 \end{bmatrix}$$

$$B(x) = \epsilon \begin{bmatrix} e^{-x_1/2} \\ e^{x_1/2} \end{bmatrix}$$

$$Q(x) = C^T(x)C(x) = \epsilon^2 \begin{bmatrix} e^{x_1} & 1 \\ 1 & e^{-x_1} \end{bmatrix}$$

The reason that we can drop the dynamics of x_1 in the FRE design is that there is no direct control acting on x_1 and it is not factored out in the performance index $q(x)$. Note that $A(x)$, $B(x)$ and $Q(x)$ are functions of only x_1 which varies on the interval $[-3, 3]$. It can be confirmed by simple calculations that $\{A(x), B(x)\}$ is controllable and $\{A(x), C(x)\}$ is observable in the linear sense for all $x_1 \in [-3, 3]$. As a result, the Riccati equation

$$A^T(x_1)P(x_1) + P(x_1)A(x_1) - P(x_1)B(x_1)B^T(x_1)P(x_1) + Q(x_1) = 0$$

should admit a unique positive definite solution for all x_1. In fact, in this case we can even get an analytical expression for $P(x_1)$:

$$P(x_1) = \begin{bmatrix} e^{x_1} & 0 \\ 0 & e^{-x_1} \end{bmatrix}.$$

And the feedback control law given by the FRE is as follows:

$$u(x) = -\epsilon(e^{x_1/2}x_2 + e^{-x_1/2}x_3). \tag{3.29}$$

The resulting closed-loop dynamics are

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2, x_3) \\ \dot{x}_2 &= -\epsilon^2 x_2 + (1-\epsilon^2)e^{-x_1}x_3 \\ \dot{x}_3 &= -(1+\epsilon^2)e^{x_1}x_2 - \epsilon^2 x_3 \end{cases} \tag{3.30}$$

In Figure 3.1, simulation results show that the closed-loop system is unstable when the FRE controller (3.29) is applied. Because x_1, whose dynamics are ignored in the design, is treated as "frozen," the solution $P(x_1)$ to the FRE does not define a global Lyapunov function. Thus, the feedback controller constructed from $P(x_1)$ fails to stabilize the system.

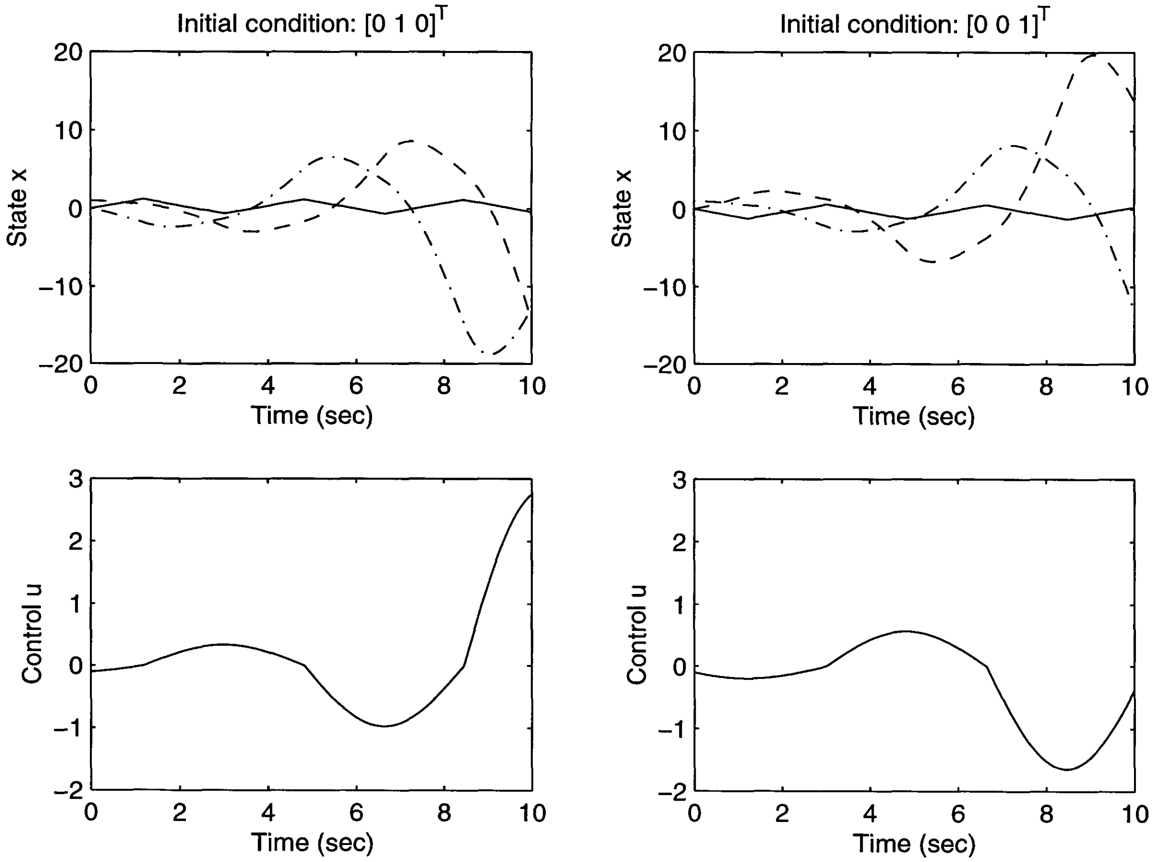

Figure 3.1: Closed-loop simulations with the FRE controller of Example 3.7 starting from two different initial conditions. The states x_1 (solid), x_2 (dashed) and x_3 (dashdot) are shown in the upper plots.

3.3.3 Existence of the Optimal SDLR

The FRE design relies on a state-dependent linear representation of the nonlinear dynamics. Even though we have seen that the performance of the FRE controller can be far off from the optimal for a particular choice of SDLR, it will be shown that not every choice of $A(x)$ is equal in producing a FRE controller. Because of the freedom one has in choosing $A(x)$, it is interesting to ask the question: does there always exist a state-dependent representation $f(x) = A(x)x$ so that the FRE design gives the optimal controller?

Some preparation is needed to establish the main result.

Lemma 3.8 *Suppose* $S(x) : \mathbf{R}^n \to \mathbf{R}^{n\times n}$ *is a continuous matrix-valued function such that* $S(x) = S^T(x)$ *and* $x^T S(x) x = 0$ *for all* x. *Then there always exists a continuous skew-symmetric matrix-valued function* $R(x) : \mathbf{R}^n \to \mathbf{R}^{n\times n}$, *i.e.,* $R(x) = -R^T(x)$, *such that*

$$(S(x) - R(x))x = 0. \tag{3.31}$$

Proof We will first show that for any x, there exists a skew-symmetric R that satisfies (3.31). Then the set-valued analysis will be used to establish the continuity property. It is obvious that any skew-symmetric matrix can satisfy (3.31) when $x = 0$. When $x \neq 0$, because $S(x)$ is symmetric, a congruence transformation can be done to diagonalize $S(x)$:

$$S(x) = U^T(x) D(x) U(x)$$

where $D(x) = diag[\lambda_1(x), \cdots, \lambda_n(x)]$ and $U(x)$ is some orthogonal matrix-valued function. Now let

$$y(x) = U(x)x =: (y_1(x), \cdots, y_n(x))^T,$$

then we have

$$x^T S(x) x = y^T(x) D(x) y(x) = \sum_{i=1}^{n} \lambda_i(x) y_i^2(x).$$

Since $x^T S(x) x = 0$ by assumption,

$$\sum_{i=1}^{n} \lambda_i(x) y_i^2(x) = 0.$$

Because $U(x)$ has full rank, $y(x)$ cannot be zero for any $x \neq 0$. Without loss of generality, let's assume $y_p \neq 0$ for some $1 \leq p \leq n$. Let

$$E(x) = \frac{1}{y_p} \begin{bmatrix} 0 & \cdots & 0 & \lambda_1 y_1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \lambda_{p-1} y_{p-1} & 0 & \cdots & 0 \\ -\lambda_1 y_1 & \cdots & -\lambda_{p-1} y_{p-1} & 0 & -\lambda_{p+1} y_{p+1} & \cdots & -\lambda_n y_n \\ 0 & \cdots & 0 & \lambda_{p+1} y_{p+1} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \lambda_n y_n & 0 & \cdots & 0 \end{bmatrix} \tag{3.32}$$

So $E(x)$ is a matrix with all zero entrees except the pth row and pth column. Notice that $E(x) = -E^T(x)$. Now, $R(x)$ can be constructed as follows:

$$R(x) = U^T(x) E(x) U(x), \tag{3.33}$$

which is obviously skew-symmetric. Also, it is easy to check that $R(x)$ satisfies (3.31):

$$\begin{aligned} (S(x) - R(x))x &= U^T(x)(D(x) - E(x))y(x) \\ &= U(x)(\lambda_1 y_1 - \lambda_1 y_1, \cdots, \frac{\sum_{i=1}^{n}(\lambda_i y_i^2)}{y_p}, \cdots, \lambda_n y_n - \lambda_n y_n)^T \\ &= 0 \end{aligned}$$

For the continuity argument, we use Michael's selection theorem of the set-valued analysis machinery [3]. Because $R(x)$ is required to be skew-symmetric, i.e., $R(x)$ is of the following form

$$R = \begin{bmatrix} 0 & r_{1,2} & \cdots & r_{1,n-1} & r_{1,n} \\ -r_{1,2} & 0 & \cdots & r_{2,n-1} & r_{2,n} \\ & & \vdots & & \\ -r_{1,n-1} & -r_{2,n-1} & \cdots & 0 & r_{n-1,n} \\ -r_{1,n} & -r_{2,n} & \cdots & r_{n-1,n} & 0 \end{bmatrix}.$$

We can use a vector r to represent R:

$$r = [r_{1,2} \ \cdots \ r_{1,n} \ r_{2,3} \ \cdots \ r_{2,n} \cdots \ r_{n-1,n}]^T.$$

Then (3.31) becomes

$$M(x)r(x) = S(x)x$$

where

$$M(x) = \begin{bmatrix} x_2 & x_3 & \cdots & x_{n-1} & x_n & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -x_1 & 0 & \cdots & 0 & 0 & x_3 & x_4 & \cdots & x_{n-1} & x_n & \cdots & 0 \\ 0 & -x_1 & \cdots & 0 & 0 & -x_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & \vdots & & & & & \vdots & & & \vdots & \\ 0 & 0 & \cdots & -x_1 & 0 & 0 & 0 & \cdots & -x_2 & 0 & \cdots & x_n \\ \underbrace{0 \quad 0 \quad \cdots \quad 0 \quad -x_1}_{n-1} & & & & & \underbrace{0 \quad 0 \quad \cdots \quad 0 \quad -x_2}_{n-2} & & & & & \cdots & \underbrace{-x_{n-1}}_{1} \end{bmatrix}$$

Notice that M(x) has n rows and $\frac{n(n-1)}{2}$ columns, and there are $n-1$ nontrivial entrees in each row and two nontrivial entrees in each column. Because of the assumption of $x^T S(x)x = 0$ for all x, we can conclude that $S(x)x$ is in the span of $M(x)$ for all x. To see this, note that $S(x)x$ is the linear combination of all the columns that contain the nonzero element x_i provided $x \neq 0$. This confirms that there exists a solution to (3.31) for each x. It is also observed that $M(x)$ has a rank of $n-1$ for all nontrivial x. At $x = 0$, $M(x)$ is simply zero.

Define the set-valued map $\mathcal{F} : \mathbf{R}^n \to \mathbf{R}^{\frac{n(n-1)}{2}}$ as

$$\mathcal{F}(x) = \begin{cases} \{r \in \mathbf{R}^{\frac{n(n-1)}{2}} : M(x)r = S(x)x\} & x \neq 0 \\ \{0\} & x = 0 \end{cases}$$

It can be shown that this map has the closed and convex subsets as its values and is lower semi-continuous. By Michael's selection theorem, there is a continuous selection $r(x)$ of $\mathcal{F}$ with $r(0) = 0$.

□

Lemma 3.9 *Suppose both $A(x)$ and $Q(x) = Q^T(x)$ are continuous matrix-valued functions, and $P(x)$ is a continuous positive definite matrix-valued function. If we have*

$$x^T(A^T(x)P(x) + P(x)A(x) + Q(x))x = 0 \tag{3.34}$$

for all x, then there always exists a continuous $\tilde{A}(x) = A(x) + N(x)$ with $N(x)x = 0$ such that

$$\tilde{A}^T(x)P(x) + P(x)\tilde{A}(x) + Q(x) = 0 \tag{3.35}$$

for all x.

Proof Let

$$S(x) = A^T(x)P(x) + P(x)A(x) + Q(x),$$

thus $S(x)$ is symmetric and $x^T S(x)x = 0$ from (3.34). So by (3.35) we need to find a $N(x)$ with $N(x)x = 0$ such that

$$S(x) + N^T(x)P(x) + P(x)N(x) = 0.$$

Since $S(x)$ is symmetric, $N(x)$ can be parameterized as

$$N(x) = -\frac{1}{2}P^{-1}(x)(S(x) - R(x)) \tag{3.36}$$

where $R(x)$ is any skew-symmetric matrix-valued function, i.e., $R(x) = -R^T(x)$. Recall that $N(x)x = 0$, so we need to find a $R(x)$ such that

$$(S(x) - R(x))x = 0$$

for all $x \in \mathbf{R}^n$. By Lemma 3.8, a continuous $R(x)$ satisfying the above equation can alway be constructed. Therefore, a continuous $N(x)$ can be obtained by (3.36) for all $x \in \mathbf{R}^n$. □

With the two preliminary lemmas, we now present the main result of this chapter.

Theorem 3.10 *Consider the input-affine nonlinear system*

$$\dot{x} = f(x) + B_1(x)w + B_2(x)u$$

where f,B_1 and B_2 are all $\mathcal{C}^1$ functions with $f(0) = 0$. Assume that the optimal value function $V(x)$ to the minimax problem (3.20) is continuously differentiable and has a gradient of the following form

$$\frac{\partial V}{\partial x}(x) = 2x^T P(x) \tag{3.37}$$

for some positive definite matrix-valued function $P : \mathbf{R}^n \rightarrow \mathbf{R}^{n\times n}$, then there always exists a parameterization $f(x) = A(x)x$ with $A(x)$ being continuous such that $P(x)$ is the stabilizing solution of the frozen Riccati equation (3.21), and defines the optimal feedback controller (3.22).

Proof Since $f(x)$ is assumed to be $\mathcal{C}^1$ and $f(0) = 0$, by Lemma 3.4 we can always find a continuous $A_o(x)$ such that $f(x) = A_o(x)x$. Because of the continuity assumptions on the dynamics, the $\mathcal{C}^2$ value function $V(x)$ has to satisfy the HJE (3.8) associated with the minimax problem. Substituting $f(x)$ and $\frac{\partial V}{\partial x}$ into (3.8), we have

$$x^T(A_o^T(x)P(x) + P(x)A_o(x) + P(x)(\frac{1}{\gamma^2}B_1(x)B_1^T(x) - B_2(x)B_2^T(x))P(x) + Q(x))x = 0$$

for all $x \in \mathbf{R}^n$. Applying Lemma 3.9, a continuous $A(x)$ satisfying $f(x) = A(x)x$ can always be found such that the following frozen Riccati equation is satisfied for all x:

$$A^T(x)P(x) + P(x)A(x) + P(x)(\frac{1}{\gamma^2}B_1(x)B_1^T(x) - B_2(x)B_2^T(x))P(x) + Q(x) = 0.$$

□

A few important remarks should be made concerning the above proofs:

- In Lemma 3.9, it is clear that the construction of the optimal $\tilde{A}(x)$ depends on the function $Q(x)$. Since the information in the performance index is included in $Q(x)$, we can expect that even for the same system, the optimal parameterization $A(x)$ may not be the same for different performance criteria.
- For two-dimensional systems, the optimal choice of $A(x)$ is uniquely determined. However, for systems of higher dimension, there may exist an infinite number of $A(x)$ such that the FRE recovers the optimal controller.

The assumption (3.37) on the optimal value of the performance index has the following interpretation: at any point of the state space, the negative gradient of the optimal cost function must point "toward" the equilibrium at the origin. More accurately, the angle of the gradient and x should be within $\pm\frac{\pi}{2}$. Note that this condition is not satisfied by every positive definite function. Here we give one counterexample:

$$V(x) = x_1^2 + x_2^2 e^{-x_1}. \tag{3.38}$$

In Figure 3.2, the contours of $V(x)$ are plotted along with an arrow illustrating the violation of $-\frac{\partial V}{\partial x}$ at $(3, 27)$. Since $V(x)$ is symmetric about $x_1 = 0$, only the positive side of x_2 is shown. The shaded area indicates where the condition (3.37) cannot be satisfied. Even though this condition excludes some potential value functions, we do not feel it is too restrictive as we can identify many classes of value functions, including quadratic functions, that do satisfy it.

Theorem 3.10 confirms that the global optimal controller can be formed from the

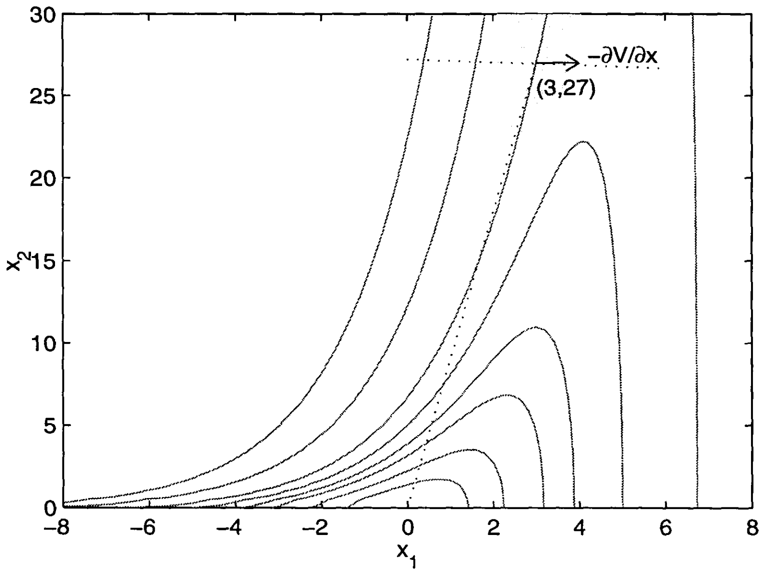

Figure 3.2: Contours of $V(x) = x_1^2 + x_2^2 e^{-x_1}$

positive definite solution to the FRE if the optimal value of the performance index satisfies (3.37) and the "right" $A(x)$ is chosen. This result provides new insights into this design methodology. While the rest of the design procedure is quite mechanical, we do have freedom in selecting a SDLR which by itself determines the closed-loop performance ranging from arbitrarily poor to optimal. However, in the above proofs, the optimal $A(x)$ is constructed with the value function assumed known a priori. In real situations, designers have no information on the optimal value function. Therefore, this result does not help in determining a proper SDLR to base the FRE design on.

3.4 Examples

A number of numerical examples will be presented to demonstrate some results of the previous sections. We will focus on the regulator case here since the $\mathcal{H}_\infty$ suboptimal problem will be discussed in more detail in Chapter 5. All the examples are created by the converse HJE method explained in Section 2.3, so the analytical solution of the optimal control is known. This is very helpful for comparing the solution of the FRE to the optimal.

Example 3.11 *Consider the optimal regulator problem for the following dynamics:*

$$\begin{cases} \dot{x}_1 &= & x_2 & + & (x_2 - 1)u \\ \dot{x}_2 &= & x_1 + x_2 & - & (x_1 + 2)u \end{cases}$$

The cost function is defined as in (3.17) with

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The Hamilton-Jacobi equation for this optimal control problem is

$$\frac{\partial V}{\partial x}(x) \begin{bmatrix} x_2 \\ x_1 + x_2 \end{bmatrix} - \frac{1}{2}\frac{\partial V}{\partial x}(x) \begin{bmatrix} x_2 - 1 \\ x_1 + 2 \end{bmatrix}^T \begin{bmatrix} x_2 - 1 & x_1 + x_2 \end{bmatrix} \frac{\partial V^T}{\partial x}(x) + x_1^2 + 2x_2^2 = 0.$$

It is easy to check that the value function of this problem is

$$V(x) = x_1^2 + x_2^2.$$

The optimal feedback is given by

$$u = -\frac{1}{2}R^{-1}\frac{\partial V}{\partial x}(x) \begin{bmatrix} x_2 - 1 \\ x_1 + 2 \end{bmatrix} = x_1 + 2x_2.$$

It is not difficult to choose a SDLR for the FRE design:

$$A(x) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B(x) = \begin{bmatrix} x_2 - 1 \\ x_1 + 2 \end{bmatrix}.$$

Figure 3.3 shows simulations of the states, control action and cost under the optimal control starting from the initial condition $x_o = [-3 \;\; 4]^T$. The final cost is 25 which

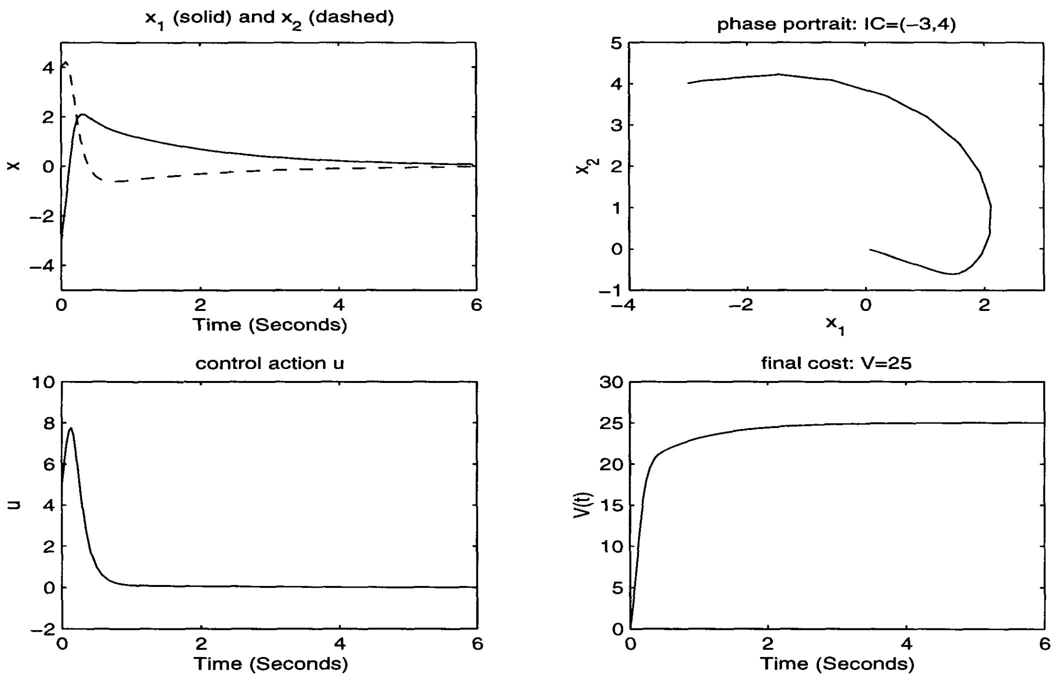

Figure 3.3: Closed-loop simulations with the optimal controller for Example 3.11

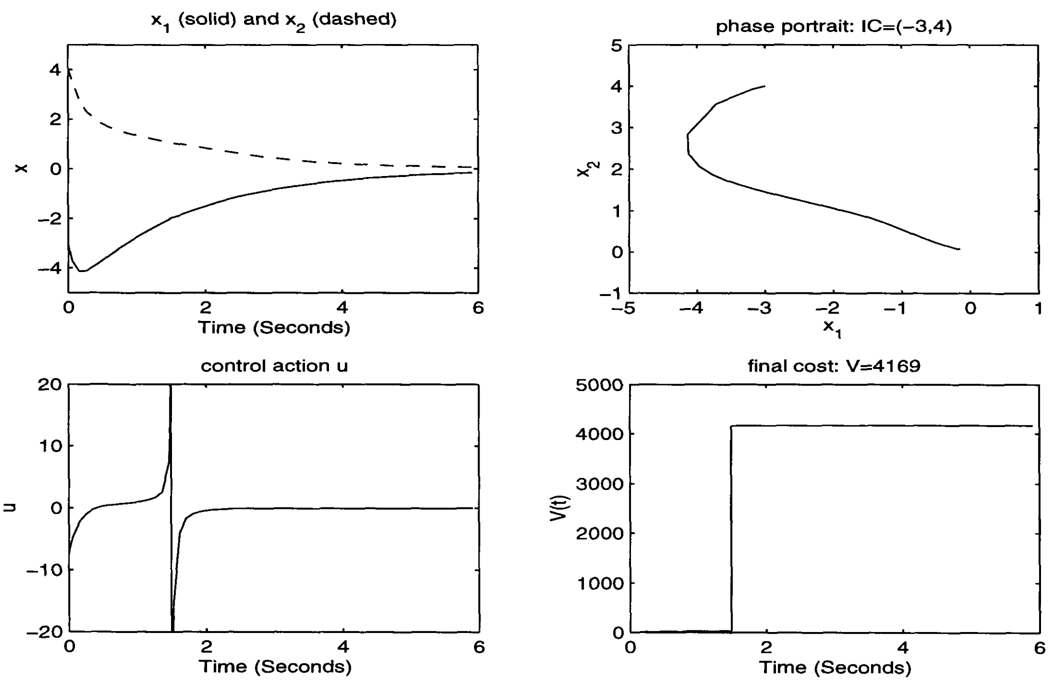

Figure 3.4: Closed-loop simulations with the FRE controller given by the obvious choice of SDLR for Example 3.11

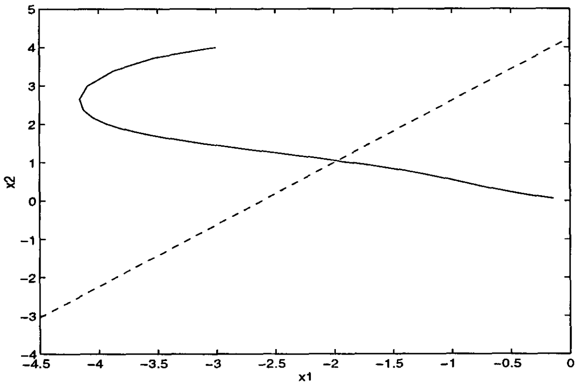

Figure 3.5: The unstabilizable line (dashed) of the chosen SDLR for Example 3.11

is consistent with the analytic solution. In comparison, Figure 3.4 plots the same simulation results but under the FRE controller calculated using the chosen SDLR. It is noticed that at $x = [-2 \;\; 1]^T$, the computed control input shoots up suddenly, which causes a big jump in the cost. A closer inspection reveals that with this particular choice of the pair $A(x)$ and $B(x)$, the "frozen" linear representation is stabilizable everywhere except on the line:

$$x_2 = \frac{1}{2}(1 + \sqrt{5})x_1 + 2 + \sqrt{5}. \tag{3.39}$$

Therefore, when the states evolve close to the point $x = [-2 \;\; 1]^T$ which is on that line, the Riccati equation returns a huge control action. Theoretically, the FRE should fail to yield a positive definite solution at the point $x = [-2 \;\; 1]^T$. Figure 3.5 illustrates that the line defined by (3.39) divides the state space into two halves: whenever the states have to cross the line, this particular FRE controller blows up. While in this example this particular $\{A(x), B(x)\}$ can be ruled out by inspection, it is not easy to identify a "valid" SDLR in general.

However, there is a fix to this problem: we can choose a different SDLR. As the optimal value function satisfies the gradient condition (3.37), the optimal $A(x)$ does exist:

$$A_{opt} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_2 - 2 \\ x_1 + 4 \end{bmatrix} \begin{bmatrix} x_2 & -x_1 \end{bmatrix}.$$

Although this $A(x)$ probably seems unnatural and complicated, it is not hard to verify that the corresponding FRE admits the solution $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The next example will show that even though a valid SDLR is selected, the FRE controller may still fail to deliver good performance.

Example 3.12 *Consider minimizing the performance index:*

$$J(x) = \int_0^\infty (\, x_2^2 + u^2 \,) dt$$

subject to

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 e^{x_1} + \frac{1}{2} x_2^2 + e^{x_1} u \end{cases}$$

Again, it is an easy exercise to confirm that the positive definite function

$$V(x) = x_1^2 + x_2^2 e^{-x_1}$$

satisfies the corresponding HJE. The optimal control law is given by

$$u = -x_2.$$

As explained before, the gradient of this value function cannot be written as $2x^T P(x)$ for some positive definite $P(x)$ for all x. Hence the globally optimal SDLR does not exist.

Let's take the obvious SDLR for this plant:

$$A(x) = \begin{bmatrix} 0 & 1 \\ -e^{x_1} & \frac{1}{2} x_2 \end{bmatrix}, \qquad B(x) = \begin{bmatrix} 0 \\ e^{x_1} \end{bmatrix}.$$

Unlike the previous example, the pair $\{A(x), B(x)\}$ is controllable for all x. However, the performance delivered by the FRE controller is still far from the optimal. Starting from $x_o = [-2\ 2]^T$, the FRE controller yields a cost of 143 (Figure 3.7), whereas the optimal cost is only 33.6 (Figure 3.6). If we compare the phase portrait of the state of the two controllers, we notice that the optimal control lets the state take a big "detour" before reaching the origin, which incurs significantly less cost than going "straight." The shaded lines in the phase portrait of Figure 3.6 are contours of the value function. It can be clearly seen that the state trajectory is consistent with level curves of the value function.

To finish on a positive note, we revisit the 2-d oscillator of Example 2.1. In this case the obvious SDLR

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{2}(1 - g^2(x)) \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ g(x) \end{bmatrix}$$

indeed leads to the optimal controller. In fact, this is also true for more general value functions which satisfy

$$\frac{\partial V}{\partial x} = 2x^T \begin{bmatrix} p_1(x) & 0 \\ 0 & p_2(x) \end{bmatrix}$$

where $p_1(x)$, $p_2(x) > 0$ for all x.

3.5 Conclusions

The frozen Riccati approach to the nonlinear minimax optimal control problem is investigated in this chapter. It is revealed that not only can the FRE controller provide performance which is far from the optimal, it may even fail to yield global stability because of its pointwise nature. In addition to the existence of a pointwise stabilizing solution, a curl type condition is needed to guarantee optimality. However, one typically cannot assess *a priori* the guaranteed performance for a particular state-dependent linear representation. On the other hand, it is confirmed that if the optimal value of the performance index satisfies a certain constraint, an SDLR always exists

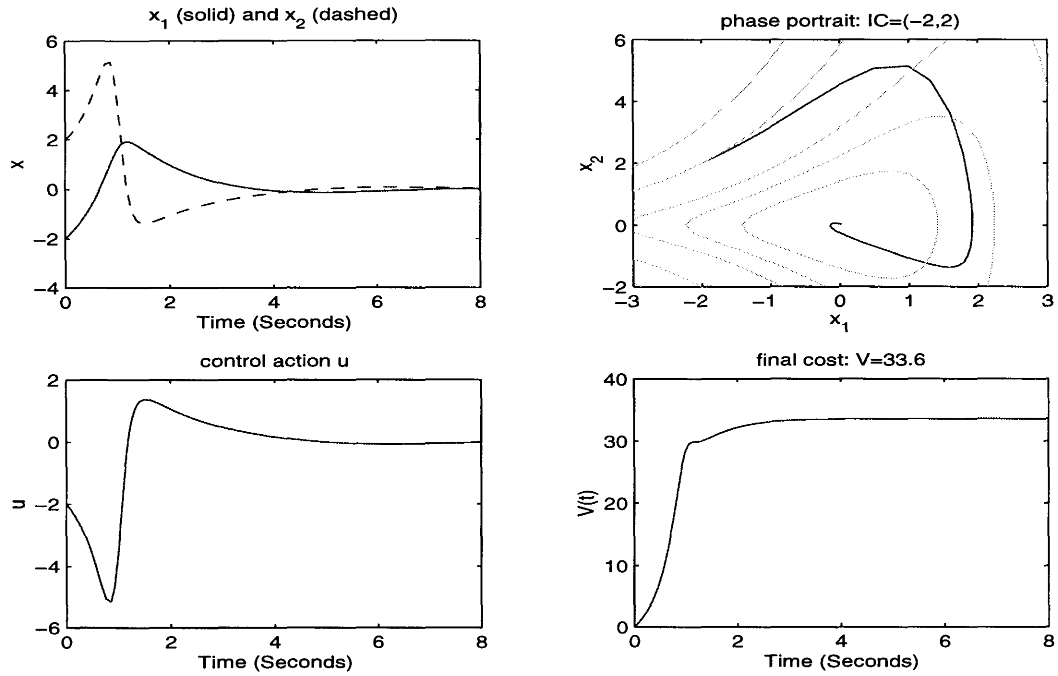

Figure 3.6: Closed-loop simulations with the optimal controller for Example 3.12

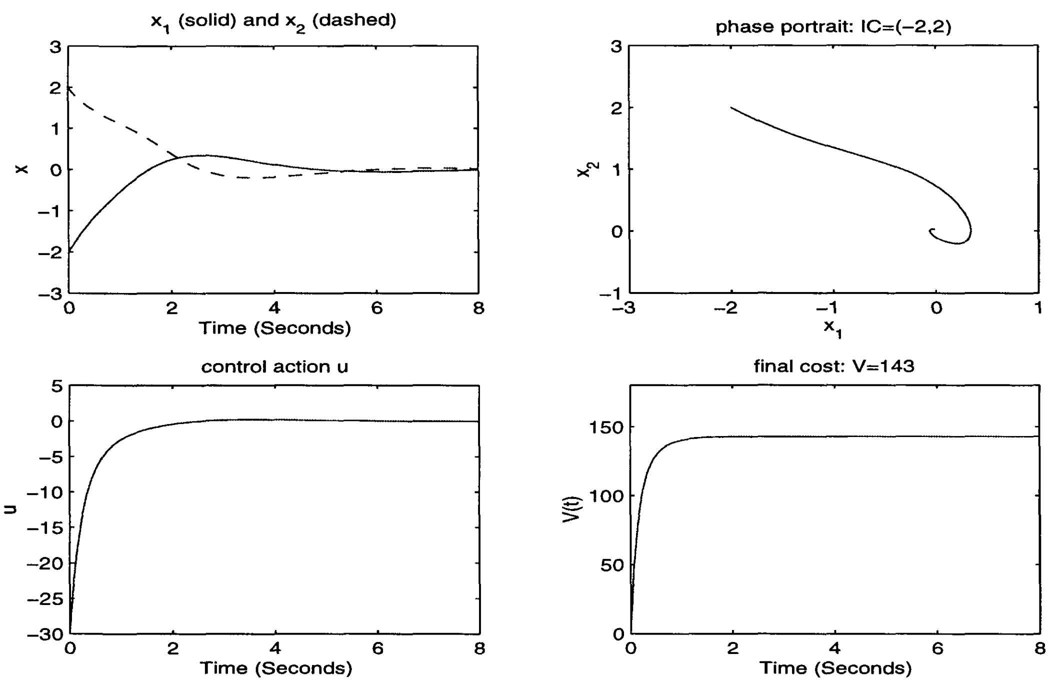

Figure 3.7: Closed-loop simulations with the FRE controller of the obvious choice of SDLR for Example 3.12

such that controller constructed from the positive definite solution to the FRE is optimal. Unfortunately, the result provides no help in identifying the optimal SDLR without knowing the value function.

Chapter 4 Regulation of Nonlinear Systems

The traditional linear quadratic regulator (LQR) problem was extensively studied in the 1950s. It can be viewed as minimizing the energy a system consumes in order to drive the plant to its equilibrium. The solution is characterized by a differential Riccati equation. For the infinite time horizon case, the solution further reduces to an algebraic Riccati equation, which can be solved efficiently. It has also been shown that the optimal regulator provides a guaranteed stability margin for the closed-loop system so that it can withstand uncertainties in the plant itself to a certain degree.

Recently, parameter-dependent Lyapunov functions have been employed to synthesize $\mathcal{H}_\infty$ type controllers for linear parameter varying (LPV) systems with bounded variation rates [39]. The solution is captured as an infinite dimensional convex optimization, and a computational scheme has been used to get an approximate solution by solving a finite set of linear matrix inequalities (LMI's). Based upon similar ideas, the LQR theory will be generalized to a class of nonlinear input-affine systems in this chapter. Using the quasi-LPV representation of the nonlinear plant, an upper bound on the optimal value of the performance criterion, which also serves as a Lyapunov function for the closed-loop system, is derived in terms of an affine matrix inequality (AMI). A globally stabilizing state feedback law can be constructed from the solution to the AMI's. A practical computation scheme is proposed to solve the optimization problem under the infinite dimensional LMI constraints. The connection of this quasi-LPV technique with the frozen Riccati equation approach is discussed and illustrated through examples.

4.1 Regulation of Quasi-LPV Plants

Consider the following input-affine nonlinear autonomous system

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ z &= h(x) + k(x)u \end{cases} \qquad x(0) = x_o \tag{4.1}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^p$ and $z \in \mathbf{R}^q$ are control input and regulated output, respectively. Functions f, g, h and k are assumed to be $\mathcal{C}^1$ with $f(0) = 0$ and $h(0) = 0$. Thus, $0 \in \mathbf{R}^n$ is the equilibrium of the system with $u = 0$. The initial state x_o is assumed given but arbitrary. Our performance objective is the same as the LQR problem:

> Find a control law $u \in \mathcal{L}_2[0, \infty)$ such that the closed-loop system is internally stable and the performance criterion $\|z\|_2$ is minimized.

We assume that all the states can be measured and available for the controller. Note that this problem is a generalized version of the optimal nonlinear regulation problem discussed in Section 2.1, where there is no crossover term between the state variable x and control input u in the performance index. As illustrated in the earlier chapters, the exact solution to this optimal control problem is characterized in terms of a Hamilton-Jacobi partial differential equation. We will, however, derive an algebraic condition which can yield a globally stabilizing suboptimal controller.

As pointed out by Lemma 3.4, system (4.1) can be rewritten in the following form:

$$\begin{cases} \dot{x} &= A(x)x + B(x)u \\ z &= C(x)x + D(x)u \end{cases} \qquad x(0) = x_o \tag{4.2}$$

for some matrix-valued functions $A(x)$ and $C(x)$. As it is often the case that a small number of functions can be chosen to represent the dependence of A, B, C and D on the state vector x, the following model will be used in this chapter

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A(\rho(x)) & B(\rho(x)) \\ C_1(\rho(x)) & 0 \\ C_2(\rho(x)) & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \qquad x(0) = x_o \tag{4.3}$$

where $\rho \in \mathbf{R}^m$ is chosen as the underlying parameter. The regulated output $z = [z_1\ z_2]^T$ is assumed to have been scaled to the above form for simplicity of derivation. This assumption can be relaxed at the expense of a more complicated formula.

The representation (4.3) is called a quasi-linear parameter varying (quasi-LPV) model because it is different from the true LPV model only in that the underlying parameter ρ is now function of the state. For the true LPV plant the underlying parameters can vary freely in time, whereas the trajectory of the parameter of the quasi-LPV model is determined by the behavior of the nonlinear system. Assume that the parameter of (4.3) varies in the admissible set

$$\mathcal{F}_{\mathcal{P}}^{\nu} := \{\rho \in \mathcal{C}^1(\mathbf{R}^+, \mathbf{R}^m) : \rho \in \mathcal{P},\ \underline{\nu}_i(\rho) \leq \dot{\rho}_i \leq \overline{\nu}_i(\rho),\ i = 1, \cdots, m\} \tag{4.4}$$

where $\mathcal{P} \subset \mathbf{R}^m$ is a compact set, and the functions $\overline{\nu}$, $\underline{\nu} : \mathcal{P} \to \mathbf{R}^m$ bound the variation rate of ρ. Note that these bounds are functions of the parameter value.

Theorem 4.1 *If there exists a differentiable matrix-valued function $X(\rho)$ such that $X(\rho) > 0$ and*

$$\begin{bmatrix} -\sum_{i=1}^{m} \underline{\overline{\nu}}_i(\rho) \frac{\partial X}{\partial \rho_i}(\rho) + \hat{A}(\rho)X(\rho) + X(\rho)\hat{A}^T(\rho) - B(\rho)B^T(\rho) & X(\rho)C_1^T(\rho) \\ C_1(\rho)X(\rho) & -I \end{bmatrix} < 0 \tag{4.5}$$

for all $\rho \in \mathcal{P}$ with $\hat{A}(\rho) = A(\rho) - B(\rho)C_2(\rho)$, then the closed-loop system is asymptotically stable with the state feedback $u(x) = F(\rho(x))x$ where

$$F(\rho) = -(B^T(\rho)X^{-1}(\rho) + C_2(\rho)). \tag{4.6}$$

Moreover, we have an upper bound on the $\mathcal{L}_2$-norm of $z(t)$ as a function of the initial state x_o:

$$\min_{u\in\mathcal{L}_2[0,\infty)} \|z\|_2^2 \leq x_o^T X^{-1}(\rho(x_o))x_o.$$

The notation $\sum_{i=1}^{m} \overline{\nu}_i \underline{\nu}_i \frac{\partial X}{\partial \rho_i}$ in (4.5) means that every combination of $\overline{\nu}_i \frac{\partial X}{\partial \rho_i}$ and $\underline{\nu}_i \frac{\partial X}{\partial \rho_i}$ should be considered in the inequality. For instance, when $m = 2$, $\overline{\nu}_1 \frac{\partial X}{\partial \rho_1} + \overline{\nu}_2 \frac{\partial X}{\partial \rho_2}$, $\overline{\nu}_1 \frac{\partial X}{\partial \rho_1} + \underline{\nu}_2 \frac{\partial X}{\partial \rho_2}$, $\underline{\nu}_1 \frac{\partial X}{\partial \rho_1} + \overline{\nu}_2 \frac{\partial X}{\partial \rho_2}$ and $\underline{\nu}_1 \frac{\partial X}{\partial \rho_1} + \overline{\nu}_2 \frac{\partial X}{\partial \rho_2}$ should be checked individually. In other words, (4.5) actually represents 2^m inequalities.

In order to prove the above theorem, we need to establish the following lemma.

Lemma 4.2 *Given matrices A, B, C_1, C_2, $G = G^T$ and $X = X^T$ with X invertible, there exists a matrix F such that*

$$G + (A+BF)X + X(A+BF)^T + X \begin{bmatrix} C_1^T & (C_2+F)^T \end{bmatrix} \begin{bmatrix} C_1 \\ C_2+F \end{bmatrix} X < 0 \quad (4.7)$$

if and only if

$$\begin{bmatrix} G + \hat{A}^T X + X\hat{A} - BB^T & XC_1^T \\ C_1 X & -I \end{bmatrix} < 0 \quad (4.8)$$

where $\hat{A} = A - BC_2$. Moreover, if (4.8) is satisfied, then $F = -(B^T X^{-1} + C_2)$ is the choice for (4.7).

The proof involves Schur complement and completion of squares, and can be found in the literature (see [17], for example.) Next, we will prove Theorem 4.1.

Proof Suppose there exists a positive definite solution $X(\rho)$ to the inequality constraint (4.5). When applying the state feedback law (4.6), the closed-loop system becomes

$$\begin{cases} \dot{x} = (A(\rho) + B(\rho)F(\rho))x \\ z = \begin{bmatrix} C_1(\rho) \\ C_2(\rho)F(\rho) \end{bmatrix} x \end{cases} \qquad x(0) = x_o$$

We will use $A_{cl}(\rho(x))$ and $C_{cl}(\rho(x))$ to denote the closed-loop state matrix and output matrix respectively. By Lemma 4.2, the inequality constraint (4.5) is equivalent to

$$-\sum_{i=1}^{m}\underline{\nu}_i(\rho)\frac{\partial X}{\partial \rho_i}(\rho) + A_{cl}(\rho)X(\rho) + X(\rho)A_{cl}^T(\rho) + X(\rho)C_{cl}^T(\rho)C_{cl}(\rho)X(\rho) < 0.$$

Let $P(\rho) = X^{-1}(\rho)$, and note $\frac{\partial P}{\partial \rho_i} = -X^{-1}(\rho)\frac{\partial X}{\partial \rho_i}X^{-1}(\rho)$. If we premultiply and postmultiply by $P(\rho)$, the above inequality becomes

$$\sum_{i=1}^{m}\underline{\nu}_i(\rho)\frac{\partial P}{\partial \rho_i}(\rho) + P(\rho)A_{cl}(\rho) + A_{cl}^T(\rho)P(\rho) + C_{cl}^T(\rho)C_{cl}(\rho) < 0. \tag{4.9}$$

Let $V(\rho(x)) = x^T P(\rho(x))x$. We will show that $V(x)$ serves as a Lyapunov function of the closed-loop system. Along any state trajectory $x(t)$, we have

$$\begin{aligned}
\dot{V}(t) &= \frac{d}{dt}x^T(t)P(\rho(t))x(t) \\
&= x^T(t)\frac{d}{dt}(P(\rho(t)))x(t) + \dot{x}^T(t)P(\rho(t))x(t) + x^T(t)P(\rho(t))\dot{x}(t) \\
&= x^T(t)(\sum_{i=1}^{m}\dot{\rho}(t)\frac{\partial P}{\partial \rho_i} + A_{cl}^T(\rho(t))P(\rho(t)) + P(\rho(t))A_{cl}(\rho(t)))x(t)
\end{aligned}$$

Because ρ is assumed to vary in the admissible set (4.4), inequality (4.9) guarantees

$$\sum_{i=1}^{m}\dot{\rho}(t)\frac{\partial P}{\partial \rho_i} + A_{cl}^T(\rho(t))P(\rho(t)) + P(\rho(t))A_{cl}(\rho(t) + C_{cl}^T(\rho)C_{cl}(\rho) < 0.$$

Hence,

$$\begin{aligned}
\dot{V}(t) &\leq -x^T(t)C_{cl}^T(\rho(t))C_{cl}(\rho(t))x(t) \\
&= -\|z(t)\|^2
\end{aligned} \tag{4.10}$$

Since $\dot{V} < 0$ for all $x \neq 0$, the the closed-loop system (4.3) is globally asymptotically stable. Now integrate (4.10) from 0 to $+\infty$ to obtain

$$\|z\|_2^2 \leq V|_{t=0} - V|_{t=\infty} = x_o^T P(\rho(x_o))x_o.$$

Thus, $x_o^T X^{-1}(\rho(x_o))x_o$ provides an upper bound on the optimal value of the performance index.

□

Note that a necessary condition for the AMI (4.5) to be feasible for some $X(\rho) > 0$ is that the pair $\{A(\rho), B(\rho)\}$ must be stabilizable in the linear sense for all $\rho \in \mathcal{P}$. Since the quasi-LPV model (4.3) of the nonlinear plant (4.1) is not unique, failing to satisfy this condition for any particular pair in no means indicates the nonlinear system cannot be stabilized.

In comparison to the quadratic Lyapunov function for the LQR problem, the parameter-dependent Lyapunov functions in the form of $x^T P(\rho)x$ are considered for the quasi-LPV plants. A sufficient condition for the regulation problem of the nonlinear system (4.3) is derived in terms of the affine matrix inequalities. Although these are still infinite dimensional constraints, they are much more appealing than the Hamilton-Jacobi equation associated with the optimal regulation problem from a computational point of view. A practical computational scheme for solving the AMI (4.5) will be discussed in the next section.

As a generalization of the LQR theory to nonlinear plants, this quasi-LPV treatment of the nonlinear system can be highly conservative because of the variation rate bounds imposed on the underlying parameters. For a nonlinear time-invariant plant, the value of the chosen parameters and their derivatives, which are functions of the state, should be determined merely by the current state of the system. However, the admissible set defined by (4.4) can include additional parameter trajectories that don't correspond to the dynamics of the system unless the rate bounds are exact. Theorem 4.1, on the other hand, actually requires that the constraint (4.5) be satisfied for all parameter trajectories in $\mathcal{F}_{\mathcal{P}}^{\nu}$. Consequently, the gap between the bound given by Theorem 4.1 and the optimum of the performance criterion can be large depending upon how good the rate bounds of the parameters are.

Since the original control objective is to minimize the performance index, and Theorem 4.1 provides an upper bound on it, we can therefore formulate the regulation problem as the minimization over the upper bound:

$$\min_{X(\rho),Z} \operatorname{tr} Z$$

subject to

$$\begin{bmatrix} -\sum_{i=1}^{m} \underline{\nu}_i(\rho)\frac{\partial X}{\partial \rho_i}(\rho) + \hat{A}(\rho)X(\rho) + X(\rho)\hat{A}^T(\rho) - B(\rho)B^T(\rho) & X(\rho)C_1^T(\rho) \\ C_1(\rho)X(\rho) & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} Z & I \\ I & X(\rho) \end{bmatrix} > 0 \tag{4.11}$$

$$X(\rho) > 0$$

The inequality (4.11) is equivalent to $X^{-1}(\rho) < Z$ by Schur complement. Since $x^T X^{-1}(\rho)x$ is an upper bound on the performance criterion, we want $X^{-1}(\rho)$ to be "small." By bounding $X^{-1}(\rho)$ by a constant matrix Z, we hope to get good performance over the whole operating region. The quantity $\operatorname{tr} Z$ also has a nice physical interpretation. Let e_i denote the *ith* standard basis vector of $\mathbf{R}^n$, and z_i be the output of the closed-loop system with the initial condition $x_o = e_i$. Then it is easy to derive that

$$\sum_{i=1}^{n} \|z_i\|_2^2 \leq \operatorname{tr} Z \tag{4.12}$$

4.2 Computational Considerations

An *ad hoc* computation scheme proposed in [39] for induced $\mathcal{L}_2$-norm control of LPV systems is adopted here to solve the state feedback regulation problem stated in Theorem 4.1. First choose a set of $\mathcal{C}^1$ functions $\{\phi_i(\rho)\}_{i=1}^M$ as a basis for $X(\rho)$, i.e.,

$$X(\rho) = \sum_{i=1}^{M} \phi_i(\rho)X_i \tag{4.13}$$

where $X_i \in \mathcal{S}^{n\times n}$ is the symmetric coefficient matrix for $\phi_i(\rho)$. If $X(\rho)$ is substituted with (4.13), the minimization scheme for the nonlinear regulation problem then reduces to

$$\min_{\{X_i\},Z} \mathrm{tr}Z$$

$$\mathcal{M}_1(X_1,\cdots,X_M,\rho) = \begin{bmatrix} -\sum_{i=1}^{m}\underline{\nu}_i(\rho)(\sum_{j=1}^{M}\frac{\partial\phi_j}{\partial\rho_i}X_j) + \sum_{j=1}^{M}\phi_j(\rho)(\hat{A}(\rho)X_j + X_j\hat{A}^T(\rho)) - B(\rho)B^T(\rho) & \sum_{j=1}^{M}\phi_j(\rho)X_jC_1^T(\rho) \\ \sum_{j=1}^{M}\phi_j(\rho)C_1(\rho)X_j & -I \end{bmatrix} < 0 \tag{4.14}$$

$$\mathcal{M}_2(X_1,\cdots,X_M,Z,\rho) = \begin{bmatrix} Z & I \\ I & \sum_{j=1}^{M}\phi_j(\rho)X_j \end{bmatrix} > 0 \tag{4.15}$$

$$\mathcal{M}_3(X_1,\cdots,X_M,\rho) = \sum_{j=1}^{M}\phi_j(\rho)X_j > 0 \tag{4.16}$$

for all $\rho \in \mathcal{P}$. The three constraints above are linear matrix inequalities in terms of the matrix variables $\{X_i\}_{i=1}^{M}$ and Z. In order to solve this infinite dimensional convex optimization problem, we will instead solve it over a finite set of parameter values. We can grid the parameter set $\mathcal{P}$ by L_k points in the kth dimension, and evaluate the above LMI's at these points. Since (4.14) consists of 2^m constraints, a total of $(2^m+2)\prod_{k=1}^{m}L_k$ LMI's in terms of $M+1$ matrix variables $\{X_i\}$ and Z have to be considered. To guarantee the global feasibility of the above LMI's over all $\rho \in \mathcal{P}$, the gridding points on which the LMI's are solved must be sufficiently dense. Next we will use the 1-d case as an example to give an approximation on the required density of the gridding points.

When $m = 1$, $\mathcal{P} \subset \mathbf{R}$. Grid the set $\mathcal{P}$ by L points $\{\rho_k\}_{k=1}^{L}$. Suppose the LMI's are feasible at these gridding points. Pick $T > 0$ and $\delta > 0$ such that

$$\|X_i\|_F \leq T \qquad i = 1,\cdots,M \tag{4.17}$$

$$\mathcal{M}_1(X_1,\cdots,X_M,\rho_k) \leq -\delta I \qquad i = 1,\cdots,L \tag{4.18}$$

$$\mathcal{M}_2(X_1,\cdots,X_M,Z,\rho_k) \;\geq\; \delta I \qquad i=1,\cdots,L \tag{4.19}$$

$$\mathcal{M}_3(X_1,\cdots,X_M,\rho_k) \;\geq\; \delta I \qquad i=1,\cdots,L \tag{4.20}$$

Lemma 4.3 *Assume that $A(\rho)$, $B(\rho)$, $C_1(\rho)$ and $C_2(\rho)$ are continuously differentiable, and the ϕ_i's are twice continuously differentiable. Let*

$$\begin{aligned} h \;=\; & \delta\cdot\min\Bigg\{\left[T\sum_{i=1}^{M}\max_{\rho\in\mathcal{P}}\left|\frac{d\underline{\nu}}{d\rho}\frac{d\phi_i}{d\rho}-\underline{\nu}\frac{d^2\phi_i}{d\rho^2}\right|+2T\sum_{i=1}^{M}\max_{\rho\in\mathcal{P}}\left\|\frac{d(\phi_i\hat{A})}{d\rho}\right\|_F\right.\\ & \left.+\max_{\rho\in\mathcal{P}}\left\|\frac{d(BB^T)}{d\rho}\right\|_F+2T\sum_{i=1}^{M}\max_{\rho in\mathcal{P}}\left\|\frac{d(\phi_iC_1)}{d\rho}\right\|_F\right]^{-1},\\ & \left[T\sum_{i=1}^{M}\max_{\rho\in\mathcal{P}}\left|\frac{d\phi_i}{d\rho}\right|\right]^{-1}\Bigg\}. \end{aligned} \tag{4.21}$$

If $|\rho_{k+1}-\rho_k|\leq h$ for all $k=1,\cdots,L-1$, and matrices $\{X_i\}_{i=1}^M$ and Z satisfies the constraints (4.17)-(4.20), then

$$\mathcal{M}_1(X_1,\cdots,X_M,\rho) \;<\; 0 \tag{4.22}$$

$$\mathcal{M}_2(X_1,\cdots,X_M,Z,\rho) \;>\; 0 \tag{4.23}$$

$$\mathcal{M}_3(X_1,\cdots,X_M,\rho) \;>\; 0 \tag{4.24}$$

for all $\rho\in\mathcal{P}$.

Proof For any $\rho\in\mathcal{P}$, there exists a k such that $\rho\in[\rho_k,\ \rho_{k+1}]$. Let

$$\begin{aligned} \Delta\mathcal{M}_1^X(\rho) &= \mathcal{M}_1(X_1,\cdots,X_M,\rho)-\mathcal{M}_1(X_1,\cdots,X_M,\rho_k)\\ \Delta\mathcal{M}_2^X(\rho) &= \mathcal{M}_2(X_1,\cdots,X_M,Z,\rho)-\mathcal{M}_1(X_1,\cdots,X_M,Z,\rho_k)\\ \Delta\mathcal{M}_3^X(\rho) &= \mathcal{M}_3(X_1,\cdots,X_M,\rho)-\mathcal{M}_1(X_1,\cdots,X_M,\rho_k) \end{aligned}$$

Note that for any $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, we have $\|M\|_F \leq \|M_{11}\|_F + \|M_{12}\|_F + \|M_{21}\|_F + \|M_{22}\|_F$. Thus,

$$
\begin{aligned}
&\left\|\Delta\mathcal{M}_1^X(\rho)\right\|_F \\
\leq\ & \sum_{i=1}^{M}\left\|(\underline{\nu}(\rho)\frac{d\phi_i}{d\rho}(\rho) - \underline{\nu}(\rho_k)\frac{d\phi_i}{d\rho}(\rho_k))X_i\right\|_F + 2\sum_{i=1}^{M}\left\|(\phi_i(\rho)\hat{A}(\rho) - \phi_i(\rho_k)\hat{A}(\rho_k))X_i\right\|_F \\
& + \left\|B(\rho)B^T(\rho) - B(\rho_k)B^T(\rho_k)\right\|_F + 2\sum_{i=1}^{M}\|(\phi_i(\rho)C_1(\rho) - \phi_i(\rho_k)C_1(\rho_k))X_i\|_F \\
\leq\ & (\rho - \rho_k)\left[\sum_{1=1}^{M}\left\|(\frac{d\underline{\nu}}{d\rho}\frac{d\phi_i}{d\rho} - \underline{\nu}\frac{d^2\phi_i}{d\rho^2})(\xi_{1i})X_i\right\|_F + 2\sum_{i=1}^{M}\left\|\frac{d(\phi_i\hat{A})}{d\rho}(\xi_{2i})X_i\right\|_F\right. \\
& \left. + \left\|\frac{d(BB^T)}{d\rho}(\xi_3)\right\|_F + 2\sum_{i=1}^{M}\left\|\frac{d(\phi_i C_1)}{d\rho}(\xi_{4i})X_i\right\|_F\right] \\
\leq\ & h\left[T\sum_{i=1}^{M}\max_{\rho\in\mathcal{P}}\left|\frac{d\underline{\nu}}{d\rho}\frac{d\phi_i}{d\rho} - \underline{\nu}\frac{d^2\phi_i}{d\rho^2}\right| + 2T\sum_{i=1}^{M}\max_{\rho\in\mathcal{P}}\left\|\frac{d(\phi_i\hat{A})}{d\rho}\right\|_F\right. \\
& \left. + \max_{\rho\in\mathcal{P}}\left\|\frac{d(BB^T)}{d\rho}\right\|_F + 2T\sum_{i=1}^{M}\max_{\rho\in\mathcal{P}}\left\|\frac{d(\phi_i C_1)}{d\rho}\right\|_F\right] \\
\leq\ & \delta
\end{aligned}
$$

where $\xi_{1i},\ \xi_{2i},\ \xi_3,\ \xi_{4i} \in [\rho_k, \rho_{k+1}]$. Similarly, we can derive

$$
\left\|\Delta\mathcal{M}_2^X(\rho)\right\|_F = \left\|\Delta\mathcal{M}_3^X(\rho,k)\right\|_F \leq hT\sum_{i=1}^{M}\max_{\rho\in\mathcal{P}}\left|\frac{d\phi_i}{d\rho}\right| \leq \delta
$$

Since

$$
\bar{\sigma}(\Delta\mathcal{M}_i^X(\rho)) \leq \left\|\Delta\mathcal{M}_i^X(\rho)\right\|_F \leq \delta \qquad \forall\, i = 1,2,3
$$

we can conclude that (4.22)-(4.24) are satisfied for all $\rho \in \mathcal{P}$. □

Although we have demonstrated that it is possible to get an estimation of the density of the gridding points to ensure the global feasibility of the LMI's, we must solve a set of more stringent LMI's (4.17) -(4.20). A more efficient way is to solve the original LMI's (4.14)-(4.16) at a chosen set of parameter values, then verify the feasibility of those LMI's at a denser set of points. If the LMI's are not feasible at some points, then we should add these points to the original set of parameter values and redo the design.

One of the difficulties when applying this quasi-LPV scheme is that one has to estimate the variation rate bounds of the underlying parameters for the closed-loop system before the control design. One way to get a good estimation is to try to find a quadratic Lyapunov function first, in which case the rate bounds play no role in the LMI's. Then get the rate bounds of the parameters from the closed-loop simulation. For the systems where the quadratic Lyapunov function cannot be found, one should first choose the rate bounds of the "uncertain" parameters to be sufficiently wide, which is equivalent to allowing very little dependence of $X(\rho)$ on those parameters. Once a stabilizing controller is found, we can get a better idea of the rate bounds from the closed-loop simulation. Then the design can be refined with the tighter rate bounds.

The quasi-LPV approach has a few obvious limitations. Associated with gridding is the "dimension curse." Fortunately, the number of varying parameters can usually be kept no more than 3. Another significant problem is the lack of guidance in choosing the basis functions for $X(\rho)$, and an inappropriate choice may greatly affect the achievable performance level. Despite these drawbacks, it provides us a tractable computation scheme to construct feedback control laws that guarantee global stability and performance for a large class of nonlinear systems.

4.3 Connections with the FRE Approach

The quasi-LPV technique aims at searching for a state-dependent Lyapunov function. By bounding the variation rate of the parameters, it guarantees performance provided that the states vary in a pre-specified set. Since the scheduling parameters in the Quasi-LPV model are actually functions of the states, they cannot vary freely as in true LPV systems. Thus, the controller designed to handle the worst parameter trajectory is inevitably conservative for the nonlinear system.

From a designer's standpoint, the variation rate bounds can be used as a design parameter that reflects the conservatism of the resulting controller. At one extreme, a quadratic Lyapunov function may be found so that the closed-loop system can withstand arbitrarily fast parameter variation. At the other extreme, the solution of the frozen Riccati approach is replicated. If the rate bounds of the parameters are set to zero, or in other words, the states are treated as though they were "frozen," the AMI (4.5) in Theorem 4.1 simply reduces to a parameter-dependent matrix Riccati inequality. Therefore, the FRE controller can be viewed as the most aggressive controller given by the quasi-LPV scheme, which comes with no guarantee on performance, or even stability.

In the previous chapter, Example 3.7 was used to show that the FRE solution can produce an unstable closed-loop system. However, using the same state-dependent linear representation, the LPV technique can generate a stable solution by taking into account the rate variation of x_1. Recall that

$$\dot{x}_1 = \tanh(10\ (e^{x_1}x_2^2 - e^{-x_1}x_3^2)),$$

and since tanh is bounded by $+1$ and -1, the variation rate bounds of x_1 are therefore determined. The allowable parameter set is then

$$\mathcal{F}_{\mathcal{P}}^{\nu} = \{\rho : \rho \in [-3, 3]\ and\ |\dot{\rho}| \leq 1\}.$$

The same $A(x_1)$, $B(x_1)$ and $C(x_1)$ as in the FRE design are used to form the AMI's at 20 uniform grid points on $[-3, 3]$. Five basis functions, $\{1, e^{\rho/2}, e^{\rho}, e^{-\rho/2}, e^{-\rho}\}$, are used to span $X(\rho)$. A feasible solution to the LMI's evaluated at the gridding points is found. The global feasibility is checked at much denser points. Figure 4.1 shows the responses of the closed-loop system starting from two initial conditions $x_o = [0\ 1\ 0]^T$ and $x_o = [0\ 0\ 1]^T$. For both initial conditions, x_2 and x_3 are driven to zero eventually whereas x_1 stays at a constant level. This is because x_1 enters the performance index in a way that does not incur extra cost as long as both x_2 and x_3 are zero.

This example also demonstrates the conservatism of the LPV controller. The closed-loop simulation results with the LPV controller designed using a rate bound of

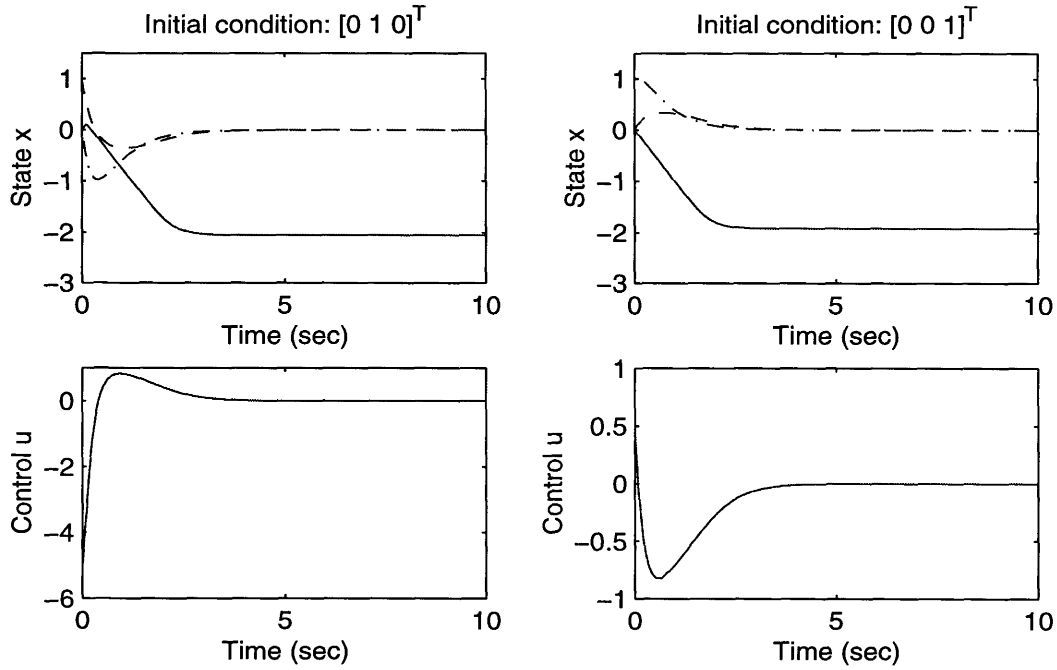

Figure 4.1: Closed-loop simulations with the LPV controller of Example 3.7 starting from two different initial conditions. Rate bound of x_1 is 1.0. x_1 (solid), x_2 (dashed) and x_3 (dashdot) are shown in the upper plots.

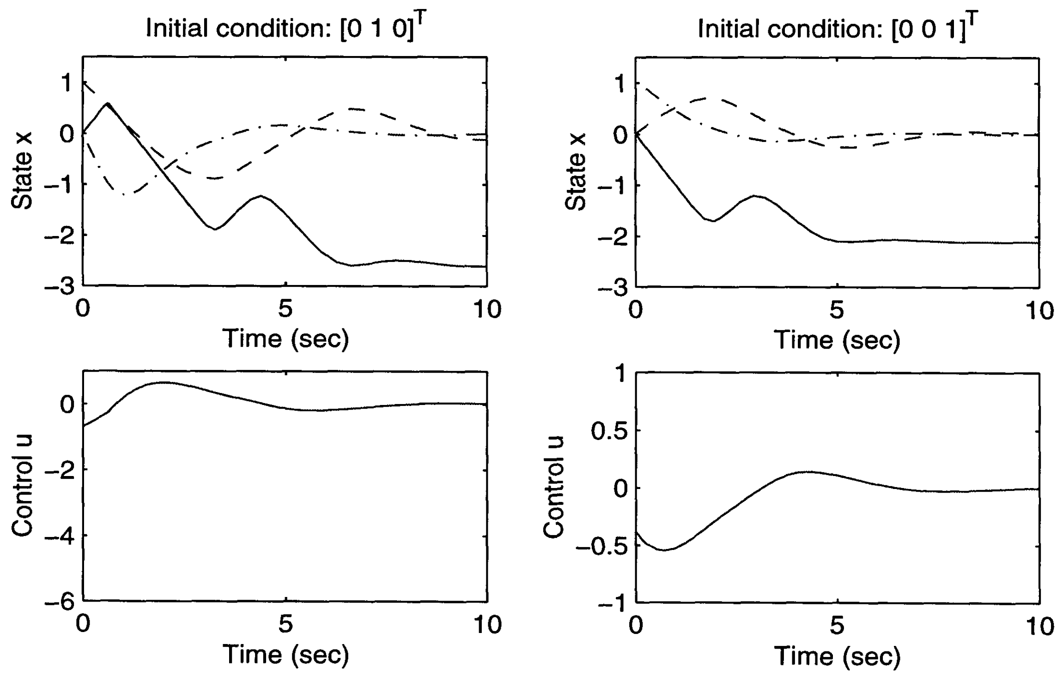

Figure 4.2: Closed-loop simulations with the LPV controller of Example 3.7 starting from two different initial conditions. Rate bound of x_1 is 0.4. x_1 (solid), x_2 (dashed) and x_3 (dashdot) are shown in the upper plots.

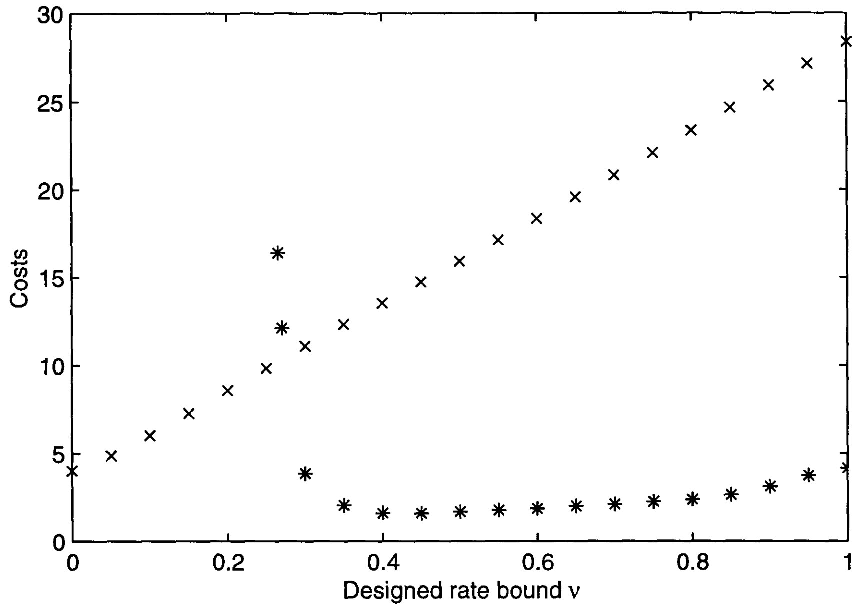

Figure 4.3: Guaranteed costs by LPV design ($\times$) vs actual costs incurred by LPV controller ($*$) of Example 3.7

0.4 are plotted in Figure 4.2. Although $\dot{x}_1$ may exceed this bound during operation, the simulations indicate that it still yields a stabilizing controller. Moreover, the relaxed design rate bound produces a less aggressive controller allowing the states to fluctuate longer before settling down. In order to illustrate the effect of the rate bound on the achievable performance, the guaranteed cost and actual cost versus rate bound used for the LPV design are drawn in Figure 4.3. The guaranteed cost is $\mathrm{tr}Z$ obtained by the minimization algorithm, and the actual cost is calculated as the sum of the cost the closed-loop system accrued starting from $[0\ 1\ 0]^T$ and $[0\ 0\ 1]^T$. The former should serve as an upper bound for the latter as long as $\dot{x}_1$ stays within the design rate bounds along the trajectories. As one can see in Figure 4.3, the guaranteed cost is indeed a monotonely increasing function of the design rate bound. The gap between the guaranteed cost and the actual cost is substantial when $\nu = 1$. The lowest actual cost occurs near $\nu = 0.4$. Further reduction on the rate bound causes the actual cost to exceed the value rendered by the LPV calculation, and the system goes unstable eventually.

Another advantage the quasi-LPV approach has over the FRE is the form of the assumed Lyapunov functions. The former searches among those of the form $x^T P(x) x$ for some positive definite $P(x)$ while the latter assumes the gradient of the value function has the form of $x^T P(x)$ for some $P(x) > 0$. It is not hard to see that the set of the value functions the FRE controllers target is only a subset of the set of the Lyapunov functions that the LPV controllers can find. As shown earlier, the positive definite function defined by (3.38) belongs to the difference of the two sets. Let us revisit Example 3.12 where the value function is given by (3.38). The same SDLR is used as in the FRE control. Here we compare two schemes where different basis functions are chosen for $X(\rho)$.

Scheme 1

• Choose the varying parameter $\rho = [x_1\ x_2]$. Set the operating range as $\mathcal{P} = [-5, 5] \times [-5, 5]$. So the allowable parameter set is

$$\mathcal{F}^{\nu}_{\mathcal{P}} = \{\rho : \rho \in \mathcal{P}, \dot{\rho}_1 = \rho_2\}.$$

• Choose the $4th$ order Legendre polynomials of ρ_1 as the basis functions for $X(\rho)$

$$\{\phi_i\}_{i=1}^4 = \{1,\ \frac{\rho_1}{5},\ (3(\frac{\rho_1}{5})^2 - 1)/2,\ (5(\frac{\rho_1}{5})^3 - 3(\frac{\rho_1}{5}))/2\}$$

• Grid $\mathcal{P}$ uniformly with 20 points in each dimension.

Scheme 2

• Choose the varying parameter $\rho = [e^{x_1}\ x_2]$. Set the operating range as $\mathcal{P} = [e^{-5}, e^5] \times [-5, 5]$. So the allowable parameter set is

$$\mathcal{F}^{\nu}_{\mathcal{P}} = \{\rho : \rho \in \mathcal{P}, \dot{\rho}_1 = \rho_2 e^{\rho_1}\}.$$

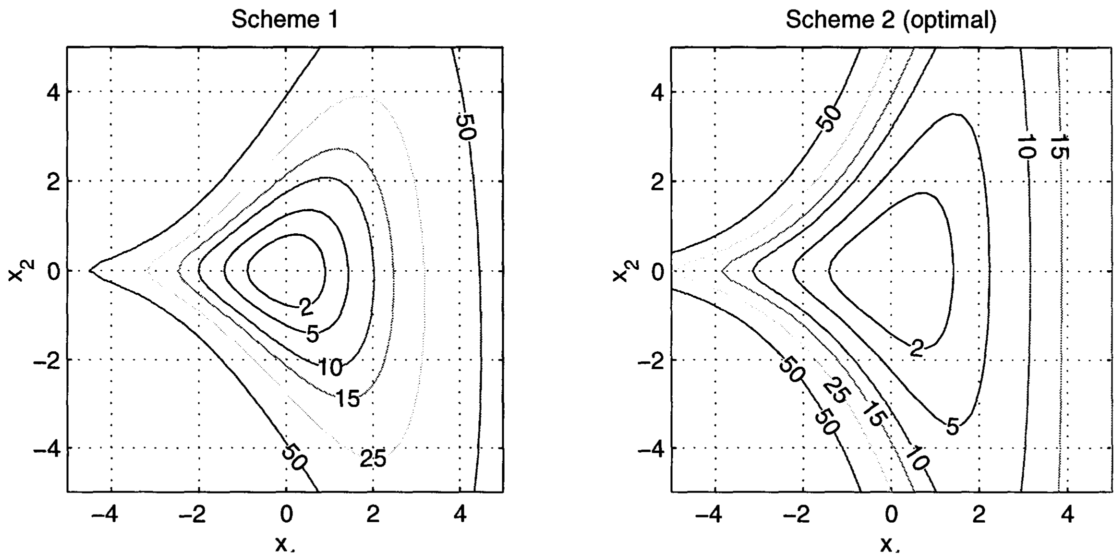

Figure 4.4: Contours of the Lyapunov functions found by two quasi-LPV schemes of Example 3.12

- Choose the basis functions for $X(\rho)$

$$\{\phi_i\}_{i=1}^{5} = \{1,\ \rho_1,\ \rho_1^{-1},\ \rho_1^{1/2},\ \rho_1^{-1/2}\}$$

- Grid $\mathcal{P}$ with 20 points uniformly on a log scale in ρ_1 and with 20 points uniformly on a linear scale in ρ_2.

Since the derivative of x_1 is given by the dynamics, we restrict $X(\rho)$ to depend solely on ρ_1 so that we don't have to estimate the variation rate of ρ_2. Scheme 1 assumes that we have no information on the value function, and $4th$ order polynomials are used to span $X(\rho)$. Scheme 2 includes the right basis functions to generate the actual value function, so the optimal value function is expected to be found since the variation rate of ρ_1 assumed in $\mathcal{F}_{\mathcal{P}}^{\nu}$ is exact.

Figure 4.4 shows the contours of the Lyapunov functions found by the two quasi-LPV schemes. As expected, Scheme 2 returns the actual value function. Scheme 1 manages to find a Lyapunov function with very similarly shaped level curves. Table 4.1 compares the actual cost incurred by the FRE and the two quasi-LPV controllers starting from four different initial conditions. As can be seen, for the two initial

Method	Initial Condition			
	$(2,2)$	$(2,-2)$	$(-2,2)$	$(-2,-2)$
FRE	4.63	4.65	143	220
LPV1	10.2	9.90	54.4	49.6
LPV2 (optimal)	4.54	4.54	33.6	33.6

Table 4.1: Cost comparison of different controllers of Example 3.12

conditions where the FRE controller does really poorly, Scheme 1 is able to reduce the cost dramatically. However, it incurs more cost on the other two initial conditions where the FRE controller is very close to optimal. This is caused by bounding $X(\rho)$ with a constant matrix Z in the minimization step.

4.4 Conclusions

The quasi-LPV treatment of nonlinear plants enables us to extend the well developed linear theory to nonlinear systems. The recent development in solving LMI's provides a viable option for synthesizing stabilizing controllers. Unlike the FRE controller, the quasi-LPV regulator guarantees global stability at a price of potential conservatism by bounding the variation rate of the chosen parameters. If the information on the variation rate is accurate, it is demonstrated through examples that quasi-LPV scheme can even generate an optimal controller. Although only the state feedback problem is considered in this chapter, the same treatment is readily applied to the output feedback case, a nonlinear analogue of $\mathcal{H}_2$ control for linear time-invariant systems.

Chapter 5 Nonlinear $\mathcal{H}_\infty$ Control

5.1 Introduction

One of the principle objectives of modern control theory is to synthesize feedback controllers for systems that have the ability to attenuate uncertain exogenous disturbances. The $\mathcal{H}_\infty$ norm (or equivalently, $\mathcal{L}_2$-induced norm) of linear time-invariant systems measures the worst case system gain over finite-power input signals, and is often used as the disturbance attenuation measurement. The $\mathcal{H}_\infty$ (sub)optimal control problem was heavily studied in the 1980's, and the state-space solutions were described in terms of two coupled algebraic Riccati equations [13]. This breakthrough in the linear theory motivated people to look at its nonlinear counterpart. In the past decade, nonlinear equivalents to $\mathcal{H}_\infty$ analysis and synthesis have been developed using differential game approaches [4, 5] as well as nonlinear dissipation theory [38, 21, 35]. In particular, sufficient conditions for the existence of global solutions were given in terms of a Hamilton-Jacobi equation/inequality (HJE/HJI). However, due to the computational complexity of the HJE/HJI, the general framework was only applied to restricted linearized $\mathcal{H}_\infty$ control problems.

In the meantime, several techniques were developed that attempted to extend the linear $\mathcal{H}_\infty$ theory to a much broader class of systems. For the class of nonlinear input-affine systems, Lu [28] proposed to characterize the solutions in terms of some algebraic conditions, namely, nonlinear matrix inequalities (NLMI's) which are in fact state-dependent LMI's. Suffering the same problem of other pointwise approaches, the pointwise solution to the NLMI's is not sufficient to guarantee a desired controller. Some additional condition is required, but the computational implication on the constraints of the solution to the NLMI's is not clear. On the other hand, $\mathcal{H}_\infty$ theory of linear time-invariant systems was generalized to the linear parameter varying plants by Wu *et al.* [39]. Searching for a parameter-dependent storage function, the LPV technique synthesizes controllers that guarantee good performance when the underlying parameters have bounded rate variation. This LPV approach can be directly applied to nonlinear systems where the varying parameters become functions of the states instead of "free" variables.

Both of these pseudo linear techniques rely on a state-dependent linear representation of the nonlinear dynamics. As revealed in Chapter 2, different linear representations used for the FRE design may lead to quite different performance levels. Since for the $\mathcal{H}_\infty$ suboptimal control the Riccati type inequalities have to be solved, we can expect a similar effect. Hence, we propose an enhanced quasi-LPV strategy that incorporates the design freedom of choosing the state-dependent linear representation into the control selection process.

In this chapter, the NLMI and standard LPV techniques will be reviewed first. Then an infinite dimensional Linear Matrix Inequality (LMI) condition is derived to compute the best achievable $\mathcal{L}_2$-gain over all the linear representations of the dynamics and parameter-dependent scaling functions. More specifically, the problem of $\mathcal{L}_2$-gain analysis can be formulated as an infinite dimensional convex optimization problem which can be solved by gridding the parameter space and using the recently developed interior-point methods. Unfortunately, the problem of state feedback synthesis is not convex in both design variables. The synthesis problem will be cast as an infinite dimensional Bilinear Matrix Inequality (BMI) problem. Because of the non-convexity of the condition in both design variables, there is no guarantee of finding the global optimum. However, analogous to the "$D-K$ iteration" for μ synthesis [40], an iterative procedure is presented that attempts to find the best achievable performance. A practical computation scheme is proposed along with examples demonstrating that the $\mathcal{L}_2$-gain of the closed loop system can be greatly improved when the choice of the quasi-LPV representation for the nonlinear dynamics is considered as one of the design variables. It is our hope that the introduction of this extra design freedom in forming the quasi-LPV model can be viewed as an enhancement to the existing LPV methodology, and will reduce the conservatism to a minimum.

To keep our focus on the basic ideas, we will only discuss the state feedback case of the nonlinear $\mathcal{H}_\infty$ control in this chapter. The very same ideas can be applied to the output feedback case with little modification.

5.2 Induced $\mathcal{L}_2$-Gain of Nonlinear Systems

Consider a nonlinear input-affine system:

$$\begin{cases} \dot{x} &= f(x) + g(x)w \\ z &= h(x) + k(x)w \end{cases} \tag{5.1}$$

where $x \in \mathbf{R}^n$ are local coordinates for a smooth state-space manifold $\mathcal{M}$, $w \in \mathbf{R}^p$ is the exogenous input, and $z \in \mathbf{R}^q$ is the regulated output. We further assume that f, g, h and k are all $\mathcal{C}^1$ functions with $f(0) = 0$ and $h(0) = 0$.

Definition 5.1 *Let $\gamma \geq 0$. System (5.1) is said to have $\mathcal{L}_2$-gain less than or equal to γ if*

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|w(t)\|^2 dt \tag{5.2}$$

for all $T \geq 0$ and all $w \in \mathcal{L}_2[0, T)$ with the initial condition $x(0) = 0$.

This definition is a special case of the general definition of dissipativity as given in [38] with a supply rate of $\gamma^2\|w\|^2 - \|z\|^2$. The following lemma, taken from [35], states that the solvability of the Hamilton-Jacobi inequality guarantees that the $\mathcal{L}_2$-gain of the nonlinear system does not exceed a certain level. It also reveals that the finite $\mathcal{L}_2$-gain implies (internal) asymptotic stability under the detectability condition.

Lemma 5.2 *[35] System (5.1) has $\mathcal{L}_2$ gain $\leq \gamma$ if $R(x) = \gamma^2 I - k^T(x)k(x) > 0$ and there exists a non-negative $\mathcal{C}^1$ solution $V(x)$ to the Hamilton-Jacobi inequality:*

$$V_x(x)f(x)+(\frac{1}{2}V_x(x)g(x)+h^T(x)k(x))R^{-1}(x)(\frac{1}{2}V_x(x)g(x)+h^T(x)k(x))^T+h^T(x)h(x) \leq 0, \tag{5.3}$$

$$V(0) = 0$$

Moreover, if V is proper (i.e., for each $c > 0$ the set $\{x \in \mathcal{M} : 0 \leq V(x) \leq c\}$ is compact) and $\{f(x), h(x)\}$ is zero-state detectable, then the system with $w = 0$ is globally asymptotically stable at 0.

Any $V(x)$ satisfying (5.3) in the above lemma is called a storage function in much of the literature [35, 37] since it satisfies the dissipation inequality

$$V(x(t_1)) - V(x(t_o)) \leq \int_{t_o}^{t_1} (\gamma^2 \|w(t)\|^2 - \|z(t)\|^2) dt \tag{5.4}$$

$$V(0) = 0.$$

Among all the storage functions, the minimal one is the available storage $V_a(x)$ which is defined as

$$V_a(x) := \sup_{u \in \mathcal{L}_2[0,T)} \int_0^T (\|z(t)\|^2 - \gamma^2 \|w(t)\|^2) dt \tag{5.5}$$

for all $T > 0$ and $x(0) = x$. Note that if the available storage function is continuously differentiable, it satisfies (5.3) with equality. For a system with $\mathcal{L}_2$-gain $\leq \gamma$, $V_a(x)$ measures to what extent the $\mathcal{L}_2$-norm of the output signal can be larger than γ^2 times the $\mathcal{L}_2$-norm of the input signal, depending on the initial state x. All the other storage functions satisfying the Hamilton-Jacobi inequality (5.3) serve as upper bounds of the available storage. The minimax problem stated in Chapter 3 attempts to find the smooth available storage function. Most tools developed for nonlinear $\mathcal{H}_\infty$ control center around the search for a storage function. While our focus is on storage functions that are at least C^1, an alternative is provided by considering viscosity solutions to (5.3) [23].

Now we consider the state feedback suboptimal $\mathcal{H}_\infty$ control problem. The system setup is as follows:

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)w + g_2(x)u \\ z &= \begin{bmatrix} h_1(x) \\ h_2(x) + u \end{bmatrix} \end{aligned} \tag{5.6}$$

with the same assumptions for (5.1) and an additional control input $u \in \mathbf{R}^r$. We seek a state feedback law $u(x)$ such that the closed-loop system has $\mathcal{L}_2$-gain $\leq \gamma$ from w to z. Note that the dependence of the regulated output z on the control input u is quite simple in this special case, but it is not very difficult to generalize the results to affine systems with a more general dependence on u. Similar results on systems with $h_2(x) = 0$ are well known [21, 35].

Theorem 5.3 *Consider system (5.6). Suppose there exists a $\mathcal{C}^1$ solution $V(x) \geq 0$ to the following Hamilton-Jacobi inequality:*

$$V_x(x)(f(x)-g_2(x)h_2(x))+\frac{1}{4}V_x(x)(\frac{1}{\gamma^2}g_1(x)g_1^T(x)-g_2(x)g_2^T(x))V_x^T(x)+h_1^T(x)h_1(x) \leq 0, \tag{5.7}$$

then the state feedback

$$u = -\frac{1}{2}g_2^T(x)V_x^T(x) - h_2(x) \tag{5.8}$$

yields a closed-loop system with $\mathcal{L}_2$-gain $\leq \gamma$.

Proof For the sake of simplicity, the dependence of the functions on x is omitted. By Lemma 5.2, system (5.6) has $\mathcal{L}_2$-gain $\leq \gamma$ if the following HJI holds

$$V_x(f + g_2 u) + \frac{1}{4\gamma^2}V_x g_1 g_1^T V_x^T + h_1^T h_1 + (h_2 + u)^T(h_2 + u) \leq 0.$$

Let $\tilde{u} = u + h_2$, the above inequality becomes

$$V_x(f - g_2 h_2) + \frac{1}{4\gamma^2}V_x g_1 g_1^T V_x^T + h_1^T h_1 + V_x g_2 \tilde{u} + \tilde{u}^T \tilde{u} \leq 0.$$

Completing the square of the last two terms of the left-hand side leads to

$$V_x(f - g_2 h_2) + \frac{1}{4\gamma^2}V_x g_1 g_1^T V_x^T + h_1^T h_1 - \frac{1}{4}V_x g_2 g_2^T V_x^T + (\tilde{u} + \frac{1}{2}V_x g_2)^T(\tilde{u} + \frac{1}{2}V_x g_2) \leq 0.$$

This inequality is equivalent to (5.7) if we take $\tilde{u} = -\frac{1}{2}V_x g_2$, which results in the state feedback law (5.8). □

This theorem establishes a sufficient condition for the existence of an $\mathcal{H}_\infty$ suboptimal state feedback controller for system (5.6). Note that any $V(x)$ satisfying (5.7) is a storage function for the closed-loop system with the control law (5.8).

The Hamilton-Jacobi inequality is a first order partial differential inequality. Although there have been some numerical schemes proposed to solve the HJI arising from the nonlinear $\mathcal{H}_\infty$ control problem based on the finite difference principle [23, 27], they all involve gridding the state space which causes exponential growth with the state dimension. Therefore, they are not practical for systems with more than a few states.

5.3 NLMI Approach

Similar to the frozen Riccati approach to the minimax problem discussed in the previous chapter, the partial differential inequalities (5.7) can be reduced to algebraic inequalities if we assume that a $\mathcal{C}^1$ solution exists and is of a certain form. Lu [28] used nonlinear matrix inequalities to characterize the solutions. Consider the nonlinear system (5.6). By Lemma 3.4 it can be written in the state-dependent linear structure for some continuous matrix-valued functions $A(x)$, $C_1(x)$ and $C_2(x)$:

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A(x) & B_1(x) & B_2(x) \\ C_1(x) & 0 & 0 \\ C_2(x) & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{5.9}$$

To keep the derivation simple, the standard assumptions are made in the above model. The feed-through term from w to z is assumed to be zero, and D_{22}, where the control u comes in, is scaled to identity. More complicated formulas can be derived if these assumptions are relaxed.

Lemma 5.4 *[28] Suppose there is a $\mathcal{C}^0$ matrix-valued function $X(x) > 0$ with $\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x)$ for some $\mathcal{C}^1$ function $V(x) : \mathcal{M} \to \mathbf{R}^+$ such that for all $x \in \mathcal{M}$, the fol-*

lowing inequality constraint holds

$$\begin{bmatrix} \hat{A}(x)X(x) + X(x)\hat{A}^T(x) - B_2(x)B_2^T(x) & X(x)C_1^T(x) & B_1(x) \\ C_1(x)X(x) & -I & 0 \\ B_1^T(x) & 0 & -\gamma^2 I \end{bmatrix} < 0 \qquad (5.10)$$

with $\hat{A}(x) = A(x) - B_2(x)C_2(x)$, *then the closed-loop system (5.9) has an* $\mathcal{L}_2$*-gain less than* γ *under the state feedback law:*

$$u(x) = -(B_2(x)X^{-1}(x) + C_2(x))x. \qquad (5.11)$$

The algebraic inequality (5.10), known as a nonlinear matrix inequality (NLMI), is sufficient to ensure the HJI (5.7) with $V_x(x) = 2x^T X^{-1}(x)$. However, it is obviously more conservative than the original HJI condition. Since the NLMI (5.10) is in fact a state-dependent LMI, the existing convex optimization methods for solving LMI's can be directly applied to seek a pointwise solution. It is also revealed in [28] that pointwise solvability of the NLMI indicates the existence of a continuous solution. However, as with the FRE results, a pointwise positive definite solution to the NLMI is not sufficient to guarantee the $\mathcal{L}_2$-gain. The additional requirement of the existence of a storage function must be imposed to ensure that the HJI (5.7) is indeed satisfied. But how to enforce this constraint on the solution to the NLMI's remains as a research topic.

5.4 Preliminary LPV Results

Let's review some results for linear parameter varying systems with bounded parameter variation rates. Readers please refer to [39] for details.

Define the parameter ν-variation set as

$$\mathcal{F}_{\mathcal{P}}^{\nu} := \{\rho \in \mathcal{C}^1(\mathbf{R}^+, \mathbf{R}^m) : \rho(t) \in \mathcal{P},\ \underline{\nu}_i(\rho) \le \dot{\rho}_i \le \overline{\nu}_i(\rho),\ i = 1, \cdots, m\}$$

where $\mathcal{P} \subset \mathbf{R}^m$ is a compact set, and $\underline{\nu}_i$ and $\overline{\nu}_i$ bound the variation rate of ρ_i. Note that the variation bounds are functions of the parameter value.

5.4.1 Analysis Test

The LPV system is governed by

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & D(\rho(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{5.12}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, and A, B, C and D are continuous functions of proper dimensions. The next lemma gives a sufficient condition for the system to have an $\mathcal{L}_2$-gain bounded by γ.

Lemma 5.5 *[39] If there exists a $\mathcal{C}^1$ positive definite function $P : \mathcal{P} \to \mathbf{R}^{n\times n}$ such that*

$$\begin{bmatrix} \sum_{i=1}^{m} \underline{\overline{\nu}}_i(\rho)\frac{\partial P}{\partial \rho_i}(\rho) + P(\rho)A(\rho) + A^T(\rho)P(\rho) & P(\rho)B(\rho) & C^T(\rho) \\ B^T(\rho)P(\rho) & -\gamma I & D^T(\rho) \\ C(\rho) & D(\rho) & -\gamma I \end{bmatrix} < 0 \tag{5.13}$$

for all $\rho \in \mathcal{P}$, then the LPV system (5.12) is asymptotically stable with $w = 0$ and has an $\mathcal{L}_2$-gain less than γ for all the admissible parameter trajectories.

Note that the notation $\sum_{i=1}^{m} \underline{\overline{\nu}}_i(\rho)\frac{\partial P}{\partial \rho_i}$ is the same as that in the nonlinear regulator case, which means (5.13) actually represents 2^m inequalities since it indicates that every combination of $\overline{\nu}_i$ and $\underline{\nu}_j$ should be considered.

The proof of the lemma is omitted here. The key point in the proof is the use of a parameter-dependent storage function $V(x,t) = x^T(t)P(\rho(t))x(t)$, in contrast to the quadratic storage functions for linear time-invariant systems. This generalization captures the varying nature of LPV plants.

5.4.2 State Feedback Synthesis

Next we consider the state-feedback synthesis problem of the LPV plant

$$\begin{bmatrix} \dot{x}(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_1(\rho(t)) & 0 & 0 \\ C_2(\rho(t)) & 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{5.14}$$

where u is the control input. Assume both the parameter value and the state variables can be measured in real time. We want to find a continuous function $F(\rho)$ such that the closed loop system has an $\mathcal{L}_2$ gain less than γ under the feedback law $u(t) = F(\rho(t))x(t)$.

Lemma 5.6 *[39] If there exists a continuously differentiable matrix function $X(\rho) > 0$ for all $\rho \in \mathcal{P}$ that satisfies*

$$\begin{bmatrix} -\sum_{i=1}^{m} \underline{\nu}_i(\rho)\frac{\partial X}{\partial \rho_i}(\rho) + \hat{A}(\rho)X(\rho) + X(\rho)\hat{A}^T(\rho) - B_2(\rho)B_2^T(\rho) & X(\rho)C_1^T(\rho) & B_1(\rho) \\ C_1(\rho)X(\rho) & -I & 0 \\ B_1^T(\rho) & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{5.15}$$

where $\hat{A}(\rho) = A(\rho) - B_2(\rho)C_2(\rho)$, then under the state feedback law

$$u(t) = -(B_2(\rho(t))X^{-1}(\rho(t)) + C_2(\rho(t)))x(t) \tag{5.16}$$

the closed-loop system (5.14) has an $\mathcal{L}_2$ gain less than γ for all $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{\nu}$.

As in the analysis test, we search for a parameter-dependent storage function $V(x,t) = x^T(t)X^{-1}(\rho(t))x(t)$ for the closed-loop system. The result is a parameter-dependent AMI condition (5.15).

5.4.3 Application to Quasi-LPV Models

Following the same treatment of the nonlinear system for the regulation problem in Chapter 4, the $\mathcal{L}_2$-gain analysis and synthesis results for linear parameter varying plants can be directly applied to nonlinear input-affine systems. After putting the nonlinear dynamics into the state-dependent linear structure (5.9), we can choose the underlying parameters $\rho(x) \in \mathbf{R}^m$ which represent the dependence of $A(x)$, $B_1(x)$, $B_2(x)$, $C_1(x)$ and $C_2(x)$ on the state vector x. For instance, $\rho(x)$ can simply be part of the state variables that the state matrices depend upon. For computational considerations, the number of parameters should be kept to a minimum. Aiming at attenuating the effect of the exogenous inputs for all the admissible parameter trajectories, the LPV theory applied to the nonlinear systems will inevitably introduce some conservatism due to the connection between the parameters and the states.

As discussed in the nonlinear regulation problem, one of the difficulties when applying Lemma 5.6 is to determine the bounds on the variation rate of the parameters since they are no longer known a priori as in the LPV plant. Accurate information on the bounds is crucial in obtaining a good performance. At one extreme, we can restrict X in Lemma 5.6 to be constant, i.e., only the quadratic storage functions are considered. Since the variation rate ν plays no role in this case, the $\mathcal{L}_2$-gain is guaranteed for all state trajectories. More often, however, it is too conservative to restrict our search to quadratic functions. A more practical approach is to use the best estimate of the operating range of the chosen varying parameters and an approximation of the parameter variation rates. These assumptions should be verified by simulations after the controller has been synthesized. At the other extreme, we can obtain the NLMI solution by ignoring the parameter variation rates, i.e., setting them to zero. Then we have no guarantee on the performance of the closed system, or even its stability.

Even though the quasi-LPV approach to nonlinear $\mathcal{H}_\infty$ control is still a fairly straightforward generalization of the linear theory, it has an obvious advantage over the pointwise NLMI characterization: by taking into account the variation of the parameters, it guarantees the performance as long as the underlying parameters stay within the pre-specified set. But this is achieved at a price of potentially severe conservatism.

5.5 Enhanced Quasi-LPV Approach

5.5.1 Motivation

As stated in Lemma 3.4, we have an infinite number of choices when constructing the quasi-LPV model from the nonlinear dynamics. However, they are not equal in terms of the achievable performance when the LPV technique is applied. Here is an illustrative example.

Example 5.7

$$\begin{cases} \dot{x}_1 &= & x_2 & +(x_2-1)u \\ \dot{x}_2 &= & x_1+x_2 & +w-(x_1+2)u \\ z &= & [x_1\ x_2\ u]^T & \end{cases}$$

Though the dynamics are linear without control, it is not easier than any other nonlinear problem if the HJI has to be solved. Since this example was created using the converse Hamilton-Jacobi method [15], we know that a closed-loop $\mathcal{L}_2$-gain of 1 can be achieved by state feedback $u = x_1 + 2x_2$ with $V(x) = x_1^2 + x_2^2$ serving as a storage function. However, if the obvious choice of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is used to form the quasi-LPV model, no finite $\mathcal{L}_2$ gain can be achieved by state feedback since the pair $\{A(x), B_2(x)\}$ is not controllable on two lines in the state space: $x_2 = \frac{1}{2}(1+\sqrt{5})x_1 + 2 + \sqrt{5}$ and $x_2 = \frac{1}{2}(1-\sqrt{5})x_1 + 2 - \sqrt{5}$, and A is simply unstable. Fortunately, not every choice of $A(x)$ has this problem. If we are lucky enough to use this one:

$$A(x) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}x_2 - 1 \\ -\frac{1}{2}x_1 - 2 \end{bmatrix} \begin{bmatrix} x_2 & -x_1 \end{bmatrix}$$

then it is easy to check that (5.15) is satisfied for any $\gamma > 1$ and $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

This example demonstrates that the differences in the $\mathcal{L}_2$-gain achieved by LPV controllers can be huge due to different choices of $A(x)$, and the "obvious" one may not lead to the best performance. In Chapter 2, the frozen Riccati approach to the nonlinear minimax problem was studied, and it has been proven that there always exists a representation $f(x) = A(x)x$ such that the solution to the frozen Riccati equation recovers the optimal control, provided that the optimal value function has a certain form. This means if the right $A(x)$ is used, then one only needs to solve the algebraic Riccati equation at any instant to get the optimal control. The bad news is that we cannot find the optimal $A(x)$ without figuring out the optimal value function. Even though the variation of the state variables are taken into account in the quasi-LPV scheme, a proper choice of $A(x)$ still can reduce the conservatism. And at the extreme, if the "optimal" $A(x)$ is found, we don't have to worry about the variation rate bounds of the underlying parameters which can be difficult to predict from the open-loop dynamics. In the sequel, we will make use of the valuable insights gained in the FRE method, and propose a systematic way to refine the original choice of $A(x)$ toward those that result in the best performance.

5.5.2 $\mathcal{L}_2$-Gain Analysis

The next theorem is based on Lemma 3.4 which gives the parameterization of the state-dependent linear representations of the nonlinear plant (5.1). Consider the following quasi-LPV model:

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A(\rho(x)) & B(\rho(x)) \\ C(\rho(x)) & D(\rho(x)) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{5.17}$$

Assume the underlying parameter ρ varies in the allowable set $\mathcal{F}_{\mathcal{P}}^{\nu}$. Note that the bounds on the rate variation of the parameter are allowed to vary with the parameter value.

Theorem 5.8 *If there exists a C^1 matrix valued function $P(\rho(x)) > 0$ and a $W(\rho(x))$*

with $W(\rho(x))x = 0$ that satisfy the following inequality constraint:

$$\begin{bmatrix} \sum_{i=1}^{m} \underline{\nu}_i(\rho)\frac{\partial P}{\partial \rho_i}(\rho) + P(\rho)A(\rho) + A^T(\rho)P(\rho) + W(\rho) + W^T(\rho) & P(\rho)B(\rho) & C^T(\rho) \\ B^T(\rho)P(\rho) & -\gamma I & D^T(\rho) \\ C(\rho) & D(\rho) & -\gamma I \end{bmatrix} < 0 \tag{5.18}$$

for all $\rho \in \mathcal{P}$, then the nonlinear system (5.17) is asymptotically stable and has an $\mathcal{L}_2$-gain less than γ.

Proof Let

$$N(\rho) = P^{-1}(\rho)W(\rho).$$

Substituting $W(\rho)$ with $P(\rho)N(\rho)$, (5.18) becomes

$$\begin{bmatrix} \sum_{i=1}^{m} \underline{\nu}_i(\rho)\frac{\partial P}{\partial \rho_i} + P(\rho)(A(\rho) + N(\rho)) + (A(\rho) + N(\rho))^T P(\rho) & P(\rho)B(\rho) & C^T(\rho) \\ B^T(\rho)P(\rho) & -\gamma I & D^T(\rho) \\ C(\rho) & D(\rho) & -\gamma I \end{bmatrix} < 0$$

Since $W(\rho(x))x = 0$, we have $N(\rho(x))x = 0$. Therefore, inequality (5.18) is just the $\mathcal{L}_2$-gain analysis test (5.13) of the same nonlinear system for a different quasi-LPV representation $\tilde{A}(\rho) = A(\rho) + N(\rho)$. □

Recognizing the freedom we have in choosing $A(\rho)$ for a nonlinear plant, we introduce an extra design variable $W(\rho)$ to the standard LPV analysis affine matrix inequality. The new test (5.18) reduces to the standard AMI condition (5.13) when $W(\rho(x))$ is set to zero. Notice that the constraint (5.18) is affine in both $P(\rho)$ and $W(\rho)$. The purpose of introducing this extra design variable $W(\rho)$ is to reduce the conservatism caused by choosing one particular quasi-LPV model for the nonlinear system. As we pointed out earlier, the achievable performance level calculated by the LPV method is greatly dependent upon the choice of $A(\rho)$. Most often, however, it is not at all trivial to pick a good $A(x)$. The design variable $W(\rho)$ provides a systematic way to make the selection process tractable.

A similar computation scheme as described in Section 4.2 is proposed to implement the new $\mathcal{L}_2$-gain analysis test in Theorem 5.8. First, select a set of $\mathcal{C}^1$ functions $\{\phi_i(\rho)\}_{i=1}^M$ as a basis for $P(\rho)$, i.e.,

$$P(\rho) = \sum_{i=1}^{M} \phi_i(\rho) P_i \tag{5.19}$$

where $X_i \in \mathcal{S}^{n\times n}$ is the symmetric coefficient matrix for $\phi_i(\rho)$. Then for $W(\rho)$, we choose a set of continuous matrix-valued basis functions $\{\Theta_i(\rho)\}_{i=1}^N$ with $\Theta_i : \mathcal{P} \to \mathbf{R}^{k\times n}$ satisfying

$$\Theta_i(\rho(x))x = 0 \qquad i = 1, \cdots, N.$$

The requirement of $W(\rho(x))x = 0$ will be enforced when this basis is used to span $W(\rho)$:

$$W(\rho(x)) = \sum_{i=1}^{N} W_i \Theta_i(\rho(x)) = 0 \tag{5.20}$$

where $W_i \in \mathbf{R}^{n\times k}$ is the coefficient matrix for $\Theta_i(\rho)$. If the chosen finite basis expansions are substituted in Theorem 5.8 for $P(\rho)$ and $W(\rho)$, the $\mathcal{L}_2$-gain analysis can be reformulated as a minimization problem over two infinite dimensional matrix inequalities:

$$\min_{\{P_i\},\{W_i\}} \gamma$$

$$\begin{bmatrix} \mathcal{M}_{11} & \sum_{j=1}^{M} \phi_j(\rho) P_j B(\rho) & C^T(\rho) \\ \sum_{j=1}^{M} \phi_j(\rho) B^T(\rho) P_j & -\gamma I & D^T(\rho) \\ C(\rho) & D(\rho) & -\gamma I \end{bmatrix} < 0 \tag{5.21}$$

$$\sum_{j=1}^{M} \phi_j(\rho) P_j > 0 \tag{5.22}$$

with

$$\mathcal{M}_{11} = -\sum_{i=1}^{m} \underline{\nu}_i(\rho) \left(\sum_{j=1}^{M} \frac{\partial \phi_j}{\partial \rho_i} P_j\right) + \sum_{j=1}^{M} \phi_j(\rho)(P_j A(\rho) + A^T(\rho) P_j) + \sum_{j=1}^{N} (W_j \Theta_j(\rho) + \Theta_j^T(\rho) W_j^T).$$

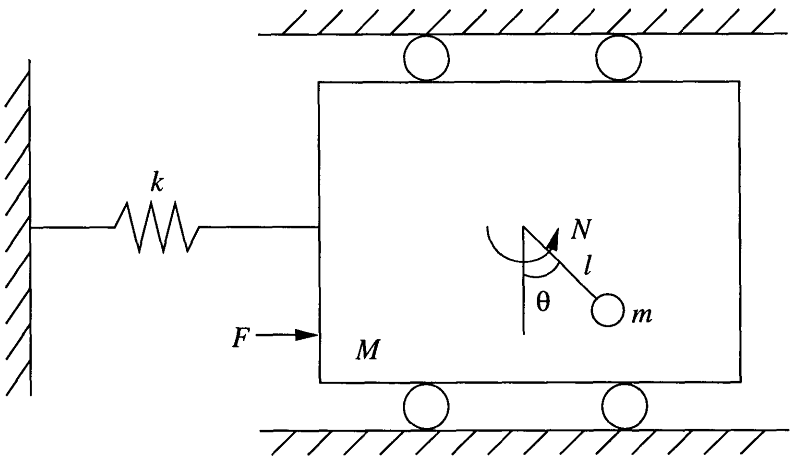

Figure 5.1: Rotational Actuator to control a Translational Oscillator

The two constraints above are LMI's in terms of the matrix variables $\{P_i\}_{i=1}^{M}$ and $\{W_i\}_{i=1}^{N}$. Again, we need to grid the parameter space and solve the LMI's on a finite set of points to get an approximate solution. Since (5.21) actually consists of 2^m inequalities, a total of $2^m + 1$ LMI's have to be evaluated at each point.

Example 5.9 *Consider the TORA example [8] which was introduced as a benchmark for nonlinear control design. The system is shown in Figure 5.1 which represents a translational oscillator with an eccentric ratational proof-mass actuator. The oscillator consists of a cart of mass M connected to a fixed wall by a linear spring of stiffness k. The cart is constrained to have one-dimensional travel. The proof mass actuator attached to the cart has mass m and moment of inertia I about its center of mass, which is located a distance l from the point about which the proof mass rotates. The motion occurs in a horizontal plane, so no gravitational forces need to be considered. N denotes the control torque applied to the proof mass, and F is the disturbance force on the cart. Let q and θ denote the translational position of the cart and the angular position of the rotational proof mass, repectively. Letting $x = [\xi\ \dot{\xi}\ \theta\ \dot{\theta}]^T$, the nondimensional equation of motion in first-order form are given by*

$$\dot{x} = f(x) + B_1(x)w + B_2(x)u$$

where

$$f(x) = \begin{bmatrix} x_2 \\ \frac{-x_1+\epsilon x_4^2 \sin x_3}{1-\epsilon^2 \cos^2 x_3} \\ x_4 \\ \frac{\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3)}{1-\epsilon^2 \cos^2 x_3} \end{bmatrix}, \; B_1(x) = \begin{bmatrix} 0 \\ \frac{1}{1-\epsilon^2 \cos^2 x_3} \\ 0 \\ \frac{-\epsilon \cos x_3}{1-\epsilon^2 \cos^2 x_3} \end{bmatrix}, \; B_2(x) = \begin{bmatrix} 0 \\ \frac{-\epsilon \cos x_3}{1-\epsilon^2 \cos^2 x_3} \\ 0 \\ \frac{1}{1-\epsilon^2 \cos^2 x_3} \end{bmatrix}$$

with normalizations

$$\xi = \sqrt{\frac{M+m}{I+ml^2}}q, \qquad u = \frac{M+m}{k(I+ml^2)}N, \qquad w = \frac{1}{k}\sqrt{\frac{M+m}{I+ml^2}}F$$

and the coupling parameter

$$\epsilon = \frac{ml}{\sqrt{(I+ml^2)(M+m)}}.$$

Based on the parameter values from [8], ϵ is assumed to be 0.3. The regulated output is defined by

$$z = [0.5x_1 \;\; x_3 \;\; u]^T.$$

The LQR state feedback gain of the linearized system at the origin is easy to compute and is given below

$$K = [-0.2380 \; 0.2248 \; 1.0000 \; 1.4638].$$

Our task is to compute the $\mathcal{L}_2$-gain of the closed system with the control law $u = -Kx$ for $x_3 \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

A natural choice of the quasi-LPV model is

$$A_o(\rho) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{1-\epsilon^2 \cos^2 x_3} & 0 & 0 & \frac{\epsilon x_4 \sin x_3}{1-\epsilon^2 \cos^2 x_3} \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon \cos x_3}{1-\epsilon^2 \cos^2 x_3} & 0 & 0 & \frac{\epsilon^2 x_4 \cos x_3 \sin x_3}{1-\epsilon^2 \cos^2 x_3} \end{bmatrix}$$

$$C_{1o} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where $\rho = [x_3 \; x_4]$ is used as the varying parameter. The closed-loop system with the given feedback gain is then of the form (5.17) with

$$A(\rho) = A_o(\rho) - B_2(\rho)K, \quad B(\rho) = B_1(\rho), \quad C(\rho) = \begin{bmatrix} C_{1o} \\ -K \end{bmatrix}, \quad D(\rho) = 0.$$

Restricting the operating range of (x_3, x_4) to be $\mathcal{P} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times [-0.5, 0.5]$, we grid $\mathcal{P}$ uniformly with 11 points on x_3 and 9 points on x_4. The range of x_4 is decided based on the simulation of the closed-loop system with the given state feedback. Since x_4 is the derivative of x_3, we are able to use the exact variation rate of x_3, which should incur no conservatism at all. Also from the simulation, a rate bound of 0.5 is tentatively set for x_4. Three sets of basis functions for $P(\rho)$ are used for comparison:

$$\beta_1 = \{1,\ \tfrac{2}{\pi}x_3,\ (3(\tfrac{2}{\pi}x_3)^2 - 1)/2\},$$

$$\beta_2 = \{1,\ \tfrac{2}{\pi}x_3,\ (3(\tfrac{2}{\pi}x_3)^2 - 1)/2,\ (5(\tfrac{2}{\pi}x_3)^3 - 3(\tfrac{2}{\pi}x_3))/2,\ 35(\tfrac{2}{\pi}x_3)^4 - 30(\tfrac{2}{\pi}x_3)^2 + 3)/2\},$$

$$\beta_3 = \{1,\ \tfrac{2}{\pi}x_3,\ (3(\tfrac{2}{\pi}x_3)^2 - 1)/2,\ 2x_4,\ (3(2x_4)^2 - 1)/2,\ x_3x_4\}.$$

Note that the above basis functions are Legendre polynomials on the operating regions. The basis in β_1 and β_2 are functions of only x_3, thus the LPV algorithm should yield a guaranteed bound on the $\mathcal{L}_2$-gain since the exact variation rate is used for x_3.

To impose the constraint of $W(\rho(x))x = 0$, we assume that $W(\rho(x))$ is of the following form

$$W(\rho(x)) = Z(\rho(x))[0 \; 0 \; x_4 \; -x_3]$$

where $Z(\rho)$ is a 4×1 vector function. Picking a basis of $\{g_i(\rho)\} = \{1,\ x_3,\ x_4\}$ for $Z(\rho)$, we have

$$W(\rho(x)) = \sum_{i=1}^{3} Z_i g_i(\rho)[0 \; 0 \; x_4 \; -x_3].$$

with $Z_i \in \mathbf{R}^4$.

Method	Constant P	Parameter-dependent $P(\rho)$		
		β_1	β_2	β_3
LPV w/o $W(\rho)$	not feasible	73.6	73.2	71.4
LPV w/ $W(\rho)$	66.4	8.2	8.0	7.0

Table 5.1: $\mathcal{L}_2$-gain of the closed-loop system of Example 5.9 with the given state feedback law

Table 5.1 lists the $\mathcal{L}_2$-gain computed using both the standard and new LPV algorithms. The LMI's (5.18) from the standard LPV method on the gridding points are not feasible for constant P. With the extra variable $W(\rho)$ in the LMI's, we obtain significantly less conservative numbers. Also notice that allowing P to be parameter-dependent greatly reduces conservatism. When β_1 is used, the coefficient matrices Z_i returned by the optimization are as follows:

$$Z_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} 2.70 \\ -1.85 \\ -0.68 \\ 1.12 \end{bmatrix}, \qquad Z_3 = \begin{bmatrix} -0.11 \\ 1.48 \\ -0.32 \\ -0.08 \end{bmatrix}.$$

Similar results are obtained for other basis functions.

As we know, the $\mathcal{H}_\infty$ norm of the linearized system at the origin provides a lower bound for the best global $\mathcal{L}_2$-gain for the nonlinear system. Not to our surprise, the linearized closed-loop system with the given state feedback law indeed has a lower $\mathcal{H}_\infty$ norm of 6.6 at the origin. In fact, the best achievable $\mathcal{H}_\infty$ norm of the original system near zero is 4.6.

5.5.3 State Feedback $\mathcal{H}_\infty$ Synthesis: $X - N$ Iteration

The following quasi-LPV representation of the nonlinear input-affine system (5.6) will be used for the synthesis problem:

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A(\rho(x)) & B_1(\rho(x)) & B_2(\rho(x)) \\ C_1(\rho(x)) & 0 & 0 \\ C_2(\rho(x)) & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{5.23}$$

Assume the underlying parameter ρ varies in the allowable set $\mathcal{F}_{\mathcal{P}}^{\nu}$. The next theorem is based on Lemma 5.6 and Lemma 3.4.

Theorem 5.10 *If there exists a $\mathcal{C}^1$ matrix-valued function $X(\rho(x)) > 0$ and an $N(\rho(x))$ with $N(\rho(x))x = 0$ that satisfy the following inequality constraint*

$$\begin{bmatrix} \begin{matrix} -\sum_{i=1}^{m} \underline{\nu}_i(\rho)\frac{\partial X}{\partial \rho_i}(\rho) + (\hat{A}(\rho) + N(\rho))X(\rho) \\ +X(\rho)(\hat{A}(\rho) + N(\rho))^T - B_2(\rho)B_2^T(\rho) \end{matrix} & X(\rho)C_1^T(\rho) & B_1(\rho) \\ C_1(\rho)X(\rho) & -I & 0 \\ B_1^T(\rho) & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{5.24}$$

with $\hat{A}(\rho) = A(\rho) - B_2(\rho)C_2(\rho)$ for all $\rho \in \mathcal{P}$, then the closed-loop system has an $\mathcal{L}_2$-gain less than γ under the state feedback law:

$$u(x) = -(B_2(\rho(x))X^{-1}(\rho(x)) + C_2(\rho(x)))x. \tag{5.25}$$

Compared to the standard LPV result stated in Lemma 5.6, a new matrix-valued function $N(\rho(x))$ is added to take advantage of the non-uniqueness of the quasi-LPV model of the nonlinear dynamics. The best γ level associated with any particular $A(\rho)$ from Lemma 5.6 serves as an upper bound of the best achievable $\mathcal{L}_2$-gain of the closed-loop nonlinear system.

Note that the constraint (5.24) is not affine in $X(\rho)$ and $N(\rho)$ in contrast to the analysis result. For any value of ρ, (5.24) is a bilinear matrix inequality (BMI) in X and N. Because the BMI can be reformulated as a nonconvex programming problem, no tractable algorithm is believed to exist for a general BMI. However, there are computational procedures suggested for solving the BMI via a global optimization [18], though they exhibit exponential growth.

Even though minimizing γ over both X and N under the BMI constraint (5.24) is not a convex optimization, when we fix X or N in (5.24) and try to minimize γ over the other, it is a convex problem. Similar to the "$D - K$" iteration for μ synthesis, an "$X - N$" iteration is proposed as a practical scheme to achieve the best closed loop performance:

1. Find a quasi-LPV representation (5.23) for the nonlinear open loop system.

2. Choose the varying parameters ρ, the range of the parameter $\mathcal{P}$ and the variation rate bound ν.

3. Calculate the best achievable γ level for the current quasi-LPV model (Lemma 5.6). If the performance is satisfactory, stop.

4. Relax γ slightly, solve (5.15) for a feasible $X(\rho) > 0$.

5. Calculate the best γ over all $N(\rho)$ with $X(\rho)$ fixed from the previous step (Theorem 5.10). If the new γ level is satisfactory or unchanged, stop.

6. Relax γ slightly, solve (5.24) with $X(\rho)$ from step 4 for a feasible $N(\rho)$.

7. Replace $A(\rho)$ with $A(\rho) + N(\rho)$. Go to step 3.

The practical computation scheme described in Section 5.5.2, which involves gridding the parameter space $\mathcal{P}$ and employing basis functions for $X(\rho)$, can be used to solve the AMI (5.15) in steps 3 and 4. At steps 5 and 6, the constraint (5.24) is reduced to an AMI in $N(\rho)$ since $X(\rho)$ is fixed. Selecting a set of basis functions for $N(\rho)$ that reflect the requirement of $N(\rho(x))x = 0$, we can get an approximate solution to (5.24) by solving the collection of the LMI's evaluated at the gridding points of $\mathcal{P}$.

This proposed procedure divides the original nonconvex optimization into two steps, and a convex problem under the LMI constraints needs to be solved at each step. Although this algorithm is not guaranteed to converge to the global optimum, it makes use of the efficient algorithms for solving LMI's. At step 4 and 6, we try to avoid using $X(\rho)$ or $N(\rho)$ associated with the minimum γ at the previous step, because it usually does not lead to the best overall γ level. From our experience, $X(\rho)$ and $N(\rho)$ that correspond to a relaxed γ are more likely to bring γ further down at the next step.

The next two examples will illustrate that optimizing over $A(\rho)$ along with the standard LPV procedure can indeed improve the closed-loop performance of a nonlinear system.

Example 5.11 *Design a state feedback law that minimizes the $\mathcal{L}_2$-gain from w to z for the following dynamics:*

$$\begin{cases} \dot{x}_1 &= x_1 + 2x_1x_2^2 + w_1 + 2u \\ \dot{x}_2 &= 4x_1^2 - x_2 + w_2 + 2x_1u \\ z &= [x_1 \ x_2 \ u]^T \end{cases}$$

Since this example is created using the converse Hamilton-Jacobi method, we know that the closed-loop $\mathcal{L}_2$-gain of 1 can be achieved by the state feedback $u = -2x_1 - 2x_1x_2$ with $V(x) = x_1^2 + x_2^2$ being a storage function.

We begin with an arbitrary choice of $A(x)$:

$$A(x) = \begin{bmatrix} 1 & 2x_1x_2 \\ 4x_1 & -1 \end{bmatrix}.$$

The other state space matrices needed in model (5.23) are as follows:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad B_2 = \begin{bmatrix} 2 \\ 2x_1 \end{bmatrix};$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Method	Best γ	X	Best γ from analysis
Standard LPV	2.4	$\begin{bmatrix} 0.95 & 0.00 \\ 0.00 & 0.09 \end{bmatrix}$	1.7
LPV w/ X-N iteration	1.0	$\begin{bmatrix} 0.99 & 0.00 \\ 0.00 & 0.95 \end{bmatrix}$	1.0

Table 5.2: Performance comparison of the standard and the enhanced LPV synthesis for Example 5.11

The two state variables are chosen as the varying parameters, i.e., $\rho = [x_1 \ x_2]^T$. $\mathcal{P} := [-10, 10] \times [-10, 10]$ is uniformly gridded. To avoid the problem caused by the rate variation estimation, we only consider the quadratic storage functions, i.e., X is restricted to be constant. Notice that in the two-dimensional case all the $N(x)$ satisfying $N(x)x = 0$ can be parameterized as $N(x) = \Theta(x)[x_2 \ \ -x_1]$ where $\Theta(x) : \mathbf{R}^2 \to \mathbf{R}^{2\times 1}$. The set of basis functions $\{1, \ x_1, \ x_2\}$ for $\Theta(x)$ is chosen for this example.

The computation results are shown in Table 5.2. With the initial choice of $A(\rho)$, the best γ level over all the positive definite scaling functions X obtained by the standard LPV scheme is 2.4, though the $\mathcal{L}_2$-gain analysis (Theorem 5.8) of the closed-loop system gives a better bound of 1.7. However, starting from the same initial $A(\rho)$, γ is reduced to 1.0 after a single "$X - N$ iteration."

5.6 Conclusions

The quasi-LPV way of dealing with the nonlinear $\mathcal{H}_\infty$ control problem greatly reduces computational complexity when compared to solving the HJI directly. It is shown through examples that the performance of the quasi-LPV controller is greatly dependent upon the choice of the model. The standard LPV analysis and state feedback synthesis tests are enhanced by incorporating a new design variable which represents the parameterization of the quasi-LPV model. Under this new framework, the $\mathcal{L}_2$-gain analysis is reformulated as a minimization problem over an affine matrix inequality constraint, and an approximate solution can be obtained by a tractable computation which involves solving a collection of LMI's. In contrast, the state feedback synthesis problem appears to be a "bilinear" constraint which is not jointly convex in both unknown variables. However, an iterative algorithm is proposed which divides it into two convex optimization problems. Although not guaranteed to converge to the global optimum, its effectiveness is demonstrated by a numerical example.

Even though conservatism is reduced by the enhanced quasi-LPV scheme, the optimal achievable performance still may not be obtained. In order to limit the computational load, the number of underlying parameters can be much less than the number of states. Consequently, the search space of $A(x)$ is rather limited in comparison to the complete parameterization.

Chapter 6 Concluding Remarks

In this last chapter we conclude the thesis with a brief summary of the results, and a discussion of related future research directions.

6.1 Summary

Gain scheduling has been a common practice among engineers when dealing with nonlinear plants. Although this pseudo linear design methodology has fundamental limitations, it is one of the very few synthesis tools that can be applied to a large class of nonlinear systems. As the newest member to the family, the frozen Riccati equation approach was recently developed.

A formal analysis of the frozen Riccati equation design methodology has been conducted in Chapter 2. As a variant form of the traditional gain scheduled control design technique for nonlinear systems, it exhibits the fundamental deficiencies inherited from its pointwise nature. Although a curl type condition is established for checking the optimality of the solution, one typically cannot assess *a priori* the guaranteed performance, or even mere stability, of the design. Rather, any such properties can only be inferred from extensive computer simulations. What differentiates this method from the traditional gain scheduled approach, which employs the linear time-invariant approximation to the nonlinear plant, is the use of the non-unique state-dependent linear representation of the dynamics. It is this extra freedom in choosing the "linear-like" model that gives the FRE method an advantage. We have proved that under a certain assumption on the value function, an optimal SDLR always exists such that FRE recovers the optimal control law. Although unable to help the FRE design in its original form, the new insights lay the groundwork for the enhanced quasi-LPV approach.

As the traditional gain scheduled control of nonlinear systems takes the form of a linear parameter-varying system, the study of the class of LPV plants provides possible remedies for overcoming the limitations of the "frozen" time design schemes. One such remedy is to take into account the information of parameter variations. It is shown in Chapter 4 and Chapter 5 that the explicit variation rate bounds of the underlying parameter can be used to derive a sufficient condition on the guaranteed performance of a nonlinear plant. However, accurate information on the rate bounds can only be obtained after the feedback controller is constructed. Even if it is verified, the control law is apt to be conservative. From a designer's standpoint, the variation rate bounds can be used as a design parameter that reflects the conservatism of the resulting controller. At one extreme, one should obtain the aggressive FRE controller which comes with no guarantee on performance as the parameter variation is completely ignored. At the other extreme, a quadratic Lyapunov function may be found so that the closed-loop system can withstand arbitrarily fast parameter variation. In most practical situations, though, we suspect that the best design comes somewhere in between. And this can be evaluated through extensive computer simulations by iterating on this design choice.

The quasi-LPV way of dealing with nonlinear systems greatly reduces computational complexity when compared to solving Hamilton-Jacobi partial differential equations/inequalities directly. Instead of gridding the entire state space, a lower dimensional parameter space is selected to capture the nonlinearity of the dynamics. Because the underlying parameters in the quasi-LPV model are functions of the state variables, they cannot vary freely as in true LPV plants. Consequently, the design may suffer severe conservatism.

The valuable insights we gained about the choice of the SDLR in the FRE method led to a new design concept in Chapter 5. It has been shown that the extra freedom of choosing a quasi-LPV representation can be incorporated into the LPV design procedure systematically. Under the enhanced LPV design framework, $\mathcal{L}_2$-gain analysis is formulated as an infinite dimensional convex optimization problem. By exploiting one more design variable than the standard LPV scheme, conservatism is reduced. The nonlinear $\mathcal{H}_\infty$ synthesis problem, unfortunately, can only be cast as an optimization over an infinite dimensional bilinear matrix inequality constraint. Although not guaranteed to converge to the global optimum, an iterative procedure is proposed where two convex optimizations are conducted in turn. Similar to the "$D - K$ iteration" for μ synthesis, this computation scheme has been shown through examples to be effective for improving the overall performance.

6.2 Future Research

Even though we have shown that under appropriate conditions optimality is possible for frozen Riccati equations, we feel that there is little hope to elevate the FRE technique in its present form. Nevertheless, the new quasi-LPV framework proposed in this thesis constitutes an enhancement to the existing LPV methodology. But its effectiveness has to be tested on a variety of nonlinear systems. To keep our focus on the basic ideas, we only discussed the state feedback problem of nonlinear $\mathcal{H}_\infty$ control. Similar results can be obtained for the output feedback case and under other performance objectives along the same line.

The existing nonlinear control techniques can be roughly divided into two groups: those that rely on on-line computation and real-time optimizations to generate control moves, and those that relegate the selection of a control law to off-line analysis and predetermination of a global scheme. From this perspective, the quasi-LPV design belongs to the latter. Since most of the time the LPV controller is not optimal, we can make use of on-line optimization algorithms to refine the control law in implementation. Research along this direction is currently being undertaken at Caltech.

As a generalization of the linear theory, the LPV techniques provide tractable ways to synthesize controllers for a large class of nonlinear systems. With the enhancement proposed in this thesis, we feel that the quasi-LPV technique has been advanced to its full potential. The future research of nonlinear control design should be focused on exploration of the special structures of the systems. As no practical design technique is likely to provide satisfatory performance for all kinds of nonlinear systems, it is of great importance to identify a well-suited technique for each specific catagory of nonlinear plants.

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