Differentially Flat Nonlinear Control Systems

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Abstract

Differentially flat systems are underdetermined systems of (nonlinear) ordinary differential equations (ODEs) whose solution curves are in smooth one-one correspondence with arbitrary curves in a space whose dimension equals the number of equations by which the system is underdetermined. For control systems this is the same as the number of inputs. The components of the map from the system space to the smaller dimensional space are referred to as the flat outputs. Flatness allows one to systematically generate feasible trajectories in a relatively simple way. Typically the flat outputs may depend on the original independent and dependent variables in terms of which the ODEs are written as well as finitely many derivatives of the dependent variables. Flatness of systems underdetermined by one equation is completely characterised by Elie Cartan's work. But for general underdetermined systems no complete characterisation of flatness exists.

In this dissertation we describe two different geometric frameworks for studying flatness and provide constructive methods for deciding the flatness of certain classes of nonlinear systems and for finding these flat outputs if they exist. We first introduce the concept of "absolute equivalence" due to Cartan and define flatness in this frame work. We provide a method of testing for the flatness of systems, which involves making a guess for all but one of the flat outputs after which the problem is reduced to the case solved by Cartan. Secondly we present an alternative geometric approach to flatness which uses "jet bundles" and present a theorem which partially characterises flat outputs that depend only on the original variables but not on their derivatives, for the case of systems described by two independent one-forms in arbitrary number of variables. Finally, for the class of Lagrangian mechanical systems whose number of control inputs is one less than the number of degrees of freedom, we provide a characterisation of flat outputs that depend only on the configuration variables, but not on their derivatives. This characterisation makes use of the Riemannian metric provided by the kinetic energy of the system.

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Chapter 1

Introduction

Historically, determined systems of ordinary differential equations (ODEs), i.e. systems where the number of dependent variables equals the number of ODEs, have been given a lot of attention and their properties have been studied in detail. However, in physics the equations governing any phenomenon are often underdetermined in their original form mainly due to the fact that we isolate a system from its environment and model the effect of the environment by some forces external to the system. For instance when one considers a particle moving in Euclidean space the laws of motion give rise to second order ODEs

$$\ddot{x}_i = P_i, \quad i = 1, 2, 3,$$

and this is an underdetermined system (three equations and six dependent variables) until one assigns the forces P_i to some known functions of time. This point becomes important in control engineering where one considers the above system as a control system where the variables F_i are under the direct control of an engineer. The purpose of analysis is to decide what functions should be assigned to these forces, in other words the question is how to vary P_i in order to achieve some desirable goals.

Underdetermined systems have been studied in the form of control systems by control theorists of the past few decades. Control theorists often separate the dependent variables into two natural groups, one being the "inputs" which are the variables that can be directly varied and the rest as "states" whose evolution is governed by a determined system of first order ODEs once the inputs are assigned to some given functions of time. The study of control systems has led to a whole set of complex properties that are only applicable to underdetermined systems. For instance the notion of controllability roughly deals with being able to find a solution that would take the system from any given initial condition to any given final condition in some given time. On the other hand for a typical determined system, given an initial condition there is only one solution that satisfies it and hence it can never be controllable in this sense. We refer to Sontag [38], Isidori [14] and Nijmeijer and van der Schaft [26] for details on nonlinear control theory. The class of underdetermined systems that are linear after an invertible (nonlinear) change of the dependent variables have also been studied in the past decades and have been classified with the aid of methods from differential geometry. Unlike determined systems, a reasonably large class of real world underdetermined systems are "linearisable" in the above sense and hence in principle, once the appropriate change of variables has been made, methods of linear control theory can be applied to control such systems. Unfortunately, the majority of control systems are still not linearisable in this way. The control of such systems is still very much the subject of research.

Another important property that an underdetermined system of ODEs may possess is "differential flatness" (often referred to as "flatness"). Roughly speaking differentially flat systems are systems whose entire set of solutions are in a smooth one-one correspondence with arbitrary curves in a space of dimension p, equal to the number of equations by which the system is underdetermined. This property allows one to systematically generate solution trajectories in a relatively simple way by translating the problem to the task of finding curves in the lower dimensional space that need to satisfy some conditions only at their end points but are otherwise free.

The notion of differential flatness has been around in some form or another since the time of Cartan and Hilbert, see [6, 7, 13]. In fact Cartan has studied a slightly more general version of the problem where the one-one correspondence may involve a change of the dependent variable as well, where as in differential flatness the independent variable is preserved, see [6]. The precise notion of differential flatness in its current form as well as the terminology was introduced by Fliess and coworkers. See [12, 10] and also Martin [20]. They have also introduced a more general notion akin to the notion of Cartan, called "orbital flatness" which allows change of the independent variable as well. In this thesis we shall only focus on differential flatness and "flatness" shall stand for differential flatness throughout this thesis.

A complete characterisation of flatness does not exist as yet for general class of systems. Cartan's work completely characterises flatness for the case of systems underdetermined by one equation.

1.1 Differentially Flat Systems

Differential flatness is a concept that applies to underdetermined systems of ODEs. A general underdetermined system of ODEs of order k may be written as

$$F^j(t, x, x^{(1)}, \ldots, x^{(k)}) = 0, \quad j = 1, \ldots, N - p, \tag{1.1}$$

where F^j are assumed to be C^∞-smooth functions, $x = (x^1, \ldots, x^N) \in \mathbb{R}^N$ are the dependent variables, t is the independent variable (usually time), $x^{(r)}$ stands for the rth time derivative of x, and p (≥ 1) is the number of equations by which the system is underdetermined.

As an example consider the planar rigid body controlled by two independent forces that pass through a fixed point P in the body as illustrated in Figure 1.1. The rigid body is assumed to move in the vertical plane in the presence of gravity and that P is different from the centre of mass G. Some practical examples that

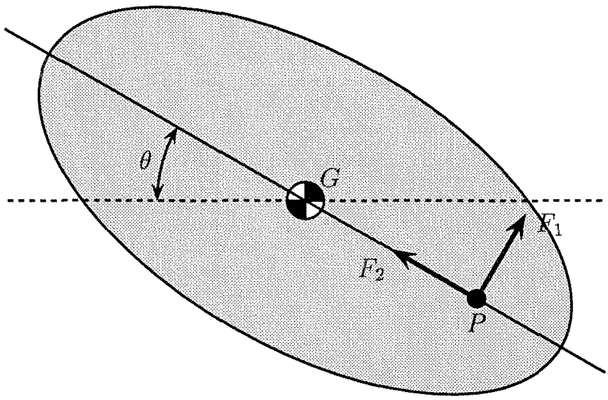


Figure 1.1 Planar rigid body controlled by two independent forces

may be approximately modelled by this system are the fan engine with flaps that can be controlled by motors [40] and the VTOL aircraft [22].

If x_1 and x_2 are the horizontal and vertical coordinates of G, θ the angular orientation, and F_1 and F_2 are the components of forces as shown in Figure 1.1 then the equations of motion can be written as

$$m\ddot{x}_1 \sin\theta + m\ddot{x}_2 \cos\theta - mg\cos\theta = F_1 \tag{1.2}$$
$$m\ddot{x}_1 \cos\theta - m\ddot{x}_2 \sin\theta + mg\sin\theta = -F_2 \tag{1.3}$$
$$I\ddot{\theta} = -F_1 R \tag{1.4}$$

where m is the mass, I the moment of inertia, g the acceleration of gravity and R is the length PG. This is a system of ODEs in the dependent variables x_1, x_2, θ, F_1 and F_2 and hence is underdetermined by two equations. Two is also the number of independent control inputs. If we set any two of the variables to arbitrary functions of time we obtain a fully determined system of ODEs. The set of solutions of this resulting system may typically depend on a number of constants corresponding to some initial conditions. For instance setting F_1 and F_2 to arbitrary functions we obtain a system whose solutions depend on six constants, which are initial conditions for x_1, x_2 and θ and their first time derivatives. Thus we may regard the entire set of solutions $(x_1(t), x_2(t), \theta(t), F_1(t), F_2(t))$ as being parametrised by two arbitrary functions (which specify $F_1(t)$ and $F_2(t)$) and six arbitrary constants (which specify initial conditions for the resulting determined system of ODEs). It is clear that the set of solutions must depend on two arbitrary functions since the system is underdetermined by two equations, but an interesting question is whether the six constants that are needed in addition are really necessary. From a mathematical point of view there is no compelling reason why F_1 and F_2 should be the variables that are assigned to the arbitrary functions. Any pair of variables may be chosen to be the "free variables," i.e. the ones that are to be assigned to arbitrary functions, provided they are "differentially independent." For instance F_1 and θ may not be chosen as the free variables since they are related by the ODE (1.4), in which no other variables are present. In other words they are "differentially dependent."

However we may choose any pair of x_1, x_2 and θ as free variables and then the number of constants needed is only two instead of six because some of the ODEs become purely algebraic equations. In order to simplify the analysis we may eliminate F_1 from Equations (1.2) and (1.4) to obtain the following equation.

$$m\ddot{x}_1 \sin\theta + m\ddot{x}_2 \cos\theta - mg\cos\theta + (I/R)\ddot{\theta} = 0. \tag{1.5}$$

This is also an underdetermined ODE in three dependent variables x_1, x_2 and θ. Furthermore it may be noted that the set of all solutions $(x_1(t), x_2(t), \theta(t), F_1(t), F_2(t))$ of the system consisting of Equations (1.2), (1.3) and (1.4) is in smooth one-one correspondence with the set of solutions $(x_1(t), x_2(t), \theta(t))$ of the system consisting of the single ODE (1.5). This is because given any solution of (1.5), F_1 and F_2 are uniquely determined by the Equations (1.2) and (1.3). Hence we might as well focus on the smaller system given by the single ODE (1.5). If we set any pair of x_1, x_2 and θ to arbitrary functions then the resulting ODE has a set of solutions that depend on two constants and hence it is also clear that the solution set of the full system consisting of Equations (1.2),(1.3) and (1.4) can also be parametrised by two arbitrary functions (which specify a pair of variables amongst x_1, x_2 and θ) and two constants (which specify the initial conditions for the other variable in the set $\{x_1, x_2, \theta\}$).

A natural question is whether there is a clever choice of the two free variables (the ones that are assigned to arbitrary functions) such that no constants are required to parametrise the solution set. This will happen if and only if after these free variables are assigned to some arbitrary functions the resulting determined system of ODEs is degenerate, in the sense that it has only discretely many solutions as opposed to the continuously many solutions for a nondegenerate system. For the above system, no pair of variables from the set $\{x_1, x_2, \theta, F_1, F_2\}$ would achieve this property. However there is no reason to limit ourselves to these five variables. We may more generally look for a pair of variables y_1 and y_2 that are functions of $x_1, x_2, \theta, F_1, F_2$. In fact, the following choice works:

$$\begin{aligned} y_1 &= x_1 - \frac{I}{mR}\cos\theta, \\ y_2 &= x_2 + \frac{I}{mR}\sin\theta. \end{aligned} \tag{1.6}$$

Two arbitrary functions assigned to y_1 and y_2 parametrise the entire set of solutions without the need for additional constants. In order to see this we may rewrite Equation (1.5) in the variables y_1, y_2 and θ. This in fact is given by

$$\ddot{y}_1 \sin\theta + \ddot{y}_2 \cos\theta + g\cos\theta = 0. \tag{1.7}$$

Once y_1 and y_2 are assigned to arbitrary functions the resulting equation for θ is a purely algebraic equation (does not have derivatives of θ). This has only two possible solutions which differ by π.

The variables y_1 and y_2 in above example are called "flat outputs." In the above example the flat outputs were functions of the original variables $x_1, x_2, \theta, F_1, F_2$. In general they are allowed to depend on finitely many derivatives of the original variables. Not all systems possess flat outputs. We may define differential flatness as follows.

Definition 1.1 The system given by (1.1) is said to be *differentially flat* or simply *flat* if there exist variables $y^1, \dots, y^p$ given by an equation of the form

$$y = h(t, x, x^{(1)}, \dots, x^{(m)}) \tag{1.8}$$

such that the original variables x may be recovered from y (locally) by an equation of the form

$$x = g(t, y, y^{(1)}, \dots, y^{(l)}). \tag{1.9}$$

The variables $y^1, \dots, y^p$ are referred to as the *flat outputs.*

Flatness may be regarded as a (local) bijective correspondence between the solutions $x(t)$ of (1.1) and arbitrary curves $y(t)$ in $\mathbb{R}^p$ that is given by the maps h and g of the Equations (1.8) and (1.9). It must be noted that the Equations (1.8) and (1.9) are only required to hold locally. For instance in the planar rigid body example, θ is given implicitly in terms of $\ddot{y}_1$ and $\ddot{y}_2$. Only locally, using implicit function theorem, can we obtain explicit equations such as (1.9). Also singularities are often present in the transformations (1.8) and (1.9). As an example, in the case of planar rigid body, curves $y(t)$ for which $\ddot{y}_1 = \ddot{y}_2 + g = 0$ do not map to unique (not even locally) solutions $x(t)$.

Remark 1.2 The flat outputs of the planar rigid body example were originally discovered by Martin [22]. In that paper the system under consideration was a VTOL aircraft which may be modelled by our planar rigid body system. The flat outputs correspond to the coordinates of a special point on the body known as the "centre of oscillation," which arose historically in the study of pendulums by Huygens.

1.2 Trajectory Generation

Given that the system (1.1) is differentially flat, the problem of generating solution curves of (1.1) that pass through given initial and final conditions $x(t_0)$ and $x(t_1)$, may be solved by translating the problem to the lower dimensional flat output space. For simplicity suppose the flat outputs y depended only on t and x. In the flat output space these conditions become conditions on $y^{(k)}(t_0)$ and $y^{(k)}(t_1)$ for k upto some finite number. We may choose any curve $y(t)$ satisfying these endpoint conditions and then using Equation (1.9) we can obtain trajectories $x(t)$ that pass through the initial and final conditions.

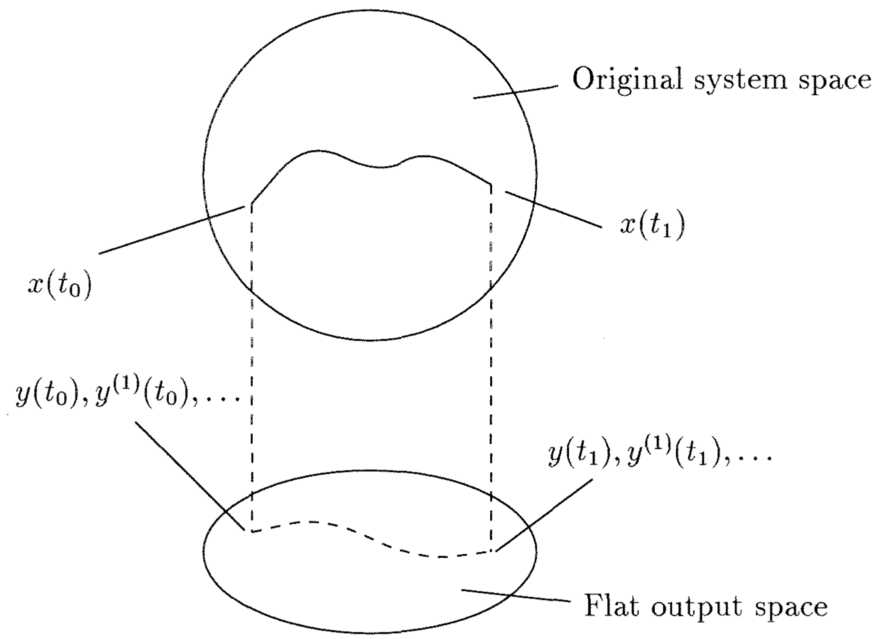


Figure 1.2 Trajectory planning using flat outputs y

We shall illustrate the idea using the planar rigid body example. Suppose we need to find a trajectory for the planar rigid body that starts from some given initial values of configuration, velocity and forces and reaches some final values of configuration, velocity and forces in some given time T. In other words the task is to find a solution $(x_1(t), x_2(t), \theta(t), F_1(t), F_2(t))$ that satisfies some prescribed values for $x_1, x_2, \theta, \dot{x}_1, \dot{x}_2, \dot{\theta}, F_1, F_2$ at time $t = 0$ and at time $t = T$. All such solutions can be mapped to some curves on the flat output space with coordinates (y_1, y_2), which are given by (1.6). It is instructive to write the derivatives of the flat outputs in terms of the original variables x_1, x_2, θ, F_1 and F_2 and their derivatives. We obtain the following equations.

$$
\begin{aligned}
y_1^{(1)} &= \dot{x}_1 + \frac{I}{mR}\dot{\theta}\sin\theta \\
y_2^{(1)} &= \dot{x}_2 + \frac{I}{mR}\dot{\theta}\cos\theta \\
y_1^{(2)} &= -\frac{F_2}{m}\cos\theta + \frac{I}{mR}\dot{\theta}^2\cos\theta \\
y_2^{(2)} &= \frac{F_2}{m}\sin\theta - \frac{I}{mR}\dot{\theta}^2\sin\theta - g
\end{aligned}
$$

$$
\begin{aligned}
y_1^{(3)} &= -\dot{\sigma}\cos\theta + \sigma\dot{\theta}\sin\theta \\
y_2^{(3)} &= \dot{\sigma}\sin\theta + \sigma\dot{\theta}\cos\theta \\
y_1^{(4)} &= -\ddot{\sigma}\cos\theta + 2\dot{\sigma}\dot{\theta}\sin\theta + \sigma\dot{\theta}^2\cos\theta - \frac{R}{I}\sigma F_1\sin\theta \\
y_2^{(4)} &= \ddot{\sigma}\sin\theta + 2\dot{\sigma}\dot{\theta}\cos\theta - \sigma\dot{\theta}^2\sin\theta - \frac{R}{I}\sigma F_1\cos\theta
\end{aligned}
$$

where $\sigma = \frac{F_2}{m} - \frac{I}{mR}\dot{\theta}^2$. It may be shown that the above relations are locally invertible, i.e. there is a (local) diffeomorphism between the variables

$$y_1, y_2, y_1^{(1)}, y_2^{(1)}, \ldots, y_1^{(4)}, y_2^{(4)}$$

and the variables

$$x_1, x_2, \theta, \dot{x}_1, \dot{x}_2, \dot{\theta}, F_1, F_2, \dot{\sigma}, \ddot{\sigma}.$$

Given the prescribed values for $x_1, x_2, \theta, \dot{x}_1, \dot{x}_2, \dot{\theta}, F_1, F_2$ at time $t = 0$ we may assign some arbitrary values for $\dot{\sigma}$ and $\ddot{\sigma}$ after which we obtain unique prescribed values for y_1, y_2 and their first four derivatives at time $t = 0$. The same may be done for time $t = T$. This leaves us with the problem of finding $y(t) = (y_1(t), y_2(t))$ that satisfy these initial and final conditions on derivatives upto fourth order, but are otherwise free. At this stage infinitely many possibilities exist. One may fit some spline curves for instance. Once such a $y(t)$ has been chosen then the corresponding solution curve in the original space may be obtained by first solving for $\theta(t)$ using Equation (1.7), then finding $x_1(t), x_2(t)$ from (1.6) and finally finding $F_1(t), F_2(t)$ from Equations (1.2) and (1.3). This solution curve will satisfy the prescribed initial and final conditions.

Remark 1.3 It is clear from above, that a solution connecting any two generic points in the original system space can be found. Thus flat systems are controllable. We have not proven this but the above procedure contains the ideas for a proof.

Remark 1.4 For a detailed description of the numerical and computational issues of using flatness to generate solution trajectories we refer to van Nieuwstadt and Murray [41] as well as van Nieuwstadt [39].

1.3 Control Systems, Feedbacks and Differential Flatness

In this section we shall give a brief description of some notions from nonlinear control theory and describe their relation to flatness. For a detailed description of these concepts we refer to [26] and [14]. Our description is fairly brief since we shall not explore the relationship between flatness and feedbacks in this thesis. This link has been explored by other researchers in the field and we shall only mention their results.

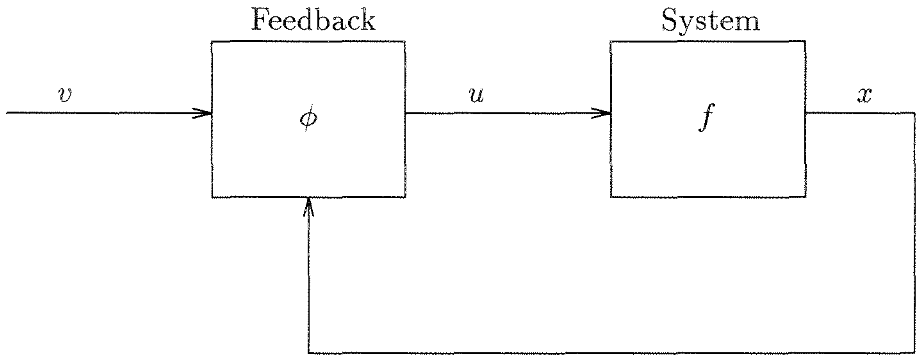


Figure 1.3 Feedback of control systems

Control systems are engineering systems where some variables called "controls" or "inputs" may be varied directly in order to effect changes in rest of the variables of the system called "states." Typically states evolve according to some dynamic principles and this depends on the input variables. We are primarily concerned with control systems whose dynamics are governed by systems of ODEs. The typical model of a nonlinear control system used by control theorists is the system of equations

$$\dot{x}^i = f^i(t, x, u), \quad i = 1, \ldots, n, \tag{1.10}$$

where f^i are assumed C^∞-smooth, $x = (x^1, \ldots, x^n)$ are the states and $u = (u^1, \ldots, u^p)$ are the inputs. Typically $p < n$. Observe that this system is underdetermined by p equations. In engineering practice as well as in analysis, this control system is often modified by adding a "feedback." This may be illustrated by a "block diagram" as in Figure 1.3. The block marked f refers to the control system and the outcome of the block are the states x. The block marked ϕ stands for the feedback. This block may be thought of as an operator that maps a new set of control inputs $v = (v^1, \ldots, v^p)$ and the states x to the old controls $u = (u^1, \ldots, u^p)$. This feedback is called a *static feedback* when the operator ϕ is given by a map $\gamma : \mathbb{R}^{1+n+p} \to \mathbb{R}^p$ in the form

$$u(t) = \gamma(t, x(t), v(t)).$$

The feedback itself may involve some dynamics governed by some ODEs as in the form

$$\begin{aligned} \dot{z}(t) &= \alpha(t, x(t), z(t), v(t)), \\ u(t) &= \beta(t, x(t), z(t), v(t)), \end{aligned} \tag{1.11}$$

where $z = (z^1, \ldots, z^m)$ are called the new states. The composite system may be thought of as a control system with v as its p inputs and (x, z) as its $n+m$ states. This kind of feedback is called a *dynamic feedback*. Observe that when $m = 0$ the dynamic feedback becomes a static feedback and hence static feedback is just a special case of dynamic feedback.

After a static feedback and a possible nonlinear transformation of the state variables some control systems may be expressed in the linear form

$$\dot{\xi} = A\xi + Bv, \tag{1.12}$$

where $\xi = F(x)$ are the new coordinates for the states. Such systems are said to be *feedback linearisable via a static feedback* and have been completely classified in literature, see [14] for instance. Systems that are not static feedback linearisable may still be *dynamic feedback linearisable* in the sense that after a dynamic feedback and a diffeomorphism of the states $\xi = F(x, z)$ they take the linear form (1.12). Classification of dynamic feedback linearisability is still an open problem though classification results exist for special classes of systems. For instance in the case of $p = 1$ dynamic feedback linearisability has been shown to be equivalent to static feedback linearisability, see [8] and [34] for instance. Feedback linearisability is a desirable property since at least in theory after the application of appropriate feedback and coordinate changes the tools of linear control theory may be used to control the system.

Fliess and coworkers have introduced the notion of an endogenous feedback which is essentially a dynamic feedback of the form (1.11) with the added requirement that z and v be uniquely determined as functions of t, x, u and finitely many derivatives of u. They have shown that feedback linearisability via endogenous feedback is equivalent to differential flatness, see [12]. More recently, they have also shown that dynamic feedback linearisability in a more general sense is still equivalent to differential flatness [9].

In terms of control applications there are two ways to regard flatness. One view is to emphasise the feedback linearisability property and develop control schemes that make use of the appropriate feedback that linearises the system. The alternative view is to make use of the fact that flatness enables one to generate feasible trajectories (at least theoretically) in a simple way. In this view point one may not necessarily feedback linearise the system. See van Nieuwstadt and Murray [41] for an example of the latter view point. Since the concern of this thesis is classification of flatness rather than its applications we shall not elaborate on this topic.

1.4 Theoretical Tools and Results

A complete classification of flatness is not available as yet and is likely to require some what sophisticated mathematical tools. For systems underdetermined by one equation ($p = 1$) the classification is complete and the theory is essentially due to Elie Cartan. His approach was to use differential geometry to study systems of ODEs. In particular the use of differential forms to describe ODEs and the use of the powerful tools of "exterior differential systems" developed by Cartan himself, see [4] and [35] for instance.

The notion of differential flatness was introduced to the control community by Fliess and coworkers, originally in the language of "differential algebra"; see [12] for a detailed description. Differential algebra is an area of mathematics which was primarily developed by Ritt. It is essentially an attempt to develop a "Galois theory" of differential equations, see [31] and [16] for instance. The differential algebraic setting requires one to assume that the ODEs are polynomials or at least meromorphic functions of the variables and their derivatives. The theory does provide an elegant setting and brings out some of the key concepts very clearly. However this theory does not provide a convenient framework for local analysis nor does it facilitate the study of singularities. A more geometric theory would be necessary in order to rectify these shortcomings.

Differential flatness may be formulated in terms of the geometric notion of "absolute equivalence" of Cartan which helped him solve the case of $p = 1$ and this is presented in Chapter 2 of this thesis. Also see van Nieuwstadt *et al.* [42] and Sluis [35] for details. But it is unclear whether the tools of Cartan that proved useful for the case of $p = 1$ could also provide a theory for the general $p > 1$ case.

Alternative geometric approaches may be found in the works of Fliess and coworkers [10, 12] and Pomet [27]. Fliess and coworkers as well as Pomet have independently proposed an "infinite dimensional jet bundle" approach to differential flatness. These infinite dimensional spaces have the variables t, x and the derivatives of x of all orders as their coordinates. They also reflect all the relations amongst the derivatives (of all orders) of the variables x (in other words all the ODEs implied by the original system of ODEs). These approaches are in some sense closer to the differential algebraic view and differ from the absolute equivalence approach in a number of ways. The notion of absolute equivalence primarily involves one-one correspondence of solution curves (though the goal is to relate this notion to properties of systems of differential forms expressed in terms of the exterior differential calculus of Cartan) whereas the infinite dimensional jet bundle approach directly talks about transformations that depend on derivatives of variables and does not necessarily use the tools of exterior calculus.

Though there is no general theory of flatness for the $p > 1$ case, several scattered results exist. In addition to the results concerning feedback linearisation already stated, we mention a few more. A known necessary condition for flatness of control systems of the form $\dot{x} = f(x, u)$ (i.e. time independent control systems) is that at every point x on the state space the set of tangent vectors $f(x, u)$ for all u should be a ruled submanifold of the tangent space at the point x and this is due to Rouchon, see [32]. Rouchon also shows with the aid of this condition that flatness is not a generic property of control systems.

Martin and Rouchon have shown that all controllable systems $\dot{x} = f(x, u)$ where f is linear in u (known as driftless systems) with n states and $n - 2$ controls are differentially flat [21]. Recently Pomet has completely classified time independent control affine systems (systems where f is affine in u) in 4 states and 2 inputs that possess flat outputs that depend only on states ($y = h(x)$) as well as those which possess flat outputs that depend only on states and controls but not on derivatives of control ($y = h(x, u)$), see [28].

1.5 Contributions

The main aim of this thesis is to provide results that classify differentially flat systems and methods for finding flat outputs. The value of using the correct tools to study a mathematical problem cannot be underestimated. In this thesis we shall use two different geometric approaches to study flatness. Both approaches have their merits.

We assume a basic knowledge of concepts from differential geometry such as manifolds, tangent spaces, vector fields, differential k-forms, distributions, codistributions, and pull backs of forms and codistributions. Suggested references for this material are Nijmeijer and van der Schaft [26] and Abraham *et al.* [2].

In Chapter 2 we explain the differential forms approach to systems of ODEs. We also explain the notions of "Cartan prolongations" and "absolute equivalence" as well as other related concepts due to Cartan and present a definition of flatness in terms of these concepts and relate it to the nominal definition 1.1. Cartan has completely solved the problem of flatness for systems underdetermined by one equation ($p = 1$) and we summarise this result in terms of concepts from exterior differential systems. We also propose a method of testing for the flatness of systems underdetermined by more than one equation ($p > 1$) which involves making a guess for all but one of the flat outputs after which the problem is reduced to the one solved by Cartan. With the aid of the theory of absolute equivalence we demonstrate the validity of this method. We illustrate the method by two examples one of which is nontrivial and was not knowe to be flat before. In this chapter, we also point out some drawbacks in the current version of the notion of a Cartan prolongation.

In Chapter 3 we present an alternative, geometrical framework for studying differential flatness which does not have the drawbacks of the theory of Chapter 2. This approach involves "jet bundles." In other words we use a sequence of spaces that have the independent variable t, the dependent variables x as well as finitely many derivatives of x upto a certain order as their coordinates. This approach is related to the approaches of Fliess and coworkers [10, 11] and of Pomet [27], a major difference is that we work with finite dimensional spaces while these other approaches use an infinite dimensional space. We introduce several concepts and lemmas that are relevant for a clear understanding of these spaces and these also help us define differential flatness. We also introduce the notion of "zero-flatness" which is a restricted form of flatness where flat outputs are allowed to depend only on the original variables t and x but not on derivatives of x. In general the flat outputs can depend on higher order derivatives of x. It is not known whether there is an upper limit (which may depend on the size of the system) such that for a flat system it is always possible to find flat outputs that do not depend on derivatives of order higher than this upper limit. We believe that studying zero-flatness may be a useful first step towards understanding differential flatness in general. In this chapter we present a theorem which characterises on an infinitesimal level the level sets of the zero-flat outputs of a system that may be modelled by two independent one-forms. In other words we find necessary and sufficient conditions on the tangent spaces of the level sets. Our theorem leads to PDEs to be solved for flat outputs and these PDEs may not always have solutions. We do not provide any theory on the existence of solutions to these PDEs.

In Chapter 4 we concentrate on Lagrangian mechanical control systems whose number of control inputs is one less than the number of degrees of freedom. With the aid of a jet bundle formulation we introduce the notion of "configuration flatness" which is the same as flatness but where the flat outputs depend only on the configuration variables q but not on their derivatives, i.e. $y = h(q)$. Under certain assumptions on the nature of the kinetic energy and of the control forces we obtain a result which completely characterises configuration flat outputs for systems with one fewer controls than number of degrees of freedom. Our result gives a constructive approach for finding configuration flat outputs and we illustrate the method by two examples.

Finally in Chapter 5 we make some concluding remarks and discuss future directions of research.

Chapter 2

Absolute Equivalence and Differential Flatness

In this chapter we shall introduce the notions of *Pfaffian systems*, *Cartan prolongations* and *absolute equivalence* and provide a definition of differential flatness in terms of absolute equivalence.

The basic approach taken here is due to Elie Cartan. Cartan took a differential forms approach to study systems of ODEs as well as PDEs. The powerful theory of exterior differential systems primarily developed by Cartan provides an elegant geometric theory of PDEs. See [4] and [5] for details. Using the same tools, Cartan also undertook the study of systems of ODEs underdetermined by one equation and solved the problem of absolute equivalence for such systems. See Chapter 2 of [35], [6] and [7] for details. The thesis of Sluis [35] explores Cartan's notion of absolute equivalence for systems of ODEs underdetermined by arbitrary number of equations (more than one).

In this chapter, with the aid of some important results due to Sluis [35], we shall establish a definition of differential flatness in terms of absolute equivalence. Our approach differs from that of Cartan's in the following way. Cartan did not distinguish between independent and dependent variables, since he was studying very general change of variables of the form $(t, x) \mapsto (\tau = \psi(t, x), X = \phi(t, x))$ where the independent variable was not necessarily preserved (i.e. $\tau \neq t$). Since the notion of differential flatness concerns transformations that keep the independent variable unchanged, we shall impose a special notion of time (independent variable) and our definitions of Cartan prolongations and absolute equivalence are accordingly more restrictive. See van Nieuwstadt, Rathinam et al. [42] and Rathinam and Sluis [30] for additional information.

We summarise the characterisation of flatness for systems of ODEs underdetermined by one equation and present a method of testing for flatness of systems underdetermined by more than one equation. This method involves making guesses for all but one of the flat outputs, after which the problem is reduced to the one solved by Cartan. We illustrate the method by a simple example and also provide an additional example of two planar coupled rigid bodies controlled by three inputs. We demonstrate the validity of this method using the definition of flatness in terms of absolute equivalence.

Cartan prolongations provide a nice geometric way to visualise differentially flat systems and the corresponding transformations. They also allow us to prove certain useful results such as the method of testing for flatness presented in this chapter. However there are important drawbacks in this approach that basically stem from an inadequate definition of Cartan prolongations. These are mentioned in remarks 2.9, 2.10 and 2.11.

Some of the results and examples in this chapter are joint work with Michiel van Nieuwstadt and Willem Sluis. See [30] and [42] for additional information.

2.1 Differential Forms Approach to Systems of ODEs

Throughout this chapter, time t denotes the standard coordinate on $\mathbb{R}$. Maps between manifolds and objects such as forms, vector fields etc. on manifolds are assumed to be C^∞-smooth and submanifolds are assumed to be regular.

A *Pfaffian system* I on a manifold M is a submodule of the module (over the ring of smooth functions on M) of all one-forms. We shall only consider finitely generated Pfaffian systems, i.e. those for which local to any point on M a finite basis of one-forms can be found. Since we are only interested in changes of coordinates that preserve time, i.e. those of the form $(t, x) \mapsto (t, X = \psi(t, x))$, we shall be dealing with manifolds M equipped with a notion of time given by a map $\pi : M \to \mathbb{R}$ which is a submersion and we shall refer to the triple (M, π, I) as a system. The time coordinate on M is $\pi^*t = t \circ \pi$, which we shall often write as t for notational ease. Also we may refer to I as a system, the underlying manifold M together with π being understood from the context. For $q \in M$, the *codimension at* q of the system is $\dim M - \dim I(p)$. A system is *trivial* if $I = \{0\}$.

Let $S = (M, \pi, I)$ be a system. A *solution* of S is a curve $c : (a, b) \to M$ such that $\pi \circ c = \mathrm{id}$ (in other words c is a section of the bundle $\pi : M \to \mathbb{R}$) and $c^*(I) = \{0\}$ (in other words $\langle \alpha, \frac{dc}{dt} \rangle = 0$ for all forms α in I). Since c is a section it follows that c is an immersion and that the image of the solution, $c(a, b)$, is a submanifold of M.

To see the connection with systems of differential equations, let us consider a coordinate system $(t, x^1, \dots, x^N)$ on an open set $U \subset M$. Suppose that $\{\omega^1, \dots, \omega^n\}$ ($n < N$) is a set of linearly independent generators of I and in local coordinates let

$$\omega^i = \sum_{j=1}^{N} a_{i,j}(t, x)dx^j + a_{i,0}(t, x)dt, \quad i = 1, \dots, n. \tag{2.1}$$

A solution is given by functions $(x^1(t), \dots, x^N(t))$ that satisfy the following underdetermined system of differential equations ($n < N$):

$$\sum_{j=1}^{N} a_{i,j}(t, x)\dot{x}^j + a_{i,0}(t, x) = 0, \quad i = 1, \dots, n. \tag{2.2}$$

In general, this system cannot be put in the familiar form of a control system $\dot{x} = f(t, x, u)$ by a coordinate change. However, $I + \mathrm{span}\{dt\}$ is locally integrable in the sense of Frobenius if and only if in suitable local coordinates $(t, x^1, \dots, x^n, u^1, \dots, u^p)$, equation (2.2) take the form $\dot{x} = f(t, x, u)$ for a control system with p inputs; see

[35]. Note that this system has codimension $N - n + 1 = p + 1$.

Remark 2.1 If in some open set, $dt \in I$, then there cannot be any solutions in that open set, since by definition, for a section c, $\langle dt, \frac{dc}{dt} \rangle = 1$. Suppose there exists some function λ, such that $\lambda dt \in I$ in some open set, then clearly all solutions in that open set lie on the subset given by $\lambda = 0$. This corresponds to one of the ODEs degenerating to an algebraic equation.

Thus we see that Pfaffian systems correspond to quasi-linear (linear in derivatives) systems of ODEs. In fact it is possible to relate any nonlinear system of ODEs to a Pfaffian system on a suitable manifold. Indeed any system of ODEs in independent variable t can be rewritten in the first order form as

$$F^k(t, x^1, \ldots, x^N, \dot{x}^1, \ldots, \dot{x}^N) = 0 \quad k = 1, \ldots, n < N.$$

In order to relate this to a system of differential forms one rewrites these as quasi-linear ODEs by introducing extra variables p^j for $j = 1, \ldots, N$ which stand for $\dot{x}^j$. Then we have the linear system of ODEs

$$\dot{x}^j = p^j, \quad j = 1, \ldots, N$$

together with algebraic equations $F^k(t, x^1, \ldots, x^N, p^1, \ldots, p^N) = 0$. Assuming the set described by $F = 0$ is a manifold, which we shall denote by M, the Pfaffian system to be studied is given by

$$\text{span}\{dx^j - p^j dt\}, \quad j = 1, \ldots, n$$

which should be restricted to manifold M. Finding suitable coordinates on M can be tedious depending on functions F^k, however. But since most systems of interest (especially mechanical systems) are already in quasi-linear form we don't have to face this difficulty often.

Simple equivalence between two systems is achieved by diffeomorphisms. More precisely two systems (M_1, π_1, I_1) and (M_2, π_2, I_2) are said to be *equivalent* if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that $\pi_1 = \pi_2 \circ \phi$ and $\phi^*(I_2) = I_1$. The first condition ensures that the notion of time is preserved by the diffeomorphism. Clearly two systems can be equivalent only if they live in spaces of the same dimension.

2.2 Cartan Prolongations and Absolute Equivalence

Cartan prolongations give rise to a more general notion of equivalence between systems that live in spaces of possibly different dimensions.

In order to explain Cartan prolongations we first introduce the notion of a *morphism* (see also [30]). Let $S_1 = (M_1, \pi_1, I_1)$ and $S_2 = (M_2, \pi_2, I_2)$ be two systems. A *morphism* from S_1 to S_2 is a surjective submersion $\phi : M_1 \to M_2$, with the following properties:

1. $\pi_2 \circ \phi = \pi_1$,

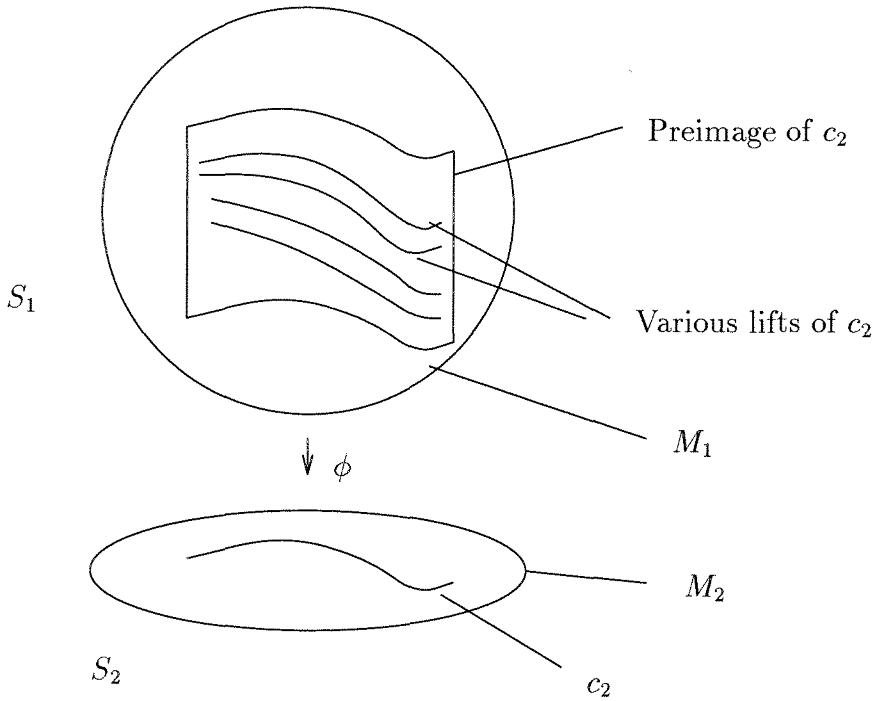


Figure 2.1 Morphism from S_1 to S_2.

2. $\phi^*(I_2) \subset I_1$.

For a curve c_2 in M_2, a *lift* of c_2 is a curve c_1 in M_1 such that $c_2 = \phi \circ c_1$.

Since ϕ is a surjective submersion it follows that M_1 is fibred over M_2. We refer to M_1 as the *full space*, M_2 as the *base space*, I_1 as the *full system* and I_2 as the *base system*. Condition 1 ensures that the notion of time is the same for both M_1 and M_2. Condition 2 says that the base system I_2 is contained in the full system I_1 after being pulled back to the full space. It also follows from the second condition that the dynamics of S_1 projects into the dynamics of S_2 in the sense that every solution curve c_1 of S_1 projects via ϕ to a solution curve c_2 of S_2. On the other hand, a solution of S_2 may have 0, 1 or many lifts to a solution of S_1, depending on the solution and the morphism ϕ. This is described in Figure 2.1.

Remark 2.2 Roughly speaking if one takes a system I_2 on manifold M_2 and adds extra differential forms and extra coordinates one gets a bigger system I_1 living in a space M_1 that is fibred over M_2. The corresponding projection is a morphism from the bigger system to the smaller one. In terms of ODEs, this is equivalent to adding extra quasi-linear ODEs and introducing extra variables.

Example 2.3 Consider the control system in one state x and single control u given by

$$\dot{x} = u.$$

This corresponds to a system $I_2 = \text{span}\{dx - udt\}$ in coordinates (t, x). Consider adding a dynamic feedback given by

$$\begin{aligned} \dot{z}^1 &= z^2 \\ \dot{z}^2 &= -z^1 \\ u &= g(z)v \end{aligned}$$

where g is a nonvanishing scalar function, z^1, z^2 are additional new states and v the new control input. The system including the feedback corresponds to the Pfaffian system

$$I_1 = \text{span}\{dx - g(z)vdt, dz^1 - z^2dt, dz^2 + z^1dt\}$$

in coordinates (t, x, v, z^1, z^2). (Note that v and u are not independent.)

Let ϕ be the projection that maps (t, x, v, z^1, z^2) in the full space to $(t, x, u = g(z)v)$ in the base space. Then clearly we see that ϕ is a morphism from the full system (with feedback) to the original base system. For any solution $(x(t), u(t))$ of the system I_2 there are infinitely many (in fact a two parameter family of) solutions $(x(t), v(t) = u(t)/g(z(t)), z^1(t), z^2(t))$ of I_1 that depend on $z^1(0)$ and $z^2(0)$.

Definition 2.4 Let ϕ be a morphism from $S_J = (B, \pi_B, J)$ to $S_I = (M, \pi_M, I)$. Then S_J is a *Cartan prolongation* of S_I *via* ϕ if every solution c of S_I has a unique lift to a solution $\tilde{c}$ of S_J. We say S_J is a *Cartan prolongation* of S_I if there exists a morphism ϕ from S_J to S_I such that S_J is a Cartan prolongation of S_I via ϕ. See figure 2.2.

Example 2.5 Consider the same control system in the Example 2.3 given by I_2 but instead add the following dynamic feedback:

$$\begin{aligned} u &= z \\ \dot{z} &= v. \end{aligned}$$

In control engineering terminology we have added an integrator in front of the control input u. Here we have turned u into a new state denoted by z and our new control v is the derivative of u. The full system including feedback is given by

$$J = \text{span}\{dx - zdt, dz - vdt\}$$

in coordinates (t, x, z, v). If ϕ is the projection that maps (t, x, z, v) to (t, z) (same as (t, u)), then firstly we see that ϕ is a morphism from the system J to I_2. Given any solution $(x(t), u(t))$ of system I_2 there is a unique solution $(x(t), z(t) = u(t), v(t) =$

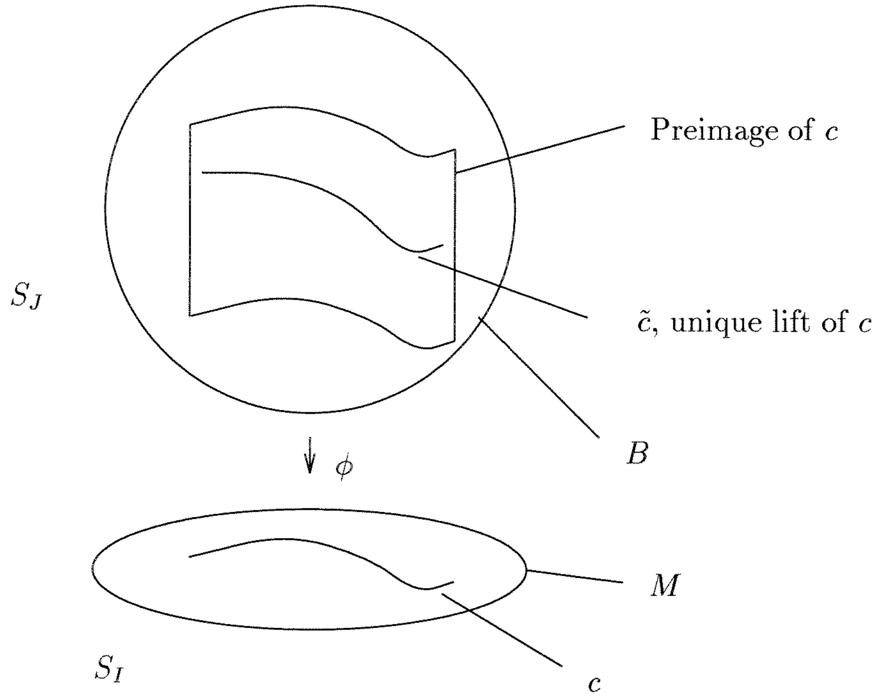


Figure 2.2 A Cartan prolongation S_J of S_I.

$\dot{u}(t)$) of J that projects down to it. Hence we see that J is a Cartan prolongation of I_2 via ϕ.

More generally, given a general control system

$$I = \text{span}\{dx^i - f^i(t,x,u)dt\}, \quad i = 1, \ldots, n \tag{2.3}$$

in n states and p controls, "adding integrators" to a partial subset of inputs $u^1, \ldots, u^s$ ($s \leq p$) gives rise to the system

$$J = \text{span}\{dx^i - f^i(t,x,u)dt, du^j - v^j dt\}, \quad i = 1, \ldots, n,\ j = 1, \ldots, s,$$

which is a Cartan prolongation of I. This type of Cartan prolongation is termed *prolongation by differentiation.* Let M denote the manifold on which I lives (M has coordinates (t, x, u)) and B denote the manifold on which J lives (B has coordinates (t, x, u, v)). We see that B is fibred over M with $(v^1, \ldots, v^s)$ being a choice of fibre coordinates. Notice that v^j for $j = 1, \ldots, s$, correspond to time derivatives of some coordinates on the base manifold M, namely u^j for $j = 1, \ldots, s$. This explains the terminology "prolongation by differentiation." Repeating this procedure by differentiating some of the v^j, say $j = 1, \ldots, s_1$ ($s_1 \leq s$) and so on also leads to Cartan prolongations which are all referred to as *repeated prolongations by differen-*

tiation. When one adds integrators to all inputs (i.e. $s = p$), the resulting Cartan prolongation is called a *total prolongation.* Total prolongations have nice geometric properties. In fact any given system (M, π, I) has a unique total prolongation, whereas its Cartan prolongations are not unique. In some of the literature these are simply referred to as *prolongations.* In Chapter 3 we will be developing the language of total prolongations where we will simply refer to them as prolongations.

Not all Cartan prolongations are prolongations by differentiation, as illustrated by the following example (see also [35]).

Example 2.6 Consider the general control system I given by (2.3) on manifold M with coordinates $(t, x^1, \ldots, x^n, u^1, \ldots, u^p)$ and consider adding the differential form $du^1 - u^2 du^3 - vdt$ and the extra coordinate v. This results in a Cartan prolongation since given any solution $(x(t), u(t))$ of the base system there is a unique lift to a solution of the full system given by $(x(t), u(t), v(t) = \dot{u}^1(t) - u^2(t)\dot{u}^3(t))$. We shall show by method of contradiction that this does not correspond to a prolongation by differentiation. Suppose this does correspond to a prolongation by differentiation. Then there exist a function $f(t, x, u)$ on the base space and a choice of fibre coordinate $z = z(t, x, u, v)$ ($\frac{\partial z}{\partial v} \neq 0$) such that z is the derivative of f along solutions of the full system. This means $df - zdt$ belongs to the full system. Which is the same as

$$df - zdt = \lambda(du^1 - u^2 du^3 - vdt) + \alpha,$$

where $\lambda(t, x, u, v)$ is a nonzero function and α is in the span of I with coefficients possibly functions on the full space. This implies

$$\lambda\mu = df + \beta$$

where $\mu = du^1 - u^2 du^3$ and $\beta \in \text{span}\{dt, dx^1, \ldots, dx^n\}$ with coefficients that may depend on (t, x, u, v). Taking d on both sides and the exterior product (wedge) with μ we see that

$$\lambda d\mu \wedge \mu = d\beta \wedge \mu.$$

Expanding β in terms of $dt, dx^1, \ldots, dx^n$ and μ in terms of du^1, du^3 we see that the above equation cannot be true since the right hand side has no term $du^1 \wedge du^2 \wedge du^3$ where as the left hand does. Hence this is a Cartan prolongation which is not a prolongation by differentiation.

In fact above reasoning shows that if we add the differential form $\mu - vdt$ to the system I given by (2.3) where μ is any form on M (i.e. is expressible in coordinates (t, x, u)) and does not belong to $I + \text{span}\{dt\}$ (in other words has some non trivial du^j components) then we obtain a Cartan prolongation. Furthermore this is a prolongation by differentiation only if

$$d\mu \wedge \mu = 0 \mod dt, dx^1, \ldots, dx^n,$$

or equivalently

$$d\mu = 0 \mod dt, dx^1, \ldots, dx^n, \mu.$$

(See Remark 2.7 for an explanation of this notation.) By applying Frobenius theorem it can be seen that this condition is also sufficient.

Remark 2.7 In exterior algebra one often uses the "mod" notation which is defined as follows. Let α and β be two differential k-forms, k being some integer. Let $\omega^1, \ldots, \omega^n$ be a set of differential forms with degrees $k_1, \ldots, k_n$ (where $k_i \leq k$ for each i) respectively. Then one writes

$$\alpha = \beta \mod \omega^1, \ldots, \omega^n,$$

to mean, that there exist forms $\gamma_1, \ldots, \gamma_n$ with degrees $k-k_1, \ldots, k-k_n$ respectively, such that

$$\alpha = \beta + \gamma_i \wedge \omega^i,$$

where summation is implied.

The type of prolongations mentioned in Example 2.6 do not exhaust all the possible Cartan prolongations where the fibre dimension is one, as illustrated by the following example.

Example 2.8 Consider adding the one-form $du^1 - vdu^2$ and the fibre coordinate v to the general control system (2.3). Every solution $(x(t), u(t))$ of the base system lifts to the unique solution $(x(t), u(t), v(t) = \frac{\dot{u}^1(t)}{\dot{u}^2(t)})$, provided $\dot{u}^2(t) \neq 0$. This corresponds to a singularity.

Remark 2.9 In practice singularities of the type in Example 2.8 are common. Strictly speaking, this example is not a Cartan prolongation according to our definition since not all base solutions have a unique lift. Specifically those for which $\dot{u}^2(t) = 0$, do not have a unique lift. But a "generic" solution has a unique lift. However encooperating this idea into the definition of Cartan prolongations would require addressing the difficult issue of genericity in the set of solution curves which we shall avoid. It is also clear that singularities do not correspond to points on the base space but to solution curves on the base space. In fact genericity and singularities are easily addressed in the jet bundle formulation in Chapter 3. It is easier in that formulation because jets of solution curves (equivalence classes of solution curves that agree upto some finite number of derivatives) is a much nicer set to deal with than the set of all solution curves or even the set of all germs of solution curves.

Remark 2.10 In practice base solution curves often have more than one lift, but yet the lifts are all isolated, i.e. they are locally (in the full space) unique. For instance adding the differential form $du^1 - g(v)dt$, where $g : \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth function, and the extra coordinate v to the control system (2.3) results in a Cartan prolongation where the fibre coordinate v of the lift of a base solution $(x(t), u(t))$ satisfies $g(v(t)) = \dot{u}^1(t)$. Hence $v(t)$ is implicitly defined and typically may have many solutions which are isolated. To make matters more complicated, for some values of $\dot{u}^1(t)$ there may be no solutions at all. For instance if $g(v) = (v)^2$ then there are two isolated lifts, a unique lift or no lifts at all, depending on whether $\dot{u}^1(t)$ is positive, zero or negative. Examples of this nature make it clear that a better definition of Cartan prolongation should address these local issues. But in this Chapter we shall work with the given definition and shall investigate an alternate approach to flatness via jet bundles in chapter 3 where such local issues are dealt with in a better way.

Remark 2.11 It must also be noted that we shall assume that Cartan prolongations preserve codimension. In other words a system and its Cartan prolongations have the same codimension except at singular points where they may drop rank. We are not aware of a proof of this and a better definition of Cartan prolongations may be a prerequisite to prove this. Most developments in this chapter rely on some important results due to Sluis [35], where codimension preservation is implicitly assumed.

We need to define the notion of a derived system of a Pfaffian system before we present a theorem due to Sluis.

Definition 2.12 Let I be a Pfaffian system on a manifold M. The *derived systems* of I are $I^{(0)} = I$ and, for each $k \geq 0$,

$$I^{(k+1)} = \{\omega \in I^{(k)} \ : \ d\omega = 0 \mod I^{(k)}\}.$$

Calculating derived systems only involves differentiation and linear algebra and poses no problems for concrete examples.

If a system (B, π_B, J) is the total prolongation of some system (M, π_M, I), then it follows that $J^{(1)} = \phi^* I$ where ϕ is the projection from B to M. For a proof see [35][Chapter 3]. Hence if a given system happens to be the total prolongation of some other, taking derived system will help "strip off" that total prolongation. But if a given system is not the total prolongation of some other system then taking derived systems does not seem to provide much information.

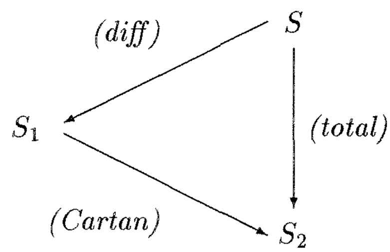
For systems with codimension 2 (i.e. $p = 1$), there is a complete theory of Cartan prolongations, due to Elie Cartan. For the general case of $p > 1$, not much is known about Cartan prolongations except for an important result due to Sluis, which is the Theorem 24 in [35], and our definition of flatness makes sense because of this theorem. Before we restate it, we need the following notion of regularity for a Pfaffian system I on some manifold M.

Definition 2.13 A Pfaffian system I on manifold M is *degree-two-regular* if the following conditions are satisfied.

1. The retraction space is maximal, i.e. one cannot find generators for I that are expressible in fewer coordinates than a full coordinate system on M.

2. The degree two part of the algebraic ideal of differential forms (of all degrees) generated by $I^{(j)}$ have constant rank for all j.

Theorem 2.14 (Sluis) *Suppose $S_1 = (B, \pi_B, J)$ is a Cartan prolongation of $S_2 = (M, \pi_M, I)$ and that Pfaffian systems I and J are degree-two-regular. Then on an open dense subset of B there exists a repeated prolongation by differentiation of S_1 that is also a repeated total prolongation of S_2, the upper bound on the number of repeated total prolongations of S_2 that are required being equal to the fibre dimension of B over M. See diagram below.*



Proof: See [35], Theorem 24. ■

An immediate corollary is the following (see also [42]):

Corollary 2.15 *Suppose $S_1 = (B, \pi_B, J)$ is a Cartan prolongation of $S_2 = (M, \pi_M, I)$ with I and J being degree-two-regular. Then on an open and dense subset of B the fibre coordinates of B over M are functions of coordinates on M and finitely many derivatives of these coordinates, the upper limit on the order of derivatives being equal to the fibre dimension of B over M.*

Remark 2.16 Theorem 2.14 essentially says that the seemingly more general requirement of Cartan prolongations having one-to-one correspondence at the level of solution curves actually means that the prolongations are in local coordinates given by differentiation and algebra. This enables us to define differential flatness in terms of Cartan prolongations and relate them to the rough definition given in Chapter 1 in local coordinates, i.e. Definition 1.1. The upper bound on the number of derivatives is also a very important result and some results presented in chapter 3 rely on this. Even though this upper bound is not mentioned as part of the theorem in [35] it is clear from the proof.

Remark 2.17 It is not clear whether the open dense subset is critical for the Theorem 2.14 to be true. We believe it is partly due to the way in which Cartan prolongations are defined. Also the importance of the degree-two-regularity conditions is not well understood.

There is a simple one-to-one correspondence between solutions of a system and its Cartan prolongations, given by the unique lifting property. Hence a system is "equivalent" to any of its Cartan prolongations in this broader sense. The most general notion of equivalence due to Cartan, called absolute equivalence is achieved by combining simple (diffeomorphic) equivalence and Cartan prolongations together and applies to systems that may live in different dimensional spaces.

Definition 2.18 Systems (M_1, π_{M_1}, I_1) and (M_2, π_{M_2}, I_2) are *absolutely equivalent* if there exist respective Cartan prolongations (B_1, π_{B_1}, J_1) and (B_2, π_{B_2}, J_2) that are equivalent.

Remark 2.19 Note that the term *equivalence* is strictly reserved for equivalence via diffeomorphisms.

2.3 Differential Flatness

In this section we present a definition of differential flatness that in local coordinates corresponds to the Definition 1.1 of Chapter 1. Our definition makes use of the concept of an absolute morphism (see [35]).

Definition 2.20 An *absolute morphism* from a system $S_1 = (M_1, \pi_1, I_1)$ to a system $S_2 = (M_2, \pi_2, I_2)$ consists of a Cartan prolongation $S_3 = (M_3, \pi_3, I_3)$ of S_1 together with a morphism ϕ from S_3 to S_2.

Definition 2.21 Two systems $S_1 = (M_1, \pi_1, I_1)$ and $S_2 = (M_2, \pi_2, I_2)$ are said to be *absolutely morphic* if there exist absolute morphisms from S_1 to S_2 and from S_2 to S_1. Suppose two systems S_1 and S_2 are absolutely morphic with S_3 and S_4 being Cartan prolongations of S_1 and S_2 respectively, via ψ_1 and ψ_2 respectively. Let $\phi_1 : S_4 \to S_1$ and $\phi_2 : S_3 \to S_2$ be the corresponding morphisms. Then S_1 and S_2 are said to be *invertibly absolutely morphic* if the following inversion property holds. Let c_1 be an integral curve of S_1 with $\tilde{c}_1$ being its unique lift to an integral curve of S_3. Let $c_2 = \phi_2 \circ \tilde{c}_1$ be the projection of $\tilde{c}_1$ and let $\tilde{c}_2$ be the unique lift of c_2 to a solution of S_4. Then we require that $\phi_1 \circ \tilde{c}_2 = c_1$. The equivalent condition must hold for solution curves of S_2 as well. See Figure 2.3.

We are now ready to give a definition of differential flatness.

Definition 2.22 (Differential Flatness) A system (M, π_M, I) is *differentially flat* if it is invertibly absolutely morphic to some trivial system $(N, \pi_N, \{0\})$. If $(t, y^1, \ldots, y^p)$ are local coordinates on N then $(y^1, \ldots, y^p)$ are a set of *flat outputs*.

In local coordinates $(t, x^1, \ldots, x^N)$ on M and local coordinates $(t, y^1, \ldots, y^p)$ on N it follows, from Lemma 2.15 and above definition that (also see [42]) the one-one correspondence between solutions is, on an open dense set, given by equations of the form

$$\begin{aligned} x(t) &= g(t, y(t), y^{(1)}(t), \ldots, y^{(l)}(t)), \\ y(t) &= h(t, x(t), x^{(1)}(t), \ldots, x^{(m)}(t)). \end{aligned} \tag{2.4}$$

Conversely starting from a one-to-one correspondence (2.4) between solutions $x(t)$ of a system (M, π_M, I) and arbitrary curves $y(t)$ that are sections of some bundle $\pi_N : N \to \mathbb{R}$, we see that the system (M, π_M, I) is invertibly absolutely morphic to the trivial system $(N, \pi_N, \{0\})$, where the Cartan prolongation of (M, π_M, I) is a m times repeated total prolongation and the Cartan prolongation of $(N, \pi_N, \{0\})$ is a l times repeated total prolongation.

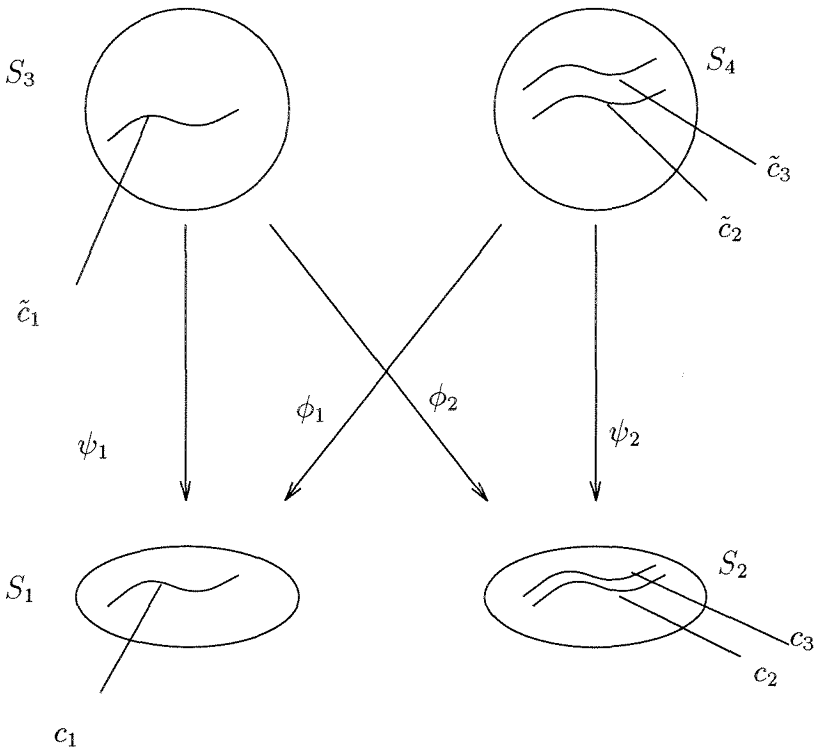


Figure 2.3 Invertibly absolutely morphic systems.

The following theorem allows us to characterise flatness in terms of absolute equivalence.

Theorem 2.23 *Two systems are invertibly absolutely morphic if and only if they are absolutely equivalent.*

Proof: Sufficiency is trivial. We shall prove necessity. Let $S_1 = (M_1, \pi_1, I_1)$ and $S_2 = (M_2, \pi_2, I_2)$ be invertibly absolutely morphic, with $S_3 = (B_1, \pi_3, J_1)$ and $S_4 = (B_2, \pi_4, J_4)$ being respective Cartan prolongations and $\phi_1 : B_2 \to M_1$ and $\phi_2 : B_1 \to M_2$ being respective morphisms.

We now argue that S_4 is a Cartan prolongation of S_1 (and hence S_1 and S_2 are absolutely equivalent). Hence need to show that every solution c_1 of S_1 has a unique lift $\tilde{c}_2$ (on B_2) which is a solution of S_4.

To show existence of a lift which is a solution of S_4, observe that for any given c_1 which is a solution of S_1, we can obtain its unique lift $\tilde{c}_1$ on B_1 (which solves J_1), and get its projection c_2 on M_2 (which solves I_2) and then consider its unique lift $\tilde{c}_2$ on B_2 (which solves J_2). The fact that $\tilde{c}_2$ is a lift of c_1 follows from the invertibility property, which states that $\phi_1 \circ \tilde{c}_2 = c_1$. In other words, $\tilde{c}_2$ projects down to c_1 and hence is a lift of c_1.

To see the uniqueness of this lift, suppose $\tilde{c}_2$ and $\tilde{c}_3$ which are solutions of S_4 on B_2, both project down to c_1 on M_1. Consider their projections c_2 and c_3 (respectively) on M_2. See Figure 2.3. When we lift c_2 or c_3 to B_2 and project down to M_1 we get c_1. Which when lifted to B_1 gives, say $\tilde{c}_1$. By the requirement of the absolute morphisms being invertible $\tilde{c}_1$ should project down to c_2 as well as c_3 (via ϕ_2). Then uniqueness of projection implies that c_2 and c_3 are the same. Which implies $\tilde{c}_2$ and $\tilde{c}_3$ are the same. Hence S_4 is a Cartan prolongation of S_1 as well. Hence S_1 and S_2 are absolutely equivalent. ■

The following corollary is obvious.

Corollary 2.24 *A system (M, π_M, I) is differentially flat if and only if it is absolutely equivalent to some trivial system $(N, \pi_N, \{0\})$.*

Remark 2.25 Observe that the number of flat outputs is p where $p+1$ is the codimension of system (M, π_M, I). If the system is a control system then p is also the number of inputs. This follows from our assumption that Cartan prolongations preserve codimension (see Remark 2.11). It must be observed that in the particular instance when $m = 0$, i.e. the flat outputs only depend on (t, x), the original system is a Cartan prolongation of the trivial system $(N, \pi_N, \{0\})$.

The absolute equivalence problem has been completely solved by Elie Cartan for codimension 2 systems. See chapter 2 of [35], for instance. All Cartan prolongations are locally equivalent to total prolongations. Starting with any system, taking derived systems enables one to "strip off" prolongations and reach the "core" system, which is not a total prolongation of any system. For differentially flat systems, the core is trivial. See [35] for a detailed discussion.

The following result characterises flatness for codimension 2 systems and we refer to [34, 35, 42] for details.

Theorem 2.26 *A system (M, π_M, I) of constant codimension 2 is flat if and only if*

1. $\dim I^{(i)} = \dim I^{(i-1)} - 1$, *for* $i = 0, \ldots, n = \dim I$. *This implies* $I^{(n)} = \{0\}$.

2. *The system* $I^{(i)} + span\{dt\}$ *is integrable for each* $i = 0, \ldots, n$.

2.4 Reduction of Higher Codimension Systems to Codimension 2

Theorem 2.26 characterises flatness for systems with codimension 2. Although some verifiable necessary conditions [32, 36] are known, no complete characterisation exists for systems with higher codimension, except for isolated results for special categories. Deciding whether such a system is flat involves making an educated guess based on the special structure of the system and experience.

We will now describe a method that determines whether a system has flat outputs of a particular form. We will only look for flat outputs that depend on the original variables (t, x) of the system and not on the derivatives of x. In other words, we will only check if the given system is a Cartan prolongation of a trivial system. This may seem restrictive, but it is not. In fact, if we suspect that the flat outputs depend on up to q derivatives of x then we first prolong the given system by total prolongation q times and then take the resulting system as our starting point.

Assume we have a system with p inputs. The first step of the method involves making a guess for $p-1$ flat outputs $y^1, \ldots, y^{p-1}$. Often, this guess will involve expressing the flat outputs as a parameterised family. A simple example will serve to illustrate the idea. Consider the system of differential equations

$$\begin{aligned} x^2\dot{x}^1 - x^1\dot{x}^2 &= x^3, \\ x^1\dot{x}^3 &= x^4, \end{aligned} \tag{2.5}$$

corresponding to a system $(\mathbb{R}^5, \pi, I)$, where $\pi(t, x) = t$ and $I = \{x^2dx^1 - x^1dx^2 - x^3dt, x^1dx^3 - x^4dt\}$. We "guess" that one of the flat outputs is given by $y = x^1 - \lambda x^4$, where λ is constant.

The second step in the method sets the outputs to free functions of time: $y^i = Y^i(t), i = 1, \ldots, p-1$. Solve for (some of) the variables x in terms of the free functions $Y(t)$, and substitute them in the system equations. This leads to a system, for which Theorem 2.26 applies. Note that the resulting system is often time dependent. For the example, set $y = Y(t)$ for arbitrary $Y : \mathbb{R} \to \mathbb{R}$. Then $x^1 = Y(t) + \lambda x^4$, and substituting this into (2.5) yields,

$$\begin{aligned} \lambda x^2\dot{x}^4 + x^2\dot{Y}(t) - (Y(t) + \lambda x^4)\dot{x}^2 &= x^3, \\ (\lambda x^4 + Y(t))\dot{x}^3 &= x^4. \end{aligned} \tag{2.6}$$

This system, which we call the reduced system, is underdetermined by 1 equation as opposed to 2 in the case of the original system.

The third step of the method checks whether the conditions of Theorem 2.26 are satisfied. In the case that they are, a flat output z for the reduced system can be calculated. In general, this flat output will depend on (t, x) and the free functions $Y^i(t)$, but in order that z is the final flat output for the original system, it is necessary that $z = h(t, x)$. For the example, the Pfaffian system of the reduced (restricted) system (2.6) is given by

$$\bar{I} = \{\lambda x^2 dx^4 - (Y(t) + \lambda x^4)dx^2 - (x^3 - x^2 Y'(t))dt,$$
$$(Y(t) + \lambda x^4)dx^3 - x^4 dt\}. \tag{2.7}$$

Calculations show that $\bar{I} + \text{span}\{dt\}$ is integrable and $\bar{I}^{(1)}$ drops rank by one, i.e. $\dim \bar{I}^{(1)} = \dim \bar{I} - 1$. In fact, $\bar{I}^{(1)} = \{\alpha\}$, where

$$\alpha = -Y(t)^2 dx^2 - \lambda Y(t) x^4 dx^2 + 2\lambda x^3 Y(t) dx^3 + 2\lambda^2 x^3 x^4 dx^3$$
$$+ \lambda x^2 Y(t) dx^4 - (2\lambda x^3 x^4 + Y(t) x^3 - Y(t) Y'(t) x^2) dt. \tag{2.8}$$

Since $d\alpha = 2\lambda^2 x^3 dx^3 \wedge dx^4 \mod \alpha, dt$, it follows that $\lambda = 0$ is the only value for which $\bar{I}^{(1)} + \text{span}\{dt\}$ is integrable. For this choice of λ, it follows that $\bar{I}^{(2)} = \{0\}$. Moreover, $\alpha = Y(t)(-Y(t)dx^2 + (x^2 Y'(t) - x^3)dt)$, indicating that x_2 is a flat output for the reduced system. We have thus found x^1 and x^2 to be a set of flat outputs.

In order to see the geometric meaning of this method, suppose we start with a system S. We are interested in knowing if S is a Cartan prolongation of some trivial system S_1 with a corresponding morphism ϕ_1. We consider a trivial system S_2 that corresponds to the subset of all but one flat output that we have guessed. There is a morphism ϕ_2 from S to S_2 that relates the flat outputs as functions of coordinates (t, x) of S. Thus in the above example, $S_2 = (\mathbb{R}^2, \pi_2, \{0\})$ and in local coordinates (t, y) on $\mathbb{R}^2$, $\phi_2 : (t, x) \to (t, y = x^1 - \lambda x^4)$. If our guess is correct, then there must be a morphism $\phi_{1,2}$ from S_1 to S_2 which just picks out the subset of flat outputs. This means, $\phi_2 = \phi_{1,2} \circ \phi_1$. See Figure 2.4.

Having decided on S_2 and ϕ_2 our method involves choosing an arbitrary solution c of S_2 and looking at the restriction of S to the fibres of ϕ_2 over the image of c. Proposition 2.28 (which is more general in that S_1 and S_2 need not be trivial systems) asserts the validity of our approach. Also since the codimension of S_2 is one less than that of S and S_1, the restriction $S|_{\phi_2^{-1} \circ c(a,b)}$ has codimension 2. By Theorem 2.26, it may then be verified whether S is a Cartan prolongation of some trivial system.

Definition 2.27 Let $S = (M, \pi, I)$ be a system and suppose $N \subset M$ is a submanifold of M such that $\pi|_N : N \to \mathbb{R}$ is a submersion. Then the *restriction of S to N* is $S|_N = (N, \pi|_N, i_N^*(I))$, where i_N is the inclusion $N \to M$.

Proposition 2.28 *Let $S = (M, \pi_M, I)$ and $S_i = (M_i, \pi_{M_i}, I_i), i = 1, 2,$ be systems. Let ϕ_1 and ϕ_2 be morphisms from S to S_1 and S_2 respectively. Furthermore suppose $\phi_{1,2}$ is a morphism from S_1 to S_2 and that $\phi_{1,2} \circ \phi_1 = \phi_2$.*

Then, S is a Cartan prolongation of S_1 if and only if for every solution $c : (a, b) \to M_2$ of S_2, $S|_{\phi_2^{-1} \circ c(a,b)}$ is a Cartan prolongation of $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$ via $\phi_1|_{\phi_2^{-1} \circ c(a,b)}$. See Figure 2.4.

The proof reduces to a series of lemmas.

Lemma 2.29 *Let $S = (M, \pi, I)$ be a system and $S|_N = (N, \pi|_N, i_N^*(I))$ be its*

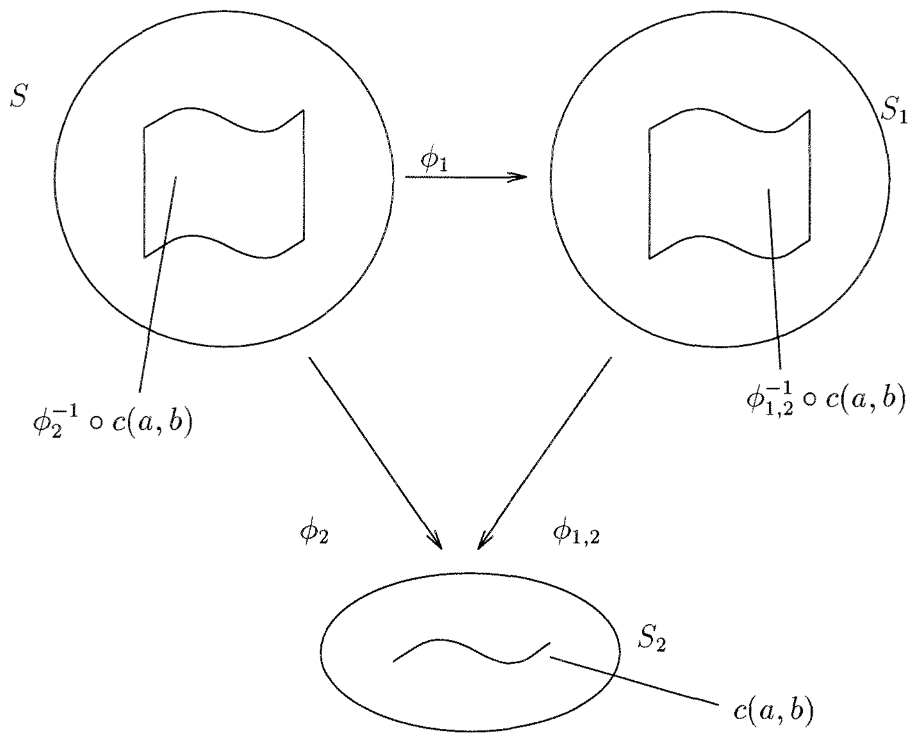


Figure 2.4 Illustration of Proposition 2.28.

restriction to $N \subset M$. Let $c : (a,b) \to N$ be a curve such that $\pi \circ c = id$. Then c is a solution of S if and only if it is a solution of $S|_N$.

Proof: Follows from $(i_N \circ c)^*(I) = c^*(i_N^* I)$. ■

Lemma 2.30 *Let ϕ be a morphism from $S_A = (\pi_A, M_A, I_A)$ to $S_B = (\pi_B, M_B, I_B)$. Let $S_B|_N$ be a restriction of S_B to $N \subset M_B$. Then $S_A|_{\phi^{-1}(N)}$ is a well defined restriction of S_1 to $\phi^{-1}(N)$ and $\phi|_{\phi^{-1}(N)}$ is a morphism from $S_A|_{\phi^{-1}(N)}$ to $S_B|_N$.*

Proof: Since ϕ is a submersion, $\phi^{-1}(N)$ is a submanifold and $\phi|_{\phi^{-1}(N)}$ is a surjective submersion onto N. ■

Remark 2.31 We need to make sure that the restricted systems in the statement of the proposition are well defined. First note that $S_2|_{c(a,b)}$ is a well defined restriction and hence from the Lemma 2.30 it follows that $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$ and $S|_{\phi_2^{-1} \circ c(a,b)}$ are well defined and $\phi_1|_{\phi_2^{-1} \circ c(a,b)}$ is a morphism from $S|_{\phi_2^{-1} \circ c(a,b)}$ to $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$.

Lemma 2.32 *Let $S_A = (\pi_A, M_A, I_A)$ be a Cartan prolongation of $S_B = (\pi_B, M_B, I_B)$ via ϕ. Let $S_B|_N$ be a restriction of S_B to $N \subset M_B$. Then $S_A|_{\phi^{-1}(N)}$ is a Cartan prolongation of $S_B|_N$ via $\phi|_{\phi^{-1}(N)}$.*

Proof: Follows from Lemmas 2.29 and 2.30. ■

Proof: (of Proposition 2.28) Necessity follows from Lemma 2.32.
Sufficiency: Let c_1 be a solution of S_1. First we show it has a lift. Let $c = \phi_{1,2} \circ c_1$. Then c is a solution of S_2 and hence $S|_{\phi_2^{-1} \circ c(a,b)}$ is a Cartan prolongation of $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$ by assumption. But, by Lemma 2.29 above, c_1 is a solution of $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$ and hence there is a unique lift $\tilde{c}$, which is a solution of $S|_{\phi_2^{-1} \circ c(a,b)}$ and hence a solution of S appealing again to Lemma 2.29. To show uniqueness of lift, suppose $\tilde{c}_2$ is another lift of c_1. Then it follows that $\tilde{c}_2$ is also a solution of $S|_{\phi_2^{-1} \circ c(a,b)}$, violating the unique lift of $S|_{\phi_2^{-1} \circ c(a,b)}$ being a Cartan prolongation of $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$. ■

So far, we have only discussed the scenario where the test succeeds for some parameter value. The test fails if the reduced system is not flat. To illustrate this, consider the same example and suppose we choose $\lambda = 1$, i.e. we guess that $x^1 - x^4$ is a flat output. Our calculations show that the reduced system cannot be flat for any choice of $Y(t)$. Hence Proposition 2.28 tells us that the system cannot be flat with $x^1 - x^4$ and a function of $(t, x^1, \ldots, x^4)$ as the flat outputs. However, one may wonder whether there exists a function of t, x^i and finitely many derivatives of x^i that together with $x^1 - x^4$ forms a set of flat outputs. The following proposition shows that this is not possible.

Proposition 2.33 *Let $S = (M, \pi_M, I)$ and $S_i = (M_i, \pi_{M_i}, I_i), i = 1, 2,$ be systems. Let ϕ_2 be a morphism from S to S_2 and $\phi_{1,2}$ be a morphism from S_1 to S_2, and suppose S is absolutely equivalent to S_1. Then, for every solution $c : (a, b) \to M_2$ of S_2, $S|_{\phi_2^{-1} \circ c(a,b)}$ and $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$ are absolutely equivalent. See Figure 2.5.*

Proof: Since S and S_1 are absolutely equivalent there exists a system $\tilde{S} = (B, \pi_B, J)$ which is a Cartan prolongation of S and S_1. Let $\tilde{\phi} : B \to M$ and $\tilde{\phi}_1 : B \to M_1$ be the corresponding morphisms. Then by the Lemmas 2.30 and 2.32 $\tilde{S}|_{\tilde{\phi}^{-1} \circ \phi_2^{-1} \circ c(a,b)}$ is a valid restriction and is a Cartan prolongation of $S|_{\phi_2^{-1} \circ c(a,b)}$. But it is also the same as $\tilde{S}|_{\tilde{\phi}_2^{-1} \circ \phi_{1,2}^{-1} \circ c(a,b)}$ and is a Cartan prolongation of $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$. ■

When S_1 and S_2 are trivial and S_2 has codimension one less than S_1 the situation corresponds to our method. Then the proposition says that S is flat with M_2 providing all but one flat output, only if every restriction $S|_{\phi_2^{-1} \circ c(a,b)}$ is flat. In our example, when $\lambda \neq 0$ the reduced system (restriction) fails to be flat and hence $x^1 - x^4$ cannot be a flat output.

2.5 Example: Two Planar Coupled Rigid Bodies with 3 Inputs

Various mechanical systems have been found to be flat with coordinates of a body fixed point providing a subset of the flat outputs; see [25] for some examples. With

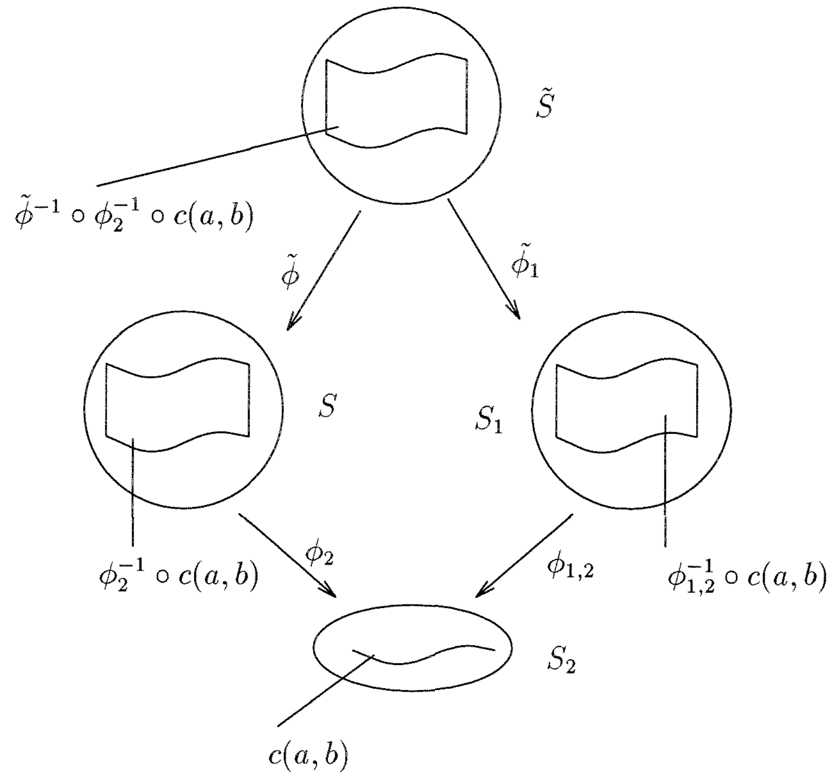


Figure 2.5 Illustration of Proposition 2.33.

the method developed in the previous section, one may systematically search for such flat outputs. We demonstrate this for a mechanical control system.

The system we consider consists of 2 planar rigid bodies hinged at a point, O, and moving under gravity, g (see Figure 2.6). Two of the inputs, f_1, f_2, are body fixed forces acting on the first body such that their lines of action intersect at a point P on the line joining the point O and G_1, the center of mass of the first body. The third input is a pure torque, τ, between the two bodies, i.e. equal and opposite torques on the two bodies. Let $OG_i = r_i$, $\sigma_i = \sqrt{J_i/m_i}$ where J_i and m_i are the moment of inertia and the mass of body i. Furthermore, assume $OP = 1$, the mass of the first body $m_1 = 1$, and $m_2 = \mu$ in order to non dimensionalise the problem. From a Lagrangian point of view, the system evolves on the configuration manifold

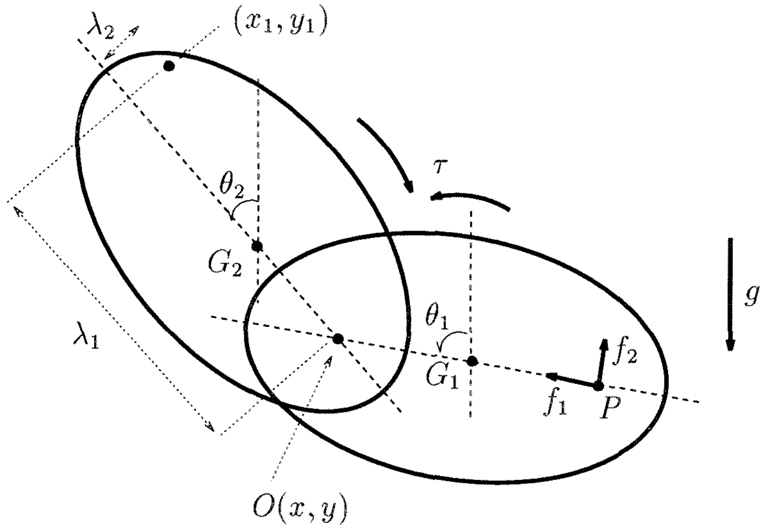


Figure 2.6 Two coupled rigid bodies in $\mathbb{R}^2$.

$\mathbb{R}^2 \times S^1 \times S^1$, with coordinates $(x, y, \theta_1, \theta_2)$. The equations of motion are given by

$$\begin{aligned}
&(1+\mu)(\ddot{x}\sin\theta_1 - \ddot{y}\cos\theta_1 - g\cos\theta_1) + r_1\dot{\theta}_1^2 \\
&\qquad +\mu r_2 \sin(\theta_2-\theta_1)\ddot{\theta}_2 + \mu r_2\cos(\theta_2-\theta_1)\dot{\theta}_2^2 = f_1 \\
&(1+\mu)(\ddot{x}\cos\theta_1 + \ddot{y}\sin\theta_1 + g\sin\theta_1) - r_1\ddot{\theta}_1 \\
&\qquad -\mu r_2\cos(\theta_2-\theta_1)\ddot{\theta}_2 + \mu r_2\sin(\theta_2-\theta_1)\dot{\theta}_2^2 = -f_2 \\
&(r_1^2+\sigma_1^2)\ddot{\theta}_1 - r_1\cos\theta_1\ddot{x} - r_1\sin\theta_1\ddot{y} - gr_1\sin\theta_1 = f_2+\tau \\
\mu(r_2^2+\sigma_2^2)\ddot{\theta}_2 &- \mu r_2\cos\theta_2\ddot{x} - \mu r_2\sin\theta_2\ddot{y} - \mu g r_2\mu\sin\theta_2 = -\tau.
\end{aligned} \tag{2.9}$$

The system can be written as a Pfaffian system of codimension 4. The single second order equation, obtained by eliminating f_2 and τ from the last three equations, corresponds to a Pfaffian system of codimension 4 in coordinates $(t, x, y, \theta_1, \theta_2, \dot{x}, \dot{y}, \dot{\theta}_1, \dot{\theta}_2)$. The full system is a Cartan prolongation of this latter system, because, given any solution of the latter, there is a unique corresponding solution for the full system, in which $(x, y, \theta_1, \theta_2, \dot{x}, \dot{y}, \dot{\theta}_1, \dot{\theta}_2)$ are the same, and (f_1, f_2, τ) are given by above equations. We look for flat outputs that only depend on configuration and velocity variables. In other words, check whether the simpler system is a Cartan prolongation of a trivial system. Our starting point is the following differential equation.

$$\begin{aligned}(r_1^2+\sigma_1^2)\ddot{\theta}_1 &- r_1\cos\theta_1\ddot{x} - r_1\sin\theta_1\ddot{y} - gr_1\sin\theta_1 \\ &+ \mu(r_2^2+\sigma_2^2)\ddot{\theta}_2 - \mu r_2\cos\theta_2\ddot{x} - \mu r_2\sin\theta_2\ddot{y} - \mu g r_2\sin\theta_2 \\ &+ (1+\mu)(\ddot{x}\cos\theta_1 + \ddot{y}\sin\theta_1 + g\sin\theta_1) - r_1\ddot{\theta}_1 \\ &- \mu r_2\cos(\theta_2-\theta_1)\ddot{\theta}_2 + \mu r_2\sin(\theta_2-\theta_1)\dot{\theta}_2^{\,2} = 0. \qquad (2.10)\end{aligned}$$

This corresponds to a Pfaffian system,

$$\begin{aligned}\{d\theta_1 - \dot{\theta}_1 dt,\ & d\theta_2 - \dot{\theta}_2 dt, \\ &(r_1^2+\sigma_1^2)d\dot{\theta}_1 - r_1\cos\theta_1 d\dot{x} - r_1\sin\theta_1 d\dot{y} - gr_1\sin\theta_1 dt \\ &+ \mu(r_2^2+\sigma_2^2)d\dot{\theta}_2 - \mu r_2\cos\theta_2 d\dot{x} - \mu r_2\sin\theta_2 d\dot{y} - \mu g r_2\sin\theta_2 dt \\ &+ (1+\mu)(d\dot{x}\cos\theta_1 + d\dot{y}\sin\theta_1 + g\sin\theta_1)dt - r_1 d\dot{\theta}_1 \\ &- \mu r_2\cos(\theta_2-\theta_1)d\dot{\theta}_2 + \mu r_2\sin(\theta_2-\theta_1)\dot{\theta}_2^{\,2}dt\}.\end{aligned}$$

We are looking for 3 flat outputs and to use the method we need to guess some form for 2 of them. We test if the system is flat with 2 of the flat outputs given by coordinates of a body fixed point in the second body (intuitively, the second body is a more reasonable guess than the first body where the forces are applied.)

Coordinates (x_1, y_1) of a body fixed point are given by,

$$\begin{aligned} x_1 &= x - \lambda_1\sin\theta_2 + \lambda_2\cos\theta_2, \\ y_1 &= y + \lambda_1\cos\theta_2 + \lambda_2\sin\theta_2, \end{aligned} \qquad (2.11)$$

where (λ_1, λ_2) are its coordinates in the body fixed frame and as such are constants. Restricting the system to $x_1 = X_1(t)$ and $y_1 = Y_1(t)$ for arbitrary $X_1, Y_1 : \mathbb{R} \to \mathbb{R}$ corresponds to substituting

$$\begin{aligned} x &= X_1(t) + \lambda_1\sin\theta_2 - \lambda_2\cos\theta_2, \\ y &= Y_1(t) - \lambda_1\cos\theta_2 - \lambda_2\sin\theta_2 \end{aligned} \qquad (2.12)$$

in the Pfaffian system. We get a codimension 2 system I that has three forms in coordinates $(t, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$.

All computations were carried out with the aid of symbolic manipulation software, and we shall not show every step for brevity. Calculations reveal that the derived system $I^{(1)}$ drops rank by 1, ($\dim I^{(1)} = \dim I - 1$), and $I + \text{span}\{dt\}$ is integrable. Further calculations show that $I^{(2)}$ drops rank by one only if certain algebraic relations amongst system parameters (μ, r_i, σ_i) and λ_1, λ_2 hold. For generic parameter values these relations are:

$$\lambda_1 = \frac{\mu r_2}{1+\mu-r_1}, \quad \lambda_2 = 0. \qquad (2.13)$$

For this choice all the necessary and sufficient conditions for flatness are satisfied.

In other words, $I^{(2)}$ and $I^{(3)}$ drop rank by one and each $I^{(i)}+\text{span}\{dt\}$ is integrable. Also a flat output can be obtained from the form $dz - wdt$ that generates $I^{(2)}$. It is given by

$$z = (-r_1^3 + 2r_1^2 - r_1 + \sigma_1^2\mu + r_1^2\mu - \sigma_1^2 r_1 - r_1\mu + \sigma_1^2)\theta_1 \\ +\mu(-\sigma_2^2 r_1 + \sigma_2^2\mu - r_2^2 r_1 + \sigma_2^2 + r_2^2)\theta_2. \tag{2.14}$$

Note that z is well-defined on the manifold of the original system. In particular, z does not depend on $f(t)$ or $h(t)$. Therefore, the original system is indeed flat.

Observe that x, y, θ_1 can be solved in terms of the flat outputs and θ_2. Substituting these in (2.10), we obtain an algebraic equation involving the flat outputs, their derivatives and θ_2, but no derivatives of θ_2. Since this equation is fairly messy we shall not show it here. Hence θ_2 can be solved from this equation in terms of the flat outputs and their derivatives. The solution, however, may not be unique, but the set of solutions is discrete.

So we conclude that the system is differentially flat. Two of the flat outputs are given by the body fixed point located on the line OG_2 and a distance (in dimensionalised form) $\frac{m_2 r_2 R}{(m_1+m_2)R - m_1 r_1}$ from O. The third output is a linear combination of the angles, as given in equation (2.14).

Chapter 3

Jet Bundle Approach to Differential Flatness

In this chapter we shall develop a framework for studying differential flatness that is slightly different from Chapter 2. Given a space M with coordinates $(t, x^1, \ldots, x^N)$ (where t is regarded as special as in Chapter 2), one geometric approach is to construct spaces $J^r(t, M)$ that have the variables (t, x) and derivatives $x^{(k)}$ up to $k \leq r$ as coordinates. Such spaces are called the jet bundles associated with the bundle $t : M \to \mathbb{R}$. We refer the reader to [33] for a discussion of jet bundles. An ODE of the form $F(t, x, \ldots, x^{(r)}) = 0$ may be regarded as a codimension one submanifold of $J^r(t, M)$ given by the zero set of the function F. A system of ODEs also corresponds to a submanifold of $J^r(t, M)$. When one starts with a Pfaffian system I on M with coordinates $(t, x^1, \ldots, x^N)$ the corresponding first order quasi-linear system of ODEs may be regarded as a submanifold $E^1 \subset J^1(t, M)$. Differentiation of all the ODEs provides additional ODEs that are second order. The combined system corresponds to a submanifold $E^2 \subset J^2(t, M)$. This procedure can be repeated and yields submanifolds $E^r \subset J^r(t, M)$. Instead of looking at the full jet bundles $J^r(t, M)$ one may only look at the submanifolds E^r that describe the system of ODEs. These spaces are the spaces of the repeated total prolongations already introduced in Chapter 2.

We shall provide an intrinsic definition of (total) prolongations, introduce the notion of a time derivative of functions and k-forms for all k and formulate flatness in terms of the time derivative operator. We believe this approach is better than the one presented in Chapter 2 in that it allows for a local definition and also the nature of the singularities become clearer. Our approach is closely related to those of Fliess et al. [11, 10] and Pomet [27]. One difference is that these other approaches work with infinite dimensional spaces while we work in a finite dimensional setting.

With the aid of the concepts introduced in this chapter we provide a theorem that partially characterises *zero-flatness* of systems that consist of two independent one-forms. We define zero-flat outputs to be those that only depend on the original variables in which the one-forms are written, but not on their derivatives. The theorem shows that it is possible to split the task of finding zero-flat outputs into two parts. The first part deals with the "infinitesimal" aspects. The theorem provides intrinsic geometric conditions that the tangent spaces to level sets (in M) of the zero-flat outputs should satisfy. The second part deals with integrability (piecing these tangent spaces together to form level sets of the flat outputs). The second part leads to PDEs and we do not provide any theory for the second part. In other words we characterise zero-flatness on an "infinitesimal" level, but do not deal with integrability. We illustrate the use of this theorem by two examples.

There are two reasons for studying zero-flatness. One is that this is relatively simpler than the general case of differential flatness where so far it is not known if there is an upper bound on the number of derivatives of x that the flat outputs y may depend on. Also conceptually this may prove to be a useful step towards understanding the more general case. Secondly, generally speaking the lower the number of derivatives that appear in the transformations between flat outputs y and the original variables x the better the numerical aspects of generating feasible trajectories. Hence there is a practical reason to look for outputs that depend on as few x derivatives as possible.

3.1 Motivation

In order to motivate some of the abstract developments in this chapter let us consider the example of a zero-flat system described by the two one-forms

$$\begin{aligned} \omega^1 &= x^2 dx^1 - x^1 dx^2 - x^3 dt \\ \omega^2 &= x^1 dx^3 - x^4 dt. \end{aligned} \tag{3.1}$$

This corresponds to the quasi-linear system of ODEs $E^1 = 0, E^2 = 0$ where

$$\begin{aligned} E^1 &= x^2 \dot{x}^1 - x^1 \dot{x}^2 - x^3 \\ E^2 &= x^1 \dot{x}^3 - x^4, \end{aligned} \tag{3.2}$$

and the system has zero-flat outputs $y^1 = x^1$ and $y^2 = x^2$. Indeed given $y = (y^1, y^2)$ as a function of time x^3 is obtained by $x^3 = y^2 \dot{y}^1 - y^1 \dot{y}^2$ which involves differentiating y once. In order to obtain x^4 a second differentiation is necessary. In other words x^4 depends on two derivatives of y. In fact for any system of n one-forms the number of zero-flat output derivatives required to express all the original variables in terms of which the one-forms are written is n and this follows from the Corollary 2.15.

The process of obtaining x^3 and x^4 from y involves a series of steps consisting of differentiation and algebra. In order to develop a general approach it may be simpler to separate the two operations. By first differentiating the ODEs sufficiently many times we may introduce sufficiently many derivatives of y. For a system of n one-forms this involves differentiating the ODEs $n - 1$ times. In our example since $n = 2$ we need to differentiate the ODEs once to introduce two derivatives of all the variables. Thus we obtain the system of four ODEs $E^1 = 0, E^2 = 0, \dot{E}^1 = 0, \dot{E}^2 = 0$, where $\dot{E}^1$ and $\dot{E}^2$ are explicitly given below.

$$\begin{aligned} \dot{E}^1 &= x^2 \ddot{x}^1 + \dot{x}^2 \dot{x}^1 - x^1 \ddot{x}^2 - \dot{x}^1 \dot{x}^2 - \dot{x}^3 \\ \dot{E}^2 &= x^1 \ddot{x}^3 + \dot{x}^1 \dot{x}^3 - \dot{x}^4. \end{aligned}$$

Having introduced sufficiently many derivatives ($\dot{y}$ and $\ddot{y}$) the condition that x^3

and x^4 be functions of $t, y, \dot{y}$ and $\ddot{y}$ alone, becomes purely an algebraic matter. In other words it must be possible to eliminate derivatives of x^3 and x^4 from the four equations to obtain two equations containing $t, y, \dot{y}, \ddot{y}, x^3$ and x^4 alone. Only then we may solve for x^3 and x^4 in terms of derivatives of y. This condition is equivalent to

$$\operatorname{rank} \frac{\partial(E, \dot{E})}{\partial(\dot{z}, \ddot{z})} \leq 2 \tag{3.3}$$

for points that satisfy $E = 0, \dot{E} = 0$, where $E = (E^1, E^2)$ and $z = (x^3, x^4)$. In our example it is easy to verify that the above condition is indeed satisfied. In addition we need the regularity condition that the matrix

$$\frac{\partial(E, \dot{E})}{\partial(z, \dot{z}, \ddot{z})}$$

has full rank for points on $E = 0, \dot{E} = 0$. This condition ensures that there are no equations involving $t, y, \dot{y}$ and $\ddot{y}$ only. In terms of one-forms, these conditions together ensure that dz^i for $i = 1, 2$ are in the span of $dt, dy, d\dot{y}$ and $d\ddot{y}$.

At first glance these conditions seem to involve the separation of the variables into two groups: y involving zero-flat outputs and z involving the rest. Hence this may seem useless since our goal is to find y in the first place. However these conditions are indeed useful in finding zero-flat outputs and this is seen by realising that the Jacobian condition (3.3) provides conditions on the tangent spaces to the level sets of the zero-flat outputs.

The submatrices $\frac{\partial E}{\partial \dot{z}}$ and $\frac{\partial \dot{E}}{\partial \ddot{z}}$ are the same and have an intrinsic meaning, but the submatrix $\frac{\partial \dot{E}}{\partial \dot{z}}$ does not. However it may be shown (see Appendix A) that the overall rank condition (3.3) does have an intrinsic interpretation. Firstly it must be noted that the span of $\frac{\partial}{\partial \dot{z}^i}$ for $i = 1, 2$ is intrinsic and equals $\ker T(t, y)$. Hence the rank condition (3.3) is really a condition on the distribution $\ker T(t, y)$. In fact (3.3) is equivalent to the condition,

$$\begin{aligned} \eta_1 \lrcorner \eta_2 \lrcorner (\omega^1 \wedge \omega^2) &= 0 \\ \eta_1 \lrcorner \eta_2 \lrcorner d(\omega^1 \wedge \omega^2) &= 0 \quad \bmod I, \end{aligned} \tag{3.4}$$

where η_1, η_2 span $\ker T(t, y^1, \ldots, y^p)$ and ω^1, ω^2 span I.

This result suggests that one may first solve for a distribution spanned by η_1 and η_2, two unknown vector fields. At this stage several possible solutions exist. Then one must look for an integrable distribution from the set of solutions. If such a distribution exists then leaves of its foliation will be the level sets of some zero-flat outputs. We shall show in Section 3.7 how this approach may be used to find zero-flat outputs.

Though these results are specific for systems of two one-forms our approach was general enough to hope to obtain similar results for systems of arbitrary number of one-forms. However this coordinate based approach becomes exceedingly messy even for systems of three one-forms since one has to consider a Jacobian matrix of size 9×9. Hence a better approach may be to look for a more intrinsic proof of the same results. The abstract notions developed in this chapter are precisely aimed at capturing the geometry behind these rank conditions.

It is also important to note that the condition (3.4) makes sense even for nonintegrable distributions spanned by η_1 and η_2 while the original Jacobian condition makes sense only for the integrable case. Hence it may be useful to develop a notion of "infinitesimal flat outputs" which correspond to a (possibly nonintegrable) distribution or equivalently its annihilator which is a codistribution. The "infinitesimal zero-flat outputs" may be thought of as one-forms $\alpha^1, \ldots, \alpha^p$ expressible in terms of original variables t and x such that for all the original variables x^i the one-forms dx^i are in the span of $dt, \alpha^1, \ldots, \alpha^p$ and finitely many (two in the case of systems of two one-forms) derivatives of α^j. But this requires a notion of time derivative of one-forms. When $dt, \alpha^1, \ldots, \alpha^p$ form an integrable (in the Frobenius sense) codistribution then the system has zero-flat outputs.

The submanifolds corresponding to $E = 0, \ldots, E^{(k)} = 0$ where $E^{(k)}$ is the kth derivative of equations E, have an intrinsic meaning and are the same as the total prolongations already encountered in Chapter 2. We shall develop the notion of (total) prolongations and time derivatives of functions as well as forms in a completely intrinsic geometric way in the next two sections. Starting with a manifold M equipped with special coordinate t we will construct the prolongation manifolds J^k for $k \geq 1$.

Some algebraic properties of the Jacobian of (3.3) that are exploited in Appendix A reflect more general geometric properties of the spaces J^k and the time derivative operator. More specifically the properties of the "vertical spaces" of the bundles J^k over J^{k-1} have a role to play. We shall explore this in Section 3.4.

3.2 Prolongations

As in Chapter 2 our basic object of study is a Pfaffian system on a manifold equipped with a special notion of time. But our viewpoint and notation are slightly different. All objects such as manifolds, functions, vector fields, forms and codistributions are assumed C^∞-smooth unless stated otherwise. In this section we shall define (total) prolongations in an intrinsic geometric way.

Our starting point is a fibre bundle $t : M \to \mathbb{R}$ (which means M is a manifold, t is a surjective submersion and the bundle is locally trivial, see [33]) where t corresponds to time, and a constant dimensional and smooth codistribution I on M. Notice that given any smooth codistribution there is a unique corresponding Pfaffian system, i.e. a submodule of the module of all one-forms and vice versa. We shall often use the same notation for both, the meaning being clear from the context. But generally we regard I as a codistribution. We shall assume $dt_q \notin I_q$ for all $q \in M$ (as seen in Chapter 2, this ensures that locally around any point there are solution curves). We refer to the triplet $S = (t, M, I)$ as a *system*. As in Chapter 2, a *solution* is a local section c, i.e. a curve $c : (a, b) \to M$ such that $t \circ c$ is the identity map on $\mathbb{R}$ and $c^*(I) = \{0\}$. Let $N = \dim M - 1$, $n = \dim I$ and $p = N - n$.

We have already seen in Chapter 2 that this system corresponds to a quasi-linear system of ODEs in any local coordinates. We shall rewrite these ODEs here in a slightly different way by splitting the coordinates (other than t) into two parts. Locally around any point $q \in M$ one can find independent functions $u^1, \ldots, u^p$ such that $dt, I, du^1, \ldots, du^p$ span T^*U, where U is an open set containing q. Let $x^1, \ldots, x^n$ be local functions such that $(dt, dx^1, \ldots, dx^n, du^1, \ldots, du^p)$ form a local coframe (i.e. a basis of T^*U). Then it is possible to find locally a basis $\omega^1, \ldots, \omega^n$ for I and functions A^i and B^i_j such that

$$\omega^i = dx^i - A^i dt - B^i_j du^j, \quad i = 1, \ldots, n. \tag{3.5}$$

Hence in this coordinate system I corresponds to the system of ODEs

$$\dot{x}^i = A^i(t, x, u) + B^i_j(t, x, u)\dot{u}^j. \tag{3.6}$$

Notice that the splitting of the coordinates into x and u is not unique in general. In the special case when $B^i_j = 0$, the system takes the familiar control system form

$$\dot{x}^i = A^i(t, x, u). \tag{3.7}$$

Geometrically this is possible if and only if the codistribution $\operatorname{span}\{dt\} + I$ is locally integrable (in the Frobenius sense) (see [35]). Then $t, x^1, \ldots, x^n$ are independent functions that "cut out" the leaves of the corresponding foliation.

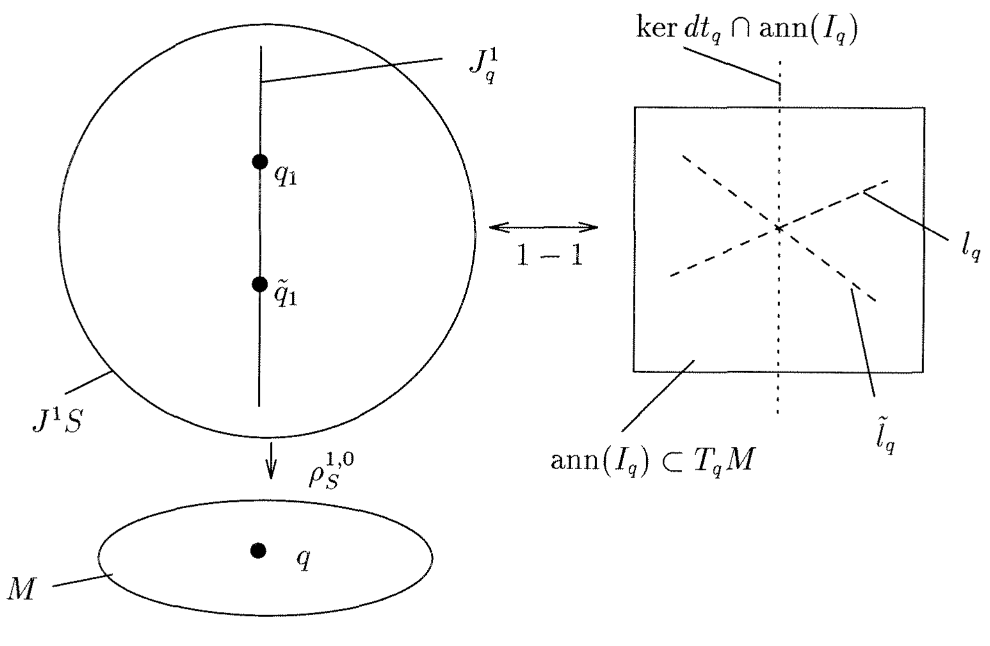


Figure 3.1 The bundle J^1S.

We shall define the notion of an *affine space* before defining the prolongation of the system $S = (t, M, I)$.

Definition 3.1 Let A be a manifold and V be a vector space. Suppose further that V regarded as a Lie Group acts on A freely and transitively. Then A is an *affine space modelled on* V.

Note that the definition implies that $\dim A = \dim V$.

The collection of all one dimensional subspaces $l_q \subset T_qM$ with the property that $l_q \subset \operatorname{ann}(I_q)$ and $dt_q(l_q) \neq \{0\}$ is denoted by J^1_qS and as q "runs over" M the collection is denoted J^1S. Note that J^1_qS is an affine space modelled on $\operatorname{ann}(I_q) \cap \ker dt_q$. Often we shall denote J^1S by J^1 and J^1_qS by J^1_q, the system S being obvious from the context. We tend to use notation q_1 for points on J^1 and use l_q when we regard that point as a one dimensional subspace of T_qM. Though strictly speaking q_1 and l_q are the same we believe it is less confusing to keep two separate notations for the same object depending on how we regard it. See Figure 3.1, where the subspaces l_q and $\tilde{l}_q$ that correspond to two different points q_1 and $\tilde{q}_1$ of J^1S that project down to the same point $q \in M$ are depicted. Let $\rho^{1,0}_S : J^1S \to M$ denote the map that takes a one dimensional subspace $l_q \subset \operatorname{ann}(I_q)$ to $q \in M$. We shall often drop S and denote the map $\rho^{1,0}$.

Let U be an open neighbourhood of q in which equation (3.5) is valid. Define functions $f^j : (\rho^{1,0})^{-1}U \subset J^1 \to \mathbb{R}$ by the criterion that $\omega^1, \ldots, \omega^n$ together with $du^j - f^j(q_1)dt$ for $j = 1, \ldots, p$ span the annihilator of l_q for all $q_1 \in (\rho^{1,0})^{-1}U$, where $q = \rho^{1,0}q_1$ and l_q is the one dimensional subspace corresponding to q_1. It is clear that the map $(t, x^1, \ldots, x^n, u^1, \ldots, u^p, f^1, \ldots, f^p)$ is a bijection of $(\rho^{1,0})^{-1}U$ onto $t(U) \times x(U) \times u(U) \times \mathbb{R}^p$. This can be used to give J^1 a fibre bundle structure over M. In fact J^1 is an affine bundle over M. The fibre dimension of J^1 over M is p. The functions f^j serve as fibre coordinates for the fibre of J^1 over M.

There is a canonical codistribution I_1 on J^1 defined as follows. Given $q_1 \in J^1$, let $l_q \subset T_qM$ be the corresponding one dimensional subspace, where $q = \rho^{1,0}q_1$. Then I_{1q_1} is simply the annihilator of $(T\rho^{1,0})^{-1}l_q$. See Figure 3.2. It follows that $\dim I_1 = \dim I + p = n + p$. It also follows that locally

$$I_1 = (\rho^{1,0})^*I + \operatorname{span}\{du^j - f^j dt \ : \ j = 1, \ldots, p\}. \tag{3.8}$$

It can also be seen that $\operatorname{span}\{(\rho^{1,0})^*dt\} + I_1$ is locally integrable (in the Frobenius sense) with $(t, x^1, \ldots, x^n, u^1, \ldots, u^p)$ "cutting out" leaves of the foliation. In fact the leaves are the fibres of J^1 over M.

The *total prolongation* or the *first prolongation* or simply the *prolongation* of the system $S = (t, M, I)$ is the system $S^1 = ((\rho^{1,0})^*t, J^1, I_1)$. Very often we will abuse the terminology and refer to J^1 or I_1 as the prolongation of M or I.

Remark 3.2 Total prolongation corresponds to introducing extra variables f^j and extra ODEs which define f^j to be the derivatives of u^j as already seen in Chapter 2.

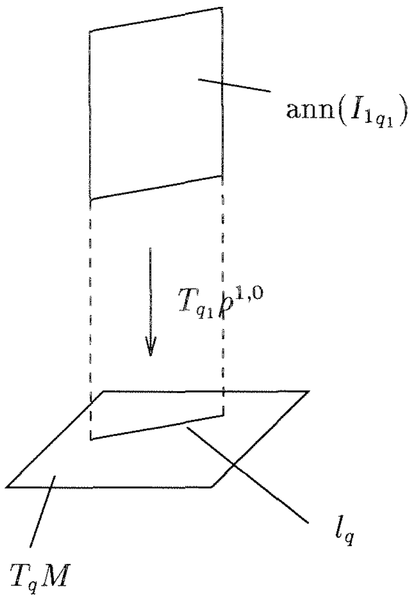
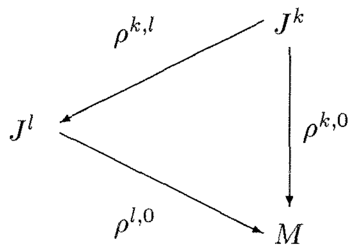


Figure 3.2 The codistribution I_1.

Higher order prolongations are defined iteratively. Hence S^2, the second prolongation of S consists of the manifold J^1S^1 which we denote by J^2S, and the codistribution $(I_1)_1$ which is denoted by I_2. The projection $\rho^{1,0}_{S^1}$ from J^2S to J^1S is denoted by $\rho^{2,1}_S$. Also define $\rho^{2,0}_S = \rho^{1,0}_S \circ \rho^{2,1}_S$. Again we may often drop S from the notation and use J^2, $\rho^{2,1}$ and $\rho^{2,0}$. The definition of J^r, S^r and $\rho^{r,l}$ where $0 \leq l < r$ is clear. It follows that $S^r = ((\rho^{r,0})^*t, J^r, I_r)$. It also follows that J^r is an affine bundle over J^{r-1} for $r \geq 1$. The commutative diagram below depicts bundles J^k, J^l and M where $0 < l < k$.



Remark 3.3 For a trivial system, i.e. a system of the form $T = (t, M, \{0\})$, it follows that the bundles J^rT correspond to the familiar jet bundles $J^r(t, M)$ of the bundle $t : M \to \mathbb{R}$ (see Saunders [33] where he uses the notation $J^r(M)$) . Also the codistributions I_r of the prolongations correspond to the contact codistributions

on $J^r(t, M)$. For a general system $S = (t, M, I)$ the bundles $J^r S$ can be naturally regarded as subbundles of $J^r(t, M)$.

Remark 3.4 Throughout this chapter, when a function f or a form Ω on J^l is pulled back to J^r $(r > l)$, we may still denote them by f or Ω instead of $(\rho^{r,l})^* f$ or $(\rho^{r,l})^* \Omega$ to avoid notational clutter. For instance we shall write t instead $(\rho^{r,0})^* t$.

Given any solution c of $S = (t, M, I)$, its *lift* to J^1 denoted $\dot{c}$ or $c^{(1)}$, is a local section of $t : J^1 \to \mathbb{R}$ and is defined by the criterion that $c^{(1)}(t) \in J^1$ correspond to the one-dimensional subspace of $\operatorname{ann}(I_{c(t)})$ spanned by $\frac{d}{dt}c(t)$. It follows that $c^{(1)}$ is a solution of $S^1 = (t, J^1, I_1)$. In fact the solutions of S^1 and S are in one-to-one correspondence via this lifting operation and this is in accordance with the developments in Chapter 2. In coordinates, if $(x(t), u(t))$ is a solution of S then its lift is given by $(x(t), u(t), f(t) = \frac{d}{dt}u(t))$ which is a solution of S^1 and furthermore all solutions of S^1 are of this form. The lift of a solution c of S to J^2 denoted $\ddot{c}$ or $c^{(2)}$, is defined as the lift of $c^{(1)}$ to J^2 and is well defined since S^2 is the total prolongation of S^1 and $c^{(1)}$ is a solution of S^1. Lifts of c to J^r for all r are defined iteratively and it follows from the definition that a local section $\tilde{c}$ of the bundle $t : J^r \to \mathbb{R}$ is a solution of S^r if and only if it is the lift of solution c of S, i.e. $\tilde{c} = c^{(r)}$.

3.3 Time Derivative

A notion of time derivative is achieved by "vector fields" $F_r : J^r \to TJ^{r-1}$. These are defined as follows. Let $q_r \in J^r$, $\rho^{r,r-1} q_r = q_{r-1}$ and $l_{q_{r-1}} \subset T_{q_{r-1}} J^{r-1}$ be the one dimensional subspace corresponding to q_r. Then we require that $F_r(q_r) \in l_{q_{r-1}}$ and $\langle dt_{q_{r-1}}, F_r(q_r) \rangle = 1$. This uniquely defines F_r. See Figure 3.3 which shows the value of F_r at two different points q_r and $\tilde{q}_r$ of J^r that project down to the same point $q_{r-1} \in J^{r-1}$. Since $\langle dt, \frac{d}{dt}c^{(r)}(t) \rangle = 1$ by definition, it follows that $F_r(c^{(r)}(t)) = \frac{d}{dt}c^{(r-1)}(t) \in T_{c^{(r-1)}(t)} J^{r-1}$.

Given any function $f : J^r \to \mathbb{R}$, we define its *time derivative* denoted by $\dot{f}$ or $f^{(1)}$, to be the function on J^{r+1} given by

$$\dot{f} = \langle df, F_{r+1} \rangle. \tag{3.9}$$

Note that the above definition makes sense since df is a form on J^r and for any $q_{r+1} \in J^{r+1}$, $F_{r+1}(q_{r+1})$ lies in TJ^r. It follows that for any solution $c^{(r)}$ of $S^r = (t, J^r, I_r)$,

$$\begin{aligned}(\dot{f} \circ c^{(r+1)})(t) &= \langle df, F_{r+1}(c^{(r+1)}(t)) \rangle \\ &= \langle df, \frac{d}{dt}c^{(r)}(t) \rangle \\ &= \frac{d}{dt}(f \circ c^{(r)})(t).\end{aligned}$$

This justifies the term "time derivative." Also it follows from the definition that

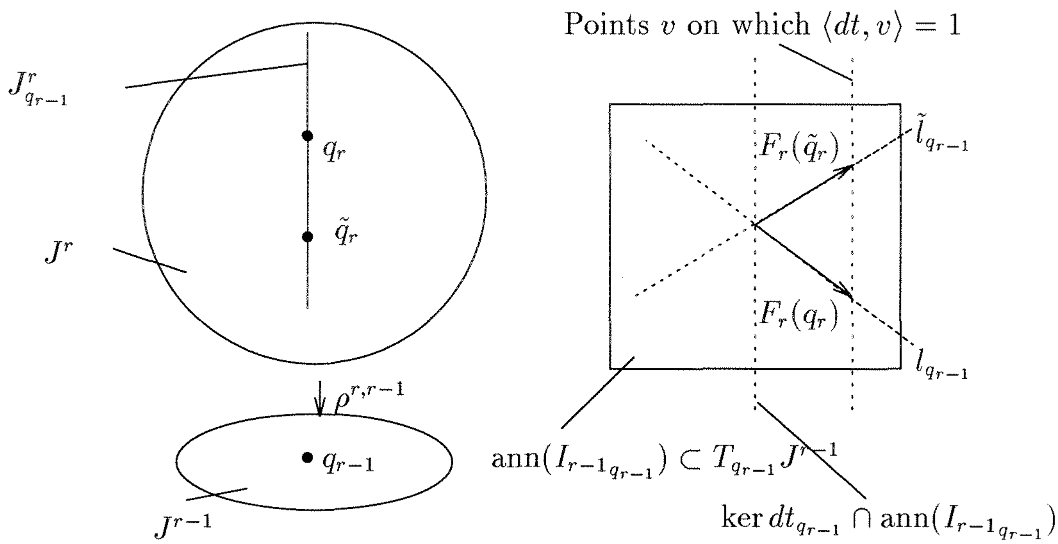


Figure 3.3 Vector field F_r.

the time derivative $\dot{f}$ is a function that is affine in the fibres of J^{r+1} over J^r. This may be easier to see in coordinates, once we derive the formula for time derivative in coordinates. Higher order time derivatives are defined in the obvious way and are denoted by $f^{(r)}$. The second order time derivative may be denoted by $\ddot{f}$ as well. Also when indices are present, as in f^j, the time derivatives are denoted by $f^{j,(r)}$ to avoid confusion.

Remark 3.5 Observe that the time derivative of $t : J^r \to \mathbb{R}$ is 1, the constant function on J^{r+1} of value 1.

It turns out that given the coordinate system on M there is a corresponding coordinate system on J^r that is arrived at by taking time derivatives of some of the coordinates on M. It can be seen that the functions f^j and u^j mentioned earlier are in fact related by $f^j = \dot{u}^j$. This is because by definition, $\langle du^j - f^j dt, F_1\rangle = 0$. Hence $\langle du^j, F_1\rangle = f^j\langle dt, F_1\rangle = f^j$. But $\langle du^j, F_1\rangle = \dot{u}^j$ by definition and hence $f^j = \dot{u}^j$. Hence $\dot{u}^j$ for $j = 1, \ldots, p$ is a set of fibre coordinates for J^1. In the coordinate system $(t, x, u, \dot{u})$ for J^1

$$I_1 = \mathrm{span}\{dx^i - (A^i + B^i_j\dot{u}^j)dt, du^j - \dot{u}^j dt \ : \ i = 1, \ldots, n, \quad j = 1, \ldots, p\},$$

where the functions A^i and B^i_j are the same as those that appear in equation (3.6). Since $dt, I_1, d\dot{u}^1, \ldots, d\dot{u}^p$ form a local coframe on J^1, $\dot{u}^j$ play the the role of u^j for system S^1. So $\ddot{u}$ form a set of fibre coordinates for the fibres of J^2 over J^1. It also follows that $u^{j,(r)}$ are fibre coordinates for the fibres of J^r over J^{r-1}. Hence if $(t, x^1, \ldots, x^n, u^1, \ldots, u^p)$ are local coordinates on M such that $dt, I, du^1, \ldots, du^p$

span T^*M locally, then $(t, x, u, u^{(1)}, \ldots, u^{(r)})$ form a coordinate system on J^r. We also see that for $r \geq 1$,

$$I_r = \text{span}\{dx^i - (A^i + B^i_j u^{j,(1)})dt, du^{j,(k)} - u^{j,(k+1)}dt \\ : i = 1, \ldots, n, \quad j = 1, \ldots, p, \quad k = 0, \ldots, r-1\}. \tag{3.10}$$

The vector field F_r is given in coordinates by

$$F_r = \frac{\partial}{\partial t} + (A^i + B^i_j u^{j,(1)})\frac{\partial}{\partial x^i} + \sum_{k=1}^{k=r} u^{j,(k)} \frac{\partial}{\partial u^{j,(k-1)}},$$

where summation over i and j is intended.

Remark 3.6 We shall frequently employ the collective notation as in above paragraph where we have referred to $(u^{1,(j)}, \ldots, u^{p,(j)})$ as $u^{(j)}$ for $j = 0, \ldots, r$ and to $(x^1, \ldots, x^n)$ as x.

Remark 3.7 The time derivatives of coordinates x^i are functions on J^1 and they are given in terms of the coordinates $(t, x, u, \dot{u})$ by the equations (3.6) which are precisely the ODEs that I corresponds to. In fact all relations amongst the time derivatives of all orders, of functions on J^r correspond to the various ODEs "implied by" the original system of ODEs (3.6). Thus the time derivative operator allows us to bypass thinking in terms of solution curves and to think in terms of dependencies of various functions (variables) and their time derivatives of various orders. This allows us to get closer to the algebraic intuition offered by differential algebra and to formulate a definition of flatness that closely resembles the differential algebraic definition but yet provides a geometric setting where tools of geometric analysis can be applied and unnecessary assumptions on analyticity can be avoided.

Remark 3.8 It follows from the definition that, in coordinates, given a function $f(t, x, u, \ldots, u^{(r)})$ on J^r its time derivative is given by

$$\dot{f} = \frac{\partial f}{\partial t} + (A^i + B^i_j u^{j,(1)})\frac{\partial f}{\partial x^i} + \sum_{k=0}^{k=r} u^{j,(k+1)} \frac{\partial f}{\partial u^{j,(k)}},$$

where we have used the equation (3.6) to write $\dot{x}$ in terms of the coordinates. Hence it follows that $\dot{f}$ is affine in the fibre coordinates $u^{j,(r+1)}$.

We shall introduce the notion of a *semibasic* form which will be used frequently in this chapter.

Definition 3.9 A form Ω on J^r is said to be *semibasic with respect to* $\rho^{r,l}$ where $0 \leq l < r$, if Ω lies in the pull back of T^*J^l to J^r.

Remark 3.10 This means in coordinates Ω has no $du^{j,(k)}$ terms for $k > l$. But the coefficients of the terms may depend on $u^{(k)}$ for $k \leq r$. For example $u^{1,(r)}dx^1 + u^{2,(r-1)}du^2$ is a one-form on J^r that is semibasic with respect to $\rho^{r,0}$.

A key step in the theory is to extend the time derivative operator to one-forms. Later we shall see that this allows us to define flatness in terms of one-forms and their time derivatives and deal with integrability (the one-forms being exact) separately. In fact we may define the time derivative for arbitrary k-forms. The time derivative of a k-form Ω on J^r is a k-form on J^{r+1} denoted by $\dot{\Omega}$ and is defined by

$$\dot{\Omega} = F_{r+1} \lrcorner d\Omega + d(F_{r+1} \lrcorner \Omega) \tag{3.11}$$

for $k > 0$. The $k = 0$ case corresponds to functions and has already been dealt with. Note that the above definition make sense since Ω and $d\Omega$ are forms on J^r and image of F_{r+1} lies in TJ^r. Also note $F_{r+1} \lrcorner d\Omega$ and $F_{r+1} \lrcorner \Omega$ are semibasic with respect to $\rho^{r+1,r}$ but $d(F_{r+1} \lrcorner \Omega)$ is generally not semibasic (with respect to $\rho^{r+1,r}$). The second derivative is denoted by $\ddot{\Omega}$. We will also use $\Omega^{(k)}$ for the kth time derivative. When there is an index as in Ω^j, we shall denote the kth derivative by $\Omega^{j,(k)}$ to avoid confusion.

If f is a function on J^r then it follows from the definition that $(df)^{(1)} = d(f^{(1)})$ and also that if α is a one form on J^r then $(f\alpha)^{(1)} = f^{(1)}\alpha + f\alpha^{(1)}$. These observations allow us to calculate time derivatives of one-forms in coordinates, which we illustrate by the following example.

Example 3.11 Consider the system of two one forms

$$\begin{aligned} \omega^1 &= dx^1 - x^2 du^1 - x^1 dt \\ \omega^2 &= dx^2 - x^1 du^2 - u^1 dt, \end{aligned}$$

in coordinates (t, x^1, x^2, u^1, u^2) on some manifold M. Let $\alpha = u^1 dx^2 + x^2 du^1$. Then

$$\begin{aligned} \dot{\alpha} &= u^1 d\dot{x}^2 + \dot{u}^1 dx^2 + x^2 d\dot{u}^1 + \dot{x}^2 du^1 \\ &= u^1 d(x^1 \dot{u}^2 + u^1) + \dot{u}^1 dx^2 + x^2 d\dot{u}^1 + (x^1 \dot{u}^2 + u^1) du^1 \\ &= u^1 x^1 d\dot{u}^2 + u^1 \dot{u}^2 dx^1 + u^1 du^1 + \dot{u}^1 dx^2 + x^2 d\dot{u}^1 + (x^1 \dot{u}^2 + u^1) du^1, \end{aligned}$$

which is a one-form given in terms of local coordinates $(t, x^1, x^2, u^1, u^2, \dot{u}^1, \dot{u}^2)$ on J^1.

Remark 3.12 The time derivative operator of forms defined here is exactly the same as the Lie derivative along a Cartan vector field as defined Fliess et al. [10, 11] and Pomet [27]. The major difference is that their approach involves working in a space which in our notation might be called J^∞. This space is not a manifold however. Also the Cartan vector field in their work may be roughly thought of as F_∞, the limiting form of the "vector fields" F_r. Proper definition of J^∞ and F_∞ requires more mathematical tools which we shall avoid.

3.4 Vertical Spaces and Extended Codistributions

In this section we shall define certain "vertical bundles" that play an important role in later developments. As mentioned in Section 3.1 some algebraic properties of the Jacobian of (3.3) that are exploited in Appendix A reflect more general geometric properties of the spaces J^k and the time derivative operator. Some of these properties are concerned with the "vertical spaces" of the bundles J^k over J^{k-1}. We shall investigate them in this section.

Also we shall introduce the notion of extended codistributions and distributions which will later be used to define flatness.

Let $V_q(t, M)$ denote the "vertical space" at $q \in M$ of the bundle $t : M \to \mathbb{R}$. In other words, $V_q(t, M) = \ker dt_q$. Then $V(t, M) \subset TM$ will stand for the corresponding bundle. Also let us define $V_q^0 S = \ker dt_q \cap \operatorname{ann}(I_q)$ and denote the corresponding bundle by $V^0 S$. Clearly $V^0 S \subset V(t, M) \subset TM$. For $0 \leq l < r$ and a point $q_r \in J^r$ we shall define $V_{q_r}^{r,l} S = \ker T_{q_r} \rho^{r,l}$. The corresponding bundle is denoted by $V^{r,l} S$. As usual we may drop S and just use V, V^0 and $V^{r,l}$.

Remark 3.13 In a coordinate system (t, x, u) on M with the property that (dt, du) complement I to a local coframe, for all $r \geq 1$, vectors $\frac{\partial}{\partial u^{j,(r)}}_{q_r}$ for $j = 1, \ldots, p$ form a basis for $V_{q_r}^{r,r-1}$. It is also clear that $\frac{\partial}{\partial u^j}_q + B_j^i \frac{\partial}{\partial x^i}_q$ for $j = 1, \ldots, p$ form a basis for V_q^0 where B_j^i are the functions that appear in (3.5).

We present a few important lemmas about these vertical spaces.

Lemma 3.14 *For all $q_1 \in J^1$, the space $V_{q_1}^{1,0}$ is canonically isomorphic to V_q^0, where $q = \rho^{1,0}(q_1)$.*

Proof: The fibre J_q^1 is an affine space modelled on $\ker dt_q \cap \operatorname{ann}(I_q) = V_q^0$ and hence $T_{q_1}(J_q^1)$ is canonically isomorphic to V_q^0. But $V_{q_1}^{1,0}$ is the same as $T_{q_1}(J_q^1)$. ■

Lemma 3.15 *For all $q_{r+1} \in J^{r+1}$ with $r \geq 0$, $V_{q_{r+1}}^{r+1,r}$ is canonically isomorphic to $V_{q_r}^{r,r-1}$, where $q_r = \rho^{r+1,r}(q_{r+1})$.*

Proof: It is clear from the definition that $V_{q_{r+1}}^{r+1,r} S$ is the same as $V_{q_{r+1}}^{1,0} S^r$. Also from the equation (3.10) it is clear that $\operatorname{span}\{dt\} + I_r$ is the annihilator of $V_{q_r}^{r,r-1} S$. But $\operatorname{span}\{dt\} + I_r$ is also the annihilator of $V_{q_r}^0 S^r$ and hence $V_{q_r}^0 S^r$ and $V_{q_r}^{r,r-1} S$ are the same. Hence the proof follows from Lemma 3.14. ■

The following corollary is immediate.

Corollary 3.16 *For all $q_r \in J^r$ and $r \geq 1$, $V_{q_r}^{r,r-1}$ is canonically isomorphic to V_q^0, where $q = \rho^{r,0}(q_r)$.*

Remark 3.17 In a coordinate system (t, x, u) on M as in Remark 3.13, vectors $\frac{\partial}{\partial u^{j,(r)}}_{q_r} \in V_{q_r}^{r,r-1}$ are mapped by the isomorphism to $\frac{\partial}{\partial u^j}_q \in V_q^0$.

We can use this isomorphism to "lift" a vertical vector and we shall use the same notation as the one for time derivatives to denote this. Thus given $\eta_{q_r} \in V_{q_r}^{r,r-1}$, $\dot{\eta}_{q_{r+1}}$ or $\eta_{q_{r+1}}^{(1)}$ denotes the corresponding vector in $V_{q_{r+1}}^{r+1,r}$ where q_{r+1} is any point such that $q_r = \rho^{r+1,r} q_{r+1}$. The meaning of $\eta^{(k)}_{q_{r+k}}$ is clear. Also given any vector field (need not be smooth) η on J^r that lies in $V^{r,r-1}$, its lift to J^{r+k}, a vector field $\eta^{(k)}$ on J^{r+k} that lies in $V^{r+k,r+k-1}$ is defined in the obvious way.

Though the same notation is used for time derivatives of forms as well as lifting of vertical vector fields the two operations are quite different from each other since the lifting of a vertical vector field is a pointwise operation that only depends on the value of the vector field at a point but not on how it varies locally, whereas the time derivative of a form depends on the first order derivative of a form. The reason for adopting the same notation is evident from the following lemmas. Since there is no notion of a time derivative for vector fields the notation does not cause any confusion.

Lemma 3.18 *Let f be a smooth function on M and η a vector field in V^0. Then $\dot{\eta}[\dot{f}] = \eta[f]$ (pull back implied).*

Proof: Let $q \in M$ and $q_1 \in J^1_q$. By definition $\dot{\eta}_{q_1} \in V^{1,0}_{q_1}$ and

$$\dot{\eta}_{q_1}[\dot{f}] = \lim_{s \to 0} \frac{\dot{f}(q_1 + s\eta_q) - \dot{f}(q_1)}{s}.$$

Note that since J^1_q is an affine space modelled on V^0_q, the sum $q_1 + s\eta_q$ makes sense. The right hand side of the equation can be rewritten as

$$\lim_{s \to 0} \frac{\langle df_q, F_1(q_1 + s\eta_q)\rangle - \langle df_q, F_1(q_1)\rangle}{s}.$$

Since F_1 is affine in the fibres of J^1 over M it follows that,

$$\langle df_q, F_1(q_1 + s\eta_q) - F_1(q_1)\rangle = \langle df_q, s\eta_q\rangle.$$

Hence

$$\dot{\eta}_{q_1}[\dot{f}] = \langle df_q, \eta_q\rangle = \eta_q[f],$$

completing the proof. ■

Lemma 3.19 *Let α be a one-form on J^r that is semibasic with respect to $\rho^{r,0}$ and let $\eta \in V^0$. Here η may be regarded as a single vector or as a vector field that lies in V^0 which need not be smooth. Then*

$$\langle \dot{\alpha}, \dot{\eta}\rangle = \langle \alpha, \eta\rangle. \qquad (3.12)$$

Proof: Choose a coordinate system (t, x) on M. Then α can be written as $\alpha = a_i dx^i + a dt$ where a_i and a are in general functions on J^r and hence

$$\dot{\alpha} = \dot{a}_i dx^i + \dot{a} dt + a_i d\dot{x}^i.$$

Since $\langle dx^i, \dot{\eta}\rangle = \langle dt, \dot{\eta}\rangle = 0$,

$$\begin{aligned}\langle \dot{\alpha}, \dot{\eta}\rangle &= a_i \langle d\dot{x}^i, \dot{\eta}\rangle \\ &= a_i \langle dx^i, \eta\rangle = \langle \alpha, \eta\rangle,\end{aligned}$$

where we have used Lemma 3.18. ■

Remark 3.20 The above lemmas are still true if one replaces 0 by l, in other words replace M by J^l, $\rho^{r,0}$ by $\rho^{r,l}$ and V^0 by $V^{l,l-1}$, where $0 < l < r$. This is seen by replacing system S with S^l.

Remark 3.21 The above lemmas are generalisations of the fact that the submatrices $\frac{\partial E}{\partial \dot{z}}$ and $\frac{\partial \dot{E}}{\partial \ddot{z}}$ in the Jacobian of (3.3) are equal.

From above lemmas we obtain local coordinates formulae for the lift of appropriate vector fields. For the basis vector fields η_j of V^0 given by

$$\eta_j = \frac{\partial}{\partial u^j} + B^i_j \frac{\partial}{\partial x^i}$$

the corresponding lifts are given by

$$\dot{\eta}_j = \frac{\partial}{\partial \dot{u}^j}.$$

For the basis vector fields $\frac{\partial}{\partial u^{j,(r)}}$ of $V^{r,r-1}$ the lifts are given by $\frac{\partial}{\partial u^{j,(r+1)}}$.

We present a few more lemmas that will be of use later.

Lemma 3.22 *Let α be a one-form on M with $\dot{\alpha}$ being semibasic (with respect to $\rho^{1,0}$). Then α lies in span$\{dt\} + I$.*

Proof: Since $\dot{\alpha}$ is semibasic it follows that for any $v \in V^{1,0}$

$$\langle \dot{\alpha}, v\rangle = 0.$$

But then by Lemma 3.19 for any $\eta \in V^0$

$$\langle \alpha, \eta\rangle = \langle \dot{\alpha}, \dot{\eta}\rangle = 0$$

and hence it follows α lies in $\mathrm{ann}(V^0) = \mathrm{span}\{dt\} + I$. ■

Lemma 3.23 *If α is a one-form on M that lies in I then $\dot{\alpha}$ lies in I_1. In other words $I_{(1)} \subset I_1$. The converse is true if we assume that the bottom derived system of I is trivial. More precisely, suppose that the bottom derived system of I is trivial and that α is a one-form on M such that $\dot{\alpha}$ lies in I_1. Then α lies in I.*

Proof: Suppose α lies in I and let $v_{q_1} \in \mathrm{ann}(I_{1q_1})$. Suppose l_q is the one dimensional subspace corresponding to q_1 where $q = \rho^{1,0} q_1$. By definition of I_1 and F_1 it follows that $T_{q_1}\rho^{1,0}v_{q_1} = \lambda F_1(q_1)$, for some $\lambda \in \mathbb{R}$. Also $\langle \alpha, F_1 \rangle = 0$ by definition of F_1. Hence

$$\begin{aligned}\langle \dot{\alpha}, v_{q_1} \rangle &= d\alpha_q(F_1(q_1), T_{q_1}\rho^{1,0}v_{q_1}) + (d(\langle \alpha, F_1 \rangle))_{q_1} \\ &= d\alpha_q(F_1(q_1), \lambda F_1(q_1)) = 0.\end{aligned}$$

It follows that $\dot{\alpha}$ lies in I_1.

Conversely if $\dot{\alpha}$ lies in I_1 from Lemma 3.22 α lies in $\text{span}\{dt\} + I$, since forms that lie in I_1 are semibasic. Let $\alpha = \omega + \lambda dt$, where ω lies in I and λ is a function on M. Hence $\dot{\alpha} = \dot{\omega} + \dot{\lambda} dt$. But $\dot{\omega}$ lies in I_1. So it follows $\dot{\lambda} = 0$ and this is possible only if $d\lambda \in I$. But since bottom derived system of I is trivial it follows $\lambda = 0$ implying that $\alpha \in I$. ■

Given a smooth codistribution $P \subset T^*M$ its *extended codistribution* is a codistribution on J^1, denoted by $P_{(1)} \subset T^*J^1$ and defined by

$$P_{(1)} = \text{span}\{\alpha, \dot{\alpha} \ : \ \alpha \in P\}, \tag{3.13}$$

where we have defined $P_{(1)}$ as a module of one-forms on J^1 and the span is over smooth functions on J^1 and α are one-forms in the module P. Successive extended codistributions are defined in the obvious way,

$$P_{(r)} = \text{span}\{\alpha, \dot{\alpha}, \ldots, \alpha^{(r)} \ : \ \alpha \in P\}. \tag{3.14}$$

Remark 3.24 Codistribution P need not be constant dimensional for above definitions to make sense.

Given a constant dimensional and smooth distribution $D \subset TM$ we define its rth extended distribution $D_{(r)} \subset TJ^r$ by

$$D_{(r)} = \text{ann}(P_{(r)}), \tag{3.15}$$

where $P = \text{ann}(D)$. Constant dimensionality of D ensures that P is smooth and hence the definition makes sense.

The following is a consequence of above definitions.

$$T_{q_r}\rho^{r,r-1}D_{(r)_{q_r}} \subset D_{(r-1)_{q_{r-1}}} \quad \forall q_r \in J^r, \tag{3.16}$$

where $q_{r-1} = \rho^{r,r-1}q_r$.

Finally we present a lemma that relates the dimensions of intersections of a distribution and its extensions with the appropriate vertical spaces.

Lemma 3.25 *Let D be a smooth constant dimensional distribution on M. Then*

$$(D \cap V^0)^{(1)} = D_{(1)} \cap V^{1,0}, \tag{3.17}$$

where $(D \cap V^0)^{(1)} \subset V^{1,0}$ is the distribution obtained by lifting all the vectors in $D \cap V^0$.

Proof: Let η be a vector field in V^0 (need not be smooth). Let α be a one-form taking values in $\mathrm{ann}(D)$. Since $\langle\dot{\alpha}, \dot{\eta}\rangle = \langle\alpha, \eta\rangle$, it follows η is in D if and only if $\dot{\eta}$ is in $D_{(1)}$. ∎

The following corollary is obtained by replacing S with S^{r-1}.

Corollary 3.26

$$(D \cap V^0)^{(r)} = D_{(r)} \cap V^{r,r-1}. \tag{3.18}$$

3.5 Differential Flatness

In this section we shall introduce the notion of a generator on M and provide a definition of differential flatness. Roughly speaking a generator is a codistribution $P \subset T^*M$ with the property that any one-form on J^r for arbitrary r is in $P_{(k)}$ for some large enough k. When integrable, a generator contains the differentials of flat outputs that only depend on (t, x, u), i.e. zero-flat outputs.

Throughout this section unless stated otherwise the system under consideration is $S = (t, M, I)$ with $\dim I = n$ and $\dim M = 1 + n + p$.

Firstly we present a useful lemma which illustrates the nature of singularities that occur along the fibres of J^r.

Lemma 3.27 *Let $\{\alpha^1, \ldots, \alpha^k\}$ be a set of one-forms on M. Let $\{d_1, \ldots, d_k\}$ be a set of integers all less than or equal to r. Consider the set*

$$Z = \{\alpha^1, \ldots, \alpha^{1,(d_1)}, \ldots, \alpha^k, \ldots, \alpha^{k,(d_k)}\}$$

of one-forms on J^r. The following are true.

1. *If Z is linearly dependent in an open set U_r of J^r then it is linearly dependent for all points in the open set $(\rho^{r,0})^{-1} \circ \rho^{r,0} U_r$.*

2. *If Z is linearly independent at a point $q_r \in J^r$ then there exists an open neighbourhood U of $q = \rho^{r,0} q_r$ such that Z is linearly independent in an open dense subset of $(\rho^{r,0})^{-1}\tilde{q}$ for all $\tilde{q} \in U$.*

Proof: Choosing local coordinates (t, x, u) on M and the corresponding coordinates $(t, x, u, \ldots, u^{(r)})$ on J^r one can relate the linear dependence or independence of Z to the rank of a certain matrix which is a function of $(t, x, u, \ldots, u^{(r)})$. Since the forms α^j are forms on M it follows that all the forms in Z and consequently the above matrix depend on $u^{(1)}, \ldots, u^{(r)}$ polynomially. Since the only polynomial that is zero in an open set is the trivial one, Statement 1 follows. Statement 2 follows by adding the continuity argument. ∎

Definition 3.28 Let $Z = \{\alpha^1, \ldots, \alpha^k\}$ be a set of one-forms on M. The set Z is said to be *differentially r-independent* at q if the set $\{dt, \alpha, \alpha^{(1)}, \ldots, \alpha^{(r)}\}$ is linearly independent in an open dense subset of points in $(\rho^{r,0})^{-1}q$. The set Z is said to be *differentially r-dependent* around q if there exists an open neighbourhood U of q such that the set $\{dt, \alpha, \alpha^{(1)}, \ldots, \alpha^{(r)}\}$ is linearly dependent for all the points in $(\rho^{r,0})^{-1}U$.

When $Z = \{dy^1, \ldots, dy^k\}$, differential r-dependence of Z around $q \in M$ implies that there exists an open neighbourhood U of q such that around any point $q_r \in (\rho^{r,0})^{-1}U$ there exists a submersive function f such that

$$f(t, y, \ldots, y^{(r)}) = 0$$

every where in an open neighbourhood of q_r.

Remark 3.29 It follows from Lemma 3.27 that the set Z in above definition is either differentially r-independent around q or differentially r-dependent around q for an open dense subset of points q in M. Also the Lemma 3.27 implies that differential independence at q implies differential independence "around" q.

The differential r-independence of a set Z is completely determined by the codistribution spanned by Z and dt. This fact leads to the following definition.

Definition 3.30 Let P be a constant dimensional smooth codistribution containing dt. Then P is said to be *r-regular* at $q \in M$ if $\dim P_{(r)} = 1 + (r+1)(\dim P - 1)$ (which is the maximum possible dimension) in an open dense subset of points of $(\rho^{r,0})^{-1}q \subset J^r$.

It follows from the definition if

$$P = \text{span}\{dt, \alpha^1, \ldots, \alpha^k\},$$

then $\{\alpha^1, \ldots, \alpha^k\}$ is differentially r-independent around $q \in M$ if and only if P is r-regular around q. Also if the codistribution P is r-regular at q then it is also s-regular at q for $0 < s < r$.

Definition 3.31 Let P be a constant dimensional smooth codistribution around $q \in M$ that contains dt. Then P is an *r-generator* for the system S around q if P is r-regular at q and there exists an open neighbourhood $U \subset M$ of q such that $(\rho^{r,0})^*(T^*M) \subset P_{(r)}$ in an open dense subset of points in $(\rho^{r,0})^{-1}U$. If P is an r-generator for some integer r then P is said to be a *generator*.

Remark 3.32 The notion of a generator is essentially the same as the "linearising Pfaffian system" introduced by Pomet [27], except Pomet's linearising Pfaffian system could be a codistribution on J^r for some r which according to our definition will be a generator of S^r instead of S. Also Pomet only considers time independent systems and hence does not have t as coordinate on his space. It has been shown that linearising Pfaffian systems always exist and one example is given by the Pfaffian system constructed during the construction of the infinitesimal Brunovsky normal form; see Aranda et al. [3]. Also it can be shown with the aid of module theory that any generator must have dimension $p+1$. See [12, 11, 27] as well. From here onwards we shall use this fact.

Remark 3.33 It is clear from the definition that if P is an r-generator then it is also an s-generator for $s > r$.

Now we can define *zero-flatness* of a system.

Definition 3.34 Let $y^1, \ldots, y^k$ be functions around $q \in M$ and let

$$P = \text{span}\{dt, dy^1, \ldots, dy^k\}.$$

Then $y^1, \ldots, y^k$ are (0,r)- *flat outputs* of the system S around q if P is an r-generator for the system S around q. If such y exist around q then the system S is said to be (0,r)-flat around q. If system S is (0,r)-flat around q for some integer r then the system S is said to be *zero-flat* around q and corresponding y are *zero-flat outputs* around q.

It follows that in local coordinates (t, x, u) on M the zero-flat outputs are given by some map h,

$$y = h(t, x, u)$$

in some open set $U \subset M$ while x and u in turn are given by some maps g_1 and g_2,

$$\begin{aligned} x &= g_1(t, y, \ldots, y^{(r)}), \\ u &= g_2(t, y, \ldots, y^{(r)}), \end{aligned} \tag{3.19}$$

in an open dense subset of points in $(\rho^{r,0})^{-1}U$. It also follows that the number of zero-flat outputs k equals p, where $p = \dim M - \dim P - 1$, by Remark 3.32.

Remark 3.35 Lemma 3.27 essentially implies that if Equation (3.19) holds in an open subset of $(\rho^{r,0})^{-1}U$ then it holds in an open dense subset of $(\rho^{r,0})^{-1}U$. This allows us to define the zero-flatness of a system S around a point $q \in M$ without having to mention points in J^r.

Definition 3.36 The system S is said to be *(l, r)-flat* around $q_l \in J^l$ if S^l is $(0, r)$-flat around q_l. The system S is *differentially flat* around $q_l \in J^l$ if S^l is zero-flat around q_l.

It follows that in local coordinates (t, x, u) on M the flat outputs are given by some map h,

$$y = h(t, x, u, \ldots, u^{(l)})$$

in some open set $U_l \subset J^l$ while x and u in turn are given by some maps g_1 and g_2 as in Equation (3.19) in some an open dense subset of $(\rho^{l+r,l})^{-1}U_l$.

The following lemma is needed to prove an important lemma concerning generators.

Lemma 3.37 *Let P be a constant dimensional codistribution of dimension $p+1$ containing dt and suppose there exists a nonvanishing one-form in $U \subset M$ that does not lie in P but lies in $P_{(1)}$ for points in an open set $U_1 \subset (\rho^{1,0})^{-1}U$. Then $P \cap I$ is nontrivial for all points in $\rho^{1,0}U_1 \subset U$.*

Proof: Let $P = \text{span}\{dt, \alpha^1, \ldots, \alpha^p\}$. By assumption there exists a one-form β on $U \subset M$ that does not lie in P but is in the span of $\dot{\alpha}^j$ for $j = 1, \ldots, p$. Let $\beta = f_j\dot{\alpha}^j$ without loss of generality, where f_j (not all vanishing at the same point) are in general smooth functions on $U_1 \subset J^1$. Then for points on $U_1 \subset J^1$

$$\begin{aligned} & \langle f_j\dot{\alpha}^j, \dot{v}\rangle = 0, \quad \forall v \in V^0 \\ \iff & f_j\langle \dot{\alpha}^j, \dot{v}\rangle = 0, \quad \forall v \in V^0 \\ \iff & f_j\langle \alpha^j, v\rangle = 0, \quad \forall v \in V^0 \end{aligned}$$

where the expressions are all functions on U_1, where we have used Lemma 3.19. But since $\langle \alpha^j, v\rangle$ is clearly a function only on M it is possible to find g_j for $j = 1, \ldots, p$ (not all vanishing at the same point) that are functions only on M such that

$$g_j\langle \alpha^j, v\rangle = 0, \quad \forall v \in V^0,$$

for all points on $\rho^{1,0}U_1$. Choosing $\gamma = g_j\alpha^j$ (so that γ lies in P) we also see that $\langle \gamma, v\rangle = 0$ for all $v \in V^0$, implying that γ lies in $P \cap I$ for all points on $\rho^{1,0}U_1$. But γ is nonvanishing in $\rho^{1,0}U_1$. ■

Now we present an important lemma about generators.

Lemma 3.38 *Let P be a constant dimensional codistribution of dimension $p+1$ containing dt and suppose P is a generator of $S = (t, M, I)$ around $q \in M$. Then $P \cap I$ and $D \cap V^0$ are both nontrivial around q.*

Proof: Let $D = \text{ann}(P)$ and recall $V^0 = \text{ann}((I + \text{span}\{dt\}))$. So $D \cap V^0 = \text{ann}(P + I)$. From linear algebra

$$\dim(P \cap I) = \dim P + \dim I - \dim(P + I)$$

for all points on M and we also know that $\dim P + \dim I = \dim M$. Hence it follows that $\dim(P \cap I) = \dim(D \cap V^0)$ pointwise. It also follows that

$$\dim(P_{(r)} \cap I_{(r)}) = \dim(D_{(r)} \cap V^{r,r-1}), \quad \forall r,$$

for all points on J^r.

Now since P is flat, there exists an open neighbourhood U of q and an integer r such that in an open dense subset $W_r \subset (\rho^{r,0})^{-1}U$ the codistribution $P_{(r)}$ spans all the forms on M and hence by lemma 3.37 it follows that $P_{(r-1)} \cap I_{r-1}$ is nontrivial for all the points in $\rho^{r,r-1}W_r$. But $\rho^{r,r-1}W_r$ is an open dense subset of $(\rho^{r-1,0})^{-1}U$, since the image under a submersion of an open dense subset is also an open dense subset. Since $P_{(r-1)}$ and I_{r-1} are smooth, it follows that if their intersection is trivial at a point in $(\rho^{r-1,0})^{-1}U$ then it must also be trivial in an open neighbourhood of that point, which violates the fact that the intersection is nontrivial in an open dense subset of $(\rho^{r-1,0})^{-1}U$. Hence it follows that $P_{(r-1)} \cap I_{r-1}$ is nontrivial for all points in $(\rho^{r-1,0})^{-1}U$. But this is equivalent to $D_{(r-1)} \cap V^{r-1,r-2}$ being nontrivial for all points in $(\rho^{r-1,0})^{-1}U$, which in turn is equivalent to $D \cap V^0$ being nontrivial for all points in U by (3.18). Hence it also follows that $P \cap I$ is nontrivial for all points in q. ∎

The above lemma is a generalisation of the fact that any Cartan prolongation of a trivial system contains at least one one-form that is semibasic (with respect to the projection onto the manifold of the trivial system). (This statement is true even for Cartan prolongations of nontrivial systems. See Sluis [35] for more details).

3.6 Zero-Flatness of a System of Two One-Forms

In this section we present a theorem that partially characterises zero-flatness of a system of two one-forms. We need the following lemma before stating the theorem.

Lemma 3.39 *Let η be a vector field in $V \subset TM$ and ω be a one-form in I. Then*

$$\langle \dot{\omega}, \eta \rangle = 0 \iff \eta \,\lrcorner\, d\omega = 0 \mod I.$$

(Note that from Lemma 3.23 it follows that $\dot{\omega}$ takes values in T^*M and hence its pairing with η makes sense and results in a function on J^1.)

Proof: Since $\langle \omega, F_1 \rangle = 0$ by definition of F_1 it follows from the definition of time derivative that $\dot{\omega} = F_1 \,\lrcorner\, d\omega$. Hence

$$\langle \dot{\omega}, \eta \rangle = d\omega(F_1, \eta) = \langle \eta \,\lrcorner\, d\omega, F_1 \rangle,$$

where all the expressions are functions on J^1. By definition of F_1, for any one form α on J^1, $\langle \alpha, F_1 \rangle = 0$ if and only if α lies in I (after I has been pulled back to J^1.) The proof follows by substituting $\alpha = \eta \,\lrcorner\, d\omega$ in above. ∎

Theorem 3.40 *Let $S = (t, M, I)$ be a system with $\dim I = 2$. Suppose locally around $q \in M$ that $I = span\{\omega^1, \omega^2\}$. Also suppose $D \subset V$ is a two dimensional distribution around q with $D = span\{\eta_1, \eta_2\}$ and that $P = \mathrm{ann}(D)$ is 2-regular around q (note that $D \subset V$ is equivalent to P containing dt). Then P is a 2-generator for system S around q if and only if*

$$\eta_1 \,\lrcorner\, \eta_2 \,\lrcorner\, (\omega^1 \wedge \omega^2) = 0 \tag{3.20}$$

and

$$\eta_1 \,\lrcorner\, \eta_2 \,\lrcorner\, d(\omega^1 \wedge \omega^2) = 0 \mod I \tag{3.21}$$

for points around q.

Proof: Observe that the conditions (3.20) and (3.21) are independent of the choice of basis for I and D and therefore it is reasonable to work with a special choice of bases. The condition (3.20) is equivalent to the 2×2 matrix with entries $\langle \omega^i, \eta_j \rangle$ being singular and hence (3.20) holds if and only if there exists a choice of ω^i and η_j such that

$$\begin{aligned} \langle \omega^i, \eta_1 \rangle &= 0, \quad i = 1,2 \\ \langle \omega^1, \eta_j \rangle &= 0, \quad j = 1,2, \end{aligned} \tag{3.22}$$

and this is equivalent to $P \cap I$ being nontrivial. Hence $P \cap I$ being nontrivial is equivalent to (3.20) pointwise.

Necessity: Suppose P is a 2-generator around q. Then by Lemma 3.38 $P \cap I$ and $D \cap V^0$ are nontrivial around q. Hence without loss of generality we can assume that (3.22) holds around q. This also means $\omega^1 \in P \cap I$ and $\eta_1 \in D \cap V^0$ around q.

Let U be an open set containing q such that P is a 2-generator in U and (3.22) is valid in U. Since $P \cap I$ is nontrivial in U, its dimension is either 1 or 2. Let $W \subset U$ denote the open set in which dim $P \cap I$ is 1. Then η_1 spans $D \cap V^0$ and ω^1 spans $P \cap I$ in W. Since η_1 spans $D \cap V^0$ it follows that $\dot{\eta}_1$ spans $D_{(1)} \cap V^{1,0}$ and that $\ddot{\eta}_1$ spans $D_{(2)} \cap V^{2,1}$ at points that project down to W by Corollary 3.18. By 2-regularity of P it follows that $D_{(2)}$ is two dimensional in an open dense subset of of $(\rho^{2,0})^{-1}W$. Choose a smooth vector field $\bar{\eta}$ on $(\rho^{2,0})^{-1}W$ that complements $\ddot{\eta}_1$ to a basis for $D_{(2)}$ in this open dense subset. Then $T_{q_2}\rho^{2,1}\bar{\eta}_{q_2} \in D_{(1),q_1}$ for points $q_2 \in (\rho^{2,0})^{-1}W$. Since P is a 2-generator in U and hence in $W \subset U$, $D_{(2)} \subset V^{2,0}$ in an open dense subset of $(\rho^{2,0})^{-1}W$. Since $\bar{\eta}$ is smooth and lies in $D_{(2)}$ it follows that $\bar{\eta}$ lies in $V^{2,0}$ for all points in $(\rho^{2,0})^{-1}W$. Hence for all points $q_2 \in (\rho^{2,0})^{-1}W$ it follows that $T_{q_2}\rho^{2,0}\bar{\eta}_{q_2} = 0$ implying that $T_{q_2}\rho^{2,1}\bar{\eta}_{q_2} \in V^{1,0}_{q_1}$ where $q_1 = \rho^{2,1}q_2$. Therefore for all points $q_2 \in (\rho^{2,0})^{-1}W$, $T_{q_2}\rho^{2,1}\bar{\eta}_{q_2} \in D_{(1),q_1} \cap V^{1,0}_{q_1}$, where $q_1 = \rho^{2,1}q_2$. Therefore by a smooth rescaling we could redefine $\bar{\eta}$ such that $T_{q_2}\rho^{2,1}\bar{\eta}_{q_2} = \dot{\eta}_{q_1}$ for all points $q_2 \in (\rho^{2,0})^{-1}W$ and $q_1 = \rho^{2,1}q_2$.

Now, since $\ddot{\omega}^1 \in P_{(2)}$ it follows that

$$\langle \ddot{\omega}^1, \bar{\eta} \rangle = 0.$$

But since $\ddot{\omega}^1$ is semibasic with respect to $\rho^{2,1}$

$$\langle \ddot{\omega}^1, \bar{\eta} \rangle = \langle \ddot{\omega}^1, \dot{\eta}_1 \rangle$$

where both sides of the equality are to be regarded as functions on $(\rho^{2,0})^{-1}W \subset J^2$. By Lemma 3.19 it follows that

$$\langle \dot{\omega}^1, \eta_1 \rangle = 0,$$

for points in $(\rho^{1,0})^{-1}W$. Then by Lemma 3.39 it follows that

$$\eta_1 \lrcorner d\omega^1 = 0 \mod I,$$

for points in W. This and (3.22) together imply (3.21) for points in W.

For points in U but not in W, $P \cap I$ is two dimensional and hence $I \subset P$ which implies $\langle \omega^i, \eta_j \rangle = 0$ for $i, j = 1, 2$ and therefore (3.21) follows at once.

Sufficiency : Let U be an open neighbourhood of q in which (3.20) and (3.21) hold and P is 2-regular. Then $P \cap I$ is nontrivial for points in U. Choose ω^i and η_j such that (3.22) holds in U.

Suppose $P \cap I$ is one dimensional in $W \subset U$. The set W is open. The equations (3.20), (3.21) and (3.22) imply $\eta_1 \lrcorner d\omega^1 = 0 \mod I$, which is equivalent to $\langle \dot{\omega}^1, \eta_1 \rangle = 0$ by Lemma 3.39. Let

$$P = \operatorname{span}\{dt, \omega^1, \alpha^2, \ldots, \alpha^p\},$$

in U. Since P is 2-regular in $W \subset U$ it follows that the set

$$A = \{dt, \omega^1, \alpha^2, \ldots, \alpha^p, \dot{\omega}^1\}$$

is linearly independent in an open dense subset of $W_2 \subset (\rho^{2,0})^{-1}W$ and hence spans a space of dimension $p + 2$. Also since all the forms in the set are semibasic with respect to $\rho^{2,0}$ by Lemma 3.23 and all of them annihilate η_1 it follows that the set spans the annihilator of η_1. Since $\langle \omega^2, \eta_1 \rangle = 0$ it follows that ω^2 is in the span of A for points in W_2, and the coefficient of $\dot{\omega}^1$ is nonvanishing in W_2. Then it also follows by time derivative operation that $\dot{\omega}^2$ is in the span of B for points in W_2, where

$$B = \{dt, \omega^1, \alpha^2, \ldots, \alpha^p, \dot{\omega}^1, \dot{\alpha}^2, \ldots, \dot{\alpha}^p, \ddot{\omega}^1\},$$

and the coefficient of $\ddot{\omega}^1$ is nonvanishing in W_2. Hence it follows that the set C defined by

$$C = \{dt, \omega^1, \alpha^2, \ldots, \alpha^p, \omega^2, \dot{\alpha}^2, \ldots, \dot{\alpha}^p, \dot{\omega}^2\}$$

is linearly independent in W_2. Therefore the set $\{dt, \omega^1, \alpha^2, \ldots, \alpha^p, \omega^2, \dot{\omega}^2\}$ is linearly independent in W_2 and also since all the forms in it take values in T^*M by Lemma 3.23 its span contains the pull back of any one-form in T^*M for points in W_2. But the set C is contained in $P_{(2)}$, implying P is a 2-generator in $W \subset U$.

For points in $U - W$, $P \cap I$ is two dimensional implying $I \subset P$. Since P is 2-regular in U it follows that $P + I_{(1)} \subset P_{(1)}$ has dimension $1 + p + 2$ in a dense subset of $(\rho^{1,0})^{-1}(U - W)$. But $I_{(1)}$ and P lie inside T^*M as codistributions making P a 1-generator and hence a 2-generator on the interior of $U - W$. It is also clear by continuity argument that P is a 2-generator around any point on the boundary of $U - W$ as well, completing the proof. ■

Lemma 3.41 *Suppose system $S = (t, M, I)$ has zero-flat outputs $y^1, \ldots, y^p$ in an open set $U \subset M$. Then y are $(0, \dim I)$-flat outputs.*

Proof: Let $V = (t(U), y(U)) \subset \mathbb{R}^{p+1}$. Then the system $S_U = (t, U, I)$ is a Cartan prolongation of the trivial system $T = (\pi_1, V, \{0\})$, where π_1 is the projection onto the first component. To see this, around any point $q \in U$ choose a coordinate system (t, y, z). Any two solution curves $(y(t), z = z_1(t))$ and $(y(t), z = z_2(t))$ that project down to the same solution curve $y(t)$ of T are isolated by the requirement that $z(t)$ is given locally uniquely by an equation of the form

$$z(t) = f(t, y(t), \ldots, y^{(r)}(t)),$$

where r is a large enough fixed integer. Hence S_U is a Cartan prolongation of T in this broader sense. Theorem 2.14 still holds for Cartan prolongations of this broader sense and the result follows from the Corollary 2.15 of this theorem. ∎

This lemma provides the following corollary of Theorem 3.40.

Corollary 3.42 *Given system $S = (t, M, I)$ with* $\dim I = 2$*, let $y^1, \ldots, y^p$ be functions around q that are differentially 2-independent around q. Let D be the annihilator of the codistribution spanned by $\{dt, dy^1, \ldots, dy^p\}$ with basis $\{\eta_1, \eta_2\}$ around q and let $\{\omega^1, \omega^2\}$ be a basis for I around q. Then y are zero-flat outputs around q if and only if (3.20) and (3.21) hold around q.*

Remark 3.43 For a system with $\dim I = 2$ that is zero-flat in an open neighbourhood $U \subset M$, the variables x and u can be given in terms of $(t, y, \dot{y}, \ddot{y})$ only in an open dense subset of points in $(\rho^{r,0})^{-1}U$. From the proof of Theorem 3.40 it is clear that the singularities in $(\rho^{r,0})^{-1}U$ occur precisely where y fail to be differentially 2-independent.

Remark 3.44 The Corollary 3.42 was first proven using a coordinate based approach in [29]. For sake of comparison we include this proof in Appendix A. If one were to extend these results to systems of arbitrary number of one-forms, the coordinate approach becomes intractable and the more intrinsic approach of this chapter may be more appropriate.

The Corollary 3.42 provides us with the following algorithm for finding zero-flat outputs of a system with $\dim I = 2$. Solve for a two dimensional distribution D spanned by unknown vector fields η_1, η_2 which satisfy the algebraic conditions (3.20), (3.21) and such that dt annihilates D. (The last condition is equivalent to $(t, y^1, \ldots, y^p)$ being independent functions.) This step involves (symbolic) differentiation and algebra. Typically at each point $q \in M$ there is a submanifold S_q (of the manifold $G_2(T_qM)$ of all two dimensional subspaces of T_qM) of possible solutions for D_q. The next step is to determine if it is possible to choose a smooth D that satisfies these conditions ($D_q \in S_q$) and in addition is integrable. This step involves solving PDEs. Generally this results in a nonlinear system of PDEs and typically this system does not have solutions. Unfortunately Theorem (3.40) does not say anything about this and hence it must be regarded as a partial characterisation of zero-flatness. Each solution gives an integrable D. Then one picks functions $y^1, \ldots, y^p$ that together with t cut out the foliations of D. Finally one needs to check the differential 2-independence of y. If this is satisfied then $y^1, \ldots, y^p$ are zero-flat outputs. If $y^1, \ldots, y^p$ are not differentially 2-independent around q then they are not flat outputs as they satisfy some ODE. One must pick another integrable D. The whole algorithm is illustrated in the next section by examples.

Corollary 3.42 also helps us decide if a control affine system

$$\dot{x}^i = f^i(x) + g^i_j(x)u^j \quad j = 1, \ldots, n-2, \quad i = 1, \ldots, n,$$

in n states and $n-2$ controls with g^i_j full rank has flat outputs that only depend on the states x. This is because the u^j can be eliminated from the n equations and this introduces two quasi-linear ODEs in x only. This corresponds to a system with $\dim I = 2$ and the zero-flat outputs of the latter system correspond to flat outputs of the original system that only depend on the states x. From the flat outputs and their derivatives x can be recovered and from x and $\dot{x}$ the controls u can be recovered.

Remark 3.45 Systems consisting of a single one-form, i.e. $I = \text{span}\{\omega\}$, are always zero-flat (locally) provided the system is not integrable in the Frobenius sense. In fact it may be proven that any vector field η that spans the (one dimensional) distribution that is tangent to the level sets of zero-flat outputs y, needs to satisfy the condition $\eta \lrcorner \omega = 0$. Note that this is similar to the condition (3.20). It may also be shown that the outputs y being differentially 1-independent is equivalent to the condition that $\eta \lrcorner d\omega$ should not lie in I. This makes it clear that locally such an η can always be found given $\omega \wedge d\omega \neq 0$, in other words I is not integrable in the Frobenius sense. This result for the single one-form case is not new however, since this corresponds to an already known result which states that all control affine systems in n states and $n-1$ controls are flat provided they are (locally) controllable, see [8, 24].

3.7 Examples

3.7.1 Example 1 : Kinematic Car

Let us consider the example of "kinematic car" (see Figure 3.4). In this model we ignore the dynamics and just consider the kinematics. The system under consideration is then a system of two one-forms (two constraints) in a 5 dimensional space with coordinates (t, x, y, θ, ϕ). The system I is given by

$$I = \text{span}\{\omega^1 = \cos\theta dx + \sin\theta dy - l\cot\phi d\theta, \\ \omega^2 = \sin\theta dx - \cos\theta dy\}.$$

This example is known to be flat with flat outputs x and y. In fact the system is zero-flat. This may also be considered as a control system with controls u and w, where u is the velocity of the vehicle (at the midpoint between rear wheels) and w

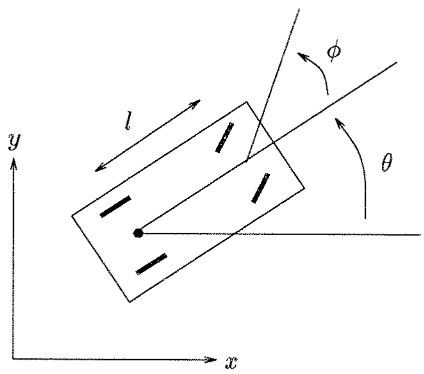


Figure 3.4 Kinematic car.

the steering velocity. Then the control system is described by

$$\begin{aligned}\dot{x} &= u\cos\theta \\ \dot{y} &= u\sin\theta \\ \dot{\theta} &= \frac{1}{l}u\tan\phi \\ \dot{\phi} &= w.\end{aligned} \tag{3.23}$$

The system I being zero-flat is equivalent to the above control system having flat outputs that depend only on states. We shall apply the theory of previous section to this example.

As outlined in Section 3.6, the first step is to solve for a two dimensional distribution $D = \operatorname{span}\{\eta_1, \eta_2\}$ that annihilates dt and satisfies conditions (3.20) and (3.21). It is easier to solve for the "2-vector field" N, which is the exterior product of the two vector fields η_1 and η_2, in other words $N = \eta_1 \wedge \eta_2$. Note that just as for one-forms it is possible to define the exterior product of vector fields, see [4] for instance. Furthermore, the interior product of a p-vector field ξ with a q-form α ($p < q$) is uniquely defined by following rules:

1. When $p = 1$ the definition is the usual one.
2. The product is linear in ξ.
3. $(v_1 \wedge v_2) \lrcorner \alpha = v_1 \lrcorner (v_2 \lrcorner \alpha)$.

Then conditions (3.20) and (3.21) can be written in terms of N as

$$\begin{aligned} N \lrcorner (\omega^1 \wedge \omega^2) &= 0 \\ N \lrcorner d(\omega^1 \wedge \omega^2) &= 0 \mod I. \end{aligned}$$

Since D must annihilate dt, in coordinates (t, x, y, θ, ϕ), N has the form

$$N = N_1 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + N_2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \theta} + N_3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \phi} + N_4 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \theta} + N_5 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \phi} + N_6 \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial \phi}.$$

Then conditions (3.7.1) give linear equations for components N_i. In addition N must be decomposable, in other words it should be the exterior product of two vector fields. N is decomposable if and only if the components N_i satisfy the "Plücker relations" which are always homogeneous quadratics, see [43] or [5]. In our case N is a 2-vector in 4 variables (x, y, θ, ϕ) and the Plücker relation is given by

$$N_1 N_6 - N_2 N_5 + N_3 N_4 = 0. \tag{3.24}$$

With the aid of Maple it is seen that the general solution depends on two arbitrary parameters λ and μ and that $N_1 = N_2 = N_4 = 0$, $N_3 = \lambda \cos\theta, N_5 = \lambda \sin\theta$ and $N_6 = \mu$.

This corresponds to a two dimensional distribution D spanned by η_1 and η_2 given by

$$\eta_1 = \frac{\partial}{\partial \theta} \tag{3.25}$$

$$\eta_2 = \mu(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}) - \lambda \frac{\partial}{\partial \phi}. \tag{3.26}$$

The next step is to find choices of λ and μ that lead to integrable D. This step typically leads to solving PDEs as seen in the next example. However one may sometimes find the solution by inspection without writing down PDEs, as in this example. If $\lambda = 0$ then this is clearly not integrable (as seen by calculating the Lie bracket of η_1 and η_2). Hence we can take $\lambda = 1$. Then the annihilator of D is given by

$$\text{ann}(D) = \text{span}\{dt, dx + \mu \cos\theta d\phi, dy + \mu \sin\theta d\phi\}. \tag{3.27}$$

It is clear that this is not integrable unless $\mu = 0$ (cannot "wedge out" $d\theta$). Hence we are left with the only solution for ann(D)

$$\text{ann}(D) = \text{span}\{dt, dx, dy\}. \tag{3.28}$$

Finally in order for x and y to be flat outputs we must verify that they are differentially 2-independent. But instead one may directly verify that these are flat outputs.

In this example we already know them to be flat outputs. This calculation shows that these are the only zero-flat outputs up to a diffeomorphism involving time.

3.7.2 Example 2

In this section we shall consider the following example of a control system in 4 states and two controls due to Sluis [37].

$$
\begin{aligned}
\dot{x}_1 &= x_2(1+v)\\
\dot{x}_2 &= x_3 + x_1 v\\
\dot{x}_3 &= u\\
\dot{x}_4 &= x_3 v.
\end{aligned}
$$

This example is known to be flat with flat outputs that depend on inputs and their derivatives. However we can eliminate inputs u and v to obtain two quasi-linear ODEs in the state variables. These can be written as a Pfaffian system of dimension 2 in coordinates $(t, x_1, \ldots, x_4)$. In fact we have

$$
\begin{aligned}
I = \operatorname{span}\{\omega^1 &= x_1 dx_4 - x_3 dx_2 + x_3^2 dt,\\
\omega^2 &= x_2 dx_4 - x_3 dx_1 + x_2 x_3 dt\}.
\end{aligned}
$$

Hence the theory of Section 3.6 can be applied to look for zero-flat outputs. If this system I is zero-flat then it means the original control system has flat outputs that depend only on states.

As in the previous example we shall first solve for a 2-vector field N. Since D must annihilate dt, in coordinates $(t, x_1, \ldots, x_4)$, N has the form

$$
\begin{aligned}
N = {} & N_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + N_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + N_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4}\\
& + N_4 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + N_5 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} + N_6 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}.
\end{aligned}
$$

As before conditions (3.7.1) give linear equations for components N_i. In this case also N is a 2-vector in 4 variables $(x_1, \ldots, x_4)$ and hence the Plücker relation is the same as in previous example and is given by (3.24). With the aid of Maple it is seen that the general solution for N_i depends on two arbitrary parameters λ and μ.

$$
\begin{aligned}
N_1 &= 0\\
N_2 &= -x_1 x_2 x_3 \lambda + x_2^3 \lambda + x_1^2 x_2 \lambda - x_1 x_2^2 \mu\\
N_3 &= x_2^2 x_3 \lambda\\
N_4 &= -x_1^2 x_3 \lambda + x_1 x_2^2 \lambda + x_1^3 \lambda - x_1^2 x_2 \mu\\
N_5 &= x_1 x_2 x_3 \lambda\\
N_6 &= x_1 x_2 x_3 \mu.
\end{aligned}
$$

Since any scalar multiple of N corresponds to the same two dimensional distribution D the solution manifold $S_q \subset G_2(T_q M)$ is 1 dimensional for each $q \in M$. A possible choice of η_1 and η_2 is as follows.

$$\eta_1 = (x_1x_2x_3\lambda - x_2^3\lambda - x_1^2x_2\lambda + x_1x_2^2\mu)\frac{\partial}{\partial x_1}$$
$$+ (x_1^2x_3\lambda - x_1x_2^2\lambda - x_1^3\lambda + x_1^2x_2\mu)\frac{\partial}{\partial x_2} + x_1x_2x_3\mu\frac{\partial}{\partial x_4} \tag{3.29}$$
$$\eta_2 = x_2^2x_3\lambda\frac{\partial}{\partial x_1} + x_1x_2x_3\lambda\frac{\partial}{\partial x_2} + x_1x_2x_3\mu\frac{\partial}{\partial x_3}. \tag{3.30}$$

The next step is to impose involutivity conditions on η_1 and η_2. Requiring $[\eta_1, \eta_2] \in \text{span}\{\eta_1, \eta_2\}$ results in the following PDE for λ and μ as functions on M.

$$- x_1^2x_2^2\mu^2 + x_1^3x_3\lambda^2 + x_1^2x_2^2x_3\lambda\frac{\partial\mu}{\partial x_2}$$
$$+ x_1^2x_2x_3^2\mu\frac{\partial\lambda}{\partial x_3} - x_1x_2^3x_3\mu\frac{\partial\lambda}{\partial x_3} - x_1^3x_2x_3\mu\frac{\partial\lambda}{\partial x_3}$$
$$- 3x_1x_2^2x_3\lambda^2 + x_2^3x_3\lambda\mu + x_1x_2^3x_3\lambda\frac{\partial\mu}{\partial x_1}$$
$$- x_1^2x_3^2\lambda^2 + x_2^2x_3^2\lambda^2 - x_1^2x_3x_2^2\mu\frac{\partial\lambda}{\partial x_2}$$
$$- x_1x_2^3x_3\mu\frac{\partial\lambda}{\partial x_1} - x_1x_2^2x_3^2\mu\frac{\partial\lambda}{\partial x_4} - x_1^2x_2x_3^2\lambda\frac{\partial\mu}{\partial x_3}$$
$$+ x_1x_2^3x_3al\lambda\frac{\partial\mu}{\partial x_3} + x_1x_2^3\lambda\mu + x_1^3x_2x_3al\lambda\frac{\partial\mu}{\partial x_3}$$
$$+ x_1^3x_2\lambda\mu + x_1x_2^2x_3^2\lambda\frac{\partial\mu}{\partial x_4} = 0. \tag{3.31}$$

Since the PDE is independent of t it is possible to seek solutions that are independent of t. Also since the PDE is homogeneous, assuming $\mu \neq 0$ we may equivalently solve for $f = \frac{\lambda}{\mu}$. The PDE for f is as follows.

$$x_1^2x_2^2 + x_1^3x_3f^2 - x_2^3x_3f - x_2^2x_3^2f^2 - x_1x_2^3f$$
$$- x_1^3x_2f + 3x_1x_2^2x_3f^2 + x_1^2x_3^2f^2 - x_1^2x_2x_3^2\frac{\partial f}{\partial x_3}$$
$$+ x_1x_2^3x_3\frac{\partial f}{\partial x_3} + x_1^3x_2x_3\frac{\partial f}{\partial x_3} + x_1^2x_2^2x_3\frac{\partial f}{\partial x_2}$$
$$+ x_1x_2^3x_3\frac{\partial f}{\partial x_1} + x_1x_2^2x_3^2\frac{\partial f}{\partial x_4} = 0. \tag{3.32}$$

Since this is a single quasi-linear PDE it has solutions. Each solution will correspond to an integrable D. One solution of the PDE is

$$f = \frac{x_1x_2(x_4 - x_2)}{-x_2^3 - x_1^2x_2 + x_1^2x_4 - x_2x_3^2 + x_2^2x_4 + 2x_1x_2x_3 - x_1x_3x_4}. \tag{3.33}$$

This leads to an integrable D spanned by η_1 an η_2 given by,

$$\begin{aligned}\eta_1 = {} & (x_1x_2^2 - x_3x_2^2)\frac{\partial}{\partial x_1} + (x_1^2x_2 - x_1x_2x_3)\frac{\partial}{\partial x_2} \\ & + (2x_1x_2x_3 - x_2^3 - x_1^2x_2 + x_1^2x_4 - x_2x_3^2 \\ & + x_2^2x_4 - x_1x_3x_4)\frac{\partial}{\partial x_4}\end{aligned} \tag{3.34}$$

$$\begin{aligned}\eta_2 = {} & (x_2^2x_4 - x_2^3)\frac{\partial}{\partial x_1} + (x_1x_2x_4 - x_1x_2^2)\frac{\partial}{\partial x_2} \\ & + (2x_1x_2x_3 - x_2^3 - x_1^2x_2 + x_1^2x_4 - x_2x_3^2 \\ & + x_2^2x_4 - x_1x_3x_4)\frac{\partial}{\partial x_3}.\end{aligned} \tag{3.35}$$

A choice of first integrals of this distribution (other than t) is

$$y_1 = x_2^2 - x_1^2 \tag{3.36}$$

$$y_2 = \frac{x_4\sqrt{x_2^2 - x_1^2}}{x_2(x_3 - x_1)} - \frac{\sqrt{x_2^2 - x_1^2}}{x_3 - x_1} - \arctan\left(\frac{x_1}{\sqrt{x_2^2 - x_1^2}}\right). \tag{3.37}$$

Finally we need to check if y_1 and y_2 are differentially 2-independent. But alternatively one may directly verify that y^1, y^2 are flat outputs. The following considerations show that these are indeed flat outputs. Firstly observe that x_2 can be solved for in terms of y_1 and x_1. Hence it follows x_4 can be solved for in terms of y_1, y_2, x_1 and x_3. The two one forms ω_1 and ω_2 correspond to the following ODEs.

$$x_1\dot{x}_4 - x_3\dot{x}_2 + x_3^2 = 0 \tag{3.38}$$

$$x_2\dot{x}_4 - x_3\dot{x}_1 + x_2x_3 = 0. \tag{3.39}$$

These two equations imply

$$x_3(x_2\dot{x}_2 - x_1\dot{x}_1) + x_1x_2x_3 - x_2x_3^2 = 0. \tag{3.40}$$

Substituting $\frac{\dot{y}_1}{2} = x_2\dot{x}_2 - x_1\dot{x}_1$ and $x_2 = \sqrt{y_1 + x_1^2}$ in above and solving for x_3 we get

$$x_3 = \frac{x_1\sqrt{y_1 + x_1^2} + \frac{1}{2}\dot{y}_1}{\sqrt{y_1 + x_1^2}}. \tag{3.41}$$

Hence we can express x_2, x_3 and x_4 in terms of $x_1, y_1, \dot{y}_1$ and y_2. Substituting for x_2, x_3 and x_4 in equation (3.38) we get an equation involving $x_1, y_1, \dot{y}_1, \ddot{y}_1, y_2$ and $\dot{y}_2$, but no derivatives of x_1 (since this expression is too big we shall not show it here). From this equation, x_1 can be solved for in terms of y_1 and y_2 and finitely many of their derivatives. This establishes that y_1 and y_2 are a set of flat outputs that depend only on states of the control system (3.7.2).

Remark 3.46 The flat outputs y_1 and y_2 were not known before. This example illustrates how nontrivial flat outputs can be found systematically using the conditions (3.20) and (3.21) and then solving for PDEs. This is different from the more familiar approach of adding integrator chains to original inputs (in this example u and v) of varying lengths and then applying the theory of static feedback linearisation to the resulting system. The previously known flat outputs (see [37]) were found by this dynamic extension approach and depend on inputs and their derivatives. The technique provided in this chapter enables one to find all flat outputs that depend only on original variables but no derivatives, provided one can find solutions to the resulting PDEs.

Chapter 4

Configuration Flatness of Lagrangian Systems

Many interesting examples of mechanical systems are differentially flat and in most known examples flat outputs have been found that depend only on the configuration variables but not on their derivatives. We refer to such flat outputs as "configuration flat outputs" and systems possessing such outputs as "configuration flat." For instance, the example of kinematic car in Section 3.7.1 is configuration flat. All Lagrangian systems that are fully actuated (number of controls equals number of degrees of freedom) are configuration flat with all the configuration variables as flat outputs. The planar rigid body example of Chapter 1 is also configuration flat. See [12] and [25] for a catalogue of other examples. The reasons for studying configuration flatness are as follows. Firstly it is a simpler case than the general case of differential flatness and is possibly the first thing to study if one were to be able to relate the mechanical structure with differential flatness. For instance configuration controllability of mechanical systems has already been studied and related to the mechanical structure (see Lewis and Murray [19]). Secondly the smaller the number of derivatives of configuration variables the flat outputs depend upon the simpler the numerical implementation of the transformations involved in trajectory generation.

In this chapter we completely characterise configuration flatness for a special class of mechanical systems. The class under consideration involves systems whose dynamics is described by Lagrangian mechanics with a Lagrangian function of the form "kinetic energy minus potential." Also the number of independent controls is assumed to be one less than the number of degrees of freedom (the simplest case next to fully actuated systems) and the possible range of control forces only depends on the configuration and not on the velocity. We describe an algorithm for deciding if such a system is configuration flat and if it is so, we describe a procedure for finding all possible configuration flat outputs. We do not consider systems with nonholonomic constraints. The kinematic car example hence does not fall into the class of systems under our consideration. Similar to the characterisation of zero-flatness of a system of two one-forms in Chapter 3, the characterisation of configuration flatness also involves conditions on the tangent spaces to the level sets of the flat outputs. Since the level sets are one dimensional for the case of systems underactuated by one control, integrability always follows (locally). Hence the theorem presented in this chapter is not as restrictive as the Theorem 3.40 of Chapter 3.

The chapter is organised as follows. Firstly we present a formulation of flatness that directly applies to higher order systems of ODEs. We introduce some concepts from Lagrangian control systems theory and also provide a definition of configuration flatness. Then we introduce some concepts from Riemannian geometry that are necessary for our theory and also state and prove the main theorem and outline an algorithm for coordinate calculations to check configuration flatness. We also explore how system symmetries relate to symmetries of the flat outputs. Finally we provide two examples to illustrate the theory.

4.1 Jet Bundle Approach to Higher Order Systems of ODES

In this section we briefly present a formulation of flatness that is closely related to the developments in Chapter 3, except that this formulation applies directly to a system of higher order ODEs. This is useful when one wants to exploit the higher order nature of the ODEs and in fact we shall use this to investigate configuration flatness of Lagrangian mechanical systems which are governed by second order ODEs. As before our manifold is equipped with a special notion of time and hence we start with a fibre bundle $t : M \to \mathbb{R}$ as in Chapter 3. But we no longer consider a Pfaffian system on M (which corresponds to first order quasi-linear system of ODEs). Suppose we have an underdetermined system of kth order ODEs in some coordinate system $(t, x^1, \ldots, x^N)$ given by

$$F^j(t, x, \ldots, x^{(k)}) = 0, \quad j = 1, \ldots, N - p.$$

This system of ODEs may be regarded as the submanifold $\mathscr{E} \subset J^k(t, M)$, which is given by the common zero set of the functions F^j on the kth order jet bundle $J^k(t, M)$. We refer the reader to [33] for a discussion of jet bundles.

In Chapter 3 we considered the bundles $J^k(t, M, I)$ that were associated to the bundle $t : M \to \mathbb{R}$ and a codistribution I on M. As mentioned in Remark 3.3 by taking $I = \{0\}$ we can apply the theory of Chapter 3 to derive properties of the jet bundles $J^k(t, M)$. More precisely $J^k(t, M)$ is the same as $J^k(t, M, \{0\})$ in Chapter 3 terminology. Sections c of the bundle $t : M \to \mathbb{R}$ may be naturally lifted to sections of $J^k(t, M)$. A section c of $t : M \to \mathbb{R}$ is called a solution of the system of ODEs if and only if its lift to $J^k(t, M)$ lies in $\mathscr{E}$. Denote by $\rho_M^{r,l}$ the projection from $J^r(t, M)$ to $J^l(t, M)$ for $r > l$. A section c_k of the bundle $(\rho_M^{k,0})^* t : J^k(t, M) \to \mathbb{R}$ that lies in $\mathscr{E}$ is a solution only if it is the lift of a section of $t : M \to \mathbb{R}$ and this is true if and only if $\frac{dc_k}{dt}$ annihilates the contact codistribution Ω_k on $J^k(t, M)$. If one were to ignore the higher order nature of the ODEs then the starting point will be the system S consisting of the manifold $\mathscr{E}$ with the time given by the pull back of t on M and the codistribution on $\mathscr{E}$ will be the restriction of the contact codistribution Ω_k on $J^k(t, M)$ to $\mathscr{E}$. In other words $S = (t \circ \rho_M^{k,0} \circ i_{\mathscr{E}}, \mathscr{E}, (i_{\mathscr{E}})^* \Omega_k)$, where $i_{\mathscr{E}}$ is the inclusion of $\mathscr{E}$ into $J^k(t, M)$. But since we want to exploit the higher order nature the approach will be slightly different. From Chapter 3 it follows that we have a notion of time derivative of functions and differential forms. It can be seen that the prolongation of above system S as defined in Chapter 3 is given by a submanifold $\mathscr{E}^1 \subset J^{k+1}(t, M)$ which is locally the common zero set of the functions F^j and $\dot{F}^j$ for $j = 1, \ldots, N-p$ and a codistribution on $\mathscr{E}^1$ which is the restriction of Ω_{k+1} to $\mathscr{E}^1$. Higher prolongations of S turn out to be given by the submanifolds $\mathscr{E}^l \subset J^{k+l}(t, M)$ which are locally the zero set of $F^j, F^{j,(1)}, \ldots, F^{j,(l)}$ for $j = 1, \ldots, N-p$ and codistributions on $\mathscr{E}^l$ which are restrictions of Ω_{k+l} to $\mathscr{E}^l$.

The following definitions are in complete accordance with Chapter 3.

Definition 4.1 Consider the system of kth order ODEs on the bundle $t : M \to \mathbb{R}$ given by $\mathscr{E} \subset J^k(t, M)$. Let $y^1, \ldots, y^p$ be functions defined around $q \in M$. Then y are said to be *differentially r-independent* around q if there exists an open neighbourhood U of q such that the set

$$\{dt, dy, dy^{(1)}, \ldots, dy^{(r)}\}$$

is differentially independent when restricted to $\mathscr{E}^r$ in an open dense subset of $(\rho_M^{k+r})^* U \cap \mathscr{E}^r$.

Definition 4.2 Consider the system of k-th order ODEs on the bundle $t : M \to \mathbb{R}$ given by $\mathscr{E} \subset J^k(t, M)$. Let $y^1, \ldots, y^p$ be functions defined around $q \in M$. Then y are said to be $(0, r)$-*flat outputs* around q if they are differentially r-independent around q and if there exists an open neighbourhood U of q such that

$$(\rho_M^{k+r})^*(T^*M) \subset \operatorname{span}\{dt, dy, dy^{(1)}, \ldots, dy^{(r)}\}$$

when restricted to $\mathscr{E}^r \subset J^{k+r}(t, M)$ in an open dense subset of $(\rho_M^{k+r})^* U \cap \mathscr{E}^r$. The functions y are said to be *zero-flat outputs* if they are $(0, r)$-flat outputs for some r.

Thus y being $(0, r)$-flat outputs means that y depend only on original variables t and x but not on derivatives of x and that the variables x may be obtained as functions of $t, y, \ldots, y^{(r)}$ alone.

4.2 Lagrangian Control Systems and Configuration Flatness

Consider a Lagrangian system with configuration manifold Q of dimension n and a Lagrangian $L : TQ \to \mathbb{R}$. When no external (generalised) forces are applied, the motion of this system satisfies the Euler-Lagrange equations, written in coordinates $(q^1, \ldots, q^n)$ as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \ldots, n. \tag{4.1}$$

In a control situation external control forces are applied and it is natural to think of forces as covectors on the manifold Q. In other words, for a configuration $q \in Q$

the total external force acting on the system can be represented by an element of T_q^*Q. This is because forces naturally pair with velocities, which can be thought of as elements of T_qQ, to give instantaneous power. The possible range of control forces lies in a subspace of T_q^*Q which may depend on position q as well as velocity v_q. In other words the control forces can be described by a horizontal valued codistribution $\bar{P} \subset T^*(TQ)$, and $p = \dim \bar{P}$ is the number of independent controls. For an interesting and wide class of systems this subspace only depends on configuration q and hence can be described by a codistribution $P \subset T^*Q$ of dimension p. For the rest of the discussion we shall only consider this case. All feasible paths (solutions) of such a system are characterised by the following underdetermined system of second order ODEs in coordinates $(q^1, \ldots, q^n)$:

$$a_k^i\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right) = 0, \quad k = 1, \ldots, n-p \tag{4.2}$$

where $a_k^i \frac{\partial}{\partial q^i}$ for $k = 1, \ldots, n-p$ span the annihilator of P, denoted $\mathrm{ann}(P)$.

As mentioned in the Section 4.1, the geometric object to consider is the associated submanifold $\mathcal{E} \subset J^2(\mathbb{R}, Q)$ of the second order jet bundle $J^2(\mathbb{R}, Q)$. Note that since we have a time independent system of ODEs our manifold is $M = \mathbb{R} \times Q$. The time t is the projection onto the first factor, and the bundle $J^k(t, M) = J^k(\mathbb{R}, Q)$. The submanifold $\mathcal{E}$ has codimension $n-p$ and in local coordinates $(t, q, \dot{q}, \ddot{q})$ is cut out by the common zeroes of the functions

$$a_k^i\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial L}{\partial q^i}\right), \quad k = 1, \ldots, n-p.$$

Definition 4.3 A Lagrangian control system with configuration space Q is said to be *configuration flat* around $q \in Q$ if there exist zero-flat outputs $y^1, \ldots, y^p$ (in the sense of Definition 4.2) that are functions on Q defined around q. (Note that y are required to be time independent.)

We present the following lemma which will be of use later.

Lemma 4.4 *Let $q \in Q$, U an open neighbourhood of q, and $y : U \to \mathbb{R}^p$ be configuration flat outputs. Then generically the set of solutions $c : \mathbb{R} \to U$ that project down to the same curve $y \circ c$ are all isolated.*

Proof: Choose local coordinates z that complement y to a full coordinate system. It follows from flatness that along typical solution curves, $z(t)$ is locally uniquely given by an equation of the form

$$z(t) = f(t, y(t), \ldots, y^{(r)}(t)).$$

■

4.3 Mechanical Systems with n Degrees of Freedom and n − 1 Controls

Consider the mechanical system whose Lagrangian is given by

$$L(v) = \frac{1}{2}g(v,v) - V \circ \tau_Q(v), \tag{4.3}$$

where g is the Riemannian metric (assumed to be non degenerate) corresponding to kinetic energy and V is the potential energy function on Q and $\tau_Q : TQ \to Q$ is the tangent bundle projection. Suppose the number of controls is $p = n - 1$, in other words $\dim P = n - 1$, where $n = \dim Q$. In this section we shall present a method for determining if this system is configuration flat. If the system is configuration flat our approach provides us with a constructive method for finding all possible (configuration) flat outputs. We assume that all the holonomic constraints have been taken into account by the configuration manifold Q and that no nonholonomic constraints are present.

Before proceeding further we present some concepts from Riemannian geometry. Given a metric g we have a notion of differentiation of objects on the manifold such as functions, vector fields, differential forms and tensors along a given vector field Z. This is the covariant derivative ∇ given by the Levi-Civita connection (see [1]). ∇_Z denotes covariant derivative along a vector field Z and is related to parallel (with respect to metric) transport of objects along the integral curves of Z. The covariant derivative of a function f along Z denoted $\nabla_Z f$ is just the familiar directional derivative $Z(f)$ or the Lie derivative. But the covariant derivative of a vector field X along Z denoted $\nabla_Z X$ is not the same as the Lie derivative $[Z, X]$. Some properties of ∇ are

$$\nabla_Z(X_1 + X_2) = \nabla_Z X_1 + \nabla_Z X_2 \tag{4.4}$$
$$\nabla_Z(fX) = \nabla_Z X + Z(f)X \tag{4.5}$$
$$\nabla_{fZ} X = f\nabla_Z X \tag{4.6}$$
$$\nabla_Z X - \nabla_X Z = [Z, X] \tag{4.7}$$

where X, X_1, X_2, Z are arbitrary vector fields and f is an arbitrary function on the manifold. In a coordinate system $(q^1, \ldots, q^n)$ on manifold Q the covariant derivatives are calculated with the aid of Christoffel symbols Γ^i_{jk} where $i, j, k = 1, \ldots, n$ and Christoffel symbols are defined by

$$\nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^k} = \Gamma^i_{jk} \frac{\partial}{\partial q^i}. \tag{4.8}$$

From the properties (4.7) of ∇ it follows that $\Gamma^i_{jk} = \Gamma^i_{kj}$. The Christoffel symbols Γ^i_{jk} can be computed from metric g by the formula

$$\Gamma^m_{jk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^i}\right)g^{im}, \quad j, m = 1, \ldots, n, \tag{4.9}$$

where $g^{ik}g_{kj} = \delta^i_j$ (g^{ik} are components of the inverse of matrix g_{ik}). Then the covariant derivative of vector field $X = X^k \frac{\partial}{\partial q^k}$ along $Z = Z^j \frac{\partial}{\partial q^j}$ is given by

$$\nabla_Z X = Z^j X^k \Gamma^i_{jk} \frac{\partial}{\partial q^i} + Z^j \frac{\partial X^k}{\partial q^j} \frac{\partial}{\partial q^k}. \tag{4.10}$$

For the mechanical system under consideration let us define an associated distribution D by

$$D = \text{span}\{\xi, \nabla_Z \xi \ : \ Z \in \mathfrak{X}(Q)\}, \tag{4.11}$$

where ξ is any vector field such that $\text{ann}(P) = \text{span}\{\xi\}$ and $\mathfrak{X}(Q)$ is the set of all smooth vector fields on Q. It is easy to check that D doesn't depend on the choice of $\xi \in \text{ann}(P)$. By the linearity of covariant derivative it follows that

$$D = \text{span}\{\xi, \nabla_{\frac{\partial}{\partial q^i}} \xi \ : \ i = 1, \ldots, n\} \tag{4.12}$$

where $(q^1, \ldots, q^n)$ are any set of coordinates. Hence D is easily calculated using equations (4.9), (4.10) and (4.12). The following theorem characterises configuration flat outputs $y^1, \ldots, y^p$ by conditions on $\ker Ty$, which in coordinates is the null space of the Jacobian of the map y.

Theorem 4.5 *Let q be a point on Q, U an open neighbourhood of q and suppose $y : U \subset Q \to \mathbb{R}^p$ is a submersion. If $y^1, \ldots, y^p$ are configuration flat outputs, then*

$$g(\ker Ty, D) = 0. \tag{4.13}$$

Conversely if $g(\ker Ty, D) = 0$ and if certain regularity condition holds at q then $y^1, \ldots, y^p$ are configuration flat outputs around q.

The regularity condition is that the ratios of functions in the following set should not all be the same at q:

$$\{\nabla_\eta(g(\xi, Z)) : g(\xi, Z), \nabla_\eta(g(\nabla_{Z_1} Z_2, \xi)) : g(\nabla_{Z_1} Z_2, \xi), \nabla_\eta(\xi(V)) : \xi(V)\}, \tag{4.14}$$

where Z, Z_1, Z_2 are arbitrary vector fields around q that are y-related to some vector field on $\mathbb{R}^p$ and ξ, η are fixed nonvanishing vector fields such that $\text{ann}(P) = span\{\xi\}$ and $\ker Ty = span\{\eta\}$.

Remark 4.6 Theorem 4.5 states the conditions for configuration flatness in intrinsic geometric terms. In coordinates the algorithm for deciding if the system is configuration flat is as follows. Calculate D using equation (4.12). If $D = TQ$ then system is not configuration flat, since for any y, one can find some vector field $Z \in D = TQ$, such that $g(\ker Ty, Z) \neq 0$. Suppose $\dim D \leq n - 1$. Then choose a one dimensional distribution, say spanned by a vector field η, that is orthogonal to D. Since a one dimensional distribution is integrable locally, one can find independent functions $y^1, \ldots, y^p$ ($p = n - 1$) around q that "cut out" the leaves of the corresponding foliation. These will be flat outputs provided the regularity conditions are met.

The regularity conditions can be checked in coordinates as follows. Choose a function z that completes $y^1, \ldots, y^p$ to a coordinate system. Then $y^1, \ldots, y^p$ will be flat outputs if the following ratios of functions are not all identically equal in a local neighbourhood:

$$\begin{aligned} &\frac{\partial}{\partial z}(g(\xi, \frac{\partial}{\partial y^j})) : g(\xi, \frac{\partial}{\partial y^j}), && j = 1, \ldots, p \\ &\frac{\partial}{\partial z}(g(\nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j}, \xi)) : g(\nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j}, \xi)), && j, k = 1, \ldots, p \\ &\frac{\partial}{\partial z}(\xi(V)) : \xi(V). \end{aligned} \tag{4.15}$$

If these are all identically equal that means $y^1, \ldots, y^p$ are differentially dependent and another one dimensional distribution must be tried.

Remark 4.7 It is readily seen that configuration flatness is determined primarily by the kinetic energy metric g since the role of potential function V only enters via the regularity conditions. This explains why in many known examples (see [25]) the presence or absence of gravity does not alter the configuration flat outputs but only the solution curves where singularities occur. However, we present an example in next section where the potential function plays a crucial role via the regularity conditions.

Proof of Theorem 4.5: Given a submersion $y : Q \to \mathbb{R}^p$, one can choose a local coordinate chart on Q such that y is the canonical submersion of $\mathbb{R}^n$ onto $\mathbb{R}^p$. Let the corresponding coordinates on Q be $(q^1, \ldots, q^n)$. Then, $y^j(q) = q^j$ for $j = 1, \ldots, p = n - 1$. Let $\xi = a^i \frac{\partial}{\partial q^i}$ span $\text{ann}(P)$. Then all solutions of the system satisfy the single ODE

$$a^i(\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i}) = 0. \tag{4.16}$$

Suppose in these coordinates g is given by g_{ij}. Then we can rewrite equation (4.16) as

$$a^i(g_{ij}\ddot{q}^j + \frac{\partial g_{ik}}{\partial q^j}\dot{q}^j\dot{q}^k - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^i}) = 0. \tag{4.17}$$

Using the formula (4.9) for the Christoffel symbols and using $q^j = y^j$ for $j = 1, \ldots, p$ to separate the terms involving $\dot{q}^n$ and $\ddot{q}^n$, we rewrite equations (4.17) as,

$$a^i(g_{ij}\ddot{y}^j + \Gamma^m_{jk} g_{mi}\dot{y}^j\dot{y}^k + \frac{\partial V}{\partial q^i} + g_{in}\ddot{q}^n + \Gamma^m_{nn} g_{mi}(\dot{q}^n)^2 + \Gamma^m_{jn} g_{mi}\dot{y}^j\dot{q}^n) = 0 \tag{4.18}$$

where range of summation of various indices is clear.

Necessity: Suppose y are flat outputs. Then it follows that the coefficient of $\ddot{q}^n$ in the above ODE must to be zero. Otherwise we can rewrite the equation as

$$\frac{d\dot{q}^n}{dt} = f(y, \dot{y}, \ddot{y}, q^n, \dot{q}^n)$$

for some smooth function f, and by existence theorem of solutions to ODEs, given any curve $y(t)$ we get a 2-parameter family of solutions $q(t)$ (parametrised by initial conditions $q^n(t_0), \dot{q}^n(t_0)$) that project to $y(t)$ and they are not isolated from each other and hence by Lemma 4.4 y cannot be flat, contradicting our assumption. So $a^i g_{in} = 0$ and this leaves us with an ODE of the form

$$A(y)(\dot{q}^n)^2 + B(y, \dot{y})\dot{q}^n + C(y, \dot{y}, \ddot{y}, q^n) = 0.$$

A similar reasoning tells us that the term $\dot{q}^n$ should be absent, in other words $A(y) = 0$ and $B(y, \dot{y}) = 0$. Here A and B are given by,

$$A = a^i \Gamma^m_{nn} g_{mi} \quad B = a^i \Gamma^m_{jn} g_{mi} \dot{y}^j.$$

Observe that B is linear in terms $\dot{y}$ with coefficients that are functions only of (y, q^n). Hence the condition $B = 0$ can be written as $n-1$ equations that set the coefficients of $\dot{y}^j$ to be zero. The equation $A = 0$ has the same form as these, and we get the following n equations:

$$a^i \Gamma^m_{jn} g_{im} = 0, \quad j = 1, \ldots, n.$$

So all together flatness of y implies the following equations,

$$\begin{aligned} a^i g_{in} &= 0 \\ a^i \Gamma^m_{jn} g_{im} &= 0, \quad j = 1, \ldots, n. \end{aligned} \tag{4.19}$$

If $\ker Ty = \operatorname{span}\{\eta\}$, then in our choice of coordinates $\eta = \lambda \frac{\partial}{\partial q^n}$ where λ is some nonvanishing function on Q. Hence, $g(\xi, \eta) = a^i g_{in} = 0$ by the first condition, where $\xi = a^i \frac{\partial}{\partial q^i}$ spans $\operatorname{ann}(P)$. Also since

$$\nabla_{\frac{\partial}{\partial q^j}} \eta = \lambda \Gamma^m_{jn} \frac{\partial}{\partial q^m} + \frac{\partial \lambda}{\partial q^j} \frac{\partial}{\partial q^n},$$

it follows that

$$g(\nabla_{\frac{\partial}{\partial q^j}} \eta, \xi) = \lambda a^i \Gamma^m_{jn} g_{im} + \frac{\partial \lambda}{\partial q^j} a^i g_{in} = 0.$$

But, by derivation property,

$$\nabla_Z(g(\xi, \eta)) = (\nabla_Z g)(\xi, \eta) + g(\nabla_Z \xi, \eta) + g(\xi, \nabla_Z \eta)$$

and since $\nabla_Z g = 0$ for any $Z \in \mathfrak{X}(Q)$ (by the property of Levi-Civita connection) and since $g(\eta, \xi) = 0$ it follows that

$$g(\nabla_{\frac{\partial}{\partial q^j}} \xi, \eta) = 0, \quad j = 1, \ldots, n.$$

By linearity of ∇ it follows that

$$g(\nabla_Z \xi, \eta) = 0, \quad \forall Z \in \mathfrak{X}(Q).$$

Hence, $\ker Ty$ is orthogonal to D.

Sufficiency: Conversely, if $\ker Ty$ is orthogonal to D, previous reasoning shows that, in the same coordinate system the equations (4.19) hold. As seen before these imply that the solution curves of the system are given by the ODE

$$E(q^n, y, \dot{y}, \ddot{y}) = 0,$$

where

$$E = a^i g_{ij} \ddot{y}^j + a^i g_{im} \Gamma^m_{jk} \dot{y}^j \dot{y}^k + a^i \frac{\partial V}{\partial q^i}.$$

This is not sufficient for flatness of $y^1, \ldots, y^p$ since it is possible that $y^1, \ldots, y^p$ are differentially dependent and this happens when E does not depend on q^n. More precisely $y^1, \ldots, y^p$ are differentially dependent around q when there exists a neighbourhood V of q such that $\frac{\partial E}{\partial q^n}$ is identically zero on $(\pi_2^{-1}(V) \cap \{E = 0\}) \subset J^2(\mathbb{R}, Q)$ where $\pi_2 : J^2(\mathbb{R}, Q) \to Q$ is the standard projection. The functions E and $\frac{\partial E}{\partial q^n}$ are both affine in $\ddot{y}$ and quadratic in $\dot{y}$ with the coefficients functions only of (y, q^n) and E depends on $\ddot{y}$ non trivially since metric g is non degenerate. Hence $\frac{\partial E}{\partial q^n}$ is either identically zero on $\pi_2^{-1}(q) \cap \{E = 0\}$ or it is non zero in an open dense subset of points on $\pi_2^{-1}(q) \cap \{E = 0\}$. Further more $\frac{\partial E}{\partial q^n}$ is identically zero on $\pi_2^{-1}(q) \cap \{E = 0\}$ if and only if it is a multiple of E as a polynomial in $\dot{y}$ and $\ddot{y}$ for points on $\pi_2^{-1}(q)$. Hence the regularity condition we impose is that $\frac{\partial E}{\partial q^n}$ is a not a multiple of E as a polynomial in $\ddot{y}$ and $\dot{y}$ for points on $\pi_2^{-1}(q)$. Then it would follow from continuity and implicit function theorem that for an open dense subset of points on $\pi_2^{-1}(V) \cap \{E = 0\}$ where V is some neighbourhood of q, q^n can be locally solved for in terms of $y, \dot{y}, \ddot{y}$, implying flatness around q.

Rest of the proof is concerned with showing that this condition translates to the regularity condition stated in the theorem. It is sufficient to show that $\frac{\partial E}{\partial q^n}$ is a multiple of E as polynomials in $\dot{y}, \ddot{y}$ with the ratio being a smooth function on Q is equivalent to the set of ratios of functions (4.14) all being identically equal in a neighbourhood of q.

Let η span $\ker Ty$. Then $\eta = \lambda \frac{\partial}{\partial q^n}$ for some nonvanishing function λ. Also let $\xi = a^i \frac{\partial}{\partial q^i}$ span $\mathrm{ann}(P)$. Suppose $\frac{\partial E}{\partial q^n} = fE$ for some function f defined in a neighbourhood of q on Q. Considering coefficients of $\ddot{y}^j$ terms we get

$$\frac{\partial}{\partial q^n}(a^i g_{ij}) = f a^i g_{ij} \quad j = 1, \ldots, p. \tag{4.20}$$

Also observe that any vector field Z on Q is y-related if and only if it has the form $Z^j(y)\frac{\partial}{\partial y^j} + Z^n(y, q^n)\frac{\partial}{\partial q^n}$. Hence

$$\begin{aligned} \nabla_\eta(g(\xi, Z)) &= \lambda \frac{\partial}{\partial q^n}(Z^j a^i g_{ij}) \\ &= \lambda Z^j \frac{\partial}{\partial q^n}(a^i g_{ij}) = \lambda f Z^j a^i g_{ij}, \end{aligned}$$

where we have used $a^i g_{in} = 0$ and equation (4.20). Hence equation (4.20) is equivalent to

$$\nabla_\eta(g(\xi, Z)) = \lambda f g(\xi, Z), \tag{4.21}$$

where Z is any arbitrary y-related vector field.

Considering coefficients of $\dot{y}^j \dot{y}^k$ we get

$$\frac{\partial}{\partial q^n}(a^i g_{im}\Gamma^m_{jk}) = f a^i g_{im}\Gamma^m_{jk}, \quad j, k = 1, \ldots, p. \tag{4.22}$$

Assuming equation (4.20), this is equivalent to

$$\nabla_\eta(g(\nabla_{Z_1} Z_2, \xi)) = f\lambda g(\nabla_{Z_1} Z_2, \xi), \tag{4.23}$$

where Z_1, Z_2 are arbitrary y-related vector fields. This is because substituting $Z_l = Z_l^j(y)\frac{\partial}{\partial y^j} + Z_l^n(y, q^n)\frac{\partial}{\partial q^n}$ for $l = 1, 2$ we get

$$g(\nabla_{Z_1} Z_2, \xi) = Z_1^j Z_2^k g(\Gamma^m_{jk}\frac{\partial}{\partial y^m}, \xi) + Z_1^j \frac{\partial Z_2^k}{\partial y^j} g(\frac{\partial}{\partial y^k}, \xi),$$

where we have used $a^i g_{in} = 0$, $a^i \Gamma^m kn g_{im} = 0$ (since $\ker Ty$ is orthogonal to D) and $\frac{\partial Z_2^k}{\partial q^n} = 0$ for $k = 1, \ldots, p$. Hence

$$\begin{aligned} &\nabla_\eta(g(\nabla_{Z_1} Z_2, \xi)) \\ &= \lambda Z_1^j Z_2^k \frac{\partial}{\partial q^n}(a^i g_{im}\Gamma^m_{jk}) + \lambda Z_1^j \frac{\partial Z_2^k}{\partial y^j}\frac{\partial}{\partial q^n}(a^i g_{ik}) \\ &= \lambda f Z_1^j Z_2^k a^i g_{im}\Gamma^m_{jk} + \lambda f Z_1^j \frac{\partial Z_2^k}{\partial y^j} a^i g_{ik} \end{aligned}$$

where we have used equations (4.20) and (4.22). This simplifies to

$$\nabla_\eta(g(\nabla_{Z_1} Z_2, \xi)) = \lambda f g(\nabla_{Z_1} Z_2, \xi). \tag{4.24}$$

Finally considering the coefficients of the terms independent of $\dot{y}$ and $\ddot{y}$ we get

$$\frac{\partial}{\partial q^n}(a^i \frac{\partial V}{\partial q^i}) = f a^i \frac{\partial V}{\partial q^i}.$$

Clearly this is equivalent to

$$\nabla_\eta(\xi(V)) = \lambda f \xi(V), \tag{4.25}$$

completing the proof. ■

4.4 Systems with n Degrees of Freedom, n − 1 Controls and Symmetry

In this section we shall consider systems of the type considered in last section that also exhibit symmetries. We shall suppose that a Lie group G acts on our configuration space Q with action Φ_h corresponding to $h \in G$ and that

$$\Phi_h^* g = g, \qquad \Phi_h^* P = P \quad \forall h \in G. \tag{4.26}$$

In other words the kinetic energy of the system as well as the range of control forces both are invariant under the group action. However we do not assume that V is invariant under the group action. Many mechanical systems fall under this category. Rigid body systems moving in Euclidean space actuated by body fixed forces are typical examples where the group is $G = SE(3)$, even though the equations of motion often do not have $SE(3)$ as a symmetry group since potential forces due to gravity break the symmetry. But since V plays a very limited role in configuration flatness we may expect that when the system is configuration flat that it would be possible to find flat outputs that reflect this symmetry. We believe this to be true and shall prove it for the case $\dim D = n - 1$. The general case $\dim D < n - 1$ has not yet been resolved completely (see Remark 4.11).

Lemma 4.8 *Consider a system satisfying (4.26). Let D be defined as in (4.11). Then $\Phi_{h_*} D = D$.*

Proof: Let ξ span $\operatorname{ann}(P)$. Clearly $\Phi_{h_*}(\operatorname{ann}(P)) = \operatorname{ann}(P)$. Hence $\Phi_{h_*}\xi = \lambda_h \xi \in D$ where λ_h is some smooth function. Since Φ_h is an isometry by (4.26), it follows that $\Phi_{h_*}(\nabla_Z \xi) = \nabla_{\Phi_{h_*} Z}(\Phi_{h_*}\xi)$ by properties of ∇ (see, for example, [15] page 161). Hence

$$\begin{aligned} \Phi_{h_*} \nabla_Z \xi &= \nabla_{\Phi_{h_*} Z}(\lambda_h \xi) \\ &= \lambda_h \nabla_{\Phi_{h_*} Z} \xi + (\nabla_{\Phi_{h_*} Z} \lambda_h) \xi \in D. \end{aligned} \tag{4.27}$$

So we have $\Phi_{h_*} D \subset D$. Since Φ_h is a diffeomorphism, the result follows by dimension count. ■

Let $y : Q \to \mathbb{R}^p$ be a map defined locally around $q \in Q$. We shall say $y^1, \ldots, y^p$ are *G-equivariant* if

$$\Phi_{h_*} \ker Ty = \ker Ty.$$

This means level sets of y are mapped to level sets by the group action.

Proposition 4.9 *Consider a system satisfying (4.26). Suppose* $\dim D = n - 1$ *and that the system is configuration flat. Then the flat outputs are G-equivariant.*

Proof: Follows from the fact that $\ker Ty$ is the orthogonal complement to D and Lemma 4.8. ■

Remark 4.10 The case $\dim D = n-1$ is not as restrictive as it may seem. Typically $\dim D = n$, implying that the system is not configuration flat. When the system is configuration flat ($\dim D \leq n - 1$), most likely $\dim D = n - 1$. In fact many examples of systems that are configuration flat fall into this category including the first example in next section as well as the "ducted fan with stand" in [41] and the "planar coupled rigid bodies" example in [30].

Remark 4.11 In the case when $\dim D < n - 1$, given the system is flat with flat outputs $y : Q \to \mathbb{R}^p$ around $q \in Q$, it is possible to construct outputs $\tilde{y} : Q \to \mathbb{R}^p$ around q that are G-equivariant and satisfy $g(\ker T\tilde{y}, D) = 0$. But it hasn't been resolved whether it is possible to construct $\tilde{y}$ in such a way that it also satisfies the regularity conditions (4.14). But we suspect that at least in typical cases this construction should work. The second example in next section falls into the case $\dim D = n - 2$ and we see that it possesses G-equivariant flat outputs.

4.5 Examples

In this section we shall consider some examples to illustrate the theory developed in the previous section.

4.5.1 Underwater Vehicle

We shall study a simple model of an underwater vehicle that is controlled by a force applied through a fixed point P on the body whose magnitude and direction can be independently controlled.

Only the motion in the vertical plane is considered and hence our configuration space is $SE(2) = \mathbb{R}^2 \times S^1$. This is reasonable when the vehicle has symmetries about 3 orthogonal planes. In addition if we assume that the centre of buoyancy is coincident with centre of mass, the kinetic energy is given by

$$\frac{1}{2}(m + \delta m)(\dot{x}_1 \cos\theta - \dot{x}_2 \sin\theta)^2 + \frac{1}{2}(m - \delta m)(\dot{x}_1 \sin\theta + \dot{x}_2 \cos\theta)^2 + \frac{1}{2}I(\dot{\theta})^2, \tag{4.28}$$

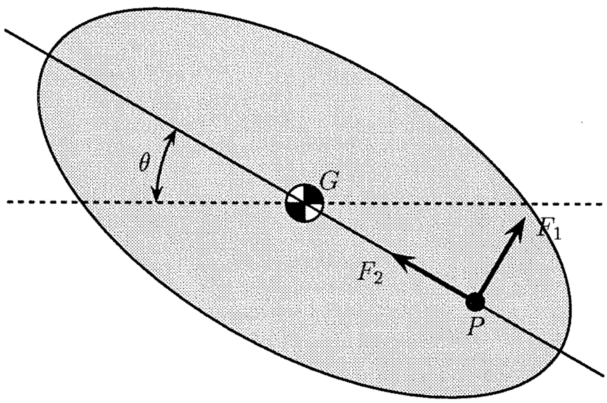


Figure 4.1 Underwater vehicle in $\mathbb{R}^2$

where (x_1, x_2) are horizontal and vertical coordinates of the centre of mass G, θ is the orientation (measured clockwise) of line PG with respect to horizontal axis, $m = M+(m_1+m_2)/2$ and $\delta m = (m_1-m_2)/2$ where M is the mass of the vehicle and m_1 and m_2 are added mass terms that take into account inertia of the fluid, and I is the effective moment of inertia taking into account the fluid. This model assumes an incompressible, irrotational flow and neglects viscosity effects. It is assumed that the motion of the fluid is entirely due to that of the solid. The body and the fluid together are considered to form a dynamical system and the kinetic energy is the combined energy of body and fluid. See [17] and [18] for details. The analysis in [18] assumes a neutrally buoyant model, but we need not make this assumption since this only alters the form of the potential function but does not affect the kinetic energy. In fact for the first part of the analysis we shall not assume any specific form for potential V. If the vehicle is in air (strictly speaking vacuum) $m_1 = m_2 = 0$, so $m = M$ and $\delta m = 0$ and the kinetic energy takes the familiar form

$$\frac{1}{2}(m(\dot{x}_1)^2 + m(\dot{x}_2)^2 + I(\dot{\theta})^2)$$

where I is the usual moment of inertia and the model is the same as that of VTOL (see [23]).

The metric g in coordinates x_1, x_2, θ is given by the matrix

$$\begin{bmatrix} m + \delta m \cos 2\theta & -\delta m \sin 2\theta & 0 \\ -\delta m \sin 2\theta & m - \delta m \cos 2\theta & 0 \\ 0 & 0 & I \end{bmatrix}.$$

The control forces lie in the codistribution

$$\begin{aligned} P &= \operatorname{span}\{d(x_1 + R\cos\theta), d(x_2 - R\sin\theta)\} \\ &= \operatorname{span}\{dx_1 - R\sin\theta d\theta, dx_2 - R\cos\theta d\theta\} \end{aligned}$$

and $\xi = \frac{\partial}{\partial\theta} + R\sin\theta\frac{\partial}{\partial x_1} + R\cos\theta\frac{\partial}{\partial x_2}$ spans ann(P) where R is the length of PG.

The Christoffel symbols Γ^i_{jk} can be computed from g using equation (4.9). Then using formula (4.10) we see that

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x_1}}\xi &= -\frac{m\delta m}{m^2-(\delta m)^2}\sin 2\theta\frac{\partial}{\partial x_1} - \frac{\delta m}{m^2-(\delta m)^2}(\delta m + m\cos 2\theta)\frac{\partial}{\partial x_2} \\
&+ \frac{R\delta m\cos\theta}{I}\frac{\partial}{\partial\theta} \\
\nabla_{\frac{\partial}{\partial x_2}}\xi &= -\frac{\delta m}{m^2-(\delta m)^2}(-\delta m + m\cos 2\theta)\frac{\partial}{\partial x_1} + \frac{m\delta m}{m^2-(\delta m)^2}\sin 2\theta\frac{\partial}{\partial x_2} \\
&- \frac{R\delta m\sin\theta}{I}\frac{\partial}{\partial\theta} \\
\nabla_{\frac{\partial}{\partial\theta}}\xi &= \frac{mR\cos\theta}{m+\delta m}\frac{\partial}{\partial x_1} - \frac{mR\sin\theta}{m+\delta m}\frac{\partial}{\partial x_2}.
\end{aligned} \tag{4.29}$$

It can be seen by computation that the above vector fields together with ξ span the full tangent space for generic points and generic parameter values $m, \delta m, I, R$. Since by equation (4.12)

$$D = \text{span}\{\nabla_{\frac{\partial}{\partial x_1}}\xi, \nabla_{\frac{\partial}{\partial x_2}}\xi, \nabla_{\frac{\partial}{\partial\theta}}\xi, \xi\},$$

it follows that $D = TQ$ for generic points on Q and for generic parameter values and hence the system is not configuration flat for generic parameter values regardless of the potential energy function.

However for the case $\delta m = 0$, we see that

$$D = \text{span}\{R\cos\theta\frac{\partial}{\partial x_1} - R\sin\theta\frac{\partial}{\partial x_2}, R\sin\theta\frac{\partial}{\partial x_1} + R\cos\theta\frac{\partial}{\partial x_2} + \frac{\partial}{\partial\theta}\}.$$

Hence $\dim D = 2$ and $\eta = \frac{\partial}{\partial\theta} - \frac{I}{mR}\sin\theta\frac{\partial}{\partial x_1} - \frac{I}{mR}\cos\theta\frac{\partial}{\partial x_2}$ spans the orthogonal complement to D. Since D has codimension 1, up to a diffeomorphism there is at most 1 set of flat outputs. One set of functions that "cut out" the foliation due to η is

$$y_1 = x_1 - \frac{I}{mR}\cos\theta, \quad y_2 = x_2 + \frac{I}{mR}\sin\theta.$$

To ensure that y_1, y_2 are indeed flat outputs we must check the regularity conditions (4.15). Let us choose $z = \theta$ as a complementary coordinate to y_1, y_2. Then,

$$\begin{aligned}
\frac{\partial}{\partial y_1} &= \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial y_2} = \frac{\partial}{\partial x_2}, \\
\frac{\partial}{\partial z} &= -\frac{I}{mR}\sin\theta\frac{\partial}{\partial x_1} - \frac{I}{mR}\sin\theta\frac{\partial}{\partial x_1} + \frac{\partial}{\partial\theta}.
\end{aligned}$$

Hence

$$\frac{\partial}{\partial z}\left(g(\xi,\frac{\partial}{\partial y_1})\right) : g(\xi,\frac{\partial}{\partial y_1}) = -\sin z : \cos z$$
$$\frac{\partial}{\partial z}\left(g(\xi,\frac{\partial}{\partial y_2})\right) : g(\xi,\frac{\partial}{\partial y_2}) = \cos z : \sin z. \tag{4.30}$$

So at any point $q = (y_1, y_2, z)$ these two ratios are unequal. This ensures that y_1, y_2 are indeed flat outputs everywhere.

When the vehicle is in air (strictly speaking vacuum) $\delta m = 0$, and in this case the model is the same as the planar rigid body considered in Chapter 1 and it is already known to be flat. We have just shown that up to a diffeomorphism these are the only configuration flat outputs. Also we have covered the case of underwater vehicle of spherical shape (since then $m_1 = m_2$) and this result is independent of any assumptions we make on the potential function V.

Now let us suppose the system is moving under gravity in air and the potential energy is given by $V = mgx_2$ where $g \approx 9.8$ m/s^2 is the acceleration due to gravity. Then the solutions of the system in coordinates y_1, y_2, z satisfy the ODE

$$\ddot{y}_1 \sin z + \ddot{y}_2 \cos z + g \cos z = 0.$$

So along generic solution curves we get,

$$z(t) = \tan^{-1} \frac{\ddot{y}_2 + g}{\ddot{y}_1}$$

or

$$z(t) = \tan^{-1} \frac{\ddot{y}_2 + g}{\ddot{y}_1} + \pi.$$

The exception being the singularity at $\ddot{y}_1 = 0, \ddot{y}_2 + g = 0$. Note that this singularity is not a point on Q but corresponds to a submanifold in the jet space $J^2(\mathbb{R}, Q)$, the space with coordinates $(t, q, \dot{q}, \ddot{q})$ and such singularities are very common in practical examples. We still want to regard such systems as flat and this is the reason why our definition of flatness refers to an open dense subset of points. Also note that though potential V does not affect the flat outputs of the system it influences where the singularities occur.

We also see that the general system (no assumptions on δm) possesses an $SE(2)$ symmetry when the potential function is ignored. If we consider translating and rotating our spatial frame of reference the expression for kinetic energy as well as the the expression for P are invariant. We may state this more precisely as follows. Consider the following action of $SE(2)$ on $Q = SE(2)$. Given $h = (\alpha_1, \alpha_2, \phi) \in SE(2)$ the action Φ_h corresponds to first rotating the spatial frame counter clockwise by ϕ about its origin and then with respect to this frame translate the frame without rotation by $(-\alpha_1, -\alpha_2)$. Hence if $q = (x_1, x_2, \theta) \in Q$ then

$$\Phi_h(q) = (x_1 \cos\phi + x_2 \sin\phi + \alpha_1, -x_1 \sin\phi + x_2 \sin\phi + \alpha_2, \theta + \phi).$$

The corresponding tangent map $T\Phi_h$ is given by

$$\begin{aligned}\frac{\partial}{\partial x_1} &\to \cos\phi\frac{\partial}{\partial x_1} + \sin\phi\frac{\partial}{\partial x_2}\\ \frac{\partial}{\partial x_2} &\to -\sin\phi\frac{\partial}{\partial x_1} + \cos\phi\frac{\partial}{\partial x_2}\\ \frac{\partial}{\partial\theta} &\to \frac{\partial}{\partial\theta}.\end{aligned} \tag{4.31}$$

It is easy to verify this preserves g. Recalling that $\xi = \frac{\partial}{\partial\theta} + R\sin\theta\frac{\partial}{\partial x_1} + R\cos\theta\frac{\partial}{\partial x_2}$ spans $\mathrm{ann}(P)$, we see that $\Phi_{h_*}\xi = \xi$, implying $\Phi_h^* P = P$. In particular these statements are true for the $\delta m = 0$ case as well. Hence by Proposition 4.9 the flat outputs are G-equivariant. This is indeed true since $\eta = \frac{\partial}{\partial\theta} - \frac{I}{mR}\sin\theta\frac{\partial}{\partial x_1} - \frac{I}{mR}\cos\theta\frac{\partial}{\partial x_2}$ spans $\ker Ty$ and $\Phi_{h_*}\eta = \eta$.

4.5.2 Particle in a Potential Field

This example does not necessarily correspond to an engineering example, but illustrates the regularity conditions. We consider a particle of unit mass moving in 3 dimensional Euclidean space in the presence of a potential field $V = x_2x_3$. Hence the kinetic energy metric is given by the 3×3 identity matrix in orthogonal coordinates x_1, x_2, x_3. Suppose we control independently the forces along x_1 and x_3 directions. Hence $P = \mathrm{span}\{dx_1, dx_3\}$ and $\xi = \frac{\partial}{\partial x_2}$ spans $\mathrm{ann}(P)$. We see that Christoffel symbols are all zero by (4.9) (which is a feature of Euclidean space) and using (4.10) and (4.12) we obtain $D = \mathrm{span}\{\frac{\partial}{\partial x_2}\}$ and hence the orthogonal complement to D is $\mathrm{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}\}$ which is two dimensional. Hence we have infinitely many "candidates" for flat outputs that are not equivalent via a diffeomorphism. But these "candidates" may not satisfy the regularity conditions (4.15). Following the method outlined in Remark refremalg we pick some η, say $\eta = \frac{\partial}{\partial x_3}$ which is orthogonal to D. Then $y_1 = x_1, y_2 = x_2$ are a possible choice of corresponding "candidates" for flat outputs (since they cut out the one dimensional foliation by η). We may choose $z = x_3$ to complete the coordinate system and then we see that the ratio of functions $\frac{\partial}{\partial z}(\xi(V)) : \xi(V)$ in the set (4.15) is $1 : x_3$ where as the ratio of $\frac{\partial}{\partial z}(g(\xi, \frac{\partial}{\partial y^2}))$ is $0 : 1$. Hence x_1, x_2 are configuration flat outputs (globally). But alternatively another choice could have been $\eta = \frac{\partial}{\partial x_1}$ with corresponding candidates $y_1 = x_2, y_2 = x_3$. Choosing $z = x_1$ we see that all the ratios in (4.15) are zero and hence equal. Hence x_2, x_3 are not flat outputs as they are differentially dependent. This example is simple enough that the above conclusions can be reached by inspecting the equations of motion for the system

$$\ddot{x}_1 - \frac{\partial V}{\partial x_1} = F_1 \tag{4.32}$$

$$\ddot{x}_2 - \frac{\partial V}{\partial x_2} = 0 \tag{4.33}$$

$$\ddot{x}_3 - \frac{\partial V}{\partial x_3} = F_3 \tag{4.34}$$

where F_1, F_3 are the forces along x_1, x_3 directions. The equation (4.33) alone characterises all solution trajectories of system and substituting $V = x_2 x_3$ we obtain,

$$\ddot{x}_2 - x_3 = 0. \tag{4.35}$$

It is clear from the equation that x_2, x_3 are differentially dependent and hence are not flat outputs. However it is also clear from the equations that x_1, x_2 are flat outputs since along solution curves,

$$x_3(t) = \frac{d^2 x_2(t)}{dt^2}$$

and x_1, x_2 do not satisfy an ODE.

Also note that the system is globally controllable since it is globally flat. However if $V = 0$ then the system is not configuration flat and not even locally accessible.

It is easy to see that translations by the group $\mathbb{R}^3$ leave g and P invariant. But Proposition 4.9 does not apply since $\dim D = n - 2$. However as mentioned in Remark 4.11 we see that G-equivariant flat outputs exist. In fact $y = (x_1, x_2)$ are G-equivariant, although not all (configuration) flat outputs are G-equivariant, since $\tilde{y} = (f(x_1, x_3), x_2)$ where f is an arbitrary smooth function with $\frac{\partial f}{\partial x_1} \neq 0$, are not G-equivariant for a typical f, but are configuration flat outputs.

Chapter 5

Conclusions and Future Work

In this dissertation we have presented two different geometric approaches for studying differential flatness and with the aid of these approaches we have obtained some results which help find flat outputs for certain classes of underdetermined systems of ODEs. We believe that though our results of Chapters 3 and 4 only apply to special classes, the results themselves suggest that it may be possible to extend them to include a wider class.

5.1 Geometric Approaches to Flatness

Our first approach to flatness was based on the notion of Cartan prolongations and absolute equivalence. In this approach the locally one-one nature of the correspondence of solution curves of a system with free curves in a lower dimensional space is emphasised. The drawbacks in this approach primarily stem from the fact that the definition of Cartan prolongation is global in nature. This makes dealing with singularities and local issues rather clumsy. However this approach proved useful in demonstrating the validity of a method that we proposed for testing for flatness of systems underdetermined by more than one equation. This method involved guessing all but one flat outputs and reducing the problem to the simpler case of a system underdetermined by one equation which has been completely solved by Cartan. If the guesses were correct the method provides the other flat output, and if the guesses were incorrect the method reveals this fact. We illustrated how this method can be used to find nontrivial flat outputs with the aid of the example of two coupled planar rigid bodies controlled by two independent forces and a differential torque between the rigid bodies.

The second approach to flatness was to use jet bundles and was presented in Chapter 3. We constructed a sequence of spaces that had the independent variable t, the dependent variables x, and derivatives of x up to some finite order as their coordinates. This approach is closely related to the infinite jet bundle approaches of Fliess and coworkers [10] and of Pomet [27]. The approach of Chapter 3 was also a differential forms approach but differed from the absolute equivalence approach of Chapter 2 in that we directly dealt with transformations that involve variables and their derivatives and not so much with solution curves.

Since the classification of differential flatness is far from being complete, it is premature to decide which particular approach or tool is most suitable. We believe, of the two approaches presented in this dissertation the jet bundle approach has proven to be more useful. However the proof of Theorem 3.42 depended on using the Corollary 2.15 which was proven in the absolute equivalence frame work. For completeness we believe this should be proven in the jet bundle frame work. We believe that it might be useful to extend the results of Theorem 3.42 to systems of arbitrary number of one-forms. If this proves successful then it will be a good indication that this is the appropriate frame work to study flatness. On the other hand another possible line of investigation may be to improve on the definition of Cartan prolongations in order to permit a local theory. Insights gained from the jet bundle approach might help here.

5.2 Zero-Flatness

The jet bundle approach also enabled us to prove a theorem on zero-flatness of a system consisting of two independent one-forms, where zero-flat outputs were defined as flat outputs that depended only on t and x but not on derivatives of x. This theorem showed that it is possible to split the task of finding zero-flat outputs into two parts. The first part deals with the "infinitesimal" aspects. We found intrinsic geometric conditions that the tangent spaces of the level sets of zero-flat outputs should satisfy. The second part deals with integrability (in other words being able to piece these tangent spaces together to form smooth level sets) and leads to a system of nonlinear PDEs and we did not provide any theory on the existence of solutions for the second part. In other words we characterised zero-flatness on an "infinitesimal" level, but did not deal with integrability. Dealing with integrability is the subject of further research. Though this is a serious limitation we believe that this approach is still useful for the following reasons. Firstly it provides a way to write down PDEs to be solved for. But more importantly the simple and elegant geometric conditions we have obtained suggest that there may be similar conditions for the general case of arbitrary number of one-forms. We suspect that it is possible to split the problem in this way for the general case of arbitrary number of one-forms and to obtain "infinitesimal" conditions and that these may provide useful insights into the general problem of differential flatness. We also believe that the theory developed in Chapter 3 will prove useful in obtaining these infinitesimal conditions (if they exist) for systems governed by arbitrary number of one-forms.

Most of the theory developed in nonlinear control applies to ideal systems. However, approximation and discretisation are inevitable in the engineering world. The theory for zero-flatness of a system of two one-forms shows that the first part (infinitesimal part) of the problem always has solutions. The solutions are families of two dimensional distributions. The second part involves PDEs to solve in order to find an integrable distribution from the family of solutions to the first part. The second part typically does not have solutions, but one may look for approximate methods at this stage to find approximate solutions leading to "approximate flat outputs". We believe that this is a research direction worth exploring.

5.3 Configuration Flatness of Lagrangian Systems

With the aid of jet bundle formalism we also presented a theorem which completely classified configuration flatness (this is a special case of flatness where the flat outputs depend only on configuration variables and not on their derivatives) of Lagrangian mechanical systems whose number of independent controls was one less than the number of degrees of mechanical freedom (i.e. systems underactuated by one control). This classification was also in terms of the tangent spaces of the level sets of the flat outputs. But in this case integrability was not an issue since one dimensional distributions are always integrable. The characterisation was in terms of the covariant derivative corresponding to the Riemannian metric which corresponds to the kinetic energy and gave rise to a constructive algorithm for finding zero-flat outputs (if they exist). We illustrated the method by two examples. Our characterisation also was able to explain why the potential energy typically did not play a role in deciding the configuration flatness of mechanical systems. We were also able to relate symmetries of mechanical systems with certain symmetries of the flat outputs.

The success of the theory of configuration flatness for Lagrangian systems underactuated by one control, suggests that there may be similar characterisations for systems with arbitrary number of controls. However an intrinsic proof of the configuration flatness result may be necessary in order to extend the theory. The current proof is coordinate based and a similar approach for the general case will be exceedingly clumsy. Also integrability becomes an issue in the general case. For the case of one fewer controls integrability was not an issue as mentioned above. It will also be interesting to extend the theory to include nonholonomic constraints as well as certain friction models.

Appendix A

An Alternative Proof of Two One-Forms Case

In this appendix we shall present an alternative jet bundle approach to a formulation of zero-flatness and an alternative proof of Theorem 3.40. This is the original approach which led to the result of Theorem 3.40 and we include it here for completeness. This is essentially an excerpt from the paper [29].

Let $t : M \to \mathbb{R}$ be a fibre bundle. In other words M is a (smooth) manifold, t is a surjective submersion (corresponds to time) and the bundle is locally trivial. Let us denote the associated kth order jet bundle $J^k(t, M)$ by J^k for brevity and the projection from J^k to M by ρ^k. We shall consider a constant dimensional Pfaffian system (in other words a system of one-forms) I on M such that $dt_q \notin I_q$ for all points $q \in M$. We shall always be concerned with coordinate systems on M of the form $(t, x^1, \ldots, x^N)$, where $\dim M = N + 1$. A section c of the bundle (i.e. a curve $c : \mathbb{R} \to M$ such that $t \circ c$ is the identity on $\mathbb{R}$) is said to be a solution of I if $c^*(I) = \{0\}$. Let $(t, x^1, \ldots, x^N)$ be a coordinate system and let $I = \operatorname{span}\{\omega^i \; : \; i = 1, \ldots, n\}$ locally, where $\omega^i = a^i_j dx^j + a^i dt$ and $n = \dim I$. Then solutions of I satisfy

$$a^i_j \frac{dx^j}{dt} + a^i = 0, \quad i = 1, \ldots, n.$$

It is natural to consider the associated set of functions E^i defined by,

$$E^i = a^i_j \dot{x}^j + a^i, \quad i = 1, \ldots, n,$$

which are functions on the first order jet space J^1. Given a one-form $a_j dx^j + a dt$ on M there is a corresponding function $a_j \dot{x}^j + a$ on J^1 which is affine in the fibre coordinates $\dot{x}^j$. Conversely given an affine (in fibre coordinates) function $a_j \dot{x}^j + a$ on J^1 there is a unique one-form $a_j dx^j + a dt$ on M. This correspondence is intrinsic. The functions E^i above that describe the ODEs are obtained from ω^i by this correspondence. Hence there is an intrinsic subbundle $\mathscr{E}^1 \subset J^1$ that corresponds to I which is locally the zero set of the functions E^i,

$$\mathscr{E}^1 = \{q_1 \in J^1 \; : \; E^i(q_1) = 0, i = 1, \ldots, n\}.$$

We refer to Section 4.1 for the notions of prolongations $\mathscr{E}^k \subset J^k$.

Let $y^1, \ldots, y^p$ (where $p = N - n$) be a set of functions on M locally defined near a point $q \in M$. $y^1, \ldots, y^p$ are said to be (0,r)-flat outputs if for any function z on M their exists a map g such that along "generic" solution curves c in a neighbourhood of q

$$(z \circ c)(t) = g(t, (y \circ c)(t), \frac{d(y \circ c)(t)}{dt}, \ldots, \frac{d^r(y \circ c)(t)}{dt^r}),$$

and if $y^1, \ldots, y^p$ never satisfy an ODE, in other words there exists no function F such that

$$F(t, (y \circ c)(t), \frac{d(y \circ c)(t)}{dt}, \ldots, \frac{d^r(y \circ c)(t)}{dt^r}) = 0$$

along "generic" solution curves c near q.

Note that we require these properties to hold only for "generic" curves. The meaning of generic curves will be made precise by translating the above definition into a more precise geometric statement involving the objects $\mathscr{E}^k \subset J^k$.

The time derivative operator defined in Chapter 3 also makes sense in this setup and we shall make use of it.

Definition A.1 The functions $y^1, \ldots, y^p$ are (0,r)-*flat outputs* near $q \in M$ if there exists a neighbourhood $U \subset M$ of q so that for any function z on M

$$dz \in \text{span}\{dt, dy^1, \ldots, dy^p, \ldots, dy^{1,(r)}, \ldots, dy^{p,(r)}\} \tag{A.1}$$

when pulled back to $\mathscr{E}^r$ for all points on $(\rho^r)^{-1}(U) \cap \mathscr{E}^r$, and

$$\{dt, dy^1, \ldots, dy^p, \ldots, dy^{1,(r)}, \ldots, dy^{p,(r)}\}$$

is linearly independent when pulled back to $\mathscr{E}^r$ for generic (open and dense subset of) points on $(\rho^r)^{-1}(U) \cap \mathscr{E}^r$.

Suppose $\{dt, dy^1, \ldots, dy^p\}$ is linearly independent around q. It is instructive to choose a coordinate system that complements $(t, y^1, \ldots, y^p)$. Let $(t, y^1, \ldots, y^p, z^1, \ldots, z^n)$ be a coordinate system. Let $I = \text{span}\{\omega^1, \ldots, \omega^n\}$ locally. Suppose E^i is the affine function on J^1 corresponding to ω^i for $i = 1, \ldots, n$. Then $\mathscr{E}^r$ is locally the zero set of $E^i, \ldots, E^{i,(r-1)}$ for $i = 1, \ldots, n$. Then condition (A.1) is equivalent to

$$dz^i \in \text{span}\{dt, dy^1, \ldots, dy^p, \ldots, dy^{1,(r)}, \ldots, dy^{p,(r)}\}, \quad i = 1, \ldots, n, \tag{A.2}$$

when pulled back to $\mathscr{E}^r$ for all points on $(\rho^r)^{-1}(U) \cap \mathscr{E}^r$.

When pulled back to $\mathscr{E}^r$,

$$dE^{i,(k)} = 0, \quad i = 1, \ldots, n, \quad k = 0, \ldots, r-1. \tag{A.3}$$

Hence it follows that (0,r)-flatness of $y^1, \ldots, y^p$ is equivalent to

$$\text{rank}\frac{\partial(E, E^{(1)}, \ldots, E^{(r-1)})}{\partial(z^{(1)}, \ldots, z^{(r)})} \leq nr - n, \tag{A.4}$$

for all points on zero set of $E, E^{(1)}, \ldots, E^{(r-1)}$ and

$$\text{rank}\frac{\partial(E, E^{(1)}, \ldots, E^{(r-1)})}{\partial(z, z^{(1)}, \ldots, z^{(r)})} = nr \tag{A.5}$$

(in other words full rank) for generic points on zero set of $E, E^{(1)}, \ldots, E^{(r-1)}$. The rank condition (A.4) says that there are n independent relations amongst

$$\{dz^1, \ldots, dz^n, dt, dy^1, \ldots, dy^p, \ldots, dy^{1,(r)}, \ldots, dy^{p,(r)}\}.$$

The rank condition (A.5) says that there are no relations (for generic points) amongst

$$\{dt, dy^1, \ldots, dy^p, \ldots, dy^{1,(r)}, \ldots, dy^{p,(r)}\}.$$

The two conditions together then imply (A.2).

Finally we have the notion of zero-flat outputs.

Definition A.2 Functions $y^1, \ldots, y^p$ are *zero-flat outputs* if they are (0,r) flat outputs for some integer r.

We shall consider the special case of $n = \dim I = 2$. Hence $\dim M = p + 3$, p arbitrary. In this case functions $y^1, \ldots, y^p$ are zero-flat outputs if and only if the following hold.

$$\text{rank}\frac{\partial(E, \dot{E})}{\partial(\dot{z}, \ddot{z})} \leq 2 \tag{A.6}$$

for all points on $\{E = \dot{E} = 0\} \subset J^2$ and

$$\text{rank}\frac{\partial(E, \dot{E})}{\partial(z, \dot{z}, \ddot{z})} = 4 \tag{A.7}$$

for generic points on $\{E = \dot{E} = 0\} \subset J^2$. Here z^1, z^2 are such that $t, y^1, \ldots, y^p, z^1, z^2$ are local coordinates. This follows by substituting $n = 2$ and using Lemma 3.41 to substitute $r = 2$ into rank conditions (A.4) and (A.5).

We present a proposition which characterises (A.6) in an intrinsic geometric sense and is essentially the same as Theorem 3.40. This characterisation is in terms of $\ker T(t, y^1, \ldots, y^p)$, which in coordinates is the null space of the Jacobian of the $\mathbb{R}^{p+1}$ valued map $(t, y^1, \ldots, y^p)$.

Proposition A.3 *Let $y^1, \ldots, y^p$ be functions such that $\{dt, dy^1, \ldots, dy^p\}$ is linearly* *independent around* $q \in M$. *Let* z^1, z^2 *be such that* $(y^1, \ldots, y^p, z^1, z^2)$ *form a local coordinate system near* q. *Then the rank condition (A.6) is satisfied around* q *if and only if*

$$\eta_1 \lrcorner \eta_2 \lrcorner (\omega^1 \wedge \omega^2) = 0 \tag{A.8}$$

$$\eta_1 \lrcorner \eta_2 \lrcorner d(\omega^1 \wedge \omega^2) = 0 \mod I \tag{A.9}$$

where η_1, η_2 *span* $\ker T(t, y^1, \ldots, y^p)$ *and* ω^1, ω^2 *span* I.

Remark A.4 Note that the conditions (A.8) and (A.9) are independent of the choice of basis for I and $\ker T(t, y^1, \ldots, y^p)$.

Proof: Choose functions z^1, z^2 so that $(t, y^1, \ldots, y^p, z^1, z^2)$ is a coordinate system. Notice that $\eta_\beta = \frac{\partial}{\partial z^\beta}$ for $\beta = 1, 2$ is a choice of basis for $\ker T(t, y^1, \ldots, y^p)$. Since (A.8) and (A.9) are independent of choice of basis, it is sufficient to work with this choice of η_β. In coordinates $(y^1, \ldots, y^p, z^1, z^2)$, let

$$\omega^\alpha = a_j^\alpha dy^j + b_k^\alpha dz^k + c^\alpha dt, \quad \alpha = 1, 2$$

be a basis for I. Then $\mathscr{E}^1 \subset J^1$ is cut out by the zero set of

$$E^\alpha = a_j^\alpha \dot{y}^j + b_k^\alpha \dot{z}^k + c^\alpha, \quad \alpha = 1, 2. \tag{A.10}$$

We shall also use the matrix notation

$$E = a\dot{y} + b\dot{z} + c,$$

where $a \in \mathbb{R}^{2 \times p}, b \in \mathbb{R}^{2 \times 2}$ and $c \in \mathbb{R}^{2 \times 1}$ are the obvious matrices. From (A.10) it follows that

$$\begin{aligned} \dot{E}^\alpha = {} & a_j^\alpha \ddot{y}^j + b_k^\alpha \ddot{z}^k + \frac{\partial a_j^\alpha}{\partial y^{j_1}} \dot{y}^j \dot{y}^{j_1} \\ & + (\frac{\partial a_j^\alpha}{\partial z^k} + \frac{\partial b_k^\alpha}{\partial y^j}) \dot{y}^j \dot{z}^k + \frac{\partial b_k^\alpha}{\partial z^{k_1}}) \dot{z}^k \dot{z}^{k_1} + (\frac{\partial a_j^\alpha}{\partial t} + \frac{\partial c^\alpha}{\partial y^j}) \dot{y}^j \\ & + (\frac{\partial b_k^\alpha}{\partial t} + \frac{\partial c^\alpha}{\partial z^k}) \dot{z}^k + \frac{\partial c^\alpha}{\partial t}. \end{aligned} \tag{A.11}$$

Let us define

$$e_\beta^\alpha = \frac{\partial \dot{E}^\alpha}{\partial \dot{z}^\beta}. \tag{A.12}$$

Then it follows that

$$\begin{aligned} e_\beta^\alpha = {} & (\frac{\partial a_j^\alpha}{\partial z^\beta} + \frac{\partial b_\beta^\alpha}{\partial y^j}) \dot{y}^j + (\frac{\partial b_\beta^\alpha}{\partial z^k} + \frac{\partial b_k^\alpha}{\partial z^\beta}) \dot{z}^k \\ & + (\frac{\partial b_\beta^\alpha}{\partial t} + \frac{\partial c^\alpha}{\partial z^\beta}). \end{aligned} \tag{A.13}$$

Since e^α_β is a function affine in the fibres of J^1 there is a corresponding one-form $\hat{e}^\alpha_\beta$ on M. This is given by

$$\hat{e}^\alpha_\beta = (\frac{\partial a^\alpha_j}{\partial z^\beta} + \frac{\partial b^\alpha_\beta}{\partial y^j})dy^j + (\frac{\partial b^\alpha_\beta}{\partial z^k} + \frac{\partial b^\alpha_k}{\partial z^\beta})dz^k + (\frac{\partial b^\alpha_\beta}{\partial t} + \frac{\partial c^\alpha}{\partial z^\beta})dt. \tag{A.14}$$

In matrix notation

$$\frac{\partial(E,\dot{E})}{\partial(\dot{z},\ddot{z})} = \begin{bmatrix} b & 0 \\ e & b \end{bmatrix}. \tag{A.15}$$

Note that

$$\eta_\beta \lrcorner \omega^\alpha = b^\alpha_\beta.$$

From the properties of exterior product, exterior derivative and interior product it follows that

$$\eta_1 \lrcorner \eta_2 \lrcorner (\omega^1 \wedge \omega^2) = \begin{vmatrix} \frac{\partial}{\partial z^1} \lrcorner \omega^1 & \frac{\partial}{\partial z^2} \lrcorner \omega^1 \\ \frac{\partial}{\partial z^1} \lrcorner \omega^1 & \frac{\partial}{\partial z^2} \lrcorner \omega^1 \end{vmatrix} = \det b. \tag{A.16}$$

Since

$$\eta_\beta \lrcorner d\omega^\alpha = \frac{\partial}{\partial z^\beta} \lrcorner d\omega^\alpha = (\frac{\partial a^\alpha_j}{\partial z^\beta} - \frac{\partial b^\alpha_\beta}{\partial y^j})dy^j + (\frac{\partial b^\alpha_k}{\partial z^\beta} - \frac{\partial b^\alpha_\beta}{\partial z^k})dz^k + (\frac{\partial b^\alpha_\beta}{\partial t} - \frac{\partial c^\alpha}{\partial z^\beta})dt, \tag{A.17}$$

it follows that

$$\hat{e}^\alpha_\beta = \eta_\beta \lrcorner d\omega^\alpha + 2d(\eta_\beta \lrcorner \omega^\alpha). \tag{A.18}$$

Necessity: Suppose (A.15) has rank less than or equal to 2 on $\{E = 0, \dot{E} = 0\}$. Then by necessity rank $b \leq 1$, since rank of (A.15) is at least twice the rank of b. Hence b is singular and $\det b = 0$ for all points on $\{E = \dot{E} = 0\} \subset J^2$. But since b^α_k are functions on M it follows $\det b = 0$ holds on an open neighbourhood of $q \in M$. Hence by (A.16) the condition (A.8) follows.

Suppose $b = 0$ (zero matrix). Then it follows $\eta_\beta \lrcorner \omega^\alpha = b^\alpha_\beta = 0$ for $\alpha, \beta = 1, 2$. Hence (A.9) follows from properties of exterior product, exterior derivative and interior product. So let us assume $b \neq 0$. Since (A.15) has rank less than 3, it follows that all of its 3×3 minors should vanish on $\{E = \dot{E} = 0\} \subset J^2$. Singularity of b implies that all 3×3 minors that do not contain e vanish. So we are left with the four 3×3 minors that contain e. They all have the form $b^\alpha_k W$ (after using the relation $b^1_1 b^2_2 = b^1_2 b^2_1$), where

$$W = b^1_1 e^2_2 + b^2_2 e^1_1 - b^1_2 e^2_1 - b^2_1 e^1_2. \tag{A.19}$$

Since $b \neq 0$ it follows that $W = 0$ for points on $\{E = \dot{E} = 0\} \subset J^2$. But W is a function on J^1 and hence it is sufficient if it is zero for points on $\{E = 0\} \subset J^1$. Also since W is a function that is affine in the fibres of J^1 (since e^α_β are) it corresponds to a one-form $\widehat{W}$ on M. W being zero on $\{E = 0\} \subset J^1$ (which is locally $\mathscr{E}^1 \subset J^1$) is equivalent to $\hat{W} = 0 \mod I$. Since

$$\widehat{W} = b^1_1 \hat{e}^2_2 + b^2_2 \hat{e}^1_1 - b^1_2 \hat{e}^2_1 - b^2_1 \hat{e}^1_2,$$

it follows (after using properties of exterior product, exterior derivative and interior product) that

$$\widehat{W} = \eta_1 \lrcorner \eta_2 \lrcorner d(\omega^1 \wedge \omega^2) + 2d(\eta_1 \lrcorner \eta_2 \lrcorner (\omega^1 \wedge \omega^2)). \tag{A.20}$$

Since we have already shown $\eta_1 \lrcorner \eta_2 \lrcorner (\omega^1 \wedge \omega^2) = 0$, equation (A.9) follows.

Sufficiency: Suppose conditions (A.8) and (A.9) hold. Then previous reasoning shows that $\det b = 0$ and that $W = 0$ for points on $\{E = 0, \dot{E} = 0\} \subset J^2$. As explained before this means all 3×3 minors of (A.15) vanish for points on $\{E = 0, \dot{E} = 0\} \subset J^2$. This is the same as (A.6). ■

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