# SOME PROBLEMS IN THE STEADY MOTION OF VISCOUS, INCOMPRESSIBLE FLUIDS; WITH PARTICULAR REFERENCE TO A VARIATION PRINCIPLE

Thesis by

C. B. Millikan

In partial fulfillment of the requirements
for the degree of Doctor of Philosophy

California Institute of Technology

Pasadena, California

1928

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## ABSTRACT

A discussion of the general equations for steady motion of a viscous incompressible fluid from the point of view of a minimum or variation principle has, as far as the writer is aware, never been given. Helmholtz, in a classical paper[1] " On the Theory of a Stationary Flow in Viscous Fluids ", has shown that if the quadratic terms in velocity be neglected, if the velocities at the boundaries of a singly-connected region be kept constant, and if the external forces have a single-valued potential, then the motion is such that the variation of the energy dissipation in the region under consideration is zero. We shall refer to this work in more detail later. Korteweg subsequently showed that under the above conditions the steady motion which is set up is unique and is stable, i.e. the dissipation for this motion is an absolute minimum. Rayleigh[3] still later showed that the dissipation is an absolute minimum whenever

$$\nabla^2 \bar{q} = \nabla H$$

where $\bar{q}$ is the vector velocity of the fluid, and H is a single-valued function subject to the condition

$$\nabla^2 H = 0$$

In this case there is no restriction upon the magnitude of the velocity. As far as the author is aware this represents practically the entire extent of the work to date on the application of a minimum principle to the steady motion of a viscous fluid.

In the present work we discuss the following problem: given an incompressible, viscous fluid with fixed ( if any ) boundaries, to find a function L, such that if we set

$$\delta \int_V L \, d\tau = 0$$

the Eulerian equations corresponding to this condition are exactly the Navier-Stokes equations for the motion of the fluid. The integral is ( for three-dimensional cases ) a volume integral and $d\tau$ represents the volume element. We impose, of course, the customary restriction that the variation of the velocity is taken to be zero at the boundaries of the region considered. We shall for simplicity refer to such function L as a LaGrangian function, in spite of the fact that the term is not strictly accurate.

In the first section we set up a LaGrangian function by generalizing the considerations of Helmholtz relative to slow motion, and from the resulting Eulerian equations are led to a proof of the following theorem:

" If L be restricted to be a function of the velocity components and their first order space derivatives only; then it is impossible to find any such L which will give the general equations for steady flow of an incompressible fluid through the application of the variation principle described above". The conditions which must be imposed on the motion in order that it may correspond to a variation principle, involving a LaGrangian function of this type, are discussed, and it is shown that all the cases of steady motion which have thus far been discovered satisfy these conditions. The possible physical consequences relative to the existence of steady motion are mentioned.

In the second section the LaGrangian functions are found for the cases of plane laminar motion, and Poiseuille flow through a circular tube.

The third and fourth sections do not strictly form a part of the general variation problem, but the results obtained in them are used in the subsequent section, and have also a certain amount of interest on their own account. In the third section formulae are given for the transformation of certain expressions from vector to curvilinear coordinate form, and a vector expression independent of coordinate systems is deduced for the dissipation function. In the fourth section a new proof is given of a result obtained by Hamel in a very beautiful paper[4)] recently published. This paper deals with the two-dimensional flow of an incompressible viscous fluid where the streamlines coincide with logarithmic spirals, and thepresent considerations appear to be a little simpler than those used by Hamel.

In the fifth section the variation problem for logarithmic spiral flow is discussed, and the LaGrangian function is exhibited.

The vector method is used whenever it appears convenient, and in such cases the notation is that of Gibbs, involving the operator $\nabla$ . Vector quantities are denoted by a bar placed above the symbol. The magnitude of a vector is denoted by omitting the bar. We shall throughout consider no body forces, since, if such forces do occur and have a single-valued potential, then they may be taken account of in the pressure terms, without in any way altering the form of the equations.

SECTION 1

We have, for steady motion of an incompressible, viscous fluid, the Navier-Stokes equations of motion, which may be combined into the single vector equation

(1) $$\rho(\bar{q}\cdot\nabla)\bar{q} + \nabla p = \mu\nabla^2\bar{q}$$

where $\bar{q}$ represents the vector velocity of the fluid, p the pressure, $\rho$ the density, and $\mu$ the coefficient of viscosity. In addition we have the equation of continuity

(2) $$\nabla\cdot\bar{q} = 0$$

Following Helmholtz we define P as the rate at which work is done by the external forces ( in our case pressure and viscosity forces ) upon the fluid in the particular volume region we happen to discuss, and Q as the rate of dissipation of energy by friction inside the region. We introduce a new quantity R which we define to be the rate at which kinetic energy is carried out of the region across the boundary. We wish later to discuss the values of the above quantities per unit volume, and hence define the " density functions " $\mathbb{P}, \mathbb{R}, \Phi$ as follows

(3) $$P = \int_V \mathbb{P}\, d\tau$$
$$R = \int_V \mathbb{R}\, d\tau$$
$$Q = \int_V \Phi\, d\tau$$

where V is the volume region considered, and $d\tau$ is the volume element. $\Phi$ is usually called the " dissipation function " and is given by (see Lamb,V,p.549)

(4) $$\Phi = \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right\}$$

where x,y,z, are Cartesian coordinates and u,v,w, are the corresponding velocity components. This is the general expression for $\Phi$ for any viscous fluid; the condition of incompressibility has not been introduced in its derivation.

In Helmholtz' case of slow motion, squares of velocity are neglected and hence $R=0$. Hence in this case the equation of conservation of energy is

$$P-Q=0$$

Helmholtz showed that in this case, even if there are immersed bodies in the fluid so that the boundary conditions are less specified than ours, if one writes

$$\delta\left(P-\tfrac{1}{2}Q\right)=0$$

one is led to the equations of motion with quadratic terms omitted

$$\nabla p=\mu\nabla^2\bar{q}$$

In our more general case the equation of conservation of energy becomes

(5) $$P-Q-R=0$$

and one is led to try as a variation principle

(6) $$\delta\left(P-\tfrac{1}{2}Q-\gamma R\right)=0$$

where $\gamma$ is a numerical coefficient to be empirically determined later. We remark first that P may be written as a surface integral extended over the boundary of the region. Hence after performing the variation we get from this term another surface integral with components of $\delta\bar{q}$ as multiplying factors. But in Helmholtz' case if we rule out immersed bodies, and in our case, the variation of velocity is assumed to be zero at the boundary. Hence the term P contributes nothing to the Eulerian equations and may be entirely omitted in the evaluation of (6). We remark next that our variation is to be taken subject to the restriction given by the continuity equation (2).

We introduce this restriction in the customary manner, namely through the use of a LaGrangian undetermined multiplier. In view of these remarks and using equations (3) we may write equation (6) in the following form

(7) $$\delta \iint\limits_V \left[\tfrac{1}{2}\Phi + \gamma \mathbb{R} - \lambda \nabla\cdot\bar{q}\right] d\tau = \delta \iint\limits_V L' d\tau$$

where $\lambda$ is the LaGrangian undetermined multiplier.

In evaluating this expression it is convenient to use Cartesian coordinates x,y,z, with the corresponding velocity components u,v,w,. The expression for $\Phi$ is then given by equation (4) and $\mathbb{R}$ remains to be evaluated. From the definition of R we have, where S is the bounding surface of the region V, ds is the surface element, and $\bar{n}$ is a unit vector in the direction of the outwardly drawn normal to ds,

$$R = \frac{\rho}{2}\int\limits_S q^2 \bar{n}\cdot\bar{q}\, ds = \frac{\rho}{2}\int\limits_S \bar{n}\cdot(q^2\bar{q})\, ds$$

and applying the divergence theorem

$$R = \frac{\rho}{2}\int\limits_V \nabla\cdot(q^2\bar{q})\, d\tau = \frac{\rho}{2}\iint\limits_V \left(q^2\nabla\cdot\bar{q} + \bar{q}\cdot\nabla q^2\right) d\tau$$

and hence because of the continuity equation

$$R = \frac{\rho}{2}\int\limits_V \bar{q}\cdot\nabla q^2\, d\tau$$

(8) $$\mathbb{R} = \frac{\rho}{2}\,\bar{q}\cdot\nabla q^2$$

In Cartesian coordinates this gives

$$\mathbb{R} = \frac{\rho}{2}\left\{u\frac{\partial u^2}{\partial x} + u\frac{\partial v^2}{\partial x} + u\frac{\partial w^2}{\partial x} + v\frac{\partial u^2}{\partial y} + v\frac{\partial v^2}{\partial y} + v\frac{\partial w^2}{\partial y} + w\frac{\partial u^2}{\partial z} + w\frac{\partial v^2}{\partial z} + w\frac{\partial w^2}{\partial z}\right\}$$

$$= \rho\left\{u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + uw\frac{\partial w}{\partial x} + uv\frac{\partial u}{\partial y} + v^2\frac{\partial v}{\partial y} + vw\frac{\partial w}{\partial y} + uw\frac{\partial u}{\partial z} + vw\frac{\partial v}{\partial z} + w^2\frac{\partial w}{\partial z}\right\}$$

Hence the function L' defined in (7) becomes

$$L' = \mu\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right\}$$

$$+ \gamma\rho\left\{u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + uw\frac{\partial w}{\partial x} + uv\frac{\partial u}{\partial y} + v^2\frac{\partial v}{\partial y} + vw\frac{\partial w}{\partial y} + uw\frac{\partial u}{\partial z} + vw\frac{\partial v}{\partial z} + w^2\frac{\partial w}{\partial z}\right\}$$

$$- \lambda\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right\}$$

The Eulerian equations which are given by equation (7) are

$$\frac{\partial}{\partial x}\left(\frac{\partial L'}{\partial \frac{\partial u}{\partial x}}\right)+\frac{\partial}{\partial y}\left(\frac{\partial L'}{\partial \frac{\partial u}{\partial y}}\right)+\frac{\partial}{\partial z}\left(\frac{\partial L'}{\partial \frac{\partial u}{\partial z}}\right)-\frac{\partial L'}{\partial u}=0$$

with two analogous equations for v and w. Substituting our expression for L' into this equation we have

$$\mu\left\{2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right\}$$
$$+\gamma\rho\left\{u\frac{\partial v}{\partial y}+u\frac{\partial w}{\partial z}-v\frac{\partial v}{\partial x}-w\frac{\partial w}{\partial x}\right\}-\frac{\partial\lambda}{\partial x}=0$$

or

$$\mu\left\{\nabla^2 u+\frac{\partial}{\partial x}(\nabla\cdot\bar{q})\right\}+\gamma\rho\left\{u\,\nabla\cdot\bar{q}-\frac{1}{2}\frac{\partial q^2}{\partial x}\right\}-\frac{\partial\lambda}{\partial x}=0$$

and introducing the continuity equation

$$\mu\nabla^2 u-\gamma\rho\frac{1}{2}\frac{\partial q^2}{\partial x}-\frac{\partial\lambda}{\partial x}=0$$

and finally, combining with the two analogous equations deduced from the other two Eulerian equations, we have the vector equation

(9) $$\gamma\rho\frac{1}{2}\nabla q^2+\nabla\lambda=\mu\nabla^2\bar{q}$$

If now we change notation and write p for $\lambda$ and if we take our numerical coefficient $\gamma$ to be 1, then equation (9) is just the Navier-Stokes equation (1) except that $(\bar{q}\cdot\nabla)\bar{q}$ is replaced by $\frac{1}{2}\nabla q^2$.

Hence whenever conditions are imposed on the motion, of such a nature that the terms $(\bar{q}\cdot\nabla)\bar{q}$ are replaced by $\frac{1}{2}\nabla q^2$, then in such cases the equations of motion may be deduced from the variation principle given in equation (6). Since we have the general vector relation

$$(\bar{q}\cdot\nabla)\bar{q}=\frac{1}{2}\nabla q^2-\bar{q}\times\operatorname{curl}\bar{q}$$

we see that the condition just mentioned is satisfied in the more or less trivial case in which $\bar{q} \times \operatorname{curl} \bar{q} = 0$, which implies either that the vorticity is everywhere zero, or that it is everywhere parallel to the velocity. We shall later discuss several other more interesting cases in which the condition is satisfied. At present we wish to continue with our more general considerations.

If we express $(\bar{q}\cdot\nabla)\bar{q}$ in Cartesian form we have

$$\left[(\bar{q}\cdot\nabla)\bar{q}\right]_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

(similarly for y and z components); and hence it appears that the terms in the equations of motion which the variation principle does not give are

(9') $$v\frac{\partial u}{\partial y},\ w\frac{\partial u}{\partial z},\ \text{and}\ w\frac{\partial v}{\partial z},\ u\frac{\partial v}{\partial x},\ u\frac{\partial w}{\partial x},\ v\frac{\partial w}{\partial y}$$

We are therefore led to a consideration of the possibility of obtaining terms of this form from any LaGrangian function involving the velocity components and their first order space derivatives. It will suffice to discuss only one of the terms mentioned, together with the single Eulerian equation corresponding to it. Then by cyclic substitution our results can be extended to all of the other terms. We state the problem as follows.

"Given the Eulerian equation

$$E.F. = \frac{\partial}{\partial x}\left(\frac{\partial L_1}{\partial \frac{\partial u}{\partial x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L_1}{\partial \frac{\partial u}{\partial y}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial L_1}{\partial \frac{\partial u}{\partial z}}\right) - \frac{\partial L_1}{\partial u} = 0$$

where we shall call the left side of this equation the " Eulerian Function ", and denote it for convenience by E.F. To find a function

(10) $$L_1 = L_1\left(u, v, w, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)$$

such that if we introduce this function into the Eulerian equation, we get

(11) $$E.F. = v\frac{\partial u}{\partial y} + \chi = 0$$

where $\chi$ is a sum of any of the other terms in the x component of the

Navier-Stokes equation (1)."

It is obvious at once, since $V\frac{\partial u}{\partial y}$ can not be obtained from any expression of the form $\frac{\partial f_1}{\partial x}$ or $\frac{\partial f_2}{\partial x}$ where $f_1$ and $f_2$ are any functions of the velocities and their space derivatives, that for the present purpose the Eulerian function may be simplified and written

$$E.F. = \frac{\partial}{\partial y}\left(\frac{\partial L_1}{\partial \frac{\partial u}{\partial y}}\right) - \frac{\partial L_1}{\partial u}$$

Consider first

(12) $$L_1 = u^{\alpha} v^{\beta} w^{\gamma} \left(\frac{\partial u}{\partial x}\right)^{\alpha_1} \left(\frac{\partial u}{\partial y}\right)^{\alpha_2} \left(\frac{\partial u}{\partial z}\right)^{\alpha_3} \left(\frac{\partial v}{\partial x}\right)^{\beta_1} \left(\frac{\partial v}{\partial y}\right)^{\beta_2} \left(\frac{\partial v}{\partial z}\right)^{\beta_3} \left(\frac{\partial w}{\partial x}\right)^{\gamma_1} \left(\frac{\partial w}{\partial y}\right)^{\gamma_2} \left(\frac{\partial w}{\partial z}\right)^{\gamma_3}$$

where the exponents are any real numbers.

The computation of the Eulerian function for this $L_1$ is given symbolically in the accompanying Table. Referring to it we see that there are four possible combinations of values of $\beta$ and $\alpha_2$ which could give $V\frac{\partial u}{\partial y}$.

1) $\beta = \alpha_2 = 1$ In this case the terms in the bracket multiplying $V\frac{\partial u}{\partial y}$ cancel and the entire term involving $V\frac{\partial u}{\partial y}$ vanishes.

2) $\beta = 1 \quad \alpha_2 = 3$ In this case there is a factor $\frac{\partial^2 u}{\partial y^2}$ multiplying $V\frac{\partial u}{\partial y}$ which cannot be made to disappear. Hence this combination does not give us an expression of the form of (11)

3) $\beta = \alpha_2 = 2$ Here the additional conditions that we get a term of the form $V\frac{\partial u}{\partial y}$ are

$$\alpha = \gamma = \alpha_1 = \alpha_3 = \beta_1 = \beta_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0 \qquad \beta_2 = -1$$

Hence

$$L_1 = \frac{V^2\left(\frac{\partial u}{\partial y}\right)^2}{\frac{\partial v}{\partial y}}$$

# Computation of $E.F. = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \frac{\partial u}{\partial y}}\right) - \frac{\partial L}{\partial u}$

## for

$$L = u^{\alpha} v^{\beta} w^{\gamma} \frac{\partial u}{\partial x}^{\alpha_1} \frac{\partial u}{\partial y}^{\alpha_2} \frac{\partial u}{\partial z}^{\alpha_3} \frac{\partial v}{\partial x}^{\beta_1} \frac{\partial v}{\partial y}^{\beta_2} \frac{\partial v}{\partial z}^{\beta_3} \frac{\partial w}{\partial x}^{\gamma_1} \frac{\partial w}{\partial y}^{\gamma_2} \frac{\partial w}{\partial z}^{\gamma_3}$$

| Function | $v$ | $\frac{\partial u}{\partial y}$ | Constant | $u$ | $w$ | $\frac{\partial u}{\partial x}$ | $\frac{\partial u}{\partial z}$ | $\frac{\partial v}{\partial x}$ | $\frac{\partial v}{\partial y}$ | $\frac{\partial v}{\partial z}$ | $\frac{\partial w}{\partial x}$ | $\frac{\partial w}{\partial y}$ | $\frac{\partial w}{\partial z}$ | Higher Derivs. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1 =$ | $\beta$ | $\alpha_2$ | $1$ | $\alpha$ | $\gamma$ | $\alpha_1$ | $\alpha_3$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | |
| $\frac{\partial L}{\partial u} =$ | | | $\alpha$ | $-1$ | | | | | | | | | | |
| $\frac{\partial L}{\partial \frac{\partial u}{\partial y}} =$ | | $-1$ | $\alpha_2$ | | | | | | | | | | | |
| $\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \frac{\partial u}{\partial y}}\right) =$ | $-1$ | $-1$ { | $\alpha_2 \beta$ | | | | | | $+1$ | | | | | } |
| $+$ | | { | $\alpha_2 \alpha$ | $-1$ | | | | | | | | | | } |
| $+$ | | $-1$ { | $\alpha_2 \gamma$ | | $-1$ | | | | | | | $+1$ | | |
| | | $+$ | $\alpha_2 \alpha_1$ | | | $-1$ | | | | | | | | $\partial^2 u/\partial x \partial y$ |
| | | $+$ | $\alpha_2 \alpha_3$ | | | | $-1$ | | | | | | | $\partial^2 u/\partial y \partial z$ |
| | | $+$ | $\alpha_2 \beta_1$ | | | | | $-1$ | | | | | | $\partial^2 v/\partial x \partial y$ |
| | | $+$ | $\alpha_2 \beta_2$ | | | | | | $-1$ | | | | | $\partial^2 v/\partial y^2$ |
| | | $+$ | $\alpha_2 \beta_3$ | | | | | | | $-1$ | | | | $\partial^2 v/\partial y \partial z$ |
| | | $+$ | $\alpha_2 \gamma_1$ | | | | | | | | $-1$ | | | $\partial^2 w/\partial x \partial y$ |
| | | $+$ | $\alpha_2 \gamma_2$ | | | | | | | | | $-1$ | | $\partial^2 w/\partial y^2$ |
| | | $+$ | $\alpha_2 \gamma_3$ | | | | | | | | | | $-1$ | $\partial^2 w/\partial y \partial z$ } |
| $+$ | | $-2$ { | $\alpha_2(\alpha_2 - 1)$ | | | | | | | | | | | $\partial^2 u/\partial y^2$ } |
| $E.F. =$ | | { | $\alpha_2 \alpha$ | $-1$ | | | | | | | | | | |
| | | $-$ | $\alpha$ | $-1$ | | | | | | | | | | } |
| $+$ | | $-1$ { | $\alpha_2 \gamma$ | | $-1$ | | | | | | | $+1$ | | |
| | | $+$ | $\alpha_2 \alpha_1$ | | | $-1$ | | | | | | | | $\partial^2 u/\partial x \partial y$ |
| | | $+$ | $\alpha_2 \alpha_3$ | | | | $-1$ | | | | | | | $\partial^2 u/\partial y \partial z$ |
| | | $+$ | $\alpha_2 \beta_1$ | | | | | $-1$ | | | | | | $\partial^2 v/\partial x \partial y$ |
| | | $+$ | $\alpha_2 \beta_2$ | | | | | | $-1$ | | | | | $\partial^2 v/\partial y^2$ |
| | | $+$ | $\alpha_2 \beta_3$ | | | | | | | $-1$ | | | | $\partial^2 v/\partial y \partial z$ |
| | | $+$ | $\alpha_2 \gamma_1$ | | | | | | | | $-1$ | | | $\partial^2 w/\partial x \partial y$ |
| | | $+$ | $\alpha_2 \gamma_2$ | | | | | | | | | $+1$ | | $\partial^2 w/\partial y^2$ |
| | | $+$ | $\alpha_2 \gamma_3$ | | | | | | | | | | $-1$ | $\partial^2 w/\partial y \partial z$ } |
| $+$ | | $-2$ { | $\alpha_2(\alpha_2 - 1)$ | | | | | | | | | | | $\partial^2 u/\partial y^2$ } |
| $+$ | $-1$ | $-1$ { | $\alpha_2 \beta$ | | | | | | $+1$ | | | | | } |

The symbolic method of indicating the computation is very simple. The expressions for the various functions listed in the first column are to be read like ordinary algebraic espressions. However, instead of the individual terms, only the exponents are listed. The variable in question is given at the top of each column. Also, instead of the complete exponent being written every time, the Greek letter referring to the exponent is placed in the second line, and then for each term the amount to be added to or subtracted from this quantity is listed. In the fourth and last columns the actual value of the multiplicative constant and second derivative are written explicitly. For esample, the fourth line of E.F., if written in full, would be

$$+ V \frac{\partial u}{\partial y}^{\alpha_2 - 1} \left\{ \alpha_2 \alpha_1 u^{\alpha} W^{\gamma} \frac{\partial u}{\partial x}^{\alpha_1 - 1} \frac{\partial u}{\partial z}^{\alpha_3} \frac{\partial v}{\partial x}^{\beta_1} \frac{\partial v}{\partial y}^{\beta_2} \frac{\partial v}{\partial z}^{\beta_3} \frac{\partial w}{\partial x}^{\gamma_1} \frac{\partial w}{\partial y}^{\gamma_2} \frac{\partial w}{\partial z}^{\gamma_3} \frac{\partial^2 u}{\partial x \partial y} \right.$$

and the Eulerian function becomes

$$E.F. = 4v\frac{\partial u}{\partial y} + 2\frac{v^2}{\frac{\partial v}{\partial y}}\frac{\partial^2 u}{\partial y^2} - 2\frac{v^2\frac{\partial u}{\partial y}}{\left(\frac{\partial v}{\partial y}\right)^2}\frac{\partial^2 v}{\partial y^2}$$

The last two terms of this expression do not appear in the equations of motion. Hence we must find a second function $L_2$ to be subtracted from $L_1$, where $L_2$ gives these extraneous terms. But both of these terms have second derivatives as multiplicative factors, and referring to Table I we see that any such term involving second derivatives occurs in the Eulerian function in a single, unique manner; i.e. the only function $L_2$ leading to either of these terms is precisely the function $L_1$ which we have already found. Therefore, in this third case also, it is impossible to get an Eulerian function of the required form (11).

4) $\beta = 1 \quad \alpha_2 = 2$ The additional conditions are here

$$\alpha = \alpha_1 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = \gamma_1 = \gamma_3 = 0 \qquad \gamma = 1 \qquad \gamma_2 = -1$$

Hence

$$L_1 = \frac{vw\left(\frac{\partial u}{\partial y}\right)^2}{\frac{\partial w}{\partial y}}$$

and

$$E.F. = 2v\frac{\partial u}{\partial y} - 2\frac{vw\frac{\partial u}{\partial y}}{\left(\frac{\partial w}{\partial y}\right)^2}\frac{\partial^2 w}{\partial y^2} + 2\frac{vw}{\frac{\partial w}{\partial y}}\frac{\partial^2 u}{\partial y^2} + 2\frac{w\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}}{\frac{\partial w}{\partial y}}$$

Here we have three extraneous terms. To the first two we apply the same argument as we did in case 3) and conclude that in this case also it is impossible to get an Eulerian function of the form (11).

Collecting the results of the four possible cases we may conclude that with a LaGrangian function of the form (12) it is impossible to get an Eulerian function of the form (11).

This result can at once be somewhat generalized as follows. It is obvious that it is impossible to get an Eulerian function of the form (11) from a LaGrangian function composed of any sum of terms of the form (12). If now we define a new variable

$$Z = u^{\alpha} v^{\beta} w^{\gamma} \left(\frac{\partial u}{\partial x}\right)^{\alpha_1} \left(\frac{\partial u}{\partial y}\right)^{\alpha_2} \left(\frac{\partial u}{\partial z}\right)^{\alpha_3} \left(\frac{\partial v}{\partial x}\right)^{\beta_1} \left(\frac{\partial v}{\partial y}\right)^{\beta_2} \left(\frac{\partial v}{\partial z}\right)^{\beta_3} \left(\frac{\partial w}{\partial x}\right)^{\gamma_1} \left(\frac{\partial w}{\partial y}\right)^{\gamma_2} \left(\frac{\partial w}{\partial z}\right)^{\gamma_3}$$

then for any function

$$L_1 = L_1\left(u, v, \cdots, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)$$

which has no singularity inside the volume region under consideration, a series

$$S_1 = \sum_{-\infty}^{+\infty} c_i Z^i$$

can be found which converges uniformely to the function $L_1$ inside the region. But we have seen that no LaGrangian function of the form $\sum c_i Z^i$ satisfies our condition of leading to an Eulerian function of the form (11). Hence no LaGrangian function of form (10), which has no singularities in the region considered, can be found which satisfies our condition. But LaGrangian functions involving singularities inside the region in which the fluid motion is taking place are ruled out for physical reasons. Finally by cyclic substitution we find that all of the above results are valid when any of the terms of (9') are substituted for $v\frac{\partial u}{\partial y}$ if only we use the Eulerian equation corresponding.

Hence the theorem stated in the Abstract has been proved. We shall here restate it in a different form employing the conditions mentioned in the Abstract.

" It is impossible to derive the Equations of steady motion of a viscous, incompressible fluid from a variation principle involving as LaGrangian function an expression in the velocity components and their first order space derivatives, unless conditions are imposed on these velocity components such that all of the terms $v\frac{\partial u}{\partial y}$, $w\frac{\partial u}{\partial z}$, $w\frac{\partial v}{\partial z}$, $u\frac{\partial v}{\partial x}$, $u\frac{\partial w}{\partial x}$, $v\frac{\partial w}{\partial y}$

disappear from their positions in the Navier-Stokes equations". This disappearance may arise either because the terms vanish identically or because they are replaced ( according to the imposed conditions ) by other expressions not so intractable. We shall refer to cases in which such conditions are satisfied as " exceptional cases ". It should be noticed that the equations of motion may still be deduced from our variation principle even though the intractable terms appear, if the latter occur in other of the component Navier-Stokes equations from those which they occupy in general. For example, if our imposed conditions allowed us to replace $v\frac{\partial u}{\partial y}$ by $u\frac{\partial v}{\partial x}$ in the x component Navier-Stokes equation, then the function $L = uv\frac{\partial u}{\partial x}$ substituted in the Eulerian function corresponding to this x component, i.e. in

$$\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \frac{\partial u}{\partial x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \frac{\partial u}{\partial y}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \frac{\partial u}{\partial z}}\right) - \frac{\partial L}{\partial u}$$

would give the " intractable " term $u\frac{\partial v}{\partial x}$. However, such cases still fall under the class of " exceptional cases ", for some imposed condition on the velocities is necessary in order to transform the equations of motion into a form in which they may be deduced from our variation principle.

The above theorem may also be formulated from the point of view of a minimum instead of a variation principle.

" Except in the ' exceptional cases ' there is no function of the velocity components, their first order space derivatives, ( and possibly the space coordinates ) which is a minimum for the steady motion of a viscous, incompressible fluid."

The conclusions which may be drawn from this theorem are of an admittedly speculative nature. However it may be well to present them in view of their considerable interest. The question of the possible existence of steady motion in the case of the general flow of a viscous incompressible fluid has exercised many mathematicians and physicists. Korteweg, in the paper referred to makes the statement. " *When, on the contrary, the squares and higher powers* of the velocity are taken into account, I have my reasons for supposing that, even in the case of a sphere moving with uniform velocity, no state of steady motion can be reached, and the motion must finally become unstable ". As far as the present writer is aware, neither Korteweg nor anyone else has amplified or explained this statement, although other authors have expressed similar opinions as to the possibility of steady solutions. The theorem just proved would seem to furnish at least an indication of a possible basis for such beliefs. If the theorem could be shown to hold for LaGrangian functions involving any order derivatives, the non-existence of steady motion except in the " exceptional cases " would be reasonably certain. It has as yet, however, not been possible to make this generalization; in fact it is to be expected that such an extension would not be true. On the other hand there are some physical reasons for restricting the considerations to LaGrangian functions involving only first order derivatives. Among these is the fact that all expressions of the nature of flux or dissipation of energy, or rate of doing work, are expressions involving only first order space derivatives, and it is out of such terms that one should expect a " minimum function " to be constructed.

In any event we may say that it appears probable, on the basis of this theorem, that except in the " exceptional cases " described above, there is no possible steady motion of a viscous, incompressible fluid.

We proceed with a discussion of various examples of the " exceptional cases " for which steady solutions have been found. There are essentially only three such cases existing. One is motion in which the entire fluid moves as a rigid body with uniform rotation. In this case we may write

$$u = Cy \qquad v = -Cx$$

so that

$$V \frac{\partial u}{\partial y} = u \frac{\partial V}{\partial x}$$

and we have the particular example of an " exceptional case " discussed explicitly on page 13. The second case includes plane laminar flow, and Poiseuille flow in a tube of uniform cross-section. In both of these cases we can choose a set of rectangular axes such that two components of velocity vanish throughout the fluid. Hence we have the " exceptional case " in which the intractable terms vanish identically. Finally there is Hamel's case of flow in logarithmic spirals, which will be considered in detail in Sections 4 and 5. Here there is only one non-vanishing component of velocity relative to a set of orthogonal curvilinear coordinates, and in the one component Navier-Stokes equation which remains to determine this velocity, the component of $(q \cdot \nabla) q$ is replaced by the corresponding component of $\frac{1}{2} \nabla q^2$. Hence all of the cases of general steady motion thus far discovered belong to the class of " exceptional cases " and , to this extent at least, our theorem is verified.

SECTION 2

In this section we shall give the explicit form of the LaGrangian function for plane laminar flow and Poiseuille flow in a tube of circular cross-section. Let us consider first plane laminar flow.

We choose Cartesian coordinates and assume two-dimensional motion, writing

$$w = \frac{\partial}{\partial z} = 0$$

In general it will be convenient when talking of two-dimensional motion to retain the three dimensional language as to volume and surface elements, introducing a unit length in the z direction to give the three dimensions i.e.

$$d\tau = dx\,dy \cdot 1$$

$$\int_S ds = \text{line integral in the } x, y \text{ plane.}$$

We consider flow between two parallel walls $x = x_1$ and $x = x_2$ and take a region bounded by these two walls and any two perpendicular planes $y = y_1$ and $y = y_2$ ( and by two planes $z = z_1$ , $z = z_2$ unit distance apart )

Then, introducing the usual restriction on the motion

$$q = u \qquad v = 0$$

and the continuity equation gives

$$\frac{\partial u}{\partial x} = 0 \qquad \text{or} \qquad u = u(y)$$

The Stokes-Navier equations reduce to the following single equation for the determination of u in terms of the pressure gradient

$$(13) \qquad \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} = 0 \qquad \text{where} \qquad u = u(y)$$

In this case the flux of kinetic energy out of our volume region is zero i.e.

$$\Pi = 0$$

Hence the equation of conservation of energy takes Helmholtz' simplified form

( see equation (5) )

$$P - Q = 0$$

and we expect Helmholtz' result to hold, namely

$$\delta\left(P - \frac{1}{2}Q\right) = \delta Q = 0$$

This equation is found to be valid, in fact Rayleigh[3)] has shown that Q is an absolute minimum. However if we take the dissipation function $\mathcal{F}$ as LaGrangian function we do not get the equation of motion (13) from a variation principle.

In order to find the proper LaGrangian function we must somewhat alter the procedure followed in Section 1. The difficulty is that here we have already introduced the continuity equation explicitly, and consequently can not introduce an arbitrary multiplier $\lambda$ into the variation equation, which multiplier is later to be identified with the pressure p. We must find some other way of introducing p into the LaGrangian function. The method adopted appears to be the one which must be employed in general, whenever the continuity equation has been explicitly used to restrict the nature of the velocity components. The method is the following.

Instead of considering P as a surface integral and as having, therefore, no effect on the variation problem, we transform P to a volume integral, just as we previously transformed R, and incorporate the " density function " $\mathbb{P}$ in the LaGrangian function. Since at the walls the velocity is zero, according to the customary assumption of no slip, there is no work done on the fluid in our region due to viscous forces acting across these boundaries. Obviously no work is done by any tractions across the planes $z = z_1$ and $z = z_2$ ( bounding the region in the z direction). Since the velocity is normal to the other boundarie $y = y_1$ and $y = y_2$, the only work done on the region is due to pressure forces and we may write, ( $\overline{n}$ = outwardly drawn normal )

$$\mathcal{P} = -\int_{S'} p\,\bar{n}\cdot\bar{q}\,ds = -\int_{S'} \bar{n}\cdot p\bar{q}\,ds$$

and applying the divergence theorem

$$\mathcal{P} = -\int_{V} \nabla\cdot p\bar{q}\,d\tau = -\int_{V} \frac{\partial}{\partial x}(pu)\,d\tau$$

and finally since $u = u(y)$

$$\mathcal{P} = -\int_{V} u\frac{\partial p}{\partial x}\,d\tau$$

$$\mathbb{P} = -u\frac{\partial p}{\partial x}$$

From equation (4)

$$\Phi = \mu\left(\frac{\partial u}{\partial y}\right)^2$$

and hence the variation principle (6) becomes for this case

$$\delta\left(P - \tfrac{1}{2}\Phi\right) = 0$$

(14)

$$\delta\iint_{V}\left[\frac{\mu}{2}\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial p}{\partial x}\right]dx\,dy = 0$$

The Eulerian equation gives

$$\mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} = 0$$

which is exactly the equation of motion (13).

Hence for plane laminar flow we have verified the fact that the correct LaGrangian function, giving the equations of motion by a variation principle, is

(15) $$L = \mathbb{P} - \frac{1}{2}\Phi - \mathbb{R}$$

in agreement with equation (6). In this particular case since $\mathbb{R}=0$ we have

(16) $$L_{\text{plane laminar}} = \mathbb{P} - \frac{1}{2}\Phi$$

The case of Poiseuille flow in a tube is obviously exactly the same in principle as the above. The imposed conditions imply that all but one component of velocity vanish, so that if the x axis be parallel with the axis of the tube,

$$q = q_x = u$$

Then the continuity equation imposes the additional restriction that u is independent of x and is a function of a single parameter representing something of the nature of the distance from the axis, so that the Navier-Stokes equations reduce to a single one determining u as a function of this parameter. R vanishes exactly as above, and $\mathbb{P}$ has obviously the same value

$$\mathbb{P} = -u\frac{\partial p}{\partial x}$$

so that the only problem in finding L is to express $\Phi$ in the proper coordinates. In the case of a circular tube, for example, where we define r as the distance from the axis,

$$u = u(r)$$

$$\Phi = \mu\left(\frac{\partial u}{\partial r}\right)^2$$

and $$L = \mathbb{P} - \frac{1}{2}\Phi = -\left[\frac{1}{2}\mu\left(\frac{\partial u}{\partial r}\right)^2 + u\frac{\partial p}{\partial x}\right]$$

Hence the variation principle takes the form

$$\delta\int L\,d\tau = \delta\int_0^{2\pi}\int_{x_1}^{x_2}\int_0^{r_0}\left[\frac{1}{2}\mu\left(\frac{\partial u}{\partial r}\right)^2 + u\frac{\partial p}{\partial x}\right] r\,dr\,dx\,d\theta = 0$$

where $r_0$ = radius of the tube,

or $2\pi(x_2-x_1)\,\delta\int_0^{r_0}\left[\frac{\mu r}{2}\left(\frac{\partial u}{\partial r}\right)^2+\mu r\frac{\partial p}{\partial x}\right]dr = \delta\int_0^{r_0} L_1\,dr = 0$

where $L_1 = 2\pi(x_2-x_1)\,r L$

The Eulerian equation

$$\frac{\partial}{\partial r}\left(\frac{\partial L_1}{\partial\frac{\partial u}{\partial r}}\right) - \frac{\partial L_1}{\partial u} = 0$$

gives

$$\mu\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - r\frac{\partial p}{\partial x} = 0$$

or

$$\frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$

which is exactly the Navier-Stokes equation for u(r).

Hence the results for this case are identical with those for the previous one as far as the form of the LaGrangian function is concerned. A new element which has appeared, however, is one which we shall find again in Section 5, namely: if curvilinear coordinates are employed then, in writing the Eulerian equations, the functions of coordinates introduced through the transformation from a Cartesian to a curvilinear system, must be taken into account.

The results of this section have been to verify, for some special cases, the correctness of the variation principle given in (6), and to demonstrate the fact that if the continuity equation be introduced to give an explicit limitation on the velocity components, then in getting the LaGrangian function, P must be considered as a volume integral and must be included in performing the variations.

## SECTION 3

In this section we shall be dealing with orthogonal curvilinear coordinates, so that it will be convenient to list certain formulae connected with such systems before proceeding with the analysis. We shall change our nomenclature and denote the orthogonal curvilinear coordinates by u, v, w. ( We shall refer to velocity components as $q_u$ , $q_v$ , $q_w$ , etc. ) Hence in terms of Cartesian coordinates we have

$$u = u(x,y,z)$$

$$v = v(x,y,z)$$

$$w = w(x,y,z)$$

Then, if ds represents the line element, we define the quantities U, V, W, by

$$ds^2 = U^2 du^2 + V^2 dv^2 + W^2 dw^2$$

where

$$U = U(u,v,w)$$
$$V = V(u,v,w)$$
$$W = W(u,v,w)$$

Then the volume element becomes

$$d\tau = UVWdudvdw$$

For convenience we list the expressions in curvilinear coordinates for the most common vector and scalar quantities normally occuring in potential theory( where $\bar{i}_u, \bar{i}_v, \bar{i}_w$ represent unit vectors in the direction of increasing u, v, w respectively, at any point u, v, w in the space):

(17)

$$\nabla\varphi = \frac{1}{U}\frac{\partial\varphi}{\partial u}\bar{i}_u + \frac{1}{V}\frac{\partial\varphi}{\partial v}\bar{i}_v + \frac{1}{W}\frac{\partial\varphi}{\partial w}\bar{i}_w$$

$$\nabla\cdot\bar{A} = \frac{1}{UVW}\left\{\frac{\partial}{\partial u}(VWA_u) + \frac{\partial}{\partial v}(WUA_v) + \frac{\partial}{\partial w}(UVA_w)\right\}$$

$$(17)\quad \begin{cases} \nabla^2\varphi = \dfrac{1}{UVW}\left\{\dfrac{\partial}{\partial u}\left(\dfrac{VW}{U}\dfrac{\partial\varphi}{\partial u}\right)+\dfrac{\partial}{\partial v}\left(\dfrac{WU}{V}\dfrac{\partial\varphi}{\partial v}\right)+\dfrac{\partial}{\partial w}\left(\dfrac{UV}{W}\dfrac{\partial\varphi}{\partial w}\right)\right\} \\ \nabla\times\bar{A} = \dfrac{1}{UVW}\begin{vmatrix} U\bar{i}_u & V\bar{i}_v & W\bar{i}_w \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ UA_u & VA_v & WA_w \end{vmatrix} \end{cases}$$

In addition to these we shall require the expressions in terms of curvilinear coordinates for $\nabla^2\bar{A}$ and $(\bar{A}\cdot\nabla)\bar{A}$. By the following simple vector transformations these can be obtained from the expressions (17)

$$\nabla^2\bar{A} = \nabla(\nabla\cdot\bar{A}) - \nabla\times(\nabla\times\bar{A})$$

$$(\bar{A}\cdot\nabla)\bar{A} = \frac{1}{2}\nabla A^2 - \bar{A}\times(\nabla\times\bar{A})$$

The resulting expressions, which the writer has not been able to find in any published work with which he is acquainted, are

$$(18)\quad \begin{cases} \nabla^2\bar{A} = \left\{\dfrac{1}{U}\dfrac{\partial}{\partial u}(\nabla\cdot\bar{A}) + \dfrac{1}{VW}\left[\dfrac{\partial}{\partial w}\left\{\dfrac{V}{WU}\left(\dfrac{\partial UA_u}{\partial w}-\dfrac{\partial WA_w}{\partial u}\right)\right\}-\dfrac{\partial}{\partial v}\left\{\dfrac{W}{UV}\left(\dfrac{\partial VA_v}{\partial u}-\dfrac{\partial UA_u}{\partial v}\right)\right\}\right]\right\}\bar{i}_u \\ \quad + \left\{\dfrac{1}{V}\dfrac{\partial}{\partial v}(\nabla\cdot\bar{A}) + \dfrac{1}{WU}\left[\dfrac{\partial}{\partial u}\left\{\dfrac{W}{UV}\left(\dfrac{\partial VA_v}{\partial u}-\dfrac{\partial UA_u}{\partial v}\right)\right\}-\dfrac{\partial}{\partial w}\left\{\dfrac{U}{VW}\left(\dfrac{\partial WA_w}{\partial v}-\dfrac{\partial VA_v}{\partial w}\right)\right\}\right]\right\}\bar{i}_v \\ \quad + \left\{\dfrac{1}{W}\dfrac{\partial}{\partial w}(\nabla\cdot\bar{A}) + \dfrac{1}{UV}\left[\dfrac{\partial}{\partial v}\left\{\dfrac{U}{VW}\left(\dfrac{\partial WA_w}{\partial v}-\dfrac{\partial VA_v}{\partial w}\right)\right\}-\dfrac{\partial}{\partial u}\left\{\dfrac{V}{WU}\left(\dfrac{\partial UA_u}{\partial w}-\dfrac{\partial WA_w}{\partial u}\right)\right\}\right]\right\}\bar{i}_w \\ \text{and} \\ (\bar{A}\cdot\nabla)\bar{A} = \left\{\dfrac{1}{2U}\dfrac{\partial A^2}{\partial u} - \dfrac{A_v}{UV}\left(\dfrac{\partial VA_v}{\partial u}-\dfrac{\partial UA_u}{\partial v}\right)+\dfrac{A_w}{WU}\left(\dfrac{\partial UA_u}{\partial w}-\dfrac{\partial WA_w}{\partial u}\right)\right\}\bar{i}_u \\ \quad + \left\{\dfrac{1}{2V}\dfrac{\partial A^2}{\partial v} - \dfrac{A_w}{VW}\left(\dfrac{\partial WA_w}{\partial v}-\dfrac{\partial VA_v}{\partial w}\right)+\dfrac{A_u}{UV}\left(\dfrac{\partial VA_v}{\partial u}-\dfrac{\partial UA_u}{\partial v}\right)\right\}\bar{i}_v \\ \quad + \left\{\dfrac{1}{2W}\dfrac{\partial A^2}{\partial w} - \dfrac{A_u}{WU}\left(\dfrac{\partial UA_u}{\partial w}-\dfrac{\partial WA_w}{\partial u}\right)+\dfrac{A_v}{VW}\left(\dfrac{\partial WA_w}{\partial v}-\dfrac{\partial VA_v}{\partial w}\right)\right\}\bar{i}_w \end{cases}$$

Using the above formulae (17) and (18) it is possible to express very easily the Navier-Stokes equations in terms of any orthogonal curvilinear coordinate system. However in searching for LaGrangian functions we must have the dissipation function $\Phi$ expressed in a similar way. The simplest method of accomplishing this, in view of our nomenclature, appears to be to first obtain an expression for $\Phi$ involving only vectors, and hence entirely independent of coordinate systems, and then to use (17) and (18) to get the final form desired. There is also some interest per se in having an expression for $\Phi$ independent of any coordinate system. The transformation of $\Phi$ as given in (4) to such a form is not at once apparent, so that it may be worth while to indicate the method used. We have (4) (using our earlier nomenclature)

$$(4)\quad \Phi = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right]$$

$$= \mu\Big[2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2 - 4\frac{\partial w}{\partial z}\frac{\partial v}{\partial y} - 4\frac{\partial u}{\partial x}\frac{\partial w}{\partial z} - 4\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}$$

$$+\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2 + 4\frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + 4\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + 4\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\Big]$$

$$(19)\quad \therefore \Phi = \mu\Big[2(\nabla\cdot\bar{q})^2 + (\nabla\times\bar{q})^2$$

$$+4\left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial z}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\right)\Big]$$

and the problem now becomes: to transform the last parenthesis. Since the calculations are laborious we shall carry them out for the two dimensional case: $w = \frac{\partial}{\partial z} = 0$. The method for the general case is identical and the results for the general case can be at once inferred from the simplified case. In the two-dimensional case the parenthesis becomes:

$$(19')\quad 4\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\right)$$

Consider the expression

$$I = \int_V 4\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\right)d\tau$$

where as mentioned earlier, we retain three-dimensional language, i.e. $d\tau = dx\,dy$ is a two-dimensional volume element and V is a two dimensional volume. Similarly we define S as the surface enclosing V, so that ds is a one-dimensional surface element. Then if we should go to the three-dimensional case the nomenclature would remain unaltered. With these conventions, then

$$I = \int_V \left( 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - 2\frac{\partial v}{\partial y}\frac{\partial u}{\partial x} \right) dx\,dy$$

and integrating the first two terms with respect to x and the second two with respect to y, we have, where l,m represent the components of $\bar{n}$, the outwardly drawn normal to ds:

$$I = \int_S \left[ 2lv\frac{\partial u}{\partial y} - 2lu\frac{\partial v}{\partial y} + 2mu\frac{\partial v}{\partial x} - 2mv\frac{\partial u}{\partial x} \right] ds$$

$$= \int_S \Big[ 2lv\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + 2mu\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + 2lv\frac{\partial v}{\partial x} + 2mu\frac{\partial u}{\partial y}$$

$$- 2lu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2mv\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2lu\frac{\partial u}{\partial x} + 2mv\frac{\partial v}{\partial y} \Big] ds$$

If for simplicity we write $\bar{\Omega} \equiv \nabla \times \bar{q}$, then

$$I = \int_S \left[ -2(lv\Omega_z - mu\Omega_z) - 2(lu + mv)\nabla\cdot\bar{q} + l\frac{\partial q^2}{\partial x} + m\frac{\partial q^2}{\partial y} \right] ds$$

$$= \int_S \left[ -2\bar{n}\cdot(\bar{q}\times\bar{\Omega}) - 2\bar{n}\cdot\bar{q}(\nabla\cdot\bar{q}) + \bar{n}\cdot\nabla q^2 \right] ds$$

and by the divergence theorem:

$$I = \int_V \left[ \nabla^2 q^2 - 2\nabla\cdot(\bar{q}\times\bar{\Omega}) - 2\nabla\cdot\{\bar{q}(\nabla\cdot\bar{q})\} \right] d\tau$$

But by elementary vector transformations

$$\nabla\cdot(\bar{q}\times\bar{\Omega}) = \Omega^2 - \bar{q}\cdot\nabla(\nabla\cdot\bar{q}) + \bar{q}\cdot\nabla^2\bar{q}$$

and

$$\nabla\cdot\left\{\bar{q}(\nabla\cdot\bar{q})\right\} = (\nabla\cdot\bar{q})^2 + \bar{q}\cdot\nabla(\nabla\cdot\bar{q})$$

$$\therefore I = \int_{\tau}\left[\nabla^2 q^2 - 2(\nabla\times\bar{q})^2 - 2(\nabla\cdot\bar{q})^2 - 2\bar{q}\cdot(\nabla^2\bar{q})\right]d\tau$$

and finally

(20) $$4\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\right) = \nabla^2 q^2 - 2(\nabla\times\bar{q})^2 - 2(\nabla\cdot\bar{q})^2 - 2\bar{q}\cdot(\nabla^2\bar{q})$$

In three dimensions the analysis is identical and leads to this same equation, where the left-hand member is replaced by the complete parenthesis of equation (19). Introducing this result into (19) and cancelling where possible we have finally

(21) $$\Phi = \mu\left[\nabla^2 q^2 - (\nabla\times\bar{q})^2 - 2\bar{q}\cdot\nabla^2\bar{q}\right]$$

which is the expression we set out to obtain. It is interesting to note that Lamb ( V, p. 549, equation 11 ) gives an expression for the dissipation of an incompressible fluid which can be shown to be identical with this general expression in which incompressibility has not been assumed.

## SECTION 4

. In the recent paper[4)] referred to, Hamel has given a very elegant discussion of a new general type of steady motion of an incompressible fluid. He considers a two dimensional motion in which the streamlines are restricted to coincide with an isometric family of curves. He then proves that this family must necessarily be a set of logarithmic spirals. He works with Stokes' stream-function, and his proof is quite abstruse. Oseen, in his later paper[5)] on the same general subject has generalized somewhat Hamels' considerations, but his method is very similar to that of the earlier paper. In this section we shall give an entirely different proof of Hamels' result, in which the reasoning seems a little less abstract.

We use the curvilinear coordinate notation explained in the first part of section 3, and since we are dealing with two-dimensional motion take

$$\frac{\partial}{\partial w} = 0 \qquad W = 1 \qquad q_w = 0$$

We define

(22) $$u + iv = \pi(x + iy) = \pi(z)$$

where $\pi$ is a function of the complex variable $z = x + iy$. This imposes the restriction:

(23) $$V = U$$

Then u, v form a set of orthogonal curvilinear coordinates such that

$$\nabla^2 u = \nabla^2 v = 0$$

Hence $u =$ constant and $v =$ constant constitute two orthogonal families of isometric curves. We assume that the velocity of the fluid is everywhere tangent to the family $u =$ constant, i.e. the streamlines coincide with this family.

(24) $$\therefore \quad q_u = q \ ; \ q_v = 0$$

The continuity equation for an incompressible fluid gives, from (17) and (23)

$$\nabla\cdot\bar{q} = 0$$

$$\therefore \frac{\partial}{\partial u}(\Pi q) = 0$$

(25) $$\therefore q = \frac{F(v)}{\Pi}$$

Under these conditions, writing

$$F' = \frac{\partial F}{\partial v} \qquad F^{2\,\prime} = \frac{\partial F^2}{\partial v} \qquad F'' = \frac{\partial^2 F}{\partial v^2} \qquad etc.$$

(17) and (19) give

$$(\bar{q}\cdot\nabla)\bar{q} = \frac{F^2}{2\Pi}\frac{\partial}{\partial u}\left(\frac{1}{\Pi^2}\right)\bar{i}_u + \left\{\frac{1}{2\Pi}\frac{\partial}{\partial v}\left(\frac{F^2}{\Pi^2}\right) - \frac{FF'}{\Pi^3}\right\}\bar{i}_v$$

$$= \frac{F^2}{2\Pi}\frac{\partial}{\partial u}\left(\frac{1}{\Pi^2}\right)\bar{i}_u + \frac{F^2}{2\Pi}\frac{\partial}{\partial v}\left(\frac{1}{\Pi^2}\right)\bar{i}_v$$

$$\nabla p = \frac{1}{\Pi}\frac{\partial p}{\partial u}\bar{i}_u + \frac{1}{\Pi}\frac{\partial p}{\partial v}\bar{i}_v$$

$$\nabla^2\bar{q} = \frac{1}{\Pi}\frac{\partial}{\partial v}\left(\frac{F'}{\Pi^2}\right)\bar{i}_u - \frac{F'}{\Pi}\frac{\partial}{\partial u}\left(\frac{1}{\Pi^2}\right)\bar{i}_v$$

and the Navier-Stokes equations (1) become

(26) $$\begin{cases} \dfrac{\partial p}{\partial u} = -\dfrac{\rho}{2}F^2\dfrac{\partial}{\partial u}\left(\dfrac{1}{\Pi^2}\right) + \mu\dfrac{\partial}{\partial v}\left(\dfrac{F'}{\Pi^2}\right) \\[2ex] \dfrac{\partial p}{\partial v} = -\dfrac{\rho}{2}F^2\dfrac{\partial}{\partial v}\left(\dfrac{1}{\Pi^2}\right) - \mu F'\dfrac{\partial}{\partial u}\left(\dfrac{1}{\Pi^2}\right) \end{cases}$$

Now differentiating the first with respect to v and the second with respect to u and subtracting we get

$$\rho FF'\frac{\partial}{\partial u}\left(\frac{1}{\Pi^2}\right) - \mu\left\{\frac{\partial^2}{\partial v^2}\left(\frac{F'}{\Pi^2}\right) + F'\frac{\partial^2}{\partial u^2}\left(\frac{1}{\Pi^2}\right)\right\} = 0$$

or expanding and factoring $\frac{1}{U^2}$ out,

$$\frac{1}{U^2}\left[\rho f f' U^2 \frac{\partial}{\partial u}\left(\frac{1}{U^2}\right) - \mu\left\{f'\left(U^2\frac{\partial^2}{\partial u^2}\left(\frac{1}{U^2}\right) + U^2\frac{\partial^2}{\partial v^2}\left(\frac{1}{U^2}\right)\right) + 2f'' U^2\frac{\partial}{\partial v}\left(\frac{1}{U^2}\right) + f'''\right\}\right] = 0$$

And since $U^2$ (which is the Jacobean of the transformation $x,y \rightarrow u,v$) cannot vanish, and remembering

$$U^2 \frac{\partial}{\partial u}\left(\frac{1}{U^2}\right) = \frac{\partial}{\partial u}\log\frac{1}{U^2} \qquad etc.$$

we have the equation of motion with pressure eliminated:

(27) $$\rho f f' \frac{\partial}{\partial u}\log\frac{1}{U^2} - \mu\left\{f'\left[U^2\frac{\partial^2}{\partial u^2}\left(\frac{1}{U^2}\right) + U^2\frac{\partial^2}{\partial v^2}\left(\frac{1}{U^2}\right)\right] + 2f''\frac{\partial}{\partial v}\log\frac{1}{U^2} + f'''\right\} = 0$$

Since the last term, $f'''$, is a function of v only and since (27) is an identity holding for all values of u and v , therefore the entire left hand member is a function of v only. Then, since the first term is the only one containing $\rho$ , it is a function of v only.

(27') $$\therefore \frac{\partial}{\partial u}\log\frac{1}{U^2} = g_1(v) \qquad (say)$$

The coefficients of $f'$ and $f''$ may either be individually functions of v only, or the expression

$$f'\left[U^2\frac{\partial^2}{\partial u^2}\left(\frac{1}{U^2}\right) + U^2\frac{\partial^2}{\partial v^2}\left(\frac{1}{U^2}\right)\right] + 2f''\frac{\partial}{\partial v}\log\frac{1}{U^2} = E \qquad (say)$$

may be a function of v only, while the separate coefficients are mixed functions. We shall show that the first mentioned condition is the one which must actually be satisfied. For if the coefficients of $f'$ and $f''$ are functions of u we may expand them in power series of the form $\sum a_n u^n$ where the $a_n$'s are functions of v. Then combining the coefficients of like powers of u, the condition that E be a function of v alone implies that the coefficient of each power of u vanishes. Hence we are led to a series of total differential equations giving f as a function of v, which must be satisfied simultaneously. We remark that these differential equations, whichwe shall call the " first equations " do not contain the constants $\rho$ and $\mu$. Now consider equation (27) which has become a total differential equation, not involving u, determining f as a function of v. This " second equation " does contain the non-vanishing constants $\rho$ and $\mu$. Hence the " first " and " second " equations can never become equivalent or identical and we have two or more non-equivalent equations, giving f as a function of v, which must be satisfied simultaneously. However it is impossible to satisfy this condition, and our assumption that the coefficients of $f'$ and $f''$ in (27) are functions involving u, is demonstrated to be false.

(27'') $$\therefore \frac{\partial}{\partial v} \log \frac{1}{U^2} = g_3(v) \qquad \text{(say)}$$

Equations (27') and (27'') are sufficient to determine uniquely the form of $U^2$ as we shall see. After this has been done we must verify the fact that the $U^2$ so determined does actually give the coefficient of $f'$ in (27) as a function of v only.

We have from (27') and (27'')

$$\log \frac{1}{U^2} = u\, g_1(v) + g_2(v) \qquad \text{(say)}$$

$$\text{and} \quad \log \frac{1}{U^2} = h(u) + g_4(v) \qquad \text{(say)}$$

$$\therefore g_1(v) = \text{constant} = -A \qquad \text{(say)}$$

$$\therefore \frac{1}{U^2} = e^{-Au + g_2(v)}$$

or we may write

(28) $$U^2 = e^{Au + g(v)}$$

But from the theory of the complex variable, if we have (22)

$$u + iv = \pi(x + iy)$$

which may be written

$$x + iy = \varpi(u + iv)$$

then

$$x - iy = \varpi_1(u - iv)$$

where $\varpi_1$ is obtained by replacing i by $-i$ throughout $\varpi$.

Also

$$\pi^2 = \varpi(u+iv)\,\varpi_1(u-iv)$$

$$\therefore \log \pi^2 = \log \varpi(u+iv) + \log \varpi_1(u-iv)$$

$$= k(u+iv) + k_1(u-iv) \qquad \text{(say)}$$

where $k_1$ bears the same relation to k as $\varpi_1$ does to $\varpi$.

In our case, therefore, we have from (28)

(29) $$Au + g(v) = k(u+iv) + k_1(u-iv)$$

Using primes to denote differentiation with respect to the complex variables indicated in (29), we have, differentiating with respect to u

$$k' + k_1' = A$$

(29') $$k'' + k_1'' = 0$$

while differentiating with respect to v

$$ik' - ik_1' = \frac{dg}{dv}$$

(29'') $$-k'' - k_1'' = \frac{d^2g}{dv^2}$$

Hence combining (29') and ( 29'')

$$\frac{d^2g}{dv^2} = 0$$

$$\therefore g = Bv + C_1 \qquad \text{(say)}$$

Hence we may write (from 28)

(30) $$\pi^2 = Ce^{Au+Bv}$$

where A, B, C are real constants. Substituting this expression in the coefficient of f' in (29) we see that it does satisfy the condition that this coefficient is a function of v only, and does not contain u.

We must now find the functions $\Pi$ or $\varpi$ corresponding to this $U^2$. The first equations of (29') and (29'') become

$$k' + k_1' = A$$

$$ik' - ik_1' = B$$

$$\therefore k' = \frac{A - iB}{2}$$

$$\therefore k = \frac{A-iB}{2}(u+iv) + C_2 = \log \varpi$$

$$\therefore \varpi = e^{\frac{A-iB}{2}(u+iv)}$$

where the $C_2$, which is of no importance, has been dropped. In order to show that this transformation does correspond to logarithmic spirals it is convenient to introduce two new real constants a and b, where

(31) $$A = \frac{2b}{a^2+b^2} \qquad B = \frac{2a}{a^2+b^2}$$

from which follows immediately

$$\varpi = e^{\frac{u+iv}{b+ia}} \quad \text{or} \quad x+iy = e^{\frac{u+iv}{b+ia}}$$

(32) $$\therefore u+iv = (b+ia)\log(x+iy)$$

Introducing polar coordinates $r, \theta$ this becomes

$$u+iv = \log(re^{i\theta})^{b+ia}$$

$$= \log(r^b e^{-a\theta}) + i\log(r^a e^{b\theta})$$

(33) $$\therefore \begin{cases} u = b\log r - a\theta \\ v = a\log r + b\theta \end{cases}$$

which represent two orthogonal families of logarithmic spirals.

Hence we have proved that if a two dimensional flow is constrained to have its streamlines coincide with a family of isometric curves, this family must belong to the unique class of logarithmic spirals. This is Hamels' result.

Before passing on to section 5 , it may be well to discuss briefly one or two points connected with this type of motion. The limiting cases, where the streamlines coincide with u = constant, i.e. $q = q_u$, are:

1) Flow in concentric circles, usually called Couette flow, which is obtained by setting b = 0.

2) Plano radial flow which is given where a = 0.

Hamel has discussed both of these cases at length in the paper referred to.

The differential equation for f, which might be called the equation of motion, is obtained by introducing the expression for $U^2$ (30), into the equation of motion with pressure eliminated (27)

$$\frac{\partial}{\partial u} \log \frac{1}{U^2} = -A \qquad U^2 \frac{\partial^2}{\partial u^2}\left(\frac{1}{U^2}\right) = A^2$$

$$\frac{\partial}{\partial v} \log \frac{1}{U^2} = -B \qquad U^2 \frac{\partial^2}{\partial v^2}\left(\frac{1}{U^2}\right) = B^2$$

$$\therefore A\rho f f' + \mu\left[(A^2+B^2)f' - 2Bf'' + f'''\right] = 0$$

or integrating once, and writing $\frac{\mu}{\rho} = \nu =$ coefficient of kinematic viscosity

(34) $$f'' - 2Bf' + (A^2+B^2)f + \frac{A}{2\nu} f^2 = \alpha$$

where $\alpha$ is an arbitrary constant of integration.

## SECTION 5

In this section we wish to apply the results of the first two sections to the case of flow in logarithmic spirals. It is very easy to see that this case belongs to the class of " exceptional cases " defined in section 1. For referring to equation (18), if we have two dimensional flow such that

$$q_w = \frac{\partial}{\partial w} = 0 \qquad W = 1$$

and if the velocity is assumed to be such that the streamlines always coincide with one family of orthogonal curvilinear coordinates, i.e.

$$q_u = q \qquad q_v = 0 \qquad \text{(say)}$$

then the u component of $(\bar{q}\cdot\nabla)\bar{q}$ becomes $\frac{1}{2U}\frac{\partial q^2}{\partial u}$ , which is identical with the u component of $\frac{1}{2}\nabla q^2$ . But the Navier-Stokes equations in such a case reduce to the single u-component equation for the determination of $q_u$. Hence in the Navier-Stokes equations $(\bar{q}\cdot\nabla)\bar{q}$ may be replaced by $\frac{1}{2}\nabla q^2$ wherever the former occurs. But this is exactly the " exceptional case " mentioned explicitly in the discussion following equation (9).

Having settled this point we pass on to a consideration of the variation problem: to find a LaGrangian function which will give the " equation of motion " (34) through the application of a variation principle. For the region over which to integrate we make a similar choice to that of section (2), i.e. we take the z boundaries unit distance apart. We consider flow between fixed walls which coincide with two members of the chosen family of logarithmic spirals and take these two walls, $u = u_1$ and $u = u_2$, as boundaries of our region. The other two boundaries are formed by any two members of the orthogonal family of curves, $v = v_1$ and $v = v_2$. Notice that in any particular problem $u_1$ and $u_2$ are specified, while $v_1$ and $v_2$ are arbitrary.

We assume $\delta q = 0$ at all these boundaries, and search for a LaGrangian function L, such that if we set

$$\delta\int_V L\,d\tau = 0$$

where V is the region specified, then the Eulerian equation is (34). From the general results of the first section it is to be expected that the variation principle will be of the form given in (6), i.e.

(35) $$\delta\int_V L\,d\tau \backsim \delta(P - \tfrac{1}{2}Q - \gamma R)$$

However, in connection with this special case two remarks must be made. First, since the continuity equation has been explicitly introduced in the expression for velocity components, we expect from section 2 that P must be considered as a volume integral and included in the variation. Second, since we have chosen a particular set of coordinates, and since this choice has introduced certain restrictions, we may expect certain functions of the coordinates to appear in the right hand member of (35). For the same reason the numerical factor $\gamma$, which appears in that member, may not be unity, as was found in the first section, to be the case in general. For these reasons it will be convenient to first try

(36) $$L' = \mathbb{P} - \tfrac{1}{2}\mathbb{Q} - \gamma\,\mathbb{R}$$

remembering that it may be necessary to introduce some functions of coordinates into the right-hand member in order that the Eulerian equation give the equation of motion (34).

In evaluating $\mathbb{P}$, the conditions are essentially the same as for plane laminar flow, discussed in section 2, and for the same reasons as there mentioned, the work of the external forces is entirely due to pressure forces. ( Notice that the velocity is perpendicular to the boundaries $v = v_1$ and

$v = v_2$ ). As in that section we retain the three dimensional nomenclature but omit the z variable as having no effect on the equations. Hence

$$P = -\int_S p\,\bar{n}\cdot\bar{q}\,ds = -\int_\tau \bar{q}\cdot\nabla p\,d\tau$$

$$\therefore \mathbb{P} = -\bar{q}\cdot\nabla p = -q_u \frac{1}{\mathcal{U}}\frac{\partial p}{\partial u}$$

(37) $$\therefore \mathbb{P} = -\frac{f}{\mathcal{U}^2}\frac{\partial p}{\partial u}$$

In the case we are considering the pressure p has been eliminated from the equation of motion (34) which we are trying to obtain, and an arbitrary constant $\alpha$ has been introduced. Hence in (37) we must replace $\frac{\partial p}{\partial u}$ by an expression involving the velocity, its first derivatives, and $\alpha$. To do this we go back to the equations of motion as given in (26). Here $\frac{\partial p}{\partial u}$ is given in terms of f and f''. The f'' is eliminated by using the equation of motion (34). Using the expression for $\mathcal{U}^2$ given in (30) and eliminating f'' we get

$$\frac{\partial p}{\partial u} = -\frac{\mu}{\mathcal{U}^2}\left[(A^2+B^2)f - Bf' - \alpha\right]$$

so that finally

(38) $$\mathbb{P} = \frac{\mu}{\mathcal{U}^4}\left[(A^2+B^2)f^2 - Bff' - \alpha f\right]$$

The general expression for $\mathbb{R}$ given in (8) holds in this case, so that

$$\mathbb{R} = \frac{\rho}{2}\,\bar{q}\cdot\nabla q^2 = \frac{\rho}{2}\frac{f}{\mathcal{U}^2}\frac{\partial}{\partial u}\left(\frac{f^2}{\mathcal{U}^2}\right) = \frac{\rho}{2\mathcal{U}^2} f^3 \frac{\partial}{\partial u}\left(\frac{1}{\mathcal{U}^2}\right)$$

(39) $$\therefore \mathbb{R} = -\frac{1}{\mathcal{U}^4}\frac{\rho}{2} A f^3$$

The vector expression for $\Phi$ is given in (21)

(21) $$\Phi = \mu\left[\nabla^2 q^2 - (\nabla\times\bar{q})^2 - 2\bar{q}\cdot\nabla^2\bar{q}\right]$$

But from (17) and (18)

$$\nabla^2 q^2 = \frac{1}{\Pi^2}\left[F^2\frac{\partial^2}{\partial u^2}\left(\frac{1}{\Pi^2}\right) + F^2\frac{\partial^2}{\partial v^2}\left(\frac{1}{\Pi^2}\right) + 4FF'\frac{\partial}{\partial v}\left(\frac{1}{\Pi^2}\right) + \frac{2}{\Pi^2}FF'' + \frac{2}{\Pi^2}F'^2\right]$$

$$(\nabla\times\bar{q})^2 = \frac{1}{\Pi^4}F'^2$$

$$\bar{q}\cdot\nabla^2\bar{q} = \frac{1}{\Pi^2}\left[FF'\frac{\partial}{\partial v}\left(\frac{1}{\Pi^2}\right) + \frac{FF''}{\Pi^2}\right]$$

$$\therefore \Phi = \frac{\mu}{\Pi^2}\left[F^2\left\{\frac{\partial^2}{\partial u^2}\left(\frac{1}{\Pi^2}\right) + \frac{\partial^2}{\partial v^2}\left(\frac{1}{\Pi^2}\right)\right\} + 2FF'\frac{\partial}{\partial v}\left(\frac{1}{\Pi^2}\right) + \frac{F'^2}{\Pi^2}\right]$$

(40) $$\therefore \Phi = \frac{\mu}{\Pi^4}\left[(A^2+B^2)F^2 - 2BFF' + F'^2\right]$$

Hence using the expression (36) for L'

(41) $$L' = e^{-2(Au+Bv)}\left[\mu\left\{\frac{A^2+B^2}{2}F^2 - \alpha F - \frac{1}{2}F'^2\right\} + \frac{\gamma}{2}\rho A F^3\right]$$

We remarked in section 2 (Poiseuille Flow) that it was necessary to form a function $L_1$ obtained by multiplying L by the Jacobean of the transformation from Cartesian to curvilinear coordinates, and that it was necessary to form the Eulerian equation using $L_1$ instead of L. This is a general rule for carrying out variation principles in any curvilinear coordinates. In our case we have

$$\delta\int_V L\,d\tau = \delta\iint_V L\,\Pi^2\,du\,dv \equiv \delta\iint_V L_1\,du\,dv = 0$$

Then the Eulerian equation is:

$$\frac{\partial}{\partial v}\left(\frac{\partial L_1}{\partial F'}\right) - \frac{\partial L_1}{\partial F} = 0$$

or

$$\frac{\partial}{\partial \nu}\left(\frac{\partial L U^2}{\partial F'}\right) - \frac{\partial L U^2}{\partial F} = 0$$

If we take $L = L'$ as defined in (41) this does not give the equation of motion (34), and it appears that in order to get the result we are seeking we must set

(42) $$L = \frac{L'}{U^2}$$

Then $L_1 \equiv LU^2 = L'$

and the Eulerian equation becomes

$$\frac{\partial}{\partial \nu}\left(\frac{\partial L'}{\partial F'}\right) - \frac{\partial L'}{\partial F} = 0$$

which gives:

$$F'' - 2BF' + (A^2+B^2)F + \frac{3\gamma}{2\nu}AF^2 = \alpha$$

which is exactly the equation of motion (34) if

$$\gamma = \frac{1}{3}$$

Hence as we suspected, the introduction of a particular coordinate system may alter the general expression for L. In the case of logarithmic spiral flow, this alteration consists in changing the value of $\gamma$ from 1 to $\frac{1}{3}$ and in changing L from the invariant function

$$L = \mathbb{P} - \frac{1}{2}\mathbb{E} - \gamma\mathbb{R}$$

to the expression depending on the coordinate system

(43) $$L = \left(\mathbb{P} - \frac{1}{2}\mathbb{E} - \gamma\mathbb{R}\right)\frac{1}{U^2}$$

where $U^2$ is the Jaccbean of the transformation from Cartesian to logarithmic spiral coordinates.

## CONCLUSIONS

The final conclusions of this thesis, insofar as they relate to a variation principle are as follows:

1) It appears probable that except in certain " exceptional cases " steady motion of a viscous incompressible fluid cannot exist.

2) All of the cases of such motion which have yet been discovered do belong to the class of " exceptional cases ", as we have defined this class.

3) The equations of motion of all known cases of steady motion may be derived from a variation principle.

4) The LaGrangian functions for all these cases are exhibited and it is seen that the corresponding variation principle is

$$\delta\left(P-\tfrac{1}{2}Q-\gamma R\right)=0$$

except for functions of the coordinates, which may be introduced through the imposing of restrictions on the velocity by the coordinate system chosen to describe the motion.

In conclusion the author wishes to express his deep appreciation of the continued assistance and inspiration of Dr. H. Bateman, who originally suggested the problem, as well as his gratitude to other members of the staff of the Norman Bridge Laboratory, too numerous to mention, who were always ready to discuss difficulties and to offer valuable suggestions for their solution.

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