Poisson structures for PDEs associated with diffeomorphism groups

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Abstract

We study Poisson and Lie-Poisson structures on the diffeomorphism groups with a smooth metric spray in connection with dynamics of nonlinear PDEs. In particular, we provide a precise analytic sense in which the time t map for the Euler equations of an ideal fluid in a region of $\mathbb{R}^n$ (or on a smooth compact n-manifold with a boundary) is a Poisson map relative to the Lie-Poisson bracket associated with the group of volume preserving diffeomorphisms. The key difficulty in finding a suitable context for that arises from the fact that the integral curves of Euler equations are not differentiable on the Lie algebra of divergence free vector fields of Sobolev class H^s. We overcome this obstacle by utilizing the smoothness that one has in Lagrangian representation and carefully performing a non-smooth Lie-Poisson reduction procedure on the appropriate functional classes.

This technique is generalized to an arbitrary diffeomorphism group possessing a smooth spray. The applications include the Camassa-Holm equation on S^1, the averaged Euler and EPDiff equations on the n-manifold with a boundary. In all cases we prove that time t map is Poisson on the appropriate Lie algebra of H^s vector fields, where $s > n/2 + 1$ for the Euler equation and $s > n/2 + 2$ otherwise.

Contents

Chapter 1

Introduction

Hamiltonian structures play a fundamental role in mathematical physics. It's enough to recall a few examples: classical mechanics, electrodynamics, quantum mechanics, hydrodynamics and general relativity. However, when applying the classical methods and technics of symplectic geometry to PDEs, one faces significant difficulties, both analytical and conceptual.

Part of the problem is that symplectic forms that arise in many applications are weak symplectic forms on infinite dimensional manifolds. More importantly, often the integral curves of PDEs are not differentiable in time in the function spaces one would normally use; in the linear case, this corresponds to the fact that the operators involved are unbounded. Stock examples include the Euler and Klein-Gordon equations. When dealing with such systems, one has to pay careful attention to the domains of definitions as many standard formulae become only formal relationships. Their justification is often cumbersome and requires some ad hoc methods.

The goal of this research is to contribute to the development of techniques that are useful for the treatment of nonlinear PDEs with non-differentiable (in time) solutions and build a framework that allows a systematic and rigorous study of such systems and is applicable to the broad range of physical phenomena. Previous work in this vein is Chernoff and Marsden [1974].

The application that provided the main inspiration and motivation for our research is the Euler equation for an ideal fluid on a compact manifold. While the work of Chernoff and Marsden developed the Hamiltonian dynamics for unbounded vector fields in a very general context, the assumptions built into the theory are too restrictive to allow its application to the Euler equation (and many other important PDEs).

Specifically, our goal was to understand in what exact sense (if any) the flow generated by the Euler equation consists of Poisson maps. It has been known since the classic work of Arnold [1966] (see also Arnold and Khesin [1998]; Marsden and Ratiu [1999]) that *formally* the Euler equation of an ideal fluid could be viewed as a Hamiltonian system. Later Ebin and Marsden [1970] showed that in appropriate functional spaces (groups of volume preserving diffeomorphisms of Sobolev class H^s with $s > n/2 + 1$ to be precise) the flow of the Euler equation in the Lagrangian representation is generated by a smooth vector field. This work also shows that one can perform a reduction to the Eulerian representation to rigorously derive that solutions obtained this way satisfy the Euler equation, however, there is a derivative loss due to the reduction procedure.

While a Poisson nature of the flow of the Euler equation was clear from the results in Ebin and Marsden [1970], it was not so clear that there is a well-defined Poisson context for that. In fact, the works of Lewis et al. [1986] (and many subsequent papers by other authors) show that in the Poisson context this derivative loss is a nontrivial issue in defining a good sense in which one has a Poisson manifold and in which the Euler equations then define a Hamiltonian system.

In the thesis we fill this gap by means of a non-smooth Lie-Poisson reduction procedure on the appropriate classes of functions. The Euler equation in its classical (Eulerian) form is the equation of the geodesic motion of L^2 metric on the group of volume preserving diffeomorphisms translated to the identity of the group via the map

$$\pi(\dot{\eta}, \eta) = \dot{\eta} \circ \eta^{-1}.$$

To justify the formal insight that the flow of Euler equation is Poisson due to the theory of Lie-Poisson reduction, one has to overcome two difficulties. The first hurdle is that the group of volume preserving diffeomorphisms is only a weak symplectic manifold and therefore does not carry a Poisson bracket in any obvious sense without special ad hoc hypotheses, such as "the needed functional derivative exists," which have long been recognized as awkward at best.

The second hurdle is that the groups of diffeomorphisms are not Lie groups in the usual sense (left multiplication is not smooth). This results in the fact that the reduction map π is not smooth, it is not even of class C^1. Therefore, the well-developed theory of Poisson and Lie-Poisson reduction is not directly applicable.

We overcome these difficulties by constructing the Poisson bracket on the tangent bundle of the diffeomorphism group in a particular way and exploiting the special form of the reduction map π.

The key technical ingredient that allows this idea to work is the result proved in the Chapter 3. There we show that the tangent bundle of a weak Riemannian manifold carries a Poisson structure in an appropriate sense, provided that the manifold possesses a smooth Riemannian connection. This structure, defined on the subalgebra of the smooth functions, retains the essential dynamical properties of a "true" Poisson bracket, including the Jacobi identity and the fact that flows of Hamiltonian vector fields are Poisson maps, and, of course, the energy is conserved. Moreover, the bracket is related to the canonical weak symplectic form in the way that one would expect.

The next important step is to introduce a Poisson structure on the Lie algebra itself. As one would expect from the bracket derived via a type of Lie-Poisson reduction, this bracket is closely related to the *formal* Lie-Poisson bracket on the dual of the Lie algebra.

Further, we extend this technique from the group of volume preserving diffeomorphisms equipped with L^2 metric to the general diffeomorphism groups and right-invariant metrics with the smooth spray. This allows us to provide the precise analytical sense in which the time t map for the several important PDEs associated with diffeomorphism groups is Poisson relative to the Lie-Poisson bracket on the appropriate space of vector fields.

We consider the following equations:

1. The Euler equation on a smooth compact n-manifold M with the boundary ∂M:

$$\frac{\partial u_t}{\partial t} + \nabla_{u_t} u_t = -\operatorname{grad} p_t,$$

where $\operatorname{div} u = 0$ and u is parallel to ∂M.

The map $u_0 \rightarrow u_t$ is Poisson on the space of H^s $(s > n/2 + 1)$ divergence free vector fields.

2. The Camassa-Holm (CH) equation on S^1—see Camassa and Holm [1993]:

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}.$$

The map $u_0 \rightarrow u_t$ is Poisson on the space of H^s $(s > 5/2)$ vector fields.

3. The averaged Euler equations (or the LAE-α equations)—see Holm, Marsden, and Ratiu [1998a,b]:

$$\partial_t (1 - \alpha^2 \Delta) u + (u \cdot \nabla)(1 - \alpha^2 \Delta) u - \alpha^2 (\nabla u)^T \cdot \Delta u = -\operatorname{grad} p\,,$$

where $\operatorname{div} u = 0$ and u satisfies appropriate boundary conditions, such as the no-slip conditions $u = 0$ on ∂M.

The map $u_0 \rightarrow u_t$ is Poisson on the space of H^s $(s > n/2 + 2)$ divergence free vector fields.

4. The EPDiff equation for the H^1 metric (also called the averaged template matching equation) on a compact manifold M—see Holm and Marsden [2003] and Hirani et al. [2001]:

$$\begin{aligned}\partial_t (u - \alpha^2 \Delta) u + u (\operatorname{div} u) - \alpha^2 (\operatorname{div} u) \Delta u + (u \cdot \nabla) u \\ - \alpha^2 (u \cdot \nabla) \Delta u + (Du)^T \cdot u - \alpha^2 (Du)^T \cdot \Delta u = 0,\end{aligned}$$

with appropriate boundary conditions, such as the no-slip conditions $u = 0$ on ∂M. The EPDiff equations reduce to the CH equations in the case $M = S^1$.

The map $u_0 \to u_t$ is Poisson on the space of H^s $(s > n/2 + 2)$ vector fields.

These equations may be derived as the reduction to the identity of the geodesic motion on the appropriate diffeomorphism group (see, for example, Camassa and Holm [1993], Khesin and Misiolek [2003] and Misiolek [1998, 2002] for the case of the CH equations), and the preceding references for the other equations. In the case of the averaged Euler equation "appropriate" is the group of volume preserving diffeomorphisms equipped with the right-invariant H^1 metric and the whole group of diffeomorphisms with the right-invariant H^1 metric for the EPDiff and CH equations. The smoothness of the spray for the CH, EPDiff and LAE-α equations on the regions with no boundary is due to Shkoller [1998] and for the regions with a boundary to Marsden et al. [2000]. The similar result for the Euler equation goes back to Ebin and Marsden [1970].

It is essential that in all cases we introduce a Lie-Poisson bracket directly on the Lie algebra and prove that the flow that the equation generates on the Lie algebra is Poisson. This is an important difference with the usual Lie-Poisson reduction performed on the cotangent bundle of the group.

While the reduction procedure itself is more natural and easier to carry out on the cotangent bundle, the flows in question are not well-defined there since the necessary existence and uniqueness theorems are missing and there is very little hope to establish them. For example, in order to rigorously apply the Lie-Poisson reduction on the cotangent bundle to the PDEs that we consider, one would need to prove the existence and uniqueness theorems in negative Sobolev spaces, which is quite unlikely. The situation might be different for the KDV equation, where the existence was shown in class $H^{-3/10+\epsilon}$ (see Colliander et al. [2001]).

It should be noted that our results apply directly to the regular solutions (i.e., the ones that are obtained as a reduction of the geodesic motion on the Lie group). It is known that the solutions of the above PDEs may lose regularity in finite time (see Camassa and Holm [1993] for the case of the CH equation). It is an important open question whether one could carry out a similar reduction procedure for these singular solutions.

This work is organized as follows. In Chapter 2 we collect various background material that we will use. These facts are common knowledge and we omit the proofs. In Chapter 3 we carry out our programm for the Euler equation. The material contained in this chapter is the subject of the article by Vasylkevych and Marsden [2003]. The construction of the Poisson bracket on a Riemannian manifold with the smooth spray is also included here in Section 3.3. This result is important to all the applications that we consider. It is included in Chapter 3 to make it self-contained. In Chapter 4 we generalize the results of the Chapter 3 to general diffeomorphism groups. In the Chapter 5 we consider the Camassa-Holm, Euler-α and EPDiff equations and prove that the flows they generate are Poisson. This is the direct application of the general theory developed in Chapter 4 and Section 3.3. We conclude with a short discussion of the results in Chapter 6.

Chapter 2

Background

In this chapter we collect the background material that will be extensively used in this work. The majority of the results is a common knowledge about Riemannian manifolds and diffeomorphism groups. This is certainly not a complete exposition, but rather a list of formulae and definitions to be used later. In Section 2.1 we recall the basic facts and concepts of Riemannian geometry, paying special attention to the differences that one encounters dealing with the weak Riemannian metrics on infinite dimensional manifolds. Most of the results here could be found in Sakai [1996]; Do Carmo [1992]; Lang [1985]. In Section 2.2 we recall how the operators of vector calculus are defined in a manifold setting and their basic properties. In particular we remind the construction of Laplace de Rham operator and Bohner's formula. In this section we follow Abraham, Marsden and Ratiu [1988] and Rosenberg [1997]. In Section 2.3 we define groups of diffeomorphisms used throughout this thesis and describe their Lie group structure. These results are from Ebin and Marsden [1970] and Marsden et al. [2000].

2.1 Riemannian manifolds

Let M be a Banach manifold., i.e., a manifold modelled on a Banach space E. The tangent space to M at a point x is denoted T_xM and T^*_xM is the dual of T_xM.

A weak Riemannian structure on M is a smooth assignment

$$x \to \langle \cdot, \cdot \rangle_x$$

of a nondegenerate inner product (not necessarily complete) to each tangent space $T_x M$. Here, the smoothness means that in the local charts the map

$$x \in U \subset E \to \langle \cdot, \cdot \rangle_x \in L_2(E \times E, \mathbb{R})$$

is smooth, where $L_2(E \times E, \mathbb{R})$ is the Banach space of bilinear maps;
nondegeneracy of an inner product means that

$$\langle v, u \rangle_x = 0 \quad \forall u \in T_x M$$

if and only if $v = 0$.

A weak Riemannian manifold is a manifold endowed with a weak Riemannian structure.

A Riemannian manifold (or strong Riemannian manifold) is a weak Riemannian manifold for which the map $T_x M \to T_x^* M$, given by

$$v \to \langle v, \cdot \rangle_x ,$$

is a surjection.

The distinction between weak Riemannian and Riemannian manifolds is important only in infinite dimensional case.

The Riemannian connection. The weak Riemannian metric defines a unique Riemannian connection (also called Levi-Civita connection) ∇ on M by the identity

$$2 \langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle + \langle Z, [X, Y] \rangle - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle \tag{2.1}$$

for arbitrary vector fields X, Y, Z on M. In equation 2.1 $X \langle Y, Z \rangle$ means differentiation of a function $x \to \langle Y(x), Z(x) \rangle_x$ by a the vector field X and $[Y, Z]$ is the usual bracket of the vector fields Y, Z.

The Riemannian connection satisfies the following properties:

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{2.2}$$

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle . \tag{2.3}$$

The vertical lift. Let $u, v \in T_x M$. The vertical lift of u with respect to v is defined as $u^l_v \in T_v T_x M$, such that

$$u^l_v = \frac{d}{dt}(v + tu).$$

In local coordinates

$$u^l_v = (x, v, 0, u).$$

The geodesic spray. Let $\tau : TM \to M$ be a natural projection. The geodesic spray on M is a vector field on TM, the tangent bundle of M, satisfying the equation

$$\frac{d}{dt}\dot\gamma_t = S(\dot\gamma_t),$$

where γ_t is a geodesic on M and $\dot\gamma_t$ is its velocity vector.

The Christoffel map $\Gamma_x : E \times E \to E$ is defined in a chart U around $x \in M$ by the relationship

$$\nabla_Z Y(x) = DY \cdot Z(x) + \Gamma_x(Z(x), Y(x)). \tag{2.4}$$

In finite dimensions one defines the Christoffel symbols by the formula

$$\Gamma^k_{ij} Z^i Y^j \frac{\partial}{\partial x^k} = \Gamma_x(Z, Y),$$

where $Z = Z^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$ are vector fields on M (as usually, the summation on repeated indexes is understood here and below).

If connection is Riemannian, the Christoffel map is symmetric.

The connector map (see Eliasson [1967]; Dombrovski [1962]). The connection ∇ on M defines a map $K : T^2M \to TM$ such that in a chart U around $x \in M$

$$K(x,z,y,w) = (x, w + \Gamma_x(y,z)) \quad \forall z,y,w \in E. \tag{2.5}$$

Conversely, the connector K uniquely defines the connection ∇ on M by the formula

$$\nabla_Z Y = K[TY \cdot Z]. \tag{2.6}$$

We call the connection on M smooth if the corresponding connector map is a smooth bundle map.

The relationship between the geodesic spray and the connection is given by equality

$$S(Y) = TY \cdot Y - (\nabla_Y Y)^l_Y \tag{2.7}$$

for any vector field Y on M. In a chart U around $x \in M$, the geodesic spray can be expressed in terms of Christoffel map in the following way:

$$S(x,v) = (x, v, v - \Gamma_x(v,v) \quad \forall v \in E. \tag{2.8}$$

Due to Formulae 2.5, 2.8 the following statements are equivalent:

1. The spray S on M is smooth;
2. The connection ∇ on M is smooth;
3. For every chart U in M, the Christoffel maps $\Gamma : U \times E \times E \to E$ are smooth.

It should be noted that the Levi-Civita connection of a weak Riemannian metric is not automatically smooth.

The Riemannian curvature tensor $\mathcal{R}$ on M is defined by the equality

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.9}$$

for arbitrary vector field X, Y, Z on M. It is easy to show that, in fact, $\mathcal{R}$ depends only on point values of X, Y, Z and the following holds:

$$\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y = 0, \tag{2.10}$$

$$\langle \mathcal{R}(X,Y)Z, W\rangle = -\langle \mathcal{R}(X,Y)W, Z\rangle, \tag{2.11}$$

$$\mathcal{R}(X,Y)Z = -\mathcal{R}(Y,X)Z, \tag{2.12}$$

$$\langle \mathcal{R}(X,Y)Z, W\rangle = \langle \mathcal{R}(Z,W)X, Y\rangle. \tag{2.13}$$

These are called Bianchi identities.

The Ricci curvature tensor is well-defined on a finite dimensional manifold as a trace of Riemannian curvature tensor, i.e.,

$$\mathrm{Ric}(y,z) = \mathrm{Tr}(w \to \mathcal{R}(w,y)z).$$

2.2 Vector calculus on Riemannian manifolds

In this section we assume that M is a compact orientable Riemannian manifold of finite dimension n. The metric is defined by a symmetric metric tensor g via

$$\langle Y, Z\rangle_x = g_{ij}(x)Y^i Z^j.$$

for arbitrary vector fields $Y = Y^i \frac{\partial}{\partial x^i}, Z = z^i \frac{\partial}{\partial x^i}$ on M.

The flat operator $\flat : Y \to Y^\flat$ maps a vector field Y on M to a 1-form $Y^\flat$, where

$$Y^\flat(v) = \langle Y(x), v\rangle_x \quad \forall v \in T_xM.$$

The sharp operator $\sharp : \beta \to \beta^\sharp$ maps a 1-form β on M to a vector field satisfying the equality

$$\left\langle \beta^\sharp, v\right\rangle_x = \beta(v) \quad \forall v \in T_xM.$$

The flat and sharp operators provide a natural isomorphism between the spaces of vector fields and 1-forms on M.

In coordinates,

$$u^\flat = g_{ij} u^j dx^i$$

$$\beta^\sharp = g^{ij} \beta_j \frac{\partial}{\partial x^i},$$

where $g^{ij} g_{jk} = \delta^i_k$ and δ^i_k is a Kronecker's symbol.

The volume form is a nowhere vanishing n-form on M. Such form is guaranteed to exist on an orientable manifold. On a Riemannian manifold there is a canonical volume form μ defined by the metric. In a chart, μ is given by the expression

$$\mu = |det[g_{ij}]|^{1/2} dx^1 \wedge \cdots \wedge dx^n, \tag{2.14}$$

where $dx^1 \wedge dx^2$ is a wedge product of the forms dx^1 and dx^2. Further we assume that the volume form μ is chosen to be canonical, i.e., given by 2.14.

The pullback operator. Suppose $\psi : M \to N$ is a C^1 map from a manifold M to a manifold N. The pullback operator ψ^* maps a k-form β on N to a k-form $\psi^*\beta$, where

$$(\psi^*\beta)(x)(v_1, \ldots, v_k) = \beta(\psi(x))(T\psi \cdot v_1, \ldots, T\psi \cdot v_k) \quad \forall v_1, \ldots, v_k \in T_x M.$$

The Lie derivative. Let Λ^k denotes the bundle of k-forms on M. For a vector field Y on M with a flow F_t, the Lie derivative $\mathcal{L}_Y : \Lambda^k \to \Lambda^k$ is defined by

$$\mathcal{L}_Y \beta = d/dt_{t=0} F_t^* \beta \quad \forall \beta \in \Lambda^k.$$

The divergence of a vector field Y on M is a function $\operatorname{div} Y$, such that

$$\mathcal{L}_Y \mu = (\operatorname{div} Y)\mu.$$

By properties of the Lie derivative, for a function $f : M \to \mathbb{R}$

$$\mathcal{L}_Y(f\mu) = (Yf)\mu + (f \operatorname{div} Y)\mu. \tag{2.15}$$

The gradient of a function f is a vector field on M, given by the formula

$$\nabla f = \operatorname{grad} f = (df)^\sharp,$$

where d denotes the exterior derivative.

The metric tensor induces the scalar product on the space of k-forms in the following way:

$$\langle \alpha, \beta \rangle = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \ldots i_k} g^{i_1 j_1} \ldots g^{i_k j_k} \beta_{j_1 \ldots j_k}$$

for all $\alpha, \beta \in \Lambda^k$.

The codifferential $\delta : \Lambda^{k+1} \to \Lambda^k$ is defined as an adjoint of d, that is,

$$\int_M \langle d\alpha, \beta \rangle \, \mu = \int_M \langle \alpha, \delta\beta \rangle \, \mu \tag{2.16}$$

for all $\alpha \in \Lambda^k$, $\beta \in \Lambda^{k+1}$, $(k > 0)$ and $\delta \equiv 0$ on functions.

The divergence and codifferential are related by formula

$$\operatorname{div} Y = -\delta Y^\flat.$$

The covariant derivative. The Riemannian connection on M defines a covariant derivative also denoted ∇, given on 1-forms by the formula

$$Y(\beta(Z)) = (\nabla_Y \beta)(Z) + \beta(\nabla_Y Z) \tag{2.17}$$

for arbitrary vector fields X, Y and a 1-form β. Hence, in coordinates

$$\nabla_Y \beta = -\Gamma^i_{jk} Y^j \beta_i dx^k, \tag{2.18}$$

where $Y = Y^i \frac{\partial}{\partial x^i}$, $\beta = \beta_i dx^i$.

Let Y be a vector field. Define the contraction operator $i_Y : \Lambda^k \to \Lambda^{k-1}$ by

$$i_Y \beta = \beta(Y, \ \cdot \ , \cdots , \ \cdot \ ) \quad \forall \beta \in \Lambda^k.$$

We will write $\nabla\beta$ for a 2-form, given by the equality

$$i_Y \beta = \nabla_Y \beta.$$

The Laplace de Rham operator $\Delta^R : \Lambda^k \to \Lambda^k$ on a k-form β is defined as

$$\Delta^R \beta = \delta d\beta + d\delta\beta. \tag{2.19}$$

Taking into account the isomorphism between the 1-forms and vector fields, the Formula 2.19 also defines the Laplacian operator Δ^R on vector fields via

$$\Delta^R Y = (\Delta^R(Y^\flat))^\sharp.$$

Notice, that $\delta Y^\flat = 0$ for a divergence free vector field Y, hence, in that particular case,

$$\Delta^R Y = \delta d Y^\flat.$$

The Rough Laplacian $\hat{\Delta} : \Lambda^1 \to \Lambda^1$ is defined on 1-forms by the equality

$$\hat{\beta} = -g^{ij} \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \beta. \tag{2.20}$$

The Ricci operator $\mathrm{Ric} : \Lambda^1 \to \Lambda^1$ on 1-forms is induced by Ricci curvature in the following way:

$$\mathrm{Ric}(\beta) = \mathrm{Ric}(\beta^\sharp, \cdot).$$

For a vector field Y,

$$\mathrm{Ric}(Y) = [\mathrm{Ric}(Y^\flat]^\sharp = [\mathrm{Ric}(Y, \cdot)]^\sharp.$$

It's easy to see that Ric is a self-adjoint operator on the vector fields with respect to the Riemannian metric. Indeed,

$$\begin{aligned}\langle \mathrm{Ric}(Y), W\rangle &= \mathrm{Ric}(Y, W)\\ &= \mathrm{Ric}(W, Y)\\ &= \langle \mathrm{Ric}(W), Y\rangle .\end{aligned}$$

It is rather troublesome to calculate the Laplace de Rham operator in coordinates directly from definition. The following theorem simplify this task by expressing Laplace de Rham operator in terms of the rough Laplacian and Ricci curvature.

Theorem 2.1. *Bohner's formula (see Rosenberg [1997]). For 1-form β*

$$\Delta^R \beta = \hat{\Delta}\beta + \mathrm{Ric}(\beta).$$

2.3 Review of diffeomorphism groups

In this section we sketch the construction of diffeomorphism groups and manifolds of mappings following Ebin and Marsden [1970] and recall some of the important properties of these groups.

The manifold structure of diffeomorphism groups. Let M be a compact oriented smooth n-dimensional manifold, possibly with C^∞ boundary ∂M. Let $\tilde{\tau} : E \to M$ be a vector bundle over M. Then, one can define in the usual way a Hilbert space $H^s(E)$ of section of E of Sobolev class s.

The Sobolev embedding theorem states that if $k > 0$ and $s > n/2 + k$, then there is a continuous inclusion $H^s(E) \subset C^k(E)$. Hence, for $s > n/2$, it makes sense to talk about H^s maps from one manifold to another. Given a Riemannian metric on a manifold N without a boundary and its associated exponential map $exp : TN \to N$, there is a natural way to construct the charts for $H^s(M, N)$. For a function $f \in H^s(M,N)$ we define

$$T_f H^s(M,N) = \{g \in H^s(M,TN) |\, \tau_N \circ g = f\},$$

where $\tau_N : TN \to N$ is the canonical projection on N. The exponential map exp_{MN} : $T_f H^s(M,N) \to H^s(M,N)$ is defined by

$$exp_{MN}(g) = exp \circ g.$$

It's easy to check that this is indeed a chart from a neighborhood of 0 in $T_f H^s(M,N)$ to a neighborhood of f in $H^s(M,N)$. This construction defines a manifold structure on $H^s(M,N)$ independent of the metric chosen.

Suppose M is a compact without boundary. The set $C^1\mathcal{D}$ of C^1 diffeomorphisms of M is open in $C^1(M,M)$ and it is a topological group. For $s > n/2+1$ one can set

$$\mathcal{D}^s = H^s(M,M) \cap C^1\mathcal{D}.$$

Then, (see Ebin [1966])

$$\mathcal{D}^s = \{\eta \in H^s(M,M) |\, \eta \text{ is bijective and } \eta^{-1} \in H^s(M,M)\} \tag{2.21}$$

is open in $H^s(M,M)$ and is a topological group.

If M has a boundary, one cannot form a smooth manifold $H^s(M,M)$ as above. However, one can still define $\mathcal{D}^s$ by Formula 2.21, modifying the argument in the following way. Embed M in a compact manifold $\tilde{M}$ without a boundary, so that the interior and the boundary of M are submanifolds of $\tilde{M}$. For example, one can take $\tilde{M}$ to be the double of M. Then $H^s(M,\tilde{M})$ is a smooth manifold as above and we set

$$\mathcal{D}^s = C^1\mathcal{D} \cap H^s(M,\tilde{M}).$$

Formula 2.21 is still true in that case as shown in Ebin and Marsden [1970].

$\mathcal{D}^s$ as a right Lie group. $\mathcal{D}^s$ is a topological group with a group operation being a composition. The right multiplication

$$R_\eta : \mathcal{D}^s \to \mathcal{D}^s,$$

$$R_\eta(\xi) = \xi \circ \eta$$

is a smooth map for each $\eta \in \mathcal{D}^s$. The tangent map of R_η, $TR_\eta : T\mathcal{D}^s \to T\mathcal{D}^s$ is given by the expression

$$TR_\eta X = X \circ \eta.$$

The left multiplication $L_\eta : D^s \to D^s$,

$$L_\eta(\xi) = \eta \circ \xi$$

is not smooth. L_η is of class C^l as a map $\mathcal{D}^{s+l} \to \mathcal{D}^s$ $(l \geq 0)$ and its tangent is given by

$$TL_\eta X = T\eta \circ X.$$

The inverse map $\eta \to \eta^{-1}$ is continuous on $\mathcal{D}^s$ and is of class C^l as a map $\mathcal{D}^{s+l} \to \mathcal{D}^s$. If η_t is a C^1 curve in $\mathcal{D}^{s+l}$, $l \geq 1$, then

$$\frac{d}{dt}\eta_t^{-1} = -T\eta_t^{-1} \circ \left[\frac{d}{dt}\eta_t\right] \circ \eta_t^{-1}.$$

As left multiplication is not smooth, $\mathcal{D}^s$ is not a Lie group. However, one can still define a Lie algebra structure at the identity of $\mathcal{D}^s$ using right-invariant vector fields.

The identity of the group is the map $e : M \to M$, $e(m) = m$. The tangent space at the identity is

$$T_e\mathcal{D}^s = \{u \in H^s(M, TM) \mid \tau \circ u = e \quad \text{and} \quad u \| \partial M\},$$

where $\tau : TM \to M$ is a canonical projection. This is the space of H^s vector fields on M that are parallel to the boundary (the later requirement is dropped if M does not have a boundary).

The tangent space at η is defined by a similar relationship:

$$T_\eta \mathcal{D}^s = \{u \in H^s(M, TM) \mid \tau \circ u = \eta \quad \text{and} \quad u \| \partial M\}.$$

For $u \in T_e\mathcal{D}^s$, we define the right-invariant vector field X_u on $\mathcal{D}^s$ as follows:

$$X_u(\eta) = TR_\eta u = u \circ \eta.$$

The bracket of vector fields on $\mathcal{D}^s$ induces a Lie bracket $[\cdot, \cdot]_L$ on $T_e\mathcal{D}^s$ in the usual way:

$$[u, v]_L \equiv [X_u, X_v](e) \quad \forall u, v \in T_e\mathcal{D}^s.$$

It's easy to check that

$$[X_u, X_v](e) = [u, v], \tag{2.22}$$

where $[u, v]$ is a usual bracket of vector fields u, v on M.

Subgroups of $\mathcal{D}^s$. We will mention several subgroups of $\mathcal{D}^s$ that are important for the treatment of PDEs with constraints, such as boundary conditions or incompressibility.

Let μ be the volume form on M. The group of volume preserving diffeomorphisms

$$\mathcal{D}^s_\mu = \{\eta \in \mathcal{D}^s | \, \eta^* \mu = \mu\}$$

is a closed submanifold of $\mathcal{D}^s$ and a right Lie subgroup. Its Lie algebra $\mathfrak{X}^s_{div}$ is a space of H^s divergence free vector fields, i.e,

$$\mathfrak{X}^s_{div} = T_e\mathcal{D}^s_\mu = \{u \in T_e\mathcal{D}^s | \, \operatorname{div} u = 0\}.$$

The other important subgroups are

$$\mathcal{D}^s_{\mu,0} = \{\eta \in \mathcal{D}^s_\mu | \, \eta(m) = m \text{ for } m \in \partial M\}$$

with the Lie algebra

$$T_e\mathcal{D}^s_{\mu,0} = \{u \in \mathfrak{X}^s_{div} |\, u = 0 \text{ on } \partial M\};$$

$$\mathcal{D}^s_0 = \{\eta \in \mathcal{D}^s |\, \eta(m) = m \text{ for } m \in \partial M\} \tag{2.23}$$

with the Lie algebra

$$T_e\mathcal{D}^s_0 = \{u \in T_e\mathcal{D}^s |\, u = 0 \text{ on } \partial M\}.$$

Chapter 3

Euler equation of an ideal fluid

This chapter is the content of the article by Vasylkevych and Marsden [2003]. Here we study the Euler equation of an incompressible fluid. We establish that the flow generated by this equation on the space of divergence free vector fields is Poisson. This is the Theorem 3.25. Another key result is the Theorem 3.4 in Section 3.3.

It has the following structure. In Section 3.1 we establish the connection between the Euler equation and manifolds of diffeomorphisms. Then, we recall the basic ideas of Poisson reduction in Section 3.2. Our results are presented in next two sections. In Section 3.3 we prove that tangent bundle of a weak Riemannian manifold carries a Poisson structure in an appropriate sense, provided that the manifold possesses a *smooth* Riemannian connection. The later requirement is fulfilled on the groups of diffeomorphisms according to the work of Ebin and Marsden [1970]. In Section 3.4 we utilize this result to show that the flow of Euler equation is Poisson in an appropriate sense.

3.1 Solutions of the Euler equation

In this section we present some classical results concerning the Euler equation that motivated our study. The notation and exposition follows Ebin and Marsden [1970].

The Euler equations on compact manifold are traditionally formulated in the following way. Let M be a compact Riemannian n-manifold possibly with boundary ∂M. Find a time dependent vector field u, (which has an associated flow denoted η_t)

such that

1. u_0 is a given initial condition with $\operatorname{div} u_0 = 0$
2. The Euler equations hold:
$$\frac{\partial u_t}{\partial t} + \nabla_{u_t} u_t = -\operatorname{grad} p_t \tag{3.1}$$
for some scalar function $p_t : M \to \mathbb{R}$ (the pressure),
3. $\operatorname{div} u_t = 0$, and
4. u is parallel to ∂M.

It is standard that above equation can be formally rewritten as an ODE on the space of divergence free vector fields with a derivative loss. But it was discovered by Ebin and Marsden [1970] that this is literally true with no derivative loss in Lagrangian representation. We recall how this proceeds. Let μ be a volume form on the manifold M. Let $H^s(M, N)$ denote the space of mappings of Sobolev class s from an n-manifold M to a manifold N. For $s > n/2 + 1$, let

$$\mathcal{D}^s = \{\eta \in H^s(M, M) \mid \eta \text{ is bijective and } \eta^{-1} \in H^s(M, M)\} \quad \text{and}$$

$$\mathcal{D}^s_\mu = \{\eta \in \mathcal{D}^s \mid \eta^* \mu = \mu\}.$$

Then both $\mathcal{D}^s, \mathcal{D}^s_\mu$ are smooth infinite dimensional manifolds and topological groups, moreover $\mathcal{D}^s_\mu$ is a closed submanifold and a subgroup of $\mathcal{D}^s$.

Let $\widetilde{\tau} : T\mathcal{D}^s_\mu \to \mathcal{D}^s_\mu$ and $\tau : TM \to M$ be the canonical projections and let $e : M \to M, \quad e(m) = m$ be the identity element of the groups $\mathcal{D}^s_\mu, \mathcal{D}^s$. Then

$$T_\eta \mathcal{D}^s = \{u \in H^s(M, TM) \mid \tau \circ u = \eta \quad \text{and} \quad u \| \partial M\},$$

$$T_e \mathcal{D}^s_\mu = \mathfrak{X}^s_{\operatorname{div}}(M) = \{u \in H^s(M, TM) \mid \tau \circ u = e,\ \operatorname{div} u = 0 \quad \text{and} \quad u \| \partial M\},$$

where $\mathfrak{X}^s_{\mathrm{div}}(M)$ denotes the space of H^s divergence free vector fields on M that are parallel to the boundary.

A given Riemannian metric on M induces a right-invariant weak Riemannian metric on $\mathcal{D}^s_\mu$ given by

$$\langle X, Y \rangle_\eta = \int_M \langle X(m), Y(m) \rangle_{\eta(m)} \mu(m) \tag{3.2}$$

for $X, Y \in T_\eta \mathcal{D}^s_\mu$ where scalar product under the integral sign is taken in M.

As was shown in Ebin and Marsden [1970], $\mathcal{D}^s_\mu$ possesses a smooth Riemannian connection and, as a consequence, a smooth spray, which we will denote S.

Proposition 3.1. (Ebin and Marsden [1970]) *For $s > (n/2)+1$, the weak Riemannian metric* (3.2) *has a C^∞ spray $S : T\mathcal{D}^s_\mu \to TT\mathcal{D}^s_\mu$. Let $F_t : T\mathcal{D}^s_\mu \to T\mathcal{D}^s_\mu$ be the (local, C^∞) flow of S. Let $v_t = F_t(u_0)$ (the material velocity field) and $\eta_t = \tilde{\tau}(v_t)$ (the particle position field). Then the solution of the Euler equation with initial condition $u(0) = u_0$ is given by*

$$u_t = v_t \circ \eta_t^{-1}.$$

From the properties of the diffeomorphism group, one sees that this result shows that the Euler equations (3.1) are well posed in H^s in Eulerian representation.

3.2 Motivation: the Poisson reduction theorem

First, recall the following basic and simple result about Poisson reduction (see, for example, Marsden and Ratiu [1999]).

Suppose that G is a Lie group that acts on a Poisson manifold P and that for each $g \in G$ the action map $\Phi_g : P \to P$ is a Poisson map. Suppose that the quotient P/G is a smooth manifold and the projection $\pi : P \to P/G$ is a submersion. Then, there is a unique Poisson structure $\{\cdot, \cdot\}$ on P/G such that π is a Poisson map. It is given by

$$\{f, k\} \circ \pi = \{f \circ \pi, k \circ \pi\}_P \quad \forall k, f \in \mathcal{F}(P/G),$$

where $\{\cdot,\cdot\}_P$ is a Poisson bracket in P and $\mathcal{F}(P/G)$ is a set of smooth functions on P/G.

If $\mathbf{X}_H$ is a Hamiltonian vector field for a G-invariant Hamiltonian $H \in \mathcal{F}(P)$, then π also induces reduction of dynamics. There is a function $h \in \mathcal{F}(P/G)$ such that $H = h \circ \pi$. Since π is a Poisson map it transforms $\mathbf{X}_H$ on P to $\mathbf{X}_h$ on P/G, that is, $T\pi \circ \mathbf{X}_H = \mathbf{X}_h \circ \pi$. Denoting the flow of $\mathbf{X}_H$ by F_t and the flow of $\mathbf{X}_h$ by $\tilde{F}_t$ we obtain commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{F_t} & P \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ P/G & \xrightarrow{\tilde{F}_t} & P/G \end{array}$$

Our strategy is to apply the above procedure to the context of fluids. To do so, define the map $\pi : T\mathcal{D}^s_\mu \to \mathfrak{X}^s_{\text{div}}$ via

$$\pi(\eta, v) = v \circ \eta^{-1},$$

where $\eta \in \mathcal{D}^s_\mu$; $(\eta, v) \in T_\eta\mathcal{D}^s_\mu$; $\tau \circ v = \eta$. Let $\tilde{F}_t : \mathfrak{X}^s_{\text{div}} \to \mathfrak{X}^s_{\text{div}}$ be given by

$$\tilde{F}_t(v) = \pi \circ F_t(v)$$

for $v \in \mathfrak{X}^s_{\text{div}}$. By Proposition 3.1, $\tilde{F}_t$ is the flow of Euler equation on $\mathfrak{X}^s_{\text{div}}$, i.e., $u_t = \tilde{F}_t(u_0)$ satisfies the Euler equations (3.1).

It is clear from the preceding developments that F_t (as a flow of a spray) is a flow of Hamiltonian vector field on $T\mathcal{D}^s_\mu$. The following commutative diagram

$$\begin{array}{ccc} T\mathcal{D}^s_\mu & \xrightarrow{F_t} & T\mathcal{D}^s_\mu \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ T_e\mathcal{D}^s_\mu & \xrightarrow{\tilde{F}_t} & T_e\mathcal{D}^s_\mu \end{array}$$

suggests that the flow of Euler equation itself, which is obtained from F_t via Poisson reduction, should be a Hamiltonian flow in the sense of Poisson manifolds and this is certainly formally true (see, for instance Lewis et al. [1986] for both the case considered here as well as the case of free boundary problems).

However, as noted in this reference and elsewhere, there are difficulties in finding the right class of functions so that one gets a Poisson structure in a precise sense. To justify the formal insight in precise function spaces, one has to overcome two hurdles.

The first hurdle is that $T\mathcal{D}^s_\mu$ is only a weak symplectic manifold, and therefore does not necessarily carry a Poisson bracket in any obvious way without special ad hoc hypotheses such as "the needed functional derivatives exist," which have long been recognized as awkward at best.

The second hurdle is that $T\mathcal{D}^s_\mu$ is not a Lie group in the usual sense (left multiplication is not smooth), and π is not a smooth map (inversion in $\mathcal{D}^s_\mu$ is not smooth). Therefore, the well-developed theory of Poisson and Lie-Poisson reduction is not directly applicable in this case, even though the loss of derivatives one suffers from these transformations is well understood.

The main point of this paper is to resolve these difficulties in what we believe is a satisfactory way. We do this in the following sections.

3.3 Poisson structures on weak Riemannian manifolds

Let Q be a weak Riemannian manifold modelled on Banach space $\mathbf{E}$ with metric $\langle\cdot,\cdot\rangle$. Then TQ possesses a canonical *weak* symplectic form that is given in charts by the following standard formula (see, e.g., Marsden and Ratiu [1999]):

$$\Omega(\eta, e)((e_1, e_2), (e_3, e_4)) = \langle e_1, e_4\rangle_\eta - \langle e_2, e_3\rangle_\eta + D_\eta \langle e, e_1\rangle_\eta \cdot e_3 - D_\eta \langle e, e_3\rangle_\eta \cdot e_1,$$

where $\eta \in Q$, $e, e_1, e_2, e_3, e_4 \in E$.

For a smooth function $f : M \to \mathbb{R}$ on a (strong) symplectic manifold (M, Ω_1), let $\mathbf{X}_f$ denote its Hamiltonian vector field. Then

$$\{f, g\} = \Omega_1(X_f, X_g) \tag{3.3}$$

makes $(M, \{\cdot,\cdot\})$ into a Poisson manifold.

Since Ω is weak, Formula 3.3 does not automatically define Poisson bracket $\{f, g\}$ for arbitrary functions $f, g \in \mathcal{F}(TQ)$ since $\mathbf{X}_f, \mathbf{X}_g$ may fail to exist and even if they do, one has to make additional hypotheses to obtain the Jacobi identity.

However, under the two additional hypotheses:

1. Q has smooth Riemannian connection;

2. The inclusion $T_\eta Q \to T^*_\eta Q$ (the literal dual space) via

$$v(u) = \langle v, u \rangle_\eta \quad \forall u \in T_\eta Q$$

   is dense,

it will be shown that one can define a Poisson bracket on the subalgebra

$$\mathcal{K}(TQ) = \left\{ f \in \mathcal{F}TQ \,\middle|\, \frac{\partial f}{\partial \eta}, \frac{\partial f}{\partial v} \in C^\infty(TQ, TQ) \right\}$$

of $\mathcal{F}(TQ)$. Here $\frac{\partial f}{\partial \eta}, \frac{\partial f}{\partial v}$ are **covariant partial derivatives** on TQ, the definition of which will be given below.

This newly defined bracket *makes $\mathcal{K}(TQ)$ into a Lie algebra and retains essential dynamical properties of a "true" Poisson bracket, including the Jacobi identity and the fact that flows of Hamiltonian vector fields are Poisson maps and, of course, energy is conserved.* Moreover, we will show that the bracket indeed is related to the canonical weak symplectic form in the way that one would expect. In the following we assume that conditions (1) and (2) are satisfied.

Covariant partial derivatives. First, we introduce covariant partial derivatives on TQ. Let $\tau : TQ \to Q$ and $\tau_1 : TTQ \to TQ$ be natural projections, $\Gamma : Q \supset U \times E \times E \to E$ be a Christoffel map and $K : TTQ \to TQ$ be a connector map. In local representation,

$$K(\eta, v, u, w) = (\eta, w + \Gamma(\eta)(v, u)).$$

Define $\Theta : TTQ \to TQ \bigoplus TQ \bigoplus TQ$ by

$$\Theta = (\tau_1, T\tau, K).$$

It is standard that Θ is a diffeomorphism (see Eliasson [1967]). For $H : TQ \to \mathbb{R}$ we set

$$\frac{\partial H}{\partial \eta}(V) \cdot W = dH \cdot \Theta^{-1}(V, W, 0) \quad \forall V, W \in T_q Q,$$
$$\frac{\partial H}{\partial v}(V) \cdot W = dH \cdot \Theta^{-1}(V, 0, W) \quad \forall V, W \in T_q Q.$$

In local representation, this reads

$$\begin{aligned}\frac{\partial H}{\partial \eta}(\eta, v) \cdot (\eta, u) &= dH \cdot \Theta^{-1}((\eta, v), (\eta, u), (\eta, 0)) \\ &= dH \cdot (\eta, v, u, -\Gamma(\eta)(v, u)),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial H}{\partial v}(\eta, v) \cdot (\eta, w) &= dH \cdot \Theta^{-1}((\eta, v), (\eta, 0), (\eta, w)) \\ &= dH \cdot (\eta, v, 0, w)).\end{aligned}$$

Similarly, for $\phi : TQ \to TQ_1$ we define $\frac{\partial \phi}{\partial \eta}, \frac{\partial \phi}{\partial v} : TQ \to L(TQ, TTQ_1)$ (here $L(TQ, TTQ_1)$ is the space of linear maps $TQ \to TTQ_1$) by

$$\frac{\partial \phi}{\partial \eta}(V) \cdot W = T\phi \cdot \Theta^{-1}(V, W, 0) \quad \forall V, W \in T_q Q,$$

$$\frac{\partial \phi}{\partial v}(V) \cdot W = T\phi \cdot \Theta^{-1}(V, 0, W) \quad \forall V, W \in T_q Q.$$

The following lemmata are readily verified.

Lemma 3.2. *Let X be a vector field on TQ, Y be a vector field on TQ_1, $\phi : TQ_1 \to TQ$. Then*

$$
\begin{aligned}
dH \cdot X &= \frac{\partial H}{\partial \eta} \cdot T\tau(X) + \frac{\partial H}{\partial v} \cdot K(X), \\
\frac{\partial (H \circ \phi)}{\partial \eta} \cdot Y &= dH \cdot \left(\frac{\partial \phi}{\partial \eta} \cdot Y \right), \\
\frac{\partial (H \circ \phi)}{\partial v} \cdot Y &= dH \cdot \left(\frac{\partial \phi}{\partial v} \cdot Y \right).
\end{aligned}
$$

Lemma 3.3. *For $H \in C^1(TQ, \mathbb{R})$, we have*

$$
\begin{aligned}
\frac{\partial H}{\partial \eta}(\eta, v) \cdot (\eta, u) &= \left. \frac{d}{dt} \right|_{t=0} H(\eta_t, v_t), \\
\frac{\partial H}{\partial v}(\eta, v) \cdot (\eta, w) &= \left. \frac{d}{dt} \right|_{t=0} H(\eta, v + tw),
\end{aligned}
$$

where (η_t, v_t) is the parallel translation of (η, v) along the curve η_t with $\eta_t'(0) = u$.

Let

$$
\mathcal{K}^k(TQ) = \left\{ f \in C^{k+1}(TQ, \mathbb{R}) \;\middle|\; \frac{\partial f}{\partial \eta}, \frac{\partial f}{\partial v} \in C^k(TQ, TQ) \right\}.
$$

Now we can define the bracket $\{\cdot, \cdot\}$ via

$$
\{f, g\}(\eta, v) = \left\langle \frac{\partial f}{\partial \eta}(\eta, v), \frac{\partial g}{\partial v}(\eta, v) \right\rangle_\eta - \left\langle \frac{\partial f}{\partial v}(\eta, v), \frac{\partial g}{\partial \eta}(\eta, v) \right\rangle_\eta . \tag{3.4}
$$

Preliminaries on the Poisson Structure. The following is the first main result.

Theorem 3.4. *The bracket* (3.4) *maps $\mathcal{K}^k \times \mathcal{K}^m$ into $\mathcal{K}^{\min(k,m)-1}$ and also maps $\mathcal{K} \times \mathcal{K}$ into $\mathcal{K}$.*

Remark 3.5. *By definition of the covariant partial derivatives, $\frac{\partial h}{\partial \eta}, \frac{\partial h}{\partial v} : T_\eta Q \to T_\eta^* Q$ for $h : TQ \to \mathbb{R}$. The theorem asserts that if $h = \{f, g\}$ then, in fact, $\frac{\partial h}{\partial \eta}(\eta, v), \frac{\partial h}{\partial v}(\eta, v) \in TQ$, i.e., there are $Z(\eta, v), Y(\eta, v) \in T_\eta Q$ such that*

$$
\frac{\partial h}{\partial \eta}(\eta, v) \cdot X = \langle Z, X \rangle, \quad \frac{\partial h}{\partial v}(\eta, v) \cdot X = \langle Y, X \rangle \quad \forall X \in T_\eta Q
$$

and the maps $(\eta, v) \to Z(\eta, v), Y(\eta, v)$ have appropriate smoothness.

Proof. Define operator $\frac{\mathbf{D}}{dt} = K \circ \frac{d}{dt}$. This definition extends the usual notion of covariant derivative from vector fields along curves on Q to arbitrary curves on TQ. Let $f, g : TQ \to \mathbb{R}$ and $h = \{f, g\}$. Choosing (η_t, v_t) as in Lemma 3.3, we obtain

$$\begin{aligned}
&\frac{\partial h}{\partial \eta}(\eta, v) \cdot (\eta, u) \\
&= \frac{d}{dt}\bigg|_{t=0} \left\langle \frac{\partial f}{\partial \eta}(\eta_t, v_t), \frac{\partial g}{\partial v}(\eta_t, v_t) \right\rangle_{\eta_t} - \frac{d}{dt}\bigg|_{t=0} \left\langle \frac{\partial f}{\partial v}(\eta_t, v_t), \frac{\partial g}{\partial \eta}(\eta_t, v_t) \right\rangle_{\eta_t} \\
&= \left\langle \frac{\mathbf{D}}{dt}\bigg|_{t=0} \frac{\partial f}{\partial \eta}(\eta_t, v_t), \frac{\partial g}{\partial v}(\eta, v) \right\rangle_{\eta} + \left\langle \frac{\partial f}{\partial \eta}(\eta, v), \frac{\mathbf{D}}{dt}\bigg|_{t=0} \frac{\partial g}{\partial v}(\eta_t, v_t) \right\rangle_{\eta} \\
&\quad - \left\langle \frac{\mathbf{D}}{dt}\bigg|_{t=0} \frac{\partial f}{\partial v}(\eta_t, v_t), \frac{\partial g}{\partial \eta}(\eta, v) \right\rangle_{\eta} - \left\langle \frac{\partial f}{\partial v}(\eta, v), \frac{\mathbf{D}}{dt}\bigg|_{t=0} \frac{\partial g}{\partial \eta}(\eta_t, v_t) \right\rangle_{\eta} \\
&= \left\langle K \frac{\partial}{\partial \eta} \frac{\partial f}{\partial \eta}(\eta, v) \cdot (\eta, u), \frac{\partial g}{\partial v}(\eta, v) \right\rangle_{\eta} + \left\langle K \frac{\partial}{\partial \eta} \frac{\partial g}{\partial v}(\eta, v) \cdot (\eta, u), \frac{\partial f}{\partial \eta}(\eta, v) \right\rangle_{\eta} \\
&\quad - \left\langle K \frac{\partial}{\partial \eta} \frac{\partial g}{\partial \eta}(\eta, v) \cdot (\eta, u), \frac{\partial f}{\partial v}(\eta, v) \right\rangle_{\eta} - \left\langle K \frac{\partial}{\partial \eta} \frac{\partial f}{\partial v}(\eta, v) \cdot (\eta, u), \frac{\partial g}{\partial \eta}(\eta, v) \right\rangle_{\eta}.
\end{aligned}$$

To proceed further, we need to calculate the quantity

$$\left\langle K \frac{\partial}{\partial \eta} \frac{\partial f}{\partial \eta}(\eta, v) \cdot (\eta, u), (\eta, w) \right\rangle_{\eta},$$

where (η, w) is an arbitrary element of $T_\eta Q$. Let (η_{ts}, v_{ts}) be a parametric surface in TQ with the following properties:

1. $\frac{d}{dt}\big|_{t=0} \eta_{t0} = u, (\eta_{00}, v_{00}) = (\eta, v)$;

2. (η_{t0}, v_{t0}) is a parallel translation of (η, v);

3. (η_{t0}, w_t) is a parallel translation of $(\eta_{00}, w_0) = (\eta, w)$;

4. $\frac{d}{ds}\big|_{s=0} \eta_{ts} = w_t$ for all t;

5. (η_{ts}, v_{ts}) is a parallel translation of (η_{t0}, v_{t0}) for all s.

Then, keeping in mind Lemmata 3.2, 3.3 and symmetry of Riemannian connection, one checks the following:

$$\left\langle K\frac{\partial}{\partial\eta}\frac{\partial f}{\partial\eta}(\eta,v)\cdot(\eta,u),(\eta,w)\right\rangle_\eta = \left\langle \frac{\mathbf{D}}{dt}\Big|_{t=0}\frac{\partial f}{\partial\eta}(\eta_{t0},v_{t0}),(\eta,w)\right\rangle_\eta$$

$$= \frac{d}{dt}_{t=0}\left\langle \frac{\partial f}{\partial\eta}(\eta_{t0},v_{t0}),(\eta_{t0},w_t)\right\rangle_{\eta_t}$$

$$= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}_{s=0}f(\eta_{ts},v_{ts}) = \frac{d}{ds}_{s=0}\frac{d}{dt}_{t=0}f(\eta_{ts},v_{ts}) = \frac{d}{ds}_{s=0}df\cdot\frac{d}{dt}\Big|_{t=0}(\eta_{ts},v_{ts})$$

$$= \frac{d}{ds}\Big|_{s=0}\left[\frac{\partial f}{\partial v}(\eta_{0s},v_{0s})\cdot K\frac{d}{dt}_{t=0}(\eta_{ts},v_{ts}) + \frac{\partial f}{\partial\eta}(\eta_{0s},v_{0s})\cdot T\tau\frac{d}{dt}_{t=0}(\eta_{ts},v_{ts})\right]$$

$$= \frac{d}{ds}\Big|_{s=0}\left[\left\langle\frac{\partial f}{\partial v}(\eta_{0s},v_{0s}),\frac{\mathbf{D}}{dt}_{t=0}(\eta_{ts},v_{ts})\right\rangle_{\eta 0s} + \left\langle\frac{\partial f}{\partial\eta}(\eta_{0s},v_{0s}),\frac{d}{dt}_{t=0}\eta_{ts}\right\rangle_{\eta_{0s}}\right]$$

$$= \left\langle\frac{\mathbf{D}}{ds}_{s=0}\frac{\partial f}{\partial v}(\eta_{0s},v_{0s}),\frac{\mathbf{D}}{dt}_{t=0}(\eta_{t0},v_{t0})\right\rangle_\eta + \left\langle\frac{\partial f}{\partial v}(\eta,v),\frac{\mathbf{D}}{ds}_{s=0}\frac{\mathbf{D}}{dt}\Big|_{t=0}(\eta_{ts},v_{ts})\right\rangle_\eta$$

$$+\left\langle\frac{\mathbf{D}}{ds}_{s=0}\frac{\partial f}{\partial\eta}(\eta_{0s},v_{0s}),\frac{d}{dt}_{t=0}\eta_{t0}\right\rangle_\eta + \left\langle\frac{\partial f}{\partial\eta}(\eta,v),\frac{\mathbf{D}}{ds}\Big|_{s=0}\frac{d}{dt}_{t=0}\eta_{ts}\right\rangle_\eta.$$

Lemma 3.6. (see Do Carmo [1992]). *Let $\mathcal{R}$ denote the Riemannian curvature tensor. Then*

$$\frac{\mathbf{D}}{ds}\frac{\mathbf{D}}{dt}(\eta_{ts},v_{ts}) = \frac{\mathbf{D}}{dt}\frac{\mathbf{D}}{ds}(\eta_{ts},v_{ts}) + \mathcal{R}(\frac{d}{dt}\eta_{ts},\frac{d}{ds}\eta_{ts})(\eta_{ts},v_{ts}),$$

$$\frac{\mathbf{D}}{ds}\frac{d}{dt}\eta_{ts} = \frac{\mathbf{D}}{dt}\frac{d}{ds}\eta_{ts}.$$

By construction of (η_{ts},v_{ts}), we have $\frac{\mathbf{D}}{dt}\big|_{t=0}(\eta_{t0},v_{t0}) = 0$. Applying Lemma 3.6 we obtain

$$\frac{\mathbf{D}}{ds}_{s=0}\frac{\mathbf{D}}{dt}\Big|_{t=0}(\eta_{ts},v_{ts}) = \mathcal{R}\left(\frac{d}{dt}\Big|_{t=0}\eta_{t0},\frac{d}{ds}\Big|_{s=0}\eta_{0s}\right)(\eta,v) = \mathcal{R}((\eta,u),(\eta,w))(\eta,v),$$

$$\frac{\mathbf{D}}{ds}_{s=0}\frac{d}{dt}\Big|_{t=0}\eta_{ts} = \frac{\mathbf{D}}{dt}_{t=0}(\eta_{t0},w_t) = 0.$$

Thus,

$$\begin{aligned}
&\left\langle K\frac{\partial}{\partial\eta}\frac{\partial f}{\partial\eta}(\eta,v)\cdot(\eta,u),(\eta,w)\right\rangle_\eta\\
&=0+\left\langle\frac{\partial f}{\partial v}(\eta,v),\mathcal{R}((\eta,u),(\eta,w))(\eta,v)\right\rangle_\eta\\
&\quad+\left\langle K\frac{\partial}{\partial\eta}\frac{\partial f}{\partial\eta}(\eta,v)\cdot(\eta,w),(\eta,u)\right\rangle_\eta+0\\
&=\left\langle K\frac{\partial}{\partial\eta}\frac{\partial f}{\partial\eta}(\eta,v)\cdot(\eta,w),(\eta,u)\right\rangle_\eta-\left\langle\mathcal{R}((\eta,v),\frac{\partial f}{\partial v}(\eta,v))(\eta,w),(\eta,u)\right\rangle_\eta
\end{aligned}$$

by Bianchi's identity. Similar calculations yield

$$\left\langle K\frac{\partial}{\partial v}\frac{\partial f}{\partial v}(\eta,v)\cdot(\eta,u),(\eta,w)\right\rangle_\eta=\left\langle K\frac{\partial}{\partial v}\frac{\partial f}{\partial v}(\eta,v)\cdot(\eta,w),(\eta,u)\right\rangle_\eta,$$

$$\left\langle K\frac{\partial}{\partial\eta}\frac{\partial f}{\partial v}(\eta,v)\cdot(\eta,u),(\eta,w)\right\rangle_\eta=\left\langle K\frac{\partial}{\partial v}\frac{\partial f}{\partial\eta}(\eta,v)\cdot(\eta,w),(\eta,u)\right\rangle_\eta.$$

Substituting this into the formulae for $\frac{\partial h}{\partial\eta}$ and using Bianchi's identity once again, we get

$$\begin{aligned}
&\frac{\partial h}{\partial\eta}(\eta,v)\cdot(\eta,u)=\left\langle K\frac{\partial}{\partial\eta}\frac{\partial f}{\partial\eta}(\eta,v)\cdot\frac{\partial g}{\partial v}(\eta,v)+K\frac{\partial}{\partial v}\frac{\partial g}{\partial\eta}(\eta,v)\cdot\frac{\partial f}{\partial\eta}(\eta,v),(\eta,u)\right\rangle_\eta\\
&-\left\langle K\frac{\partial}{\partial\eta}\frac{\partial g}{\partial\eta}(\eta,v)\cdot\frac{\partial f}{\partial v}(\eta,v)+K\frac{\partial}{\partial v}\frac{\partial f}{\partial\eta}(\eta,v)\cdot\frac{\partial g}{\partial\eta}(\eta,v),(\eta,u)\right\rangle_\eta\\
&\quad+\left\langle\mathcal{R}(\frac{\partial f}{\partial v},\frac{\partial g}{\partial\eta})\cdot(\eta,v),(\eta,u)\right\rangle_\eta.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{\partial h}{\partial v}(\eta,v)\cdot(\eta,u)=\left\langle K\frac{\partial}{\partial\eta}\frac{\partial f}{\partial v}(\eta,v)\cdot\frac{\partial g}{\partial v}(\eta,v)+K\frac{\partial}{\partial v}\frac{\partial g}{\partial v}(\eta,v)\cdot\frac{\partial f}{\partial\eta}(\eta,v),(\eta,u)\right\rangle_\eta\\
&\qquad-\left\langle K\frac{\partial}{\partial\eta}\frac{\partial g}{\partial v}(\eta,v)\cdot\frac{\partial f}{\partial v}(\eta,v)+K\frac{\partial}{\partial v}\frac{\partial f}{\partial v}(\eta,v)\cdot\frac{\partial g}{\partial\eta}(\eta,v),(\eta,u)\right\rangle_\eta.
\end{aligned}$$

As K is smooth, the statement of the theorem follows. $\square$

Hamiltonian Vector Fields. The smoothness structure of Hamiltonian vector fields is given as follows.

Proposition 3.7. *The vector field* $\mathbf{X}_H$ *is a* C^k *Hamiltonian vector field (with respect to canonical weak symplectic form) on* TQ *of class* C^k *if and only if* $H \in \mathcal{K}^k(TQ)$*. Moreover,*

$$\mathbf{X}_H(\eta, v) = \left(\eta, v, \frac{\partial H}{\partial v}, -\frac{\partial H}{\partial \eta} - \Gamma(\eta)(v, \frac{\partial H}{\partial v})\right). \tag{3.5}$$

Proof. In local representation, we have

$$\Omega(\eta, e)((e_1, e_2), (e_3, e_4)) = \langle e_1, e_4\rangle_\eta - \langle e_2, e_3\rangle_\eta + \langle \Gamma(\eta)(e, e_3), e_1\rangle_\eta - \langle \Gamma(\eta)(e, e_1), e_3\rangle_\eta . \tag{3.6}$$

Indeed,

$$D_\eta \langle e, e_1\rangle \cdot e_3 = \langle \Gamma(\eta)(e_3, e_1), e\rangle_\eta + \langle \Gamma(\eta)(e_3, e), e_1\rangle_\eta .$$

Substituting this expression into the formula for Ω and using the symmetry of Γ we obtain the desired result.

Let $\mathbf{X}_H = (\eta, v, e_1, e_2)$ be a Hamiltonian vector field, $Z = (\eta, v, u, w) \in T_{(\eta,v)}TQ$ be arbitrary. Then

$$\Omega(\mathbf{X}_H, Z) = \langle w + \Gamma(\eta)(v, u), e_1\rangle - \langle e_2 + \Gamma(\eta)(v, e_1), u\rangle .$$

On the other hand, by Lemma 3.2

$$\Omega(\mathbf{X}_H, Z) = dH \cdot Z = \frac{\partial H}{\partial \eta} \cdot T\tau Z + \frac{\partial H}{\partial v} KZ = \frac{\partial H}{\partial \eta} \cdot (\eta, u) + \frac{\partial H}{\partial v} \cdot (\eta, w + \Gamma(u, v)).$$

Setting $u = 0$ and comparing the above expressions we see that $\frac{\partial H}{\partial v}(\eta, v) \cdot (\eta, w) = \langle e_1, w\rangle \ \forall w \in \mathbf{E}$. Similarly, setting $w = 0$ yields

$$\frac{\partial H}{\partial \eta}(\eta, v) \cdot (\eta, u) = -\langle e_2 + \Gamma(\eta)(v, e_1), u\rangle \ \forall u \in \mathbf{E}.$$

Thus, $H \in \mathcal{K}^k$.

Conversely, let $H \in \mathcal{K}^k$. Defining a vector field $\mathbf{X}_H$ by Formula 3.5 and substituting into Formula 3.6 one obtains for arbitrary vector $Z \in T_{(\eta,v)}TQ$

$$\Omega(\mathbf{X}_H, Z) = \left\langle \frac{\partial H}{\partial v}, KZ \right\rangle + \left\langle \frac{\partial H}{\partial \eta}, T\tau Z \right\rangle = dH \cdot Z. \quad \square$$

Proposition 3.8. *Let $f, g \in \mathcal{K}^k$ be arbitrary. Then*

$$\{f, g\} = \Omega(\mathbf{X}_f, \mathbf{X}_g).$$

Proof. By Proposition 3.7, the vector fields $\mathbf{X}_f, \mathbf{X}_g$ are defined whenever $\{f, g\}$ is. Then

$$\Omega(\mathbf{X}_f, \mathbf{X}_g) = df \cdot \mathbf{X}_g = \frac{\partial f}{\partial \eta} \cdot T\tau \mathbf{X}_g + \frac{\partial f}{\partial v} K\mathbf{X}_g = \frac{\partial f}{\partial \eta} \cdot \frac{\partial g}{\partial v} - \frac{\partial g}{\partial \eta} \cdot \frac{\partial f}{\partial v} = \{f, g\}. \quad \square$$

Theorem 3.9. *The bracket $\{\cdot, \cdot\}$ is antisymmetric, bilinear, derivation on each factor and makes $\mathcal{K}$ into a Lie-algebra.*

Proof. Antisymmetry, linearity and derivation property follows directly from the definition of the bracket. By Theorem 3.4 $\{\cdot, \cdot\}$ leaves $\mathcal{K}$ invariant. Then, Jacobi identity follows from Proposition 3.8 in the usual way, for example as in Marsden and Ratiu [1999]. $\square$

Now, TQ has both symplectic and Poisson structures, and therefore two generally different definitions of Hamiltonian vector fields. We need to check that in our case these coincide. To do so, let $\mathbf{X}_f^P$ temporarily denote the Hamiltonian vector field with respect to Poisson structure $\{\cdot, \cdot\}$ and $\mathbf{X}_f$ denotes the Hamiltonian vector field with respect to canonical symplectic form corresponding to function the f. Recall, that $\mathbf{X}_f^P$ is defined as a vector field such that

$$\mathbf{X}_f^P[h] = \{h, f\} \quad \forall h \in \mathcal{K}.$$

Thus, for all $h \in \mathcal{K}$,

$$\begin{aligned}\mathbf{X}_f^P[h] &= \frac{\partial h}{\partial \eta} \cdot \frac{\partial f}{\partial v} - \frac{\partial h}{\partial v} \cdot \frac{\partial f}{\partial \eta} \\ &= dh \cdot \mathbf{X}_f^P = \frac{\partial h}{\partial \eta} \cdot T\tau \mathbf{X}_f^P + \frac{\partial h}{\partial v} \cdot K\mathbf{X}_f^P\end{aligned}$$

and therefore, $T\tau\mathbf{X}_f^P = \frac{\partial f}{\partial v}$ and $K\mathbf{X}_f^P = -\frac{\partial f}{\partial \eta}$. Comparing this with Formula 3.5, we see that $\mathbf{X}_f \equiv \mathbf{X}_f^P$. Finally, from the coordinate expression, it is easy to see that $\mathbf{X}_f$ is a well-defined C^k vector field for any $f \in \mathcal{K}^k$.

Previously we established that classes $\mathcal{K}^k$ are preserved under bracketing. Unfortunately, for $f \in \mathcal{K}^k$ and a diffeomorphism $\psi : TQ \to TQ$ the composition $f \circ \psi$ does not have to be in any class $\mathcal{K}^m$. One can, however, compose with *symplectic diffeomorphisms.*

Proposition 3.10. *Let ψ be a symplectic C^{k+1} diffeomorphism, $f \in \mathcal{K}^k$. Then $f \circ \psi \in \mathcal{K}^k$.*

Proof. We have

$$\mathbf{X}_{f\circ\psi} = \psi^*(\mathbf{X}_f),$$

and so by Proposition 3.7, $f \circ \psi \in \mathcal{K}^k$. □

Proposition 3.11. *Let F_t be a flow of a smooth Hamiltonian vector field on TQ. Then F_t is a Poisson, i.e., for all $f, g \in \mathcal{K}$*

$$\{f \circ F_t, g \circ F_t\} = \{f, g\} \circ F_t.$$

Proof. F_t is symplectic with respect to the weak Riemannian form. Since F_t preserves class $\mathcal{K}$, the statement follows from Jacobi identity by the usual argument. □

3.4 Geometric properties of the flow of the Euler equations

As we stated earlier, in Ebin and Marsden [1970] it is shown that $\mathcal{D}^s_\mu$ carries a smooth Riemannian connection, and therefore the results of the previous section apply. Therefore, by those results, the space $T\mathcal{D}^s_\mu$ carries a Poisson structure (in the precise sense given there) which we denote $\{\cdot,\cdot\}$. Let K, $\widehat{K}$, $\widetilde{K}$ stand for the corresponding connector maps on the underlying manifold M, on $\mathcal{D}^s$ and $\mathcal{D}^s_\mu$, respectively, while ∇, $\widehat{\nabla}$, $\widetilde{\nabla}$ are the corresponding connections and Γ, $\widehat{\Gamma}$, $\widetilde{\Gamma}$ are the corresponding Christoffel maps. In the following $\langle\cdot,\cdot\rangle$ denotes the Riemannian metric on M, $\mathcal{D}^s$, $\mathcal{D}^s_\mu$ and an induced scalar product on $\mathfrak{X}^s_{\mathrm{div}} = T_e\mathcal{D}^s_\mu$ depending on the context. The relationship between these metrics is given by 3.2.

Recall the notation from Section 3.2. Namely, let F_t be the flow of the spray on $T\mathcal{D}^s_\mu$, $\tilde{F}_t$ denote the flow of Euler equation on $\mathfrak{X}^s_{\mathrm{div}}$ and $\pi : T\mathcal{D}^s_\mu \to \mathfrak{X}^s_{\mathrm{div}}$, $\pi(\eta, v) = v \circ \eta^{-1}$. Recall also that we have the commutative diagram.

Proposition 3.12. *The following diagram is commutative:*

$$\begin{array}{ccc} T\mathcal{D}^s_\mu & \xrightarrow{F_t} & T\mathcal{D}^s_\mu \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ \mathfrak{X}^s_{\mathrm{div}} & \xrightarrow{\tilde{F}_t} & \mathfrak{X}^s_{\mathrm{div}} \end{array}$$

Now we prepare and recall from Ebin and Marsden [1970] some useful lemmata.

Lemma 3.13. *Let $\xi \in \mathcal{D}^s_\mu$. Define $R_\xi : \mathcal{D}^s_\mu \to \mathcal{D}^s_\mu$ via $R_\xi(\eta) = \eta \circ \xi$, $\forall \xi$. Then*

$$TR_\xi \circ F_t(v) = F_t \circ TR_\xi(v) \quad \forall v \in T\mathcal{D}^s_\mu.$$

Proof. Indeed, notice that

$$\begin{aligned}\frac{d}{dt}(\eta_t \circ \xi, \dot{\eta}_t \circ \xi) &= (\eta_t \circ \xi, \dot{\eta}_t \circ \xi, \dot{\eta}_t \circ \xi, \ddot{\eta}_t \circ \xi) \\ &= TTR_\xi(\eta_t, \dot{\eta}_t, \dot{\eta}_t, \ddot{\eta}_t) = TTR_\xi S(F_t(v)) \\ &= S(TR_\xi F_t(v)) = S(\eta_t \circ \xi, \dot{\eta}_t \circ \xi)\end{aligned}$$

by right invariance of the spray. Thus, $TR_\xi F_t(v) = (\eta_t \circ \xi, \dot{\eta}_t \circ \xi)$ is an integral curve of S. Since $TR_\xi F_0(v) = TR_\xi(v)$, the statement of the Lemma follows from uniqueness of integral curves. $\square$

Recall that by definition, $\tilde{F}_t(V) = \pi \circ F_t(V)$ for all $V \in T_e\mathcal{D}^s_\mu = \mathfrak{X}^s_{\mathrm{div}}$. Let $V = (\eta, v) \in T\mathcal{D}^s_\mu$. Then, using the preceding Lemma, we obtain

$$\begin{aligned}\tilde{F}_t \circ \pi(V) &= \pi \circ F_t(\pi(V)) \\ &= \pi \circ F_t \circ TR_{\eta^{-1}}(V) = \pi \circ TR_{\eta^{-1}} \circ F_t(V).\end{aligned}$$

Notice, that $\pi \circ TR_\xi = \pi$ for any $\xi \in \mathcal{D}^s_\mu$. Indeed,

$$\begin{aligned}\pi \circ TR_\xi(\eta, v) &= \pi(\eta \circ \xi, v \circ \xi) = (e, v \circ \xi \circ (\eta \circ \xi)^{-1}) \\ &= (e, v \circ \xi \circ \xi^{-1} \circ \eta^{-1}) = (e, v \circ \eta^{-1}) = \pi(\eta, v).\end{aligned}$$

Thus $\pi \circ TR_{\eta^{-1}} = \pi$ and the Proposition is proved. $\square$

A Poisson Structure on the Lie Algebra. Now, we construct a Poisson bracket $\{\cdot,\cdot\}_+$ on $\mathfrak{X}^s_{\mathrm{div}}$ so that π is a Poisson map. For $f, g : \mathfrak{X}^s_{\mathrm{div}} \to \mathbb{R}$ such that $df, dg : \mathfrak{X}^s_{\mathrm{div}} \to \mathfrak{X}^r_{\mathrm{div}}$ define

$$\{f, g\}_+(v) = \left\langle dg(v), \nabla_{df(v)} v \right\rangle - \left\langle df(v), \nabla_{dg(v)} v \right\rangle .$$

As in Section 3.3, define

$$\mathcal{K}^{k,s} = \{f \in C^{k+1}(\mathfrak{X}^s_{\mathrm{div}}, \mathbb{R}) \mid df \in C^k(\mathfrak{X}^s_{\mathrm{div}}, \mathfrak{X}^s_{\mathrm{div}})\}$$

and

$$\mathcal{K}^{k,s}_{r,t} = \{f \in C^{k+1}(\mathfrak{X}^s_{\mathrm{div}}, \mathbb{R}) \mid df \in C^k(\mathfrak{X}^r_{\mathrm{div}}, \mathfrak{X}^t_{\mathrm{div}})\}.$$

Theorem 3.14. *Let $s > n/2+1$. Then $\{\cdot,\cdot\}_+$ is a bilinear map $\mathcal{K}^{k,s}\times\mathcal{K}^{k,s} \to \mathcal{K}^{k-1,s}_{s+1,s-1}$ and a derivation on each factor. Moreover, it satisfies Jacobi identity on $\mathfrak{X}^{s+1}_{\mathrm{div}}$, that is for all $f, g, h \in \mathcal{K}^{k,s}$, and $v \in \mathfrak{X}^{s+1}_{\mathrm{div}}$,*

$$O(v) := \left\{f, \{g,h\}_+\right\}_+(v) + \left\{h, \{f,g\}_+\right\}_+(v) + \left\{g, \{h,f\}_+\right\}_+(v) = 0.$$

Proof. Let $f, g \in \mathcal{K}^{k,s}$. Recall, that for $r > n/2$, $H^r(M,\mathbb{R})$ is an algebra. Thus, $(u,v) \to \nabla_u v$ is a bilinear bounded map $\mathfrak{X}^s_{\mathrm{div}} \times \mathfrak{X}^s_{\mathrm{div}} \to \mathfrak{X}^{s-1}$ (and $\mathfrak{X}^s_{\mathrm{div}} \times \mathfrak{X}^{s+1}_{\mathrm{div}} \to \mathfrak{X}^s$), hence smooth. This implies that

$$z = \{f,g\}_+ \in C^k(\mathfrak{X}^s_{\mathrm{div}}, \mathbb{R}).$$

Bilinearity and derivation property of $\{\cdot,\cdot\}_+$ trivially follows from properties of d, ∇ and $\langle\cdot,\cdot\rangle$.

Now we calculate dz. Let $v, u \in \mathfrak{X}^{s+1}_{\mathrm{div}}$. Since $z \in C^k(\mathfrak{X}^s_{\mathrm{div}}, \mathbb{R})$, the Fréchet derivative of z exists and coincides with its Gateaux derivative. Thus, by bilinearity of scalar product and ∇,

$$\begin{aligned} dz(v)\cdot u &= \left.\frac{d}{dt}\right|_{t=0} z(v+tu) \\ &= \left\langle Ddg(v)\cdot u, \nabla_{df(v)}v\right\rangle + \left\langle dg(v), \nabla_{Ddf(v)\cdot u}v\right\rangle \\ &\quad + \left\langle dg(v), \nabla_{df(v)}u\right\rangle - \left\langle Ddf(v)\cdot u, \nabla_{dg(v)}v\right\rangle \\ &\quad - \left\langle df(v), \nabla_{Ddg(v)\cdot u}v\right\rangle - \left\langle df(v), \nabla_{dg(v)}u\right\rangle. \end{aligned}$$

Lemma 3.15. *Let $X \in \mathfrak{X}^s_{\mathrm{div}}$, $s > n/2+1$, and let Y, W be H^s vector fields on M.*

Then

$$\langle Y, \nabla_X W \rangle = - \langle \nabla_X Y, W \rangle .$$

Proof. By the Sobolev theorems, X is a C^1 vector field on M. By properties of the Riemannian connection, for all $m \in M$

$$\langle Y, \nabla_X W \rangle_m = - \langle \nabla_X Y, W \rangle_m + X \langle Y, W \rangle_m .$$

Thus,

$$\langle Y, \nabla_X W \rangle = - \langle \nabla_X Y, W \rangle + \int_M X \langle Y, W \rangle_m \mu.$$

Let G_t be a flow of X on M. Since X is divergence free, μ is G_t invariant, i.e., $G_t^*(\mu) = \mu$, where G_t^* denotes a pullback by G_t. Then

$$\begin{aligned}
\int_M X \langle Y, W \rangle_m &= \int_M \frac{d}{dt}_{t=0} \langle Y, W \rangle_{G_t(m)} \mu \\
&= \frac{d}{dt}_{t=0} \int_M \langle Y, W \rangle_{G_t(m)} G_t^*(\mu) \\
&= \frac{d}{dt}_{t=0} \int_M G_t^*(\langle Y, W \rangle_m \mu) \\
&= \frac{d}{dt}_{t=0} \int_M \langle Y, W \rangle_m \mu = 0. \quad \square
\end{aligned}$$

Lemma 3.16. *Let $df \in C^k(\mathfrak{X}^s_{\mathrm{div}}, \mathfrak{X}^t_{\mathrm{div}})$, $s, t \geq 0$. Then for all $u, v, w \in \mathfrak{X}^s_{\mathrm{div}}$*

$$\langle Ddf(v) \cdot u, w \rangle = \langle Ddf(v) \cdot w, u \rangle .$$

Proof. We compute as follows:

$$\begin{aligned}
\langle Ddf(v) \cdot u, w \rangle &= \frac{d}{dt}_{t=0} \langle df(v + tu), w \rangle \\
&= \frac{d}{dt}_{t=0} \frac{d}{ds}_{s=0} f(v + tu + sw) \\
&= \frac{d}{ds}_{s=0} \frac{d}{dt}_{t=0} f(v + tu + sw) \\
&= \langle Ddf(v) \cdot w, u \rangle . \quad \square
\end{aligned}$$

Lemma 3.17. (The Hodge Decomposition; see Ebin and Marsden [1970]). *Let X be an H^s vector field on M, $s \geq 0$. There is an H^{s+1} function θ and an H^s vector field Y with Y divergence free, such that*

$$X = \operatorname{grad}\theta + Y.$$

Further, the projection maps

$$P_e(X) = Y$$

$$Q(X) = \operatorname{grad}\theta$$

are continuous linear maps on $H^s(M, TM)$. The decomposition is orthogonal in L^2 sense, that is for all $Z \in \mathfrak{X}^s_{\mathrm{div}}$

$$\langle Z, X\rangle = \langle Z, Y\rangle = \langle Z, P_e X\rangle \tag{3.7}$$

Lemma 3.18. *There is a bilinear continuous map $B : \mathfrak{X}^s_{\mathrm{div}} \times \mathfrak{X}^{s+1}_{\mathrm{div}} \to \mathfrak{X}^s_{\mathrm{div}}$ $(s > n/2)$ such that for all $Z \in \mathfrak{X}^s_{\mathrm{div}}, W \in \mathfrak{X}^{s+1}, Y \in C(M, TM)$*

$$\langle Z, \nabla_Y W\rangle = \langle B(Z, W), Y\rangle$$

Proof. Fix coordinate system $\{x_i\}$ on M and let g_{ij} denote components of metric tensor, Z^i denote components of vector field Z in the chosen system. Let $g_{ij}g^{jk} = \delta_i^k$ (as usually, the summation on repeated indexes is understood). Then

$$\begin{aligned}
\langle Z, \nabla_Y W\rangle &= \int_M g_{ij} Z^i \left(\frac{\partial W^j}{\partial x_k} Y^k + \Gamma^j_{kr} Y^k W^r\right)\mu \\
&= \int_M g_{sm} g^{mk} g_{ij} Z^i \left(\frac{\partial W^j}{\partial x_k} + \Gamma^j_{kr} W^r\right) Y^s \mu \\
&= \langle V, Y\rangle,
\end{aligned}$$

where

$$V^m = g^{mk} g_{ij} Z^i \left(\frac{\partial W^j}{\partial x_k} + \Gamma^j_{kr} W^r\right).$$

Since H^s is an algebra for $s > n/2$ it follows that V is an H^s vector field. Now we set

$$B(Z, W) = P_e V$$

and use 3.7. □

By Lemmata 3.15-3.18, we have

$$\begin{aligned} dz(v) \cdot u = & \left\langle Ddg(v) \cdot P_e \nabla_{df(v)} v, u \right\rangle + \langle Ddf(v) \cdot B(dg(v), v), u \rangle - \left\langle \nabla_{df(v)} dg(v), u \right\rangle \\ & - \left\langle Ddf(v) \cdot P_e \nabla_{dg(v)} v, u \right\rangle - \langle Ddg(v) \cdot B(df(v), v), u \rangle + \left\langle \nabla_{dg(v)} df(v), u \right\rangle . \end{aligned}$$

Thus for any $v \in \mathfrak{X}_{\mathrm{div}}^{s+1}$,

$$\begin{aligned} d\{f, g\}_+ (v) = & P_e[\nabla_{dg(v)} df(v) - \nabla_{df(v)} dg(v)] \\ & + Ddf(v) \cdot B(dg(v), v) - Ddg(v) \cdot B(df(v), v) \\ & + Ddg(v) \cdot P_e \nabla_{df(v)} v - Ddf(v) \cdot P_e \nabla_{dg(v)} v, \end{aligned}$$

and hence $d\{f, g\}_+ \in C^{k-1}(\mathfrak{X}_{\mathrm{div}}^{s+1}, \mathfrak{X}_{\mathrm{div}}^{s-1})$ and $\{f, g\}_+ \in \mathcal{K}_{s+1,s-1}^{k-1,s}$.

Now we prove the Jacobi identity. To simplify notation, we set

$$B_f(v) = B(df(v), v), \quad \nabla_f(v) = P_e \nabla_{df(v)} v.$$

Moreover, since in the following argument all functions are evaluated at the same point $v \in \mathfrak{X}_{\mathrm{div}}^{s+1}$, we will write B_f, ∇_f, df instead of $B_f(v)$, etc. By Lemmata 3.16-3.18, we obtain

$$
\begin{aligned}
O_{fgh}(v) &= \left\{f, \{g,h\}_+\right\}_+(v) \\
&= \left\langle d\{g,h\}_+, \nabla_f\right\rangle - \left\langle B_f, d\{g,h\}_+\right\rangle \\
&= \left\langle d\{g,h\}_+, \nabla_f - B_f\right\rangle \\
&= \langle P_e[\nabla_{dh}dg - \nabla_{dg}dh], \nabla_f - B_f\rangle + \langle Ddg \cdot (B_h - \nabla_h), \nabla_f - B_f\rangle \\
&\quad + \langle Ddh \cdot (\nabla_g - B_g), \nabla_f - B_f\rangle \\
&= \langle [dh, dg], \nabla_f - B_f\rangle + D_{ghf} - D_{hfg},
\end{aligned}
$$

where $D_{ghf} = \langle Ddg \cdot (B_h - \nabla_h), \nabla_f - B_f\rangle$ and $[\cdot,\cdot]$ is a Lie bracket of vector fields on M. Notice that Lie bracket of divergence free vector fields is divergence free.

For $s > n/2+2$

$$
\langle [dh, dg], \nabla_f - B_f\rangle = \langle [[dh(v), dg(v)], df(v)], v\rangle .
$$

Since terms of type D_{fgh} cancel out in the Jacobi cycle

$$
O(v) = O_{fgh}(v) + O_{hfg}(v) + O_{ghf}(v),
$$

and so the Jacobi identity for bracket $\{\cdot,\cdot\}_+$ follows from the Jacobi identity for vector fields. However, for $n/2+1 < s \le n/2+2$ Lie bracket of $dh(v)$ and $dg(v)$ is an $\mathfrak{X}^{s-1}_{\mathrm{div}}$ vector field, hence merely continuous and therefore $[[dh(v), dg(v)], df(v)]$ may fail to exist. Therefore, in this case more care is needed.

Let

$$
A_{fgh} = \langle df, \nabla_{dg}\nabla_{dh}v\rangle ,
$$

$$
C_{fgh} = \left\langle df, \nabla_{[dg,df]}v\right\rangle .
$$

With this notation in mind, by Lemma 3.15 and the Hodge decomposition

$$
\langle [dh, dg], \nabla_f\rangle = \langle \nabla_{dh}dg - \nabla_{dg}dh, \nabla_{df}v\rangle = -A_{ghf} + A_{hgf}.
$$

Similarly, by definition of B

$$\langle [dh, dg], B_f \rangle = C_{fhg}.$$

By a well-known formula for Riemannian connection,

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z,$$

for all sufficiently smooth vector fields X, Y, Z. Thus,

$$A_{fgh} - A_{fhg} = \langle df, \nabla_{dg} \nabla_{dh} v - \nabla_{dh} \nabla_{df} v \rangle = \left\langle df, \nabla_{[dg,dh]} v \right\rangle = C_{fgh}.$$

Thus,

$$\left\{f, \{g, h\}_+\right\}_+ = -A_{ghf} + A_{hgf} - C_{fhg} + D_{ghf} - D_{hfg},$$

and so

$$\begin{aligned}
&\left\{f, \{g, h\}_+\right\}_+ + \left\{h, \{f, g\}_+\right\}_+ + \left\{g, \{h, f\}_+\right\}_+ \\
&= -A_{ghf} + A_{hgf} - C_{fhg} + D_{ghf} - D_{hfg} \\
&\qquad - A_{fgh} + A_{gfh} - C_{hgf} + D_{fgh} - D_{ghf} \\
&\qquad - A_{hfg} + A_{fhg} - C_{gfh} + D_{hfg} - D_{fgh} \\
&= (A_{gfh} - A_{ghf} - C_{gfh}) + (A_{fhg} - A_{fgh} - C_{fhg}) \\
&\qquad + (A_{hgf} - A_{hfg} - C_{hgf}) = 0. \quad \square
\end{aligned}$$

Remark 3.19. *If $df(v), dh(v) \in \mathfrak{X}^s_{\mathrm{div}}$, $s > n/2 + 1$, then by Lemma 3.15*

$$\{f, h\}_+ (v) = \langle [dh(v), df(v)], v \rangle .$$

This shows that bracket $\{\cdot, \cdot\}_+$ is naturally related to Lie-Poisson bracket on $(\mathfrak{X}^s_{\mathrm{div}})^*$.

Now we establish the relationship between Poisson bracket $\{\cdot, \cdot\}_+$ on $\mathfrak{X}^s_{\mathrm{div}}$ that we just introduced and Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{D}^s_\mu$. For $f, h : \mathfrak{X}^s_{\mathrm{div}} \to \mathbb{R}$ define

$$f_R = f \circ \pi.$$

Theorem 3.20. *Define the function spaces*

$$C^k_r(\mathfrak{X}^s_{\mathrm{div}}) = \left\{ f \in C^k(\mathfrak{X}^s_{\mathrm{div}}, \mathbb{R}) \middle| \, df(v) \in \mathfrak{X}^r_{\mathrm{div}} \forall v \in \mathfrak{X}^s_{\mathrm{div}} \right\}, \tag{3.8}$$

and

$$C^k_r(T\mathcal{D}^s_\mu) = \left\{ f \in C^k(T\mathcal{D}^s_\mu, \mathbb{R}) \middle| \frac{\partial f}{\partial \eta}(v), \frac{\partial f}{\partial v}(v) \in T\mathcal{D}^r_\mu \forall v \in T\mathcal{D}^s_\mu \right\}. \tag{3.9}$$

Then $f_R \in C^k_r(T\mathcal{D}^{s+k}_\mu)$ *for* $f \in C^k_r(\mathfrak{X}^s_{\mathrm{div}})$ $(r, s > n/2 + 1, k \geq 1)$ *and for all* $f, h \in C^1_r(\mathfrak{X}^s_{\div}), v \in \mathfrak{X}^{s+1}_{\mathrm{div}}$

$$\{f, h\}_+(v) = \{f_R, h_R\}(v) = \{f \circ \pi, h \circ \pi\}(v).$$

Proof. Without loss of generality $s \geq r$. Since π is not even a C^1 function $T\mathcal{D}^s_\mu \to \mathfrak{X}^s_{\mathrm{div}}$ it is not obvious that $\{f_R, h_R\}$ is defined. However, differentiating f_R and h_R as functions $T\mathcal{D}^{s+k}_\mu \to T\mathcal{D}^s_\mu$ one obtains the required result.

Lemma 3.21. *Under the assumptions of the Theorem,*

$$\frac{\partial f_R}{\partial v}(\eta, v) = TR_\eta df(\pi(\eta, v)).$$

Proof. It is well-known (Ebin and Marsden [1970]) that $\pi \in C^k(T\mathcal{D}^{s+k}_\mu, T\mathcal{D}^s_\mu)$. Notice, that for $(\eta, u) \in T\mathcal{D}^{s+k}_\mu$,

$$\frac{\partial \pi}{\partial v}(\eta, v) \cdot (\eta, u) = \frac{d}{dt}_{t=0} \pi(\eta, v + tu) = (e, v \circ \eta^{-1}, 0, u \circ \eta^{-1})$$

where time derivative is taken in $T\mathcal{D}^s_\mu$. By Lemma 3.2

$$\frac{\partial f_R}{\partial v} = df \cdot \widetilde{K} \frac{\partial \pi}{\partial v}.$$

Thus, by right invariance of the metric on $\mathcal{D}^s_\mu$

$$\begin{aligned}\frac{\partial f_R}{\partial v}(\eta, v)\cdot(\eta, u) &= df(v\circ\eta^{-1})\cdot\widetilde{K}(e, v\circ\eta^{-1}, 0, u\circ\eta^{-1})\\ &= df(v\circ\eta^{-1})\cdot(u\circ\eta^{-1})\\ &= \left\langle df(\pi(\eta, v), u\circ\eta^{-1}\right\rangle_e\\ &= \langle TR_\eta df(\pi(\eta, v)), (\eta, u)\rangle_\eta . \quad \square\end{aligned}$$

Lemma 3.22. *Under the assumptions of the Theorem*

$$\frac{\partial f_R}{\partial \eta}(\eta, v)\cdot(\eta, u) = -\left\langle df(v\circ\eta^{-1}), \widetilde{K}[T(v\circ\eta^{-1})\circ(u\circ\eta^{-1})]\right\rangle_e ,$$

that is,

$$\frac{\partial f_R}{\partial \eta}(\eta, v) = -TR_\eta B_f(\pi(\eta, v)).$$

Proof. First, we calculate $\frac{\partial\pi}{\partial\eta}$. Let $(\eta, u)\in T\mathcal{D}^{s+k}_\mu$, (η_t, v_t) be a parallel translation of (η, v) with $\frac{d}{dt}_{t=0}\eta_t = u$. Recall that

$$\frac{d}{dt}{\eta_t}^{-1} = -T{\eta_t}^{-1}\circ\frac{d}{dt}\eta_t\circ{\eta_t}^{-1}.$$

Then, by Lemma 3.3,

$$\begin{aligned}\frac{\partial\pi}{\partial\eta}(\eta, v)\cdot(\eta, u) &= \frac{d}{dt}_{t=0}\pi(\eta_t, v_t) = \frac{d}{dt}_{t=0} v_t\circ{\eta_t}^{-1}\\ &= Tv_0\circ\frac{d}{dt}_{t=0}{\eta_t}^{-1} + \left(\frac{d}{dt}_{t=0}v_t\right)\circ\eta_0^{-1}\\ &= -Tv\circ T\eta^{-1}\circ u\circ\eta^{-1} + \left(\frac{d}{dt}_{t=0}v_t\right)\circ\eta^{-1}.\end{aligned}$$

Since connection on $\mathcal{D}^s_\mu$ is right-invariant, i.e.,

$$\widetilde{K}\circ TTR_\xi = TR_\xi\circ\widetilde{K} \quad \forall\xi\in\mathcal{D}^s_\mu$$

we have

$$\widetilde{K}\left[\frac{d}{dt}_{t=0} v_t \circ \eta^{-1}\right] = \left[\widetilde{K}\frac{d}{dt}_{t=0} v_t\right] \circ \eta^{-1} = 0.$$

By Lemma 3.2

$$\frac{\partial f_R}{\partial \eta} = df \cdot \widetilde{K}\frac{\partial \pi}{\partial \eta}.$$

Combining above equalities together, we get

$$\frac{\partial f_R}{\partial \eta}(\eta, v) \cdot (\eta, u) = -df \cdot \widetilde{K}\left[T(v \circ \eta^{-1}) \circ (u \circ \eta^{-1})\right].$$

$$= -\left\langle df(v \circ \eta^{-1}), \widetilde{K}[T(v \circ \eta^{-1}) \circ (u \circ \eta^{-1})]\right\rangle_e .$$

We claim that for all $X, Y, Z \in \mathfrak{X}^s_{\mathrm{div}}$

$$\left\langle Z, \widetilde{K}[TX \circ Y]\right\rangle = \langle Z, \nabla_Y X\rangle . \tag{3.10}$$

Recall that by construction (see Ebin and Marsden [1970]),

$$\widetilde{K} = P \circ \widehat{K},$$

$$P = TR_\eta \circ P_e \circ TR_\eta^{-1},$$

$$\widehat{K}(Y) = K \circ Y,$$

By a well-known formula of differential geometry, we have

$$K \circ TX \circ Y = \nabla_Y X,$$

and hence

$$\widetilde{K}[TX \circ Y] = P_e[\nabla_Y X].$$

By the Hodge decomposition

$$\left\langle Z, \widetilde{K}[TX \circ Y]\right\rangle = \langle Z, \nabla_Y X\rangle = \langle B(Z, X), Y\rangle .$$

By the above developments and right invariance of metric on $\mathcal{D}^s_\mu$, we have

$$\frac{\partial f_R}{\partial \eta}(\eta, v)\cdot(\eta, u) = -\left\langle B_f(v\circ\eta^{-1}), u\circ\eta^{-1}\right\rangle = -\left\langle TR_\eta B_f(\pi(\eta, v)), u\right\rangle_\eta . \quad \square$$

Calculating $\{f_R, h_R\}$ at $v \in \mathfrak{X}^{s+1}_{\mathrm{div}}$ by Lemmata 3.21,3.22, we obtain

$$\begin{aligned}\{f_R, h_R\}(v) &= -\left\langle B_f(v), dh(v)\right\rangle + \left\langle B_h(v), df(v)\right\rangle \\ &= -\left\langle df(v), \nabla_{dh(v)}v\right\rangle + \left\langle dh(v), \nabla_{df(v)}v\right\rangle \\ &= \{f, h\}_+(v). \quad \square\end{aligned}$$

Proposition 3.23. *Map $\pi : T\mathcal{D}^s_\mu \to \mathfrak{X}^s_{\mathrm{div}}$ is a Poisson map, i.e., for all $f, h \in C^1_r(\mathfrak{X}^s_{\mathrm{div}})$ pointwise in $T\mathcal{D}^{s+1}_\mu$ $(r, s > n/2+1)$*

$$\{f\circ\pi, h\circ\pi\} = \{f, h\}_+\circ\pi.$$

Proof. Since π is the identity on $\mathfrak{X}^s_{\mathrm{div}}$, the statement follows immediately from Theorem 3.20. $\square$

Proposition 3.24. *Let $v \in T\mathcal{D}^r_\mu$ and $f, g \in C^1(T\mathcal{D}^r_\mu, \mathbb{R})$ are such that $\frac{\partial f}{\partial v}(F_t(v))$, $\frac{\partial f}{\partial \eta}(F_t(v))$, $\frac{\partial g}{\partial v}(F_t(v))$, $\frac{\partial g}{\partial \eta}(F_t(v)) \in T\mathcal{D}^s_\mu$, $r, s > n/2+1$. Then*

$$\{f\circ F_t, g\circ F_t\}(v) = \{f, g\}(F_t(v)).$$

In particular, F_t preserves $C^1_s(T\mathcal{D}^r_\mu)$ and for $f, h \in \mathcal{K}^{1,s}$ pointwise in $T\mathcal{D}^{s+1}_\mu$

$$\{f\circ\pi\circ F_t, h\circ\pi\circ F_t\}(v) = \{f\circ\pi, h\circ\pi\}(F_t(v)).$$

Proof. Without loss of generality $r \geq s$. First, we notice that covariant partial derivatives of $f\circ F_t, g\circ F_t$ at v are elements of $T\mathcal{D}^s_\mu$. Indeed,

$$\frac{\partial}{\partial \eta}(g\circ F_t)(v)\cdot u = \left\langle \frac{\partial g}{\partial \eta}(F_t(v)), T\widetilde{\tau}\frac{\partial F_t}{\partial \eta}(v)\cdot u\right\rangle + \left\langle \frac{\partial g}{\partial v}(F_t(v)), \widetilde{K}\frac{\partial F_t}{\partial \eta}(v)\cdot u\right\rangle .$$

There is a function $\tilde{g} \in \mathcal{K}(T\mathcal{D}^s_\mu)$ such that

$$\frac{\partial g}{\partial v}(F_t(v)) = \frac{\partial \tilde{g}}{\partial v}(F_t(v)), \quad \frac{\partial g}{\partial \eta}(F_t(v)) = \frac{\partial \tilde{g}}{\partial \eta}(F_t(v)).$$

Thus,

$$\frac{\partial}{\partial \eta}(g \circ F_t)(v) \cdot u = \frac{\partial}{\partial \eta}(\tilde{g} \circ F_t)(v) \cdot u.$$

However, by Proposition 3.10 $\tilde{g} \circ F_t \in \mathcal{K}(T\mathcal{D}^s_\mu)$ for any $\tilde{g} \in \mathcal{K}(T\mathcal{D}^s_\mu)$, hence there is $Z_g \in C^\infty(T\mathcal{D}^s_\mu, T\mathcal{D}^s_\mu)$ such that for all u,

$$\frac{\partial}{\partial \eta}(g \circ F_t)(v) \cdot u = \frac{\partial}{\partial \eta}(\tilde{g} \circ F_t)(v) \cdot u = \langle Z_g(v), u \rangle .$$

In a similar sense, one shows that $\frac{\partial}{\partial v}(f \circ F_t)(v) \in T\mathcal{D}^s_\mu$.

Thus, $\{f \circ F_t, g \circ F_t\}(v)$ is well-defined and depends only on values of $\frac{\partial f}{\partial v}, \frac{\partial f}{\partial \eta}, \frac{\partial g}{\partial \eta}, \frac{\partial g}{\partial v}$ calculated at point $F_t(v)$. However, $\{f, g\} \circ F_t(v)$ also depends only on values of covariant partial derivatives at $F_t(v)$. Then, we choose $\tilde{f}, \tilde{g} \in \mathcal{K}(T\mathcal{D}^s_\mu)$ such that

$$\begin{aligned}
\frac{\partial f}{\partial v}(F_t(v)) &= \frac{\partial \tilde{f}}{\partial v}(F_t(v)), \\
\frac{\partial f}{\partial \eta}(F_t(v)) &= \frac{\partial \tilde{f}}{\partial \eta}(F_t(v)), \\
\frac{\partial g}{\partial v}(F_t(v)) &= \frac{\partial \tilde{g}}{\partial v}(F_t(v)), \\
\frac{\partial g}{\partial \eta}(F_t(v)) &= \frac{\partial \tilde{g}}{\partial \eta}(F_t(v)).
\end{aligned}$$

The equality

$$\left\{\tilde{f} \circ F_t, \tilde{g} \circ F_t\right\}(v) = \left\{\tilde{f}, \tilde{g}\right\} \circ F_t(v)$$

follows from Proposition 3.11. By the preceding arguments, the same holds if we replace $\tilde{f}, \tilde{g}$ with f, g. This concludes the first part of the Proposition. The second part then follows. $\square$

Theorem 3.25. *The map $\tilde{F}_t$ is Poisson with respect to the bracket $\{\cdot, \cdot\}_+$.*

Proof. Let $f, h \in \mathcal{K}^{k,s}$. Then $f \circ \pi \in C^1_s(\mathfrak{X}^{s+1}_{\mathrm{div}})$. By Proposition 3.24.

$$f \circ \tilde{F}_t = f \circ \pi \circ F_t \in C^1_s(\mathfrak{X}^{s+1}_{\mathrm{div}})$$

and we have pointwise in $\mathfrak{X}^{s+2}_{\mathrm{div}}$:

$$\begin{aligned}
&\left\{f \circ \tilde{F}_t, h \circ \tilde{F}_t\right\}_+ \quad \text{(Theorem 3.20)} \\
&\quad = \left\{f \circ \tilde{F}_t \circ \pi, h \circ \tilde{F}_t \circ \pi\right\} \quad \text{(Proposition 3.12)} \\
&\quad = \{f \circ \pi \circ F_t, h \circ \pi \circ F_t\} \quad \text{(Proposition 3.24)} \\
&\quad = \{f \circ \pi, h \circ \pi\} \circ F_t \quad \text{(Proposition 3.23)} \\
&\quad = \{f, h\}_+ \circ \pi \circ F_t = \{f, h\}_+ \circ \tilde{F}_t. \quad \square
\end{aligned}$$

Chapter 4

Reduction on general diffeomorphism groups

In this chapter we will adapt the reduction technique developed in Section 3.4 for the group of volume preserving diffeomorphisms to the more general setting. Considering other subgroups of diffeomorphism group or choosing a different Riemannian structure on the group, one obtains an appropriate configuration space for various important PDE's that will be presented in the next chapter. Here we keep the exposition as general as possible in order to treat these applications simultaneously rather than on case by case basis.

Such generalization is possible since in all cases the difficulties arise from the non-differentiability of the reduction map

$$\pi(\eta, v) = v \circ \eta^{-1}, \tag{4.1}$$

which is the same for all subgroups and metrics, hence could be treated uniformly.

Similarly to the case of the Euler equation, we assume that the subgroup (or the whole group) of $\mathcal{D}^s$ is equipped with a weak right-invariant Riemannian metric, which possesses a smooth spray. This ensures the existence of the Poisson bracket on the tangent bundle of the subgroup by Theorem 3.9.

We will show that one can carry out the reduction procedure and obtain a well-defined bracket on the Lie algebra of the subgroup, provided the weak Riemannian connection is transposable (see definition 4.3). The bracket is given by

$$\{f, g\}_+ = \{f \circ \pi, g \circ \pi\}$$

for members of an appropriate functional class on the Lie algebra. The reduction of dynamics also follows.

It turns out that natural L^2 and H^1 metrics on the diffeomorphism groups induce transposable connections, hence this restriction is not significant for practically important applications.

4.1 Construction of the reduced bracket

Here and further in this chapter we will denote by Q^s a Lie subgroup of a right Lie group $\mathcal{D}^s$ $(s > n/2 + 1)$ and let $\mathfrak{X}^s$ be a Lie algebra of Q^s.

Given a subgroup Q^s, for $r \geq s$ we define $Q^r = Q^s \cap \mathcal{D}^r$ and endow Q^r with the topology induced from $\mathcal{D}^r$. Trivially, Q^r is a subgroup and a submanifold of Q^s. The Lie algebra of Q^r will be denoted as $\mathfrak{X}^r$. Since for any $u \in \mathfrak{X}^s \cap H^r(TM)$ the vector field $X_u(\eta) = u \circ \eta$ is a right-invariant vector field on Q^r, it's clear that $\mathfrak{X}^r = \mathfrak{X}^s \cap H^r(TM)$.

Moreover, we suppose that Q^s carries a weak right-invariant Riemannian metric, which we denote $\langle \cdot, \cdot \rangle$ (not necessarily given by 3.2), and the metric defines a smooth spray on Q^s. The smoothness of the spray is equivalent to the smoothness of the connector map $\tilde{K}$.

Recall, that the map π mapping the tangent bundle of Q^s to its Lie algebra is continuous, but not differentiable. However, it is a C^k as a map $Q^{s+k} \to \mathfrak{X}^s$. Similarly to Chapter 3.4 define functional spaces $C^k_s(\mathfrak{X}^r)$ and

$$f_R = f \circ \pi$$

for $f : \mathfrak{X}^r \to \mathbb{R}$. Define a Poisson bracket on $\mathfrak{X}^r$ by

$$\{f, g\}_+ (v) = \frac{\partial f_R}{\partial \eta}(e, v) \cdot \frac{\partial g_R}{\partial v}(e, v) - \frac{\partial g_R}{\partial \eta}(e, v) \cdot \frac{\partial f_R}{\partial v}(e, v). \tag{4.2}$$

We will call this bracket *the reduced bracket.*

Proposition 4.1. *Let $f, g \in C^1_s(\mathfrak{X}^r)$. Then $\{f, g\}_+$ is a well-defined function on $\mathfrak{X}^{r+1}$*

Proof. Note that f_R is a C^1 function on TQ^{r+1}. Calculating as in Lemmata 3.21,3.22, one readily obtains the following two results:

$$\frac{\partial f_R}{\partial v}(\eta, v) = TR_\eta df(v \circ \eta^{-1}), \tag{4.3}$$

$$\frac{\partial f_R}{\partial \eta}(\eta, v) \cdot (\eta, u) = -\left\langle df(v \circ \eta^{-1}), \tilde{K}[T(v \circ \eta^{-1}) \cdot (u \circ \eta^{-1})] \right\rangle. \tag{4.4}$$

By above, for any $v \in \mathfrak{X}^{r+1}$ the covariant partial derivatives $\frac{\partial f_R}{\partial v}(e, v)$, $\frac{\partial g_R}{\partial v}(e, v)$, are the elements of $\mathfrak{X}^r$, hence Formula 4.2 makes sense and

$$\{f, g\}_+ (v) = \left\langle dg(v), \tilde{K}[Tv \cdot df(v)] \right\rangle - \left\langle df(v), \tilde{K}[Tv \cdot dg(v)] \right\rangle. \quad \square \tag{4.5}$$

Since the weak metric on Q^s defines a smooth spray, it also defines a Poisson bracket $\{\cdot, \cdot\}$ by Formula 3.4. Rewriting the expression 3.4 as

$$\{f_R, g_R\} (\eta, v) = \frac{\partial f_R}{\partial \eta}(\eta, v) \cdot \frac{\partial g_R}{\partial v}(\eta, v) - \frac{\partial g_R}{\partial \eta}(\eta, v) \cdot \frac{\partial f_R}{\partial v}(\eta, v)$$

we extend the bracket on Q^s to the functions f_R, g_R.

Theorem 4.2. *The function $\pi(\eta, v) = v \circ \eta^{-1}$ is a Poisson map with respect to the brackets $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}_+$. Precisely, let $f, g \in C^1_s(\mathfrak{X}^r)$, $v \in \mathfrak{X}^{r+1}$. Then*

$$\{f \circ \pi, g \circ \pi\} (v) = \{f, g\}_+ \circ \pi(v).$$

Proof. The bracket $\{f \circ \pi, g \circ \pi\}$ is defined as above. The statement follows from the definition 4.2 and the fact that π is an identity on the Lie algebra. $\square$

4.2 Properties of the reduced bracket

In order to obtain the Jacobi identity and the desired dynamical properties of the reduced bracket, one has to assume some additional hypotheses on the weak Riemannian connection on the right Lie group Q^s.

Definition 4.3. *We will call a connector $\tilde{K}$ on Q^s* (r,s)-transposable if there is a bounded bilinear operator $B : \mathfrak{X}^s \times \mathfrak{X}^r \to \mathfrak{X}^s$ such that for all $w, u \in \mathfrak{X}^s$ $v \in \mathfrak{X}^r$

$$\left\langle w, \tilde{K}[Tv \cdot u] \right\rangle = \langle B(w,v), u \rangle .$$

In this case we will also say that the connection on Q^s is (r,s)-transposable.

Remark 4.4. *The above condition is equivalent to the equality*

$$\left\langle X_w, \tilde{\nabla}_{X_u} X_v \right\rangle (e) = \langle B(w,v), u \rangle ,$$

where X_w, X_u, X_v denote the right-invariant vector fields corresponding to w, u, v, respectively and $\tilde{\nabla}$ is a weak Riemannian connection on Q^s.

Define the spaces

$$\mathcal{K}^{k,s} = \{ f \in C^{k+1}(\mathfrak{X}^s, \mathbb{R}) \mid df \in C^k(\mathfrak{X}^s, \mathfrak{X}^s) \}$$

and

$$\mathcal{K}^{k,s}_{r,t} = \{ f \in C^{k+1}(\mathfrak{X}^s, \mathbb{R}) \mid df \in C^k(\mathfrak{X}^r, \mathfrak{X}^t) \}.$$

The following is the analog of Theorem 3.14.

Theorem 4.5. *Suppose the connector $\tilde{K}$ is $(r, s+1)$-transposable. Let $r_0 = max(s+1, r)$. Then $\{\cdot,\cdot\}_+$ is a bilinear map $\mathcal{K}^{k,s+1} \times \mathcal{K}^{k,s+1} \to \mathcal{K}^{k-1,s+1}_{r_0,s}$ and a derivation on each factor. Moreover, it satisfies the Jacobi identity on $\mathfrak{X}^{r_0}$.*

Proof. Bilinearity and derivation property are trivial. Without loss of generality

$r \geq s+1$ and $r_0 = r$. Let, $f, g \in \mathcal{K}^{k,s+1}$. Denote

$$z(v) = \{f, g\}_+ (v).$$

From Formula 4.5 it's clear that $z \in C^k(\mathfrak{X}^{s+1}, \mathbb{R})$. Differentiating this expression, one obtains

$$\begin{aligned} dz(v) \cdot u = &\left\langle Ddg(v) \cdot u, \tilde{K}[Tv \cdot df(v)] \right\rangle + \left\langle dg(v), \tilde{K}[Tu \cdot df(v)] \right\rangle \\ &+ \left\langle dg(v), \tilde{K}[Tv \cdot (Ddf(v) \cdot u)] \right\rangle - \left\langle Ddf(v) \cdot u, \tilde{K}[Tv \cdot dg(v)] \right\rangle \\ &- \left\langle df(v), \tilde{K}[Tu \cdot dg(v)] \right\rangle - \left\langle df(v), \tilde{K}[Tv \cdot (Ddg(v) \cdot u)] \right\rangle. \end{aligned}$$

Lemma 4.6. *For all $x, y \in \mathfrak{X}^{s+1}$, $w \in \mathfrak{X}^s$*

$$\left\langle x, \tilde{K}[Ty \cdot w] \right\rangle = -\left\langle y, \tilde{K}[Tx \cdot w] \right\rangle.$$

Proof. Recall that

$$\tilde{\nabla}_{X_w} X_y(\eta) = \tilde{K}[T(y \circ \eta) \cdot (w \circ \eta)].$$

Hence,

$$\left\langle x, \tilde{K}[Ty \cdot w] \right\rangle = \left\langle X_x, \tilde{\nabla}_{X_w} X_y \right\rangle (e). \tag{4.6}$$

However, by properties of Riemannian connection and right invariance of metric

$$\left\langle X_x, \tilde{\nabla}_{X_w} X_y \right\rangle = X_w \langle X_x, X_w \rangle - \left\langle X_y, \tilde{\nabla}_{X_w} X_x \right\rangle = -\left\langle X_y, \tilde{\nabla}_{X_w} X_x \right\rangle. \quad \square$$

Repeating the argument of Lemma 3.16, we obtain for any $w \in \mathfrak{X}^s$

$$\langle Ddf(v) \cdot u, w \rangle = \langle Ddf(v) \cdot w, u \rangle. \tag{4.7}$$

Thus, taking into account Lemma 4.6, Formula 4.7 and the fact that connection is $(r, s+1)$-transposable, we get

$$\begin{aligned} dz(v) = & Ddg(v) \cdot \tilde{K}[Tv \cdot df(v)] - Ddf(v) \cdot \tilde{K}[Tv \cdot dg(v)] \\ & + \tilde{K}[Tdf(v) \cdot dg(v) - Tdg(v) \cdot df(v)] \\ & + Ddf(v) \cdot B(dg(v), v) - Ddg(v) \cdot B(df(v), v). \end{aligned}$$

From the above expression is clear that $dz \in C^{k-1}(\mathfrak{X}^r, \mathfrak{X}^s)$, hence $z \in \mathcal{K}_{r,s}^{k-1,s+1}$.

Formula for dz could be simplified.

Lemma 4.7. *For $x, y \in \mathfrak{X}^{s+1}$*

$$\tilde{K}[Tx \cdot y - Ty \cdot x] = [x, y].$$

Proof. By identities 4.6 and 2.22 we have

$$\begin{aligned} \tilde{K}[Tx \cdot y - Ty \cdot x] &= \tilde{\nabla}_{X_y} X_x(e) - \tilde{\nabla}_{X_x} X_y(e) \\ &= [X_y, X_x](e) \\ &= [x, y]. \quad \square \end{aligned}$$

Define

$$B_f(v) = B(df(v), v),$$

$$\tilde{\nabla}_f(v) = \tilde{K}[Tv \cdot df(v)].$$

Then, after applying Lemma 4.7, we rewrite the expression for dz as

$$dz(v) = Ddg(v) \cdot (\tilde{\nabla}_f(v) - B_f(v)) - Ddf(v) \cdot (\tilde{\nabla}_g(v) - B_g(v)) + [df(v), dg(v)]. \quad (4.8)$$

Now, the proof of Jacobi identity repeats the argument of Theorem 3.14. $\square$

It's easy to show that the bracket $\{\cdot, \cdot\}_+$ is related in a natural way to the canonical Lie-Poisson bracket on the dual of the Lie algebra.

Proposition 4.8. *If $v, df(v), dg(v) \in \mathfrak{X}^{s+1}$, then*

$$\{f, g\}_+ (v) = \langle [df(v), dg(v)], v \rangle .$$

Proof. We apply successively Lemma 4.6 and Lemma 4.7 to the identity 4.5. □

4.3 The reduction of dynamics

Here we study the flows on the Lie algebra which are the reduction of the Hamiltonian flows on the Lie group Q^s. These flows describe the evolution in the spatial or Eulerian representation of the partial differential equations that we discuss. They are (in general) non-differentiable both in time and initial conditions, however, as we show below, these flows are Poisson with respect to the Poisson bracket on the Lie algebra. As we mentioned earlier, our methods yield results provided the Riemannian connection is transposable.

Let F_t be a flow of a right-invariant vector field X on TQ^s. Define a mapping on the Lie algebra by

$$\tilde{F}_t(v) = \pi(F_t(v)).$$

The main result of this section is that $\tilde{F}_t$ is a Poisson map if X is a smooth Hamiltonian vector field. First we establish an analog of proposition 3.12.

Proposition 4.9. *Let X be C^1. Then the flow F_t commutes with the reduction map π, i.e., for all $v \in TQ^s$.*

$$\tilde{F}_t \circ \pi(v) = \pi \circ F_t(v).$$

Proof. The proof is based on the fact that F_t commutes with right shifts because of the right invariance of its generator.

Lemma 4.10. *Let F_t be a flow of a C^1 right-invariant vector field on a tangent bundle of a (right) Lie group Q. Then for all $\xi \in Q, v \in TQ$*

$$TR_\xi \circ F_t(v) = F_t \circ TR_\xi(v)$$

Proof. Fix $v \in TQ$ and let

$$c(t) = TR_\xi \circ F_t(v).$$

Differentiating c at $t = 0$ and using the right invariance of X we obtain

$$\begin{aligned} c'(0) &= TTR_\xi \left[\frac{d}{dt}|_{t=0} F_t(v) \right] \\ &= TTR_\xi X(v) \\ &= X(TR_\xi(v)). \end{aligned}$$

Hence, c is an integral curve of X through v, starting with velocity $TR_\xi(v)$. The claim of the lemma follows from the uniqueness of integral curves. □

Now, the argument found after Lemma 3.13 completes the proof of proposition 4.9. □

The following analog of proposition 3.24 holds:

Proposition 4.11. *Let $v \in TQ^r$ and there are TQ^s-neighborhoods U of v and V of $\psi(v)$ such that $\psi : U \to V$ is a symplectic diffeomorphism. Suppose $f, g \in C^1(TQ^r \restriction V, \mathbb{R})$ are such that $\frac{\partial f}{\partial v}(\psi(v))$, $\frac{\partial f}{\partial \eta}(\psi(v))$, $\frac{\partial g}{\partial v}(\psi(v))$, $\frac{\partial g}{\partial \eta}(\psi(v)) \in TQ^s$. Then $\frac{\partial (f\circ\psi)}{\partial v}(v)$, $\frac{\partial (f\circ\psi)}{\partial \eta}(v)$, $\frac{\partial (g\circ\psi)}{\partial v}(v)$, $\frac{\partial (g\circ\psi)}{\partial \eta}(v) \in TQ^s$ and*

$$\{f \circ \psi, g \circ \psi\}(v) = \{f, g\}(\psi(v)).$$

Proof. Same as the proof of proposition 3.24, where $T\mathcal{D}^r_\mu$ is replaced with TQ^r and $T\mathcal{D}^s_\mu$ is replaced with TQ^s. □

Corollary 4.12. *Suppose X is a smooth Hamiltonian vector field. Then F_t preserves the classes $C^1_s(TQ^r)$. Moreover, if the weak Riemannian connection is (r, s)-transposable then for $f, g \in \mathcal{K}^{1,s}$, $v \in \mathfrak{X}^{r_0+1}$*

$$\{f \circ \pi \circ F_t, g \circ \pi \circ F_t\}(v) = \{f \circ \pi, g \circ \pi\}(F_t(v)),$$

where $r_0 = max(r, s + 1)$.

Proof. The flow F_t is a symplectic diffeomorphism of TQ^s, hence the first part trivially follows from proposition 4.11. Without loss of generality $r \geq s+1$ and $r_0 = r$. Taking into account the assumption that $\tilde{K}$ is (r,s)-transposable and equalities 4.3, 4.4,

$$\frac{\partial(f \circ \pi)}{\partial v}(\eta, v) = TR_\eta df(v),$$

$$\frac{\partial(f \circ \pi)}{\partial \eta}(\eta, v) = TR_\eta B(df(v), v).$$

Hence $f \circ \pi, g \circ \pi \in C^1_s(TQ^r)$. Now, we may apply proposition 4.11 to $f \circ \pi, g \circ \pi$. $\square$

Theorem 4.13. *Suppose F_t is a flow of a smooth right-invariant Hamiltonian vector field on TQ^s. Moreover suppose the weak Riemannian connection on Q^s is (r,s)-transposable. Then the reduced flow $\tilde{F}_t$ is Poisson with respect to the bracket $\{\cdot,\cdot\}_+$. Precisely, for $f, h \in \mathcal{K}^{1,s}$, $v \in \mathfrak{X}^{r_0+1}$*

$$\left\{f \circ \tilde{F}_t, h \circ \tilde{F}_t\right\}_+ (v) = \{f,h\}_+ \circ \tilde{F}_t(v),$$

where $r_0 = max(r, s+1)$.

Proof. Let $f, h \in \mathcal{K}^{1,s}$. Then $f \circ \pi \in C^1_s(\mathfrak{X}^{r_0})$. By Proposition 4.11.

$$f \circ \tilde{F}_t = f \circ \pi \circ F_t \in C^1_s(\mathfrak{X}^{r_0})$$

and we have pointwise in $\mathfrak{X}^{r_0+1}$:

$$\begin{aligned}
&\left\{f \circ \tilde{F}_t, h \circ \tilde{F}_t\right\}_+ \quad \text{(proposition 4.1)}\\
&= \left\{f \circ \tilde{F}_t \circ \pi, h \circ \tilde{F}_t \circ \pi\right\} \quad (\ 4.9\ )\\
&= \{f \circ \pi \circ F_t, h \circ \pi \circ F_t\} \quad \text{(Corollary 4.12)}\\
&= \{f \circ \pi, h \circ \pi\} \circ F_t \quad \text{(Theorem 4.2)}\\
&= \{f,h\}_+ \circ \pi \circ F_t = \{f,h\}_+ \circ \tilde{F}_t. \quad \square
\end{aligned}$$

Chapter 5

Applications

In this chapter we will apply the general methods developed in Section 3.3 and Chapter 4 to the several partial differential equations. The equations that we consider are given by the geodesic flow on a suitable group of diffeomorphisms in the same sense as the Euler equation is given by the geodesic flow on the group of volume preserving diffeomorphisms. Essential for the technique that we employ is the fact that the weak Riemannian metric on the group defines a smooth spray. Because of that, the methods are not applicable to the KDV equation.

For each application we introduce an appropriate Lie-Poisson bracket directly on the Lie algebra and prove that the flow, generated by the equation, on the Lie algebra is Poisson. This is an important difference with the usual Lie-Poisson reduction performed on the cotangent bundle of the group.

While the reduction procedure itself is more natural and easier to carry out on the cotangent bundle, the flows in question are not well-defined there since the necessary existence and uniqueness theorems are missing and there is a very little hope to establish them. For example, in order to rigorously apply the Lie-Poisson reduction on the cotangent bundle to the PDEs that we consider, one would need to prove the existence and uniqueness theorems in negative Sobolev spaces, which is quite unlikely. The situation might be different for the KDV equation, where the existence was shown in class $H^{-3/10+\epsilon}$ (see Colliander et al. [2001]).

Further, we consider the following applications:

- The Camassa-Holm (CH) equation on S^1—see Camassa and Holm [1993]:

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}, \tag{5.1}$$

where u_t and u_x denote, respectively, time and spacial derivatives of an unknown function u.

The equation is generated by the geodesic flow of the right-invariant H^1 metric on the whole group $\mathcal{D}^s(S^1)$ of H^s diffeomorphisms.

- The averaged Euler equations (or the LAE-α equations)—see Holm, Marsden, and Ratiu [1998a,b]:

$$\partial_t(1-\alpha^2\Delta)u + (u\cdot\nabla)(1-\alpha^2\Delta)u - \alpha^2(\nabla u)^T\cdot\Delta u = -\operatorname{grad} p\,, \tag{5.2}$$

where $\operatorname{div} u = 0$ and u satisfies appropriate boundary conditions, such as the no-slip conditions $u = 0$ on ∂M.

The equation is generated by the geodesic flow of right-invariant L^2 metric on the $\mathcal{D}^s_\mu$. The map $u_0 \to u_t$ is Poisson on the space of H^s $(s > n/2+2)$ divergence free vector fields.

- The EPDiff equation for the H^1 metric (also called the averaged template matching equation) on a compact manifold M—see Holm and Marsden [2003] and Hirani et al. [2001]:

$$\begin{aligned}\partial_t(u-\alpha^2\Delta u) + u(\operatorname{div} u) - \alpha^2(\operatorname{div} u)\Delta u + (u\cdot\nabla)u \\ - \alpha^2(u\cdot\nabla)\Delta u + (Du)^T\cdot u - \alpha^2(Du)^T\cdot\Delta u = 0,\end{aligned}$$

with the appropriate boundary conditions, such as the no-slip conditions $u = 0$ on ∂M. The EPDiff equation reduces to the CH equations in the case $M = S^1$.

The equation is generated by the geodesic flow of the right-invariant H^1 metric on the whole group $\mathcal{D}^s(S^1)$ of H^s diffeomorphisms. The map $u_0 \to u_t$ is Poisson on the space of H^s $(s > n/2 + 2)$ vector fields.

In all cases, the smoothness of the spray plays a crucial role. First it guarantees that the flow maps are well-defined both on the Lie group and Lie algebra. What is even more important, it guarantees the existence of the Poisson structure on the Lie group in the sense of Section 3.3. Hence, in order to apply Theorem 4.13 to each of the applications, we only need to show that the appropriate Riemannian connection is transposable.

5.1 Camassa-Holm equation

The Camassa-Holm (CH) equation was derived by Cammassa and Holm (see Camassa and Holm [1993]) as an approximation to the Euler equation in the shallow water regime. In the same article also was given a formal Lie-Poisson description of the equation. Since then the equation has been studied extensively (for example, Misiolek [1998]; Danchin [2001]; Kouranbaeva [1998]; Khesin and Misiolek [2003], and found to possess many remarkable properties.

In a sense, CH combines features of the Euler equation and the Korteveg de Vries equation

$$u_t + uu_x + u_{xxx} = 0.$$

Like KDV, Camassa-Holm equation is a completely integrable system, has a bi-hamiltonian structure, the same symmetry group (the Bott-Virasoro group or $\mathcal{D}^s(S^1)$, depending on the form in which equations are formulated), solitons and infinitely many first integrals. From the other side, similarly to Euler equation (and contrary to KDV), CH is generated by the smooth spray on the diffeomorphism group, yet the solutions of CH may lose regularity in finite time (Camassa and Holm [1993]).

Our goal in this section is to establish in what sense the map $\tilde{F}_t : u_0 \to u_t$, which maps the initial condition u_0 to the solution of equation 5.1 at time t, is a Poisson map.

First, we establish the relationship between the solutions of Cammassa-Holm equation and the geodesic flow on $\mathcal{D}^s(S^1)$, following Misiolek [1998]; Kouranbaeva [1998].

Let $s > 3/2$, so that $\mathcal{D}^s \equiv \mathcal{D}^s(S^1)$ is an infinite dimensional manifold. It is a right Lie group and its Lie algebra is the space of all H^s vector fields, that we will denote $\mathfrak{X}^s$.

Define the weak Riemannian structure at the identity of $\mathcal{D}^s$ by

$$\langle v, u \rangle_e = \int_{S^1} (uv + u_x v_x) dx$$

and extend it to the right-invariant metric on the group. That is, for $V, W \in T_\eta D^s$

$$\begin{aligned} \langle V, W \rangle_\eta &= \langle TR_{\eta^{-1}} \cdot V, TR_{\eta^{-1}} \cdot W \rangle_e && (5.3) \\ &= \left\langle V \circ \eta^{-1}, W \circ \eta^{-1} \right\rangle_e . && (5.4) \end{aligned}$$

Theorem 5.1. (see Kouranbaeva [1998], Theorem IV.1). *Let $t \to \eta_t$ be a curve in the diffeomorphism group $\mathcal{D}^s$. Then η_t is a geodesic of the metric if and only if the time-dependent vector field $u_t = \dot{\eta}_t \circ \eta_t^{-1}$ satisfies the CH equation 5.1.*

The smoothness of the spray on the $\mathcal{D}^s$ is due to Shkoller.

Theorem 5.2. (see Shkoller [1998], Remark 3.5). *For $s > 3/2$ the spray of the metric 5.3 is smooth on $\mathcal{D}^s$.*

The Theorem 5.2 guarantees the existence of a local flow F_t of the right-invariant H^1 spray on the diffeomorphism group. Together with Theorem 5.1, it shows that the solutions of Camassa-Holm equations are well-defined in H^s spaces (for short time). Hence, there is a well-defined local flow

$$\tilde{F}_t : \mathfrak{X}^s \to \mathfrak{X}^s,$$

given by

$$\tilde{F}_t(u_0) = u_t = \pi \circ F_t(u_0),$$

where u_t is a solution of CH equation 5.1 with u_o as an initial condition and π is given by 4.1.

As we explained above, the Theorems 5.2 and 5.1 provide all the existence and uniqueness theory for Camassa-Holm equation that we need. More detailed exposition of this subject can be found in Misiolek [2002]; Himonas and Misiolek [2001]; Danchin [2001] and references therein.

In order to apply the Theorem 4.13, we need to show that the connection on $\mathcal{D}^s$ is transposable.

Theorem 5.3. *For $s > 5/2$ the Riemannian connection of the weak right-invariant H^1 metric 5.3 on $\mathcal{D}^s$ is (s,s)-transposable.*

Proof. Let $\tilde{K}$ be a connector map of the metric 5.3, $\tilde{\nabla}$ be the corresponding connection and $\tilde{\Gamma}$ be the Christoffel map. Let u, v, w be the vector fields on S^1 and X_u, X_w, X_w be the corresponding right-invariant vector fields on the diffeomorphism group, e.g.

$$X_u(\eta) = u \circ \eta$$

and similarly for X_w, X_v. First we calculate

$$\left\langle v, \tilde{K}[Tw \cdot u] \right\rangle = \left\langle X_v, \tilde{\nabla}_{X_u} X_w \right\rangle (e).$$

Recall, that for the right-invariant Riemannian connection

$$2 \left\langle X_v, \tilde{\nabla}_{X_u} X_w \right\rangle (e) = - \langle v, [u,w] \rangle + \langle u, [w,v] \rangle + \langle w, [u,v] \rangle . \tag{5.5}$$

However, on S^1

$$[u, w] = u w_x - u_x v.$$

Substituting that into expression 5.5 and integrating by parts, we obtain

$$\left\langle X_v, \tilde{\nabla}_{X_u} X_w \right\rangle (e) = \int [w_x - v_x w + \frac{1}{2}(v_x w_{xx} + v_{xx} w_x)] u. \tag{5.6}$$

Define operator

$$\Lambda = 1 - \frac{\partial^2}{\partial x^2}.$$

Then, Λ is a linear isomorphism $H^s(S^1) \to H^{s-2}(S^1)$ (see Rosenberg [1997]). Thus, for $u \in H^s$, $z \in H^{s-2}$.

$$\int uz = \left\langle u, \Lambda^{-1} z \right\rangle.$$

With that in mind, Formula 5.6 implies

$$\left\langle v, \tilde{K}[Tw \cdot u] \right\rangle = \langle B(v,w), u \rangle ,$$

where

$$B(v,w) = \Lambda^{-1}[w_x - v_x w + \frac{1}{2}(v_x w_{xx} + v_{xx} w_x)].$$

For $s > 5/2$ the functions $v_x w_{xx}$ are the elements of H^{s-2}, hence $B(v,w) \in H^s$. □

Theorem 5.4. *The flow $\tilde{F}_t$ is Poisson with respect to the Lie-Poisson bracket given by Formula 4.2. Precisely, for $f, g \in \mathcal{K}^{1,s}(\mathcal{D}^s(S^1))$, $s > 3/2$, $v \in \mathfrak{X}^{s+2}$*

$$\left\{ f \circ \tilde{F}_t, g \circ \tilde{F}_t \right\}_+ (v) = \{f, g\}_+ \circ \tilde{F}_t(v).$$

Proof. Follows immediately from Theorems 5.4 and 4.13. □

5.2 Averaged Euler equations

The averaged Euler equations, also known as Lagrange averaged Euler equations or Euler-α equations, were introduced by Holm, Marsden, and Ratiu [1998a] as a new class of models for ideal incompressible fluids. These models provided a nonlinear filtering mechanism below the length scale α, which enhanced the stability and regularity of the solutions without compromising conservation laws and the large scale behavior of the fluid.

Interestingly, Euler-α equations can be derived directly by appropriately averaging the Lagrange-d'Alambert principle. Another description of averaged Euler equation is that of the geodesic motion of H^1 metric on the group of volume preserving diffeomorphisms. As such, the solutions of Euler-α equations preserve H^1 metric, unlike the Euler equation, which preserves L^2 metric.

The geometry of the diffeomorphism groups with right-invariant H^1 metric was studied by Shkoller [1998]. In the same article was established the smoothness of the H^1 spray on the group of volume preserving diffemorphisms. This result was extended in Marsden et al. [2000] to the groups required to treat various boundary conditions, such as no-slip and free slip.

Consider the equation

$$\partial_t(1-\alpha^2\Delta)u + (u\cdot\nabla)(1-\alpha^2\Delta)u - \alpha^2(\nabla u)^T\cdot\Delta u = -\operatorname{grad} p\,, \tag{5.7}$$

$$\operatorname{div} u = 0, \tag{5.8}$$

$$u = 0 \text{ on } \partial M \tag{5.9}$$

on the compact, orientable n-dimensional manifold M with a boundary ∂M. In equation 5.7 α is a real number, $(\nabla u)^T$ denotes the transposition of matrix ∇u (we will give an invariant geometric interpretation of this operator later on), operator Δ denotes the *negative* of Laplace-de Rham operator, i.e.,

$$\Delta = -\Delta^R = -(d\delta + \delta d).$$

Let $\langle\cdot,\cdot\rangle_m$ be the Riemannian metric on T_mM and $g(\cdot,\cdot)$ be the metric tensor, i.e.,

$$\langle u, v\rangle_m = g(u,v) = g_{ij}u^iv^j \quad \forall u, v \in T_mM$$

and μ be the associated volume form.

The diffeomorphims groups relevant for the system 5.7-5.9 are

$$\mathcal{D}^s = \{\eta \in H^s(M,M) | \eta^{-1} \in H^s(M,M), \eta(x) = x \text{ for } x \in \partial M\},$$

$$\mathcal{D}^s_\mu = \{\eta \in \mathcal{D}^s | \eta * \mu = \mu\}.$$

Their respective Lie algebras are

$$\mathfrak{X}^s = \{u \in H^s(TM) | u = 0 \text{ on } \partial M\},$$

$$\mathfrak{X}^s_{div} = \{u \in \mathfrak{X}^s | \operatorname{div} u = 0\}.$$

Set

$$\Lambda = 1 - \alpha^2 \Delta$$

and define a weak Riemannian metric at identity of $\mathcal{D}^s$ by

$$\langle u, v \rangle_e = \int_M \langle \Lambda u, v \rangle_m \mu. \tag{5.10}$$

This metric extends by right invariance to the whole diffeomorphism group, hence restricts to the right-invariant H^1 metric on the submanifold $\mathcal{D}^s_\mu$ and defines a smooth orthogonal projection $P_\eta : T_\eta \mathcal{D}^s \to \mathcal{D}^s_\mu$ given by

$$P_\eta(V) = (P_e(V \circ \eta^{-1})) \circ \eta \quad \forall V \in T_\eta \mathcal{D}^s,$$

where P_e is the H^1 orthogonal projection defined by the Hodge decomposition.

The spray S of Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathcal{D}^s_\mu$ is smooth. We refer the reader to Shkoller [1998] for the proofs of smoothness of $\langle \cdot, \cdot \rangle$, P and S.

Theorem 5.5. *(see Marsden et al. [2000]) For $s > n/2 + 1$, let η_t be the geodesic of the right-invariant H^1 metric on D^s_μ. Then $u_t = \dot{\eta}_t \circ \eta_t^{-1}$ is a vector field which satisfies the system 5.7-5.9, where the pressure p_t is determined by the Hodge decomposition and*

$$(Du)^T \cdot v(m) \equiv [\langle v, \nabla_{(\cdot)} u \rangle_m]^\sharp$$

(here and thereafter ∇ is the Riemannian connection on M.)

By proposition 5.6, the flow F_t of S is Poisson with respect to the bracket 3.4 on

$\mathcal{D}^s_\mu$. By above Theorem 5.5, the flow $\tilde{F}_t$ of Eulear-α equation is given by

$$u_0 \to u_t = \tilde{F}_t(u_0) = \pi \circ F_t(u_0),$$

where π is given by 4.1 as usually.

In order to establish the Poisson nature of the flow $\tilde{F}_t$, we need to verify that the Riemannian connection $\tilde{\nabla}$ of a weak H^1 metric is transposable on $\mathcal{D}^s_\mu$.

Proposition 5.6. *For $s > n/2 + 2$ the connection $\tilde{\nabla}$ is (s,s)-transposable on $\mathcal{D}^s_\mu$.*

Proof. Let $\tilde{K}$ be the connector map that defines $\tilde{\nabla}$. Then, for arbitrary $u, v, w \in \mathfrak{X}^s_{div}$

$$\begin{aligned}\left\langle v, \tilde{K}[Tw \cdot u]\right\rangle &= -\langle v, [u,w]\rangle + \langle u, [w,v]\rangle + \langle w, [u,v]\rangle \\ &= \langle v, \nabla_u w - \nabla_w u\rangle + \langle u, [w,v]\rangle + \langle w, \nabla_u w - \nabla_v u\rangle .\end{aligned}$$

Let P^0 be the L^2 Hodge decomposition projector and B_0 be defined as in Lemma 3.17, i.e.,

$$\int_M \langle B_0(Z,Y)(m), W(m)\rangle_m \mu = \int \langle Z(m), \nabla_Y W(m)\rangle_m \mu$$

for sufficiently smooth vector fields W, Y and divergence free vector field Z.

Then, we obtain on $\mathfrak{X}^{s+1}_{div}$

$$\begin{aligned}\left\langle v, \tilde{K}[Tw \cdot u]\right\rangle = &\int_M \left\langle u, B_0(P^0\Lambda v, w) - B_0(P^0\Lambda w, v)\right\rangle_m \mu \\ &+ \int_M \langle u, \nabla_v \Lambda w - \nabla_w \Lambda v + \Lambda[w,v]\rangle_m \mu.\end{aligned}$$

Consider an operator

$$A(v,w) = \nabla_v \Lambda w - \nabla_w \Lambda v - \Lambda[w,v]. \tag{5.11}$$

We claim that A extends to the bounded bilinear operator $\mathfrak{X}^s \times \mathfrak{X}^s \to \mathfrak{X}^{s-2}$. Even though the individual terms defining A involve the partial derivatives of the third order, the terms of the highest order cancel each other out. First we notice, that

$$A(v,w) = \nabla_v \Delta^R w - \nabla_w \Delta^R v + \Delta^R[w,v].$$

Then, recall that for vector field Z.

$$\Delta^R Z = (\hat{\Delta} Z^\flat)^\sharp + \mathrm{Ric}(Z),$$

where $\hat{\Delta}$ is a rough Laplacian and Ric is the Ricci curvature tensor on M.

Let

$$\beta = \beta_i dx^i$$

be a 1-form on M. Calculating $\hat{\Delta}\beta$ in coordinates by Formula 2.20, we get

$$\begin{aligned}\hat{\Delta}\beta &= -g^{ij}\nabla_{\partial/\partial x^i}\nabla_{\partial/\partial x^j}\beta_k dx^k \\ &= -g^{ij}\left[\frac{\partial^2 \beta_k}{\partial x^i \partial x^j} - \frac{\partial \beta_l}{\partial x^i}\Gamma^l_{jk}\frac{\partial \beta_l}{\partial x^j}\Gamma^l_{ik} + \beta_l \Gamma^l_{im}\Gamma^m_{jk} - \beta_l \frac{\partial \Gamma^l_{ik}}{\partial x^j}\right] dx^k,\end{aligned}$$

where Γ^k_{ij} are the Christoffel symbols on M. Hence, for a vector field $Z = Z^i \frac{\partial}{\partial x^i}$

$$\hat{\Delta}(Z) = -g^{ij}\frac{\partial^2 z^k}{\partial x^i \partial x^j}\frac{\partial}{\partial x^k} + A^1(Z),$$

where A^1 is a bounded linear operator $\mathcal{D}^s \to \mathcal{D}^{s-1}$, in fact differential operator of the first order.

Then,

$$\begin{aligned}\nabla_v \Lambda w &= -v^l \frac{\partial}{\partial x^l}\left(g^{ij}\frac{\partial^2 w^k}{\partial x^i \partial x^j}\right)\frac{\partial}{\partial w^k} + \Gamma^m_{lk} g^{ij}\frac{\partial^2 w^k}{\partial x^i \partial x^j}\frac{\partial}{\partial x^m} + \nabla_v A^1(Z) \\ &= -g^{ij}\frac{\partial^3 w^k}{\partial x^i \partial x^j \partial x^l} v^l + A^2(w,v),\end{aligned}$$

where $A^2 : \mathcal{D}^s \times \mathcal{D}^{s-2} \to \mathcal{D}^{s-2}$ is bilinear and bounded (here we used the fact that

H^{s-2} is a Schauder ring). Similarly,

$$\nabla_w \Lambda v = -g^{ij} \frac{\partial^3 v^k}{\partial x^i \partial x^j \partial x^l} v^l + A^2(v,w)$$

and

$$\begin{aligned}\hat{\Delta}[w,v] &= -g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \left(w^l \frac{\partial v^k}{\partial x^l} - v^l \frac{\partial w^k}{\partial x^l} \right) \frac{\partial}{\partial x^k} + A^1([w,v]) \\ &= -g^{ij} \left(\frac{\partial^3 v^k}{\partial x^i \partial x^j \partial x^l} w^l \right) \frac{\partial}{\partial x^k} + g^{ij} \left(\frac{\partial^3 w^k}{\partial x^i \partial x^j \partial x^l} v^l \right) \frac{\partial}{\partial x^k} + \tilde{A}^2(w,v),\end{aligned}$$

where $\tilde{A}^2 : \mathcal{D}^s \times \mathcal{D}^s \to \mathcal{D}^{s-2}$ is bounded bilinear operator.

Therefore,

$$A(v,w) = A^2(w,v) - A^2(w,v) + \tilde{A}^2(w,v) + \nabla_v \operatorname{Ric}(w) - \nabla_w \operatorname{Ric}(v) + \operatorname{Ric}([w,v])$$

extends to the bounded bilinear operator $\mathcal{D}^s \times \mathcal{D}^s \to \mathcal{D}^{s-2}$.

Hence,

$$\left\langle v, \tilde{K}[Tw \cdot u] \right\rangle = \langle u, B_1(v,w) \rangle \,,$$

where $B_1 : \mathfrak{X}^s_{div} \times \mathfrak{X}^s_{div} \to \mathfrak{X}^s_{div}$ is defined by

$$B_1(v,w) = P_e \Lambda^{-1} \left[B_0(P^0 \Lambda v, w) - B_0(P^0 \Lambda w, v) + A(v,w) \right]. \quad \square$$

Theorem 5.7. *For $s > n/2 + 2$ the flow map generated by Euler-α equation on the space of divergence free vector fields is Poisson with respect to the bracket $\{\cdot,\cdot\}_+$ given by 4.2, i.e., for any $v \in \mathfrak{X}^{s+2}_{div}$ and $f, h \in \mathcal{K}^{1,s}(\mathfrak{X}^{s+2}_d iv)$*

$$\left\{ f \circ \tilde{F}_t, h \circ \tilde{F}_t \right\}_+ (v) = \{f,h\}_+ \circ \tilde{F}_t(v).$$

Proof. Follows immediately from Theorem 4.13 and proposition 5.6.

5.3 EPDiff equation

EPDiff equation is the equation of the geodesic motion of H^1 metric on the whole group of diffeomorphisms. Due to its applications to the computer vision and medical imaging, it's also called the averaged template matching equation. See Mumford [1998] for the derivation of template matching equation using Euler-Poincare reduction. The version of template matching equation that we study here is derived in Hirani et al. [2001] from Lagrange-d'Alembert principle on the diffeomorphism group.

Consider the following PDE on the compact manifold M $(dim(M) = n)$ with the boundary ∂M:

$$\begin{aligned}\partial_t(u - \alpha^2 \Delta u) + u(\operatorname{div} u) - \alpha^2(\operatorname{div} u)\Delta u + (u \cdot \nabla)u \\ - \alpha^2(u \cdot \nabla)\Delta u + (Du)^T \cdot u - \alpha^2(Du)^T \cdot \Delta u = 0,\end{aligned}$$

with the no-slip boundary conditions

$$u = 0 \text{ on } \partial M.$$

Here, we using the same notation as in the previous section for operators Δ, ∇ and $(Du)^T$.

Let the diffeomorphism group D^s be defined by 2.23 and the weak right-invariant metric $\langle \cdot, \cdot \rangle$ on D^s be given by 5.10. The geodesic spray S of $\langle \cdot, \cdot \rangle$ on $\mathcal{D}^s$ is smooth by the argument found in Shkoller [1998], hence defines a smooth local flow $F_t : T\mathcal{D}^s \to T\mathcal{D}^s$.

Theorem 5.8. *Let u_t be a solution of EPDiff equation with no-slip boundary conditions. Then*

$$u_t = \pi \circ F_t.$$

Proof. Let

$$\tilde{F}_t = \pi \circ F_t.$$

Similarly to Ebin and Marsden [1970], we get the following result:

Lemma 5.9. *The map $\tilde{F}_t : \mathfrak{X}^s \to \mathfrak{X}^s$ is a local flow generated by Y, where*

$$Y(v)^l_v = S(v) - Tv \cdot v.$$

Conversely, if v_t is an integral curve of Y, then

$$v_t = \pi \circ F_t(v_0).$$

Proof. Let $v \in T_e\mathcal{D}^s$ and η_t be the projection of $F_t(u)$ to the diffeomorphism group, i.e.,

$$v_t = F_t(v)$$

$$\eta_t = \tilde{\tau}(v_t).$$

It's easy to check that $\tilde{F}_t$ is a flow. Then,

$$\begin{aligned} \frac{d}{dt}\tilde{F}_t(v) &= \frac{d}{dt} v_t \circ \eta^{-1} \\ &= -TF_t(v) \circ T\eta_t^{-1} \circ \frac{d\eta_t}{dt} \circ \eta_t^{-1} + S(v) \circ \eta_t^{-1}. \end{aligned}$$

Evaluated at $t = 0$, this expression yields

$$\frac{d}{dt}\tilde{F}_t|_{t=0} = -Tv \cdot v + S(v) = Y(v).$$

Since

$$\frac{d\eta_t}{dt} = v_t$$

by the properties of geodesic flow, the rest of the lemma follows from the uniqueness of integral curves. $\square$

Now we show that SPdiff equation can be rewritten as

$$\frac{\partial u_t}{\partial_t} = Y(u_t).$$

First, we notice that by Formula 2.7

$$Y(u) = -\tilde{K}[Tu \cdot u],$$

where $\tilde{K}$ is a connector map of the weak Riemannian metric on $\mathcal{D}^s$. By Formula 5.5 for arbitrary $v \in \mathcal{D}^s$

$$\left\langle \tilde{K}[Tu \cdot u], v \right\rangle = \langle u, [u, v] \rangle . \tag{5.12}$$

In Euclidean coordinates the expression 5.12 yields (after integration by parts)

$$\begin{aligned}
\left\langle \tilde{K}[Tu \cdot u], v \right\rangle &= \int_M (\Lambda u)^i (u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j})\mu \\
&= -\int_M u_j \frac{\partial}{\partial x^j} (\Lambda u)^i v^i \mu - \int_M (\operatorname{div} u)(\Lambda u)^i v^i \mu - \int_M \frac{\partial u^i}{\partial x^j} (\Lambda u)^j \mu \\
&= -\int_M \left\langle (u \cdot \nabla)\Lambda u + \operatorname{div} u \Lambda u + (Du)^T \cdot \Lambda u, v \right\rangle_m \mu.
\end{aligned}$$

Hence,

$$Y(u) = \Lambda^{-1} \left[(u \cdot \nabla)\Lambda u + \operatorname{div} u \Lambda u + (Du)^T \cdot \Lambda u \right] . \tag{5.13}$$

If M is not flat, more care is needed.

Lemma 5.10. *For an arbitrary C^1 function $f : M \to \mathbb{R}$,*

$$\int_M \mathcal{L}_u (f \mu) = 0$$

($\mathcal{L}_u$ denotes a Lie derivative).

Proof. Let ψ_t be a flow generated by u on M. Since u vanishes on ∂M, ψ_t maps the interior $\dot{M}$ of M into itself. Therefore, by the change of coordinates formula

$$\begin{aligned}\int_M \mathcal{L}_u(f\mu) &= \int_{\dot{M}} \frac{d}{dt}\psi_t^*(f\mu)\\ &= \frac{d}{dt}\int_{\dot{M}} \psi_t^*(f\mu)\\ &= \frac{d}{dt}\int_{\psi_t(\dot{M})} f\mu\\ &= 0. \quad \square\end{aligned}$$

Keeping in mind this lemma and the formula

$$\mathcal{L}_u(f\mu) = (u \cdot f)\mu + (f \operatorname{div} u)\mu,$$

we obtain from 5.12

$$\begin{aligned}\left\langle \tilde{K}[Tu \cdot u, v\right\rangle &= \langle u, \nabla_u v - \nabla v u\rangle\\ &= \int_M \langle \Lambda u, \nabla_u v\rangle_m \mu - \int_M \langle \Lambda u, \nabla_v u\rangle_m \mu\\ &= \int_M u(\cdot \langle \Lambda u, v\rangle_m - \langle \nabla_u \Lambda u, v\rangle_m)\mu - \int_M \left\langle (Du)^T \cdot \Lambda u, v\right\rangle_m \mu\\ &= \int_M \mathcal{L}_u(\langle \Lambda, \mu\rangle) - \int_M \operatorname{div} u \langle \Lambda u, v\rangle_m \mu - \left\langle \Lambda^{-1}\left[\nabla_u \Lambda u + (Du)^T \cdot \Lambda u\right], v\right\rangle\\ &= -\left\langle \Lambda^{-1}\left[\operatorname{div} u \Lambda u + \nabla_u \Lambda u + (Du)^T \cdot \Lambda u\right], v\right\rangle.\end{aligned}$$

Hence, again, Formula 5.13 holds.

Comparing the expression 5.13 with the EPDiff equation, we see that

$$\Lambda \frac{\partial u_t}{\partial t} = -Y(u_t)$$

as desired. $\square$

Proposition 5.11. *The weak Riemannian connection on $\mathcal{D}^s$ is (s,s)-transposable for $s > n/2 + 2$.*

Proof. Let $u, v, w \in \mathcal{D}^s$ be arbitrary. Calculating as in proposition 5.6 we obtain

$$\begin{aligned}\left\langle v, \tilde{K}[Tw \cdot u]\right\rangle &= \int_M \langle \Lambda v, \nabla_u w\rangle_m \mu - \int_M \langle \Lambda w, \nabla_u v\rangle + \langle A(w,v), u\rangle \\ &= \left\langle \Lambda^{-1} B_1(\Lambda v, w), u\right\rangle - \left\langle \Lambda^{-1} B_1(\Lambda w, v), u\right\rangle + \left\langle \Lambda^{-1} A(w,v), u\right\rangle,\end{aligned}$$

where A, given by 5.11, is a bilinear bounded operator $\mathfrak{X}^s \times \mathfrak{X}^s \to \mathfrak{X}^{s-2}$, B_1 is given in coordinates by

$$B_1(Z,W) = g^{mk} g_{ij} Z^i \left(\frac{\partial W^j}{\partial x^k} + \Gamma^j_{kl} W^l\right) \quad \forall Z, W \in \mathfrak{X}^s.$$

It's clear that $B : \mathfrak{X}^s \times \mathfrak{X}^s \to \mathfrak{X}^s$,

$$B(v,w) = \Lambda^{-1}\left[B_1(v,w) - B_1(w,v) + A(v,w)\right]$$

is bounded. □

Theorem 5.12. *The flow $\tilde{F}$ generated by EPDiff equation on the space $\mathfrak{X}^s$ ($s > n/2+2$) is Poisson with respect to the Lie Poisson bracket 4.2, i.e., for all $v \in \mathfrak{X}^{s+2}$, $f, h \in \mathcal{K}^{1,s}(\mathfrak{X}^s)$*

$$\left\{f \circ \tilde{F}_t, h \circ \tilde{F}_t\right\}_+ (v) = \{f, h\}_+ \circ \tilde{F}_t(v).$$

Proof. As we showed above,

$$\tilde{F}_t = \pi \circ F_t,$$

where F_t is a flow of the smooth geodesic spray on $\mathcal{D}^s$. Since the connection on $\mathcal{D}^s$ is (s,s)-transposable, the claim follows from the Theorem 3.24. □

Chapter 6

Conclusions

In the thesis we successfully implemented the non-smooth Poisson reduction technique for the study of the equations that, in the Lagrangian representation, are generated by the smooth spray on the various groups of diffeomorphisms. This enabled us to find a precise sense in which the flows generated by these equations on the corresponding Lie algebras are given by Poisson maps. The existence of such Poisson structures is quite remarkable since the flows under the investigation are not differentiable both in time and initial conditions (on the functional spaces natural for the problem).

An important point is that we study the flows directly on the tangent bundle, where there are existence and uniqueness theorems for the corresponding PDEs. Hence, the results presented in this work translate directly into the statements on the behavior of integral curves of the equations. This might be very useful for the study of the global behavior and in particular stability of the PDEs (see Mielke [2002]).

We develop the non-smooth reduction theory for the right action of a diffeomorphism group on itself, utilizing the special form of the reduction map. There is a hope to generalize the results to the cases where a diffeomorphism group acts on an arbitrary Riemannian manifold by right composition. That would allow applications to the free boundary problems, a notoriously difficult case for infinite dimensional Poisson structures even at the formal level.

An interesting system for which our results are not applicable directly is the KDV equation on S^1. It's a known fact (Ovsienko and Khesin [1987], Khesin and Misiolek

[2003]) that the KDV equation is a reduction to the identity of the geodesic flow of the right-invariant L^2 metric on the Bott-Virasoro group. Unfortunately, the Levi-Civita connection of this metric is not smooth.

The solutions that we study are, in a sense, regular. Even though they are not differentiable, the energy remains finite at all times. One issue which need to be addressed in the future is that of singular solutions. For example, the solutions of Camassa-Holm equation do lose regularity in finite time (Camassa and Holm [1993]), which is related to the fact that diffeomorphism groups are not geodesically complete manifolds. Even for the ideal Euler equations, this is interesting because of the singular solutions, such as point vortices, vortex filaments and sheets. They clearly have themselves an interesting Poisson structure, as was investigated by Marsden and Weinstein [1983] and Langer and Perline [1991]. There are similar singular solutions for the EPDiff equations, whose geometry is investigated in Holm and Marsden [2003]. It would be very interesting if the smooth spray property still holds (on the spaces appropriate for these classes of singular solutions), and, if that is the case, whether or not one could then carry out the program in this thesis.

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