Descriptive Properties of Measure Preserving Actions and the Associated Unitary Representations

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Abstract

Let Γ be a countable group and X a Borel Γ-space with invariant Borel probability measure μ. Let $E = E_\Gamma$ be the countable equivalence relation defined by

$$xEy \iff \exists\gamma \in \Gamma(\gamma \cdot x = y).$$

This thesis consists of two independent parts, Chapter 2 and Chapter 3:

In Chapter 2, we study the descriptive complexity of the full group $[E]$. The main result of this chapter is

i) If E is not smooth, then $[E]$ is $\mathbf{\Pi}^0_3$-complete;

ii) If E is smooth, then $[E]$ is closed.

We also study the descriptive complexity of $N[E]$, the normalizer of $[E]$. It turns out that $N[E]$ has the same complexity as $[E]$, i.e., $N[E]$ is $\mathbf{\Pi}^0_3$-complete iff E is not smooth and is closed if E is smooth.

In Chapter 3, we study descriptive properties of π^X, the Koopman unitary representation associated with the action. Consider the induced Polish Γ action on $L^2(X)$, i.e., $\gamma \cdot f = \pi^X(\gamma)(f)$. Denote by $E_\Gamma^{L^2(X)}$ the induced countable Borel equivalence relation on $L^2(X)$, i.e.,

$$fE_\Gamma^{L^2(X)}g \iff \exists\gamma \in \Gamma(g = \gamma \cdot f).$$

Let Γ act on the measure algebra of μ, $MALG_\mu$, by $\gamma \cdot A = \gamma(A)$ and on $Aut(X,\mu)$ by $\gamma \cdot T = \pi^X(\gamma)T$. We have the induced countable equivalence relations $E_\Gamma^{MALG_\mu}$ and $E_\Gamma^{Aut(X,\mu)}$ respectively. We relate the descriptive complexity of $E_\Gamma^{L^2(X)}$ to that of E_Γ^X. We show that the smoothness of $E_\Gamma^{L^2(X)}$ is equivalent to the smoothness of $E_\Gamma^{MALG_\mu}$ and the compressibility of the nonconstant part of $E_\Gamma^{L^2(X)}$ is equivalent to the compressibility of $E_\Gamma^{MALG_\mu \setminus \{X,\emptyset\}}$. We also connect the smoothness and compressibility of $E_\Gamma^{L^2(X)}$ to mixing properties of the action of Γ on X. Finally, we will show that the amenability of E_Γ^X implies a certain weak containment property of π^X.

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CHAPTER 1
Introduction

Let Γ be a countable group and X a standard Borel Γ-space with an invariant (nonatomic) Borel probability measure μ. Γ induces an equivalence relation E^X_Γ on X, which is defined by $x E^X_\Gamma y \iff \exists \gamma \in \Gamma(\gamma \cdot x = y)$. By a theorem of Feldman-Moore (see **[KM]**, Theorem 1.3), every countable Borel equivalence relation on a standard Borel space X is induced by some Borel action of some countable group Γ.

Denote by $Aut(X, \mu)$ the group of μ-measure preserving automorphisms of X (modulo null sets). For each $T \in Aut(X, \mu)$, we can define a corresponding unitary operator $U_T \in U(L^2(X))$, $U_T(f) = f \circ T^{-1}$. And by identifying T and U_T, we can view $Aut(X, \mu)$ as a subgroup of $U(L^2(X))$. The weak topology of $Aut(X, \mu)$ is the subspace topology of the weak topology on $U(L^2(X))$. $Aut(X, \mu)$ is a closed subspace of $U(L^2(X)$ in the weak topology, hence Polish.

In Chapter 2, we will study the descriptive complexity of an important invariant of E, namely the *full group* of E. The full group of E, denoted by $[E]$, is defined by

$$[E] = \{T \in Aut(X, \mu) : xETx \text{ a.e.}\}.$$

The main result of this chapter is

i) If E is not smooth, then $[E]$ is $\mathbf{\Pi}^0_3$-complete.

ii) If E is smooth, then $[E]$ is closed.

And the same result holds for $N[E]$:

i) If E is not smooth, then $N[E]$ is $\mathbf{\Pi}^0_3$-complete.

ii) If E is smooth, then $N[E]$ is closed.

Define

$$\pi^X : \Gamma \to Aut(X,\mu) \subset U(L^2(X)),$$

$$\gamma \mapsto T,$$

where T is the element in $Aut(X,\mu)$ such that $T(x) = \gamma \cdot x$ for μ-almost every $x \in X$. Or equivalently $T(f) = U_T(f) = f(\gamma^{-1}\cdot)$ for every $f \in L^2(X)$. Clearly π^X is a homomorphism of Γ into $U(L^2(X))$, i.e., $\pi^X$is a unitary representation of Γ on the Hilbert space $L^2(X)$.

The unitary representation π^X induces a natural Polish Γ action on $L^2(X)$, i.e., $\gamma \cdot f = \pi^X(\gamma)(f)$. Denote by $E_\Gamma^{L^2(X)}$ the induced countable Borel equivalence relation on $L^2(X)$, i.e.,

$$f E_\Gamma^{L^2(X)} g \iff \exists \gamma \in \Gamma (g = \gamma \cdot f).$$

In Chapter 3, we will study relations between E_Γ^X and $E_\Gamma^{L^2(X)}$.

We will obtain characterizations of the smoothness and compressibility of $E^{L^2(X)}$ and reducibility results. Denote by $MALG_\mu$ the measure algebra of μ (see Section 3.1.2). Let Γ act on $MALG_\mu$ by $\gamma \cdot A = \gamma(A)$. Similarly, we have the induced equivalence relation $E_\Gamma^{MALG_\mu}$ defined by

$$A E_\Gamma^{MALG_\mu} B \iff \exists \gamma \in \Gamma (B = \gamma \cdot A).$$

We will show that $E_\Gamma^{L^2(X)}$ is smooth iff $E_\Gamma^{MALG_\mu}$ is smooth and the nonconstant part of $E_\Gamma^{L^2(X)}$ is compressible iff $E_\Gamma^{MALG_\mu\setminus\{X,\emptyset\}}$ is compressible. These descriptive properties of $E_\Gamma^{L^2(X)}$ also have connections with mixing properties of the Γ action:

i) If Γ action is mildly mixing, then $E_\Gamma^{L^2(X)}$ is smooth.

ii) The nonconstant part of $E_\Gamma^{L^2(X)}$ is compressible iff the Γ action is weakly mixing.

Furthermore, we will show that smoothness is related to rigid factors and compressibility is related to isometric factors.

At the end of this chapter, we will study some embedding properties, containment properties, and their applications. Denote by λ_Γ the regular unitary representation of Γ and by $\lambda_{\Gamma/\Gamma_x}$ the quasi-regular unitary representation on $\lambda_{\Gamma/\Gamma_x}$. If the Γ action is amenable, then $\pi^X \prec \lambda_\Gamma$ (see [**Kuhn**]). We show that if E_Γ^X is amenable, then $\pi^X \prec \int_X^\oplus \lambda_{\Gamma/\Gamma_x} d\mu(x)$.

CHAPTER 2

Descriptive Complexity of Full Groups

2.1. Full groups and their normalizers

Let μ be a Borel probability measure defined on a standard Borel space X. Recall that $Aut(X,\mu)$ denotes the group of all measure preserving Borel isomorphisms (modulo null sets) on X.

There are two frequently used topologies defined on $Aut(X,\mu)$, namely the uniform topology and the weak topology.

Let's use $\mathbf{B}(X)$ to denote the set of Borel subsets of X and $\mathcal{A} = \{A_n\}$ an algebra generating $\mathbf{B}(X)$. The uniform topology has as basis the sets of the form

$$V_{T,\epsilon} = \{S \in Aut(X,\mu) : \sup\{\mu(S(A)\Delta T(A)) : A \in \mathbf{B}(X)\} < \epsilon\}.$$

It has two compatible metrics:

$$d_1(S,T) = \mu\{x|S(x) \neq T(x)\}$$

and

$$d_2(S,T) = \sup\{\mu(S(A)\Delta T(A))|A \in \mathbf{B}(X)\} = \sup_{n\in\mathbb{N}}\{\mu(S(A_n)\Delta T(A_n))\}.$$

$Aut(X,\mu)$ is not separable in the uniform topology.

The weak topology has as sub-basis the sets of the form

$$W_{T,A_n,\varepsilon} = \{S|\mu(S(A_n)\Delta T(A_n)) < \varepsilon\}$$

and a complete and compatible metric:

$$\rho(S,T) = \sum 2^{-n}(\mu(S(A_n)\Delta T(A_n)) + \mu(S^{-1}(A_n)\Delta T^{-1}(A_n))).$$

$Aut(X,\mu)$ is a Polish group in the weak topology.

Now consider a countable Borel equivalence relation E defined on a standard Borel space X. By Theorem 1.3 in [**KM**], E is induced by Borel action of a countable group G acting on X, i.e., $E = E_G$, $xEy \iff \exists g \in G(g \cdot x = y)$. We call a Borel measure μ on X *E-invariant* if μ is G-invariant for every countable Borel group G such that $E_G = E$.

Define for μ that which is $E-$invariant

$$[E] = \{T \in Aut(X,\mu) : xETx \text{ a.e.}\}$$

and let $N(E) \subseteq Aut(X,\mu)$ be the normalizer of $[E]$, i.e.,

$$N(E) = \{T \in Aut(X,\mu) : T^{-1}[E]T = [E]\}.$$

The goal of this chapter is to determine the descriptive complexity of the full groups of countable Borel equivalence relations and their normalizers in the weak topology.

2.2. Upper bound of the complexity of $[E]$ and $N(E)$

Let $\{f_n\} \subseteq [E]$ be a Cauchy sequence in the uniform topology and assume $\forall n \forall m > n \; d_1(f_n, f_m) < 2^{-n}$. For each $n \in N$, let

$$Y_n = \{x | \forall m > n (f_m(x) = f_n(x))\}.$$

We have $\mu(Y_n) > 1 - 2^{-n}$, and define $f : X \to X$ by $f(x) = f_n(x)$ if $x \in Y_n$. Clearly f is well-defined μ-almost everywhere and is in $[E]$. Since $d_1(f, f_n) < 2^{-n}$, we have $f = \lim f_n$. Therefore $[E]$ is complete in the uniform topology

In general, $[E]$ is not closed in the weak topology. For example, consider the Vitali equivalence relation E_0 on $(X, \mu) = ([0,1], m)$, xE_0y iff $2^n(x-y) \in \mathbb{N}$ for some $n \in \mathbb{N}$. Since E_0 is ergodic, for all measurable subsets $A, B \subseteq X$, $\exists T \in [E_0] \, (T(A) = T(B)$ iff $\mu(A) = \mu(B)$. So clearly $\overline{[E_0]} = Aut(X, \mu)$ in the weak topology. But $[E_0] \neq Aut(X, \mu)$ (for example $x \mapsto x + \pi \, mod \, 1$ is in $Aut(X, \mu) \setminus [E_0]$), thus $[E_0]$ is not closed in the weak topology.

More generally, let E be an ergodic countable Borel equivalence relation. Then $\overline{[E]} = Aut(X, \mu)$ by the same reason above. We may assume $(X, \mu) = ([0,1], m)$ and consider $f_r : x \mapsto x + r \;\, mod \, 1$. We have $f_r \in Aut(X, \mu)$ and $d_1(f_{r_1}, f_{r_2}) = 1$ if $r_1 \neq r_2$, therefore $Aut(X, \mu)$ is not separable in the uniform topology. However, by the following proposition, we can show that $[E]$ is separable in the uniform topology, hence $[E] \neq Aut(X, \mu) = \overline{[E]}$ in the weak topology.

Proposition 2.2.1. *If E is a countable Borel equivalence relation on a standard Borel space X with invariant probability measure μ, then $[E]$ is separable in the uniform topology.*

PROOF. Let $T, S : X \to X$ be Borel maps. Consider the equivalence relation

$$T \sim S \iff \forall^\mu x (T(x) = S(x)).$$

Denote by $[T]$ the equivalence class of T. When T is an automorphism, it is customary to write T instead of $[T]$ if there is no danger of confusion. To avoid the confusion with the [ ] notation of full groups, we will write the equivalence of T as T instead $[T]$ for all Borel maps too. So when we write *T is Borel (modulo null)*, we mean the equivalence class of T.

Let

$$\mathcal{A} = \{T | T : X \to X \text{ is Borel (modulo null).}\}.$$

We can view $(Aut(X, \mu), d_1)$ as a metric subspace of $(\mathcal{A}, d_1)$. We only need to show that there exists a countable subset $\mathcal{S} \subseteq \mathcal{A}$, such that $[E] \subseteq \overline{\mathcal{S}}$.

Construct $\mathcal{S}$ by the following steps. First, let $\{P_m\}$ be a sequence of finite Borel partitions of X where $P_m = \{A_n^m\}$, such that for any finite Borel partition $\{X_n\}$ and $\varepsilon > 0$, there is a $P_m = \{A_n^m\}$ with $\forall n \, (\mu(A_n^m \Delta X_n) < \varepsilon)$.

Then since E is countable, we may assume X is a countable Borel G-space and $E = E_G$. Let $\{g_i\}$ be a finite subset of G and $|\{g_i\}| \geq |P_m|$, define $S_{\{g_i\},m}$ by $S_{\{g_n\},m} | A_n^m = g_n | A_n^m$. Let

$$\mathcal{S} = \{S_{\{g_n\},m}\}_{|\{g_n\}| < \infty, \{g_n\} \subseteq G}.$$

Then clearly $S \subseteq \mathcal{A}$ and is countable.

We only need to show $[E] \subseteq \overline{\mathcal{S}}$, that is

$$\forall \varepsilon > 0 \forall T \in [E] \exists S \in \mathcal{S}\,(d_1(S,T) < \varepsilon).$$

Since $T \in [E]$, we can find a finite subset $\{g_n\}_{n \leq N} \subseteq G$ and a finite partition $\{X_n\}_{n \leq N}$ such that $T|X_n = g_n|X_n$, if $n < N$, and $\mu(X_N) < \frac{\varepsilon}{2}$. Fix an m so that $\mu(A_n^m \Delta X_n) < \frac{\varepsilon}{2N}$. Clearly

$$d_1(T, S_{\{g_n\},m}) < \mu(X_N) + \sum \mu(A_n^m \Delta X_n) < \varepsilon.$$

Therefore, $[E]$ is separable in the uniform topology. □

The following proposition gives an upper bound on the Borel complexity of $[E]$.

Proposition 2.2.2. *If E is a countable Borel equivalence relation on a standard Borel space X with invariant probability measure μ, then $[E]$ is $\mathbf{\Pi}^0_3$ in the weak topology.*

PROOF. Since $[E]$ is separable in the uniform topology by the above proposition, we can fix a countable dense subset $\{T_n\}_{n \in \mathbb{N}} \subseteq [E]$ in the uniform topology. And since $[E]$ is closed in the uniform topology, we can write

$$[E] = \bigcap_{m=1}^{\infty} \bigcup_{n=0}^{\infty} \{S : d_2(S, T_n) < 2^{-m}\}.$$

Note that $\{S : d_2(S, T_n) < 2^{-m}\}$ is an open ball in the d_2 metric. Let $\mathcal{A} = \{A_n\}$ be an algebra generating $\mathbf{B}(X)$. We can write an open ball in the d_2 metric as

$$\begin{aligned}\{S : d_2(S,T) < r\} &= \bigcup_{m=1}^{\infty} \{S : \sup_n \{\mu(SA_n \Delta TA_n)\} \leq r - 2^{-m}\} \\ &= \bigcup_{m=1}^{\infty} \bigcap_{n=0}^{\infty} \{S : \mu(SA_n \Delta TA_n) \leq r - 2^{-m}\},\end{aligned}$$

which is clearly $\mathbf{\Sigma}_2^0$ in the weak topology. Hence $[E]$ is $\mathbf{\Pi}_3^0$. □

We can use this to show that $N(E)$ is in $\mathbf{\Pi_3^0}$ in the weak topology.

Proposition 2.2.3. *If E is a countable Borel equivalence relation on a standard Borel space X with invariant probability measure μ, and if $[E] \in \mathbf{\Pi_3^0}$ in the weak topology, then $N(E) \in \mathbf{\Pi_3^0}$ in the weak topology too. In particular, by Proposition 2.2.2, $N(E) \in \mathbf{\Pi_3^0}$ in the weak topology.*

Proof. Let G be a countable group of Borel automorphisms generating E.

First note that if $T \in Aut(X, \mu)$ and $\forall g \in G\,(TgT^{-1} \in [E])$, then if xEy, $y = g(x)$ for some $g \in G$,

$$T(y) = T(g(x)) = TgT^{-1}T(x).$$

Since $TgT^{-1} \in [E]$, there is a conull subset $Y \subseteq X$, such that

$$\forall x, y \in Y\,(xEy \Rightarrow T(y)ET(x)).$$

Similarly if $\forall g \in G\,(T^{-1}gT \in [E])$ then there is a conull subset $Y \subseteq X$, $\forall x, y \in Y\ (T(x)ET(y) \Rightarrow xEy)$.

So if $\forall g \in G\,(TgT^{-1} \in [E])$ and $\forall g \in G\,(T^{-1}gT \in [E])$, then $T \in N(E)$.

On the other hand, if $T \in N[E]$, then

$$T^{-1}[E]T = [E] = T[E]T^{-1},$$

so $\forall g \in G \subseteq [E]$, $TgT^{-1} \in [E]$, and $T^{-1}gT \in [E]$.

Therefore,

$$N(E) = \bigcap_{g \in G} \{T | TgT^{-1} \in [E]\} \cap \{T | T^{-1}gT \in [E]\}.$$

Clearly $T \mapsto TgT^{-1}$ and $T \mapsto T^{-1}gT$ (for any $g \in G$) are continuous maps from $Aut(X, \mu)$ to itself, and reduce $\{T | TgT^{-1} \in [E]\}$ and $\{T | T^{-1}gT \in [E]\}$ to $[E]$, respectively. Therefore $N(E)$ is $\mathbf{\Pi}^0_3$. □

2.3. Smooth equivalence relations and the closure of $[E]$

To determine the exact complexity of $[E]$, the simplest case is when E is μ-smooth, i.e., $E|Y = F|Y$, where F is smooth and $Y \subseteq X$ is μ-conull. We can even slightly loosen our conditions. Consider a (not necessarily countable) Borel equivalence relation F on X, andconsider μ a (not necessarily F-invariant) Borel probability measure on X. We define

$$[F] = \{T \in Aut(X, \mu) | \forall^*_\mu x \, (T(x) F x)\}$$

(which extends the concept of the full group to general Borel equivalence relations). We have the following simple proposition:

Proposition 2.3.1. *Let F be a Borel equivalence relation on a standard Borel space X with a Borel probability measure μ. If F is smooth, then $[F]$ is closed in the weak topology of $Aut(X,\mu)$.*

Proof. F is smooth, hence $F \leq_B \Delta([0,1])$. Let $f : X \to [0,1]$ be a Borel function such that xFy iff $f(x) = f(y)$. Then the assignment $T \mapsto f \circ T$ is a continuous map from $Aut(X,\mu)$ to $L_2(X)$. Note that $T \in [F]$ iff $fT = f$ (modulo null sets), hence $[F]$ is closed. □

Notice that if two Borel equivalence relations F_1 and F_2 agree a.e. in the sense that $F_1|Y = F_2|Y$ for some co-null subset $Y \subseteq X$, then $[F_1] = [F_2]$. We have:

Corollary 2.3.2. *Let F be a Borel equivalence relation on a standard Borel space X with a Borel probability measure μ. If F is μ-smooth, then $[F]$ is closed in the weak topology of $Aut(X,\mu)$.*

It remains to find the complexity of E when it is not μ-smooth.

Lemma 2.3.3. *If E is a countable Borel equivalence relation on a standard Borel space X with invariant probability measure μ, then $[E]$ is closed or $\mathbf{\Sigma}^0_2$-hard.*

Proof. If $[E]$ is not closed, then $\exists T \in \overline{[E]} \setminus [E]$. If $[E]$ is $\mathbf{\Pi}^0_2$, then since $[E]$ is dense in $\overline{[E]}$, $[E]$ is comeager in $\overline{[E]}$. Hence the coset $[E]T$ is also comeager in $\overline{[E]}$, but $[E] \cap ([E]T) = \varnothing$, a contradiction. Therefore $[E]$ is not $\mathbf{\Pi}^0_2$, so it is $\mathbf{\Sigma}^0_2$ -hard (see **[Kechris 2]**, Theorem 22.10). □

It makes sense to see what $\overline{[E]}$ is. Recall the uniform ergodic decomposition for invariant measures of E. Denote by $P(X)$ the set of probability measures on X, by $\mathcal{I}_E \subseteq P(X)$ the set of E-invariant Borel probability measures on X and by

$\mathcal{EI}_E \subseteq P(X)$ the set of E-invariant ergodic Borel probability measures on X. We have (see [**KM**], Theorem 3.3)

Theorem 2.3.4. *(Farrell, Varadarajan) Let E be a countable Borel equivalence relation on a standard Borel space X. Assume $\mathcal{I}_E \neq 0$. Then there is a unique (up to null sets) Borel surjection $\pi : X \to \mathcal{EI}_E$ such that*

(1) $\pi(x) = \pi(y)$ if xEy;

(2) If $X_e = \{x : \pi(x) = e\}$, for $e \in \mathcal{EI}_E$, then $e(X_e) = 1$;

(3) For any $\mu \in \mathcal{I}_E$, $\mu = \int \pi(x) d\mu(x)$.

So we can write $\mathcal{EI}_E = \{e_x\}$, where $e_x = \pi(x)$.

From now on, we will use the above notations: π, X_e to denote the unique ergodic decomposition of (X, E) and F to denote the Borel equivalence relation on X, which is defined by xFy iff $\pi(x) = \pi(y)$. Since F is smooth, $[F]$ is closed by 2.3.2. and clearly $[E] \subseteq [F]$. Furthermore, we can show that $[F]$ is the closure of $[E]$.

Theorem 2.3.5. *Let E be a countable Borel equivalence relation on a standard Borel space X with invariant probability measure μ. Given $S \in [F]$ and a Borel set A, then there is a $T \in [E]$, such that $T(A) = S(A)$.*

Proof. Let Y be an F- invariant Borel set. Then

$$T(A \cap Y) = T(A) \cap T(Y) = T(A) \cap Y,$$

hence $\mu(A \cap Y) = \mu(T(A) \cap Y)$.

Consider the set $Y = \{x : e_x(A) > e_x(T(A))\}$. If $\mu(Y) > 0$, then $\mu(A \cap Y) > \mu(T(A) \cap Y)$. But Y is F-invariant, so this contradicts that $\mu(A \cap Y) = \mu(T(A) \cap Y)$. So $\mu(Y)$=0. We may assume therefore that $\forall x \in X,\ \ e_x(A) = e_x(T(A)) > 0$.

By a well-known lemma (see [**Kechris 3**], p.117, Lemma 4.50), there are disjoint E-invariant sets P ,Q, and R, such that $[A] \cup [T(A)] = P \cup Q \cup R$, and $A \cap P \prec T(A) \cap P$, $T(A) \cap Q \prec A \cap Q$, $A \cap R \approx T(A) \cap R$. But P, Q are also F-invariant, $\mu(A \cap P) = \mu(T(A) \cap P)$, $\mu(A \cap Q) = \mu(T(A) \cap Q)$, so $\mu(P) = \mu(Q) = 0$, $A \approx T(A)$. □

Corollary 2.3.6. *$[F]$ is the closure of $[E]$.*

Proof. We only need to show that $[E]$ is dense in $[F]$.

Recall that weak topology of $Aut(X, \mu)$ has as basis the sets of the form

$$U_{S,P,\varepsilon} = \bigcap_{A \in P} \{T : \mu(T(A) \Delta S(A)) < \varepsilon\},$$

where $S \in Aut(X, \mu)$, P is a finite Borel partition of X, and $\varepsilon > 0$ is a real.

Fix an arbitrary $S \in [F]$, a finite Borel partition $P = \{A_i\}$ and an $\varepsilon > 0$. By Theorem 2.3.5, there is a $T_i \in [E]$ such that $T_i(A_i) = S(A_i)$ for each $A_i \in P$. Let $T = \cup(T_i|A_i)$. Since S is an automorphism, $\{S(A_i)\}$ is a Borel partition. Thus $T \in Aut(X, \mu)$. Since $T(x) = T_i(x)Ex$ for some i, $T \in [E]$. Finally,

$$\mu(T(A_i) \Delta S(A_i)) = \mu(T_i(A_i) \Delta S(A_i)) = 0 < \varepsilon.$$

Therefore, $\overline{[E]} = [F]$. □

For a countable Borel E, we can divide E into the periodic part and the aperiodic part, $E = E_{aperiodic} \cup E_{periodic}$. The periodic part is smooth, $E = F$ in this part. So we only need to deal with the case that E is aperiodic. We have the following isomorphism theorem for (X, F, μ):

Lemma 2.3.7. *Let E be a countable Borel equivalence relation on a standard Borel space X with a nonatomic invariant probability measure μ.*

If E is aperiodic, then $(X, \mu, F) \cong (\mathcal{EI}_E \times [0,1], \pi\mu \times m, \Delta \times I)$.

Proof. By the isomorphism theorem of standard Borel spaces with nonatomic probability measures (see **[Kechris 2]**, Theorem 17.41), we can assume $(X, \mu) = ([0,1], m)$.

Define

$$f : X \to \mathcal{EI}_E \times [0,1]$$

$$x \mapsto (\pi(x), \pi(x)([0,x]))$$

and

$$g : \mathcal{EI}_E \times [0,1] \to X$$

$$(e, r) \mapsto \inf\{y | e([0,y]) \geq r\}.$$

Clearly, f and g are Borel, and since

$$\begin{aligned} gf(x) &= g(\pi(x), \pi(x)([0,x])) \\ &= \inf\{y | \pi(x)([0,y]) \geq \pi(x)([0,x])\} \\ &= \inf\{y | \pi(x)([y,x]) = 0\}. \end{aligned}$$

Let $A = \{x|gf(x) \neq x\}$, we have

$$\begin{aligned} A &= \{x|\exists y < x\, \pi(x)([y,x]) = 0\} \\ &= \bigcup_{n\in\mathbb{N}} \{x|\pi(x)([x-2^{-n},x]) = 0\} \end{aligned}$$

Also let $A_n = \{x|\pi(x)([x-2^{-n},x]) = 0\}$. It is easy to see that $e(A_n) = e(A_n \cap X_e) = 0$ for all $e \in \mathcal{EI}$, therefore $\mu(A_n) = 0$, hence $\mu(\bigcup A_n) = \mu(A) = 0$. That is $gf(x) = x$ almost everywhere.

It is easy to check that $f\mu = \pi\mu \times m$ and note that $f(X_e) = \{e\} \times [0,1]$, hence $(X, \mu, F) \cong (\mathcal{EI}_E \times [0,1], \pi\mu \times m, \Delta \times I)$. □

2.4. The Descriptive Complexity of $[E]$ and $N(E)$

By 2.3.3 we know that $[E]$ is either closed or $\mathbf{\Sigma}^0_2$-hard. In fact, we can show that in the case that $[E]$ is not closed, $[E]$ is not only $\mathbf{\Sigma}^0_2$-hard but also $\mathbf{\Pi}^0_3$-complete:

Proposition 2.4.1. *If E is a countable Borel equivalence relation on a standard Borel space X with invariant probability measure μ, then $[E]$ is closed or $\mathbf{\Pi}^0_3$-complete.*

Proof. Assume $[E]$ is not closed. We claim that we can find a Borel partition $\{Y_i\}_{i\in N}$ of X, such that $\mu(Y_i) > 0$ and $[E|Y_i]$ are not closed (hence $\mathbf{\Sigma^0_2}$-hard by 2.3.3) in $Aut(Y_i, \mu|Y_i)$.

Granting this, since $[E|Y_i]$ is $\mathbf{\Sigma}^0_2$-hard, we have $Q_2 \leq_c [E|Y_i]$ (where

$$Q_2 = \{x \in \mathcal{C} :\, \exists m \forall n > m\, (x(n) = 0)\}$$

is a $\mathbf{\Sigma}^0_2$-complete set see [**Kechris 2**], p.179).

Define

$$\mathcal{F} : \prod_{i\in\mathbb{N}} Aut(Y_i, (\mu|Y_i)/\mu(Y_i)) \to Aut(X, \mu),$$

$$(f_i)_{i\in\mathbb{N}} \mapsto f = \bigcup f_i$$

, i.e, $f|Y_i = f_i$. It is easy to see that $\mathcal{F}$ is continuous and

$$\begin{aligned}(f_i)_{i\in\mathbb{N}} \in \prod [E|Y_i] &\iff \forall i(f_i \in [E|Y_i]) \\ &\iff \forall i(((\cup f_i)|Y_i \in [E|Y_i]) \\ &\iff \bigcup f \in [E]\end{aligned}$$

where $f_i \in Aut(Y_i, (\mu|Y_i)/\mu(Y_i)$. Therefore $\prod_{i\in\mathbb{N}}[E|Y_i] \leq_c [E]$, hence

$$P_3 = \{x \in 2^{\mathbb{N}\times\mathbb{N}} : \forall m\,(x_m \in Q_2)\} = Q_2^{\mathbb{N}} \leq_c \prod_{i\in\mathbb{N}}[E|Y_i] \leq_c [E].$$

Since P_3 is $\mathbf{\Pi}^0_3$-complete (see [**Kechris 2**], p.179), $[E]$ is $\mathbf{\Pi}^0_3$-complete.

It remains to show that the claim is true. Let $X_0 = X$. We will define X_i and $Y_i = X_i \backslash X_{i+1}$ inductively. Suppose we already have X_i such that $\mu(X_i) > 0$ and $[E|X_i]$ is not closed (which is clearly true in the base case that $i = 0$). We can therefore fix a $T \in \overline{[E|X_i]} \backslash [E|X_i]$, and the non-null Borel subset $A = X_i \backslash \{x \in X_i | T(x)Ex\}$. Then we simply define X_{i+1} and Y_{i+1} to be any Borel partition of X_i such that both $X_{i+1} \cap A$ and $Y_{i+1} \cap A$ are not null. That is $X_{i+1} \sqcup Y_{i+1} = X_i$ and $0 < \mu(X_{i+1} \cap A) < \mu(A)$, $0 < \mu(Y_{i+1} \cap A) < \mu(A)$.

By Theorem 2.3.5, there is an $S \in [E|X_i]$ such that $S(X_{i+1}) = T(X_{i+1})$. Since

$$S^{-1}T(x)ET(x)\not{E}x$$

for almost every $x \in A \cap X_i$ and $S^{-1}T \in \overline{[E|X_i]}$, we have

$$S^{-1}T|X_{i+1} \in \overline{[E|X_{i+1}]} \backslash [E|X_i],$$

hence $[E|X_{i+1}]$ is not closed. Similarly, $[E|Y_{i+1}]$ is not closed.

And since $X_i = X_{i+1} \sqcup Y_{i+1}$and $X = X_0$, $\{Y_i\}$ is a Borel partition of X.

This completes the proof. □

We are ready to prove our main theorem.

Theorem 2.4.2. *Let E be a countable Borel equivalence relation on a standard Borel space X with invariant probability measure μ.*

If E is μ-smooth, then $[E]$, $N(E)$ are closed.

If E is not μ-smooth, then $[E]$ and $N(E)$ are $\mathbf{\Pi}^0_3$-complete .

Proof. If E is μ-smooth, the result follows from 2.3.2 and 2.2.3.

For non μ-smooth E, since the periodic part is smooth, we can assume E is aperiodic and, furthermore, by 2.3.7, we can assume that

$$(X, \mu, F) = (\mathcal{EI}_E \times [0,1], \pi\mu \times m, \Delta \times I).$$

Define $h_s : (e, r) \mapsto (e, r + s \, mod \, 1)$. We have $h_s \in [\Delta \times I] = [F]$ and $d_0(h_s, h_t) = 1 - \delta_{s,t}$. So $[F]$ is not separable in the uniform topology, while $[E]$ is, hence $[E] \neq [F]$, $[E]$ is not closed, and $[E]$ is $\mathbf{\Pi}^0_3$-complete from 2.4.1.

To determine the complexity of $N[E]$, by 2.3.7 and 2.3.5 we can find a Borel subset $Y \subset X$, $S \in [E]$, such that $S^2 = 1$ and $\{Y, S(Y)\}$ is a partition of X (modulo null sets). For example, $Y = g(\{(e,r)|r < \frac{1}{2}\})$, $S = g \circ (\lambda(e,r).(e, r + \frac{1}{2} \, mod \, 1)) \circ f$ with notations as in Lemma 2.3.7. Define

$$h : Aut(Y, 2\mu|Y) \to Aut(X, \mu),$$

$$h(T)(x) = \begin{cases} T(x) & \text{if } x \in Y \\ x & \text{if } x \in S(Y) \end{cases}.$$

It is easy to check that h is continuous. If $T \in [E|Y]$, then $h(T) \in [E] \subseteq N(E)$. Conversely, if $h(T) \in [E]$, then there is a null subset $N \subseteq X$, such that

$$\forall x, y \in X \backslash N \, (xEy \iff h(T)xEh(T)y).$$

We may also assume $S(N) = N$, otherwise, just replace N with $N \cup S(N)$. If $x \in Y \backslash N$, then $Sx \in S(Y) \backslash N \subset X \backslash N$. We have then $T(x) = h(T)(x)Eh(T)S(x) = S(x)E(x))$ for all $x \in Y$, hence $T \in [E|Y]$. Therefore $T \in [E|Y]$ iff $h(T) \in N([E])$, so $[E|Y] \leq_c N[E]$. Since $E|Y$ is not smooth almost everywhere, $[E|Y]$ is $\mathbf{\Pi}^0_3$-complete. Combining with 2.2.3, $N(E)$ is also $\mathbf{\Pi}^0_3$-complete. □

2.5. $N(E)$ as a Polishable group

Let E be a countable Borel equivalence relation on X and μ an E-invariant ergodic Borel probability measure. Consider $[E]$ as a standard Borel subgroup of $Aut(X, \mu)$ in the weak topology. Since $[E]$ is a Polish group in the uniform topology and the identity map is a Borel isomorphism between the weak topology and the uniform topology, $[E]$ is Polishable. $N(E)$ is also a subgroup of $Aut(X, \mu)$. But $N(E)$ is in general not separable, hence it is not Polish in the uniform topology. However, we can define a new topology (see [**HO**], p.91) in the normalizer group $N(E)$ by saying that $T_n \to T$ in $N(E)$ iff T_n converge to T weakly and $T_nST_n^{-1}$ converges to TST^{-1} uniformly for all $S \in [E]$. It is easy to check that $N(E)$ is a topological group with this topology. In fact $N(E)$ is a Polish group in this topology (see [**HO**], Lemma 53). It is easy to check that the identity map is a Borel isomorphism between the weak topology and this topology, so $N(E)$ is also Polishable.

CHAPTER 3

Descriptive Properties of Measure Preserving Actions and the Associated Unitary Representations

3.1. Introduction

3.1.1. Measure Preserving Actions and Unitary Representations. Let Γ be a countable group and X a standard Borel Γ-space with an invariant (nonatomic) Borel probability measure μ. Γ induces an equivalence relation E_Γ^X on X, which is defined by $xE_\Gamma^X y \iff \exists \gamma \in \Gamma(\gamma \cdot x = y)$. On the other hand, by a theorem of Feldman-Moore (see [**KM**], Theorem 1.3), every countable Borel equivalence relation on a standard Borel space X is induced by some Borel action of some countable group Γ.

Denote by $Aut(X, \mu)$ the group of μ-measure preserving automorphisms of X. For each $T \in Aut(X, \mu)$, we can define a corresponding unitary operator $U_T \in U(L^2(X))$, $U_T(f) = f \circ T^{-1}$. And by identifying T and U_T, we can view $Aut(X, \mu)$ as a subgroup of $U(L^2(X))$. The weak topology of $Aut(X, \mu)$ is the subspace topology of the weak topology defined on $U(L^2(X))$. $Aut(X, \mu)$ is a closed subspace of $U(L^2(X)$ in the weak topology, hence it is Polish.

Define $\pi^X : \Gamma \to Aut(X, \mu) \subset U(L^2(X))$, $\gamma \mapsto T$, where T is the element in $Aut(X, \mu)$ such that $T(x) = \gamma \cdot x$ for μ-almost every $x \in X$. Or equivalently $T(f) =$

$U_T(f) = f(\gamma^{-1}\cdot)$ for every $f \in L^2(X)$. Clearly π^X is a homomorphism of Γ into $U(L^2(X))$, i.e., $\pi^X$is a unitary representation of Γ on the Hilbert space $L^2(X)$.

The unitary representation π^X induces a natural Polish Γ action on $L^2(X)$, i.e., $\gamma \cdot f = \pi^X(\gamma)(f)$. Denote by $E_\Gamma^{L^2(X)}$ the induced countable Borel equivalence relation on $L^2(X)$, i.e., $f E_\Gamma^{L^2(X)} g \iff \exists \gamma \in \Gamma(g = \gamma \cdot f)$. If there is no danger of confusion, we will set $E^{L^2(X)} = E_\Gamma^{L^2(X)}$.

In this chapter, we will study relations between E_Γ^X and $E_\Gamma^{L^2(X)}$. We will obtain some characterizations of the smoothness and compressibility of $E^{L^2(X)}$ and some reducibility results.

3.1.2. The Γ action on MALG_μ and $Aut(X, \mu)$. Let X be a standard Borel space with Borel probability measure μ. Denote by MEAS_μ the σ-algebra of measurable sets and for $A, B \in \mathrm{MEAS}_\mu$, let $A =_\mu^* B \iff \mu(A \Delta B) = 0$ and denote by $[A]$ the equivalence class of A. Let $\mathrm{MALG}_\mu = \{[A] : A \in \mathrm{MEAS}_\mu\}$. Let $[A]\Delta[B] = [A\Delta B]$ and $\delta([A], [B]) = \mu(A\Delta B)$. Then $(\mathrm{MALG}_\mu, \Delta)$ is an abelian Polish group with invariant metric δ. For every measure preserving automorphism $T \in Aut(X, \mu)$, $[A] \mapsto [T(A)]$ is a measure algebra automorphism of MALG_μ. Conversely, every measure algebra automorphism of MALG_μ is of the form $[A] \mapsto [T(A)]$ for some $T \in Aut(X, \mu)$. Therefore, we can canonically identify $Aut(X, \mu)$ and the group of measure algebra preserving automorphisms of MALG_μ (see **[Kechris 2]**, p.118). If μ is continuous, MALG_μ is independent of μ and called the *Lebesgue Measure Algebra.*

Consider Γ a countable group and X a Borel Γ-space with invariant Borel probability measure μ. There is a Γ action on MALG_μ defined by $\gamma \cdot [A] = [\gamma(A)]$. This action is continuous. The map $[A] \mapsto \chi_A$ is a continuous Γ-space embedding of MALG_μ into $L^2(X,\mu)$.

Assume now that Γ acts on $Aut(X,\mu)$ by left translation and $Aut(X,\mu)$ acts on $L^2(X)$ by $T \cdot f = U_T(f)$. Let $f \in L^2(X)$ and denote by $Aut(X,\mu)_f$ the *stabilizer* of f, i.e.,

$$Aut(X,\mu)_f = \{T \in Aut(X,\mu) : T \cdot f = f\}.$$

If $Aut(X,\mu)_f = \{1\}$, then $T \mapsto T \cdot f$ is a continuous Γ-space embedding of $Aut(X,\mu)$ into $L^2(X)$. Many such f exist, for example any $f \in L^2(X)$ that is an injection of X into $\mathbb{C}$ has a trivial $Aut(X,\mu)$-stabilizer.

3.1.3. Borel Reducibility. Suppose we have Borel equivalence relations E, F on X, Y respectively. If there is a Borel map $\alpha : X \to Y$ such that $x_1 E x_2 \iff \alpha(x_1) F \alpha(x_2)$, for all $x_1, x_2 \in X$, we say E is *Borel reducible* to F. Put $E \leq_B F$ if E is Borel reducible to F. When α is a reduction and also an injection, E is said to be *Borel embedded* in F, in symbols $E \sqsubseteq_B F$. If, moreover, the image of α, $\alpha(X)$, is F-invariant, E is said to be *Borel invariantly embedde*d in F or $E \sqsubseteq_B^i F$.

B stands for *Borel* in the above symbols. We can replace Borel by another class of maps, say in a class A, and generalize the concept and notations to A-reducible, $\leq_A$, etc. For example, for the class of continuous maps, we will denote continuous reducible, embedded, invariantly embedded by $\leq_c$, $\sqsubseteq_c$, $\sqsubseteq_c^i$, respectively, where c stands for continuous.

3.2. Smoothness of $E_\Gamma^{L^2(X,\mu)}$

Smooth equivalence relations are the simplest relations in the reduction hierarchy. We first review some basic properties of smooth relations and show there are simple characterizations of the smoothness of $E_\Gamma^{Aut(X,\mu)}$ and $E_\Gamma^{MALG_\mu}$. Then Theorem 3.2.5 shows that the smoothness of $E_\Gamma^{L^2(X)}$ is equivalent to the smoothness of $E_\Gamma^{MALG_\mu}$. It turns out the smoothness of these equivalence relations has interesting connections with the concept of *rigid factors*, which we will discuss in 3.2.3. In 3.2.4, we will discuss some relations between mixing properties and smoothness. In the rest of this section, we will develop some techniques to deal with smoothness in some special kinds of Γ-spaces by using the Peter-Weyl theorem, which is sometimes easier than using Theorem 3.2.5 directly.

3.2.1. General Facts on Smooth Relations. Recall that an equivalence relation E on X is *(Borel) smooth* if $E \leq_B \Delta(Y)$, where $\Delta(Y)$ is the equivalence relation defined on some Polish space Y by $y_1 \Delta(Y) y_2 \iff y_1 = y_2$.

A *selector* is a map $s : X \to X$ such that $xEy \iff s(x) = s(y)$. A *transversal* for E is a set $T \subseteq X$ that meets each E-class at exactly one point. Having a Borel selector is equivalent to having a Borel transversal and implies smoothness. The converse is not true in general. But in the case that the E is generated by a discrete Borel group action, i.e., $E = E_\Gamma$ for some countable group Γ, the smoothness of E implies the existence of Borel selectors for E_Γ (see [**KM**], Proposition 6.4). Moreover, we have:

Theorem 3.2.1. *Let X be a Borel Γ-space, where Γ is a countable group. Then the following are equivalent:*

i) E_Γ *is smooth;*

ii) E_Γ *has a Borel selector;*

iii) $X/\Gamma = X/E_\Gamma$ *is standard Borel (in its quotient* σ*-algebra);*

iv) There is a Polish topology $\mathcal{T}$ *on* X *compatible with its Borel structure such that* E_Γ *is closed in the product topology* $(X, \mathcal{T}) \times (X, \mathcal{T})$*;*

v) There is a countable sequence (A_n) *of Borel* E_Γ*-invariant subsets of* X *separating* E *i.e.* $[\forall n(x \in A_n \iff y \in A_n)] \Rightarrow [xEy]$*;*

vi) E_0*cannot be Borel embedded in* E_Γ^X*, where* E_0 *is the equivalence relation on the Cantor space* $\mathcal{C} = 2^{\mathbb{N}}$ *defined by* $sE_0t \iff \forall^\infty n \in \mathbb{N}(s(n) = t(n))$*.*

Proposition 3.2.2. *Let* E *be a countable Borel equivalence relation on a uncountable Polish space* X*.*

i) If every equivalence class of E *is* G_δ*, then* E *is smooth.*

ii) If E *admits a nonatomic ergodic measure, then* E *is not smooth.*

iii) If E *is generically ergodic (i.e., every invariant Borel subset is either meager or comeager and every class is meager), then* E *is not smooth.*

Let X be a Polish metric space. In this case, we have a converse to Proposition 3.2.2 (i).

Proposition 3.2.3. *Let* X *be a Polish metric space,* Γ *a countable group acting on* X *by homeomorphisms. The following are equivalent:*

i) E_Γ *is smooth;*

ii) Every orbit is discrete;

iii) $\forall x \in X$*,* x *is an isolated point in* $\Gamma \cdot x$*;*

iv) The closure of each orbit is not perfect.

Proof. iii) $\iff$ ii): If $\gamma_i \cdot x \to \gamma \cdot x$ for some $x \in X$ $\gamma_i, \gamma \in \Gamma$, then $(\gamma^{-1}\gamma_i)\cdot x \to x$. So these two conditions are clearly equivalent.

ii) $\Rightarrow$ i): Proposition 3.2.2 (i).

i) $\Rightarrow$ iv): Proof by contradiction. Suppose $\overline{\Gamma \cdot x}$ is perfect for some $x \in X$. Note that $\Gamma \cdot x$ is dense in $\overline{\Gamma \cdot x}$ which is equivalent to the condition that $E_\Gamma|(\overline{\Gamma \cdot x})$ is generically ergodic. By Proposition 3.2.2 (iii), $E_\Gamma|(\overline{\Gamma \cdot x})$ is not smooth. Therefore E_Γ is not smooth.

iv) $\Rightarrow$ iii): If x is a limit point in $\Gamma \cdot x$, then $\forall \gamma \in \Gamma$ ($\gamma \cdot x$ is a limit point in $\Gamma \cdot x$). Therefore $\overline{\Gamma \cdot x}$ is perfect. □

Assume now Γ acts on (X, μ) by measure preserving automorphisms. Recall that MALG_μ and $Aut(X, \mu)$ can be embedded into $L^2(X)$ as Γ-spaces. We have

Corollary 3.2.4. *$E_\Gamma^{MALG_\mu}$ is smooth iff for every Borel subset $A \subseteq X$, there is an $r > 0$ such that $\mu(\gamma(A)\Delta A) \notin (0, r)$ for every $\gamma \in \Gamma$.*

$E_\Gamma^{Aut(X,\mu)}$ is smooth iff $\pi^X(\Gamma)$is discrete in the weak topology.

3.2.2. The Characterization of Smoothness of $E_\Gamma^{L^2(X)}$. Clearly the smoothness of $E_\Gamma^{L^2(X)}$ implies the smoothness of $E_\Gamma^{\mathrm{MALG}_\mu}$. The converse is also true.

Theorem 3.2.5. *$E_\Gamma^{MALG_\mu}$ is smooth iff $E_\Gamma^{L^2(X)}$ is smooth.*

Before we prove the theorem, it would be first convenient to prove the following lemma:

Lemma 3.2.6. *Let X be a standard Borel space with Borel probability measure μ and*

$$g, f_0, f_1, f_2 \cdots \in L^2(X, \mu).$$

If $f_i\mu = f_j\mu$ for all i, j, then the following are equivalent:

i) $f_i^{-1}(A) \to g^{-1}(A)$ *for every analytic* $A \subseteq \mathbb{C}$*;*

ii) $f_i^{-1}(A) \to g^{-1}(A)$ *for every Borel* $A \subseteq \mathbb{C}$*;*

iii) $f_i^{-1}(A) \to g^{-1}(A)$ *for every basic open set* $A \subseteq \mathbb{C}$*;*

iv) $f_i \to g$.

Proof. Obviously, i) $\Rightarrow$ ii) $\Rightarrow$ iii). Put $f_\omega = g$.

(ii)$\Rightarrow$i)) Assume ii) is true. We may assume f_i is Borel for all $i \leq \omega$. Let $B_i \subseteq X \backslash f_i^{-1}(A)$ be a Borel set such that $B_i = X \backslash f_i^{-1}(A)$ (modulo null). So $f_i(B_i)$ is analytic and $f_i(B_i) \cap A = \emptyset$. Let $B = \bigcup_{i \leq \omega} f_i(B_i)$. By the separation theorem (see [**Kechris 2**] 28.B), there is a Borel set A', such that $A \subseteq A' \subseteq \sim B$. We have

$$f_i^{-1}(A) \subseteq f_i^{-1}(A') \subseteq f_i^{-1}(\sim B) \subseteq f_i^{-1}(\sim B) = X \backslash B_i.$$

Since $X \backslash B_i = f^{-1}(A)$ (modulo null), we have $f_i^{-1}(A) = f_i^{-1}(A')$ for all $i \leq \omega$. Therefore $f_i^{-1}(A) = f_i^{-1}(A') \to g^{-1}(A') = g^{-1}(A)$.

(iii)$\Rightarrow$ii)) It is easy to check that the convergence property is preserved under complement and finite union, that is

$$[f_i^{-1}(A) \to g^{-1}(A)] \Rightarrow [f_i^{-1}(\sim A) \to g^{-1}(\sim A)]$$

and

$$[\forall_{k<n}(f_i^{-1}(A_k) \to g^{-1}(A_k))] \Rightarrow [f^{-1}(\bigcup_{k<n} A_k) \to f^{-1}(\bigcup_{k<n} A_k)].$$

So we only need to check that the convergence property is preserved under countable union. Suppose now $f_i^{-1}(A_k) \to g^{-1}(A_k)$ for all $k \in \mathbb{N}$. We may assume (A_k) is increasing and put $A = \bigcup_{k \in \mathbb{N}} A_k$. For any $\varepsilon > 0$, we can find an n large enough so that

$$(f_0\mu)(A\backslash A_n) + (g\mu)(A\backslash A_n) < \varepsilon.$$

Combine with the condition $f_i\mu = f_j\mu$ for all $i, j < \omega$, so we actually have $(f_i\mu)(A\backslash A_n) < \varepsilon$ for all $i \leq \omega$. We can also find m large enough so that

$$\mu(f_i^{-1}(A_n)\Delta g^{-1}(A_n)) < \varepsilon$$

for all $i > m$. Therefore, for every $i > m$

$$\begin{aligned} \mu(f_i^{-1}(A)\Delta g^{-1}(A)) &\leq \mu(f_i^{-1}(A)\backslash f_i^{-1}(A_n)) + \mu(f_i^{-1}(A_n)\Delta g^{-1}(A_n)) \\ &\quad +\mu(g^{-1}(A)\backslash g^{-1}(A_n)) \\ &\leq \mu(f_i^{-1}(A\backslash A_n)) + \mu(f_i^{-1}(A_n)\Delta g^{-1}(A_n)) \\ &\quad +\mu(g^{-1}(A\backslash A_n)) \\ &\leq 3\varepsilon. \end{aligned}$$

So $f_i^{-1}(A) \to g^{-1}(A)$.

(ii)$\Rightarrow$ iv)) By simple function approximation, we can find Borel sets $A_k \subseteq \mathbb{C}$ so that

$$\left\| f_0 - \sum a_k \chi_{f_0^{-1}(A_k)} \right\| < \varepsilon$$

and

$$\left\| g - \sum a_k \chi_{f_0^{-1}(A_k)} \right\| < \varepsilon$$

for some $a_k \in A_k$. We can use finitely many A_k and assume them to be bounded. Then, because of the condition $f_i\mu = f_j\mu$ for all $i, j < \omega$, we have

$$\begin{aligned}
\int_{f_i^{-1}(A_k)} |f_i(x) - a_k|^2 \, d\mu(x) &= \int_{s \in A_k} |s - a_k|^2 \, d(f_i\mu)(s) \\
&= \int_{s \in A_k} |s - a_k|^2 \, d(f_0\mu)(s) \\
&= \int_{f_0^{-1}(A_k)} |f_0(x) - a_k|^2 \, d\mu(x).
\end{aligned}$$

So we actually have

$$\left\| f_i - \sum a_k \chi_{f_i^{-1}(A_k)} \right\| < \varepsilon$$

for all $i \leq \omega$. Since $f_i^{-1}(A_k) \to g^{-1}(A_k)$, we can find m large enough so that

$$\sum |a_k|^2 \cdot \mu(f_i^{-1}(A_k) \Delta g^{-1}(A_k)) < \varepsilon^2$$

for all $i > m$. Therefore, for every $i > m$

$$\begin{aligned}
\|f_i - g\| &\leq \left\| f_i - \sum a_k \chi_{f_i^{-1}(A_k)} \right\| + \left\| g - \sum a_k \chi_{g^{-1}(A_k)} \right\| \\
&\quad + \left\| \sum a_k \chi_{g^{-1}(A_k)} - \sum a_k \chi_{f_i^{-1}(A_k)} \right\| \\
&< 2\varepsilon + \left\| \sum a_k \chi_{g^{-1}(A_k) \setminus f_i^{-1}(a_k)} \right\| + \left\| \sum a_k \chi_{f_i^{-1}(A_k) \setminus g^{-1}(a_k)} \right\| \\
&= 2\varepsilon + \sqrt{\sum |a_k|^2 \cdot \mu(f_i^{-1}(A_k) \Delta g^{-1}(A_k))} \\
&< 3\varepsilon.
\end{aligned}$$

We have $f_i \to g$.

((iv)⇒(iii)) Since $f_i \to g$, we have $f_i\mu \to g\mu$, hence $f_i\mu = g\mu$ for all i. Let $A = B_{a,r} = \{s \in \mathbb{C} : |s - a| < r\}$ be an open ball with arbitrary radius r and center a. If $f_0^{-1}(A)$ is null, then $g^{-1}(A)$ is null. So assume $\mu(f_0^{-1}(A)) > 0$. Toward a contradiction, assume $f_i^{-1}(A) \not\to g^{-1}(A)$. Pick $t > 0$ so that $\mu(f_i^{-1}(A)\Delta g^{-1}(A)) > t$ for all i. Since $\mu(f_i^{-1}(A)) = \mu(g^{-1}(A))$, we have

$$\mu(f_i^{-1}(A)\backslash g^{-1}(A)) = \mu(g^{-1}(A)\backslash f_i^{-1}(A)) > \frac{t}{2}.$$

Let $A_s = \overline{B_{a,r-s}}$ and fix an s such that $\mu(A\backslash A_s) < \frac{t}{4}$. Then we have

$$\begin{aligned} \mu(f_i^{-1}(A_s)\backslash g^{-1}(A)) &\geq \mu(f_i^{-1}(A)\backslash g^{-1}(A)) - \mu(A\backslash A_s) \\ &> \frac{t}{4}. \end{aligned}$$

Finally we have

$$\|f_i - g_i\|^2 \geq \int_{f_i^{-1}(A_\varepsilon)\backslash g^{-1}(A)} |f_i - g|^2 \, d\mu > \frac{s^2 t}{4},$$

contradicting the assumption $f_i \to g$. □

Remark. So if $f_i\mu = f_j\mu$ for all i, j, then (f_i) converge iff $f_i^{-1}(A)$ converge for all basic open sets A, as the limit g in the above lemma can be easily constructed by simple function approximation.

We are ready to prove the theorem:

PROOF. We need to show that if $E_\Gamma^{L^2(X)}$ is not smooth, then $E_\Gamma^{\mathrm{MALG}_\mu}$ is not smooth. Assume $E_\Gamma^{L^2(X)}$ is not smooth. By Proposition 3.2.3, there is an $f \in L^2(X)$

such that f is a limit point of $(\Gamma \cdot f)\backslash\{f\}$. We can find a countable sequence $\{\gamma_i\} \subseteq \Gamma\backslash\Gamma_f$ such that $\gamma_i \cdot f \to f$.

By Lemma 3.2.6, we may assume $f(X) \subseteq \mathcal{C} \subseteq [0,1]$. Otherwise, replace f by $\alpha \circ f$ where α is a Borel isomorphism of $\mathbb{C}$ to the Cantor set $\mathcal{C}$.

In order to prove this by contradiction, assume $E_\Gamma^{\mathrm{MALG}_\mu}$ is smooth. So for every Borel subset $A \subseteq X$, there is an $r_A > 0$ such that $\mu(\gamma(A)\Delta A) \notin (0, r_A)$ for every $\gamma \in \Gamma$. And since $\mu(\gamma_i(f^{-1}(B))\Delta B) \to 0$ for every Borel $B \subseteq \mathcal{C}$, there is a number $m_B \in \mathbb{N}$ such that $\mu(\gamma_{m_B}(f^{-1}(B))\Delta B) \geq r_{f^{-1}(B)}$ and $\mu(\gamma_i(f^{-1}(B))\Delta B) = 0$ for all $i > m_B$.

Let $S = \{s \in \mathcal{C} : \mu(f^{-1}(s)) > 0\}$.

Suppose $S = \emptyset$. Let $P_n = \{S_{s,n} : s \in \mathcal{C}\}$, where $S_{s,n} = \{t \in \mathcal{C} : t|n = s|n\}$.

Fix an $s \in \mathcal{C}$, $\bigcap_{n\in\mathbb{N}} f^{-1}(S_{s,n}) = f^{-1}(s)$, $\lim_{n\to\infty} \mu(f^{-1}(S_{s,n})) = 0$.

Define s_i, n_i inductively. To simplify the notations, let $A_i = S_{s_i,n_i}$, $r_i = r_{A_i}$, $m_i = m_{S_{s_i,n_i}}$.

Find s_0, n_0 such that $0 < \mu(f^{-1}(S_{s_0,n_0})) < 1$. Let $A_0 = f^{-1}(S_{s_0,n_0})$.

Suppose we have s_i, n_i.

Find s_{i+1}, $n_{i+1} > n_i$ such that $A_{i+1} = f^{-1}(S_{s_{i+1}})$, $0 < \mu(A_{i+1}) < \frac{r_i}{6}$ and $m_{i+1} > m_i$.

We can always find such s_{i+1}, n_{i+1} because $\forall^\infty n(\#\{S \in P_n : \mu(f^{-1}(S)) > r\} = 0)$ for every $r > 0$. And $\#\{S \in P_n : \mu(f^{-1}(S)) \neq 0\} > 0$.

Let $A'_n = A_0\Delta A_1 \ldots \Delta A_{n-1}$, and $[A] = \lim_{n\to\infty}[A'_n]$ in MALG_μ.

Clearly $\gamma_{m_i} \cdot A \to A$. Let $T_n = A \backslash A'_n$. Since $r_i < 2(A_i)$, we have $\mu(A_{i+1}) < \frac{\mu(A_i)}{3}$ and

$$\mu(T_i) \;\; \leq \;\; \sum_{j=i}^{\infty} \mu(A_j) < \frac{3}{2}\mu(A_i).$$

Therefore

$$\begin{aligned} \mu((\gamma_{m_i} \cdot A)\Delta A) &= \mu((\gamma_{m_i}(A_i))\Delta(\gamma_{m_i}(T_{i+1}))\Delta A_i \Delta T_{i+1}) \\ &\geq \mu(\gamma_{m_i}(A_i))\Delta A_i) - \mu(\gamma_{m_i}(T_{i+1})) - \mu(T_{i+1}) \\ &\geq r_i - 2\mu(T_{i+1}) \\ &> r_i - 3\mu(A_{i+1}) \\ &> 0, \end{aligned}$$

contradicting the assumption that $E_{\Gamma}^{\mathrm{MEAS}_\mu}$ is smooth.

Suppose $S \neq \emptyset$, then S is countable. Let $f = f_1 + f_2$, where $f_1(X) \subseteq S$ and $f_2(X) \subseteq \mathcal{C} \backslash S$. Note that $\gamma_i \cdot f_1 \to f_1$ and $\gamma_i \cdot f_2 \to f_2$ because S and $\mathcal{C}\backslash S$ are Borel. If $\exists^\infty i(\gamma_i \cdot f_2 \neq f_2)$, then replacing f by f_2, we are back to the case that $S = \emptyset$. So we may assume $f = f_1$.

If S is finite, let $m = \max_{s \in S}\{m_{\{s\}}\}$. Thus $\gamma_i \cdot f = f$ for all $i > m$, contradicting the assumption $\gamma_i \cdot f \neq f$ for all i.

If S is countably infinite, define $s_i \in S$ inductively. To simplify the notation, let $A_i = f^{-1}(s_i)$, $r_i = r_{A_i}$, $m_i = m_{\{s_i\}}$.

Pick s_0 such that $0 < \mu(A_0) = \mu(f^{-1}(s_0)) < 1$. Suppose we have s_i. Pick s_{i+1} such that

$$0 < \mu(A_{i+1}) < \frac{r_i}{6}$$

and $m_{i+1} > m_i$.

Let $A = \bigcup_{i=0}^{\infty} A_i$, $T_i = \bigcup_{j=i}^{\infty} A_i$. We have $\gamma_{m_i} \cdot A \to A$, and since $r_i < 2(A_i)$, we have $\mu(A_{i+1}) < \frac{\mu(A_i)}{3}$ and $\mu(T_i) < \frac{3}{2}\mu(A_i)$. Therefore,

$$\begin{aligned}
\mu((\gamma_{m_i} \cdot A)\Delta A) &= \mu((\gamma_{m_i}(A_i))\Delta(\gamma_{m_i}(T_{i+1}))\Delta A_i \Delta T_{i+1}) \\
&\geq \mu(\gamma_{m_i}(A_i))\Delta A_i) - \mu(\gamma_{m_i}(T_{i+1})) - \mu(T_{i+1}) \\
&\geq r_i - 2\mu(T_{i+1}) \\
&> r_i - 3\mu(A_{i+1}) \\
&> 0,
\end{aligned}$$

contradicting the assumption that $E_\Gamma^{\mathrm{MEAS}_\mu}$ is smooth.

So $E_\Gamma^{\mathrm{MEAS}_\mu}$ is not smooth, when $E_\Gamma^{L^2(X)}$ is not smooth. □

3.2.3. Rigid factors and smoothness of $E_\Gamma^{L^2(X)}$. The smoothness of $E_\Gamma^{Aut(X,\mu)}$ does not imply the smoothness of $E_\Gamma^{L^2(X,\mu)}$. We will study some connections between rigid factors, mixing properties, and the smoothness of $E_\Gamma^{L^2(X,\mu)}$.

Definition 3.2.7. A Γ action on (X, μ) is *faithful* iff $\pi^X : \Gamma \to Aut(X, \mu)$ is injective.

The following notion of rigid factor is from [**SW**].

Definition 3.2.8. Let X be a Γ-space with invariant probability measure μ. The *rigid factor* is the set

$$\mathcal{R}(\Gamma, X, \mu) = \{f \in L^1(X, \mu, S^1) : \liminf_{\gamma \in \Gamma} \|\gamma \cdot f - f\|_{L^1} = 0\}.$$

The Γ action on X is said to be *rigid* iff the rigid factor is $L^1(X, \mu, S^1)$. The Γ action on X is said to have *no rigid factor* if $\mathcal{R}(\Gamma, X, \mu)$ contains only constant functions.

There are many equivalent definitions of mildly mixing. For an action of countable group, no rigid factor is equivalent to mildly mixing.

Assume now Γ acts on (X, μ) faithfully and μ is a Γ-invariant Borel probability measure. We have:

Proposition 3.2.9. *$E_\Gamma^{Aut(X,\mu)}$ is smooth iff the Γ action on X is not rigid.*

PROOF. Let f be a Borel isomorphism of X to $[0, 1]$. Put

$$d(T, S) = \|T \cdot f - S \cdot f\|$$

for all $S, T \in Aut(X, \mu)$ so that d is a (left Γ-invariant) metric of the weak topology.

Suppose $E_\Gamma^{Aut(X,\mu)}$ is not smooth, then $\liminf_{\gamma\in\Gamma} d(\gamma, 1) = 0$. Since $L^2(X, \mu)$ is a Polish $Aut(X, \mu)$-space, the map $\gamma \mapsto \gamma \cdot f$ is continuous for the weak topology restrict on Γ for every $f \in L^2(X, \mu)$. We have $\liminf_{\gamma\in\Gamma} \|\gamma \cdot f - f\| = 0$ for all $f \in L^2(X, \mu)$.

Assume now the Γ action on X is rigid. We have $\liminf_{\gamma\in\Gamma} \|\gamma \cdot f - f\| = 0$. Since the action is faithful, Γ_f is trivial and $\pi^X(\Gamma)$ is discrete. □

In fact, the smoothness of $E_\Gamma^{L^2(X)}$ is strictly between X having no rigid factors and being nonrigid. Let

$$\begin{aligned}
\text{(NM)} &\iff \text{the } \Gamma \text{ action on } X \text{ is mildly mixing.}\\
\text{(NRF)} &\iff \text{the } \Gamma \text{ action on } X \text{ has no rigid factors.}\\
\text{(LS)} &\iff E_\Gamma^{L^2(X,\mu)} \text{ is smooth.}\\
\text{(MS)} &\iff E_\Gamma^{MALG_\mu} \text{ is smooth.}\\
\text{(AS)} &\iff E_\Gamma^{Aut(X,\mu)} \text{ is smooth.}\\
\text{(NR)} &\iff \text{the } \Gamma \text{ action on } X \text{ is not rigid.}
\end{aligned}$$

Then we have the following picture:

$$\begin{array}{ccc}
\text{(MM)} & \iff & \text{(NRF)}\\
\Downarrow \; \not\Uparrow & & \Downarrow \; \not\Uparrow\\
\text{(LS)} & \iff & \text{(MS)}\\
\Downarrow \; \not\Uparrow & & \Downarrow \; \not\Uparrow\\
\text{(AS)} & \iff & \text{(NR)}
\end{array}$$

In 3.2.4, we will show (MM) $\Rightarrow$ (LS) (Proposition 3.2.10 (iii)) and give an example to show (AS) $\not\Rightarrow$ (LS). Example 3.2.28 will show (LS) $\not\Rightarrow$ (NM).

3.2.4. Mixing Properties and Smoothness of $E_\Gamma^{L^2(X)}$. Consider the countable Borel Γ-space X with invariant probability measure μ and recall the induced

Γ-action on $L^2(X)$ by unitary operators. Denote by $L_0^2(X)$ the set

$$\{f \in L^2(X) : \langle f, 1\rangle = 0\}.$$

Then $L_0^2(X)$ is an invariant subspace of $L^2(X)$. Therefore, we have a Γ action on $L_0^2(X)$ and $\pi_0^X := \pi^X | L^2(X)$, the subrepresentation of π^Xon $L_0^2(X)$.

The mixing properties, including (strong) mixing, mild mixing, weak mixing, ergodicity can be read from the Γ action on $L_0^2(X)$. Let us review these properties from the strongest one to the weakest one:

1) (Strongly) Mixing:

An action is strongly mixing if and only if for any Borel $A, B \subseteq X$,

$$\lim_{\gamma\to\infty} \mu(\gamma \cdot A \cap B) = \mu(A)\mu(B).$$

Considering the Γ action on $L_0^2(X)$, this is equivalent to

$$\forall f, g \in L_0^2(X)(\lim_{\gamma\to\infty} \langle \gamma \cdot f, g\rangle = 0).$$

In the language of unitary representation theory, it is equivalent to saying that π_0^X is a c_0-representation.

2) Mildly mixing:

The Γ action on X is mildly mixing iff

$$\liminf_{\gamma\to\infty} \mu((\gamma \cdot A)\Delta A) > 0$$

for any Borel subset $A \subset X$ that is neither null or conull.

It is equivalent to that for all $f \in L_0^2(X)\backslash\{0\}$,

$$\limsup_{\gamma\to\infty} |\langle \gamma \cdot f, f\rangle| < \|f\|^2 .$$

3) Weakly mixing:

This is equivalent to saying that the Γ action on $L_0^2(X)$ has no finite dimensional invariant subspace. Or in the language of unitary representation theory, π_0^X has no finite dimensional subrepresentation.

4) Ergodic:

The Γ action is ergodic iff every Γ-invariant Borel subset of X is either μ-null or μ-conull.

This is equivalent to saying that $\forall f \in L_0^2(X)\backslash\{0\}(\Gamma_f \neq \Gamma)$, where $\Gamma_f = \{\gamma \in \Gamma : \gamma \cdot f = f\}$, the *stabilizer* of f. Or in the language of unitary representations, π_0^X does not contain the trivial one-dimensional representation.

Proposition 3.2.10. *i) $E^{L^2(X)}$ is smooth iff $E^{L_0^2(X)}$ is smooth.*

ii) If $E^{L^2(X)}$ is smooth, then $\pi^X(\Gamma) \subseteq Aut(X,\mu)$ is discrete in the weak topology.

iii) If E_H^X is ergodic for every infinite subgroup $H \subseteq \Gamma$, then the Γ action on X is mildly mixing iff $E^{L_0^2(X)}$ is smooth.

In particular mildly mixing implies the smoothness of $E_\Gamma^{L_0^2(X)}$.

iv) Assume E_H is ergodic for every subgroup $H \subseteq \Gamma$ such that $[\pi^X(\Gamma) : \pi^X(H)] < \infty$. If $E_\Gamma^{L_0^2(X)}$ is smooth, then the Γ action on X is weakly mixing.

Proof. i) It is easy to check that

$$E^{L^2(X)} = E^{L_0^2(X)} \times \Delta(\mathbb{C}).$$

Therefore $E^{L^2(X)}$ is smooth iff $E^{L^2_0(X)}$ is smooth.

ii) This follows directly from Corollary 3.2.4 and $E_\Gamma^{Aut(X,\mu)} \sqsubseteq_c^i E_\Gamma^{L^2(X,\mu)}$.

iii) Suppose the Γ action on X is mildly mixing. Let $A \in MALG_\mu$ If $\mu(A) = 0$ or $\mu(A) = 1$, $\forall \gamma \in \Gamma[\mu((\gamma \cdot A)\Delta A) = 0]$. Assume A is neither null or conull. Since Γ is mildly mixing, $R = \liminf_{\gamma\to\infty} \mu((\gamma \cdot A)\Delta A) > 0$. So there $S = \{\gamma : 0 < \mu((\gamma \cdot A)\Delta A) < R\}$ is a finite set. Let $r = \min_{\gamma\in S}\{\mu((\gamma \cdot A)\Delta A))\}$. We have $\mu((\gamma \cdot A)\Delta A) \notin (0, \min(r, R))$ for all γ. Therefore, by Theorem 3.2.5 and Corollary 3.2.4, $E_\Gamma^{L^2(X)}$ is smooth.

On the other hand, assume $E_\Gamma^{L^2(X)}$ is smooth. Suppose Γ is not mildly mixing on X. Then there is an $A \in MALG_\mu$ such that $\liminf_{\gamma\to\infty} \mu((\gamma \cdot A)\Delta A) = 0$ and $0 < \mu(A) < 1$. Therefore, $S_r = \{\gamma \in \Gamma : \mu((\gamma \cdot A)\Delta A) < r\}$ is infinite for every $r > 0$.

By Theorem 3.2.5 and Corollary 3.2.4, there is an $R > 0$ such that $\mu(\gamma(A)\Delta A) \notin (0, R)$ for all $\gamma \in \Gamma$. So

$$S_R = \{\gamma \in \Gamma : \mu((\gamma \cdot A)\Delta A) < r\} = \{\gamma \in \Gamma : \mu((\gamma \cdot A)\Delta A) = 0\}.$$

Let H be the subgroup of Γ generated by S_R. E_H^X is not ergodic because A is H-invariant. This contradicts the condition that E_H^X is ergodic, when H is infinite. So the Γ action on X is mildly mixing.

iv) Proof by contradiction.

Suppose $E_\Gamma^{L^2(X)}$is smooth and there is a finite dimensional Γ-invariant subspace $V \subset L^2_0(X)$. Pick an arbitrary $f \in V$. Since $E_\Gamma^{L^2(X)}$ is smooth, $\Gamma \cdot f \subseteq V$ is discrete.

Therefore $\Gamma \cdot f$ is finite. Thus

$$|\pi^X(\Gamma)/\pi^X(\Gamma_f)| < \infty.$$

But Γ_f fixes $f \in L^2_0(X)$, contradicting the hypothesis that E_{Γ_f} is ergodic. $\square$

Example 3.2.11. Let X, Y be faithful Borel Γ-spaces with invariant probability measures μ, ν respectively. Assume the Γ action on X is mildly mixing, and $E_\Gamma^{L^2(Y,\nu)}$ is not smooth. Consider Γ acting on $X \times Y$ by the diagonal action. $\pi^{X\times Y}(\Gamma)$ is discrete in $Aut(X \times Y, \mu \times \nu)$, hence $E_\Gamma^{Aut(X\times Y,\mu\times\nu)}$ is smooth. In fact, pick any $A \subseteq X$ that is neither null or conull. There exists an $r > 0$ such that $(\mu\times\nu)((\gamma\cdot(A\times Y)\Delta(A\times Y)) < r$ for almost every γ. But $E_\Gamma^{L^2(X\times Y)}$ is clearly nonsmooth because we can find a $B \subseteq Y$ and a sequence of $\gamma_i \in \Gamma$ so that $\gamma_i \cdot (X \times B) \to X \times B$ and $\gamma_i \cdot (X \times B) \neq X \times B$ for all i.

For example, let $\Gamma = S_{<\infty}$ and $G = 2^\Gamma \times 2^\mathbb{N}$. Consider the Γ action on G defined by

$$\gamma \cdot ((a_g), (b_i)) = ((a_{\gamma^{-1}g}), (b_{\gamma^{-1}(i)})).$$

Then the Γ action on 2^Γ is mixing, so $E_\Gamma^{Aut(G,\mu)}$ is smooth. But $E_\Gamma^{L^2(G)}$ is not smooth because $E_\Gamma^{L^2(2^\mathbb{N})}$ is not smooth.

3.2.5. The Peter-Weyl Theorem. We are going to develop several other techniques to determine the smoothness and nonsmoothness of $E_\Gamma^{L^2(X)}$. In some situation, it is easier to use them than directly check the conditions in Theorem 3.2.5. Most of these techniques involve the Peter-Weyl theorem.

Recall the Peter-Weyl theorem from unitary representation theory (see [**Folland, Kechris 1**]).

Theorem 3.2.12. *(Peter-Weyl) Let G be a compact Polish group. Then*

(i) Every irreducible unitary representation of G is finite dimensional;

(ii) $\hat{G}$ is countable;

(iii) Every unitary representation of G is a direct sum of irreducible unitary representations.

Consider a compact Polish group G with the (normalized) Haar measure μ. For each irreducible unitary representation π of G, denote by $\hat{\pi}$ the isomorphism class of π and by $\mathcal{H}_\pi$ the Hilbert space of it. Also denote by $\hat{G}$ the dual of G, which is the countable set $\{\hat{\pi} : \pi \text{ is an irreducible unitary representation of } G\}$.

Denote by $\rho_G : G \to U(L^2(G))$ the right regular representation of G, which is the unitary representation defined by

$$(\rho_G(g))(f(h)) = f(hg)$$

for all $g, h \in G$ and $f \in L^2(G)$.

Fix a representative π for each $\hat{\pi} \in \hat{G}$ and an orthonormal basis $\{e_i^\pi\}_{1 \le i \le d_\pi}$, where $d_\pi = \dim(\mathcal{H}_\pi)$. Let

$$\pi_{ij}(g) = \left\langle \pi(g) e_j^\pi, e_i^\pi \right\rangle$$

be the matrix coefficients of π in this basis. $\pi_{ij} \in L^2(G)$ and denote by $\mathcal{E}_\pi$ the linear span of $\{\pi_{ij}\}$. Clearly this space is independent of the choice of π, thus we can write $\mathcal{E}_{\hat{\pi}} = \mathcal{E}_\pi$.

Theorem 3.2.13. *(Peter-Weyl) Let G be a compact Polish group. Then* *(i)* $L^2(G) = \oplus_{\hat{\pi}\in\hat{G}}\mathcal{E}_{\hat{\pi}}$.

(ii) $\{\sqrt{d_\pi}\pi_{ij}\}_{1\leq i,j\leq d_\pi}$ *is an orthonormal basis for* $\mathcal{E}_{\hat{\pi}}$*, so* $\dim(\mathcal{E}_{\hat{\pi}}) = d_\pi^2$.

(iii) For $i = 1, \ldots, d_\pi$*, the subspace* $\mathcal{E}_{\hat{\pi},i}$ *of* $\mathcal{E}_{\hat{\pi}}$ *spanned by the ith row of the matrix* $(\sqrt{d_\pi}\pi_{ij})$ *is invariant under the right regular representation, and the subrepresentation of* ρ_G *determined by* $\mathcal{E}_{\hat{\pi},i}$ *is isomorphic to* π.

Define the *character* χ_π by $\chi_\pi(g) = \mathrm{trace}(\pi(g))$. Trace is also independent of the choice of the basis and the representative π of each isomorphic class, so we put $\chi_{\hat{\pi}} = \chi_\pi$. $\{\chi_{\hat{\pi}} : \hat{\pi} \in \hat{G}\}$ is an orthonormal set in $L^2(G)$ (see [**Folland**], 5.23).

Suppose Γ is a countable group acting by (topological group) automorphisms on G. Clearly Γ preserves μ. There is a natural Γ action on $\hat{G}$. For a $\gamma \in \Gamma$, define

$$(\gamma \cdot \pi)(g) = \pi(\gamma^{-1} \cdot g).$$

Since π is irreducible, $\gamma \cdot \pi$ is also irreducible. And again, $\widehat{\gamma \cdot \pi}$ is independent of the choice of π in each isomorphic class. So we can define $\gamma \cdot \hat{\pi} = \widehat{\gamma \cdot \pi}$. Also note that

$$(\gamma \cdot \chi_{\hat{\pi}})(g) = \chi_{\hat{\pi}}(\gamma^{-1} \cdot g) = \mathrm{trace}(\pi(\gamma^{-1} \cdot g)) = \mathrm{trace}((\gamma \cdot \pi)(g)) = \chi_{\gamma\cdot\pi}(g).$$

So $\gamma \cdot \chi_{\hat{\pi}} = \chi_{\gamma\cdot\hat{\pi}}$, finally $\gamma \cdot \mathcal{E}_{\hat{\pi}} = \mathcal{E}_{\gamma\cdot\hat{\pi}}$.

3.2.6. Some Characterizations of Smoothness.

Corollary 3.2.14. *1) Assume* $X = G$ *is a compact Polish group and* Γ *acts on* G *by automorphisms. If* $E_\Gamma^{L^2(G)}$ *is smooth, then* $\Gamma_{\hat{\pi}} \cdot \pi$ *is finite for every irreducible unitary representation* π *of* X*. Or equivalently,* $\Gamma_{\hat{\pi}}/\Gamma_\pi$ *is finite for every irreducible unitary representation* π *of* X*.*

2) Assume Γ acts on a compact Polish space X by isometries and X has a Borel probability measure that is invariant under any isometry. Then $E_\Gamma^{L^2(X)}$ is smooth iff E_Γ is uniformly periodic μ-a.e., i.e.,

$$\exists N < \infty \forall^\mu x \in X(|[x]_{E_\Gamma}| < N).$$

3) Assume Γ acts on X by topological group automorphisms where $X = G$ is a connected semisimple Lie group with an invariant Borel probability measure μ. Then $E_\Gamma^{L^2(X)}$ is smooth iff E_Γ^X is uniformly periodic μ-a.e.

PROOF. 1) Recall that $\Gamma_{\hat{\pi}}$ is the stabilizer of $\hat{\pi}$, i.e.,

$$\Gamma_{\hat{\pi}} = \{\gamma \in \Gamma : \widehat{\gamma \cdot \pi} = \hat{\pi}\}.$$

Hence

$$\Gamma_{\hat{\pi}} \cdot \pi = \{\gamma \cdot \pi | \gamma \in \Gamma_{\hat{\pi}}\} = \{\gamma \cdot \pi | \widehat{\gamma \cdot \pi} = \hat{\pi}\}.$$

By the Peter-Weyl Theorem, the subrepresentation of ρ_G determined by $\mathcal{E}_{\hat{\pi},i}$ is isomorphic to π. If $\Gamma_{\hat{\pi}} \cdot \pi$ is infinite, then $\Gamma_{\hat{\pi}} \cdot \pi_{ij}$ is infinite for some j. But $\mathcal{E}_{\hat{\pi},i}$ is finite dimensional, so $\overline{\Gamma_{\hat{\pi}} \cdot \pi_{ij}}$, which is contained in the unit sphere of $\mathcal{E}_{\hat{\pi},i}$, is compact. Therefore $\Gamma_{\hat{\pi}} \cdot \pi_{ij}$ has at least one limit point and therefore is not closed. By Proposition 3.2.3 (iii), $E_\Gamma^{L^2(G)}$ is not smooth.

2) ($\Leftarrow$) Since $\gamma^{N!} \cdot f = f$, $E_\Gamma^{L^2(G)}$ is smooth.

($\Rightarrow$) Since $Iso(X)$ is compact, by the Peter-Weyl theorem, π^X is the direct product of irreducible finite dimensional unitary $\Gamma-$representations, say $L^2(X) = \prod_{i \in \mathbb{N}} V_i$, and V_i are finite dimensional π^X invariant subspaces.

For every $f \in L^2(X)$, we can write $f = \sum f_i$ where $f_i \in V_i$. Suppose $\Gamma \cdot f$ is infinite. If $\Gamma \cdot f_i$ is infinite for some i, then $E_\Gamma^{L^2(X)}$ is not smooth. Assume now $\Gamma \cdot f_i$ is finite for every i. Since $\forall i(\Gamma \cdot f_i)$ is finite and $\Gamma \cdot f$ is infinite, $\bigcap_{i<M} \Gamma_{f_i}$ is infinite for every $M < \infty$. Therefore, we can find a sequence $g_M \in \bigcap_{i<M} \Gamma_{f_i} \cdot f \backslash f$, and $g_M \to f$, hence $E_\Gamma^{L^2(X)}$ is not smooth.

So if $E_\Gamma^{L^2(X)}$ is smooth, then every Γ orbit of $f \in L^2(X)$ is finite. Therefore, we must have a finite upper bound of Γ-orbit on X μ-a.e.

3) Since $Aut(G)$ is compact, this is from the proof of 2). $\square$

3.2.7. Stabilizers. Consider the Γ action on $L^2(X)$. We can also describe the stabilizer of $f \in L^2(X)$ in terms of the Γ action on X and its *full group*.

Definition 3.2.15. Let F be a (not necessarily countable) Borel equivalence relation defined on X and μ a (not necessarily F-invariant) Borel probability measure on X. Denote by $[F] = \{T \in Aut(X, \mu) | \forall^\mu x\, (T(x) E x)\}$ the *full group* of F.

This is a straightforward generalization of the usual concept of full group, which is usually defined in the case that μ is E-invariant and in the context that E is countable.

We have the following simple proposition:

Proposition 3.2.16. *Let F be a Borel equivalence relation on a standard Borel space X with a Borel probability measure μ. If F is smooth, then $[F]$ is closed in the weak topology of $Aut(X, \mu)$.*

Proof. F is smooth, hence $F \leq_B \Delta([0,1])$. Let $f : X \to [0,1]$ be a Borel function such that xFy iff $f(x) = f(y)$. Then the assignment $T \mapsto f \circ T$ is a continuous map from $Aut(X,\mu)$ to $L_2(X)$. Note that $T \in [F]$ iff $f \circ T = f$ (modulo null sets), hence $[F]$ is closed. □

Recall the uniform ergodic decomposition for invariant measures of E. Denote by $P(X)$ the set of probability measures on X, by $\mathcal{I}_E \subseteq P(X)$ the set of E-invariant Borel probability measures on X, and by $\mathcal{EI}_E \subseteq P(X)$ the set of E-invariant ergodic Borel probability measures on X. We have (see **[KM]**, Theorem 3.3)

Theorem 3.2.17. *(Farrell, Varadarajan) Let E be a countable Borel equivalence relation on a standard Borel space X. Assume $\mathcal{I}_E \neq 0$. Then there is a unique (up to null sets) Borel surjection $\pi : X \to \mathcal{EI}_E$ such that*

(1) $\pi(x) = \pi(y)$ if xEy;

(2) If $X_e = \{x : \pi(x) = e\}$, for $e \in \mathcal{EI}_E$, then $e(X_e) = 1$;

(3) For any $\mu \in \mathcal{I}_E$, $\mu = \int \pi(x) d\mu(x)$.

So we can write $\mathcal{EI}_E = \{e_x\}$, where $e_x = \pi(x)$.

Until the end of this subsection, we will use the above notations: π, X_e to denote the unique ergodic decomposition of (X, E) and F (F^X_Γ if $E = E^X_\Gamma$) to denote the Borel equivalence relation on X, which is defined by xFy iff $\pi(x) = \pi(y)$. Since F is smooth, $[F]$ is closed by Proposition 3.2.16 and clearly $[E] \subseteq [F]$. Furthermore, we can show that $[F]$ is the closure of $[E]$.

Theorem 3.2.18. *Let E be a countable Borel equivalence relation on a standard Borel space X with invariant probability measure μ. Given $S \in [F]$ and a Borel set A, then there is a $T \in [E]$, such that $T(A) = S(A)$.*

Proof. Let Y be an F-invariant Borel set. Then $T(A \cap Y) = T(A) \cap T(Y) = T(A) \cap Y$, hence $\mu(A \cap Y) = \mu(T(A) \cap Y)$.

Consider the set

$$Y = \{x : e_x(A) > e_x(T(A))\}.$$

If $\mu(Y) > 0$, then $\mu(A \cap Y) > \mu(T(A) \cap Y)$. Since Y is F-invariant, this contradicts that $\mu(A \cap Y) = \mu(T(A) \cap Y)$. So $\mu(Y)$=0. We may assume therefore that $\forall x \in X,\ e_x(A) = e_x(T(A)) > 0$.

By a well-known lemma, there are disjoint E-invariant sets P,Q, and R, such that $[A] \cup [T(A)] = P \cup Q \cup R$, and $A \cap P \prec T(A) \cap P$, $T(A) \cap Q \prec A \cap Q$, $A \cap R \approx T(A) \cap R$. But P, Q are also F-invariant, $\mu(A \cap P) = \mu(T(A) \cap P)$, $\mu(A \cap Q) = \mu(T(A) \cap Q)$, so $\mu(P) = \mu(Q) = 0$, $A \approx T(A)$. □

Corollary 3.2.19. *$[F]$ is the closure of $[E]$.*

We have the following characterization of a subgroup $H \leq \Gamma$ being a stabilizer of some $f \in L^2(X)$.

Proposition 3.2.20. *Let Γ be a countable group and X be a standard Borel Γ-space with invariant probability measure μ and $H \subseteq \Gamma$ a subgroup. Then $H = \Gamma_f$ (stabilizer of f) for some $f \in L^2(X)$ iff $H = [F_H^X] \cap \Gamma$, where F_H^X is the equivalence relation induced by the ergodic decomposition of E_H^X.*

Proof. ($\Rightarrow$) Assume $H = \Gamma_f$ for some $f \in L(X, \mu)$. Since $H = \Gamma_f$, $S \cdot f = f$ for every $S \in [E_H^X]$. By Corollary 3.2.19, $T \cdot f = f$ for every $T \in \overline{[E_H^X]} = [F_H^X]$. Therefore, $[F_H^X] \cap \Gamma \subseteq \Gamma_f = H$. On the other hand, $H = [E_H^X] \cap \Gamma \subseteq [F_H^X] \cap \Gamma$. So $H = [F_H^X] \cap \Gamma$. ($\Leftarrow$) Suppose $H = \Gamma \cap [F_H^X]$. Let $f = \beta \circ \pi_H$ where π_H is the unique ergodic decomposition of E_H^X and β is any Borel embedding of $\mathcal{EI}_{(X,H)}$ onto a subset of $[0, 1]$.

If $\gamma \in \Gamma_f$, $\pi_H(x) = \beta^{-1} \circ f(x) = \beta^{-1} \circ (\gamma \cdot f)(x) = \beta^{-1} \circ f(\gamma^{-1} \cdot x) = \pi_H(\gamma^{-1} \cdot x)$. In other words, $(\gamma \cdot x) F_H^X x$ for every $x \in X$, $\gamma \in \Gamma_f$. Thus, $\Gamma_f \subseteq \Gamma \cap [F_H^X] = H$.

On the other hand, $\forall \gamma \in H, x \in X$,

$$\gamma \cdot f(x) = f(\gamma^{-1} \cdot x) = \beta \circ \pi_H(\gamma^{-1} \cdot x) = \beta \circ \pi_H(x) = f(x).$$

Therefore , $H \subseteq \Gamma_f$. We then have $H = \Gamma_f$. □

Corollary 3.2.21. *Let Γ be a countable group and X be a standard Borel Γ-space with invariant probability measure μ and $H \subseteq \Gamma$ a subgroup. If E_Γ^X is ergodic and E_H^X is not, then we can find some $f \in L^2(X)$, such that $H \subseteq \Gamma_f \subsetneq \Gamma$, and Γ_f is minimum in the sense that $H \subseteq \Gamma_g \Rightarrow \Gamma_f \subseteq \Gamma_g$.*

Proof. Let $H' = \Gamma \cap [F_H^X]$. Since $H \leq H' \leq [F_H^X]$, we have $[F_H^X] \leq [F_{H'}^X] \leq [F_H^X]$, i.e., $[F_{H'}^X] = [F_H^X]$. By the proposition, $\Gamma' = \Gamma_f$ for some f. Suppose now $H \leq \Gamma_g$. Then we have $[F_H^X] \leq [F_{\Gamma_g}^X]$. Therefore,

$$\Gamma_f = \Gamma \cap [F_H^X] \subseteq \Gamma \cap [F_{\Gamma_g}^X] = \Gamma_g.$$

□

3.2.8. Actions by Compact Group Automorphisms. Now assume that $X = G$ is a compact Polish group and Γ acts on X by topological group automorphisms.

Recall from Corollary 3.2.14 that $|\Gamma_{\hat{\pi}}/\Gamma_\pi| < \infty$ for all irreducible unitary representations π is the necessary condition for $E_\Gamma^{L^2(G)}$ being smooth. Call a Γ action on $\hat{G}$ *locally finite* if it satisfies this condition. We have the following characterizations of smoothness:

Theorem 3.2.22. *Assume Γ acts on a compact Polish group G by automorphisms. If $E_\Gamma^{L^2(G)}$ is smooth, then* *(i) There is no infinite sequence* $(f_i) \subset L^2(G)$ *such that* (Γ_{f_i}) *is a strict decreasing sequence;*

(ii) There is no infinite sequence π_i *of irreducible unitary representations of* G *such that* $\Gamma_n = \bigcap_{i \le n} \Gamma_{\pi_i}$ *is a strict decreasing sequence;*

(iii) For any set A *which contains non-isomorphic irreducible unitary representations of* G*, there is a finite subset* $S \subset A$ *such that* $\Gamma_S = \Gamma_A$*. Here* $\Gamma_A = \bigcap_{\pi \in A} \Gamma_\pi$*.*

Moreover, if we assume that the Γ *action on* $\hat{G}$ *is locally finite, then (i)* $\iff$ *(ii)* $\iff$ *(iii)* $\iff$ $E_\Gamma^{L^2(G)}$ *is smooth .*

Proof. (smooth$\Rightarrow$(i)) Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in $L^2(G)$ such that Γ_{f_i} is strictly decreasing and we can assume f_i are chosen from the unit sphere. Let $\Gamma_n = \bigcap_{i \le n} \Gamma_{f_i}$. Choose $\gamma_n \in \Gamma_{n-1} \backslash \Gamma_n$ for each n and let $d_n = \prod_{i=1}^{n-1} 2^{-1} ||f_i - \gamma_i \cdot f_i||$. Define

$$f = \sum_{n \ge 1} 6^{-n} d_n \cdot f_n.$$

Note that (d_n) is a positive decreasing sequence and clearly, $||f - \gamma_n \cdot f|| \to 0$. Furthermore,

$$\begin{aligned} ||f - \gamma_n \cdot f|| &\ge 6^{-n} d_n \cdot ||f_n - \gamma_n \cdot f_n|| - 2 \cdot \sum_{i>n} 6^{-i} d_i \cdot ||f_i|| \\ &\ge 6^{-n} d_{n+1} - 4 \cdot 6^{-n} d_{n+1} \sum_{i>n} 6^{n-i} \frac{d_i}{d_{n+1}} \\ &\ge 6^{-n} d_{n+1} - 6^{-n} d_{n+1} \cdot \frac{4}{5} > 0\,. \end{aligned}$$

Which means f is a nontrivial limit point of $\{\gamma_n \cdot f\}$. $E_\Gamma^{L^2(G)}$ is not smooth.

((i)$\Rightarrow$(ii)) It is easy to see that $\Gamma_\pi = \Gamma_f$ for some $f \in \mathcal{E}_{\hat{\pi}}$. So (i) clearly implies (ii).

((ii)⇒(iii)) Assume (iii) is not true. Then there is an S that contains nonisomorphic irreducible unitary representations of G such that $\Gamma_s \neq \Gamma_A$ for every finite subset of $S \subset A$. Let S_i be an increasing sequence of subsets of A. Γ_{S_i} is a decreasing sequence of subgroups of Γ and $\bigcap \Gamma_{S_i} = \Gamma_A$. Since $\Gamma_{S_n} \neq \Gamma_A$, we can find some $m > n$ such that $\Gamma_{S_m} \subsetneq \Gamma_{S_n}$. So we can assume that Γ_{S_n} is strictly decreasing. Choose $\pi_n \in S_{n+1} \backslash S_n$. $\Gamma_n = \bigcap_{i \leq n} \Gamma_{\pi_i} = \Gamma_{S_{n+1}}$ is strictly decreasing.

(Assume locally finite, (iii)⇒Smooth)

Assume local finiteness of the Γ action of $\hat{G}$ and (iii).

Suppose $E_{\pi^G(\Gamma)}$ is not smooth.

Fix an arbitrary index on $\hat{G}$, that is $\hat{G} = \{\hat{\pi}_i\}$ and $\hat{\pi}_i \neq \hat{\pi}_j$ unless $i = j$. Also let $\mathcal{E}_i = \mathcal{E}_{\pi_i}$.

There is an $f \in L^2(X)$ and a sequence $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n \cdot f \neq g$ and $\gamma_n \cdot f \to g$. We can write

$$f = \sum_{i=0}^{\infty} a_i f_i$$

and

$$g = \sum_{i=0}^{\infty} a_i g_i$$

where $f_i \in \mathcal{E}_{k_i}, g_i \in \mathcal{E}_{l_i}$ and $k_i = k_j \iff i = j \iff l_i = l_j$. Due to the local finiteness, we can assume $\forall i (a_i > 0)$. Choose the corresponding $\{\pi_{k_i}\}$ to be our A and, with some reordering, we may assume $S = \{\pi_{k_i}\}_{i<M}$ as the witness of condition (iii), i.e., $\Gamma_S = \Gamma_A$.

Now suppose $n < M - 1$ and $\forall i < n (\gamma_i \cdot f_i = f_i)$. $A_n = \{g_i | a_i = a_n\}$ is finite. $\gamma_i \cdot f_n \in A_n$ almost everywhere. So there is a $g_j \in A_n$ such that $\exists^\infty i (g_j = \gamma_i \cdot f_n)$. Furthermore, due to the local finiteness, we have $\exists^\infty i (\pi' = \gamma_i \cdot \pi_{k_n})$ for some π' such that $\hat{\pi'} = \hat{\pi_{l_j}}$. Let $\{\alpha_i\}$ be the subsequence of $\{\gamma_i\}$ such that $\pi' = \alpha_i \cdot \pi_{k_n}$. Define $\beta_i = \alpha_0^{-1}\alpha_i$.Clearly β_i fixes f_i for $i \leq n$ and $\beta \cdot f$ has a nontrivial limit point $\alpha_0^{-1} \cdot g$. So, by replacing g with $\alpha_0^{-1} \cdot g$ and $\{\gamma_i\}$ with $\{\beta_i\}$ inductively if necessary, we can assume that γ_n are chosen from $\Gamma_S = \Gamma_A$. But this means $\Gamma_A \cdot f = f$, a contradiction. This proves the smoothness. □

Corollary 3.2.23. *Assume local finiteness. Then* $E_\Gamma^{L^2(G)}$ *is not smooth iff there is a sequence of normal subgroups* $N_i \subseteq G$ *such that* Γ_{G/N_i} *is an infinite strictly decreasing sequence of* Γ*, where*

$$\Gamma_{G/N} = \{\gamma \in \Gamma : \forall g \in G(\gamma \cdot g \in Ng)\},$$

i.e., the subset of coset preserving automorphisms.

Proof. ($\Rightarrow$) Suppose $E_{\pi^G(\Gamma)}$ is not smooth. By the Theorem, we can find an infinite sequence of irreducible π_i such that Γ_i is strict decreasing. Let $N_i = \bigcap_{i \leq n} ker(\pi_i)$. N_i is a normal subgroup. Since $\Gamma_{G/N_i} = \Gamma_i$, we have our decreasing sequence Γ_{G/N_i}.

($\Leftarrow$) Suppose we can find such (N_i). Note N_i is the kernel of some unitary representation π. For example $\pi = \lambda_{G/N_i} \circ P_{G/N_i}$, where $P_{G/N_i} : G \to G/N_i$ is the natural projection. Since π is the direct product of irreducible unitary representations, either we can find finitely many irreducible $\pi_{i,j}$ such that $\Gamma_{G/N_i} = \bigcap_j \Gamma_{\pi_{i,j}}$ or we cannot. The negative of condition (ii) holds in each case, hence $E_\Gamma^{L^2(G)}$ is not smooth. □

Corollary 3.2.24. $E_\Gamma^{L^2(G)}$ *is smooth if and only if* $E_{\Gamma'}^{L^2(G)}$ *is smooth for every subgroup* $\Gamma' \leq \Gamma$.

3.2.9. Examples.

Example 3.2.25. (Locally finite and not smooth) Let G be a compact abelian Polish group and Γ act on G by automorphisms. Let π be an irreducible unitary representation of G, and we may assume $\mathbb{C}$ is its underlying Hilbert space. If $\gamma\cdot\hat{\pi} = \hat{\pi}$, then

$$\begin{aligned}
(\gamma\cdot\pi)(g) &= \langle(\gamma\cdot\pi)(g), 1\rangle \\
&= \langle(\gamma\cdot\pi)(g), (\gamma\cdot\pi)(1_G)\rangle \\
&= \langle\pi(g), \pi(1_G)\rangle \\
&= \langle\pi(g), 1\rangle = \pi(g).
\end{aligned}$$

Thus $\Gamma_\pi = \Gamma_{\hat{\pi}}$ and the Γ action on $\hat{G}$ is locally finite.

Let $G = K^{\mathbb{N}}$, where K is a compact Polish group and $\Gamma = S_{<\infty}$, the group of finite permutations on $\mathbb{N}$. Consider the Γ action on G by $\gamma\cdot(k_i) = (k_{\gamma^{-1}(i)})$. Let π be an irreducible unitary representation of K and define $\pi_n((k_i)) = \pi(k_n)$. We have $\gamma\cdot\pi_i = \pi_{\gamma(i)}$. Let $\Gamma_n = \bigcap_{i\le n}\Gamma_{\pi_i}$. (Γ_n) is strictly decreasing. Hence $E_\Gamma^{L^2(G)}$is non-smooth. If furthermore, K is abelian, then the Γ action on $\hat{G}$ is locally finite but $E_\Gamma^{L^2(G)}$ is not smooth.

Example 3.2.26. (Simple compact groups) Consider a compact Polish group G and let Γ act on G by automorphisms. It is easy to check that $\Gamma_\pi = \{\gamma\in\Gamma : \gamma\cdot\ker\pi = \ker\pi\}$. If G is simple, then $\Gamma_\pi = \{1\}$ for all nontrivial irreducible unitary representations π. So local finiteness is equivalent to smoothness when G is simple.

Assume now G is the finite direct product of some simple groups, i.e., $G = G_1 \times G_2 \times \cdots \times G_n$, where every G_i is simple and compact. Similarly there are only finitly many possible Γ_π. So in this case, we also have that local finiteness is equivalent to smoothness.

Example 3.2.27. Let $\Gamma \leq G$ act on a compact Polish group G by conjugations. Then $\Gamma_{\hat{\pi}} = \Gamma$ for every $\hat{\pi} \in \hat{G}$. It is easy to check that $E_\Gamma^{L^2(G)}$ is smooth iff Γ is finite by either Theorem 3.2.5 or Theorem 3.2.22.

Example 3.2.28. (Weakly mixing, smooth, but not mildly mixing) Let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, $G = \mathbb{T}^n$ and the usual Γ action on G by matrix multiplication. The Γ action on G is weakly mixing but not mildly mixing.

The dual $\hat{G}$ is $\mathbb{Z}^n$, where $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ is identified with the character

$$\lambda(z_1, \ldots z_n) = \prod_{i=1}^{n} z_i^{k_i}.$$

The Γ action on $\hat{G}$ is defined by

$$\gamma \cdot (k_1, \ldots, k_n) = (\gamma^{-1})^t \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix},$$

where $\gamma \in \Gamma = \mathrm{SL}_n(\mathbb{Z})$ is a matrix. Since G is abelian, $\Gamma_{\hat{\pi}} = \Gamma_\pi$ for every irreducible unitary representation π and in particular, the Γ action on $\hat{G}$ is locally finite. Let $\{\bigcap_{i \leq m} \Gamma_{\pi_i}\} = \{\bigcap_{i \leq m} \Gamma_{\hat{\pi}_i}\}$ be a strictly decreasing sequence, where $\hat{\pi}_i = (k_1^i, \ldots k_n^i) \in \hat{G} = \mathbb{Z}^n$. View $\hat{\pi}_i = (k_1^i, \ldots k_n^i)$ as real vectors and let $R_m = \langle \hat{\pi}_i \rangle_{i \leq m}$ be the linear span of $\{\hat{\pi}_i\}_{i \leq m}$ in $\mathbb{R}^n$. If $\{\Gamma_\pi\}$ is an infinite sequence, then there is an m such that $R_m = R_{m+1}$.

But

$$\begin{aligned}\gamma \in \bigcap_{i \le m} \Gamma_{\pi_i} &\iff \forall i \le m(\gamma \cdot \hat{\pi}_i = \hat{\pi}_i)\\ &\iff \forall v \in R_m((\gamma^{-1})^t v^t = v^t)\\ &\iff \forall v \in R_{m+1}((\gamma^{-1})^t v^t = v^t)\\ &\iff \forall i \le m+1(\gamma \cdot \hat{\pi}_i = \hat{\pi}_i)\\ &\iff \gamma \in \bigcap_{i \le m+1} \Gamma_{\pi_i},\end{aligned}$$

contradicting that $\{\bigcap_{i\le m} \Gamma_{\hat{\pi}_i}\}$ is strictly decreasing. So every strictly decreasing sequence $\{\bigcap_{i\le m} \Gamma_{\pi_i}\}$ is finite, hence $E_\Gamma^{L^2(G)}$ is smooth.

Note that for an action by automorphisms on a compact group, mixing is equivalent to mildly mixing and weakly mixing is equivalent to ergodic (see [**Kechris 1**]).

3.3. Compressibility

Compressibility is another important descriptive property of equivalence relations. A countable Borel equivalence relation E is *compressible* if there is a Borel injection $\phi : X \to X$ such that $xE\phi(x)$ for every $x \in X$ and $X \backslash \phi(X)$ is a complete section. In this paper, we only use this terminology as an alias of *nonexistence of invariant Borel probability measure.* The equivalence of these two conditions is due to Nadkarni (see [**Nadkarni**]). Like smoothness, compressibility is also a notion of noncomplexity. While saying a Borel equivalence relation E is not smooth is the same as saying that $E_0 \sqsubseteq_B E$, for a nonsmooth E, being noncompressible is equivalent to $E_0 \sqsubseteq_B^i E$. It turns out that once again the compressibility of $E_\Gamma^{L^2(X)}$ and $E_\Gamma^{MALG_\mu}$ coincides and is strictly stronger than the compressibility of $E_\Gamma^{Aut(X,\mu)}$.

3.3.1. Compressibility of $E_\Gamma^{L^2(X,\mu)}$ and $E_\Gamma^{MALG_\mu}$. Let X, Y be Borel Γ-spaces and recall that the *diagonal* Γ *action* on $X \times Y$ is defined by $\gamma \cdot (x, y) = (\gamma \cdot x, \gamma \cdot y)$. Also assume that μ, ν are Γ-invariant Borel probability measures on X, Y respectively. The Borel probability measure $\mu \times \nu$ defined on $X \times Y$ is also Γ- invariant . Similarly, let $(X_i)_{i \in \mathbb{N}}$ be a sequence of Γ-spaces. The diagonal Γ action on $\prod_{i \in \mathbb{N}} X_i$ is defined by $\gamma \cdot (x_i) = (\gamma \cdot x_i)$. Denote by $E_\Gamma^{\prod_i X_i}$ the orbit equivalence relation of diagonal Γ action on $\prod X_i$.

Let X be a Borel Γ-space with invariant (nonatomic) Borel probability μ. Then being mildly mixing is equivalent to saying that for any Borel Γ-space Y with ergodic non-Γ-atomic invariant Borel measure ν, the diagonal Γ action on $(X \times Y, \mu \times \nu)$ is ergodic. And being weakly mixing is equivalent to saying that for any Borel Γ-space Y with ergodic non-Γ-atomic invariant Borel probability measure ν, the diagonal Γ action on $(X \times Y, \mu \times \nu)$ is ergodic.(see [**SW, Glasner**])

In particular if the Γ action on X is mildly mixing, then no Borel Γ-space Y with non-Γ-atomic invariant ergodic Borel measure can be embedded in $L^2(X)$, which means $E^{L^2(X)}$ is smooth (see Proposition 3.2.10 (iii)).

Similarly, there is also a connection between the compressibility of $E_\Gamma^{L^2(X)}$ and weakly mixing action on X. To be precise, notice first that $E_\Gamma^{L^2(X)}$ is never compressible because every constant function in $L^2(X)$ is fixed by Γ. So it only make sense to describe the compressibility of the nonconstant part of $L^2(X)$. Denote by

$L^2_{nc}(X) = L^2(X)\backslash\{\mathbb{C}\cdot 1\}$ the Γ-invariant subspace of $L^2(X)$ of nonconstant elements. We have:

Theorem 3.3.1. *Let X be a Borel Γ-space with Γ-invariant Borel probability measure μ. Then*

(i) Γ action on X is weakly mixing if and only if $E_\Gamma^{L^2_{nc}(X)}$ is compressible;

(ii) $E_\Gamma^{L^2_{nc}(X)}$ is compressible if and only if $E_\Gamma^{MALG_\mu\backslash\{X,\emptyset\}}$ is compressible;

(iii) $E_\Gamma^{Aut(X,\mu)}$ is compressible if and only if $\overline{\pi(\Gamma)}$ is not compact.

Proof. (i) If $E_\Gamma^{L^2_{nc}(X)}$ is not compressible, then it has a Γ-invariant ergodic Borel probability measure ν. Let $p : L^2(X) \to L^2_0(X)$ be the projection. Since $p\nu$ is a Γ-invariant ergodic probability measure on $L_0(X)$, supp $p\nu = p(\text{supp } \nu)$ is a compact Γ-invariant subset (see Proposition 3.3.2). So the Γ action on X is not weakly mixing.

Conversely, assume now Γ action on X in not weakly mixing. Then we can find a Borel Γ-space Y with ergodic invariant Borel probability measure ν so that $E_\Gamma^{X\times Y}$ is not ergodic. Let $F \in L^2(X\times Y, \mu\times\nu)$ be a nonconstant Γ-invariant function. Let $f_y(x) = F(x,y)$. Notice that f_y is not a constant function for ν-a.e. y, otherwise by the ergodicity of Γ action on Y, F is a constant function. So we have a ν-measurable Γ-homomorphism from Y into $L^2_{nc}(X)$ defined by $y \mapsto f_y$. Clearly, the image measure $f_*\nu$ is a Γ invariant Borel probability measure of $L^2_{nc}(X)$.

(ii) Since $E_\Gamma^{MALG_\mu\backslash\{X,\emptyset\}} \sqsubseteq^i_B E_\Gamma^{L^2_{nc}(X)}$, the compressibility of $E_\Gamma^{L^2_{nc}(X)}$ implies the compressibility of $E_\Gamma^{MALG_\mu\backslash\{X,\emptyset\}}$. On the other hand, if $E_\Gamma^{L^2_{nc}(X)}$ is not compressible, then we can find an $f \in L^2_{nc}(X)$ such that $\overline{\Gamma\cdot f}$ is compact. Let $A = f^{-1}(B)$ for some Borel set $B \subseteq \mathbb{C}$ such that $0 < \mu(A) < 1$. By Lemma 3.2.6, $\overline{\Gamma\cdot A}$ is compact. Since $Iso(\overline{\Gamma\cdot A})$ is compact, in particular amenable, there is a $Iso(\overline{\Gamma\cdot A})$ invariant

Borel probability measure ν on $\overline{\Gamma \cdot A}$. Clearly ν is Γ-invariant. So $E_\Gamma^{MALG_\mu \setminus \{X,\emptyset\}}$ is not compressible.

(iii) The existence of Γ-invariant Borel probability measure is equivalent to the existence of the Haar probability measure on $\overline{\pi(\Gamma)}$, so this statement is obviously true. □

3.3.2. Compressibility And Isometric Factors. Like the connection of rigid factors to the smoothness of $E_\Gamma^{L^2(X)}$ and $E_\Gamma^{Aut(X,\mu)}$, the notion *of isometric factors* has connections to the compressibility of $E_\Gamma^{L^2_{nc}(X)}$ and $E_\Gamma^{Aut(X,\mu)}$.

Let us first study the spectral characterization of isometric Γ-spaces. A Borel Γ-space X is said to be *isometrizable* if there is an Γ-invariant metric d on X that induces a separable topology and the same Borel structure. If moreover there is an Γ-invariant Borel probability measure μ, then we say (X, μ) is *isometrizable* if there is a conull invariant subset of X that is isometrizable.

Assume now (X, μ) is isometrizable with witness metric d. Let $X' = \text{supp}\mu$. Clearly X' is invariant conull and $\forall x \in X'$, $r > 0$, the open ball $B_{x,r}$ is non-null. X' might not be complete (with respect to the metric d), but taking the completion $\overline{X'}$, d can be (uniquely) continuously extended to $\overline{d}$ on $\overline{X'}$. We can also extend μ to $\overline{\mu}$ on $\overline{X'}$ by letting $\overline{\mu}(A) = \mu(A \cap X')$ for every Borel $A \subseteq \overline{X'}$, so that Γ acts on $(\overline{X'}, \overline{d})$ by isometries with invariant Borel probability measure $\overline{\mu}$. Clearly $(X, \mu) \cong (\overline{X'}, \overline{\mu})$, so if the Γ-space X is isometrizable, we can assume Γ acts on a Polish space (X, d) by isometries. We have the following easy connection between ergodicity and topological properties.

Proposition 3.3.2. *Let Γ act on Polish space (X, d) by isometries with an invariant Borel probability measure μ.*

i) If μ is ergodic, then X is compact;

ii) μ is ergodic iff there exists a dense orbit (iff every orbit is dense).

We can tell whether a Borel Γ space X with invariant probability measure is isometrizable from its unitary representation on $L^2(X)$.

A Γ action on X is said to have *discrete spectrum* if π^X is the direct sum of finite dimensional irreducible unitary representations. Or in other words, $L^2(X)$ is the direct sum of finite dimensional Γ-invariant subspaces.

Suppose now Γ acts by isometries on a compact Polish space X. Since $Iso(X)$ is compact, by the Peter-Weyl theorem, the Γ action on X has discrete spectrum. On the other hand, assume μ is ergodic and X has discrete spectrum; Mackey has shown that the Γ action on (X, μ) is isomorphic to the left translation of some homogeneous space, in particular, isometrizable (see [**Mackey, Furman, FK**]).

In general, we have:

Theorem 3.3.3. *Let Γ be a countable group and X a Borel Γ space with invariant probability measure μ. Then Γ action on X has discrete spectrum if and only if (X, μ) is isometrizable.*

PROOF. ($\Leftarrow$) We can assume Γ acts Polish space (X, d) by isometries and $X = \text{supp}\mu$. We can also assume the Γ action is faithful, otherwise we can replace Γ by $\Gamma / \ker \pi^X$ because their invariant subspaces are exactly the same. Notice that by Proposition 3.3.2 and Theorem 3.2.17, for μ a.e. $x \in X$, $\overline{\Gamma \cdot x}$ is compact. Since every nonempty open ball is non-null, we can pick a countable dense subset $\{x_i\} \subseteq X$ such that $\overline{\Gamma \cdot x_i}$ is compact for every i. Let $(\gamma_k) \subseteq \Gamma$ be an arbitrary sequence. We can find a converging subsequence $(\gamma'_k) \subseteq (\gamma_k)$. Since $x'_i \in \overline{\Gamma \cdot x_i}$ is compact for every x_i, we can pick γ'_k and x'_k inductively so that

$$\forall i \leq k(d(x'_i, \gamma'_k \cdot x_i) < 2^{-k}).$$

Let $T : X \to X$ defined by $T(x) = \lim_{n\to\infty} x'_{i_n}$, where (x_{i_n}) is an arbitrary subsequence of (x_i) that converge to x. It is straightforward to check that T is well defined, μ, d are Γ-invariant, and $\gamma'_k \to T$ point wisely and in the weak topology of $Aut(X, \mu)$. Therefore $\overline{\Gamma}$ is a compact group and X is a Polish $\overline{\Gamma}$ space. The conclusion of X having discrete spectrum follows from the Peter-Weyl theorem.

($\Rightarrow$) Let $L^2(X) = \prod V_n$, where V_n are finite dimensional Γ-invariant subspaces.

Let $f = \sum f_n$, where $f_n \in V_n$, $f \in L^2(X)$ and injective. Suppose $\gamma'_i \cdot f \to g$ for some $g \in L^2(X)$. Since $\overline{\Gamma \cdot f}$ is compact, there is a subsequence $\{\gamma_i\}$ such that $\gamma_i^{-1} \cdot f \to g'$ for some $g' \in L^2(X)$. So $\gamma_i^{-1}(A) \to g^{-1}(f(A))$, $\gamma_i^{-1}(A) \to g'^{-1}(f(A))$ for every Borel subset $A \subseteq X$. Therefore $\gamma_i \to \gamma \in \overline{\Gamma} \leq Aut(X, \mu)$ (in the weak topology) and $\overline{\Gamma} \cdot f = \overline{\Gamma \cdot f}$. Since $Aut(X, \mu)_f = \{1\}$, we have $\overline{\Gamma}$ is compact (in the weak topology). Let ν be the Haar measure on $\overline{\Gamma}$. So we have a Γ acting on $L^2(\overline{\Gamma})$ defined by $(\gamma \cdot \phi)(h) = \phi(\gamma^{-1}h)$.

Consider $F : \overline{\Gamma} \times X :\to \mathbb{C}$, which is defined by $F(x, \gamma) = F_x(\gamma) = \gamma \cdot f(x)$. By the Fubini-Tonelli theorem, $F_x \in L^2(\overline{\Gamma})$ μ-a.e. Since

$$\forall \lambda \in \overline{\Gamma} \forall \gamma \in \Gamma \forall^*_\mu x \in X(\lambda \cdot f(\gamma \cdot x) = (\gamma^{-1}\lambda) \cdot f(x)),$$

we have

$$\forall^*_\mu x \in X \forall \gamma \in \Gamma \forall^*_\nu \lambda \in \overline{\Gamma}(F_{\gamma\cdot x}(\lambda) = \lambda \cdot f(\gamma \cdot x) = (\gamma^{-1}\lambda) \cdot f(x) = \gamma \cdot F_x(\lambda)),$$

i.e., $\forall^*_\mu x \in X \forall \gamma \in \Gamma(F_{\gamma\cdot x} = \gamma \cdot F_x)$. Since f is injective and every $\gamma \in \overline{\Gamma} \subseteq Aut(X, \mu)$, has a pointwise realization, we have

$$\forall \gamma \in \overline{\Gamma} \forall x \forall y((\gamma \cdot f)(x) = (\gamma \cdot f)(y) \Rightarrow x = y).$$

So $\forall x \forall y(F_x = F_y \Rightarrow x = y)$. Therefore $x \mapsto F_x$ is a μ-a.e. μ-measurable Γ-space embedding of X into $L^2(\overline{\Gamma})$. So there is a μ-conull Γ-invariant Borel set $Y \subseteq X$ such that $E^X_\Gamma|Y \sqsubseteq^i_B E^{L^2(\overline{\Gamma})}_\Gamma$. $d(x, y) = \|F_x, F_y\|_2$ is a Γ-invariant metric defined on Y. □

Corollary 3.3.4. *Let X be a Borel Γ-space with invariant Borel probability measure μ. Then X is isometrizable if and only if $\overline{\pi^X(\Gamma)}$ is compact in the weak topology.*

Proof. If X is isometrizable, then $\overline{\pi^X(\Gamma)}$ is compact follows from the fist part of the proof of Theorem 3.3.3.

Suppose $\overline{\pi^X(\Gamma)}$ is compact. Since every Γ-invariant closed subspace of $L^2(X)$ is also $\overline{\pi^X(\Gamma)}$-invariant, X has discrete spectrum by the Peter-Weyl theorem. □

Corollary 3.3.5. *(Mackey) Let Γ be a countable group and X a faithful Borel Γ space with invariant probability measure μ. If the Γ action on X has discrete spectrum, then Γ can be embedded onto a dense subgroup of a compact group G such that Γ-space (X, μ) is isomorphic to $(G/K, \pi(\nu))$, where K is a closed subgroup, ν is the normalized Haar measure on G, and $\pi : G \to G/K$ is the natural projection.*

PROOF. We can assume Γ acts faithfully on Polish space (X, d) by isometries and $X = \text{supp}\mu$.

Let $G = \overline{\Gamma} \leq Aut(X, \mu)$ with Haar measure ν. Fix an arbitrary x_0 in X and let $K = G_{x_0}$. Since G is compact and Γ is dense in G, $G \cdot x_0 = \overline{\Gamma \cdot x_0} = X$. It is easy to check that the map $\alpha : gK \mapsto g \cdot x_0$ is a Γ-space isomorphism of G/K to X.

It remains to show that $\mu = (\alpha \circ \pi)(\nu)$, where $\pi : G \to G/K$ is the natural projection. Let ν_0 be the normalized Haar measure on K. Consider $A \subseteq gK$, define $\phi(A) = \nu_0(g^{-1}A)$. Since

$$\nu_0((gh)^{-1}A) = \nu_0(h^{-1}(g^{-1}A)) = \nu_0(g^{-1}A),$$

the definition is independent of the choice of g and is welldefined. Define a finite measure ν' on G by

$$v' = \int_{x \in X} \phi_y d\mu(x),$$

where $\phi_x(A) = \phi(A \cap \alpha^{-1}(x)) = \nu_0((h^{-1}A) \cap K)$ for some $h \in \alpha^{-1}(x)$. We have $\phi_x(gA) = \nu_0((h^{-1}gA) \cap K) = \phi(A \cap (g^{-1}hK)) = \phi_{g^{-1} \cdot x}(A)$ for some $h \in \alpha^{-1}(x)$.

Note that μ is Γ-invariant, so it is $\overline{\Gamma} = G$-invariant. Therefore for arbitrary $g \in G$,

$$\begin{aligned} v'(gA) &= \int_{x \in X} \phi_x(gA) d\mu(x) \\ &= \int_{x \in X} \phi_{g^{-1} \cdot x}(A) d\mu(x) \\ &= \int_{x \in X} \phi_x(A) d\mu(gx) \\ &= \nu'(A). \end{aligned}$$

By the uniqueness of the Haar measure, $\nu' = \nu$. Finally, for any Borel set X, we have

$$\begin{aligned}(\alpha \circ \pi)(\nu)(A) &= \nu'(\pi^{-1}(\alpha^{-1}(A))) \\ &= \nu'(\alpha^{-1}(A)K) \\ &= \int_{x \in X} \phi_x(\alpha^{-1}(A)K) d\mu(x) \\ &= \int_{x \in A} \nu_0(K) d\mu(x) \\ &= \mu(A).\end{aligned}$$

□

Example 3.3.6. No weak mixing action (with invariant Borel probability) is isometrizable.

Let (X, μ), (Y, ν) be Γ-spaces. If there is a Γ-map $\alpha : X \to Y$, which is onto and $\nu = f\mu$, we say that X is an *(Γ-)extension* of Y and Y is a *(Γ-)factor* of X. There is a canonical embedding of $L^2(Y, \nu)$ into $L^2(X, \mu)$, namely $f \mapsto f \circ \alpha$. So it is easy to check that the Borel Γ-space (X, μ) has no rigid factors (in the sense of Definition 3.2.8) if and only if it has no nontrivial rigid Γ-factors. Call a factor (Y, ν) of (X, μ) an *isometric factor* if it isometrizable. We have $E_\Gamma^{Aut(X,\mu)}$ is compressible iff (X, μ) is not isometrizable by Theorem 3.3.1 (iii) and Corollary 3.3.4. We also have $E_\Gamma^{L^2_{nc}(X)}$ is compressible iff (X, μ) has no nontrivial isometric factors. Because if α is a surjective Γ-map $\alpha : X \to Y$ and d is a Γ-invariant metric on Y, then $y \mapsto d(y, \alpha(\cdot))$ is an Γ-space embedding of Y into $L^2(X)$. So $L^2(X)$ is not compressible. Conversely, if

$F \in L^2(X \times Y, \mu \times \nu)$ is nonconstant Γ-invariant, X is embedded into a nontrivial subset of $L^2(Y, \nu)$ by the map $x \mapsto F(x, \cdot)$.

3.4. Some Embedding and Containment Results

3.4.1. Translation and Conjugation. Let $X = G$ be a compact Polish group with Haar measure μ and $\Gamma \subset G$ a countable subgroup. There are two natural ways for Γ to act on G: left translation

$$(\gamma, g) \mapsto \gamma g,$$

and conjugation

$$(\gamma, g) \mapsto g^\gamma = \gamma g \gamma^{-1}.$$

To distinguish these actions, let E_l^G, E_c^G be the induced equivalence relation by the Γ left translation and conjugation, respectively and let $\gamma \cdot f(g) = f(\gamma^{-1} g)$, $f^\gamma(g) = f(g^{\gamma^{-1}}) = f(\gamma^{-1} g \gamma)$ for $f \in L^2(G)$. And denote by $E_l^{L^2(G)}$, $E_c^{L^2(G)}$ the induced equivalence relation on $L^2(G)$ respectively.

Fix an $f \in L^2(G)$ that is injective everywhere, so that $Aut(G, \mu)_f$ is trivial.

Define $f_g(h) = f(g^{-1}h)$. Since

$$\gamma \cdot f(h) = f_g(\gamma^{-1} h) = f(g^{-1} \gamma^{-1} h) = f_{\gamma g}(h)$$

and $Aut(G, \mu)_f$ is trivial, $g \mapsto f_g$ is a continuous invariant embedding of E_l^G into $E_l^{L^2(G)}$. Hence $E_l^G \sqsubseteq_c^i E_l^{L^2(G)}$.

Define $\tau_g(h) = f(g^{h^{-1}}) = f(h^{-1}gh)$. We have

$$\gamma \cdot \tau_g(h) = \tau_g(\gamma^{-1}h) = f(h^{-1}\gamma g \gamma^{-1} h) = f(h^{-1}g^{\gamma}h) = \tau_{g^{\gamma}}(h)$$

and since f is injective everywhere, $\tau_{g_1} = \tau_{g_2} \iff g_1 = g_2$. So $g \mapsto \tau_g$ is a continuous invariant embedding of E_c^G into $E_l^{L^2(G)}$. Hence we have $E_c^G \sqsubseteq_c^i E_l^{L^2(G)}$.

Define $\lambda_g(h) = f(g^{-1}hg)$. We have

$$\lambda_g^{\gamma}(h) = \lambda_g(h^{\gamma^{-1}}) = \lambda_g(\gamma^{-1}h\gamma) = f(g^{-1}\gamma^{-1}h\gamma g) = \lambda_{\gamma g}(x).$$

$\lambda_{g_1} = \lambda_{g_2} \iff \forall_{\mu}^{*} h \in G(g_1^{-1}hg_1^{-1} = g_2^{-1}hg_2^{-1}) \iff g_2 g_1^{-1} \in Z(G)$, where $Z(G)$ is the center of G.

If $Z(G) \subseteq \Gamma$, then $g \mapsto \lambda_g$ is a continuous Borel reduction of E_l^G into $E_c^{L^2(G)}$, so $E_l^G \leq_B E_c^{L^2(G)}$. It is an embedding if $Z(G) = \{1\}$.

Suppose a Borel Γ space X with invariant Borel probability is properly isometrizable (see next section) with witness d. Define $f_x(y) = d(x, y)$. It is easy to check that $f_{x_1} = f_{x_2} \iff x_1 = x_2$ and $\gamma \cdot f_x = f_{\gamma \cdot x}$. Thus $E_{\Gamma}^X \sqsubseteq_c^i E_{\Gamma}^{L^2(X)}$. Since conjugations are isometries, $E_c^G \sqsubseteq_c^i E_c^{L^2(G)}$ for a compact Polish group G.

We can also show that $E_c^{L^2(G)} \sqsubseteq_c^i E_l^{L^2(G)}$. Let $\Phi : L^2(G) \to L^2(G)$ be the map defined by

$$\Phi(f)(g) = \int_{h \in G} f(ghg^{-1}) d\mu(h),$$

where f is any element in $L^2(G)$ (we no longer need f to be a fixed injective function). Since

$$\Phi(f^\gamma)(g) = \int_{h\in G} f(\gamma^{-1}ghg^{-1}\gamma)d\mu(h) = \Phi(f)(\gamma^{-1}g) = \gamma\cdot\Phi(f)(g),$$

Φ is a Γ-map. And

$$\begin{aligned}\Phi(f_1)=\Phi(f_2) &\iff \forall^*_\mu g(\int_{h\in G} f_1(ghg^{-1})d\mu(h) = \int_{h\in G} f_2(ghg^{-1})d\mu(h))\\ &\iff \forall^*_\mu g(\int_{h\in G}(f_1-f_2)(ghg^{-1})d\mu(h)=0\\ &\iff f_1=f_2.\end{aligned}$$

Combining the above results, we have

Proposition 3.4.1. *Let G be a compact Polish group and Γ be a subgroup of G. We have $E_c^G \sqsubseteq_c^i E_c^{L^2(G)} \sqsubseteq_c^i E_l^{L^2(G)}$ and $E_l^G \sqsubseteq_c^i E_l^{L^2(G)}$. If $Z(G)\leq\Gamma$, $E_l^G \leq_c E_c^{L^2(G)}$ and if $Z(G)=\{1\}$, $E_l^G \sqsubseteq_c^i E_c^{L^2(G)}$.*

Example 3.4.2. Let $X=G$ be a simply connected compact semisimple Lie group with a trivial center, $\Gamma\subset Aut(G)$ a countable subgroup acting on G by automorphisms. Identify G and $Inn(G)\leq Aut(G)$, which is a Γ invariant finite index normal subgroup. We have $E_\Gamma^G \sqsubseteq_c^i E_c^{Aut(G)}$. Therefore $E_\Gamma^G \sqsubseteq_c^i E_l^{L^2(Aut(G))}$ and $E_\Gamma^G \sqsubseteq_c^i E_c^{L^2(Aut(G))}$.

3.4.2. Embeddings related to hyperfiniteness.

Proposition 3.4.3. *Let X be a Borel Γ-space with invariant Borel probability measure μ. Then $E_\Gamma^{L^2(X)} \cong_B E_\Gamma^{(MALG_\mu)^{\mathbb{N}}} \cong_B E^{L^2(X)^{\mathbb{N}}}$.*

PROOF.

$$MALG_\mu \sqsubseteq_B L^2(X),$$

so

$$(MALG_\mu)^{\mathbb{N}} \sqsubseteq_B L^2_B(X)^{\mathbb{N}}.$$

To see that

$$L^2(X) \sqsubseteq_B (MALG_\mu)^{\mathbb{N}},$$

let

$$\alpha : L^2(X) \to (MALG_\mu)^{\mathbb{N}}$$

be the map defined by

$$\alpha(f) = (f^{-1}(A_n))_{n\in\mathbb{N}},$$

where $\{A_n\}_{n\in\mathbb{N}}$ is an enumeration of basic open subsets of $\mathbb{C}$. Clearly α is an injective Borel Γ-map. Since

$$L^2(X) \sqsubseteq_B (MALG_\mu)^{\mathbb{N}},$$

we also have

$$L^2(X)^{\mathbb{N}} \sqsubseteq_B (MALG_\mu)^{\mathbb{N}\times\mathbb{N}}.$$

Therefore, we have

$$E_\Gamma^{L^2(X)} \sqsubseteq_B^i E_\Gamma^{(MALG_\mu)^{\mathbb{N}}} \sqsubseteq_B^i E^{L^2(X)^{\mathbb{N}}} \sqsubseteq_B^i E_\Gamma^{(MALG_\mu)^{\mathbb{N}\times\mathbb{N}}}.$$

Notice that

$$E_\Gamma^{(MALG_\mu)^{\mathbb{N}}} \cong E_\Gamma^{(MALG_\mu)^{\mathbb{N}\times\mathbb{N}}}.$$

So

$$E_\Gamma^{L^2(X)} \cong_B E_\Gamma^{(MALG_\mu)^{\mathbb{N}}} \cong_B E^{L^2(X)^{\mathbb{N}}}.$$

□

Let X be a Borel Γ-space with invariant Borel probability measure μ. Consider the following property:

(*) For every sequence of Borel subsets (A_i), there is an $N \in \mathbb{N}$ such that $\bigcap_{i \leq N} \Gamma_{A_i} = \bigcap_{i \in \mathbb{N}} \Gamma_{A_i}$.

Property (*) holds for mixing and mild mixing action. In fact, it is easy to check that the smoothness of $E_\Gamma^{L^2(X)}$ implies property (*). If Γ acts freely on $MALG_\mu \backslash \{\emptyset, X\}$, i.e. , γ is μ-ergodic for all $\gamma \in \Gamma$, then it satisfies (*). If there exists an $r < 1$ such that

$$\gamma \cdot A = A \Rightarrow \mu(A) > r$$

for every $A \in MALG_\mu$ and $\gamma \in \Gamma \backslash \{1\}$, then (*) holds. More generally, if

$$\sup_{\gamma \in \Gamma \backslash \{1\}} |\{A \in MALG_\mu : \gamma \cdot A = A\}| < \infty,$$

then (*) holds.

Corollary 3.4.4. *Let X be a Borel Γ-space with invariant Borel probability measure μ. If the above (*)-property holds, then $E_\Gamma^{L^2(X)}$ is hyperfinite $\iff$ $E_\Gamma^{MALG_\mu}$ is hyperfinite.*

Proof. Let

$$P_N = \{(A_n) \in (MALG_\mu)^{\mathbb{N}} : \bigcap_{i<N} \Gamma_{A_i} \neq \bigcap_{i \leq N} \Gamma_{A_i} = \bigcap_{i \in \mathbb{N}} \Gamma_{A_i}\}$$

so that each P_N is Γ-invariant Borel and $\{P_N\}$ is a partition of $(MALG_\mu)^{\mathbb{N}}$. We only need to show that $E_\Gamma^{P_N} = E_\Gamma^{(MALG_\mu)^{\mathbb{N}}}|P_N$ is hyperfinite for each N. Fix an arbitrary N. Notice that

$$E_\Gamma^{(MALG_\mu)^N} \subseteq (E^{MALG_\mu})^N$$

is hyperfinite. So

$$E_\Gamma^{(MALG_\mu)^N} = E_{<T>}$$

for some Borel automorphism $T : (MALG_\mu)^N \to (MALG_\mu)^N$. Define

$$S : (MALG_\mu)^{\mathbb{N}} \to (MALG_\mu)^{\mathbb{N}}$$

by

$$S((A_n)_{n\in\mathbb{N}}) = (\gamma \cdot A_n)_{n\in\mathbb{N}}$$

for some $\gamma \in \Gamma$ such that $(\gamma \cdot A_n)_{n\leq N} = T((A_n)_{n\leq N})$. Notice that if

$$(\gamma_1 \cdot A_n)_{n\leq N} = (\gamma_2 \cdot A_n)_{n\leq N},$$

then

$$\gamma_1 \in \gamma_2(\bigcap_{i\leq N} \Gamma_{A_i}) = \gamma_2(\bigcap_{i\in\mathbb{N}} \Gamma_{A_i}).$$

So clearly S is well-defined. Since

$$E_\Gamma^{(MALG_\mu)^N} = E_{<T>},$$

for every $\gamma_1 \in \Gamma$, we can find an $n \in \mathbb{N}$ so that

$$(\gamma_1 \cdot A_n)_{n \leq N} = T^n((A_n)_{n \leq N}).$$

It it easy to check that

$$S^n((A_n)_{n \in \mathbb{N}})|N = T^n((A_n)_{n \leq N}),$$

so

$$S^n((A_n)_{n \in \mathbb{N}}) = (\gamma_2 \cdot A_n)_{n \in \mathbb{N}}$$

for some $\gamma_2 \in \gamma_1(\bigcap_{i \leq N} \Gamma_{A_i})$. Using the condition

$$(\bigcap_{i \leq N} \Gamma_{A_i}) = (\bigcap_{i \in \mathbb{N}} \Gamma_{A_i})$$

again, we have

$$S^n((A_n)_{n \in \mathbb{N}}) = (\gamma_1 \cdot A_n)_{n \in \mathbb{N}}$$

and

$$E_{<S>} = E_\Gamma^{P_N}.$$

Therefore, $E_\Gamma^{P_N}$ is hyperfinite. □

Let X be a Γ-space and μ a quasi-invariant measure. E_Γ^X is *amenable* iff E_Γ^X is hyperfinite on an invariant μ-conull subset (see [**Zimmer**] or [**Kaimanovich**]). Recall that $1_\Gamma \prec \lambda_\Gamma$ iff Γ is amenable (see [**BHV**]). If an ergodic Γ-space (X, μ) is *amenable* (see [**Zimmer**] for definition), then $\pi^X \prec \lambda_\Gamma$ (see [**Kuhn**]). The converse is in general not true (see [**AD**]). Let

$$\lambda^X = \lambda_\Gamma^X = \int_X^\oplus \lambda_{\Gamma/\Gamma_x} d\mu(x).$$

We have in analogy to the amenability of E_Γ^X:

Theorem 3.4.5. *Let X be a Borel Γ-space with quasi-invariant Borel probability measure μ. If $E = E_\Gamma^X$ is amenable, then $\pi^X \prec \lambda^X$.*

PROOF. We can find a sequence $\lambda^n : E \to \mathbb{R}$ of non-negative Borel functions such that

(i) $\lambda_x^n \in \ell^1([x]_E)$, where $\lambda_x^n(y) = \lambda^n(x,y)$, for xEy;

(ii) $\|\lambda_x^n\|_1 = 1$; and

(iii) There is a μ-conull Borel E-invariant set $A \subseteq X$, such that $\left\|\lambda_x^n - \lambda_y^n\right\| \to 0$ for all $x, y \in A$ (see [**KM**] or [**Kaimanovich**]).

Let $\phi^n = \sqrt{\lambda^n}$. We have $\phi_x^n \in \ell^2([x]_E)$, $\|\phi_x^n\| = 1$ and

$$\left\|\phi_x^n - \phi_y^n\right\|^2 \leq \left\|\lambda_x^n - \lambda_y^n\right\|_1 \to 0.$$

Fix an arbitrary $f \in L^2(X,\mu)$. Let $f_n : E \to \mathbb{C}$, $(x,y) \mapsto \phi^n(x,y)f(x)$ and $\gamma \cdot (x,y) = (\gamma \cdot x, y)$ for xEy. Note that $f(x)\phi_x^n \in \ell^2([x]_E)$ for every $x \in X$. For any $\gamma \in \Gamma$,

$$\begin{aligned} |\langle \gamma \cdot f_n, f_n\rangle - \langle \gamma \cdot f, f\rangle| &= \left| \int (\langle \phi_{\gamma^{-1}\cdot x}^n, \phi_x^n\rangle - 1) \sqrt{\frac{d(\gamma\mu)}{d\mu}(x)} f(\gamma^{-1}\cdot x)\overline{f(x)} d\mu(x) \right| \\ &\leq \left| \int \frac{\left\|\phi_{\gamma^{-1}\cdot x}^n - \phi_x^n\right\|^2}{2} \sqrt{\frac{d(\gamma\mu)}{d\mu}(x)} f(\gamma^{-1}\cdot x)\overline{f(x)} d\mu(x) \right|. \end{aligned}$$

Since $4 \geq \left\|\phi^n_{\gamma^{-1}\cdot x} - \phi^n_x\right\|^2 \to 0$ a.e. and

$$(\frac{d(\gamma\mu)}{d\mu}(x))^{\frac{1}{2}} f(\gamma^{-1}\cdot x)\overline{f(x)} \in L^1(X,\mu),$$

$|\langle \gamma \cdot f_n, f_n\rangle - \langle \gamma \cdot f, f\rangle| \to 0$ for every $\gamma \in \Gamma$. So for any finite subset $F \in \Gamma$ and $\varepsilon > 0$, we can find an n such that

$$|\langle \gamma \cdot f_n, f_n\rangle - \langle \gamma \cdot f, f\rangle| < \varepsilon$$

for all $\gamma \in F$. Since $[y]_E \cong_B \Gamma/\Gamma_y$, it is easy to check that

$$f_n \in \int_{y\in X}^{\oplus} \ell^2([y]_E)d\mu(y) = \int_{y\in X}^{\oplus} \ell^2(\Gamma/\Gamma_x)d\mu(y)$$

and the Γ action on E is Borel isomorphic to the left translation of Γ on

$$\int_{y\in X}^{\oplus} \Gamma/\Gamma_x d\mu(y).$$

Therefore every positive definite function realized in π^X is the pointwise limit of a sequence of positive definite functions realized in λ^X. In particular, we have $\pi^X \prec \lambda^X$. □

Call a Γ-space X *coamenable* if Γ_x is amenable for every $x \in X$. A Γ-space X is said to be *coamenable* μ-a.e. if an invariant μ-conull subset of X is coamenable.

Corollary 3.4.6. *Let X be a Borel Γ-space with quasi-invariant Borel probability measure μ. If X is coamenable μ-a.e. and E^X_Γ is amenable, then $\pi^X \prec \lambda_\Gamma$.*

PROOF. Assume that E_Γ^X is amenable and X is coamenable μ-a.e. Since Γ_x is amenable μ-a.e., we have $\lambda_{\Gamma/\Gamma_x} \prec \lambda_\Gamma$ μ-a.e. Notice that the condition $\pi_1 \prec \pi_2$ for unitary representations π_1, π_2 is equivalent to $\|\pi_1(a)\| \leq \|\pi_2(a)\|$ for all $a \in \ell^1(\Gamma)$, where $\pi(a) = \sum a_\gamma \pi(\gamma)$ for every unitary representation π (see Section F.4 of [**BHV**]). So we have $\left\|\lambda_{\Gamma/\Gamma_x}(a)\right\| \leq \|\lambda_\Gamma(a)\|$ for every μ-a.e. $x \in X$ and $a \in \ell^1(\Gamma)$. It is easy to see that

$$\begin{aligned}
\left\langle \lambda^X(a) \cdot f, f \right\rangle &= \sum_{\gamma \in \Gamma} a_\gamma \int_X \left\langle \lambda_{\Gamma/\Gamma_x} \cdot f_x, f_x \right\rangle d\mu(x) \\
&= \int_X \sum_{\gamma \in \Gamma} a_\gamma \left\langle \lambda_{\Gamma/\Gamma_x} \cdot f_x, f_x \right\rangle d\mu(x) \\
&= \int_X \left\langle \lambda^X(a) \cdot f_x, f_x \right\rangle d\mu(x) \\
&\leq \int_X \|\lambda_\Gamma\|^2 \|f_x\|^2 \, d\mu(x) \\
&= \|\lambda_\Gamma\|^2 \|f\|^2
\end{aligned}$$

for every $f \in \int_{x \in X}^{\oplus} \ell^2(\Gamma/\Gamma_x) d\mu(x)$ and $a \in \ell^1(\Gamma)$. So $\left\|\lambda_{\Gamma/\Gamma_x}(a)\right\| \leq \|\lambda_\Gamma(a)\|$ for every $a \in \ell^1(\Gamma)$ and hence by Theorem 3.4.5,

$$\pi^X \prec \lambda^X \prec \lambda_\Gamma.$$

$\square$

Corollary 3.4.7. *Let X be a Borel Γ-space with invariant Borel probability measure μ.*

If $\exists_\mu^* x(\Gamma_x$ is amenable) and E_X^Γ is amenable, then Γ is amenable.

Proof. Since amenability is preserved under subsets, we may assume that E_Γ^X is amenable and X is coamenable μ-a.e. (otherwise, replace X by an non-null invariant Borel subset $X' \subseteq X$, which is coamenable $\mu|X'$-a.e.)

By Corollary 3.4.6,

$$1 \leq \pi^X \prec \lambda^X \prec \lambda_\Gamma.$$

So Γ is amenable. □

Corollary 3.4.8. *Let X be a Borel Γ-space and Γ a free nonabelian group. If X is coamenable, then $E = E_\Gamma^{X'}$ is hyperfinite and compressible, where X' is the nonfree part of X.*

In particular, if Γ is a free nonabelian group and X is coamenable, then the Γ action is free μ-a.e. for every Γ-invariant Borel probability measure μ on X.

Proof. By replacing X with X', we may assume Γ_x is amenable and nontrivial for every $x \in X$. Thus by the Nielsen-Schreier theorem, Γ_x is cyclic for every x. Fix a well-ordering on Γ. Let

$$\rho(\gamma) = \min_{\alpha \in \Gamma} \{\alpha\gamma\alpha^{-1}\}$$

and put $\gamma_1 <_\rho \gamma_2$ if

$$\rho(\gamma_1) \leq \rho(\gamma_2) \vee (\rho(\gamma_1) = \rho(\gamma_2) \Rightarrow \gamma_1 < \gamma_2).$$

$<_\rho$ is also a well-ordering on Γ. Consider the assignment $x \mapsto a_x \in \Gamma$, where a_x is the smallest generator of Γ_x respect to $<_\rho$, i.e.,

$$\langle a_x \rangle = \Gamma_x \wedge a_x \leq_\rho a_x^{-1}.$$

This assignment is clearly Borel and $a_x = a_y$ if $\Gamma_x = \Gamma_y$. Furthermore, if $y \in [x]_E$, then $a_y E_c^\Gamma a_x$ (recall that $\gamma_1 E_c^\Gamma \gamma_2$ iff $\gamma_1 = \gamma\gamma_2\gamma^{-1}$ for some $\gamma \in \Gamma$). This is because $\gamma a_x \gamma^{-1}$ is a generator of Γ_y, for some $\gamma \in \Gamma$ such that $\gamma \cdot x = y$. If $\gamma a_x \gamma^{-1} \neq a_y$, then $a_x E_c^\Gamma a_y^{-1}$. Notice that ρ is a selector of E_c^Γ. We have

$$\rho(a_y^{-1}) = \rho(a_x) \leq \rho(a_x^{-1}) = \rho(a_y) \leq \rho(a_y^{-1}).$$

and hence $a_y E_c^\Gamma a_x$.

Let

$$Y = \{y \in X : a_y = \rho(a_y)\}.$$

Since

$$\rho(\rho(a_x)^{-1}) = \rho(a_x^{-1}) \leq \rho(a_x)$$

and $\rho(a_x)$ is a generator of $\Gamma_{\gamma \cdot x}$ for some $\gamma \in \Gamma$, $\rho(a_x) = a_y$ for some $y \in [x]_E$. So Y is a complete section.

Also if xEy, then $xE_c^\Gamma y$. Therefore if $x, y \in Y$ and xEy, then $a_y = \rho(a_x) = a_x$. By a basic fact from combinatorial group theory, two elements of a free group commute if and only if they are powers of a common element, that is, $\gamma_1\gamma_2 = \gamma_2\gamma_1$ iff $\gamma_1, \gamma_2 \in \langle\gamma\rangle$ for some $\gamma \in \Gamma$. Fix an arbitrary $x \in Y$. It is easy to check that we can uniquely write a_x as $a_x = \alpha\beta^n\alpha^{-1}$, where $n \in \mathbb{N}$, α is reduced, β is cyclically reduced (i.e., $\beta\beta$ is a reduced word) and β is not a nontrivial power of other elements, i.e.,

$$\forall\gamma \in \Gamma\forall m > 0(\gamma^m \neq \beta).$$

Put $b_x = \alpha\beta\alpha^{-1}$. We have

$$x(E|Y)y \iff \exists n \in \mathbb{Z}(x = b_x^n \cdot y).$$

Let $Y_\gamma = \{x \in Y : b_x = \gamma\}$. We have $E|Y_\gamma = E_{\langle\gamma\rangle}^{Y_\gamma}$. So

$$E|Y = \oplus_{\gamma\in\Gamma} E|Y_\gamma$$

is hyperfinite. Since Y is a complete section of E, E is hyperfinite. By Corollary 3.4.7, there is no Γ-invariant Borel probability measure on X. □

The following is an analog to a property of amenable groups (Corollary G.3.8, [**BHV**]) and amenable actions (Lemma 4.5.1, [**AD**]), which is related to Theorem 3.4.5.

Proposition 3.4.9. *Let X be a Borel Γ-space with quasi-invariant Borel probability measure μ. Suppose $H \leq \Gamma$ and $1_\Gamma \prec \lambda_{\Gamma/H}$. Then E_H^X is amenable iff E_Γ^X is amenable.*

Proof. $E_H^X \subseteq E_\Gamma^X$. So the amenability of E_Γ^X implies the amenability of E_H^X.

So assume E_H^X is amenable from now on. Let m be a Γ-invariant mean on $\lambda_{\Gamma/H}$ and $\{p_x\}$ a set of H-invariant local means on E_H^X (see [**Kaimanovich**]) such that $x \mapsto p_x(F)$ is measurable for any $F \in L^\infty(X,\mu)$.

Define $g_{f,x}$ and q_x by

$$g_{f,x}(\gamma_i H) = p_{\gamma_i^{-1}\cdot x}(f|_{[\gamma_i^{-1}\cdot x]_{E_H^X}})$$

and $q_x(f) = m(g_{f,x})$, where $\{\gamma_i\}$ is a representative of Γ/H for all $f \in \ell^\infty([x]_{E^X_\Gamma})$. It is straightforward to check that $g_{f,x} \in \ell^\infty(\Gamma/H)$, $q_x \in (\ell^\infty)^*[x]_{E^X_\Gamma}$ and $x \mapsto q_x(F)$ is measurable for any $F \in L^\infty(X,\mu)$. Check

$$\begin{aligned}
g_{f,\gamma\cdot x}(\gamma_i H) &= p_{(\gamma^{-1}\gamma_i)^{-1}\cdot x}(f|_{[(\gamma^{-1}\gamma_i)^{-1}\cdot x]_{E^X_H}}) \\
&= p_{h^{-1}\gamma_j^{-1}\cdot x}(f|_{[\gamma_j^{-1}\cdot x]_{E^X_H}}) \\
&= p_{\gamma_j^{-1}\cdot x}(f|_{[\gamma_j^{-1}\cdot x]_{E^X_H}}) \\
&= g_{f,x}(\gamma_j H) \\
&= g_{f,x}(\gamma^{-1}\gamma_i H) \\
&= \gamma \cdot g_{f,x}(\gamma_i H),
\end{aligned}$$

where $\gamma^{-1}\gamma_i = \gamma_j h$ for some $h \in H$ and

$$q_{\gamma\cdot x}(f) = m(g_{f,\gamma\cdot x}) = m(\gamma \cdot g_{f,x}) = m(g_{f,x}) = q_x(f).$$

So $q_x.$ is Γ-invariant. E^X_Γ is amenable. $\square$

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