Chapter 9

Rate (Upper Limit) Calculation

Whether the search results in a small number of detection candidates or more, we can bound the rate of CBCs in the (nearby) universe. Depending on the significance of the loudest event, the confidence band we establish may or may not exclude zero. In either case, we can establish bounds on the rate for different source mass ranges.

In this discussion, we follow the loudest event formalism described in [132, 133, 130, 131]. One advantage of calculating bounds on a rate using the loudest event formalism is that for searches that have steep background distributions, the expected upper limit you could get is always more stringent than in fixed threshold formalisms. Another advantage is that one does not need to fix a threshold before looking at the in-time triggers. It should be noted that in the end, the formalism chosen does not affect the significance of the in-time triggers and is only necessary for calculating rate bounds.

In previous CBC searches [17, 19, 18], we calculated rates of CBCs in terms of Milky Way Equivalent Galaxies (MWEGs). However, as the horizon distances of our detectors expanded to include more galaxies, we found it was prudent [134] to report rates in terms of L_{10}s, where $1\,L_{10} \equiv 10^{10}\,L_{\odot,B}$, $L_{\odot,B}$ is the solar blue-light luminosity, and 1 MWEG $\approx 1.7\,L_{10}$. We have used these units for our rate calculations in [21, 22], and we shall use it here as well. The reason for this change is because L_{10} track the rate of star formation in a given galaxy and we think CBCs are proportional to this [135]. One drawback of this is that old elliptical galaxies are ignored since they do not have significant blue-light luminosity [136].

In order to calculate an upper limit on a rate of gravitational wave signals due to a particular search, there are several things we need to know. These are related to how sensitive the search is to signals from the universe, the background of the search, and the loudest event from the search. Below we detail the calculations performed in arriving at a rate (figure 9.1).

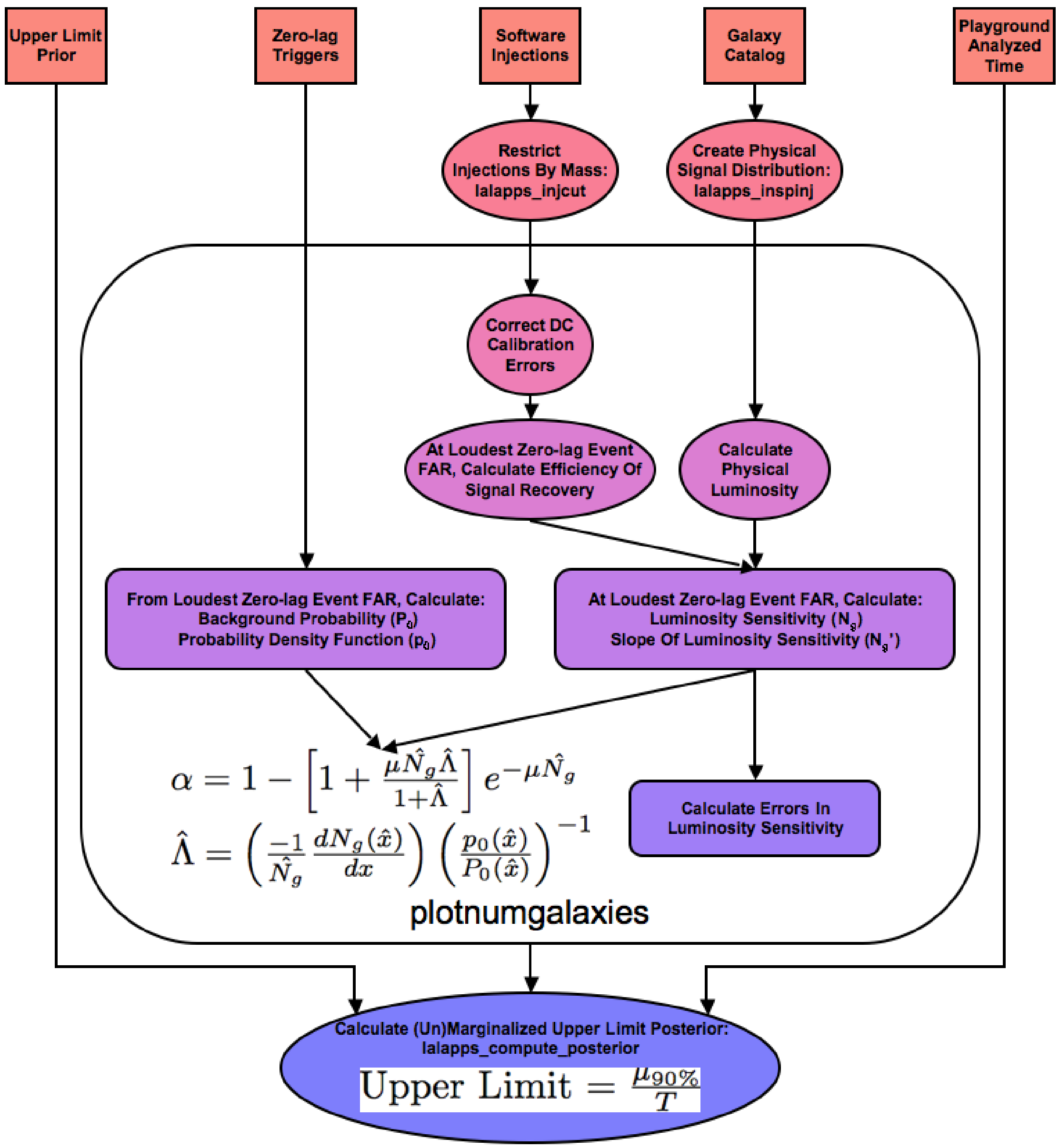

Figure 9.1: Upper Limit Calculation Pipeline
The stages of the upper limit calculation, in which we information from the top of the figure to establish bounds on the CBC rate in the nearby universe in terms of events $\text{yr}^{-1}\ L_{10}^{-1}$.

9.1 Posterior and Upper Limit Calculation

Calculating an upper limit on a rate of coalescences in the loudest-event formalism requires knowledge of the cumulative luminosity to which the search is sensitive and a measure of the likelihood that the loudest event was due to the observed background, described in section 9.3. We combine these with the time analyzed to calculate the posterior on the rate for the search. Assuming a prior on the rate $p_0(\mu)$, the posterior is given by [130]

$$p\left(\mu|\mathcal{C}_L, T, \Lambda\right) = p_0(\mu)\frac{\mathcal{C}_L T}{1+\Lambda}\left(1+\mu\mathcal{C}_L T\Lambda\right)e^{-\mu\mathcal{C}_L T} \ , \tag{9.1}$$

where μ is the rate in events yr^{-1} L_{10}^{-1}, $\mathcal{C}_L$ is the cumulative luminosity in L_{10}, T is the analyzed time in years, and Λ is a measure of the likelihood of detecting a single event with loudness parameter x (which in this analysis is FARc, cf., sections 7.3 and 8.4) versus such an event occurring due to the experimental background, given by [130]

$$\Lambda\left(x\right) = \left(\frac{-1}{\mathcal{C}_L}\frac{d\mathcal{C}_L}{dx}\right)\left(\frac{1}{P_0}\frac{dP_0}{dx}\right)^{-1} \ , \tag{9.2}$$

and P_0 the background probability given by equation (7.6b). In this search, $P_0 = dP_0/dx = e^x$, thus the term involving these quantities disappears, which we will discuss in section 9.3.2.

In the limit that $\Lambda \to 0$ (i.e., when the loudest event is absolutely due to background), the 90% confidence limit we set on the rate $\mu_{90\%}$ is given by

$$90\% = \int_0^{\mu_{90\%}} p(\mu)d\mu \Rightarrow \mu_{90\%} = \frac{2.303}{C_L T} \ , \tag{9.3}$$

and in the limit that $\Lambda \to \infty$ (i.e., when the loudest event is absolutely due to signal), the 90% confidence limit we set on the rate $\mu_{90\%}$ is given by

$$\mu_{90\%} = \frac{3.9}{C_L T} \ . \tag{9.4}$$

9.2 Incorporating Systematic Errors

The posterior (9.1) assumes a known value of $\mathcal{C}_L$ associated with the search. In reality, $\mathcal{C}_L$ has associated with it systematic uncertainties, which we model as unknown multiplicative factors ζ (fractional errors), each log-normally distributed about 1 with errors described in section 9.4. In order to marginalize over the effects of the systematic errors on the cumulative luminosity, we first calculate a probability distribution for the cumulative luminosity for the search, $p_d\left(\mathcal{C}_L\right)$. This is calculated using the log-normal error distributions of the cumulative luminosity multiplier $p_{d,\text{error}}\left(\zeta\right)$ given by

$$p_{d,\text{error}}\left(\zeta\right) = \frac{1}{\zeta\sigma_{\text{error}}\sqrt{2\pi}} e^{\frac{-\ln(\zeta)}{2\sigma^2_{\text{error}}}} \ , \tag{9.5}$$

where σ_{error} is the fractional error of the cumulative luminosity calculated in section 9.4. The $p_{d,\text{error}}\left(\zeta\right)$ distributions are then convolved with each other to obtain a combined distribution $p_{d,\text{comb}}\left(\zeta\right)$. The distribution of the cumulative luminosity is then given as

$$p_d\left(\mathcal{C}_L\right) = \int \delta\left(\mathcal{C}_L - \zeta\mathcal{C}_{L,\text{m}}\right) p_{d,\text{comb}}\left(\zeta\right) d\zeta \ , \tag{9.6}$$

where $\mathcal{C}_{L,\text{m}}$ is the measured cumulative luminosity. To obtain a marginalized posterior, we integrate the above posterior (equation (9.1)) times the distribution of cumulative luminosities (equation (9.6)) over the cumulative luminosity [130]

$$p\left(\mu|T,\zeta\right) = \int p_d\left(\mathcal{C}_L\right) p\left(\mu|\mathcal{C}_L,T,\zeta\right) d\mathcal{C}_L \ . \tag{9.7}$$

The results of several experiments (e.g., different types of S5 observing time and previous runs such as S3 and S4) can be combined by taking the product of their likelihood functions; in the case of uniform priors, this is equivalent to taking the product of their posteriors, allowing us to define the rate upper limit μ at a confidence level α by numerically solving

$$\alpha = \int_0^\mu \prod_i p_i\left(\mu'\right) d\mu' \ , \tag{9.8}$$

where the $p_i(\mu')$ are the marginalized posteriors from different experiments calculated using a uniform prior on the rate. This is equivalent to using the posterior from previous observations as the prior for subsequent ones, sequentially.

9.3 Cumulative Luminosity and Background Probability

The standard quantities we need to have before we can calculate an upper limit on the rate of coalescences are the sensitivity of the search (i.e., the cumulative luminosity), and the probability that no background triggers are louder than the loudest event.

9.3.1 Cumulative Luminosity and its Derivative

We measure the *cumulative luminosity* that the search is sensitive to in units of L_{10}. The cumulative luminosity quantifies the potential sources of observable CBC, as measured by blue-light luminosity of the galaxies containing CBCs, which can be detected by our search. This can be calculated by integrating the efficiency $\epsilon(D)$ and physical luminosity $L(D)$ both as a functions of distance over distance

$$\mathcal{C}_L \equiv \int \epsilon(D)L(D)dD \, . \tag{9.9}$$

In practice, we do this as a discrete sum

$$\mathcal{C}_L = \sum_i \epsilon_i L_i \, , \tag{9.10}$$

where ϵ_i is the efficiency in distance bin i, and L_i is the luminosity in distance bin i.

We calculate the *efficiency* ϵ_i in distance bin i as

$$\epsilon_i = \frac{N_{\mathrm{found},i}}{N_{\mathrm{injected},i}} \, , \tag{9.11}$$

where $N_{\mathrm{injected},i}$ is the number of injections injected with distance $D \in (D_{i,\mathrm{min}}, D_{i,\mathrm{max}})$, $D_{i,\mathrm{min}}$ is the minimum distance in distance bin i, $D_{i,\mathrm{max}}$ is the maximum distance in distance bin i, and $N_{\mathrm{found},i}$

is the number of found injections in that distance bin. Distance bins in which there are no injections are taken to have $\epsilon_i = 0$. An injection is counted as found if there is a trigger associated with the injection with a detection statistic louder than the detection statistic of the loudest in-time trigger.

The *physical luminosity* is calculated by creating an "injection" file using a galaxy catalog. The "injections" are distributed according to the locations and luminosities of galaxies in the catalog. More luminous galaxies end up have more "injections" associated with them. The luminosity weight W each "injection" carries is determined by the most luminous galaxy in the catalog, galaxy j. W is given by

$$W = N_j / L_j \ , \tag{9.12}$$

where N_j is the number of "injections" in galaxy j, and L_j is the luminosity of galaxy j. The physical luminosity L_i in distance bin i is then given as

$$L_i = n_i W \ , \tag{9.13}$$

where n_i is the number of "injections" from all galaxies with distance $D \in (D_{i,\min}, D_{i,\max})$.

The *derivative of the cumulative luminosity* is computed numerically by recalculating $\mathcal{C}_L$ using several values of the detection statistic x around the largest value from the in-time triggers.

9.3.2 Background Probability

The *background probability* takes a simple form with the choice of the FARc for the detection statistic. Assuming a Poisson distribution, the probability of getting zero background triggers louder than the loudest in-time trigger (i.e., with a FAR lower than the FAR of the loudest in-time trigger since low FAR are more significant) is given by equation (7.6b) where, as before, $x = -\mathrm{FAR} \times T$, FAR is the FAR of the loudest in-time trigger, and T is the total analyzed time searching for in-time triggers.

Since more significant triggers are given by lower FAR, triggers with higher values of x are louder, so the derivative with respect to the detection statistic in the direction of louder triggers is in the positive x direction (d/dx). The *derivative of the background probability* with respect to the detection statistic in the direction of louder triggers dP_b/dx is

$$\frac{dP_b}{dx} = e^x \ . \tag{9.14}$$

Combining equations (7.6b) and (9.14), we find that $(1/P_0)(dP_0/dx) = 1$, and this term in equation (9.2) goes away.

We therefore can compute all of the quantities in equation (9.1) and 9.2 needed to determine the posterior on the rate μ, before marginalizing over systematic errors.

9.4 Systematic Error Calculation

Systematic errors associated with CBC searches for GW signals include errors associated with detector calibrations, simulation waveforms, Monte Carlo statistics, and galaxy catalog distances and magnitudes. Calculating these errors in terms of the cumulative luminosity is described below [131]. We can marginalize over these uncertainties using equation (9.7), to obtain the final posterior on the rate, and use equation (9.8) to obtain the upper limits. This analysis can be repeated as function of source mass.

9.4.1 Monte Carlo Errors

We refer to statistical errors associated with the efficiency calculation as *Monte Carlo errors.* Since we calculate the efficiency as a function of distance, we calculate the error for a particular distance bin i using the binomial formula, which gives an error of zero when the efficiency is zero or one, or when there are no injections in that bin:

$$\epsilon_{\mathrm{Monte\,Carlo},i} = \sqrt{\frac{N_{\mathrm{found},i}\left(N_{\mathrm{injected},i} - N_{\mathrm{found},i}\right)}{N^3_{\mathrm{injected},i} + 10^{-5}}} \ , \tag{9.15}$$

where $\epsilon_{\mathrm{Monte\,Carlo},i}$ is the error in the efficiency, $N_{\mathrm{injected},i}$ is the number of injections injected into distance bin i, and $N_{\mathrm{found},i}$ is the number of injections found in distance bin i. Finally, the associated fractional luminosity error $\sigma_{\mathrm{Monte\,Carlo}}$ is given in terms of $\epsilon_{\mathrm{Monte\,Carlo},i}$, the physical luminosity in each distance bin L_i, and the measured cumulative luminosity $\mathcal{C}_{L,\mathrm{m}}$ as

$$\sigma_{\mathrm{Monte\,Carlo}} = \frac{1}{\mathcal{C}_{L,\mathrm{m}}} \sum_i \epsilon_{\mathrm{Monte\,Carlo},i} L_i \ . \tag{9.16}$$

9.4.2 Calibration Errors

Calibration errors in the detectors are errors due to uncertainties in the absolute calibration of the detector response to differential displacements caused by GWs [137]. These errors affect the amplitude, and in turn the distance, at which we made injections to calculate the efficiency of our search, since the injections we made assuming a specific value of the noise floor. During S5, the one-sigma uncertainty in the amplitude (and thus the distance) associated with the calibration was 8.1% for H1, 7.2% for H2, and 6.0% for L1 [137]. To calculate the luminosity error due to a calibration error $\Delta_{\mathrm{calibration}}$, all of the injections are moved further in that detector's distance by $\Delta_{\mathrm{calibration}}$

$$D' = D\left(1 + \Delta_{\mathrm{calibration}}\right) \ , \tag{9.17}$$

and the efficiency is recalculated using equation (9.11) where $N_{\mathrm{injected},i}$ is the number of injections injected with distance $D' \in (D_{i,\mathrm{min}}, D_{i,\mathrm{max}})$, $D_{i,\mathrm{min}}$ is the minimum distance in distance bin i, $D_{i,\mathrm{max}}$ is the maximum distance in distance bin i, and $N_{\mathrm{found},i}$ is the number of found injections with distance $D' \in (D_{i,\mathrm{min}}, D_{i,\mathrm{max}})$. This gives us $\epsilon_{\mathrm{calibration},i}$ where the i denotes a particular bin in distance. The standard efficiency ϵ_i is then subtracted from $\epsilon_{\mathrm{calibration},i}$ on a bin-by-bin basis

$$d\epsilon_{\mathrm{calibration},i} = \epsilon_{\mathrm{calibration},i} - \epsilon_i \ . \tag{9.18}$$

Finally, the associated fractional luminosity error $\sigma_{\mathrm{calibration}}$ is given in terms of $d\epsilon_{\mathrm{calibration},i}$, L_i, and $\mathcal{C}_{L,\mathrm{m}}$ as

$$\sigma_{\mathrm{calibration}} = \frac{1}{\mathcal{C}_{L,\mathrm{m}}} \left(\sum_i d\epsilon_{\mathrm{calibration},i} L_i \right) \ . \tag{9.19}$$

9.4.3 Waveform Errors

Waveform errors are associated with how different the true signals are from the signals we use to measure the efficiency of our pipeline (i.e., the mismatch between the true signals and our injections). This error effectively reduces the distances in our efficiency calculation since we do not recover all of the power available in the signal due to the mismatch between the signal and our injections. To calculate this, all of the injections are moved further by the error $\Delta_{\mathrm{waveform}}$

$$D' = D\left(1 + \Delta_{\mathrm{waveform}}\right) \ , \tag{9.20}$$

and the waveform fractional error is calculated as above (section 9.4.2). We calculate the waveform error in units of luminosity assuming a waveform mismatch of 10%. This value is obtained by a detailed comparison of a variety of PN waveform approximants [8, 138], and it represents a conservative estimate of this theoretical error on the true waveform.

9.4.4 Galaxy Errors

Galaxy errors are errors associated with our galaxy catalog [134] used to construct the physical luminosity. Galaxy errors come in two types: *distance errors* and *magnitude errors*.

Future searches will be normalized per unit volume (Mpc^3) to obtain a rate density, rather than per unit L_{10}, in order to avoid these errors (which are not intrinsic to this search method). In the large volume limit (i.e., above a physical distance of 30 Mpc), there is a $0.02\, L_{10}\, \mathrm{Mpc}^{-3}$ conversion that can be used to compare the two rate units.

9.4.4.1 Distance Errors

To calculate the error on the luminosity due to distance errors, the physical luminosity calculation is changed such that the distance to each of the "injections" in a galaxy is increased

$$D' = D\left(1 + \Delta_D\right) \ , \tag{9.21}$$

where Δ_D is the fractional distance error for a given galaxy. Also, the "injections" in a particular galaxy's weighting is changed to be

$$W' = W\left(1+\Delta_D\right)^2 \ , \tag{9.22}$$

where W is the original weighting for the galaxy. This form of weighting is chosen because the galaxy's luminosity is only known in terms of its magnitude and distance. If there is an error in the distance, the luminosity must change by $\left(1+\Delta_D\right)^2$ in order to maintain the same magnitude. The luminosity $L_{\text{distance},i}$ is calculated with weighting W' and with "injections" at distances D' as

$$L_{\text{distance},i} = \sum_{i=1}^{n'} W' \ , \tag{9.23}$$

where n' is the number of "injections" from all galaxies with distance $D' \in (D_{i,\text{min}}, D_{i,\text{max}})$. $L_{\text{distance},i}$ is then multiplied by the standard efficiency ϵ_i as a function of distance and integrated over distance to obtain the cumulative luminosity $\mathcal{C}_{L,\text{distance}}$ at 1σ error in distance

$$\mathcal{C}_{L,\text{distance}} = \left(\sum_i \epsilon_i L_{\text{distance},i}\right) \ . \tag{9.24}$$

The associated fractional luminosity error σ_{distance} in terms of $\mathcal{C}_{L,\text{m}}$ as

$$\sigma_{\text{distance}} = \frac{1}{\mathcal{C}_{L,\text{m}}}\left(\mathcal{C}_{L,\text{distance}} - 1\right) \ . \tag{9.25}$$

9.4.4.2 Magnitude Errors

To calculate the error on the luminosity due to magnitude errors, the physical luminosity calculation is changed such that the "injections" in a particular galaxy are weighted by

$$W' = W 10^{\Delta_M/2.5} \ , \tag{9.26}$$

where Δ_M is the (absolute) magnitude error for a given galaxy. This follows from the formula used to calculate the luminosity of a galaxy

$$L_{B,i} = L_{B,\odot} \left(\frac{D_i}{10\text{pc}} \right)^2 10^{(M_{B,\odot} - m_{B,i})/2.5} , \tag{9.27}$$

where $L_{B,\odot}$ is the blue solar luminosity, D_i is the distance to the galaxy, $M_{B,\odot}$ is the (absolute) blue solar magnitude, and $m_{B,i}$ is the (measured) apparent blue magnitude of the galaxy. The luminosity $L_{\text{magnitude},i}$ is calculated with weighting W' as

$$L_{\text{magnitude},i} = \sum_{i=1}^{n} W' , \tag{9.28}$$

where n is the number of "injections" from all galaxies with distance $D \in (D_{i,\text{min}}, D_{i,\text{max}})$. The magnitude fractional error is calculated as above (section 9.4.4.1).