Chapter 2

Gravitational Waves and Their Sources, Including Compact Binary Coalescences

In this chapter we give a brief introduction to General Relativity, focusing on GW emission. We then focus our attention on GWs from CBCs. The following presentation uses the conventions found in [1].

2.1 Pillars of Relativity

Einstein developed the pillars of Relativity Theory in the early 20th Century, which can be summarized by two thought experiments, or "gedankenexperiments." These gedankenexperiments reveal different implications of a single assumption, that the Laws of Physics are the same for everyone observing an experiment.

The first application of this assumption was to the speed of light. Einstein assumed that the speed of light was a constant, independent of an observer's frame of reference. The conclusions Einstein drew from this led him to develop the Special Theory of Relativity, which is useful in Lorentz reference frames (i.e., those without gravity).

The second application of this assumption was the universality of the motion of objects in gravitational freefall. Einstein' gedankenexperiment went as follows: suppose there are two observers observing the motion on a object in freefall in their local reference frame. Over small distances and times, they will observe the same motion of the object whether the motion is due to gravitational acceleration towards a gravitating body (e.g., for an observer whose frame is resting on the Earth), or due to a uniform acceleration of the reference frame (e.g., for an observer whose frame is in a rocket ship that is constantly accelerating in a specific direction). From this gedankenexperiment, Einstein concluded that the acceleration around a gravitating body is produced by a curvature of spacetime in the object's vicinity. Using this framework, Einstein developed the General Theory of Relativity.

2.2 Einstein's Equations

In the General Theory of Relativity, the equations that link the curvature of spacetime to the presence of matter and energy are called the Einstein Field Equation (EFE) given as

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \ , \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein Tensor given by $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$, $R_{\mu\nu} = R^{\gamma}{}_{\mu\nu\gamma}$, $R^{\gamma}{}_{\mu\nu\gamma}$ is the Riemann Curvature Tensor, $g_{\mu\nu}$ is the spacetime metric, and $T_{\mu\nu}$ is the stress-energy tensor. These equations tell matter how to move due to the curvature of spacetime and spacetime how to curve due to the presence of matter.

2.3 Gravitational Wave Solutions

The EFEs not only describe how spacetime curvature is influenced in the presence of matter, it also predicts how the curvature should propagate away from accelerating objects. Just as electromagnetic waves are emitted and travel away from accelerating electric charges, GWs are emitted and travel away from accelerating masses. If we are far away from the accelerating source, the EFEs simplify to

$$G_{\mu\nu} = 0 \ . \tag{2.2}$$

In this region, the spacetime is essentially Minkowski except for the perturbations do to GWs: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. Assuming these perturbations are small so that all components of $h_{\mu\nu} \ll 1$, we can calculate how these perturbations propagate using a linearized version of the EFEs. In the following discussion, we follow closely the description found in reference [1].

2.3.1 Linearized Gravity

To first order in $h_{\mu\nu}$ the EFE become

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} + \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = 0 \, , \tag{2.3}$$

where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and $h \equiv \eta^{\alpha\beta}h_{\alpha\beta}$. We can eliminate all of the terms after the first by imposing the tensor analogue of the Lorentz gauge condition of electromagnetic theory, namely $\bar{h}_{\mu\alpha,}{}^{\alpha} = 0$. This leaves us with the wave equation

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0 \, . \tag{2.4}$$

This wave equation has plane wave solutions

$$\bar{h}_{\mu\nu} = \Re\left[A_{\mu\nu}e^{ik_\alpha x^\alpha}\right] \, , \tag{2.5}$$

where $\Re[\ldots]$ means the real part of $[\ldots]$. From the wave equation we find $k_\alpha k^\alpha = 0$, which means $\mathbf{k}$ is a null vector and the waves travel at the speed of light. From the gauge condition $\bar{h}_{\mu\alpha,}{}^{\alpha} = 0$ we find $A_{\mu\alpha}k^\alpha = 0$, which means the amplitude tensor $\mathbf{A}$ is orthogonal to $\mathbf{k}$.

At this point $\mathbf{A}$ has six degrees of freedom, however four of these can be pinned down with an additional gauge specification. Let us choose $A_{\mu\nu}u^\nu = 0$ where u^ν is a 4-velocity of a particular frame. This fixes an additional three degrees of freedom. For the final gauge choice, let us choose

$A^{\mu}{}_{\mu} = 0$. All of these constraints in a Lorentz frame with $u^0 = 1$ and $u^j = 0$ give

$$h_{\mu 0} = 0 \;, \tag{2.6}$$

$$h_{kj,}{}^{j} = 0 \;, \tag{2.7}$$

$$h_{kk} = 0 \;. \tag{2.8}$$

The first of these equations implies only the spatial components of $h_{\mu\nu}$ are nonzero. The second implies that the spatial components are divergence free. Finally, the third implies that $h_{\mu\nu}$ is trace free. This gauge is called the transverse-traceless (TT) gauge because in this gauge $h_{\mu\nu}$ is transverse to the time direction and the direction of propagation, and is trace free.

These gauge choices leave us with two degrees of freedom for the propagating waves. This can be interpreted as two polarizations of GWs, the plus (+) and cross ($\times$) polarizations, named for their affect on test particles. In terms of the these polarizations and in the TT gauge, $A_{\mu\nu}$ takes the form

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;. \tag{2.9}$$

2.3.2 Geodesic Deviation

Objects undergoing free motion will follow geodesics described by the geodesic equation

$$\frac{d^2 x^{\lambda}}{dt^2} + \Lambda^{\lambda}{}_{\mu\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} = 0 \;. \tag{2.10}$$

Consider two particles, A and B, separated by n^j in a coordinate system that is a local Lorentz frame along A's world line. In this coordinate system the separation vector becomes $n^j = x_B{}^j - x_A{}^j = x_B{}^j$, since A is at the origin. In a TT coordinate system that moves with particle A and with its proper reference frame to first order in the metric perturbation h^{TT}_{jk}, the geodesic deviation equation is written as

$$\frac{d^2 n^j}{\partial t^2} = -R^{\mathrm{TT}}_{j0k0} n^k \ . \tag{2.11}$$

In the TT gauge the Riemann curvature tensor takes the simple form

$$R_{j0k0} = -\frac{1}{2} h^{\mathrm{TT}}_{jk,00} \ . \tag{2.12}$$

Using this along with the separation vector's definition, the geodesic deviation equation becomes

$$\frac{d^2 x_B{}^j}{\partial t^2} = \frac{1}{2} \frac{\partial^2 h^{\mathrm{TT}}_{jk}}{\partial t^2} x_B{}^k \ . \tag{2.13}$$

This equation of motion can be integrated, assuming particles initially at rest relative to each other, to obtain the location of B as a function of time in the proper reference frame of A

$$x_{Bk}(t) = x_B{}^j(0) \left(\delta_{jk} + \frac{1}{2} h^{\mathrm{TT}}_{jk} \right) \ , \tag{2.14}$$

where h^{TT}_{jk} should be evaluated at the location of A. The second in the parentheses on the right-hand side can be viewed as a strain and is proportional to the metric perturbation h_{jk}.

2.3.3 Gravitational-Wave Polarizations

As mentioned above, there are two degrees of freedom for a GW, which can be seen as distinct polarizations. Let us consider a plane, monochromatic wave propagating in the z direction. In the TT gauge of this wave, the gauge constraints (equation (2.6)) leave the only nonzero components of $h^{\mathrm{TT}}_{\mu\nu}$

$$h^{\mathrm{TT}}_{xx} = -h^{\mathrm{TT}}_{yy} = \Re \left[A_+ e^{-i\omega(t-z)} \right] \ , \tag{2.15}$$

$$h^{\mathrm{TT}}_{xy} = h^{\mathrm{TT}}_{yx} = \Re \left[A_\times e^{-i\omega(t-z)} \right] \ , \tag{2.16}$$

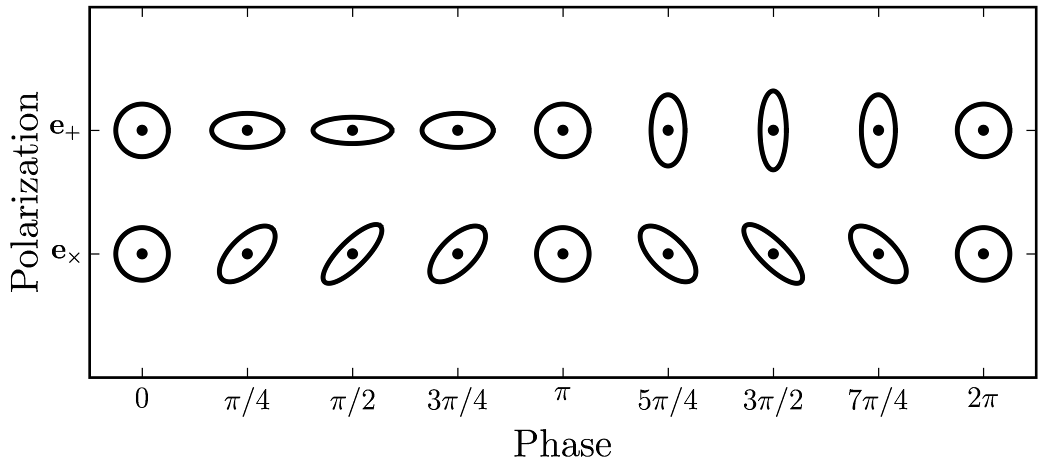

Figure 2.1: Effects of Passing Gravitational Waves
The effect of the + and × polarization GW passing perpendicular to a ring of test particles.

where A_+ and $A_\times$ are the amplitudes of the + and × polarizations, respectively, and ω is the frequency of the GW.

Defining the polarization tensors $\mathbf{e}_+ = (\mathbf{e}_x \oplus \mathbf{e}_x) - (\mathbf{e}_y \oplus \mathbf{e}_y)$ and $\mathbf{e}_\times = (\mathbf{e}_x \oplus \mathbf{e}_y) + (\mathbf{e}_y \oplus \mathbf{e}_x)$ we find a GW can be decomposed into its independent polarizations as

$$h_{jk}^{\mathrm{TT}} = h_+ e_{+jk} + h_\times e_{\times jk} \ . \tag{2.17}$$

The effects of each polarization can be seen independently by looking at a circular ring of test particles surrounding a central particle: let all of the particles lie in the plane perpendicular to the direction of propagation of the incident GW. For an incident +-polarized wave, the polarization tensor $\mathbf{e}_+$ shows that the test particles will oscillate about their circular equilibrium positions in such a way that as the distance from the origin along the x direction increases, the distance from the origin along the y direction decreases. This results in the circle being stretched and squeezed along the x and y axes at one moment, and vice versa half a wavelength later. For a ×-polarized wave, a similar thing happens, except rotated by 45°. This effect is visualized in figure 2.1.

2.4 Gravitational Wave Sources

In the case of electromagnetic radiation, a multipolar analysis reveals that electromagnetic waves are sourced by, in order of significance, accelerating electric-charge monopoles, electric-charge dipoles and electric-current dipoles, electric-charge quadrupoles and electric-current quadrupoles, etc. In the case of gravitational radiation, the conservation of energy prevents radiation due to the acceleration of mass monopoles, the conservation of linear momentum prevents radiation due to the acceleration of mass dipoles, and the conservation of angular momentum prevents radiation due to the acceleration of current dipoles. This results in the leading-order radiation coming from the acceleration of mass quadrupoles.

In reference [2], Thorne gives the relationships between GW radiation and its source. Radiation can be related to the source through the multipolar expansion. For Newtonian sources the relevant multipolar moments are given by

$$\mathcal{I}_{A_l} = \left[\int \rho X_{A_l} d^3x\right]^{\mathrm{STF}} , \tag{2.18a}$$

$$\mathcal{S}_{A_l} = \left[\int \left(\epsilon_{a_l pq} x_p \rho v_q\right) X_{A_{l-1}} d^3x\right]^{\mathrm{STF}} , \tag{2.18b}$$

where $X_{A_l} = x_{a_1} x_{a_2} \ldots x_{a_l}$, ρ is the Newtonian mass density, v is the velocity, ϵ_{ijk} is the Levi-Civita tensor, and STF stands for the symmetric trace-free part.

For these sources, the radiation h^{TT}_{jk} is given by

$$\begin{aligned} h^{\mathrm{TT}}_{jk} = & \left[\sum_{l=2}^{\infty} \frac{4}{l!} r^{-1\,(l)}\mathcal{I}^{\mathrm{rad}}_{jkA_{l-2}}(t-r) N_{A_{l-2}} \right. \\ & \left. + \sum_{l=2}^{\infty} \frac{8}{(l+1)!} r^{-1} \epsilon_{pq(j}\,{}^{(l)}\mathcal{S}^{\mathrm{rad}}_{k)pA_{l-2}}(t-r) n_q N_{A_{l-2}}\right]^{\mathrm{TT}} , \end{aligned} \tag{2.19}$$

where ${}^{(l)}\mathcal{I}$ and ${}^{(l)}\mathcal{S}$ represent the lth time derivative of these moments, and $N_{A_l} = n_1 n_2 \ldots n_l$. The superscript "rad" denotes this as the moment in the radiation field, which is the same as the moment in the near zone if we ignore nonlinear corrections proportional to $\ln(\lambda/2\pi R)$ where R is the background curvature of the local universe.

The energy and momentum carried away from a radiating system by GWs is given simply in the form of the stress-energy tensor for GWs:

$$T_{\mu\nu}^{\mathrm{GW}} = \frac{1}{32\pi} \left\langle h_{jk,\mu}^{\mathrm{TT}} h_{jk,\nu}^{\mathrm{TT}} \right\rangle \ . \tag{2.20}$$

Expanding the energy term (i.e., $\mu = \nu = 0$) in terms of the multipole moments, we find the energy flux carried by the waves to be

$$\begin{aligned} \frac{dE}{dt} &= \sum_{l=2}^{\infty} \frac{(l+1)(l+2)}{(l-1)l} \frac{1}{l!(2l+1)!!} \left\langle {}^{l+1}\mathcal{I}_{A_l} {}^{l+1}\mathcal{I}_{A_l} \right\rangle \\ &\quad + \sum_{l=2}^{\infty} \frac{4l(l+2)}{(l-1)} \frac{1}{(l+1)!(2l+1)!!} \left\langle {}^{l+1}\mathcal{S}_{A_l} {}^{l+1}\mathcal{S}_{A_l} \right\rangle \ , \end{aligned} \tag{2.21}$$

where $l! = l(l-1)\cdots 2\cdot 1$ and $l!! = l(l-2)(l-4)\cdots(2\,\mathrm{or}\,1)$.

GWs can come any source with an accelerating mass or current multipole, however, in most situations the dominant term in the multipole expansion is the $l = 2$ mass quadrupole. Different types of sources can be characterized into the following categories: known-transient waveforms (e.g., compact binary signals), unknown-transient waveforms (i.e., burst signals), discernible and narrow-band continuous wave sources (e.g., pulsar and white-dwarf binary signals), and indiscernible or broad-band continuous wave sources (e.g., stochastic and confused white-dwarf binary sources). In section 2.4.1 we derive the waveforms associated with compact binary signals. In section 2.4.2 we briefly discuss additional examples and how each type of waveform is searched for.

2.4.1 Compact Binaries

One of the families of known transient waveforms is associated with compact binary systems (i.e., binary systems whose objects are either neutron stars or black holes). As these objects orbit each other, they act as a source for GWs.

Compact binaries are interesting signals, in part, because they are so well understood. These signals can act as standard candles for measuring astrophysical distances due to the fact that their amplitude is uniquely determined by their phase evolution and the luminosity distance to the source. In addition, GW signals from compact binaries are considered some of the most extreme probes of General Relativity as the signals originate from objects that massively curve space-time and travel at speeds approaching the speed of light.

2.4.1.1 Compact Binary Inspiral Waveforms

In this section we go over the leading-order effects in calculating a compact binary inspiral waveform. To aid in this calculation, we assume the objects are Newtonian point particles with masses m_A and m_B. In this case, the mass density term in equation (2.18a) become Dirac delta functions and the quadrupole moment $\mathcal{I}_{jk}$ is given as

$$\begin{aligned} \mathcal{I}_{jk} &= \left[\sum_A m_A x_{Aj} x_{Ak}\right]^{\mathrm{STF}} \\ &= \sum_A m_A \left(x_{Aj} x_{Ak} - \frac{1}{3}\delta_{jk} r_A^2\right) . \end{aligned} \tag{2.22}$$

The leading-order term in the GW radiation (equation (2.19)) is due to the mass quadrupole and given as

$$h_{jk}^{\mathrm{TT}} = \frac{2}{D}\ddot{\mathcal{I}}_{jk}^{\mathrm{TT}}(t-D) , \tag{2.23}$$

where D is the distance from the source.

Let us assume these objects are in quasicircular orbits around each other described by Kepler's Laws where their orbital speed is related to their separation by

$$\Omega = \sqrt{\frac{M}{r^3}} , \tag{2.24}$$

where $M = m_A + m_B$ is the total mass of the system, and r is the separation.

As these objects orbit each other, in a coordinate system whose origin is at the center of mass with the plane of the orbit in the x-y plane, their locations are given by

$$x_A(t) = \frac{\mu}{m_A} r \cos(\Omega t) \ , \tag{2.25a}$$

$$y_A(t) = \frac{\mu}{m_A} r \sin(\Omega t) \ , \tag{2.25b}$$

$$x_B(t) = -\frac{\mu}{m_B} r \cos(\Omega t) \ , \tag{2.25c}$$

$$y_B(t) = -\frac{\mu}{m_B} r \sin(\Omega t) \ , \tag{2.25d}$$

where $\mu = m_A m_B / M$ is the reduced mass of the system.

Using these trajectories and equation (2.22) for $\mathcal{I}_{jk}$, we find quadrupole moments for this system as

$$\mathcal{I}_{xx} = r^2 \mu \left(\cos^2(\Omega t) - \frac{1}{3} \right) \ , \tag{2.26a}$$

$$\mathcal{I}_{yy} = r^2 \mu \left(\sin^2(\Omega t) - \frac{1}{3} \right) \ , \tag{2.26b}$$

$$\mathcal{I}_{jz} = r^2 \mu \left(-\delta_{jz} \frac{1}{3} \right) \ , \tag{2.26c}$$

$$\mathcal{I}_{xy} = \mathcal{I}_{xy} = r^2 \mu \sin(\Omega t) \cos(\Omega t) \ . \tag{2.26d}$$

As GWs leave the system, they carry away energy, which has a back-reaction effect on the orbiting binary. The energy carried away by quadrupole radiation is given from equation (2.21) as

$$\frac{dE}{dt}_{\text{mass quadrupole}} = \frac{1}{5} \langle \dddot{\mathcal{I}}_{jk} \dddot{\mathcal{I}}_{jk} \rangle \ , \tag{2.27}$$

which, substituting these moments in, we find to be

$$\begin{aligned} \frac{dE}{dt}_{\text{mass quadrupole}} &= \frac{32 r^4 \mu^2 \Omega^6}{5} \left\langle 4 \cos^2(\Omega t) \sin^2(\Omega t) + \left(2 \sin^2(\Omega t) - 1 \right)^2 \right\rangle \\ &= \frac{32 r^4 \mu^2 \Omega^6}{5} \ . \end{aligned} \tag{2.28}$$

Using the Keplerian relation equation (2.24), we find this to be

$$\frac{dE}{dt}_{\text{mass quadrupole}} = \frac{32\mu^2 M^3}{5r^5} \ . \tag{2.29}$$

The energy contained in the system as a function of their separation is given by the Newtonian formula

$$E = -\frac{\mu M}{2r} \ . \tag{2.30}$$

As energy leaves, the orbiting objects sink further into their respective gravitational potentials, causing their orbital separation to decrease. Assuming this energy loss occurs adiabatically, we can find the orbital separation as a function of time. First, we find the derivative of the separation with respect to time as

$$\begin{aligned} \frac{dr}{dt} &= \left(\frac{dE}{dt}\right) / \left(\frac{dE}{dr}\right) \\ &= -\frac{64\mu M^2}{5r^3} \ . \end{aligned} \tag{2.31}$$

Integrating this, we find the separation as a function of time to be

$$r(t) = \left(\frac{256\mu M^2}{5}\right)^{\frac{1}{4}} (t_c - t)^{\frac{1}{4}} \ , \tag{2.32}$$

where t_c denotes the time when $r = 0$.

We can use this result along with equation (2.24) to find the orbital frequency $\Omega(t)$ as a function of time

$$\Omega(t) = \left(\frac{256\mathcal{M}^{\frac{5}{3}}}{5}\right)^{\frac{-3}{8}} (t_c - t)^{\frac{-3}{8}} \ , \tag{2.33}$$

where $\mathcal{M} = \eta^{3/5} M$ is the chirp mass and $\eta = \mu/M$ is the symmetric mass ratio.

We can now use equation (2.23) to find the quadrupole radiation as

$$\begin{aligned} h_c(t) \equiv -h_{xx}^{\mathrm{TT}} = h_{yy}^{\mathrm{TT}} &= \frac{4r^2(t)\mu\Omega^2(t)}{D}\left(\cos^2(\Omega(t)t) - \sin^2(\Omega(t)t)\right) \\ &= \frac{4r^2(t)\mu\Omega^2(t)}{D}\cos(2\Omega(t)t) \ , \end{aligned} \tag{2.34a}$$

$$\begin{aligned} h_s(t) \equiv -h_{xy}^{\mathrm{TT}} = -h_{yx}^{\mathrm{TT}} &= \frac{8r^2(t)\mu\Omega^2(t)}{D}\sin(\Omega(t)t)\cos(\Omega(t)t) \\ &= \frac{4r^2(t)\mu\Omega^2(t)}{D}\sin(2\Omega(t)t) \ . \end{aligned} \tag{2.34b}$$

We can split the radiation into separate parts for the amplitude evolution $A(t)$ and the phase evolution $\Phi(t)$. The radiation then takes the form

$$h_c(t) = A(t)\cos(2\Phi(t)) \ , \tag{2.35a}$$

$$h_s(t) = A(t)\sin(2\Phi(t)) \ . \tag{2.35b}$$

Using the formulae for the separation and orbital frequency evolutions, we find $A(t)$ and $\Phi(t)$ to be

$$\begin{aligned} A(t) &= \frac{4r^2(t)\mu\Omega^2(t)}{D} \\ &= \frac{4\mu M}{Dr(t)} \\ &= \frac{5^{\frac{1}{4}}}{D}\mathcal{M}^{\frac{5}{4}}\left(t_c - t\right)^{\frac{-1}{4}} \ , \end{aligned} \tag{2.36a}$$

$$\begin{aligned} \Phi(t) &= \int \Omega(t)dt \\ &= \phi_0 - \left(\frac{t_c - t}{5\mathcal{M}}\right)^{\frac{5}{8}} \ , \end{aligned} \tag{2.36b}$$

where ϕ_0 is the orbital phase at coalescence.

The signal described by equations (2.36a and 2.36b) is characterized by a sinusoidal oscillation

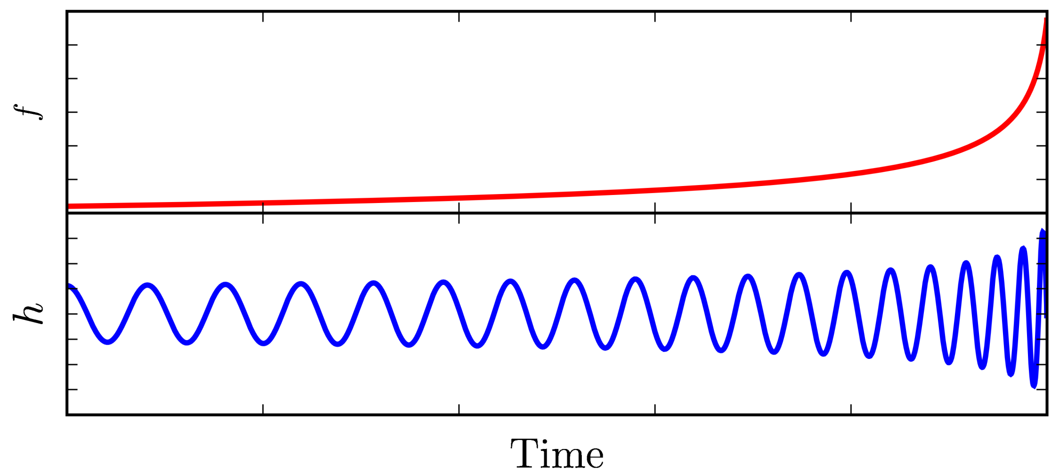

Figure 2.2: Inspiral Waveform
The frequency f and strain h evolution of an inspiral waveform given in arbitrary units.

that increase in both amplitude and frequency as a function of time (i.e., a chirp signal). This is illustrated in figure 2.2, which shows both the frequency evolution and the strain evolution over a few cycles near the end of an inspiral waveform.

This continues as long as there is a local minimum in the effective potential of the system. For the Schwarzschild potential, which is the unique potential outside a spherically symmetric mass distribution, there exists a local minimum for a test particle in a circular orbit around the object down to a separation of $6M$, called the innermost stable circular orbit (ISCO). At this point the above derivation breaks down and we terminate the calculation of the waveform. This occurs at a $(t_c - t)$ value of

$$(t_c - t) = \frac{405M}{16\eta} \ , \tag{2.37}$$

and a frequency of

$$f_{\mathrm{ISCO}} = \frac{1}{6\sqrt{6}\pi M} \ . \tag{2.38}$$

The bodies will then plunge together, merging, leaving a final perturbed black hole, which relaxes through quasi-normal mode oscillations. These final stages of the coalescence we call the "merger" and "ringdown."

2.4.1.2 Inspiral Waveform Approximants

The above derivation was a rather simple one using only the leading-order energy-loss terms and the quadrupole approximation. All of these arguments have been expanded in Post-Newtonian (PN) theory where matching between the near-zone gravitational field and the wave-zone gravitational field is done. In various studies different assumptions and expansions are used to calculate these waveforms to varying orders in v/c, where an nPN order calculation is a controlled Taylor expansion up to $(v/c)^n$. The following are a few examples: reference [3] computes waveforms to Newtonian order in amplitude and 2PN order in phase, references [4, 5, 6, 7] use the Effective One Body (EOB) approximation for the Hamiltonian to compute waveforms to Newtonian order in amplitude and 3PN order in phase, references [8, 9] use the Padé resummation technique to compute waveforms to Newtonian order in amplitude and 3.5PN order in phase. All of the previous waveforms ignored spin effects associated with angular momentum of the bodies. The SpinTaylor approximant is computed to Newtonian order in amplitude and 3.5PN order in phase using formulae from references [10] and based on references [11, 12, 3, 13, 14, 15, 8, 16].

An example waveform constructed at Newtonian order in amplitude and 2PN order in phase is given similarly as above, however, in order to simply the expressions, we introduce a dimensionless time variable

$$\Theta = \left(\frac{\eta}{5M} \left(t_c - t \right) \right)^{1/8} . \tag{2.39}$$

In terms of this new variable, the orbital phase is given by

$$\begin{aligned} \Phi(t) \quad = \quad & \phi_0 - \frac{1}{\eta} \left[\Theta^5 + \left(\frac{3715}{8064} + \frac{55}{96}\eta \right) \Theta^3 - \frac{3\pi}{4} \Theta^2 \right. \\ & \left. + \left(\frac{9\,275\,495}{14\,450\,688} + \frac{284\,875}{258\,048}\eta + \frac{1855}{2048}\eta^2 \right) \Theta^1 \right] . \end{aligned} \tag{2.40}$$

In this notation, equation (2.36b) is given by the first two terms on the right hand side of equation (2.40).

2.4.1.3 Inspiral Waveform Stationary Phase Approximation

When searching for these signals in GW detector data, it is convenient to convert these waveforms to the frequency domain using the Stationary Phase Approximation (SPA)

$$\tilde{h}(f) = \frac{A(t)}{\sqrt{\dot{f}(t)}} e^{i\Psi(f)} \tag{2.41}$$

$$= \mathcal{A}(f) e^{i\Psi(f)} \, . \tag{2.42}$$

In this approximation, the Newtonian amplitude $\mathcal{A}(f)$ is given as

$$\mathcal{A}(f) = \sqrt{\frac{5}{24}} \left(\frac{\mathcal{M}^{5/6}}{\pi^{2/3} D f^{7/6}} \right) \, , \tag{2.43}$$

and the 2PN phase $\Psi(f)$ is given in terms of a dimensionless frequency variable,

$$\tau = (\pi M f)^{1/3} \, , \tag{2.44}$$

as

$$\begin{aligned} \Psi(f) = {} & 2\pi f t_c + \phi_0 - \frac{3}{128\eta} \left[\tau^{-5} + \left(\frac{3715}{756} + \frac{55}{9}\eta \right) \tau^{-3} - 16\pi\tau^{-2} \right. \\ & \left. + \left(\frac{15\,293\,365}{508\,032} + \frac{27\,145}{504}\eta + \frac{3085}{72}\eta^2 \right) \tau^{-1} \right] \, . \end{aligned} \tag{2.45}$$

2.4.2 Other Types of Gravitational Waves

Additional examples of transient situations in which the gravitational waveform is well modelled include GWs from cusps traveling along cosmic strings, hyperbolic encounters between compact objects, and ringdowns from perturbed black holes. Since the waveform is well modeled, GW signals from these transients can be searched for using matched filtering, which we discuss in chapter 5, as was used in references [17, 18, 19, 20, 21, 22, 23].

We call transient GW events “bursts” when their waveform is not well modeled. Examples of situations generating these types of signals include GWs from supernova and neutron star glitches. In order to search for these types of signals, since the waveform is unknown, GW data is searched for transient excess power. In order to ensure that these transients are not just noise in the detectors, these types of searches require either coherent or coincident signals to be seen in multiple detectors. Examples of searches that have been performed from the LIGO Scientific Collaboration can be found in [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 23, 34].

Stochastic waves come from sources that are continuously emitting in either an incoherent fashion or in such a large number that the signals at the detectors are confused and inseparable. Examples of these types of sources include relic GWs from inflation, GWs from Galactic binary white-dwarf systems, and GWs from slow-spinning Galactic pulsars. These types of signals are searched for by looking for correlations between multiple GW detectors. Examples of these searches can be found in [35, 36, 37, 38, 39, 40, 41, 42, 43].

Continuous wave sources are are defined as signals that are quasimonochromatic and are separable from other sources. Such signals are commonly emitted by asymmetric spinning neutron stars, such as pulsars. These signals are emitted from their source at a constant frequency, however when they detected, they have been modified by Doppler shifts from motion due to the Earth's rotation, the Earth's orbit around the sun, the orbit of the source (e.g., when the source is part of a binary system), etc. Searches for these types of signals then correct for these shifts, coherently or incoherently sum the data over larger periods of time, and also correlate data from multiple detectors in order to reduce noise. Examples can be found in [44, 45, 46, 47, 48].

The remainder of this thesis focuses on GWs from CBCs.