Essays on Uncertainty: an Axiomatization and Economic Applications

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Abstract

This thesis deals with individual decision making under uncertainty and extends standard economic and financial models with risk in order to model attitudes towards ambiguity. Empirical violations of the leading theories of choice have pointed out the relevance of ambiguity on individual choices. Much empirical evidence, inspired by Ellsberg's experiment, shows that people prefer bets whose odds of winning are known, thus suggesting aversion to ambiguity.

The relevance of uncertainty is pervasive in economic literature as well as in the real world. Investment decisions and asset pricing, insurance contracts, voting in elections, gambling, buying a car or planning a trip can all be thought as choices under both risk and ambiguity.

The aim of this work is normative and descriptive. In the first part of the work, I concentrate on a theoretical model of focused regret as an extension of the classical paradigm of choice theory. From the alternatives in this literature, I focus on the multiple prior model, originally axiomatized by Gilboa and Schmeidler, where ambiguity is formalized as a set of plausible probability distributions to represent agents' beliefs. Then I turn to economic and financial applications and challenge the descriptive power of classical economic models when explaining, for example, asset pricing or insurance contracting. Lastly, the most innovative part of this work is an experimental investigation of the multiple priors model in a financial market, through which I find evidence for both risk and ambiguity to affect individual decision making and asset pricing.

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Chapter 1 Introduction

This thesis deals with individual decision making under uncertainty and extends standard economic and financial models with risk in order to model attitudes towards ambiguity. Decisions are made under risk when probabilities of a lottery are known, under ambiguity when probabilities are unknown.

Empirical violations of the leading theories of choice, the Expected Utility theory of von Neumann and Morgenstern and the Subjective Expected Utility theory of Savage, have pointed out the relevance of ambiguity on individual choices. Much empirical evidence, inspired by Ellsberg's experiment, shows that people prefer bets whose odds of winning are known, thus suggesting aversion to ambiguity.

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First, it is worth noticing that all of the applications deal with competitive markets, thus distinguishing this work from most of the known existing literature on game-theoretical applications for the model. Secondly, the common denominator for all the different applications is the use of the same multiple priors model, extended to capture different attitudes toward ambiguity. The reader is thus invited to capture the intuition behind the directions of the main results, as it follows through the chapters to explain the historical equity premium, overinsurance and prices spreads in financial experiments.

The present thesis contains six chapters, including the present introduction. The object of Chapter 2 is to axiomatize a decision theoretic model where choices are made under uncertainty and with respect to a non-fixed reference point. The model is an extension of Gilboa and Schmeidler's maximin expected utility with multiple priors to include a reference point and study focused regret. Even though such an extension may be valuable from a descriptive point of view, the concern in this first chapter is normative as I mainly discuss the mathematical properties of the axiomatization and its connection to the numerical representation of preferences. By defining prospects as joint distributions over incremental consequences, I depart from the existing literature for prospect theory, where the reference point is either given or is dependent on the choice set, so that transitivity is usually lost.

In Chapter 3, I apply the multiple priors model to a simple, static version of Lucas economy, and investigate the effects of incorporating ambiguity and heterogeneity into a capital asset pricing model. Thanks to the model's mathematical simplicity, I am able to provide closed-form solutions for the equilibrium stock and bond prices, and disentangle ambiguity effects from heterogeneity effects over the equity premium. One of the main findings is that hheterogeneity leads to a higher ex-ante equity premium than homogeneity. Moreover, this difference is a strictly increasing function of the variance of expectations about the dividend. Comparable results can be found in the existing financial literature, where a certain degree of pessimism is required in order to match the historical data. To my knowledge, all of the existing contributions share this ex-ante perspective and only partially solve the equity premium puzzle. Instead of proposing calibration exercises to compare the extent of my findings, I include an ex-post perspective and compare the predictions under ambiguity to the true (historically observed) realizations of the equity premium. The second and original finding in this chapter is that the ex-post equity premium will always be higher than its ex-ante prediction under ambiguous beliefs, thus providing an interestingly different, but still "rational," resolution of the equity premium puzzle.

In Chapter 4, I study equilibrium prices and allocations in a simple financial market where the dividend distribution of Arrow-Debreu securities is not known. First, I derive equilibrium prices and allocations using a parameterized version of the multiple priors model, which includes both MinMax and MaxMax EU models as extreme cases. As in Lucas, I assume a single investor with ambiguity parameter α, and leave the interesting case of heterogeneity for future investigation. The main departure from the previous experimental financial literature is the assumption that the investor does not know the probability distribution for tomorrow's states of the world. Second, I test the model in experimental financial markets and find evidence for both risk and ambiguity aversion. This result is not surprising from a decision-theoretic point of view, given the large volume of studies replicating the Ellsberg experiment and its variations. However, its relevance in the economic experimental literature is twofold: first, it sheds a different light on previous financial experiments testing the Capital Asset Pricing model under risk. Secondly, my results are in sharp contrast to what recently found for experimental first-price auctions, where ambiguity aversion is rejected in a pure game-theoretical setting. While a deeper understanding of the differences in the theoretical and experimental setups is required, the mixed nature of the results suggests that ambiguity attitudes might be context-dependent.

In Chapter 5, I study extrinsic uncertainty in complete markets as modeled in Cass and Shell's seminal paper on sunspot economies. By allowing for agents to hold multiple beliefs about the probability of a sunspot occuring, I confirm their result that sunspots cannot matter in equilibrium and extend it to the case of heterogeneous beliefs. Robustness follows from non-differentiability properties of ambiguity-averse preferences. However, when at least one agent in the economy is ambiguity-loving, then sunspots matter in equilibrium even in complete markets.

Finally, in Chapter 6, I apply the generalized multiple prior model to an insurance competitive market. By including the optimism-pessimism index to represent attitudes towards ambiguity, I investigate optimal insurance from a decision-theoretic perspective and provide results which call for interesting empirical investigations. At fair odds, some risk averse agents might not buy full insurance, whilst even at unfavorable odds there are agents buying full coverage. Following Rothschild and Stiglitz model in competitive markets, I then study existence and welfare properties of equilibrium contracts and show that, whenever a separating equilibrium exists, ambiguity aversion benefits low risk consumers in the insurance market. Under the assumption that, due to data gathering, insurance companies have more precise information on the probability of loss across the population, ambiguity aversion implies a rather unintuitive social welfare result, that is, it is not beneficial to ask insurance companies to reveal their information.

The thesis is presented as a collection of articles. While making the notation sometimes repetitive, its advantage is self-contained chapters which the reader may pick to best fit his/her interests.

Chapter 2 Reference choice and multiple priors

Empirical violations to the leading theories of choice, the Expected Utility theory of von Neumann and Morgenstern [10] and the Subjective Expected Utility theory of Savage [7], have pointed out the relevance of ambiguity effects and reflection effects in individual decisions. Much empirical evidence, inspired by Ellsberg [2], shows that people prefer bets whose odds of winning are known, thus suggesting aversion to ambiguity (i.e., uncertainty about probabilities).[1] Kahneman and Tversky [23] report a variety of experiments in which subjects tend to exhibit a reflection effect (they are risk averse in the gains and risk seeking in the losses with respect to their initial wealth), suggesting a utility representation based on a reference point. The existing literature offers several formal models to accommodate mainly the ambiguity effect or the reflection effect.

This chapter attempts to axiomatize a decision theoretic model where choices are made under uncertainty and with respect to a non-fixed reference point. The model we use is an extension of Gilboa and Schmeidler's [4] maximin expected utility with multiple priors. We characterize preference relations over prospects that have a numerical representation by the functional $J(f) = min\{\int u \circ (f - \phi)\, dP | P \in \mathcal{C}\}$, where f is an act, ϕ is a given reference point, u is a von Neumann-Morgenstern utility over outcomes and $\mathcal{C}$ is a closed and convex set of finitely additive probability measures on the states of nature.

By defining prospects as joint distributions over incremental consequences, I depart from the existing literature for prospect theory, where the reference point is either given, e.g. the status quo, or is dependent on the choice set, e.g. the optimal consequence, so that transitivity is lost. In my model, the reference point is only required to be any state-dependent act, which can be interpreted as a subjective aspiration

[1]For a comprehensive review of the empirical evidence about uncertainty and ambiguity in decision making, see Camerer and Weber [4].

level of the decision maker. Due to the new construction of prospects, the proof of the maximin representation needs to be checked for the existence of a mathematical representation of preferences (Lemma 1 and 2). Also, the proof of Lemma 5 is kept in the main body of the chapter as an interesting theoretical note per se.[2]

The chapter proceeds as follows. The first four sections discuss the existing literature and provide the rationale for extending the SEU model to include multiple priors and a reference point. The following sections offer the theoretical development (theorem and proof). Concluding remarks and suggestions for future research are contained in Section 7.

2.1 The challenged paradigm

2.1.1 Subjective expected utility (SEU)

The most compelling justification for representing beliefs about uncertain outcomes through a unique (subjective) prior probability was proposed by Savage [7].[3] The decision maker has to compare different acts in terms of their consequences that depend on which of several states of the world will occur. His preferences over outcomes are characterized by a utility function, assigning each outcome a real number (à la von Neumann-Morgenstern). The criterion of choice is to maximize expected utility, where expectation is carried out with respect to a prior probability derived from the decision maker's preferences over acts.

Formally, let Ω be the state space, $\mathcal{X}$ the set of possible consequences, and $\mathcal{F}$ the set of possible acts. Acts are defined as maps from states into consequences (so $\mathcal{F} \subseteq \mathcal{X}^{\otimes}$). an SEU maximizer chooses the act $f \in \mathcal{F}$ that maximizes

[2] See [19].

[3] As defined by Kreps, Savage's model is "the crowning glory of choice theory." Its elegance comes from the fact that it combines the von Neumann-Morgenstern Expected Utility approach with the de Finetti's calculus of subjective probabilities, thus restoring the unitary nature of the decisionmaking problem. In other words, preferences simultaneously reveal beliefs about the probability of events and the utility of the consequences of events.

$$U(f) \equiv \int_\Omega u(f(\omega))\, P(d\omega), \tag{2.1.1}$$

where $P \in \Delta(\Omega)$, the set of all the finitely additive probability measures on $(\Omega, 2^\Omega)$. The measure P is derived from the decision maker's preferences among bets on subsets of Ω (events), so that it represents his subjective confidence (à la de Finetti) on the plausibility of each event happening.

2.1.2 The Ellsberg paradox

One of the first objection to Savage's paradigm as a descriptive theory was raised by Ellsberg [2]. In one of the "mind experiments" he proposed, subjects are shown an urn containing 90 balls, of which 30 are red while 60 are blue and yellow with no additional information about their relative proportion. Subjects are then asked to compare the following four bets about the color of one ball randomly drawn from the urn:

either Bet 1: win \$100 if the ball is red, \$0 otherwise;

or Bet 2: win \$100 if the ball is blue, \$0 otherwise;

either Bet 3: win \$100 if the ball is either red or yellow, \$0 otherwise;

or Bet 4: win \$100 if the ball is either blue or yellow, \$0 otherwise.

Typically people express the following preferences:

$$\text{Bet } 1 \succ \text{Bet } 2 \text{ and Bet } 4 \succ \text{Bet } 3.$$

So people prefer acts with a known, i.e., objective probability of winning. This is clearly inconsistent with the sure-thing principle of SEU (Savage's P2), as it requires the ranking between acts to be independent of states in which acts yield the same consequence. In the paradox, such a state is the event "the ball is yellow."

More formally, it is easy to see that there is no probability measure supporting these preferences through SEU maximization. Let $P(r)$, $P(b)$, and $P(y)$ be respectively the probabilities that the ball drawn is red, blue, or yellow. Then under SEU:

$$\text{Bet } 1 \succ \text{Bet } 2 \Leftrightarrow P(r)u(100) + P(b \cup y)u(0) > P(b)u(100) + P(r \cup y)u(0)$$

$$\Leftrightarrow P(r) > P(b)$$

but:

$$\text{Bet } 4 \succ \text{Bet } 3 \Leftrightarrow P(b \cup y) > P(r \cup y)$$

and by the additivity of P:

$$P(b) + P(y) > P(r) + P(y) \Leftrightarrow P(b) > P(r),$$

a contradiction. So SEU cannot rationalize these preferences.

The idea that cannot be captured by the standard approach is that the decision maker can be averse to the ambiguity associated with events, or more precisely to the uncertainty about their likelihood, which informally depends on the quality and quantity of information at his disposal.[4] As defined by Ellsberg, ambiguity is the "quality depending on the amount, type, reliability, and 'unanimity' of information." The major limit of SEU is that it assumes preferences well-specified enough to derive a unique additive probability measure so that an SEU maximizer, by construction, does not mind ambiguity.

[4]This ambiguity aversion can be informally related to some dose of pessimism, as embodied in the so-called "Murphy's laws" (e.g., "if anything can go wrong, it will," or "when it rains it pours;" in the Italian tradition, "Fortune is blind, but bad luck can see pretty well").

2.2 Why multiple priors? A model for ambiguity

The task is then to construct a plausible model of behavior that allows for the possibility that the ambiguity associated with events affects behavior. There are two alternatives in the literature to capture this same idea.

Kahnemann and Tversky [23] first suggested that, under uncertainty, a lottery f is perceived by the decision maker in a "distorted" way as a $g = G \circ f$, where $G : [0,1] \to [0,1]$ is a strictly increasing and continuous transformation. So the spirit of this first approach, later formalized by Schmeidler [23], is to retain the uniqueness of the probability measure representing beliefs, whilst relaxing its additivity (that is, possibly $P(A \cup B) \neq (P(A) + P(B) - P(A \cap B))$.[5] The more non-additive the belief, the less confident the decision maker is about his beliefs.

Example 1 This approach can be used to explain the empirical evidence for the Ellsberg's experiment. Consider for example the following non-additive beliefs: $\nu(r) = \frac{1}{3}$, $\nu(b) = \frac{1}{6}$, $\nu(y) = \frac{1}{6}$, $\nu(b \cup y) = \frac{2}{3}$, $\nu(r \cup b) = \frac{1}{2}$, $\nu(r \cup y) = \frac{1}{2}$, $\nu(r \cup b \cup y) = 1$.

Then, after renormalizing u such that $u(0) = 0$ and $u(100) = 1$, we can compute the expected utility of each bet by applying the finite version of the Choquet integral:[6]

$$\text{EU(Bet 1)} = u(100)\tfrac{1}{3} + u(0)(\tfrac{1}{2} - \tfrac{1}{3}) + u(0)(1 - \tfrac{1}{2}) = \tfrac{1}{3}$$

$$\text{EU(Bet 2)} = \text{u}(100)\ \tfrac{1}{6} + u(0)(\tfrac{1}{2} - \tfrac{1}{6}) + u(0)(1 - \tfrac{1}{2}) = \tfrac{1}{6}$$

$$\text{EU(Bet 3)} = u(100)\tfrac{1}{2} + u(0)(1 - \tfrac{1}{2}) = \tfrac{1}{2}$$

$$\text{EU(Bet 4)} = u(100)\tfrac{2}{3} + u(0)(1 - \tfrac{1}{3}) = \tfrac{2}{3}.$$

□

[5]More formally, a set-function ν on (Ω, Σ) is called a *capacity* if it is normalized and monotone, that is, $\nu : \Sigma \to [0,1]$ is such that: $\nu(0) = 0$, $\nu(1) = 1$; and for $A, B \in \Sigma$, $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$.

[6]This method allows to compute expected utilities with respect to a non-additive prior. For lotteries f with finitely many outcomes, first rank the states $\omega_i \in \Omega$ (where $i = 1, \ldots, n$) based on $u(f(\omega_i))$. Then compute the Choquet Expected Utility for act f as $CEU(f) = u(f(\omega_1))\nu(\omega_1) + \sum_{i=1}^{n} u(f(\omega_i))[p(\cup_{j=1}^{i}\omega_j) - p(\cup_{j=1}^{i-1}\omega_j)]$.

The alternative is the multiple priors model, as axiomatized by Gilboa and Schmeidler [4], where people are assumed to have probabilities that are additive but not unique and to choose according to a minimax decision rule.[7] This idea was not new in the literature. Hurwicz [22] showed an example of statistical analysis where some degree of ignorance prevents the statistician from forming a unique prior, but still does not lead him to apply Wald's decision rule with respect to all the priors.

Since the decision maker has not enough information to confidently form a single probability distribution, he uses a set of priors $\mathcal{C}$, that is, all the priors "compatible" with his limited information. In order to rank acts, he calculates each act's expected utility for every probability distribution in $\mathcal{C}$ and then, being uncertainty averse, chooses the act that gives the largest minimum of these expectations:[8]

$$\max_{f \in \mathcal{F}} U(f) \equiv \max_{f \in \mathcal{F}} \min_{P \in \mathcal{C}} \int_{\Omega} u(f(\omega))\, P(d\omega). \tag{2.2.1}$$

The larger the set $\mathcal{C}$, the less confident the decision maker is about his information, i.e., the more ambiguity there is in the decision problem.[9]

Example 2 In the Ellsberg's experiment, consider the following beliefs. The probabilities for the unambiguous events are equal to the known odds of winning: $P(r) = \frac{1}{3}$, $P(b \cup y) = \frac{2}{3}$, $P(r \cup b \cup y) = 1$. In the case of the colors whose proportion in the urn is unknown, the subject has too little information to form a prior. Hence he considers a set of priors as possible: $P(b) = [0, \frac{2}{3}]$, $P(y) = [0, \frac{2}{3}]$, $P(r \cup b) = [\frac{1}{3}, 1]$,

[7] As stated by Wald [34], "a minimax (loss) solution seems, in general, to be a reasonable solution of the decision problem when an a priori distribution in Ω does not exist or is unknown to the experimenter." Given the equivalence of the minimax loss criterion to the maximin utility criterion, Gilboa and Schmeidler consider their result as an axiomatic foundation of Wald's criterion.

[8] Notice that the use of the minimum reflects a cautious, or better pessimistic, attitude of the decision maker. This pessimism captures the ambiguity aversion needed to explain behavior in situations à la Ellsberg.

[9] In order to capture ambiguity within an SEU context, it has been proposed to express ambiguity over probabilities as a second order probability (SOP) distribution. That is, the decision maker has a belief as to which of his multiple (first order) priors is more correct. The main criticism to this approach is that it might not capture ambiguity anymore, as it reduces (if not eliminates) vagueness in beliefs. The principle of "reduction of compound lotteries" (for which compound lotteries can be reduced to the equivalent single-stage lotteries) implies that the decision maker behaves as if he had a precise belief, that is, as if he was an SEU maximizer. However this principle is often violated in experiments, so that the SOP approach may not be able to distinguish between violations of the reduction principle and ambiguity effects.

$P(r \cup y) = [\frac{1}{3}, 1]$. Then the observed preferences are compatible with the maximin expected utility decision rule: the unambiguous bets, bet 1 and 4, have respectively a (minimum) expected utility equal to $\frac{1}{3}$ and $\frac{2}{3}$, whilst the ambiguous bets 2 and 3 have minimum expected utility of 0 and $\frac{1}{3}$. □

Notice that these two approaches capture the uncertainty of beliefs in a similar fashion and in many situations yield the same behavioral predictions. As shown in Schmeidler [23], the two approaches coincide when the set of possible priors is the core of a convex capacity.[10] But neither approach is nested in the other.[11]

We have chosen the multiple priors minimax expected utility model mainly because of its more immediate mathematical representation, which allows for some straightforward comparative statics and possibly a dynamic extension:

1. it is straightforward to interpret the extreme cases: when $\mathcal{C}$ is a singleton, the decision maker is actually an SEU maximizer; when $\mathcal{C} = \cdot(\otimes)$, we would define him "completely ignorant" about the plausibility of any event.

   Example 3 Consider a simple dichotomous event (such as it will or will not rain tomorrow). an SEU decision maker would summarize his beliefs by a probability, e.g., $p = 1$; a less informed decision maker could summarize his beliefs by an interval, say between $\frac{1}{2}$ and 1; if completely ignorant, his belief set would be the whole $[0, 1]$. □

2. it is straightforward to extend this model to a dynamic choice problem, where dynamics are captured by possible different levels of information: $\mathcal{C} = \mathcal{C}(\text{information})$.

[10] The *core* of a capacity ν is defined as the (possibly empty) set of all finitely additive probability measures that dominate ν pointwise. A capacity ν is said to be *convex* (or *supermodular*) if for $A, B \subseteq \Omega, \nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$.

[11] As pointed out in Gilboa [3], relaxing the additivity of probability creates some problems. For example, maximizing $u(x)$ does not necessarily coincide with minimizing $-u(x)$; and some appealing properties of standard conditional probabilities are violated. According to Gilboa, these problems might be interpreted as normative reasons to prefer approaches with additive probabilities.

However, as this requires a model for the process of acquiring information and translating it into beliefs, it goes beyond the purposes of this work.

2.3 Why a reference point?

One of the most common assumptions about choice behavior is risk aversion. However there are some serious difficulties in reconciling this assumption with empirical evidence. As first shown by Friedman and Savage [11], a person's well-being seems to depend not only on his current consumption of goods, but also on how his current consumption compares to his past consumption and (expected) total income.[12] In order to deal with this evidence, Markowitz [27] first suggested that risk attitudes are based upon deviations in consumption from a reference point.

This intuition of reference-dependent preferences was revived and consistently documented by Kahneman and Tversky [23]. In experiments on individual's choices between monetary gambles, most agents exhibit the so-called reflection effect: they are risk averse in terms of gains and risk seeking in terms of losses with respect to their reference point (in this case, the current level of wealth).[13] Kahneman and Tversky showed how this evidence may be accounted for by a utility function defined on deviations from the status quo (called "prospects"), concave for gains and convex for losses.[14]

An open question is thus whether these reference-dependent preferences have a normative justification. A partial answer is offered by the literature of regret theory. This approach models the desire of decision makers to avoid consequences in which they will appear (when the real state ω is known) to have made the wrong decision, even if in advance it appeared correct given the information available at the time. In the case of "full regret" (à la Savage), an action is thus judged to be a mistake if it

[12] For a comprehensive survey of this empirical evidence, see Kahneman and Tversky [24] and Camerer [5].

[13] Other closely related effects well documented in the literature are the framing effect (see Kahneman and Tversky [23]), the status quo bias (see Samuelson and Zeckhauser [21]), and the endowment effect (see Kahneman, Knetsch and Thaler [16]).

[14] For a more formal extension of prospect theory, see Tversky and Kahneman [25], and Wakker and Tversky [27].

is not ex-post optimal, so that the reference point is the most preferable outcome for each state of the world.[15]

In line with this approach, the case in which a given (not necessarily optimal) action seems to be a more appropriate reference point might be interpreted as a model of focused regret.[16] So an action is ex-post considered a mistake only if it fares worst than the reference point (alternatively, only if it yields a worse result than what would be obtained by sticking to the reference point, i.e., not deciding). This way of modeling a reference point can then capture the intuition that there exists a cost to make decisions, as illustrated in the following application.

2.3.1 An application of focused regret

In order to explain the so-called "roll-off" phenomenon, i.e., selective abstention in multiple elections, Ghirardato and Katz [12] introduce information costs as generated by differences in ambiguity about electoral outcomes. In their model, abstention is chosen as a reference choice ϕ and the payoff to each act f is renormalized to $f(\omega) - \phi(\omega)$ for every state ω.[17]

Formally, their model assumes that the decision maker's preferences are represented as follows: there exist a reference choice ϕ and a non-empty, closed and convex set of positive finitely additive measures such that he chooses among acts in order to

$$\max_{f \in \mathcal{F}} U(f) \equiv \max_{f \in \mathcal{F}} \min_{P \in \mathcal{C}} \int_{\Omega} [f(\omega) - \phi(\omega)]\, P(d\omega). \tag{2.3.1}$$

Notice that the model captures the idea of people caring more about losses than

[15]For an axiomatization of regret in decision making, see Bell [2]. He postulates a multiattribute utility function (as a function of both monetary assets and quantity of decision regret) and uses it to explain behavior anomalies such as the reflection effect, coexistence of insurance and gambling, and preference reversals.

[16]As an example for this case, consider the status quo in prospect theory. See also the discussion in the following sections.

[17]An intuitive justification for this choice of ϕ comes from the "anedoctal observation that abstention is viewed by most people as an option that is "safe": it does not entail the same costs as voting for a candidate, which could lead to a situation of discomfort ("regret"), if one's vote determines the election and the candidate turns out to be the wrong person."

gains, as it allows the decision maker to use different beliefs in evaluating different actions. The most pessimistic probability distribution from $\mathcal{C}$ will assign relatively higher weight to states where the difference between any act and the reference point is negative and lower when positive.

Remark 1 Renormalization by itself (without multiple priors) is not sufficient to capture the decision cost argument. If the decision maker is an SEU maximizer, then he forms a belief $P \in \Delta(\Omega)$ and chooses $f \in \mathcal{F}$ that maximizes

$$U(f) \equiv \int_{\Omega} [f(\omega) - \phi(\omega)]\, P(d\omega). \tag{2.3.2}$$

By linearity of the integral, for any two acts f and g,

$$U(f) \geq U(g) \Leftrightarrow \int_{\Omega} f(\omega)\, P(d\omega) \geq \int_{\Omega} g(\omega)\, P(d\omega), \tag{2.3.3}$$

that is, an SEU maximizer's preferences are not affected at all by the existence of a reference point.

Remark 2 On the other extreme, consider a completely ignorant decision maker, in the sense that his beliefs are represented by $\mathcal{C} = \cdot(\otimes)$. This implies that, for every state $\omega \in \Omega$, there is a $P \in \mathcal{C}$ such that $P(\omega) = 1$. So by applying equation 2.3.1:

$$\max_{f \in \mathcal{F}} U(f) \equiv \max_{f \in \mathcal{F}} \int_{\Omega} [f(\omega) - \phi(\omega)]\, \tilde{P}(d\omega), \tag{2.3.4}$$

where $\tilde{P}$ is the prior assigning probability one to the state ω that minimizes $f(\omega) - \phi(\omega)$. Thus the completely ignorant DM behaves according to the Wald's maximin criterion.

2.4 Which reference point?

For now, think of the reference point as exogenous to the problem. This is equivalent to assume that the decision maker has a yardstick in his mind, which might be interpreted in many different ways, such as:

- *threshold interpretation:* ϕ is the situation the decision maker considers "normal" for the kind of choice problem he is facing (for example, the status quo or prior expectations).[18] Similarly from a computational point of view, the decision maker may think of ϕ as a particularly desirable (aspiration level) or undesirable situation.

- *decision cost interpretation:* ϕ is the (aleatory) value for dropping out of the choice problem.

The only thing that matters is that the decision maker considers a reference point ϕ in order to compare and rank acts. The meaning he assigns to ϕ is in his head and is somehow irrelevant to our attempt to describe how he orders acts ("as-if" model).

Remark 3 If ϕ is a constant act (so that it comes out of the integral and cancels out when comparing two acts), then we are back to SEU. The introduction of a reference point is thus meaningful only if ϕ is a non-constant, i.e, state-dependent act. In this way, we allow for the case in which the decision maker does not know exactly his aspiration level. Alternatively, we allow for "not fixed" (i.e., not independent of action and information) decision costs.

2.5 Axiomatizing prospects

The empirical violations discussed in the previous sections provide a challenge to adapt standard models in order to incorporate ambiguity aversion and feelings of

[18] This provides a clearer (at least to me) meaning to the concept of utility. If asked "what is u($10) to you?", I would not know what to answer. If I could set the question in a more familiar (or "normal") problem, such as a return rate from investing $100 on the security market, then I know I am unhappy of my $10 if, even in the worst possible (according to my evaluation) contingency, my reference investment plan would (more often or more likely) fare better than them.

regret. Even though such an extension may be valuable from a descriptive and prescriptive point of view, our main concern is normative, as we are interested in the mathematical properties of the axioms and their connection to the numerical representation of preferences. The model we present is an extension of Gilboa and Schmeidler's [4] maximin expected utility, where the preference relation is defined over prospects (instead of acts).

As in Gilboa and Schmeidler, we use a simplified version of Savage's [7] model. The simplification consists of assuming the existence of objective probabilities, or equivalently, of an independent randomizing device such as a roulette wheel. Such a model containing both objective and subjective probabilities was first suggested by Anscombe and Aumann [1]. In their setup, an act f is defined as a "horse" lottery that associates a consequence to each possible event, where the set of events is not assigned a given probability distribution. Each consequence can be a "roulette" lottery $y_f(\omega)$ over outcomes with objective probabilities. So the final outcome depends first on which state ω occurs, then on which outcome the (second-stage) lottery yields.

This lottery-act formulation relies on the distinction between two kinds of randomness: one that the decision maker feels he can "dominate" by assigning probabilities (as on a roulette wheel); in the other case, instead, it may not be clear "what the odds are," so that there is no obvious probability distribution to confidently capture randomness (as in horse races). Notice that the distinction between objective and subjective probabilities is not necessary for Anscombe and Aumann's result to work: what matters is that second-stage lotteries can be constructed of additive unique probabilities.[19] These probabilities can be subjectively derived upon (à la Savage). Anscombe and Aumann's result is thus a way to assess complicated probabilities from people's preferences over simpler lotteries, in which probabilities are already known.

[19] This gives meaning to objective mixtures of acts or prospects (defined as eventwise convex combinations) and to our definition of prospect (as joint distribution over incremental consequences).

2.5.1 Some notation

Let $\mathcal{X}$ be a (non-empty) space of consequences, which we will assume to be a compact interval $[a, b]$ in $\Re$, and $\mathcal{Y}$ the set of distributions over $\mathcal{X}$ with finite supports:

$$\mathcal{Y} = \{y : \mathcal{X} \to [0,1] \mid y(x) \neq 0 \text{ for finitely many x's in } \mathcal{X} \text{ and } \sum_{x \in \mathcal{X}} y(x) = 1\}$$

so that $\mathcal{X}$ can be identified with the subset $\{y \in \mathcal{Y} \mid y(x) = 1 \text{ for some } x \in \mathcal{X}\}$ of $\mathcal{Y}$.

Let Ω be a (non-empty) state space and Σ be an algebra of subsets of Ω (events). Denote by $\mathcal{F}_0$ the set of all Σ-measurable (finite) step functions from Ω to $\mathcal{Y}$ and denote by $\mathcal{F}_c$ the constant functions in $\mathcal{F}_0$.[20] Let $\mathcal{F}$ be a convex subset of $\mathcal{F}_0$ that includes $\mathcal{F}_c$.[21] Elements of $\mathcal{Y}$ can thus be interpreted as (roulette) lotteries, while elements of $\mathcal{F}$ are acts (or horse lotteries).

Given a $\phi \in \mathcal{F}$, denote by $\hat{\mathcal{X}}$ the set of all possible incremental consequences, that is, all the possible differences between the outcomes of any act f and the outcomes of the reference point ϕ. So $\hat{\mathcal{X}} \subseteq [a-b, b-a]$. Define a prospect $\hat{f} = f - \phi$ as a function from Ω into the set of joint distributions $\hat{\mathcal{Y}}$ over $\hat{\mathcal{X}}$ and denote by $\hat{\mathcal{F}}$ the set of all prospects. We find it helpful to illustrate our definition of prospects with an example.

Example 4 For a given $\omega \in \Omega$ let f and ϕ induce the following lotteries over $\mathcal{X}$:

[20]Notice that any act $f \in \mathcal{F}$ induces a probability distribution $y_f(\omega)$ over $\mathcal{X}$ for every $\omega \in \Omega$, whilst any constant act $f_c \in \mathcal{F}_c$ induces the same y_f for every $\omega \in \Omega$.

[21]Convex combination in $\mathcal{F}_0 \subseteq \mathcal{Y}^{\otimes}$ are performed pointwise: for f and g in $\mathcal{F}_0$ and α in $[0,1]$, $\alpha f + (1-\alpha) g = h$, where $h(\omega) = \alpha f(\omega) + (1-\alpha) g(\omega)$ for $\omega \in \Omega$.

ϕ, f	-1	0	1	2	
-1	1/6	1/6	0	1/6	1/2
0	0	0	0	0	0
1	1/6	1/6	0	1/6	1/2
2	0	0	0	0	0
	1/3	1/3	0	1/3	

Table 2.1: Joint distribution $y_{f,\phi}$

$$f(\omega) = \begin{cases} -1 & \text{with probability } 1/3 \\ 0 & \text{with probability } 1/3 \\ 2 & \text{with probability } 1/3 \end{cases}$$

$$\phi(\omega) = \begin{cases} -1 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/2. \end{cases}$$

Then, given ω, f and ϕ induce a joint distribution $P_{f,\phi}$ over $\mathcal{X} \times \mathcal{X}$ as shown in Table 2.1, so that $\hat{f} = f - \phi$ induces the following probability distribution over $\hat{\mathcal{X}}$:

$$(f - \phi)(\omega) = \begin{cases} -2 & \text{with probability } 1/6 \\ -1 & \text{with probability } 1/6 \\ 0 & \text{with probability } 1/6 \\ 1 & \text{with probability } 1/3 \\ 3 & \text{with probability } 1/6. \end{cases}$$ □

This way of defining prospects seems to be the most natural, within a model of focused regret, as it captures the idea of *how likely* an act is to *perform relatively* to the reference point.

More formally, denote by $x_f(\omega) = \{x \in \mathcal{X} : y_f(\omega)(x) \neq 0\}$. Then for every $\omega \in \Omega$ $\hat{f}$ induces a distribution $y_{\hat{f}}(\omega) = y_{f,\phi}(\omega)$ over $\hat{\mathcal{X}}$ such that

$$y_{\hat{f}}(\omega)(\hat{x}) = \sum_{\{x_f(\omega), x_\phi(\omega):\, x_f(\omega) - x_\phi(\omega) = \hat{x}\}} y_{f,\phi}(\omega)(x_f, x_\phi).$$

Let $\hat{\mathcal{F}}_c$ denote the set of prospects $\hat{f}$ such that, for each state ω, f's final outcomes differ from ϕ's by the same constant.[22] So constant prospects map Ω into a subset of $\hat{\mathcal{Y}}$ that we will denote by $\hat{\mathcal{Y}}_c$.

Finally let $\mathcal{P}$ denote the set of all finitely additive probability measures $P : \Sigma \to [0, 1]$, endowed with the product topology (i.e., the weak* topology), so that $\mathcal{P}$ is compact.

2.5.2 The axioms

The primitive of the model is a binary (preference) relation $\succeq$ over $\hat{\mathcal{F}}$.[23] We are going to assume the following properties of the preference relation:

A.1 *Weak Order.*

- (a) For all f and g in $\hat{\mathcal{F}}$, $f \succeq g$ or $g \succeq f$.
- (b) For all f, g and h in $\hat{\mathcal{F}}$, if $f \succeq g$ and $g \succeq h$ then $f \succeq h$.

A.2 *Continuity.* For all f, g and h in $\hat{\mathcal{F}}$, if $f \succ g$ and $g \succ h$, then there are α and β in $(0, 1)$ such that:

$$\alpha f + (1 - \alpha)h \succ g \text{ and } g \succ \beta f + (1 - \beta)h.$$

[22] We find this is the most natural way to define "constant" prospects, given that the decision maker uses ϕ as a reference point. Just observe that $\hat{\phi} = \phi - \phi$ induces for each state a joint probability distribution such that, when we prove the existence of a state-dependent EU representation u, it will turn out that $u(\hat{\phi}(\omega)) = 0$ for each ω. Similarly, given a constant prospect $\hat{f}_c$, for all $\omega \in \Omega$ $u(\hat{f}_c(\omega)) = c$ for some consequence c.

[23] As usual, $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succeq$. Notice that the relation $\succeq$ on $\hat{\mathcal{F}}$ induces a relation also denoted by $\succeq$ on $\hat{\mathcal{Y}}$. Let f_x denote the prospect such that $f_x(\omega) = x$ for all $\omega \in \Omega$, where $x \in \hat{\mathcal{Y}}$. Then $x \succeq y \Leftrightarrow f_x \succeq f_y$.

A.3 *Monotonicity.* For all f and g in $\hat{\mathcal{F}}$, if $f(\omega) \succeq g(\omega)$ on Ω then $f \succeq g$.

A.4 *Uncertainty Aversion.* For all $f, g \in \hat{\mathcal{F}}$ and $\alpha \in (0,1)$, $f \sim g$ implies $\alpha f + (1-\alpha)g \succeq f$.

A.5 $\hat{C}$*-Independence.* For all f, g in $\hat{\mathcal{F}}$, h in $\hat{\mathcal{F}}_c$ and α in $(0,1)$:

$$f \succ g \Leftrightarrow \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h.$$

A.6 *Non-degeneracy.* There exist f and g in $\hat{\mathcal{F}}$ such that $f \succ g$.

Notice that weak order, continuity and the weakened version of independence are essentially the von Neumann-Morgenstern axioms. $\hat{C}$-Independence is weaker than (standard) independence, as it restricts h to constant prospects (rather than any prospect in $\hat{\mathcal{F}}$). By recalling our definition of constant prospects, this is equivalent to restricting independence to mixtures with prospects with "sure" outcomes.[24] Violations of independence can happen, for instance, when the decision maker sees hedging opportunities in mixing prospects. $\hat{C}$-independence does not suffer this problem since constants do not allow hedging.

Uncertainty aversion states that "objective mixtures" make the decision maker better off.[25]

Finally, monotonicity is equivalent to state-independence of preferences (Savage's P5) and non-degeneracy is a purely structural axiom that makes the representation nontrivial.

[24] Constant prospects in fact have the same u for each ω, that is, the same EU for any probability distribution over Ω. So mixing any prospect $\hat{f}$ with a constant prospect does not affect which probability distribution minimizes $EU(\hat{f})$.

[25] Notice that this is consistent with the use of a maximin rule, as minimizing EU of a mixture of prospects can be no worst than minimizing EU of each prospect separately.

2.5.3 The representation theorem

Theorem 1 *Let $\succeq$ be a binary relation on $\hat{\mathcal{F}}$. $\succeq$ satisfies assumptions A.1-A.5 if and only if there exist an affine function $u : \hat{\mathcal{Y}} \to \Re$ and a non-empty, closed and convex subset $\mathcal{C}$ of $\mathcal{P}$ such that for all $\hat{f}, \hat{g} \in \hat{\mathcal{F}}$:*

$$\hat{f} \succeq \hat{g} \Leftrightarrow \min_{P \in \mathcal{C}} \int u \circ (f - \phi)\, dP \geq \min_{P \in \mathcal{C}} \int u \circ (g - \phi)\, dP. \tag{2.5.1}$$

Furthermore, the function u is unique up to a positive affine transformation and the set $\mathcal{C}$ is unique if and only if $\succeq$ satisfies A.6, too.

2.6 Proof of the theorem

The crucial part of the proof is the "if" part. Notice first that if A.6 does not hold, then any constant u and any subset $\mathcal{C}$ will satisfy 2.5.1. Hence from now on we are going to assume A.1-A.6.

The proof consists of the following several lemmata.

Lemma 1 *There exists an affine $u : \hat{\mathcal{Y}} \to \Re$ such that for all $y, z \in \hat{\mathcal{Y}}$*

$$y \succeq z \Leftrightarrow u(y) \geq u(z).$$

Furthermore u is unique up to a positive linear transformation.

Proof: The lemma holds on $\hat{\mathcal{Y}}_c$ as an immediate consequence of the von Neumann-Morgenstern theorem, as $\hat{C}$-independence implies the independence axiom for $\hat{\mathcal{F}}_c$.

By standard arguments for extensions of the von Neumann-Morgenstern theorem, we can extend u to $\hat{\mathcal{Y}}$. Take $\bar{y}_c$, $\underline{y}_c \in \hat{\mathcal{Y}}_c$ such that $\bar{y}_c \succeq y \succeq \underline{y}_c$ for any given $y \in \hat{\mathcal{Y}}$. By the continuity axiom there exists a unique $\alpha \in [0,1]$ such that $y \sim \alpha \underline{y}_c + (1-\alpha)\bar{y}_c$. Define $u(y) = u(\alpha \underline{y}_c + (1-\alpha)\bar{y}_c)$. So by construction u satisfies the condition in the lemma. ∎

Lemma 2 *Given $u : \hat{\mathcal{Y}} \to \Re$ from Lemma 1, then there exists a unique $J : \hat{\mathcal{F}} \to \Re$ such that*

(i) for all $f, g \in \hat{\mathcal{F}}$, $f \succeq g \Leftrightarrow J(f) \geq J(g)$;

(ii) for all $f \in \hat{\mathcal{F}}$ such that $f(\omega) = y$ for all $\omega \in \Omega$, $J(f) = u(y)$.

Proof: On the subset of $\hat{\mathcal{F}}$ defined in (ii), J is uniquely determined by (ii) and lemma 1.

Analogously to the extension of u of lemma 1, we can extend J to all $\hat{\mathcal{F}}$. Take $\bar{y}$, $\underline{y} \in \hat{\mathcal{Y}}$ such that $\bar{y} \succeq f \succeq \underline{y}$ for any given $f \in \hat{\mathcal{F}}$. By the continuity axiom there exists a unique $\alpha \in [0,1]$ such that $f \sim \alpha \underline{y} + (1-\alpha)\bar{y}$. Define $J(f) = J(\alpha \underline{y} + (1-\alpha)\bar{y})$. By construction J satisfies (i) and is unique. ∎

Since u represents $\succeq$ on $\hat{\mathcal{Y}}$ up to a positive linear transformation, from now we will consider u such that there are $y_*, y^* \in \hat{\mathcal{Y}}$ for which $u(y_*) < -1$ and $u(y^*) > 1$. Such a choice of u is possible thanks to the nondegeneracy axiom (for which there exist $f_*, f^* \in \hat{\mathcal{F}}_0$ with $f^* \succ f_*$) and the monotonicity axiom (so that there exists a state $\omega \in \Omega$ such that $f^*(\omega) = y^* \succ f_*(\omega) = y_*$).

Let $\mathcal{B}_0$ denote the set of Σ-measurable real-valued simple functions on Ω and $\mathcal{B}$ the closure of $\mathcal{B}_0$ with respect to the supnorm $\| \cdot \|$.[26] Intuitively, one can think of every $b \in \mathcal{B}$ as the composite function $u \circ f$, i.e., the "evaluation of prospect f." For $c \in \Re$, let $c^* \in \mathcal{B}_0$ be the constant function on Ω with value c.[27][28]

Lemma 3 *There exists a functional $I : \mathcal{B}_0 \longrightarrow \Re$ such that*

(a) I is the normalized representation operator: for all $f \in \hat{\mathcal{F}}$, $I(u \circ f) = J(f)$ (hence $I(1^*) = 1$);

(b) I is monotonic: for all $a, b \in \mathcal{B}_0$, $a \succeq b \Rightarrow I(a) \geq I(b)$;

(c) I is superlinear, i.e.:

- *superadditive: for all $a, b \in \mathcal{B}_0$, $I(a + b) \geq I(a) + I(b)$;*
- *homogeneous of degree 1;*

(d) I is $\hat{C}$-independent: for all $a \in \mathcal{B}_0$ and $c \in \Re$, $I(a + c^*) = I(a) + I(c^*)$.

Lemma 4 *There exists a unique continuous extension of I to $\mathcal{B}$. Furthermore, this extension is monotonic, superlinear and $\hat{C}$-independent.*

Lemma 5 *Let $I : \mathcal{B} \longrightarrow \Re$ satisfy:*

[26] I.e., $b \in \mathcal{B}$ iff there exists a sequence $\{b_n\}_{n \geq 1} \subseteq \mathcal{B}_0$ such that $\| b - b_n \| \rightarrow 0$ as $n \rightarrow \infty$. So $\mathcal{B}$ is the space of all bounded Σ-measurable real-valued functions on Ω.

[27] Notice that each constant prospect $f_c \in \hat{\mathcal{F}}_c$ is evaluated c for every ω, so that it is mapped (through u) into the corresponding c^*.

[28] The proofs for the following two lemmata can be found in the appendix. Instead, we keep the proof of lemma 5 in the main body of the chapter, as it constitutes an interesting theoretical note per se.

(a') $I(0^*) = 0$ and $I(1^*) = 1$;

(b) for all $a, b \in \mathcal{B}$, $a \succeq b \Rightarrow I(a) \geq I(b)$;

(c') for all $a, b \in \mathcal{B}$, $I(a + b) \geq I(a) + I(b)$;[29]

(d') for all $b \in \mathcal{B}$, $\alpha \geq 0$, and $\beta \geq 0$, $I(\alpha b + \beta 1^*) = \alpha I(b) + \beta$.

Then there exists a convex and w-compact set $\mathcal{C} \subseteq \mathcal{P}$ such that for all $b \in \mathcal{B}$:*

$$I(b) = \min_{P \in \mathcal{C}} \int b\, dP. \tag{2.6.1}$$

Uniqueness follows from standard separation results. The proof of existence is directly based on the classical Hahn-Banach theorem.

Proof: Let $\mathcal{L}$ be the set of all linear functionals on $\mathcal{B}$ and let $\mathcal{C}$ be defined as

$$\mathcal{C} = \{L \in \mathcal{L} | L(b) \geq I(b) \text{ for all } b \in \mathcal{B} \text{ and } L(1^*) = I(1^*)\}.$$

Notice that by (a') and (b), $\mathcal{C}$ is the set of all positive linear functionals on $\mathcal{B}$. Let $\tilde{b} \in \mathcal{B}$ and denote by $\mathcal{B}_{\tilde{b}}$ the linear subspace spanned by $\tilde{b}$, that is,

$$\mathcal{B}_{\tilde{b}} = \{b | b = \alpha \tilde{b} + \beta 1^* \text{ for some } \alpha, \beta \in \Re\}.$$

Denote by $L_{\tilde{b}}$ the linear functional on $\mathcal{B}_{\tilde{b}}$ defined by:

$$L_{\tilde{b}}(\alpha \tilde{b} + \beta 1^*) = \alpha I(\tilde{b}) + \beta I(1^*) \text{ for } \alpha, \beta \in \Re.$$

Notice that $L_{\tilde{b}}$ satisfies the following:

(i) $L_{\tilde{b}}(\alpha \tilde{b} + \beta 1^*) = I(\alpha \tilde{b} + \beta 1^*)$ if $\alpha \geq 0, \beta \geq 0$;

[29] *Notice that $I(b) \leq -I(-b)$ for all $b \in \mathcal{B}$ since, by applying (c') and then (a'), $I(b) + I(-b) \leq I(b - b) = I(0^*) = 0$.*

(ii) $L_{\tilde{b}}(\alpha\tilde{b}+\beta 1^*) \geq I(\alpha\tilde{b}+\beta 1^*)$ otherwise;

where (i) follows from (d') and (ii) from the fact that:

$$-L_{\tilde{b}}(-\alpha\tilde{b}-\beta 1^*) = -I(-\alpha\tilde{b}-\beta 1^*) \geq I(\alpha\tilde{b}+\beta 1^*).$$

Thus $L_{\tilde{b}}(b) \geq I(b)$ for all $b \in \mathcal{B}_{\tilde{b}}$. Then by the Hahn-Banach theorem there exists a linear functional $\tilde{L}$ on $\mathcal{B}$ such that

$$\tilde{L}(b) = L_{\tilde{b}}(b) \text{ for all } b \in \mathcal{B}_{\tilde{b}}$$

and

$$\tilde{L}(b) \geq I(b) \text{ for all } b \in \mathcal{B}$$

so that $\tilde{L} \in \mathcal{C}$ and $I(\tilde{b}) = \tilde{L}(\tilde{b}) = \min_{L \in \mathcal{C}} L(\tilde{b})$. ∎

Lemmas 1 through 5 prove the "if" part of the theorem. To prove the "only if," assume the existence of u and $\mathcal{C}$ as stated in the theorem and define $I(b) = min\{\int b\, dP | P \in \mathcal{C}\}$. Then it is immediate to verify that I is monotonic, superlinear, $\hat{\mathcal{C}}$-independent and continuous. So the preference relation defined in 2.5.1 satisfies axioms 1 through 5.

In order to conclude the proof, we need to prove the uniqueness properties of u and $\mathcal{C}$. Lemma 1 implies the uniqueness of u up to a positive linear transformation.

Notice that, if axiom A.6 does not hold, then u can be any constant function. This implies that K, the range of u, is a singleton and $\mathcal{C}$ can be any non-empty, closed and convex set.

So we need to show that, if A.6 holds, then $\mathcal{C}$ is unique. Suppose the contrary: there exist $\mathcal{C}_1 \neq \mathcal{C}_2$, both non-empty, closed and convex such that $\succeq$ is represented by both the following functions defined on $\hat{\mathcal{F}}_0$:

$$J_1(f) = \min\{\int u(f)\, dP | P \in \mathcal{C}_1\}$$

$$J_2(f) = \min\{\int u(f)\, dP | P \in \mathcal{C}_2\}.$$

Take $P_1 \in \mathcal{C}_1 \backslash \mathcal{C}_2$. By a separation theorem, there exists $b \in \mathcal{B}$ such that

$$\int b\, dP_1 < \min\{\int b\, dP | P \in \mathcal{C}_2\}.$$

Assume without loss of generality that $b \in \mathcal{B}_0(K)$. So there exists $f \in \hat{\mathcal{F}}_0$ such that $J_1(f) < J_2(f)$. Let $y \in \mathcal{Y}$ be such that $y \sim f$, then

$$J_1(y) = J_1(f) < J_2(f) = J_2(y),$$

a contradiction.

2.7 Concluding remarks

Instances of systematic violations of standard models' axioms are worthy of study for the insight they may give in improving decision analysis. In particular, when faced with complex decision problems involving ambiguity or reference-dependence (such as framing in terms of gains or losses), people are very poor intuitive statisticians and use procedures that bias their choices in ways that cannot be captured by the standard models. Furthermore, when their choices are illustrated to be "paradoxical" when confronted with the EU axioms, people commonly agree with the step-by-step logic but nonetheless feel uncomfortable with the conclusions. This common observation led us to believe that aversion to ambiguity and feelings of regret play a role in informal evaluation of alternatives by decision makers and provided us with the rationale for building a utility model that incorporates both these effects.

Even though this extension may be desirable for descriptive or prescriptive purposes, our main interest is in the mathematical properties of axioms and their connection with the numerical representation of preferences. So, at a cost of a too extreme simplification, we would like to point out the role of the axioms for the representation properties:

- weak order, continuity, $\hat{C}$-independence $\Rightarrow$ state-dependent EU representation;
- monotonicity $\Rightarrow$ restricts u to be the same for all states (so that if f has a higher "evaluation" than g in each state, then f will be preferred to g);
- uncertainty aversion $\Rightarrow$ "min" functional in the representation (i.e., superadditivity of the representation operator);
- non-degeneracy $\Rightarrow$ uniqueness properties.

Notice that, given the correspondence between acts and prospects, it is straightforward to extend the axioms to the case in which the primitive of the model is a preference relation on $\mathcal{F}$ (instead of $\hat{\mathcal{F}}$). An important extension would be to find axioms that endogenize the reference point, so that ϕ can be part of the representation theorem.

As future research, we would like to look more closely at some alternatives that hopefully will provide us with insights for endogenizing the reference point:

- *keep a static framework but use a setup à la Savage.* All the axiomatizations of regret theory we found in the literature (see Bell [2], Loomes and Sudgen [18], and Fishburn [10]) are restricted to pairwise comparisons of prospects and abandon the transitivity axiom. By extending their axioms, it might be possible to consider richer choice sets (thus getting closer to our purposes of a weak order representation) and restore transitivity at least within any choice set (but probably not across choice sets).

- *use a dynamic framework.* In an intertemporal setting, it is possible to endogenize the reference point and make it depend on the outcomes of past choices.[30] It might also be interesting to study in depth the information processing people do when making decisions under ambiguity. An alternative might be a revealed preference approach.[31] Illustrating a pattern of choice which is compatible with our model might be the most fruitful way to get further insights on preferences' properties.

[30]Notice that some care is needed when choosing the assumptions about how the reference point changes over time. As proved in DellaVigna and LiCalzi [8], for given adjustment rules the choice behavior in the long-run converges to EU maximization.

[31]See Border [3] for this approach applied to the EU hypothesis.

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2.8 Appendix

Proof of lemma 3: Let $K = u(\hat{\mathcal{Y}})$ and let $\mathcal{B}_0(K) \subset \mathcal{B}$ denote the subset of functions with values in K. We first define I on $\mathcal{B}_0(K)$ by (a). Notice that I is welldefined thanks to lemma 2 and the monotonicity axiom.

Next we show that I is homogeneous of degree 1 on $\mathcal{B}_0(K)$. So we need to show that, for $a, b \in \mathcal{B}_0(K)$ such that $a = \alpha b$ with $\alpha \in (0, 1]$, $I(a) = \alpha I(b)$.

Define:

$$f = \alpha g + (1 - \alpha)h,$$

where $g \in \hat{\mathcal{F}}_0$ is such that $u \circ g = b$ and $h \in \hat{\mathcal{Y}}_c$ is such that $J(h) = 0$. Then

$$u \circ f = \alpha u \circ g + (1 - \alpha)u \circ h = \alpha b = a,$$

so that $I(a) = J(f)$. Let $y \in \hat{\mathcal{Y}}_c$ satisfy $y \sim g$, so that $J(y) = J(g) = I(b)$. By

$\hat{\mathcal{C}}$-independence,

$$\alpha y + (1-\alpha)h \sim \alpha g + (1-\alpha)h = f$$

and this implies the result:

$$I(a) = J(f) = J(\alpha y + (1-\alpha)h) = \alpha J(y) + (1-\alpha)J(h) = \alpha J(y) = \alpha I(b).$$

Next we extend I to all of $\mathcal{B}_0$ by homogeneity. So I is monotone and homogeneous of degree 1 on $\mathcal{B}_0$.

Next we show that I is $\hat{\mathcal{C}}$-independent. Assume without loss of generality (thanks to homogeneity) that $\frac{1}{\alpha}a, \frac{1}{1-\alpha}c^* \in \mathcal{B}_0(K)$ and define $\beta = I(\frac{1}{\alpha}a) = \frac{1}{\alpha}I(a)$.

Let $f \in \mathcal{F}_0$ satisfy $u \circ f = \frac{1}{\alpha}a$, $y \in \hat{\mathcal{Y}}$ satisfy $u \circ y = \beta^*$ and $f_c \in \hat{\mathcal{F}}_c$ satisfy $u \circ f_c = \frac{1}{1-\alpha}$. By the definition of β, it follows that $f \sim y$. Applying $\hat{\mathcal{C}}$-independence of $\succeq$,

$$\alpha f + (1-\alpha)f_c \sim \alpha y + (1-\alpha)f_c$$

So, I is $\hat{\mathcal{C}}$-independent, as

$$I(a + c^*) = I(\alpha\beta^* + c^*) = \alpha\beta + c = I(a) + c.$$

Finally we prove that I is superadditive. Take $a, b \in \mathcal{B}_0$. Without loss of generality (thanks to homogeneity), we can assume that $a, b \in \mathcal{B}_0(K)$ and all that we need to prove is that

$$I(\frac{1}{2}a + \frac{1}{2}b) \geq \frac{1}{2}I(a) + \frac{1}{2}I(b).$$

Take $f, g \in \hat{\mathcal{F}}_0$ such that $u \circ f = a$ and $u \circ g = b$. We need to distinguish two possible cases:

- If $I(a) = I(b)$, then $f \sim g$ and, by uncertainty aversion,

$$\frac{1}{2}f + \frac{1}{2}g \succeq f$$

which implies

$$I(\frac{1}{2}a + \frac{1}{2}b) \geq I(a) = \frac{1}{2}I(a) + \frac{1}{2}I(b).$$

- If $I(a) > I(b)$, let $\delta = I(a) - I(b)$ and $c = b + \delta^*$. Note that, by $\hat{\mathcal{C}}$-independence of I,

$$I(c) = I(b) + \delta = I(a).$$

Using I's $\hat{\mathcal{C}}$-independence, superadditivity (for the case above) and $\hat{\mathcal{C}}$-independence, respectively, we get

$$I(\frac{1}{2}a + \frac{1}{2}b) + \frac{1}{2}\delta = I(\frac{1}{2}a + \frac{1}{2}c) \geq \frac{1}{2}I(a) + \frac{1}{2}I(c) = \frac{1}{2}I(a) + \frac{1}{2}I(b) + \frac{1}{2}\delta$$

which completes the proof. ∎

Proof of lemma 4: First recall that $\mathcal{B}$ is a Banach space with the supnorm $\| \cdot \|$ and $\mathcal{B}_0$ is a norm-dense subspace of $\mathcal{B}$. To show existence of a unique continuous extension of I we need to show that for each $a, b \in \mathcal{B}_0$, $\mid I(a) - I(b) \mid \leq \| a - b \|$. As

$$a = b + a - b \leq b + \| a - b \|^*$$

then, by monotonicity and $\hat{\mathcal{C}}$-independence of I,

$$I(a) \leq I(b + \| a - b \|^*) = I(b) + \| a - b \| .$$

This implies the existence of a unique continuous extension of I from $\mathcal{B}_0$ to its closure $\mathcal{B}$. And this extension mantains the same properties, i.e., superlinearity, monotonicity and $\hat{\mathcal{C}}$-independence. ∎

Chapter 3 Asset pricing and multiple priors

3.1 Introduction

In this chapter, I investigate the effects of incorporating ambiguity and heterogeneity into a capital asset pricing model. The model is a simple, static version of Lucas' economy where agents hold different multiple beliefs about the realization of next period's dividend. Thanks to mathematical simplicity, I am able to provide closed-form solutions for the equilibrium stock and bond prices, and disentangle ambiguity effects and heterogeneity effects over the equity premium. First, heterogeneity per se (to be thought as a mean-preserving spread in the cross-sectional distribution of beliefs) leads to a higher risk-free rate and ex-ante equity premium than homogeneity. Thus heterogeneity per se can only explain part of the puzzle, as posed by Mehra and Prescott. However, the result is interesting as the extent of the underestimation of the equity premium under homogeneity is a strictly increasing function of the variance of expectations about the dividend. The cross-sectional variance of beliefs thus represents a well-defined measure of an "heterogeneity premium," that can be easily estimated and tested in an experimental setting.

Second, ambiguity aversion always leads to a lower interest rate and a "generally" higher equity premium than in the classical case. But the findings about the ex-ante equity premium only hold for either high values of the subjective risks or low values of the subjective means, as it would be the case when there is "enough" ambiguity in the economy. Comparable results can be found in the existing financial literature, where uncertainty increases the equity premium and decreases the risk free rate. Similarly, a certain degree of pessimism is required in order to match the historical data. To our knowledge, all of the existing contributions share this ex-ante perspective and only partially solve the equity premium puzzle.

Instead of proposing calibration exercises to compare the extent of my findings, I include an ex-post perspective and compare the predictions from the ambiguity model to the "true" (historically observed) realizations of the equity premium. In terms of my model, and more generally for the class of models incorporating subjective ambiguity-averse beliefs, this distinction is crucial as people act on the basis of "incorrect" beliefs in the sense of being consistently biased towards the worst case. I find that the ex-post equity premium will always be higher than its ex-ante prediction under ambiguous beliefs, thus providing an interestingly different, but still "rational," resolution of the equity premium puzzle.

In what is considered one of the classical models for capital markets, Lucas [26] studies the stochastic behavior of equilibrium asset prices in a dynamic one-good, pure exchange economy with a single consumer, to be interpreted as a "representative" agent for a large number of identical consumers. When assuming that productivity fluctuates stochastically over time, his model predicts fluctuations for equilibrium asset prices as well, and is aimed at understanding the relationship between such movements. Under his assumptions, there is only one way for the economy to be in competitive equilibrium: all the output is consumed, all asset shares are held, and the asset price function (solving the dynamic problem of choosing investment and consumption over time) is unique. The great simplicity, and possible weakness, of Lucas' model stems from the "assumption" that agents know a great deal about the structure of the economy, and they all agree about the available information. When discussing the stability of the equilibrium, Lucas himself doubts whether agents may know the distribution of the price shocks.

In this light, it may not come as a surprise that, when empirically tested, Lucas' model may fail to predict actual asset prices. In their famous paper, Mehra and Prescott [28] study a variation of Lucas' pure exchange economy, as constructed to display equilibrium consumption growth rates with the same mean, variance and serial correlation as those observed for the US economy between 1889 and 1978.[1]

[1]They assume that the growth rate of endowment, rather than endowment levels, follows a Markov process. This extension attempts to capture the non-stationarity in the consumption series associated with the large increase in per capita consumption over the period under study.

They find that for such economies the predicted average real annual yield on equity is a maximum of 0.35 percent higher than that on short-term debt, in sharp contrast to the six percent premium observed. So the puzzle is twofold: a low risk-free rate, and a high average equity premium. They conclude that the restrictions that this class of general equilibrium models places upon the average returns of equity and treasury bills are strongly violated by the US data in the 1889–1978 period. Among the many plausible relaxations of Lucas' assumptions, Mehra and Prescott posit agents' heterogeneity, non-time-additivity separable preferences, and other frictions and/or features making certain types of intertemporal trades among agents infeasible.

In this chapter, I incorporate heterogeneity of agents in a simple version of Lucas' economy by allowing agents to hold different beliefs about the risky dividend. Also, my model incorporates another interesting aspect of the economy, that is, the presence of uncertainty, or equivalently the vagueness of information available to investors.

This distinction between uncertainty and risk, as first pointed out by Knight [24], affects the observed behavior of traders and institutions in a way that is qualitatively different than Savage rationality would suggest. Savage's independence, or sure-thing axiom, implies that preferences should not depend on the source of the risk. In a financial context, this means that traders should not distinguish between uncertainty about the correctness of their beliefs or their pricing model and the risk inherent in the assumed stochastic prices.

However, in experimental settings, decision makers consistently violate the independence axiom. Much empirical evidence, inspired by Ellsberg [2], shows that people prefer bets whose odds of winning are known, thus revealing aversion to ambiguity, that is, uncertainty about probabilities. [2] In the context of financial markets, not knowing the realization of an asset payoff (consumption risk) and not knowing the probability measure for its dividend (model uncertainty) have different behavioral implications.

Portfolio selection thus seems a natural context for ambiguity averse behavior

[2]For a comprehensive review of the empirical evidence about uncertainty and ambiguity in decision making, see Camerer and Weber [4].

because of the enormous amount of information, both direct (financial statements, corporate spending plans, and so on) and indirect (market prices of investments, yielding information about other investors' beliefs), that an investor must incorporate into his belief. Plausible sources for ambiguity are differences in 1) the information available to each individual; 2) the way information is processed and incorporated in probability judgements; or 3) attitudes towards model uncertainty. For the purposes of my static model, it is not relevant whether these differences relate to opinions or information, as they can commonly be represented by modeling each investor's beliefs as a set of priors.

If information is missing, the investor can not confidently assess a single distribution for dividend. Alternatively, under model uncertainty, an investor forms, by one means or another, a probability distribution P for the risky dividend. However he feels uncomfortable acting on its basis, as he recognizes that, because of the difficulties involved in properly using all the available information, his probability distribution might not be exactly right. In either case, this is formalized by assuming that agents consider multiple priors as plausible description of their beliefs about the risky dividend.

The model I use is Gilboa and Schmeidler's [4] maximin expected utility with multiple priors. Each investor evaluates a portfolio z according to the functional $V(z) = min\{\int u \circ (z)\, dP | P \in N\}$, where u is a von Neumann-Morgenstern utility and N is a closed and convex set of normal probability distributions over the states of nature, that is, the realizations of next-period dividend. The investor then chooses his portfolio so as to provide the highest guaranteed level of expected utility: an investment-consumption plan is preferred to another if and only if it has larger minimum expected utility, where the minimum is computed over the set of plausible priors.

The use of the maxmin criterion captures the discomfort that many decision makers feel when forced to make choices based on a subjective probability distribution, particularly when this distribution was formed on the basis of vague information. In such cases, it is not surprising if a decision maker, cautiously or pessimistically, seeks to protect himself against the possibility his distribution is incorrect.

In a simple, static version of Lucas' economy, extended to allow for multiple priors, I characterize equilibrium returns for the risky and riskless assets, and show that the presence of uncertainty and heterogeneity generates predictions for equilibrium holdings and prices different from the classical case. Under subjective beliefs about the variance of the dividend, heterogeneity "disappears" and we are back to the representative agent model. Under subjective expectations, instead, heterogeneity and ambiguity enter the equilibrium conditions in an almost "additive" way and thus allow to separate their effect on the equilibrium interest rate and the equity premium. Heterogeneity per se leads to a higher interest rate and equity premium, and is only partially able to solve the puzzle. The effect of ambiguity instead goes in the "right" directions, that is, a lower interest rate and a higher equity premium. Despite of the descriptive appeal of including both heterogeneity of agents and aversion to lack of information, these differences in predictions call for a test of my model, both empirically and experimentally. The main interest of this chapter is theoretical: however, by keeping the model simple and sufficiently general, I provide closed-form solutions for the unique equilibrium conditions, which provide a promising setting for both calibration exercises and experimental investigation.

The chapter proceeds as follows. The related literature is informally summarized in the next subsections, after a brief outline of non-expected utility models. A simple version of the capital asset price model is formally described in Section 3.2. Section 3.3 applies ambiguity and heterogeneity to this model and solves for equilibrium prices under two different specifications for the information about the risky dividend. In the last part of this section, I show how these extensions affect the equity premium. Concluding remarks and suggestions for future research are contained in Section 3.4.

3.1.1 Decision theoretical literature

The most compelling justification for representing beliefs about uncertain outcomes through a unique subjective prior probability was proposed by Savage [7]. The decision maker has to compare different lotteries in terms of their consequences that depend on which of several states of the world will occur. His preferences over outcomes are characterized by a utility function, assigning each outcome a real number (à la von Neumann-Morgenstern). The criterion of choice is to maximize expected utility, where expectation is carried out with respect to a prior probability derived from the decision maker's preferences over lotteries.

Formally, let Ω be the state space, X the set of possible consequences and F the set of possible lotteries, defined as maps from states into consequences (so $F \subseteq X^{\Omega}$). A Subjective Expected Utility (SEU) maximizer chooses the lottery $f \in F$ that maximizes

$$U(f) \equiv \int_{\Omega} u(f(\omega))\, P(d\omega),$$

where $P \in \Delta(\Omega)$, the set of all the finitely additive probability measures on $(\Omega, 2^{\Omega})$. The measure P is derived from the decision maker's preferences among bets on subsets of Ω (events), so that it represents his subjective confidence (à la de Finetti) on the plausibility of each event happening.

One of the first objections to Savage's paradigm as a descriptive theory was raised by Ellsberg [2]. In one of the "mind experiments" he proposed, subjects are shown an urn containing 90 balls, of which 30 are red while 60 are blue and yellow with no additional information about their relative proportion. There is a widely exhibited preference to bet on drawing red over any other color, and blue and yellow over other combinations. So people prefer lotteries with a known, i.e., objective probability of winning. Even though intuituve, these rankings are clearly inconsistent with the sure-thing principle of SEU, as it requires the ranking between acts to be independent of states in which lotteries yield the same consequence.

The idea that cannot be captured by the standard approach is that the decision maker can be averse to the ambiguity associated with events, or more precisely to the uncertainty about their likelihood, which informally depends on the quality and quantity of information at his disposal. As defined by Ellsberg, ambiguity is the "quality depending on the amount, type, reliability, and 'unanimity' of information." The major limit of SEU is that it assumes preferences wellspecified enough to derive a unique additive probability measure so that an SEU maximizer, by construction, does not mind ambiguity.

The task is then to construct a plausible model of behavior that allows for the possibility that the ambiguity associated with events affects behavior. There are two alternatives in the literature to capture this same idea.

Kahnemann and Tversky [23] first suggested that, under uncertainty, a lottery f is perceived by the decision maker in a "distorted" way as a $g = G \circ f$, where $G : [0,1] \to [0,1]$ is a strictly increasing and continuous transformation. So the spirit of this first approach, later formalized by Schmeidler [8] and known as Choquet Expected Utility (CEU), is to retain the uniqueness of the probability measure representing beliefs, whilst relaxing its additivity (that is, possibly $P(A \cup B) \neq (P(A) + P(B) - P(A \cap B))$. The more non-additive the belief, the less confident the decision maker is about his beliefs.

The alternative is the multiple priors model, as axiomatized by Gilboa and Schmeidler [4], where people are assumed to have probabilities that are additive but not unique and to choose according to a minimax decision rule.[3] Since the decision maker has not enough information to confidently form a single probability distribution, he uses a set of priors C, that is, all the priors "compatible" with his limited information. In order to rank lotteries, he calculates each lottery's expected utility for every probability distribution in C and then, being uncertainty averse, chooses the lottery that gives the largest minimum of these expectations:

$$\max_{f \in F} V(f) \equiv \max_{f \in F} \min_{P \in C} \int_{\Omega} u(f(\omega))\, P(d\omega).$$

The larger the set C, the less confident the decision maker is about his information,

[3] This idea was not new in the literature. Hurwicz [22] showed an example of statistical analysis where some degree of ignorance prevents the statistician from forming a unique prior, but still does not lead him to apply Wald's minimax decision rule with respect to all the priors.

i.e., the more ambiguity there is in the decision problem.

Notice that these two approaches capture the uncertainty of beliefs in a similar fashion and in many situations yield the same behavioral predictions. As shown in Schmeidler [8], the two approaches coincide when the set of possible priors is the core of a convex capacity. But neither approach is nested in the other.[4] I have chosen the multiple priors Minimax Expected Utility (MEU) model mainly because of its more immediate mathematical representation, which allows for some straightforward comparative statics and possibly a dynamic extension. Furthermore, in most cases the nexus between information and beliefs is easy to interpret, and this will turn out to be helpful in experimental settings.[5]

3.1.2 Related literature

Krasker [25] first proposes minimax portfolios as an alternative to traditional theory of portfolio selection, assuming that investors choose their portfolio so as to provide the highest guaranteed expected utility, with the true distribution lying in an ε-neighborhood of their belief P. He characterizes the optimal portfolio by proving that it corresponds to the minimax portfolio, and it cannot feature shortsales. The reason is that the problem is set up as a zero-sum game in which the investor chooses a portfolio in his budget set and Nature chooses a distribution for the risky asset values, and these values are assumed to lie in the non-negative orthant. If the investor then sells an asset short, even low-probability events can greatly lower his expected utility. According to Krasker, this explains why the apparent diversity of traders' beliefs does not lead to observe an otherwise reasonably extensive use of short-sales in real markets. The results of his paper, showing that minimax behavior leads to reasonable (broadly diversified) portfolios of the sort we observe in practice, then lend

[4] As pointed out in Gilboa [3], relaxing the additivity of probability creates some problems. For example, maximizing $u(x)$ does not necessarily coincide with minimizing $-u(x)$; and some appealing properties of standard conditional probabilities are violated. According to Gilboa, these problems might be interpreted as normative reasons to prefer approaches with additive probabilities.

[5] As an illustration, consider a simple dichotomous event (such as it will or will not rain tomorrow). an SEU decision maker would summarize his beliefs by a probability, e.g. $p = 1$; a less informed decision maker could summarize his beliefs by an interval, say between $\frac{1}{2}$ and 1, if he just knows that rainy is more likely than sunny; if completely ignorant, his belief set would be the whole $[0, 1]$.

some support to minimax hypothesis.

Dow and Werlang [1] consider a portfolio choice problem, and show that there is an interval of exogenously given prices at which a CEU agent holding a riskless position neither buys nor sells short a risky asset. Epstein and Wang [13] extend this partial equilibrium indeterminacy result to a CEU version of Lucas' asset pricing model. Despite the strong incentives to insure in the CEU model (due to pessimism), price indeterminacy and no trade may be features of equilibria in economies without aggregate risk.

Rigotti and Shannon [6] use a different approach, as in Bewley [2], where the distinction between uncertainty and risk is formalized by assuming agents have incomplete preferences over state-contingent consumption bundles. Individual decision making still depends on a set of probability distributions, but now a bundle is preferred to another if and only if it has larger expected utility for all probabilities in the set. In this setting, the presence of uncertainty generates robust indeterminacies in equilibrium prices and allocations for *any* initial endowment.

The robust control framework of Hansen, Sargent, and Tallarini [21] is similar to the Epstein and Wang's approach. They use the recursive intertemporal formulation of uncertainty aversion, and this specification facilitates dynamic programming and preserves dynamic consistency. Maenhout [27] includes model uncertainty and portfolio rules' robustness along the lines of Hansen, Sargent, and Tallarini, but for a methodological modification, namely homotetic robustness. Robustness dramatically decreases the demand for equities and is observationally equivalent to stochastic differential utility, the continuous-time version of recursive preferences in the sense of Epstein and Wang. Robustness can thus be interpreted as increasing risk-aversion without changing the preference for intertemporal substitution. Furthermore, when deriving equilibrium asset pricing implications, robustness increases the equity premium and lowers the risk-free rate. However, in order to match the historically observed equity premium and risk-free rate, an excessive degree of pessimism is required, as the endogenous worst-case scenario turns out to be the equilibrium equity premium under expected utility. Thus the equity premium puzzle shows up again, even though in a different form.

Another partial solution to the puzzle is proposed by Epstein and Zin [15], where they integrate a different nonexpected utility model (Rank-Dependent EU) into a multiperiod representative agent model and examines its implication on the equity premium. The relevant property of such preferences is that they exhibit "first-order" risk aversion, so that the risk premium for small bets is proportional to the standard deviation rather than the variance. Given the smoothness of aggregate consumption data, the growth rate's standard deviation is considerably larger than its variance, and the model is able to predict both a small risk-free rate (2.6) and a moderate equity premium (1.6). Thus their model only partially resolves the puzzle.

Routledge and Zin's [30] setup is the closest to my model, but their approach is very different. Similarly to my work, they use Knightian uncertainty to describe model uncertainty, and use MEU preferences to characterize investors' aversion to this uncertainty. Their approach, though, is mainly computational, and as closed-form solutions for the optimal policies of their dynamic program are unavailable, they offer a computational algorithms that can be used to solve numerical versions of this model. In a specific case, where traders' beliefs differ only about the variance of the distribution for future consumption and there are only two possible values for it, they show that an increase in uncertainty can lead to a reduction in liquidity as measured by the bid-ask spread set by a monopoly market maker. In addition, "hedge preferences" for the market maker can look very different from those implied by a model without Knightian uncertainty.

3.2 The classical model as a benchmark

Consider a two-period version of Lucas [26] economy. Output is produced only by capital and is completely perishable. Production is stochastic and entirely exogenous, in the sense that no resources are utilized and there is no possibility of affecting the output of any unit at any time. The total capital stock z is assumed to be constant over time and so is the per capita $\overline{z}$. Ownership in these productive units is determined each period in a competitive stock market, at the price p. Each unit has outstanding one perfectly divisible equity share, which entitles its owner of all of the unit's output in the same period. Also, risk-free bonds b are traded in the market at a (normalized) price of 1.

Each investor is firstly endowed with some amount of the risky capital z_1, and each unit of capital produces d_1 units of the consumption good. During the first period, both the capital and the consumption good are traded in the market. Wealth, as represented by $(p + d_1) z_1$, can be allocated to consumption in period 1 c_1, risky capital z_2 or risk-free bonds b:

$$(p + d_1) z_1 = c_1 + pz_2 + b \tag{3.2.1}$$

In period two, resources are represented by the capital income plus the value of bonds, as given by the interest rate r. All the available wealth is now allocated to consumption c_2:

$$c_2 = d_2 z_2 + rb \tag{3.2.2}$$

The "classical" economy has a single representative household, which orders its preferences over consumption plans by the following additive (across periods) utility function:

$$U = u(c_1) + \beta E [u(c_2)]$$

where u is the current period utility function, β is the subjective time discount factor and the expectation E is taken with respect to the distribution of d_2. Along with the literature, I assume that $u' > 0$, $u'' < 0$, and $0 < \beta < 1$.

In period 1, each agent chooses c_1, b and z_2 which maximize U subject to the constraints 3.2.1 and 3.2.2, that is

$$\max_{c_1, b, z_2} U = u(c_1) + \beta E [u(d_2 z_2 + rb)] \text{ st } (p + d_1) z_1 = c_1 + pz_2 + b$$

Assume further that u exhibits constant absolute risk aversion: $u(x) = -e^{-\alpha x}$ where α, measuring the curvature of the utility function, is the coefficient of risk aversion. For notational simplicity, I assume that α is the same for every agent, even though I will show how this restriction can be easily relaxed. For now, I also assume just risk and no ambiguity, that is, $d_2 \backsim N(\mu, \sigma^2)$.

Under these last two assumptions, the maximization problem becomes

$$\max_{c_1, b, z_2} -e^{-\alpha c_1} + \beta E\left[-e^{-\alpha(d_2 z_2 + rb)}\right] \text{ st } (p + d_1) z_1 = c_1 + p z_2 + b \tag{3.2.3}$$

In the appendix I solve for the optimal holding of risky assets and consumption, and find

$$pr = \mu - \alpha\sigma^2 z_2 \tag{3.2.4}$$

yielding the following result as a special case of Lucas' model:

Lemma 6 *The demand functions for each agent j in the classical economy are given by*

$$z_{2j}^* = \frac{\mu - pr}{\alpha\sigma^2} \tag{3.2.5}$$

$$c_{1j}^* = -\frac{1}{\alpha} \ln \beta r + rb + \mu z_2^* - \frac{\alpha\sigma^2 z_2^{*2}}{2} \tag{3.2.6}$$

Equation 3.2.5 shows that the optimal amount of risky capital is an increasing function of the expected value for the dividend and a decreasing function of the price of the risky asset, the risk-free rate of return, the variance of the risky payoff and the coefficient of risk aversion.

When the capital market is in equilibrium, p and r are such that the sum of the net demands over the population is equal to 0:

$$\sum_j \left(z_{2j}^* - z_{1j}\right) = 0 \quad \text{and} \quad \sum_j \left(b_j^* - b_{1j}\right) = \sum_j b_j^* = 0$$

thus implying the following equilibrium conditions for the average demands:

Lemma 7 *The equilibrium conditions for average demands in the classical economy are*

$$\overline{z}_2^* = \frac{\mu - pr}{\alpha\sigma^2} = \overline{z} \tag{3.2.7}$$

$$\overline{b}^* = 0 \tag{3.2.8}$$

$$\overline{c}_1^* = -\frac{1}{\alpha}\ln\beta r + \mu\overline{z} - \frac{\alpha\sigma^2\overline{z}^2}{2} \tag{3.2.9}$$

Notice that in this case $\overline{z}_2^* = z_{2j}$ as everyone is equal (so far). When allowing for different coefficients of risk aversion for each agent, Equation 3.2.4 becomes $pr = \mu - \overline{\alpha}\sigma^2\overline{z}_2^*$ and the results that follow can be easily generalized in a similar way in order to account for different α_j's. Given this correspondence with the case of a representative agent with a properly weighted coefficient α_j, from now on I keep α to be the same for every agent and gain notational simplicity.

Also, given the assumption that the consumption good is perishable, in the first period the average (and aggregate) consumption for the population is equal to the average (respectively, aggregate) output of the consumption good:

$$\overline{c}_1^* = d_1\overline{z} \tag{3.2.10}$$

Substituting Equation 3.2.10 into Equation 3.2.9 and solving for the equilibrium interest rate and price, I obtain

Lemma 8 *The equilibrium interest rate and price in the classical economy are given by*

$$r^* = \frac{1}{\beta} e^{\alpha \overline{z}(\mu - d_1) - \frac{\alpha^2 \sigma^2 \overline{z}^2}{2}} \tag{3.2.11}$$

$$p^* = (\mu - \alpha \sigma^2 \overline{z}^2) \frac{1}{\beta} e^{-\alpha \overline{z}(\mu - d_1) + \frac{\alpha^2 \sigma^2 \overline{z}^2}{2}} \tag{3.2.12}$$

Notice that, in the special case when $\sigma^2 = 0$ (no risk) and $\mu = d_1$ (that is, the agents expect the same dividend over time), the equilibrium interest rate is equal to the rate of time preference: $r^* = \frac{1}{\beta}$. So Equation 3.2.11 indicates that under risk $r^* \neq \frac{1}{\beta}$ because of two distinct reasons:

- $\mu \neq d_1$: when $\mu > d_1$ people (on average) expect to consume more in period 2 than in period 1. So they try to borrow in order to increase c_1, thus driving r up.

- $\frac{\alpha^2 \sigma^2 \overline{z}^2}{2}$, reflecting precautionary savings. As riskiness of the future income increases, savings in period 1 tend to increase and r decreases.

3.3 Introducing ambiguity

I introduce ambiguity in this capital market framework as modeled in Gilboa and Schmeidler's multiple priors model. When an agent does not have enough information to confidently form a single probability distribution for the risky dividend, he uses a set N_j of all the priors compatible with his limited information. In this case, I assume that the lack of confidence about mean and variance of the dividend distribution induces an agent to think of intervals as possible ranges for such values. So assume $d_2 \backsim N\left(\mu_j, \sigma_j^2\right)$, with support $m_j \leq \mu_j \leq M_j$, $s_j^2 \leq \sigma_j^2 \leq S_j^2$, so that N_j is a rectangle in the mean-variance space. Heterogeneity of agents is modeled by allowing different agents to have different supports for the dividend distribution.[6]

As in Gilboa and Schmeidler, I assume ambiguity aversion. When computing his expected utility from a given investment and consumption plan, an agent uses the prior minimizing such an expectation: $V(x) = \min_{N \in N_j} \int u(x) N(dx)$. The optimization problem then becomes

$$\max_{c_1, b, z_2} -e^{-\alpha c_1} + \beta V\left[-e^{-\alpha(d_2 z_2 + rb)}\right] \text{ st } (p + d_1) z_1 = c_1 + p z_2 + b$$

where

$$\begin{aligned} V\left[-e^{-\alpha(d_2 z_2 + rb)}\right] &= e^{-\alpha rb} \min_{N \in N_j} \int -e^{-\alpha d_2 z_2} N(dd_2) \\ &= e^{-\alpha rb} \min_{\mu_j, \sigma_j^2} \int -e^{-\alpha d_2 z_2} \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{\left(d_2 - \mu_j\right)^2}{2\sigma_j^2}} dd_2 \end{aligned} \qquad (3.3.1)$$

Notice that a minimizing solution exists, as the correspondence is continuous and, by assumption, N_j is closed and convex. Also, it is simple to check that the last integral in Equation 3.3.1 is minimized at m_j, S_j^2. Such values for mean and variance correspond to the dividend distribution that is second-order stochastically dominated by all the other distributions in the support N_j. So Equation 3.3.1 becomes

$$V\left[-e^{-\alpha(d_2 z_2 + rb)}\right] = -e^{-\alpha\left(rb + m_j z_2 - \frac{\alpha}{2} S_j^2 z_2^2\right)}$$

The problem then collapses from multiple priors to a unique, though possibly different among agents, prior distribution. So the model can be interpreted as a model

[6] In this section there is no need to impose any restriction on agents' supports. However, when comparing my results to the empirical findings about the equity premium, I will need to impose some kind of structure, which I model in a general and economically intuitive way. That is, I will assume that, whenever a "true" distribution exists (else, an objective distribution, as assumed in the classical model), it belongs to the intersection of all the agents' supports. This guarantees that no one is ever "so wrong" as not to include the true distribution among his possible beliefs. Notice also that this assumption is quite general, in the sense that *any* point in the intersection of the supports share the same properties as needed for the comparative exercises proposed later, that is, they all have higher mean and lower variance than agents' subjective beliefs.

of heterogeneous beliefs: differences in information, as captured by different supports for the plausible dividend distributions, turn into different beliefs. Ambiguity plays a role in the sense that, among all the plausible beliefs, agents act on the basis of the worst one, that is, they compute expectations with respect to the most pessimistic, or cautious, probability distribution in their beliefs' support.

All the first-order conditions as computed above are the same, but for replacing μ with m_j and σ^2 with S_j^2. I thus obtain

Lemma 9 *Under heterogeneous, ambiguity-averse beliefs, the individual demand function for the risky capital is given by*

$$\widehat{z}_{2j} = \frac{m_j - pr}{\alpha S_j^2} \tag{3.3.2}$$

As noted earlier, it is possible to compare the optimal demand under ambiguity to the classical case only when assuming the existence of an objective distribution $N(\mu, \sigma^2)$ for the dividend and some restriction about agents' information (that is, the objective mean and variance are always included in the everyone's support). Then, it is always the case that $\widehat{z}_2 \leq z_2^*$. For this section only, I keep a completely subjective probability framework, but will turn to this assumption in later sections.

Along with the analysis in Section 3.2, I study capital market equilibrium conditions by considering the equilibrium averages for all the decision variables. Let n denote the number of agents in the economy. I incur in the following difficulty:

$$\begin{aligned}
\overline{\widehat{z}}_2 &= \frac{1}{n}\sum_{j=1}^{n}\frac{m_j - pr}{\alpha S_j^2} = \frac{1}{\alpha n}\sum_{j=1}^{n}\left(\frac{m_j}{S_j^2}\right) - \frac{pr}{\alpha n}\sum_{j=1}^{n}\left(\frac{1}{S_j^2}\right) = \\
&= \frac{1}{\alpha n}\left[\frac{\sum_j m_j \prod_{i\neq j} S_i^2}{\prod_j S_j^2} - pr\frac{\sum_j S_j^2}{\prod_j S_j^2}\right]
\end{aligned}$$

so that it is not straightfoward to provide any simple closed form solution or comparative static results. This computational difficulties are not uncommon in this literature (see, for example, Routledge and Zin [30]).

In order to simplify the analysis and be able to say more about the equilibrium conditions, I split the problem and assume that agents lack relevant information about either the variance or the mean of the dividend distribution, thus holding (respectively) subjective risk or subjective expectation beliefs.

3.3.1 Subjective risk

Let us start assuming that all investors have identical subjective expectations, but different risk assessments: $d_2 \backsim N\left(\mu, \sigma_j^2\right)$, with support $s_j^2 \leq \sigma_j^2 \leq S_j^2$. The demand function as of Equation 3.3.2 then simplifies to

$$\widetilde{z}_{2j} = \frac{\mu - pr}{\alpha S_j^2} \tag{3.3.3}$$

Let $av\left(\frac{1}{S_j^2}\right) = \frac{1}{n}\sum_j \frac{1}{S_j^2}$. Under subjective risk only, heterogeneity does not play a role anymore and we are back to a representative (but still ambiguity averse) agent economy:

Theorem 2 *Under ambiguity about the variance of the risky assets, the equilibrium demand for risky asset and consumption are respectively:*

$$\widetilde{z}_{2j} = \frac{\frac{1}{S_j^2}}{av\left(\frac{1}{S_j^2}\right)}\overline{z} \tag{3.3.4}$$

$$\overline{\widetilde{c}}_1 = -\frac{1}{\alpha}\ln\beta r + \mu\overline{z} - \frac{\alpha}{2}\overline{z}^2 \frac{1}{av\left(\frac{1}{S_j^2}\right)} \tag{3.3.5}$$

that is, $av\left(\frac{1}{S_j^2}\right)$ *is a sufficient statistic for the cross-sectional distribution of subjective risk beliefs.*

Proof: Computing the equilibrium average condition for risky investment yields

$$\overline{\widetilde{z}}_2 = \frac{\mu - pr}{\alpha} av\left(\frac{1}{S_j^2}\right) = \overline{z}$$

so that in equilibrium

$$pr = \mu - \frac{\alpha\overline{z}}{av\left(\frac{1}{S_j^2}\right)}$$

Substituting for pr in Equation 3.3.3 I find that the demand for risky capital is a decreasing function of the subjective risk assessment:

$$\widetilde{z}_{2j} = \frac{\overline{z}}{S_j^2 av\left(\frac{1}{S_j^2}\right)} = \frac{\frac{1}{S_j^2}}{av\left(\frac{1}{S_j^2}\right)}\overline{z}$$

Similarly, computing the equilibrium average consumption yields

$$\begin{aligned}
\overline{\overline{c}}_1 &= -\frac{1}{\alpha}\ln\beta r + av\left(\mu\widetilde{z}_2\right) - av\left(\frac{\alpha S_j^2 \widetilde{z}_2^2}{2}\right) \\
&= -\frac{1}{\alpha}\ln\beta r + \mu\overline{z} - \frac{\alpha}{2}av\left(\widetilde{z}^2 S_j^2\right) \\
&= -\frac{1}{\alpha}\ln\beta r + \mu\overline{z} - \frac{\alpha}{2}\overline{z}^2\frac{1}{av\left(\frac{1}{S_j^2}\right)}
\end{aligned}$$

∎

Asset prices in this case will then correspond to the homogeneous economy where all investors have identical risk assessments equal to $\sigma^2 = \frac{1}{av\left(\frac{1}{S_j^2}\right)}$. The introduction of heterogeneity of σ^2 does not allow then for interesting departures from the traditional representative agent model. Ambiguity still plays a role in the sense that consumption in the first period will always be smaller than what predicted in the classical model, thus driving the equilibrium interest rate down as well. Also, the ex-ante equity premium will be higher than the classical prediction, but only for larger values of the subjective variance assessment.[7]

[7] More precisely, the ratio of the ex-ante ambiguity-averse and classical equity premia is equal to

However, the equilibrium allocations only depend on the first moment of the cross-sectional distribution of subjective risk beliefs, and the economy can be summarized by a representative ambiguity-averse agent. For this reason, in the following sections I ignore cross-sectional variation in σ^2 , and separately analyze the effects of ambiguity and heterogeneity over equilibrium allocations for the case of subjective expectations.

3.3.2 Subjective expectation

Let us turn to the case of identical subjective risk assessments, but different subjective expectations. So assume $d_2 \backsim N(\mu_j, \sigma^2)$, with support $m_j \leq \mu_j \leq M_j$. The demand function for this case simplifies to

$$\widehat{z}_{2j} = \frac{m_j - pr}{\alpha\sigma^2} \tag{3.3.6}$$

Denote by $\overline{m}$ and σ_m^2 respectively the mean and the variance of the cross-sectional distribution of m_j. Homogeneity with ambiguity aversion simply corresponds to the case in which $m_j = \overline{m}$ for every j, and thus $\sigma_m^2 = 0$. Furthermore, homogeneity without ambiguity is obtained whenever the support is degenerate: $\mu_j = m_j = \overline{m}$, and we are thus back to the classical case.[8] In this context, heterogeneity can be thought simply as a mean preserving spread on the cross-sectional distribution of the mean of the dividend. Under heterogeneity, capital market equilibrium requires that

$$\overline{\widehat{z}}_2 = \frac{\overline{m} - pr}{\alpha\sigma^2} = \overline{z}$$

yielding:

the weighted ratio of the equilibrium prices for the risky asset: $\frac{E\overline{\pi}^a}{E\overline{\pi}^*} = \frac{S^2P^*}{\sigma^2P^a}$, where the equilibrium price is a decreasing function of S^2 whenever S^2 is large enough (precisely for $S^2 > \frac{\alpha\mu\overline{z}-2}{\alpha^2\overline{z}^2}$).

[8] Notice that it is not necessarily the case that $\mu_j = m_j$, all that matters is that the support is degenerate: $\mu_j = \nu_j = \overline{\nu}$. However, in the first part of this section the notation is kept simpler on purpose, so to induce to think of heterogeneity just as a mean-preserving spread, no matter whether the mean is subjective or objective.

Keeping this in mind, all the following comparative results relating *only* to the presence of heterogeneity ($\sigma_m^2 > 0$) hold with either of the homogeneous cases, with or without ambiguity. We will turn to ambiguity per se by the end of this section.

$$pr = \overline{m} - \alpha\sigma^2\overline{z} \tag{3.3.7}$$

The equilibrium condition for pr to be positive is a condition on the distribution of m_j's:

$$pr > 0 \Leftrightarrow \overline{m} > \alpha\sigma^2\overline{z} \tag{3.3.8}$$

In other words, for the risky capital to have a positive price, on average investors have to believe that the expected payoff to z_2 is sufficiently large. Notice that this does not imply that everyone will have a positive demand for risky capital in equilibrium. Pessimistic investors, that is, investors with the lowest m_j's, will sell short. On average, though, there will be a positive demand for z_2. Except for condition 3.3.8, I do not assume any other restriction on the cross-sectional distribution of m_j.

Substituting for pr in Equation 3.3.6 I find the equilibrium optimal holding of z_2:

$$\widehat{z}_{2j} = \frac{m_j - \overline{m}}{\alpha\sigma^2} + \overline{z} \tag{3.3.9}$$

Two facts about Equation 3.3.9 are worth noticing. Firstly, $\widehat{z}_2$ is a decreasing function of the subjective average expectation $\overline{m}$ whilst it does not depend on σ_m^2.[9] Secondly, it differs from both the classical and the homogeneous ambiguity-averse optimal holding. Moreover, the extent of such difference is an increasing function of the difference between the subjective expectation assessment and the average over the population:

$$\widehat{z}_2 - \overline{z} = \widehat{z}_2 - z_2^* = \frac{m_j - \overline{m}}{\alpha\sigma^2} \gtrless 0 \Longleftrightarrow m_j \gtrless \overline{m}$$

We thus proved the first part of the following:

Theorem 3 *Under heterogeneous beliefs about the mean of the risky dividend, and*

[9]This is an immediate result of the fact that the equilibrium pr is a function of $\overline{m}$ whilst it does not depend on σ_m^2. Notice then that if r is exogenous to the model, the equilibrium price for the risky capital p would only depend on the first moment of the cross-sectional distribution of beliefs, and not be affected by a mean-preserving spread in the beliefs.

condition 3.3.8, the equilibrium risky holding and average consumption are given by

$$\widehat{z}_{2j} = \frac{m_j - \overline{m}}{\alpha\sigma^2} + \overline{z}$$

$$\overline{\widehat{c}}_1 = -\frac{1}{\alpha}\ln\beta r + \overline{mz} + \frac{\sigma_m^2}{2\alpha\sigma^2} - \frac{\alpha\sigma^2\left(\overline{z}\right)^2}{2}$$

Proof: Turning to the equilibrium condition for consumption:

$$\overline{\widehat{c}}_1 = -\frac{1}{\alpha}\ln\beta r + av\left(m_j\widehat{z}^2\right) - av\left(\frac{\alpha\sigma^2\widehat{z}_2^2}{2}\right) \tag{3.3.10}$$

where

$$\begin{aligned} av\left(m_j\widehat{z}^2\right) &= av\left(\frac{m_j\left(m_j - \overline{m}\right)}{\alpha\sigma^2} + m_j\overline{z}\right) = \\ &= \frac{\sigma_m^2}{\alpha\sigma^2} + \overline{mz} \end{aligned} \tag{3.3.11}$$

and

$$\begin{aligned} av\left(\frac{\alpha\sigma^2\widehat{z}_2^2}{2}\right) &= \frac{\alpha}{2}av\left(\frac{\sigma^2\left(m_j - \overline{m}\right)^2}{\alpha^2\sigma^4} + \sigma^2\left(\overline{z}\right)^2 + 2\sigma^2\overline{z}\frac{m_j - \overline{m}}{\alpha\sigma^2}\right) = \\ &= \frac{\alpha}{2}av\left(\frac{\left(m_j - \overline{m}\right)^2}{\alpha^2\sigma^2} + \sigma^2\left(\overline{z}\right)^2 + 2\frac{\overline{z}\left(m_j - \overline{m}\right)}{\alpha}\right) = \\ &= \frac{\alpha}{2}\left(\frac{\sigma_m^2}{\alpha^2\sigma^2} + \sigma^2\left(\overline{z}\right)^2 + 0\right) = \\ &= \frac{\sigma_m^2}{2\alpha\sigma^2} + \frac{\alpha\sigma^2\left(\overline{z}\right)^2}{2} \end{aligned} \tag{3.3.12}$$

Substituting for these expressions into Equation 3.3.10, the equilibrium average consumption becomes

$$\overline{\widehat{c}}_1 = -\frac{1}{\alpha}\ln\beta r + \overline{mz} + \frac{\sigma_m^2}{2\alpha\sigma^2} - \frac{\alpha\sigma^2\left(\overline{z}\right)^2}{2} \tag{3.3.13}$$

∎

Notice that the introduction of heterogeneity into the economy leads to greater consumption and thus higher interest rate, than in the traditional, homogeneous case. More formally, recall that in equilibrium it is the case that $\overline{\widehat{c}}_1 = d_1\overline{z}$, given the assumption that the consumption good is completely perishable. By substituting this into Equation 3.3.13, and solving for the equilibrium risk free rate:

$$\widehat{r} = \frac{1}{\beta} e^{\alpha\overline{z}(\overline{m}-d_1)+\frac{\sigma_m^2}{2\sigma^2}-\frac{\alpha^2\sigma^2(\overline{z})^2}{2}} \tag{3.3.14}$$

By letting r° denotes the homogeneous equilibrium interest rate, it is always the case that

Theorem 4

$$\widehat{r} = r^{\circ} e^{\frac{\sigma_m^2}{2\sigma^2}} \tag{3.3.15}$$

Notice that the effect of $\sigma_m^2 > 0$ on the equilibrium interest rate comes from two different, opposite sources, as it can be seen in Equations 3.3.11 and 3.3.12, respectively. Firstly, an increase in σ_m^2 increases the average of the subjective expected value of d_2 's income, increasing the current consumption. Secondly, an increase in σ_m^2 increases the average value of the variance of d_2's income, decreasing c_1 because of precautionary savings. With constant absolute risk aversion, as in my case, the first effect dominates the second one. The total effect will then be an increase in the current consumption, thus raising r relative to what predicted under homogeneity. Thus heterogeneity per se cannot explain the risk-free rate puzzle.

As for the equilibrium price for risky capital I find that

$$\widehat{p} = (\overline{m} - \alpha\sigma^2\overline{z})\frac{1}{\beta} e^{-\alpha\overline{z}(\overline{m}-d_1)-\frac{\sigma_m^2}{2\sigma^2}+\frac{\alpha^2\sigma^2(\overline{z})^2}{2}}$$

so that $\widehat{p}$ is a decreasing function of the cross-sectional variation in the m_j's. Despite of the fact that an increase in σ_m^2 increases both the expected returns on bonds and stocks, I will compare these effects in the next section and show that they sum up to a well-specified prediction about the effect of heterogeneity on the equity premium.

On the contrary, when disregarding heterogeneity ($\sigma_m^2 = 0$), the effect of ambiguity per se is to decrease the first-period consumption relative to the classical model, and thus predicting a lower interest rate in equilibrium. Also, in a similar spirit to what I found under subjective risk, the ex-ante equity premium will be higher than the classical one, with a caveat: this is true only for $\overline{m}$ small enough.[10] More interestingly, though, under the assumption of the existence of a "true" distribution (as of historical realizations), the effect of ambiguity aversion on the equity premium is well-specified for any value of the parameters, thus allowing a discussion about the difference in predicted and realized values of the equity premium.

3.3.3 Equity premium

The introduction of ambiguity and heterogeneity in beliefs has important implications for the so-called equity premium puzzle as discussed by Mehra and Prescott [28]. The equity premium is the excess rate of return on equities relative to risk-free bonds. When computing the implied equity premium for the representative agent asset pricing model calibrated to the US economy, they predicted that the equity premium was less than 35 basis points. However, the observed value for the US economy over the period 1889–1978 was slightly more than 600 basis points. This failure of the asset pricing model to predict an empirically plausible equity premium has been held as evidence against it.

Mehra and Prescott's model assumed homogeneous beliefs across agents. In the first part of this section I show how introducing heterogeneous beliefs can increase the equity premium.

The ex-ante equity premium is defined as $E\pi = \frac{Ed_2}{p} - r$. With subjective heterogeneous expectations, the equity premium for agent j is given by $E\widehat{\pi} = \frac{m_j}{\widehat{p}} - \widehat{r}$, whilst the cross-sectional ex-ante average equity premium is $E\overline{\widehat{\pi}} = \frac{\overline{m}}{\widehat{p}} - \widehat{r}$. Recall the equilibrium relation for p and r as in Equation 3.3.7: $\widehat{pr} = \overline{m} - \alpha\sigma^2\overline{z}$. Substituting

[10] More precisely, the ratio of the ex-ante ambiguity-averse and classical equity premia is equal to the ratio of the equilibrium prices for the risky asset: $\frac{E\overline{\pi}^a}{E\overline{\pi}^*} = \frac{P^*}{P^a}$, where the equilibrium price is an increasing function of $\overline{m}$ whenever $\overline{m}$ is small enough (precisely for $\overline{m} < \frac{1+\alpha^2\sigma^2\overline{z}^2}{\alpha\overline{z}}$).

for r in the expression for $E\overline{\widehat{\pi}}$ yields

$$E\overline{\widehat{\pi}} = \frac{\alpha\sigma^2\overline{z}}{\widehat{p}}$$

For the case of homogeneous beliefs (that is, when $m_j = \overline{m}$ for every j, $\sigma_m^2 = 0$), denote by p° and π° respectively the equilibrium price for risky capital and the equity premium. Recall that, in a similar fashion, $p^\circ r^\circ = \overline{m} - \alpha\sigma^2\overline{z}$ and $E\overline{\pi}^\circ = \frac{\alpha\sigma^2\overline{z}}{p^\circ}$. The relationship between the homogeneous and heterogeneous equilibrium interest rates is given by Equation 3.3.15: $\widehat{r} = r^* e^{\frac{\sigma_m^2}{2\sigma^2}}$. It immediately follows that

$$\frac{E\overline{\widehat{\pi}}}{E\overline{\pi}^\circ} = \frac{p^\circ}{\widehat{p}} = \frac{\widehat{r}}{r^\circ} = e^{\frac{\sigma_m^2}{2\sigma^2}} \tag{3.3.16}$$

Corollary 1 *The equity premium is greater under heterogeneous beliefs than under homogeneity:*

$$E\widehat{\overline{\pi}} > E\overline{\pi}^\circ$$

that is, a mean-preserving spread in the cross-sectional distribution of beliefs leads to a higher ex-ante equity premium.

This means that, whenever the equity premium is calculated by assuming homogeneous beliefs, it will be underestimated. Moreover, the extent of such an underestimation is a strictly increasing function of the variance of expectations.

The results presented so far are not uncommon in the existing literature, where departures from standard assumptions are used to predict a greater ex-ante equity premium. However they do not explicitly solve the equity premium puzzle: strictly speaking, the puzzle concerns the spread of observed returns on equity portfolios over the observed returns of a riskless security, and should thus be analyzed from an ex-post perspective. [11]

[11] Maenhout [27] also investigates an ex-post perspective, by comparing the ex-post equity premium to the model that the decision maker uses to determine optimal investment in stock, which includes an assessment of the ex-ante risk premium. He finds that, in order to explain the ex-post equity premium, the ex-ante risk premium should have been extremely low (0.03. His analysis of the ex-

In terms of my model, and more generally for the class of models incorporating subjective beliefs and aversion to ambiguity, this distinction is crucial as people act on the basis of "incorrect" beliefs in the sense of being consistently biased towards the worst case. That is, we should compare the predictions for the ambiguity model to the true (historically observed) realizations of the equity premium.

So consider the ambiguous, homogeneous case. People's beliefs are represented by a common $N(m, \sigma^2)$, and they trade at equilibrium prices p^a and r^a. So they expect the equity premium to be equal to $E\overline{\pi}^a = \frac{m}{p^a} - r^a$. However, the realized (ex-post) equity premium will always be higher: under the assumption that a true distribution exists, and people are averse to ambiguity, the "true" distribution will be $N(\mu, \sigma^2)$ with $\mu > m$. By thinking of μ as the realized historical mean for the stock price, it would always be the case that the realized equity premium is higher than what investors could predict on the basis of their beliefs and thus trading prices. Whenever consistently biasing their priors due to aversion to not knowing the "true" probabilities (that is, the ones that they would be able to measure ex-post), investors (and empiricists) may have been "surprised" by the equity returns they receive: this provides a different, still rational resolution of the equity premium puzzle.

3.4 Some concluding remarks

Some of the model's restrictions call for appealing extensions, mainly to introduce multiplicity of periods and/or risky assets. I am also interested in different ways to think about the priors' support, by modeling some relation between beliefs about the mean and beliefs about the variance. Multiplicity of periods with heterogeneous ambiguity-averse beliefs can be problematic, as it poses questions about updating (still an open debate in the related decision theoretical literature) and about the appropriateness of the equilibrium concept.[12] This is strictly related to the distinction

post equity premium is thus close in spirit to the results that follow, but do not share our model's mathematical simplicity and a solid axiomatic foundation.

[12] See Bossaerts [4], proving that rational expectations equilibrium is debatable as an equilibrium concept when people disagree.

whether differences in beliefs are explained by different information or different beliefs. The origin of differences in beliefs does not matter whenever investors do not learn from prices, which is the case when they simply hold different opinions (rather than information) about the distribution of the dividend. Under this specification, the multiperiod extension could be straightforward, and plausibly confirm and strenghten the results of this work.

This direction would be fruitful in light of the calibration exercises and experiments I plan to perform soon on my model. The equity premium and risk-free rate puzzles are quantitative puzzles, and require an investigation whether the model's predicted order of magnitude is close to what had been historically documented, or at least closer than what has been proposed by the theoretic literature so far. As for calibration exercises, the direction is promising: even though heterogeneity per se is not enough, [13] some help might come from the proposed extensions and/or the introduction of a random, positive ex-dividend price in period 2 as a further source of ambiguity. The effects of aversion to the lack of relevant information about the economy go in the "right" directions and call for both an empirical and an experimental test of whether any plausible degree of ambiguity could predict sizeable risk premia for low levels of risk aversion.

[13] To attribute the observed equity premium to heterogeneous beliefs would require an increase in the risk-free rate by a factor of approximately 17.

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3.5 Appendix

The maximization problem in Section 3.2 is given by

$$\max_{c_1,b,z_2} -e^{-\alpha c_1} + \beta E\left[-e^{-\alpha(d_2 z_2 + rb)}\right] \text{ st } (p+d_1)\, z_1 = c_1 + pz_2 + b$$

where

$$E\left[-e^{-\alpha(d_2 z_2 + rb)}\right] = e^{-\alpha rb} E\left[-e^{-\alpha d_2 z_2}\right] = -e^{-\alpha \quad rb+\mu z_2 - \frac{\alpha\sigma^2 z_2^2}{2}}$$

Thus the Lagrangean function for the problem is given by

$$L(c_1, b, z_2, \lambda) = -e^{-\alpha c_1} - \beta e^{-\alpha \quad rb+\mu z_2 - \frac{\alpha\sigma^2 z_2^2}{2}} + \lambda\left[c_1 + pz_2 + b - (p+d_1)\, z_1\right]$$

and the first-order conditions for the problem are computed as:

$$\frac{\partial L}{\partial c_1} = \alpha e^{-\alpha c_1} + \lambda = 0 \tag{3.5.1}$$

$$\frac{\partial L}{\partial b} = (-\alpha r)\left(-\beta e^{-\alpha \quad rb+\mu z_2 - \frac{\alpha\sigma^2 z_2^2}{2}}\right) + \lambda = 0 \tag{3.5.2}$$

$$\frac{\partial L}{\partial z_2} = \left(-\alpha\left(\mu - \alpha\sigma^2 z_2\right)\right)\left(-\beta e^{-\alpha \quad rb+\mu z_2 - \frac{\alpha\sigma^2 z_2^2}{2}}\right) + \lambda p = 0 \tag{3.5.3}$$

$$\frac{\partial L}{\partial \lambda} = c_1 + pz_2 + b - (p+d_1)\, z_1 = 0 \tag{3.5.4}$$

By setting 3.5.1 equal to 3.5.3 and 3.5.2 respectively, I obtain

$$e^{-\alpha c_1} = \frac{\beta}{p}\left(\mu - \alpha\sigma^2 z_2\right) e^{-\alpha \quad rb+\mu z_2 - \frac{\alpha\sigma^2 z_2^2}{2}} \tag{3.5.5}$$

$$e^{-\alpha c_1} = e^{-\alpha \quad rb+\mu z_2 - \frac{\alpha\sigma^2 z_2^2}{2}} \tag{3.5.6}$$

Dividing 3.5.5 by 3.5.6:

$$1 = \frac{1}{pr}\left(\mu - \alpha\sigma^2 z_2\right)$$

that is,

$$pr = \mu - \alpha\sigma^2 z_2$$

as claimed in Section 3.2.

Chapter 4 Financial markets with ambiguity: an experimental approach

4.1 Equilibrium prices and ambiguity

This section develops the theoretical model which I will use as a benchmark for my experimental design and data analysis. I used an α-MEU framework to allow for a test of ambiguity attitude, where $\alpha = 0$ indicates ambiguity aversion (MinMax EU, as axiomatized in Gilboa and Schmeidler) and $\alpha = 1$ indicates ambiguity loving (MaxMax EU). The original model was conceived under the assumption of risk-neutrality. Given evidence of some degree of risk aversion in the financial experiments, I extended the model as to include a generic functional form for the utility function.[1] The quadratic specification will be assumed only for calibration purposes in the experimental data analysis.

Formally, suppose there are three states of the world X, Y, and Z, and three Arrow-Debreu securities denoted x, y, and z after the states of the world, for notational simplicity. Each investor's initial endowment is x^0, y^0, z^0, and the per capita capital stock is denoted by $\overline{x}, \overline{y}$, and $\overline{z}$. Given his budget constraint, every investor chooses a portfolio of securities that maximizes his expected utility:

$$\max_{x,y,z} EU(x, y, z) \text{ subject to } p_x x + p_y y + p_z z \leq p_x x^0 + p_y y^0 + p_z z^0$$

where p_i represents the market price for security i, the expectation operator E is taken with respect to the probability distribution over states of the world, and U is additive, separable and strictly increasing.

As in Lucas, I assume a single investor with ambiguity parameter α, and leave the interesting case of heterogeneity for future investigation.

[1]See Berg, Dickhaut, and Rietz [1] for induced risk-neutrality experiments.

The main departure from the previous experimental financial literature is the assumption that the investor does not know the probability distribution for tomorrow's states of the world $\Pi = (\pi_x, \pi_y, 1 - \pi_x - \pi_y)$. This assumption can be modeled in many different ways, but for the purposes of testing experimental data I opted for a mathematically simple and flexible model, such as Gilboa and Schmeidler's EU with multiple priors. The investor's lack of information is represented by a set of priors, capturing his beliefs about the plausibility of each event occurring. To make it even simpler, assume that the investor knows the probability of X occuring, e.g., $\pi_x = \pi$, but knows nothing about either Y or Z. In this case, the set of priors representing his beliefs is simply the interval $\pi_y = [0, 1 - \pi]$ (with $\pi_z = 1 - \pi - \pi_y$). Assuming ambiguity aversion, the investor evaluates each portfolio by its minimum (i.e., guaranteed) expected utility, where the minimum is computed over his belief set:

$$\max_{x,y,z} \min_{\pi_y \in [0,1-\pi]} \pi u(x) + \pi_y u(y) + (1 - \pi - \pi_y) u(z)$$

After few computations the problem reduces to:

$$\max_{x,y,z} \pi u(x) + (1 - \pi) \min\{y, z\}$$

A second departure from previous models is a tractable generalization of Minimax EU so as to include a parameter that captures attitude towards uncertainty. In this model, the investor evaluates a portfolio using α times the maximum and $(1 - \alpha)$ times the minimum expected utility over the set of priors:

$$\begin{aligned} \max_{x,y,z} \quad & \{\pi u(x) + \alpha \max_{\pi_y \in [0,1-\pi]} [\pi_y u(y) + (1 - \pi - \pi_y) u(z)] + \\ & +(1 - \alpha) \min_{\pi_y \in [0,1-\pi]} [\pi_y u(y) + (1 - \pi - \pi_y) u(z)]\} \end{aligned}$$

which is equivalent to

$$\max_{x,y,z} \{\pi u(x) + (1-\pi)\min[y,z] + \alpha(1-\pi)|u(y)-u(z)|\}$$

From now on, I will refer to strict ambiguity aversion for the case of $\alpha = 0$, and ambiguity aversion for $\alpha < \frac{1}{2}$ (where $\alpha = \frac{1}{2}$ corresponds to EU or neutrality to ambiguity).[2] First-order conditions give the following equations for relative prices:[3]

$$\begin{aligned}
\frac{p_y}{p_x} &= \frac{1-\pi}{\pi}\frac{u'(y)}{u'(x)}[\mathbb{I}_{\{y<z\}} + \alpha\ sign(y-z)] \\
\frac{p_z}{p_x} &= \frac{1-\pi}{\pi}\frac{u'(z)}{u'(x)}[\mathbb{I}_{\{y>z\}} - \alpha\ sign(y-z)]
\end{aligned}$$

where

$$sign(y-z) = \begin{cases} +1 & \text{if } y > z \\ 0 & \text{if } y = z \\ -1 & \text{if } y < z \end{cases}$$

Given the usual condition of net excess demand being equal to zero, equilibrium prices are computed under further assumptions about the ranking for the endowment of y versus z across the population. As in my experimental setup, by setting the total (and therefore average) endowment of y greater than the one of z, equilibrium prices are such that

$$\begin{aligned}
\frac{p_y}{p_x} &= \alpha\frac{1-\pi}{\pi}\frac{u'(\overline{y})}{u'(\overline{x})} \\
\frac{p_z}{p_x} &= (1-\alpha)\frac{1-\pi}{\pi}\frac{u'(\overline{z})}{u'(\overline{x})}
\end{aligned} \tag{4.1.1}$$

[2]This holds true only in our setting, where the support for ambiguity is linear as there are only two ambiguous states.

[3]Closed form solutions for optimal demand can easily be derived when imposing more structure on the utility function. Also, second-order conditions for a maximum are satisfied whenever $u''(x) < 0$, which is true for the specific quadratic utility function assumed in the next sections.

In general, whenever the optimal quantity of y exceeds that of z, equilibrium prices need to satisfy

$$\frac{p_y}{p_z} < \frac{\alpha}{1-\alpha} \tag{4.1.2}$$

and conversely, whenever the optimal holding of z is larger,

$$\frac{p_y}{p_z} > \frac{\alpha}{1-\alpha} \tag{4.1.3}$$

If the optimal holdings of y and z are equal, then neither inequality holds. Equilibrium prices in my setting need to satisfy the first inequality, that is, some agents need to be compensated for holding an unequal number of ambiguous state securities, and in particular a greater number of y. This hypothesis over equilibrium prices is only ordinal and will be tested in later sections.

4.2 Experimental design

The experiments are set up as a replication of four markets over time, where each replication is called period. Each of the two sessions I ran (henceforth labelled 020529 and 020619) involved 23–25 subjects and eight periods. At the beginning of every period, subjects are allocated a certain number of three state securities and cash, as working capital. During the period, subjects can trade assets in four continuous open book markets on the web (called Marketscape) for up to fifteen minutes, after which the dividends were paid according to final holdings.

With the exception of Notes, each asset provides a stochastic dividend. The dividend is determined at the end of the period by a random drawing of a state out of three possible states, referred to as X, Y, and Z. Each asset is labelled according to the state in which it pays 100, and it would pay 0 otherwise. The fourth asset, called Notes, is riskfree, is in zero net supply and can be sold short up to four units.

Risky assets can not be sold short. The dividend table is reported in the experiment instruction in the Appendix. Also notice that markets are complete.

Subjects were given cash at the beginning of each period, and could hold more cash by selling risky assets or borrow through shortsales of Notes. At the end of the period though they have to pay their 'loan' back to the experimenter, and it is thus possible for some subjects to make negative earnings. Subjects who incurred losses in more than two periods in a row were excluded from further trading and would earn nothing.[4]

Earnings were computed at the end of each period by using an experiment currency, called francs, and converted to dollars at the end of the experiment according to a pre-announced exchange rate.

The experiments are conducting using a trading system, called Marketscape, developed at Caltech.[5] It had been successfully used in the past for a wide range of experiments including tests of competitive equilibrium theory, and it provides with a rich dataset including bids/asks prices and quantities, trading prices and quantities, portfolio holdings and traders' identity.[6]

Subjects are recruited through a database made available by the Caltech Experimental Economics and Political Science Laboratory, and mainly includes Caltech undergraduate students in the sciences and engineering. Most of the students are experienced, in the sense that they took at least one class at the Department of Social Science and already participated in market experiments beforehand.

The main departure from previous financial experiments is the nature and range of information about the randomness of the states. Subjects are given precise and

[4]This bankruptcy rule induces risk aversion, due to potential gains in future trading periods, up to the second last period. This incentive diminishes over time, and so does the induced risk aversion, thus increasing the effect of ambiguity aversion and the power to discriminate between the two effects over time.

[5]To try out the trading system, the reader could visit http://eeps3.caltech.edu/market-demo/.

[6]All this information is publicly disclosed in the open book market, except for subjects' holdings and identity. ID numbers are assigned instead, in order to keep anonimity and prevent subjects from revealing their own parameters and endowments. This is part of the experimental design: in order to test a model, it is necessary to prevent subjects from deducing the aggregate parameters and using any of the standard models to price securities. Also, this setup is the closest to real markets, where most of the available information is limited to market prices.

common information about the probability of state X, but no information about Y or Z. The experiment starts with a bin with 18 balls, 6 of which are marked X. Each of the remaining 12 is marked either Y or Z. Balls are drawn randomly without replacement, as detailed in the instructions.

Two things are worth noticing. First, the composition of the bin (number of Y balls versus Z balls) is randomly determined before the experiment. By using a random device each of the 12 unknown balls is assigned either a Y or a Z, and this labelling is preserved throughout the whole experiment. Second, none of the subjects is told the composition of the bin or the random device used, so that no subject has better information, and no hints are given during the experiment. All the information about the bin is common, and restricted to what is disclosed in the instructions.

This experimental design is essential and enables me to use the parameterized multiple priors model for data analysis, and to calibrate the parameters in a controlled and effective way. Given the information provided to subjects, the set of priors available to subjects at the beginning of the experiments is $\pi_y \in [0, \frac{2}{3}]$ as $\pi_x = \frac{1}{3}$ and for notational simplicity I will refer to it for the next section only. When estimating data, π_x is updated accordingly to the actual draws observed in each experiment and the information available at each period.

4.2.1 Predictions

The following predictions hold when average endowments are ranked as in 020529 and 020619, that is, $\overline{y} > \overline{z} > \overline{x}$:

1. EU with risk aversion: $p_y < p_z < p_x$.

2. MEU with strict ambiguity aversion and risk neutrality: $0 = p_y < p_x < p_z = \frac{2}{3}$, with p_x equal to its expected value $\pi = \frac{1}{3}$ and in between p_y and p_z.

3. MEU with ambiguity aversion and risk neutrality: $\frac{2}{3}\alpha = p_y < p_x < p_z = \frac{2}{3}(1 - \alpha)$, with p_x equal to $\pi = \frac{1}{3}$ and in between p_y and p_z.

4. MEU with ambiguity aversion and small risk aversion: rankings are preserved as in the previous case, but shrunk in a way proportional to marginal rates of substitutions: p_x close to $\pi = \frac{1}{3}$ and in between p_y and p_z, and $\frac{p_z}{p_x} = 2(1-\alpha)\frac{u'(\overline{z})}{u'(\overline{x})} < 2(1-\alpha)$.

Different types of risk-averse utility functions have been studied and applied to financial studies, and even though my model does not impose any restriction, I chose the quadratic function $u(w) = w - \frac{b}{2}w^2$ to be able to estimate a coefficient b and capture attitudes toward risk. [7] Apart from its analytical tractability, this assumption serves two purposes: it allows to separate risk effects from ambiguity effects over prices; and it allows for comparisons with different financial experiments, whose estimates can then be used as a benchmark and stress test for my model.

The model provides indeterminate predictions for final allocations, as the following types and portfolios are possible:

- Ambiguity averse agents hold an equal number of ambiguous state securities;
- Ambiguity lovers hold all of one ambiguous state security, and none or little of the other;
- Expected utility maximizers hold a mix of ambiguous state security, with more y than z to compensate for the excess supply.

4.3 Results

4.3.1 Data patterns

Conclusion 5 *Models 1–3 are rejected, whilst there is support for Model 4.*

[7] Other functional forms can be assumed and tested to find the best fit to experimental data.

Standard risk aversion seems supported by the finding of p_x as the highest price, but the prevalence of $p_y \geq p_z$ is inconsistent with it, as illustrated in Figures 4.1 and 4.2. As of Figures 4.3 and 4.4, ambiguity averse preferences plus risk-neutrality can be rejected as well, as $p_x > \pi_x$.

When plotting the ratio of prices over probabilities with initial uniform probabilities $\pi_y = \pi_z = \frac{1}{3}$, there is support for risk averse and ambiguity averse preferences.[8] For example, in the experiment 020529, as reported in Figure 4.5, p_x is always above its probability; p_y follows its probability and drops from period 5 on; and p_z is above its probability from period 4 on, and dramatically above it in periods 7 and 8. In Figure 4.6 for the experiment on 020619, this trend is even more dramatic when considering that state Z was drawn 5 times out of 8 periods. Still y did not become too expensive and z remains relatively expensive, thus excluding speculation. Also, standard expected utility with uniform priors and risk aversion is violated by the fact that the state price density for x is less than that for z. In particular, when updated by π_z, p_z is often above the other prices, and consistently above after period 6.

In conclusion, the state price density is eventually much bigger for z than for x. This fact has never been reported in the experimental literature on financial markets, as previous setups always included known probability distributions (thus accounting for risk only, and not ambiguity). Also, the state price density of y is consistently below the one for x, so that state price densities are ranked exactly as predicted under ambiguity aversion.

Lastly, the data confirms the predictions on indeterminacy of final allocations. Some trends are worth noticing though. First, very few subjects held highly unbalanced portfolios. Most of the subjects' final portfolio were consistent with either ambiguity aversion or expected utility maximization. Secondly, the subjects' behavior tended to be consistent across periods, and only 9 subjects out of a total of 46

[8] A justification for the use of the uniform prior is twofold: the uniform prior is a focal point (and also a natural choice for a EU maximizer who is neutral to ambiguity); and it complies with Bernoulli's "principle of insufficient reason." Furthermore, in a recent experimental study by Chen, Katuscak, and Ozdenoren [5], when asked to report their beliefs subjects predominantly reported the uniform prior.

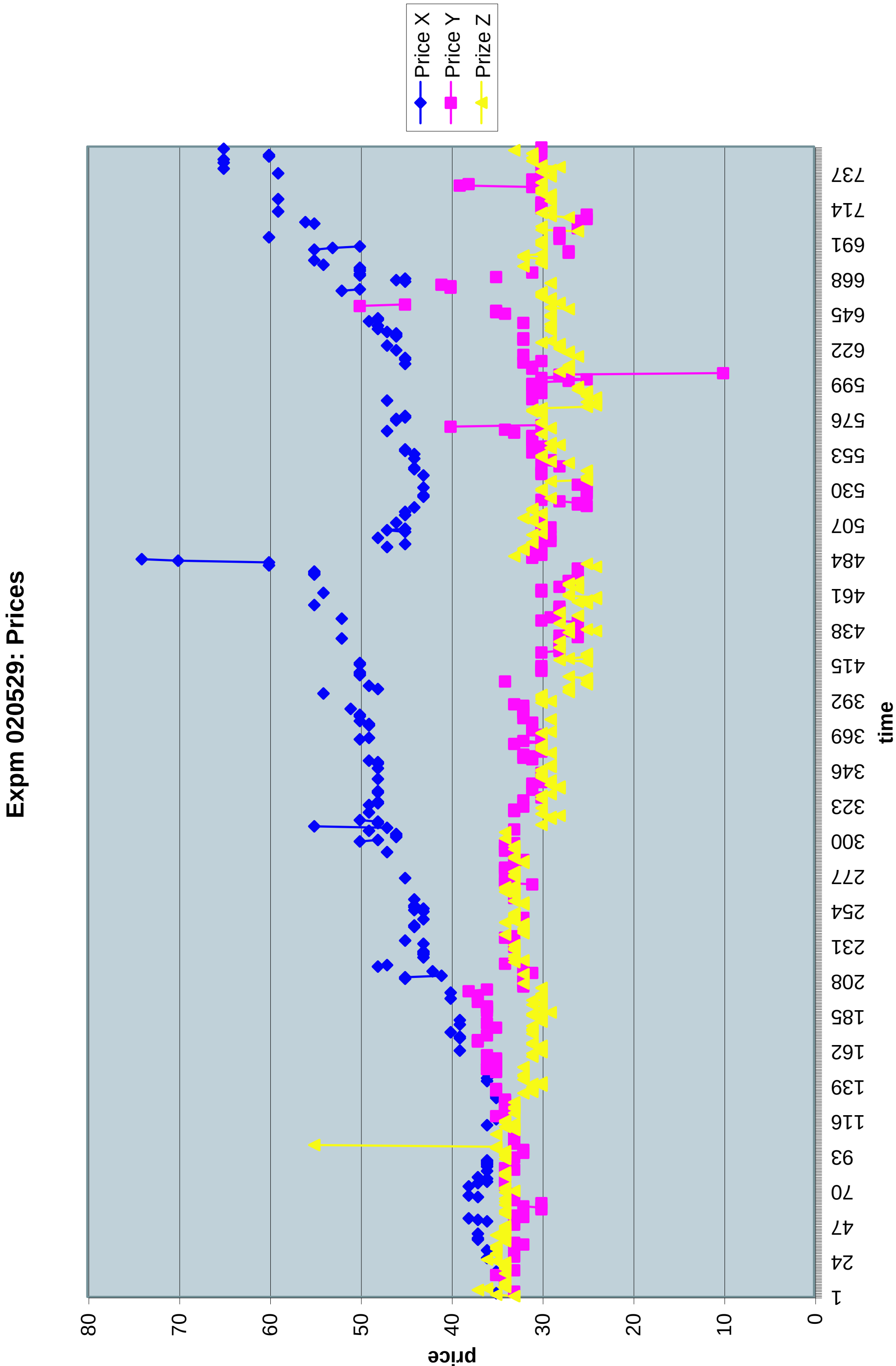

Figure 4.1: Prices in the experiment on May 29th, 2002.

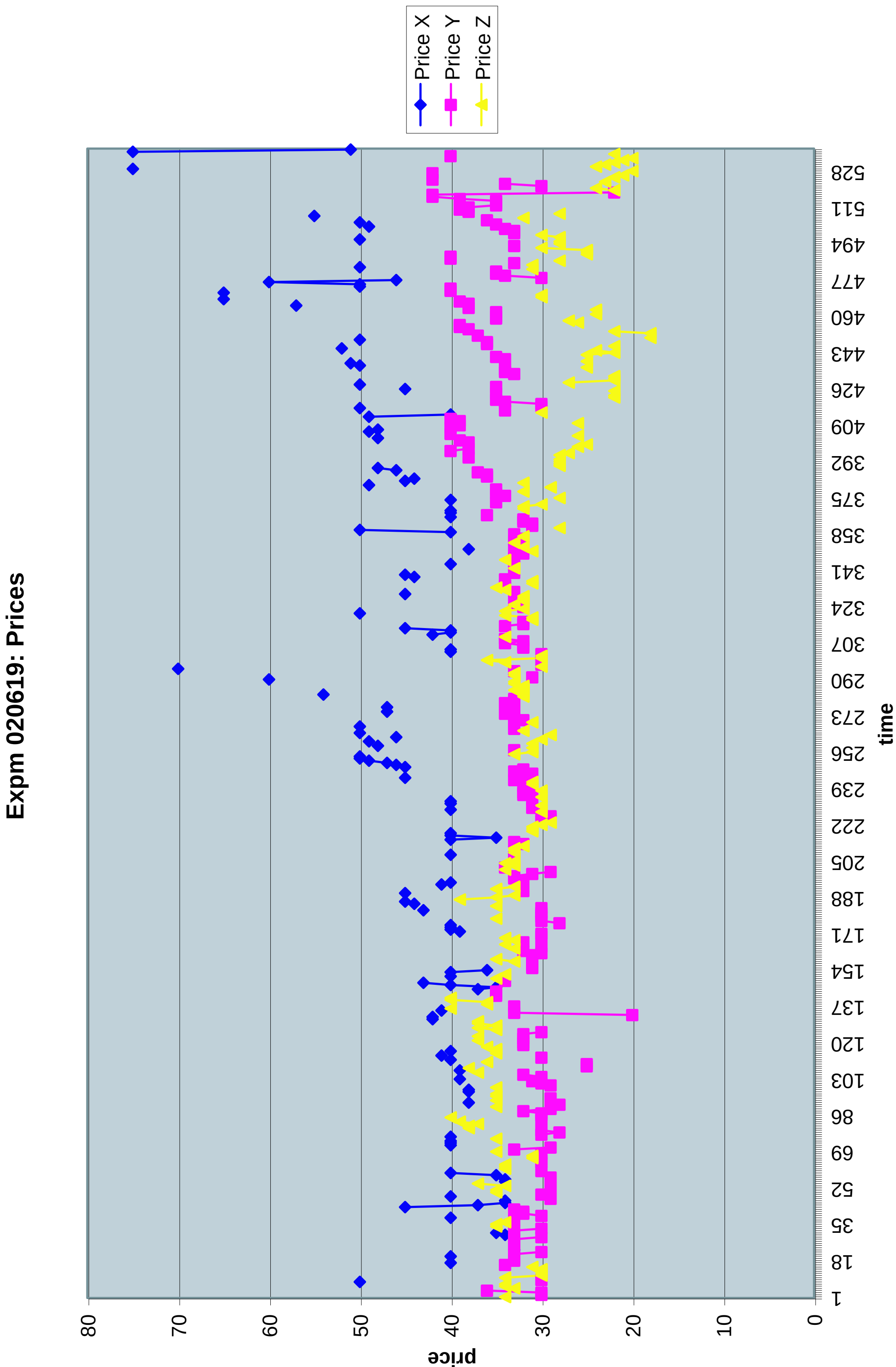

Figure 4.2: Prices in the experiment on June 19th, 2002.

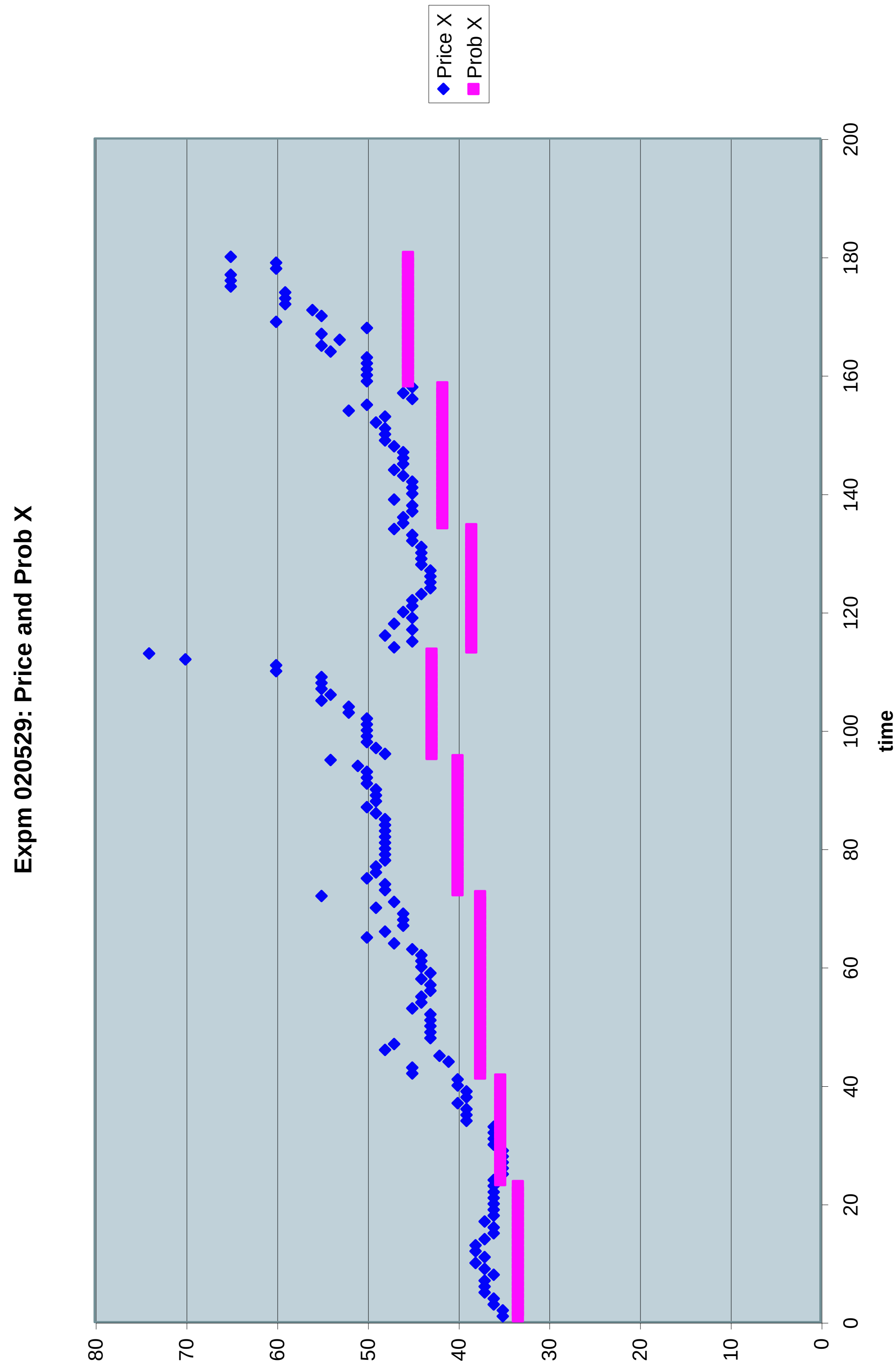

Figure 4.3: Price and probability of x in the experiment on May 29th, 2002.

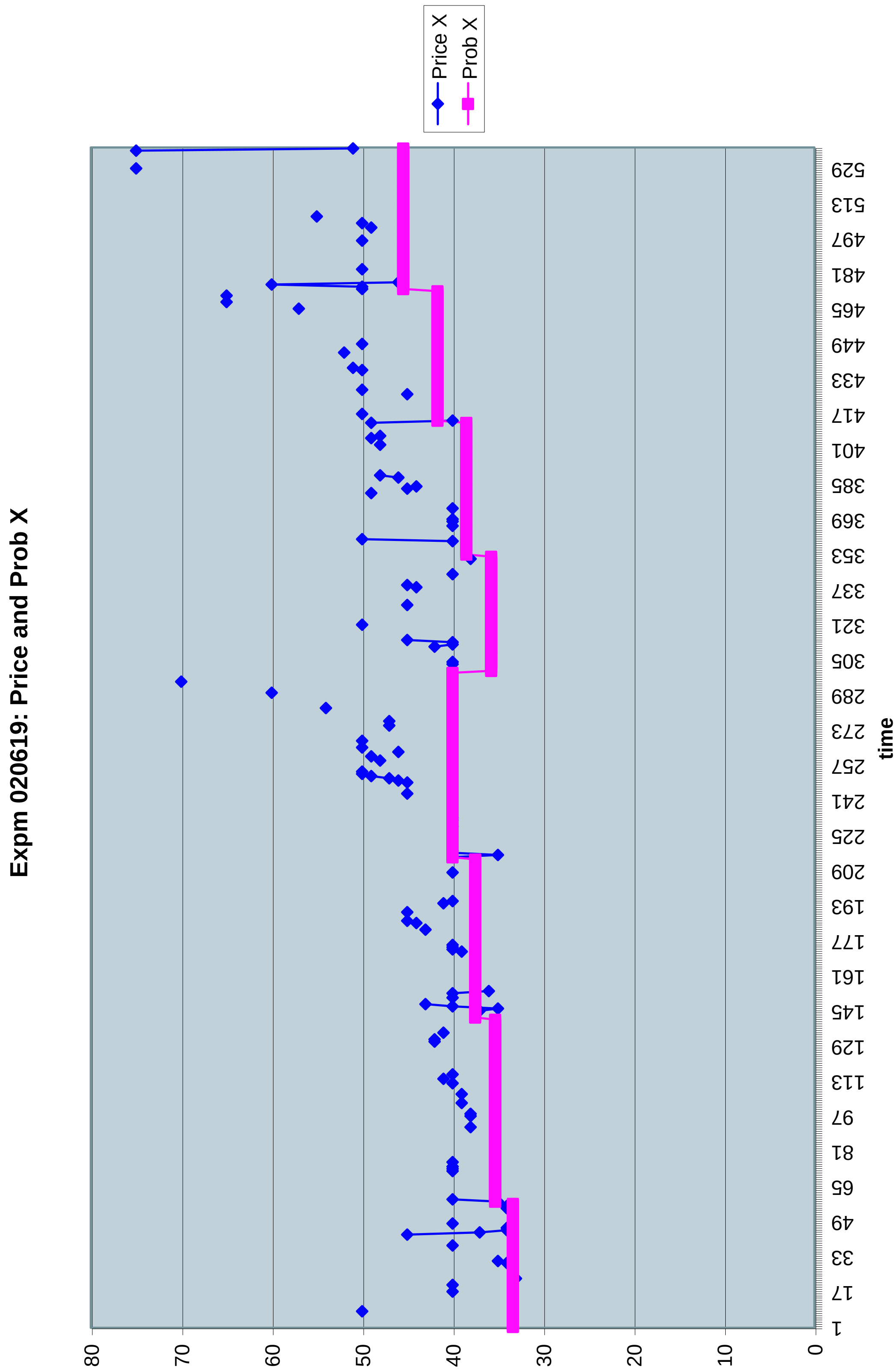

Figure 4.4: Price and probability of x in the experiment on June 19th, 2002.

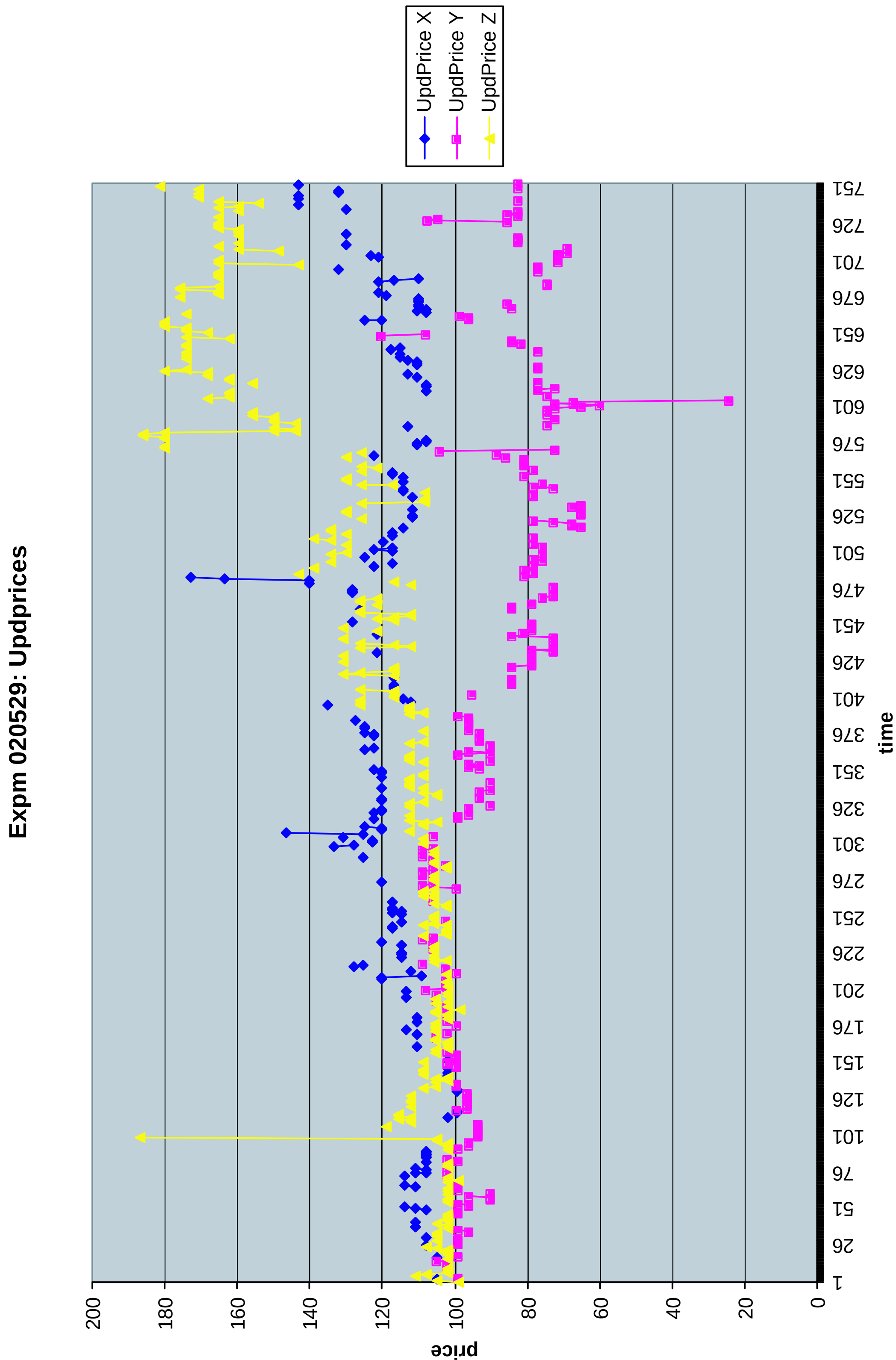

Figure 4.5: State prices in the experiment on May 29th, 2002.

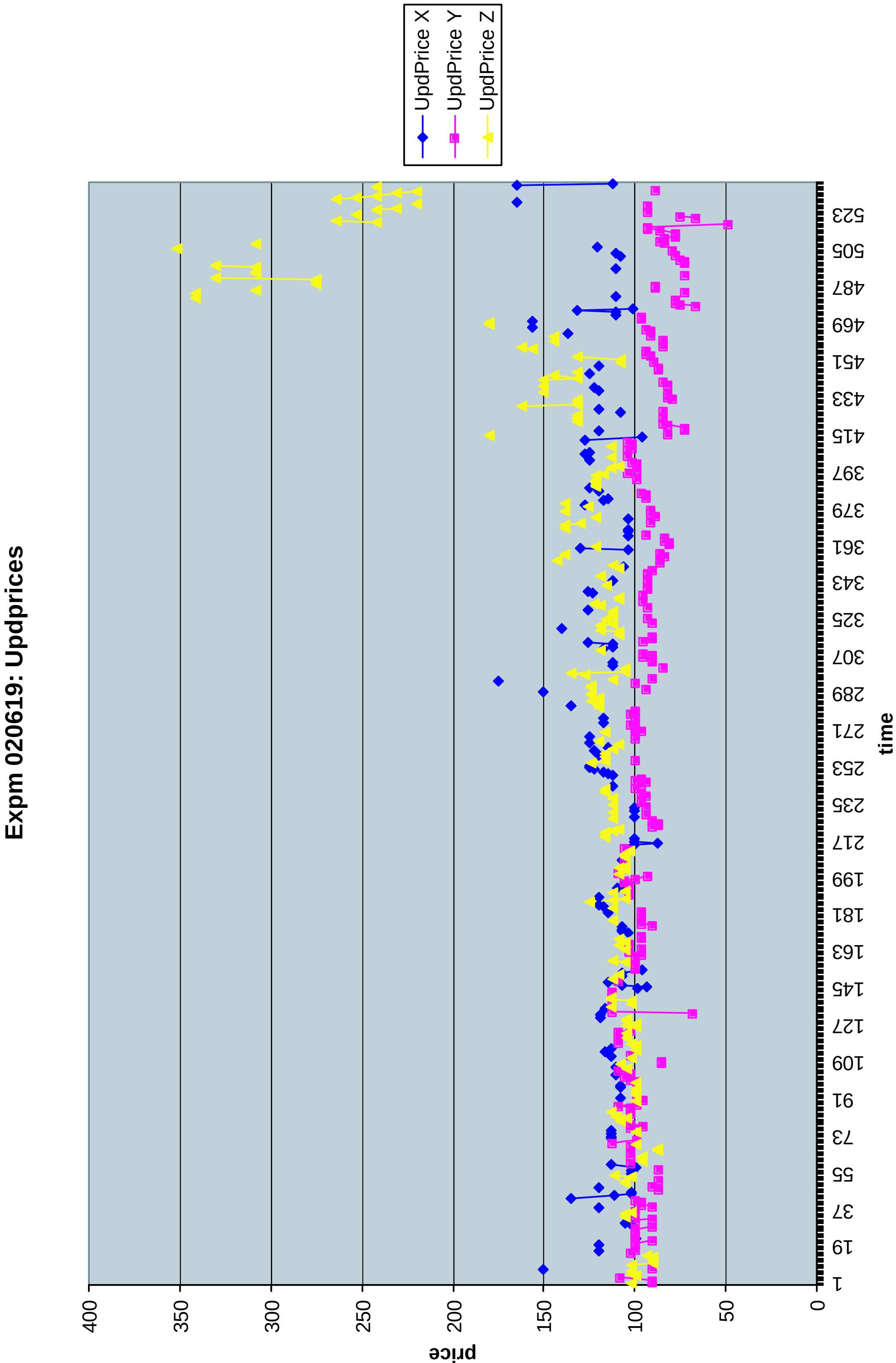

Figure 4.6: State prices in the experiment on June 19th, 2002.

switched from ambiguity aversion to ambiguity loving or viceversa. While confirming predictions on final holdings, these observations call for the introduction of heterogeneity of attitudes toward ambiguity across subjects, in order to capture some of this complexity.

4.3.2 Statistical analysis

Since the directional effects of ambiguity aversion on asset prices are similar to the ones of risk aversion, it is important to separate the two effects. In my setting, ambiguity does not play any role on the pricing of asset X since subjects know the probability distribution over outcomes $\{0, 100\}$. So I can first compute the smallest risk aversion coefficient $\widehat{b}$ that explains the observed p_x. Then, by using $\widehat{b}$ to calibrate u, I estimate α for the observed p_y and p_z.

Conclusion 6 *There is support for both risk aversion and ambiguity aversion. The estimated parameters are* $\widehat{b} = 0.0026$ *and* $\widehat{\alpha} = .0319$ *for the first experiment, and* $\widehat{b} = 0.0024$ *and* $\widehat{\alpha} = .0415$ *for the second one.*

In order to estimate attitudes toward risk, I use the observed prices for the asset involving no ambiguity and regress the following:

$$\frac{p_x}{100} - \pi_x = \widehat{a} - \widehat{b}\pi_x\overline{x}$$

where p_x is the price of x normalized due to experiment currency, $\overline{x}$ is the market portfolio, and $\widehat{b}$ is the estimate for the risk aversion coefficient b. Table 4.1 summarizes the results.

Notice that the estimated risk parameters are consistent with the estimates of previous literature in financial experiments. In the subsequent analysis, I will use the estimated b's to isolate the effects of ambiguity.

Expm	020529	020619	020529	020619
Beta	.0026 (.0003)	.0024 (.0007)	.0007 (.0001)	.0007 (.0000)
Constant	$-.1830$ (.0295)	$-.1555$ (.0708)	restr=0	restr=0
Standard errors in parentheses.				

Table 4.1: Risk coefficient estimates.

Expm	020529	020619	020529	020619
Alpha	.0873 (.2432)	.8158 (.2892)	.0105 (.0001)	.0111 (.0002)
Constant	-8.4507 (24.1666)	-439.583 (28.7812)	restr=0	restr=0
Standard errors in parentheses.				

Table 4.2: Ambiguity coefficient estimates for relative prices.

I ran two different regressions to test for attitudes toward ambiguity, both involving relative prices as in Equation 4.1.1 where $u'(w) = 1 - bw$. The first regression involves prices for the two "ambiguous" assets:

$$\frac{p_y}{p_z} = \widehat{c} + \widehat{d}\frac{1 - b\overline{y}}{1 - b\overline{z}}$$

where $\widehat{d}$ is the estimate for the ratio $\frac{\alpha}{1-\alpha}$, so that in the following table $\alpha = \frac{\widehat{d}}{\widehat{d}+1}$:

The regression results are poor, especially for the second experiment, due to lack of variability in the regressor. Next, I use relative state prices densities (instead of relative prices) and regress, for example:

$$\frac{1 - b\overline{x}}{2(1 - b\overline{y})}\frac{p_y}{p_x} = \widehat{e} + \widehat{f}\frac{\pi_{y \cup z}}{\pi_z}$$

so that $\widehat{f}$ can be directly interpreted as α. The next table reports the estimates.

The estimated ambiguity parameters are always less than $\frac{1}{2}$, whenever significant, thus suggesting ambiguity aversion. This result is not surprising from a decision-

Expm	020529	020619
Alpha	.0319 (.0039)	.0415 (.0041)
Constant	−.2112 (.0100)	−.7425 (.0136)
Standard errors in parentheses.		

Table 4.3: Ambiguity coefficient estimates for weighted relative prices.

theoretic point of view, given the large volume of studies replicating the Ellsberg experiment and its variations. However, its relevance in the economic experimental literature is twofold: first, it sheds some light on previous financial experiments testing the Capital Asset Pricing model under risk. However, I would like to run an experiment involving my parameters and risk only, in order to make the comparison more direct and the calibration exercises more effective. Secondly, my results are in sharp contrast to what found by [5], who reject ambiguity aversion in experimental first-price auctions. While a deeper understanding of the differences in the theoretical and experimental setups is required, the mixed nature of the results suggests that ambiguity attitudes might be context-dependent.

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4.4 Appendix

I. OVERVIEW

The notes given here are examples. The actual number of securities and number of states may be different.

1. The markets

Your version of Marketscape consists of four different markets that will be conducted in a sequence of trading periods. In three of these markets securities will be traded. These three securities, SECURITY X, SECURITY Y, and SECURITY Z, have a one period life. That is, the SECURITIES pay a single dividend and are removed from the system at the end of the period. In the fourth market instruments called NOTES will be traded. The NOTES are very special instruments that allow you to borrow and loan money. The currency used in all markets is called francs. Each franc is worth ______ (possibly sent to you by email).

2. Timing of events

At opening of each period you will be given, as "working capital", a portfolio of units of SECURITY X, units of SECURITY Y, units of SECURITY Z and some francs (cash). The time remaining in the period will be shown at the top of the main page in Marketscape. You will be able to trade in the markets while the period is open. At the end of the period all markets will be closed. The dividend levels will be determined randomly and distributed according to the portfolio you hold at the close of the period. Your income for the period consists of the dividends on the securities you hold at the close, as well as the cash you hold at that point, minus a predetermined payment (a "loan" repayment) for the working capital you were given at the beginning of the period. This income will be recorded as your earnings for the period. Your working capital will be automatically refreshed and a new period will open. That is, there is no carryover of Securities, Notes, or francs from period to period. You are free to purchase and sell as many securities as you want within the limits imposed by your working capital. Marketscape will not allow you to purchase securities if you do not have the francs to pay for them. Furthermore, Marketscape will not allow you to sell securities unless you actually have them in inventory. That is, you cannot go short in securities and you cannot purchase securities beyond the limits of your francs. However, as will be explained below, you will be able to undertake limited borrowing in addition to the working capital.

3. Dividend Determination and Trading Of Notes (Borrowing)

The actual dividends paid by Security X, Security Y and Security Z will depend on a randomly drawn STATE. There are three possible states, referred to as X, Y, and Z. The likelihood of these states will be detailed below. Securities are named after the state in which they pay 100, and they pay 0 in the other two states. The dividend that a security pays in each state is uniquely determined from the table that follows. Notes have a special status as will be explained.

Dividend Table

	X	Y	Z
Security X	100	0	0
Security Y	0	100	0
Security Z	0	0	100
Notes	100	100	100

For example, if the state is X for some period, then the holder of Security X will receive 100 francs for each unit held at the end of that period and the holder of Security Y will receive 0 francs for each unit held at the end of that period. If the state drawn for the period is Y then the numbers are 0 and 100 respectively, and if the state is Z they are both 0.

Notes are like bonds or IOUs that allow you to borrow and loan francs. Selling a Note is like borrowing the amount of the sale price. Buying a Note is like loaning the amount of the sale price. For each Note you sell you must pay the holder 100 francs at the end of the period. This payment will automatically be deducted from your francs and dividend payments at the end of the period. That is, if you sell a

Note for 75 francs you have borrowed 75 francs and for this loan you repay a total of 100 at the end of the period. Effectively "you repay the loan" of 75 francs plus a 25 franc "interest" payment. If you buy a note for 75 francs then you have loaned the seller 75 francs until the end of the period and for this loan you will be repaid a total of 100 francs, in essence the 75 francs plus a 25 franc "interest" payment. Clearly no one should buy a Note for more than 100 francs because it means that the buyer loaned more than would be repaid. At the beginning of the period you will be given no Notes. If you sell Notes your inventory will be listed as negative, indicating that you must pay 100 francs on each of these, and if you buy Notes your inventory will become positive indicating that you will be paid 100 francs for each Note you hold. However, Marketscape will not allow you to sell over four (4) Notes (net) in any given period.

4. Determination of States

There are 3 possible states: X, Y, and Z. We start with a bin with 18 balls, 6 of which are marked X. Each of the remaining 12 balls is marked either Y or Z.

Two things to notice about the bin: first, the composition of the bin is randomly determined BEFORE the experiment. By using a random device, each of the 12 unknown balls was assigned either a Y or a Z, and this labeling will not change throughout the experiment. Second, none of the participants in todays experiment was told about the composition of the bin, and no hints will be given at any point of the experiment. All the information is common, and restricted to the bin as illustrated above.

States are drawn RANDOMLY WITHOUT REPLACEMENT, as follows. After the first period, we draw one of the balls at random. Imagine that the draw is X. The next draw, after the second period, is from the same bin, but now with 17 balls left, namely, 5 of X, and 12 of Y and Z in unknown proportions. This means that the likelihood that we draw X is smaller than before (that is, $5/17<6/18$), and the likelihood of drawing either Y or Z is bigger (that is, $12/17>12/18$) but no information is gained about the likelihood of Y alone or Z alone. Now imagine that we draw Z from the bin. The draw after the third period will be from a bin that contains 16 balls, 5 of which are marked X, and 11 of which are marked either Y or Z. The likelihood of drawing X now improved, and the likelihood of drawing either Y or Z worsened, but once again no information is gained about the likelihood of Y alone or Z alone. Etc.

5. Initial Portfolio and Working Capital

At the beginning of each period you will be given working capital, consisting of shares of Security X, shares of Security Y, shares of Security Z, and francs cash. You will not be given any Notes. The working capital you receive can be different from the working capital received by others. For this working capital you must repay a predetermined number of francs (a "working capital loan" repayment) at the end of the period. An illustration of how to compute the payoff of your working capital is provided below (see example 1).

II. HOW YOU MAKE MONEY

The following examples will help you understand the various possibilities.

Example 1. Do nothing.

Suppose you start and end with 4 units of Security X, 2 units of Security Y, and 3 units of Security Z. At the end of the period, if the state is X, Y, or Z, your dividends would be 400, 200, 300 respectively. Including your cash holding, say 200 francs, and subtracting the required working capital loan repayment for the capital advance, say 300 francs, your earnings would be 300, 100, 200 francs respectively. Suppose you start and end instead with 2 units of Security X and 7 units of Security Z (and no Y) plus 200 francs cash, for which you are asked to pay 300 francs at the end. A similar calculation will demonstrate that your earnings would be 100, -100 (loss), 600 francs respectively.

Example 2. Play it safe.

This means that you sell all of your securities and loan out the money (buy Notes) you receive in payment. Of course, in this case your earnings depend upon the prices at which you transact. Assume that you are given a working capital of 5 Securities X, 5 Securities Y, 1 Security Z, and 200 francs cash. Assume your working capital loan is 300 francs. That is, you are required to repay 300 francs for the working capital at the end of the period. Assume that you are able to sell all securities at 33 francs for Securities X, at 17 for Securities Y, and at 50 for Securities Z. This would produce 300 francs revenue. Adding the 200 francs cash with which you started gives you a total of 500 francs. This guarantees you an income of 200 francs (500 cash, minus the 300 francs repayment for working capital). Now, these 500 francs can be invested in Notes. You could buy 5 Notes if you paid the full repayment value of 100 each (of course you would make no profit in this case), but if you purchase at less than 100 you make the difference on each Note you buy. Suppose you could buy 5 Notes at 90 each (a total investment of 450 francs of your 500). Each repays 100 so you make 10 on each for a profit of 50. Your period earnings would be 200+50 = 250 francs for the period. Of course this calculation makes important assumptions about the prices you received for the securities and the prices that you paid for the Notes.

Example 3. Speculate on price changes.

Anything you buy can be resold. If you sell at a price higher than you paid you make the difference. This difference is in the form of increased francs which you can then invest in either Securities or Notes and receive the returns.

Example 4. Leveraging.

Sell all of your Securities and use the francs to purchase one Security only. In addition use your initial francs to purchase the same Security. Finally, sell all the Notes you are able and use the francs to purchase the same Security. As an exercise assume that prices are at 33 francs for X, 50 for Y, and 17 for Z, and that the price of Notes is 90. Suppose you have 2 Securities X, 3 Securities Y, and 2 Securities Z initially, and that you were given 200 francs cash. You are required to pay 300 francs at the end of the period, as compensation for the working capital. Selling all of Y and Z generates 184 francs; selling 2 Notes generates 180 francs. Together with the initial cash allocation, you now hold 564 francs in cash. With it, you can purchase 17 Securities X, at a cost of 561 francs, leaving you with 3 francs cash. At the end of the period, you will be holding 19 Securities X and 3 francs in cash. You will be required to pay 200 francs for the (short) sale of the 2 Notes, in addition to the 300 francs you pay for the working capital. If the state is X, Y, or Z, your dividends would be 1900, 0, 0, respectively; including your cash holdings of 3 francs, this would become 1903, 3, 3, respectively; the corresponding period earnings would be 1403 (gain), 497 (loss), 497 (loss), respectively.

Example 5. Partially invested hedge.

Security Y pays 100 when state Y occurs, but its likelihood is not known. Similarly for Z. However the likelihood of either Y or Z is known (as it equals 1 minus likelihood of X, which is known throughout the experiment). By purchasing and/or selling Y and/or Z so as to hold an equal number of each of these Securities will allow you to compute expected payoffs for your working capital. That is, if in the first period you have 2 Securities X, 3 Securities Y, and 3 Securities Z, your expected earnings would be 33 for each Security X (=1/3*(100+0+0)) and 67 for each pair of Y and Z (=2/3*(0+100+100)). Your total earning is 267 (on average). By not doing anything, your realized earnings will be 200, 300, 300 respectively. Example 6. Fully invested hedge. You might use your francs cash to purchase an equal number of X, Y, and Z, and also sell Notes and use the proceeds to purchase even more securities. Since each security pays dividends in a different state, and the three securities together cover for all the possible cases, by buying an equal number of X, Y, and Z your overall variability of returns is eliminated. That is, the period earnings from each triplet (1 Security X, 1 Security Y, and 1 Security Z) are 100, 100, 100, respectively, that is, 100 no matter which state is drawn.

III. HIGHLIGHTS OF PAGES AND FORMS

Order Form:

Found in the Market Summary page. Details will be found below.

Cancellations:

By clicking on the My Offers link for the appropriate markets in the Market Summary page you can cancel orders.

The Book:

Click on the name of the security and you will view all buy orders and all sell orders in that market.

Inventory Page:

This page lists your current holdings of Securities, Notes, and francs. It also displays your cumulative earnings.

History:

These links provide lists of all trades, or exclusively your trades.

Graphs:

These links provide graphs of the historical trading prices.

Dividend Summary:

This page provides the dividend payoff table for each security and detailed information of the actual state, personal end-of-period holdings and income for all previous periods.

Announcements:

Check this for messages.

IV. GLOSSARY OF IMPORTANT MARKET ELEMENTS

These are generic definitions and concepts used in Marketscape. Some of this language might not apply to the markets that are currently operating.

ORDERS—orders are accompanied by a price. When you submit an order you are saying that "you are willing to trade at the price stated or at some better price." (more)

If you submit a SELL ORDER, you are saying that you are willing to sell to anyone at the price you state, or at a higher price.

By submitting a BUY ORDER you are saying that you are willing to buy from anyone at the price you state, or at a lower price.

If no one accepts the order, it goes unfilled. Your unfilled order remains in the order book until it expires, is canceled by by you, or until it is taken by another person.

An order LIMITS the amount you receive or the amount you pay by the price you state.

ORDER QUANTITIES—You may place orders for any number of units. The computer will automatically fill orders if possible. If your order is for more units than can be bought or sold at your price, the computer will partially fill your order. Partially filled orders will remain on the order book until they are completely filled, canceled, or the market terminates.

THE SELL ORDER BOOK—is a listing of the sell orders. The orders are listed from highest order at the top of the page, down to the lowest. The lowest is the BEST sell order from the buyers' points of view and it is found at the bottom of the sell order book. Buyers will examine this book when deciding whether or not to buy. Selling always starts at the bottom of the book (the BEST) and then moves up. The market automatically matches the cheapest existing order in the sell order book with an incoming buy order to determine if a trade is possible. (more)

THE BUY ORDER BOOK—is a listing of the buy orders. The orders are listed from lowest at the bottom of the page, up to the highest. The highest is the BEST buy order from the sellers' points of view and it is found at the top of the buy order book. Sellers will examine this book when deciding whether or not to sell. Buying always starts at the top of the book (the BEST) and then moves down. The market automatically matches the highest existing order in the buy order book with an incoming limit sell order to determine if a trade is possible. (more)

Should you send a SELL ORDER with a price lower than the best buy order in the buy order book, then a trade will occur and your order will be filled at the buy price, which is to your favor. For example, the best buy order is 600, and you send in a sell order for 500. You will sell at 600 (instead of sell at 500). The situation is analogous if you send a buy order. If the best sell order is 500 and you send a buy order for 600, you will buy at 500 (instead of buy at 600).

Chapter 5 Sunspots and multiple priors: a note

In this chapter, I study the effect of sunspots in a rational expectation equilibrium model, as first modeled in Cass and Shell [1], and extend the results to a more general setup by modeling attitudes towards uncertainty. My interest in the subject was initially raised by the welfare analysis in Cass and Shell:

> "The upshot of this analysis is that in a rational expectations, general equilibrium world, the presence of extrinsic uncertainty—sunspots, waves of pessimism/optimism, and so forth—may well have real effects. The lesson for macroeconomics is that, even if one assumes the most favorable informational and institutional conditions imaginable, there may be a role for the government to stabilize fluctuations arising from seemingly noneconomic disturbances." (p. 196)

The subsequent literature on sunspots concentrated either on uniqueness of market equilibrium or on extensions to Cass and Shell's results.[1] [2] Most of the extensions explore nonconvexities in the agents' preferences, as Cass and Shell results rely on the strict concavity of the utility functional which represents preferences. My approach is different, as I model pessimism/optimism of agents as being separate by their risk attitudes. That is, while keeping concavity of the utility index, I introduce nonconvexities and non-differentiability by allowing agents to hold multiple beliefs about tomorrow's states of the world. The set of plausible beliefs captures informational conditions in the economy, while the use of a Min/Max EU operator reflects aversion/loving for uncertainty. The model in this paper is an extension of Gilboa

[1] For an example of either stream of research, see Hens [4] and Guesnerie and Laffont [3].

[2] Although in writing this chapter I was initially unaware of earlier work in this direction, I recently became aware of work of Tallon [6]. Even though the spirit of the model and the main results look similar, the proofs include some technical differences due to choices for modelling behavior towards ambiguity. Choquet EU (as in Tallon) and Minimax EU coincide when the set of possible priors is the core of a convex capacity, but neither approach is nested in the other.

and Schmeidler's [4] where a parameter α is introduced to capture agents' attitudes towards uncertainty, including optimism and pessimism as the extreme cases.

The results in this paper are mixed news for rational expectation equilibrium and its pareto optimality: the good news is that, whenever agents are pessimistic, sunspots do not matter. Furthermore, the (pareto optimal) certainty equilibrium is robust to heterogeneity in beliefs. However, whenever there exists at least one optimistic agent, then sunspots are bound to matter even in complete markets.

5.1 The model

Consider a standard, two-period, competitive exchange economy. There are H agents who can consume two goods at period $t = 1$ and in each of the $s = 1, ..., S$ possible states of the world which may occur at $t = 2$.[3] Furthermore, they can trade all the possible contingent claims at $t = 1$, so that markets are assumed to be complete. Let $x_h = [x_h(0), x_h(1), \ldots, x_h(S)]$ denote the consumption vector, where $x_h(s) = [x_h^1(s), x_h^2(s)]$ for good 1 and 2 in state s. Similarly, let $\omega_h(s)$ and $p(s)$ denote the endowment and price for every s.

Assume $u_h : \Re^2 \to \Re$ to be differentiable, strictly increasing, and strictly concave. Every agent h chooses a consumption plan x_h as to maximize his (time-additive) utility:

$$\max_{x_h} V(x_h) = \ u(x_h(0)) \ + \ Eu(x_h(1), \ldots, x_h(S))$$

where the expectation operator E is computed with respect to h's beliefs on tomorrow's states of the world. As in Gilboa and Schmeidler, I allow for agent h to hold different plausible beliefs, that is, E can be computed with respect to any belief $\pi_h \in C_h$, a closed and convex set of probability distributions over 2^S. Assuming uncertainty aversion is equivalent to the agent choosing the most pessimistic belief,

[3] The assumption about the number of commodities can be easily generalized. I will restrict my analysis to two goods only for notational simplicity.

that is, the π_h that minimizes $Eu(x_h(1), \ldots, x_h(S))$. This reflects a pessimistic or cautious attitude, as the minimum EU can be interpreted as the guaranteed level of utility for $t = 2$. In order to allow for a more general setup and capture attitudes towards uncertainty, I use a parameterized version:

$$V(x_h) = u(x_h(0)) + \alpha \max_{\pi_h \in C_h} \sum_{s=1}^{S} \pi_h(s)\, u(x_h(s)) + (1-\alpha) \min_{\pi_h \in C_h} \sum_{s=1}^{S} \pi_h(s)\, u(x_h(s))$$

so that $\alpha = 0$ corresponds to the case of ambiguity aversion. Let then V_* (V^*) represent preferences under pessimism (respectively, optimism). Next, I introduce "extrinsic uncertainty" in the economy, as modeled by Cass and Shell:

Definition 1 *A sunspot economy is defined as an economy without aggregate uncertainty, that is* $\sum_h \omega_h(s) = \sum_h \omega_h(s')$ $\forall s, s'$.

Definition 2 *A Rational Expectation Equilibrium is defined as a vector of prices* $p^* = [p^*(0), p^*(1), \ldots, p^*(S)]$ *such that*

1) given p^*, $\forall h$:

$$x_h^* = \arg\max_{x_h} V_h(x_h) \textit{ subject to } \sum_{s=0}^{S} p(s)[x_h(s) - \omega_h(s)] = 0 \textit{ and } x_h \geq 0$$

2) $\forall s$:

$$\sum_h [x_h(s) - \omega_h(s)] = 0$$

Definition 3 *Sunspots matter if there exist* $s, s' > 0$ *and* h *such that* $x_h(s) \neq x_h(s')$.

Cass-Shell's (henceforth CS) results correspond to the case of C_h being a singleton:

Lemma 10 (CS1) *In complete markets, sunspots do not matter in equilibrium.*

The assumption of common beliefs about the probability of sunspots is critical. Differing beliefs motivate trading in contingent claims:

Lemma 11 (CS2) *In complete markets, when beliefs differ across individuals, sunspots matter in equilibrium.*

In this paper I show that, whenever agents are pessimistic, CS1 is robust to heterogeneity in beliefs (which is "good news" for REE). However, whenever there exists at least one optimistic agent, then sunspots may matter even in complete markets.

5.2 Results

All the following results hold whenever markets are complete, and each agent holds multiple beliefs about tomorrow's state of the world. Also, in order to keep the notation simple and to help intuition, I will restrict attention to two states $s = 1, 2$, but the results can be easily generalized.

5.2.1 The pessimistic case

Theorem 7 *If $C_h = C$ for every h, then sunspots do not matter at equilibrium.*

Proof: (Similarly to CS) By contradiction: suppose that at equilibrium there exists an agent h such that $x_h^*(1) \neq x_h^*(2)$. In particular, without loss of generality let $x_h^*(1) < x_h^*(2)$. Let $C_h = \{\overline{\pi} \geq \pi_1 \geq \underline{\pi}\}$, with $\overline{\pi} > \underline{\pi}$. Then:

$$\begin{aligned} V_*(x_h^*) &= \min_{\pi_1 \in C_h} u_h(x_h^*(0)) + \pi_1 u_h(x_h^*(1)) + (1-\pi_1) u_h(x_h^*(2)) = \\ &= u_h(x_h^*(0)) + \overline{\pi} u_h(x_h^*(1)) + (1-\overline{\pi}) u_h(x_h^*(2)) \end{aligned}$$

Now let $\overline{x}_h = \sum_{s=1,2} \kappa(s) x_h^*(s)$, where $\kappa \in C_h$ such that $\kappa(2) > 0$. The existence of such κ is guaranteed by the assumption that C_h is not a singleton. Notice that $\overline{x}_h$ is feasible, that is, $\forall s$,

$$\begin{aligned}\sum_h \overline{x}_h(s) &= \sum_h \sum_{s=1,2} \kappa(s) x_h^*(s) = \sum_{s=1,2} \kappa(s)[\sum_h x_h^*(s)] \\ &\leq \sum_{s=1,2} \kappa(s)[\sum_h \omega_h(s)] = \sum_h \omega_h(s)\end{aligned}$$

as x_h^* is feasible and $\sum\limits_h \omega_h(s) = \sum\limits_h \omega_h(s')$ $\forall s, s'$. Furthermore:

$$\begin{aligned}V_*(\overline{x}_h) &= \min_{\pi_1 \in C_h} u_h(\overline{x}_h(0)) + \pi_1 u_h(\overline{x}_h(1)) + (1-\pi_1) u_h(\overline{x}_h(2)) = \\ &= u_h(x_h^*(0)) + u_h(\overline{x}_h) = u_h(x_h^*(0)) + u_h(\sum_{s=1,2} \kappa(s) x_h^*(s))\end{aligned}$$

By strict concavity and uncertainty aversion:

$$u_h(\sum_{s=1,2} \kappa(s) x_h^*(s)) > \overline{\pi} u_h(x_h^*(1)) + (1-\overline{\pi}) u_h(x_h^*(2))$$

so that $V_*(\overline{x}_h) > V_*(x_h^*)$, a contradiction. ∎

The basic intuition is that ambiguity aversion works in the same direction as risk aversion. More precisely, concavity of u implies that the utility of a lottery is better than the same lottery of utilities; uncertainty aversion involves computing the utility of an even better lottery, as the weights implied by κ are always "more favorable" than the ones implied by $\overline{\pi}$, the most pessimistic belief. Under ambiguity aversion, finding "better lotteries" is always possible as for any given lottery, every prior in C_h first-order stochastically dominates $\overline{\pi}$. So, as long as C_h is a non-degenerate set, ambiguity aversion adds some bite to CS1. Furthermore, it is immediate to find the conditions that make the result robust to heterogeneity of beliefs:

Theorem 8 *If $\cap C_h \neq \emptyset$, then sunspots do not matter at equilibrium.*

Proof: In order to extend the previous proof to this case, all that is needed is 1) to use the same prior κ $\forall h$, as required by feasibility; and 2) that $\kappa \in C_h$, for it to first-order stochastically dominate the most pessimistic belief $\overline{\pi}$. These two conditions are immediately satisfied whenever $\cap C_h \neq \emptyset$. ∎

Robustness of the result CS1 to heterogeneity stems from non-differentiability of the EU functional in our case. Whenever $\alpha \neq \frac{1}{2}$ (corresponding to the standard EU model), the indifference curves have a kink at the certainty line. Then, as long as the intersection of beliefs (and thus supporting prices) is non-empty, sunspots do not matter and the endowment is an equilibrium for a continuum of prices, rather than one price only.[4]

5.2.2 Allowing for optimism

On the other hand, optimism, or uncertainty loving, implies dislike for reduction of uncertainty. Axiomatically, preferences are such that "objective lotteries" make the decision maker worst off. In a sunspot economy, the presence of at least one such optimistic agent is enough to make sunspots matter at equilibrium:

Theorem 9 *If there exists an equilibrium and if at least one agent is ambiguity loving, then sunpots matter at equilibrium.*

Proof: By contradiction: suppose that at equilibrium $x^*_h(s) = \overline{x}_h$ for every s, h (including the ambiguity loving agent denoted by a) and $p^* = [p^*(1), p^*(2)]$. Let $C_a = \{\overline{\pi} \geq \pi_1^a \geq \underline{\pi}\}$, with $\overline{\pi} > \underline{\pi}$. The optimistic agent a chooses as to maximize her expected utility given by

$$V^*(x_a) = \; u(x_a(0)) \; + \; \max_{\pi_1^a \in C_a} \pi_1^a \, u(x_a(1)) + (1 - \pi_1^a) \, u(x_a(2))$$

At equilibrium, $V^*(x^*_a) = V^*(\overline{x}_a) = u(x^*_a(0)) \; + u(\overline{x}_a)$. Now construct the alternative allocation x'_a such that $x'_a(1) = \overline{x}_a + \varepsilon$, $x'_a(2) = \overline{x}_a - \varepsilon \frac{p^*(1)}{p^*(2)}$. For simplicity of notation, denote $\frac{p^*(1)}{p^*(2)}$ by ψ.Notice that x'_a sastisfies a's budget constraint. Its expected utility

[4] In the extreme case of strict ambiguity aversion, indifference curves correspond to the ones for Leontief preferences, and certainty equilibria are the only equilibria for any price vector.

can be computed as

$$V^*(x_a') = u(x_a^*(0)) + \begin{cases} \overline{\pi}u(x_a'(1)) + (1-\overline{\pi})u(x_a'(2)) & \text{for } \varepsilon > 0 \\ \underline{\pi}u(x_a'(1)) + (1-\underline{\pi})u(x_a'(2)) & \text{for } \varepsilon < 0 \end{cases}$$

Optimality of x_a^* implies $V^*(x_a^*) \geq V^*(x_a')$, that is

- when $\varepsilon > 0$:

$$\begin{aligned}
&\iff u(\overline{x}_a) \geq \overline{\pi}u(x_a'(1)) + (1-\overline{\pi})u(x_a'(2)) \\
&\iff \overline{\pi}u(\overline{x}_a) + (1-\overline{\pi})u(\overline{x}_a) \geq \overline{\pi}u(\overline{x}_a+\varepsilon) + (1-\overline{\pi})u(\overline{x}_a-\varepsilon\psi) \\
&\iff \overline{\pi}\tfrac{u(\overline{x}_a)-u(\overline{x}_a+\varepsilon)}{\varepsilon} \geq (1-\overline{\pi})\psi\tfrac{u(\overline{x}_a-\varepsilon\psi)-u(\overline{x}_a)}{\varepsilon\psi} \quad \text{(taking limit for } \varepsilon \to 0^+\text{)} \\
&\implies -\overline{\pi}u'(\overline{x}_a) \geq -(1-\overline{\pi})\psi u'(\overline{x}_a) \\
&\iff \psi \geq \tfrac{\overline{\pi}}{1-\overline{\pi}}
\end{aligned}$$

- when $\varepsilon < 0$:

$$\begin{aligned}
&\iff u(\overline{x}_a) \geq \underline{\pi}u(x_a'(1)) + (1-\underline{\pi})u(x_a'(2)) \\
&\iff \underline{\pi}u(\overline{x}_a) + (1-\underline{\pi})u(\overline{x}_a) \geq \underline{\pi}u(\overline{x}_a+\varepsilon) + (1-\underline{\pi})u(\overline{x}_a-\varepsilon\psi) \\
&\iff \underline{\pi}\tfrac{u(\overline{x}_a)-u(\overline{x}_a+\varepsilon)}{-\varepsilon} \geq (1-\underline{\pi})\psi\tfrac{u(\overline{x}_a-\varepsilon\psi)-u(\overline{x}_a)}{-\varepsilon\psi} \quad \text{(taking limit for } \varepsilon \to 0^-\text{)} \\
&\implies \underline{\pi}u'(\overline{x}_a) \geq (1-\underline{\pi})\psi u'(\overline{x}_a) \\
&\iff \psi \leq \tfrac{\underline{\pi}}{1-\underline{\pi}}
\end{aligned}$$

In the limit, the two inequalities imply that $\underline{\pi} \geq \overline{\pi}$, a contradiction. ∎

Notice that, due to nonconvexities of preferences, the existence of equilibrium is not guaranteed. Starr [5] investigates more general non-convex preferences and finds that divergence from equilibrium is bounded in a fashion independent of the number of trades. Thus, with a sufficiently large number of traders and at least one ambiguity loving agent, the equilibrium is such that sunspots always matter.

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Chapter 6 Insurance contracts and multiple priors

In the adverse selection literature, a well-known result is that the market achieves a second-best. In an effort to separate different types, the "bad type" is guaranteed his optimal outcome, whilst the "good type" has to get content with his second-best.[1] This result was first applied to competitive insurance markets by Rothschild and Stiglitz [6]. From the point of view of a consumer, the optimal contract would offer fair and full insurance, where fairness is guaranteed by free entry for insurance companies. However, full insurance is hampered by the presence of asymmetric information. Under Rothschild and Stiglitz definition of equilibrium (Cournot-Nash), only high risk types receive their optimal contract. In order to satisfy incentive compatibility constraints, low types are offered fair but partial insurance. The cost of asymmetric information is thus borne by low type agents only, as they cannot be distinguished ex-ante from high type agents.

This chapter shows that, in the presence of ambiguity, low types may benefit from ambiguity aversion of high types. By overestimating their chance of incurring a loss, high types are less willing to misrepresent themselves, so that the low types can now be guaranteed a better second-best outcome. The less information is available, the better for the low types, as "looseness" of the incentive compatibility constraints increases with ambiguity. Under the assumption that, due to data gathering, insurance companies have more precise information on the probability of loss across the population, ambiguity aversion implies a rather unintuitive social welfare result, that is, it is not beneficial to ask insurance companies to reveal their information.

In order to shed light on the individual behavior behind the model, I also investigate the optimal insurance from a decision-theoretic perspective, and find that

[1] If he had to receive his first-best, then there would be an incentive for the "bad" type to misrepresent himself

ambiguity averse agents may buy full insurance even at unfavorable odds. Differently put, they perceive any possible odds ratio as relatively more favorable, which leads high types to be less likely to imitate low types in equilibrium, makes the incentive compatibility constraints "looser," and thus explains the game-theoretic findings.

Rothschild and Stiglitz (henceforth RS) assume that all consumers are identical, except for their probability of accident. An implicit assumption is the existence of an objective (or commonly agreed) probability, that is, both consumers and insurance companies base their decisions about buying and selling contracts on the same odds. I use RS's model as a benchmark, and show how subjective variations in consumers' beliefs affect the properties of optimal contract, and the nature and existence of equilibrium. In my model, subjective variations are the result of subjective uncertainty, rather than risk, where uncertainty stems from informational and cognitive limits.

The main justification is, people may find it difficult to assess small probabilities such as the occurence of any kind of accident. So they may overstimate its chances as a result of two behavioral facts: 1) ambiguity aversion, as found in the experimental literature; 2) difficulties in assessing small numbers.[2] However, these observations rest on behavioral assumptions that require experimental or empirical investigation. So I chose not to assume ambiguity aversion and use a more general framework by extending Gilboa and Schmeidler's [4] multiple prior model to include a pessimism-optimism index α. Different values of α captures different attitudes towards uncertainty, including the optimism of smokers underestimating their probability of cancer.

On the other hand, in my model insurance companies all agree on the probability assessment for accidents, and are assumed to be right on average. The idea behind this assumption is that in reality insurance companies have enough experience and data to assess the "objective" probability of accident, for any type of risk and for any class of consumers. Or at least they can do so more accurately than any single consumer. "Objective" probability in this context can be thought as the empirical frequency of accident across the population. Notice that, despite companies' better

[2] That is, what is a small probability? It is hard to think that people would come up with numbers like .0000327. Rather they know what 1/1000 is.

accuracy in probability estimates, asymmetry of information is not eliminated from the RS model. That is, insurance companies most likely cannot determine the type of consumer they are facing when offering their contracts.

6.1 Optimal insurance

Consider an individual with income of size W, which may be subject to a random loss $L = \{0, D\}$. His income is thus $(W, W - D)$ in the two states, no accident and accident, respectively. The individual can insure himself against the accident and purchase a contract $(-\kappa_1, \widehat{\kappa}_2 - \kappa_1)$, which involves paying a premium κ_1 and being paid compensation $\widehat{\kappa}_2$ in case of an accident. With insurance, income is given by $(W - \kappa_1, W - D + \kappa_2)$, where $\kappa_2 = \widehat{\kappa}_2 - \kappa_1$ is the net compensation.

6.1.1 Demand for insurance

In general let (W_1, W_2) be the final wealth in case of no accident and in case of accident, respectively, where $W_i > 0$ for $i = 1, 2$. A consumer's preferences for income in the two states are represented by the expected utility functional:

$$EU(p_i, W_1, W_2) = (1 - p_i)u(W_1) + p_i u(W_2)$$

where u represents the utility of money income and p_i the subjective probability of an accident. All consumers are identical in all respects, except for their probability of having an accident, and u is assumed to be increasing and (unless differently stated) strictly concave. For simplicity, assume $u(0) = 0$.

Without insurance, a consumer is endowed with income $E = (W, W - D)$. Alternatively, he can purchase at most one insurance contract $\kappa = (-\kappa_1, \kappa_2)$, which is worth $V(p_i, \kappa) = EU(p_i, W - \kappa_1, W - D + \kappa_2)$. Among all the contracts offered, the individual chooses the one yielding maximum expected utility, given his individual rationality constraint:

$$\max_{\kappa} V(p_i, \kappa) \quad s.t. \quad V(p_i, \kappa) \geq V(p_i, 0)$$

Notice that $W_1 = W_2$ corresponds to full insurance (no risk and no uncertainty). Given that $\frac{1-p_i}{p_i}$ is the marginal rate of substitution of wealth across states along the full insurance line, "fair odds" from the insured's point of view correspond to the case of the insurance company making no profits ex-ante, that is $\frac{1-p_i}{p_i} = \frac{\kappa_2}{\kappa_1}$, the individual return on insurance.

6.1.2 Information and ambiguity

Ambiguity is formalized by a set of priors $p_i \in C = [\underline{p}_i, \overline{p}_i]$ to represent the insured's beliefs. While studying individual choice for insurance, I can drop the index i and keep notation simpler for the next section. In order to capture attitudes towards ambiguity, I use a parameterized version of Gilboa and Schmeidler's multiple priors model, by including Hurwicz's α index for pessimism:

$$\begin{aligned} MEU_\alpha(C, W_1, W_2) &= \{\alpha \min_{p \in C} + (1-\alpha) \max_{p \in C}\} \circ \{EU(p, W_1, W_2)\} = \\ &= \alpha \min_{p \in C} [(1-p)u(W_1) + pu(W_2)] + (1-\alpha) \max_{p \in C} [(1-p)u(W_1) + pu(W_2)] = \\ &= \begin{cases} EU(\alpha \underline{p} + (1-\alpha)\overline{p}, W_1, W_2) & W_1 < W_2 \\ u(W_1) & W_1 = W_2 \\ EU(\alpha \overline{p} + (1-\alpha)\underline{p}, W_1, W_2) & W_1 > W_2 \end{cases} \end{aligned}$$

For the natural case of $W_1 > W_2$, $\alpha = 1$ ($\alpha = 0$) corresponds to ambiguity aversion (resp. loving), as among all the plausible beliefs it assigns the highest (resp. lowest) probability $\overline{p}$ to an accident occuring. Under the assumption that the objective probability $\widetilde{p}$ of accident belongs to the priors support, there exists some degree of pessimism-optimism $\widetilde{\alpha}$ whose representing belief corresponds to $\widetilde{p}$. Just to keep things simple, I will assume that the support is symmetric around $\widetilde{p}$ so that $\widetilde{\alpha} = \frac{1}{2}$.

Furthermore, $\widetilde{\alpha}$ corresponds to EU, to be interpreted as absence of ambiguity or neutrality to ambiguity.

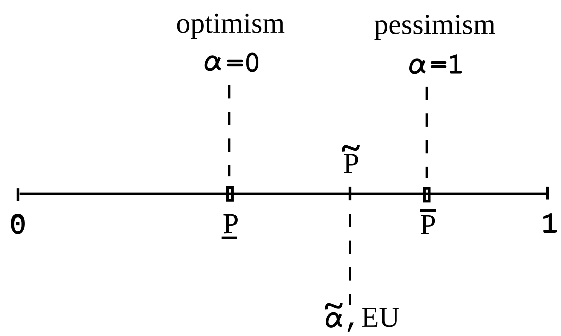

Figure 6.1: Ambiguity and attitudes for $W_1 > W_2$.

From now on, I will refer to the case of $\alpha > \frac{1}{2}$ as ambiguity aversion, as any $\alpha > \frac{1}{2}$ implies a higher probability of loss than $\widetilde{p}$.

6.1.3 The benchmark model

I will use RS model as a benchmark. Denote the fair odds by $\varphi = \frac{1-\widetilde{p}}{\widetilde{p}}$. At fair odds, it is optimal to fully insure, which corresponds to contract F in Figure 6.2. If the contract is offered at odds better (worse) than fair, that is, if $\frac{\kappa_2}{\kappa_1} > \frac{1-\widetilde{p}}{\widetilde{p}}$ (resp. $<$), then it is optimal to overinsure (resp. underinsure), as illustrated by contract O (resp. U) in Figure 6.2.

6.1.4 Optimal insurance under ambiguity

This section assumes ambiguity aversion, that is, $\alpha > \frac{1}{2}$. This can be done without loss of generality as the case of $\alpha < \frac{1}{2}$ simply obtains symmetric results. For notational simplicity, define $p_* = \alpha\underline{p} + (1-\alpha)\overline{p}$ and $p^* = \alpha\overline{p} + (1-\alpha)\underline{p}$, so that

$$MEU_{\alpha>\frac{1}{2}}(C, W_1, W_2) = \begin{cases} EU(p_*, W_1, W_2) & W_1 < W_2 \\ u(W_1) & W_1 = W_2 \\ EU(p^*, W_1, W_2) & W_1 > W_2 \end{cases}$$

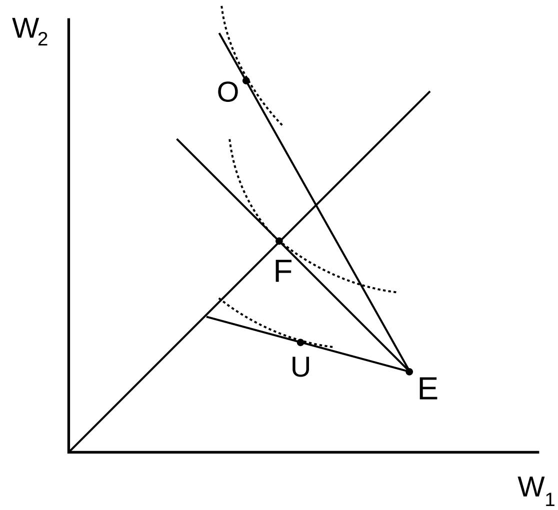

Figure 6.2: Optimal contract F in the benchmark model.

Since it is always the case that $p_* < \widetilde{p} < p^*$, it is optimal to fully insure at fair odds and furthermore at any odds between $\frac{1-p^*}{p^*}$ and $\frac{1-p_*}{p_*}$. This result intuitively follows from the shape of the indifference curves under MEU, and in particular from the kink along the certainty line. Indifference curves are flatter (resp. steeper) than EU ones for any point below (resp. above) the certainty line, as illustrated in Figure 6.3 where the thick line refers to MEU and the dotted line to EU.

Also, this result is similar in spirit to the analysis of Segal and Spivak [9], as uncertainty aversion as modeled here implies their notion of first-order risk aversion.

6.1.5 Maximal premium and optimal coverage under ambiguity

In this section I formalize the considerations made so far on individual behavior and characterize the choice of optimal insurance in a decision theoretic setup that follows Mossin [5]. Assume for this section $W_1 > W_2$, so that for any α:

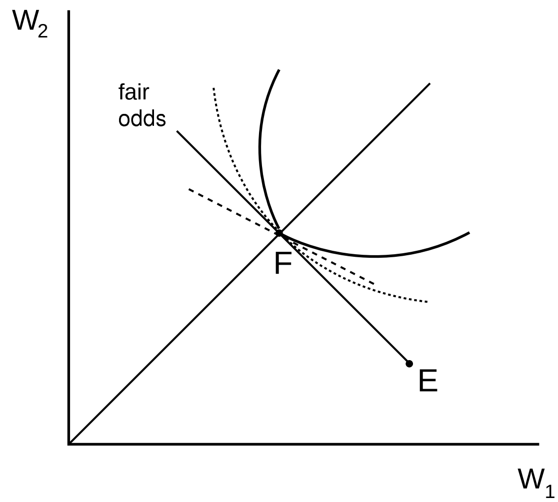

Figure 6.3: Optimal contract F under ambiguity aversion.

$$\begin{aligned} MEU_\alpha(C, W_1, W_2) &= EU(p^*, W_1, W_2) = \\ &= [1 - \alpha\overline{p} - (1-\alpha)\underline{p}]u(W_1) + [\alpha\overline{p} + (1-\alpha)\underline{p}]u(W_2) \end{aligned}$$

As above, final income for an individual is either W or $W - L$, in case of a loss L. Let us now assume that, when buying insurance, a consumer can specify the desired amount of coverage C or, equivalently, the desired proportion of coverage c. In the terminology of the previous sections, $c = 0$ is the default option of no insurance, while $c = 1$ corresponds to buying full insurance. I assume that $c \in [0, 1]$, which is not necessary for the following results but sensible.

Both the premium and the company's compensation in case of loss will now be in proportion to c. That is, if π is the premium rate, the individual's final wealth after buying an insurance contract is given by

$$W_{1,2} = \begin{cases} W - \pi c L & \text{with probability } 1 - p^* \\ W - L + (1-\pi)cL & \text{with probability } p^* \end{cases}$$

where πcL is the insurance premium, and cL the compensation from the company.

Also, as defined in previous sections, the premium is actuarially fair iff $\pi = \widetilde{p}$. Notice the one-to-one correspondence between income pairs (W_1, W_2) (the domain of EU functional) and insurance linear contracts (c, π) (the domain of V functional).

Theorem 10 *For a risk-neutral consumer, the maximal premium* $\widehat{\pi}$ *for full insurance is* p^*.

Proof: By definition, the maximal premium $\widehat{\pi}$ is the maximum premium a consumer is willing to pay for the insurance, that is, the premium that makes him indifferent between insuring or not:[3]

$$V(1, \widehat{\pi}) = V(0, \widehat{\pi})$$

where

$$\begin{aligned} V(1, \pi) &= (1 - p^*)u(W - \pi L) + p^* u(W - \pi L) = u(W - \pi L) \\ V(0, \pi) &= (1 - p^*)u(W - L) + p^* u(W) \end{aligned}$$

Under risk-neutrality, we can explicitly solve for $\widehat{\pi}$, which gives the result: $\widehat{\pi} = p^*$. ∎

Under risk-aversion all that can be said is $\widehat{\pi} > p^*$, unless a specific functional form for u is specified (see Figure 6.4). The results in Mossin thus easily translate to my setup when substituting subjective beliefs for the objective ones.

Theorem 11 *Optimal coverage is*

- *full insurance whenever* $p^* > \pi$;
- *no insurance whenever* $p^* \leq \frac{\pi u'(W)}{(1-\pi)u'(W-L)+\pi u'(W)} \stackrel{def}{=} \gamma$;
- *partial insurance for the cases in between, that is* $\gamma \leq p^* \leq \pi$.

[3]Existence of such premium is one of the axioms of von Neumann-Morgenstern Expected Utility theory.

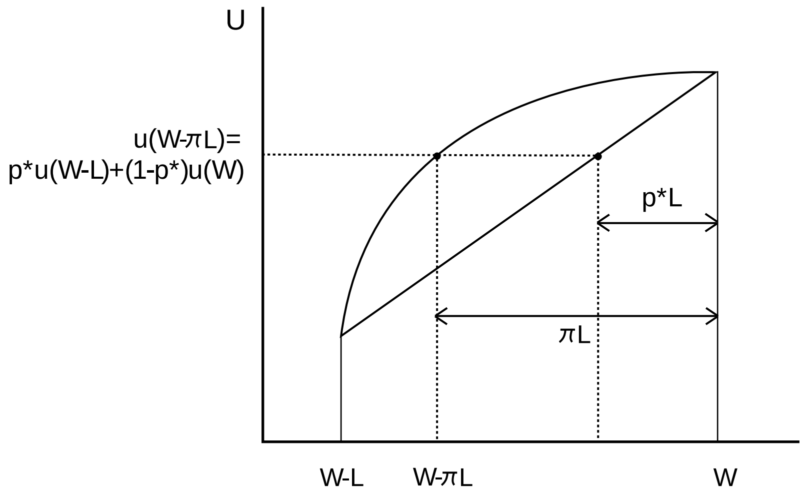

Figure 6.4: Maximal premium under risk aversion.

Proof: The optimal coverage is a solution to the problem:

$$\max_c (1-p^*)u(W-\pi cL) + p^*u(W-L+(1-\pi)cL) \text{ subject to } 0 \leq c \leq 1$$

SOC: satisfied by strict concavity of u, which guarantees a unique maximum

FOC:

$$\frac{\partial}{\partial c} = p^*(1-\pi)Lu'(W_2) - (1-p^*)\pi Lu'(W_1) = 0$$

For full insurance to be optimal:

$$\frac{\partial}{\partial c}|_{c=1} \geq 0$$

that is

$$p^*(1-\pi)Lu'(W-\pi L) - (1-p^*)\pi Lu'(W-\pi L) \geq 0$$

implying, as u is increasing and $L > 0$:

$$p^* > \pi$$

Similarly, for no insurance to be optimal:

$$\frac{\partial}{\partial c}|_{c=0} \leq 0$$

which is equivalent to:

$$p^*(1-\pi)Lu'(W-L) - (1-p^*)\pi Lu'(W) \leq 0$$

implying, as $L > 0$:

$$p^* \leq \frac{\pi u'(W)}{(1-\pi)u'(W-L) + \pi u'(W)} \stackrel{def}{=} \gamma$$

Partial coverage is optimal for the cases in between, that is, whenever $\gamma \leq p^* \leq \pi$.

∎

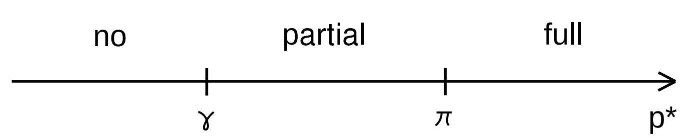

Figure 6.5: Optimal insurance and subjective beliefs.

Notice that γ can be interpreted in relation to perceived contract prices. More precisely $\frac{1}{\gamma}$ is equal to one plus the marginal rate of substitution according to the price implicit in he insurance contract.

Some interesting lessons from this exercise call for empirical investigations:

- even at unfair price $\pi > \widetilde{p}$, there are some consumers who are pessimistic enough to buy full insurance. An example would be consumers who purchase insurance on flight accidents.

- at fair price $\pi = \widetilde{p}$, there exist risk-averse agents who optimistically do not fully insure. An example might be smokers and their optimism in underestimating their probability of getting cancer.

6.2 Market equilibrium

6.2.1 Supply of insurance

As in RS, I assume that insurance companies are risk-neutral and maximize their profits, that is, they offer contracts in order to:

$$\max_{\kappa} \Pi(p, \kappa) = (1-p)\kappa_1 - p\kappa_2$$

When markets are competitive, in the sense that there is free entry, the solution to maximization problem is given by

$$\Pi = 0 \Longleftrightarrow \frac{1-p}{p} = \frac{\kappa_2}{\kappa_1}$$

so that, whenever consumer types can be observed, each type would receive full and fair insurance.

6.2.2 Information

Next, I model ambiguity as before, by assuming that each individual knows $p_i \in [\underline{p}_i, \overline{p}_i]$. On the other hand, the insurance company knows $\widetilde{p}$, to be thought as the empirical frequency (alternatively, the objective distribution) of accident. The reason behind this asymmetry in precision of information is that the insurance company can collect statistical data about the occurrence of accidents across the population, and thus come up with a more precise estimate for the objective probability of accident. An example would be to estimate the average probability, so that the insurance is right on average, but not necessarily so. A more general and reasonable assumption is for $\widetilde{p}$ to be admissible, in the sense that it belongs to each consumer's support. This implies that no agent is ever "too wrong" as not to include the empirical frequency of accident among his plausible beliefs.

As in RS, there are two types of agents, distinguished by their probability of having an accident: the low type $i = L$, and the high type $i = H$. To extend this assumption to my model, all that is needed is to make sure the labelling of types is consistent, that is, $\underline{p}_L < \underline{p}_H, \overline{p}_L < \overline{p}_H$. Notice that this condition implies that, for any fixed α, $p^*_L < p^*_H, p^*_L < p^*_H$. Finally, let λ denote the proportion of high types in the population.

6.2.3 Rothschild and Stiglitz's equilibrium under ambiguity

The benchmark model corresponds to the beliefs support being a singleton: $[\underline{p}_i, \overline{p}_i] = \{p_i\}$, for $i = H, L$. Just for reference, I report RS main results:

- RS1. There cannot be a pooling equilibrium.
- RS2. The set (κ^H, κ^L), as illustrated in Figure 6.6, is the only possible equilibrium.
- RS3. There may be no equilibrium.

For now, assume the existence of a separating equilibrium and ambiguity aversion, that is, $\alpha < \frac{1}{2}$. Indifference curves for both types are kinked at the certainty line as in Figure 6.3. Notice first that ambiguity aversion for low types does not affect the equilibrium, as κ_L is determined by the intersection of high types' indifference curves with low types' fair odds line. For high types, κ_H is still an equilibrium, but ambiguity aversion makes incentive compatibility constraints a bit "looser," so that κ_L moves up along the fair odds line. Even though low types are still far from their optimal full insurance, they are better off by the presence of ambiguity in the economy.

Under the assumption of α being the same across the population, RS1 still holds true. For a pooling equilibrium to exist, low types need to be more ambiguity averse

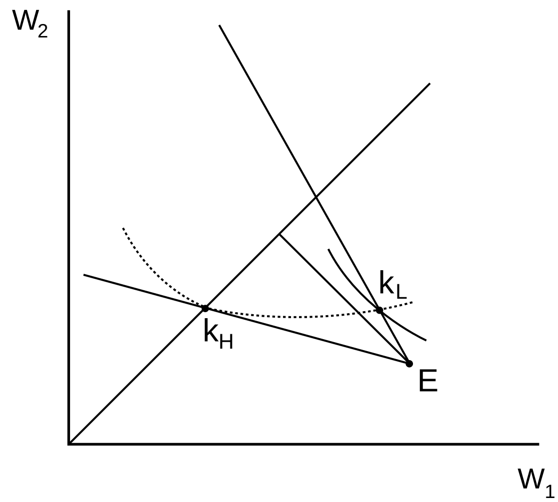

Figure 6.6: Equilibrium in Rothschild and Stiglitz.

than high types to the point that their indifference curves are flatter than the high type's ones at any point below the certainty line. Even thought there is no support for any assumption about relative differences across types, the opposite case (the high types being more averse) might seem more reasonable when considering the width of the support as a measure of ambiguity.

RS2 is true, but its validity is inversely related to the amount of ambiguity in the economy. The more ambiguity, the less likely the separating equilibrium to exist. The intuition stems from the limit case of maximum ambiguity: $\underline{p} = 0$ and $\overline{p} = 1$. In this case, indifference curves are shaped as for Leontief preferences, and it always pays to pool.

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