Chapter 2. The Consistency and Effectiveness of Mandatory District Compactness Rules

2.1. Gerrymandering And District Appearance

> The regular distribution of power into distinct departments; the introduction of legislative balances and checks; the institution of courts composed of judges holding their offices during good behavior; the representation of the people in the legislature by deputies of their own election:...
>
> They are means, and powerful means, by which the excellences of republican government may be retained and its imperfections lessened or avoided.
>
> \- *Federalist 9*, Hamilton

The rules that we use to choose representatives lie at the heart of government. In the United States, some of the most controversial of these rules govern electoral districting. Some legal scholars have claimed that gerrymandering can be virtually eliminated by requiring that districts be geographically "compact." In recent cases, the Supreme court has evidently agreed.

Most proponents of compactness standards explicitly offer them as prophylactics against gerrymandering. Mathematical functions that describe the regularity of a district's geography or population distribution can be as simple as a measure of the length of a district's borders, or as complicated as a calculation of the population-weighted moment of inertia for the district. Many scholars have proposed ways to measure compactness — the literature contains more than thirty different measures — but few have systematically analyzed these measures. What, exactly, are different compactness criteria measuring? To what extent do different measures of compactness agree?

In this chapter, I use an axiomatic analysis to join formal measures of compactness with common intuitions about how gerrymandering is carried out. I find that, contrary to the claims of some previous researchers, it is impossible for a single index to capture all recognized forms of geographic manipulations. I then develop a methodology that uses small-case analysis to quantify the agreement among different measures of compactness and to quantify how strictly each measure limits geographic manipulation. Last, I reevaluate some of the previous empirical research on district compactness. I find that the compactness measures are often inconsistent when we use them to evaluate real, as well as hypothetical, districts.

2.2. Legal Controversy And Academic Debate

While the court has made its decisions about compactness (Chapter 1), the academic world is still debating the subject. Polsby and Popper, and other strong proponents of compactness, claim in the *Yale Law and Policy Review* that such a standard could virtually eliminate gerrymandering (Stern 1974; Wells 1982), or, at the least, "make the gerrymanderer's life a living hell." (Polsby and Popper 1991: 353)[29] At the same time,

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[29] In addition, Wells and Stern make claims which are equally, or nearly, as strong (Stern 1974; Wells 1982).

opponents of compactness measures claim that these standards are at best ineffective (Grofman 1985; Musgrove 1977), or at worst often contrary to substantive representative principles.[30] (Cain 1984; Lijphart 1989; Lowenstein and Steinberg 1985; Mayhew 1971)

Those who believe that compactness measures have some effect argue over which (if any) measurement is best. On one hand, Polsby and Popper, while also advocating a particular measure, claim that practically any of the proposed measures will do: "Compactness that constrains gerrymandering is compactness enough." (Polsby and Popper 1991: 340) On the other hand, Young argues that no compactness measure is acceptable — all are fatally inconsistent with each other: "This reliance on formulas has the semblance, but not the substance, of justice." (Young 1988: 113)

2.2.1. Previous Research on Compactness Standards

Unfortunately, current redistricting theory offers no resolution to the debate over compactness standards. While there is a significant literature of varying degrees of mathematical formality, on the theory of redistricting, the vast majority of that literature

[30] Another set of authors argue, more moderately, that particular compactness standards can be used to signal manipulation of district lines (Grofman 1985; Niemi et al. 1991) — ill-compactness is a warning signal that requires justification, or that compactness is a useful, neutral, and objective criterion for limiting gerrymanders. (Morill 1990), but "compactness alone does not make a redistricting plan good." (Niemi, et al. 1991: 1177)

ignores the spatial distribution of voters and institutional constraints on gerrymandering. Even the three papers that model the spatial distributions of voters, (Musgrove 1977), (Snyder 1989), and (Sherstyuk 1993), do not formally evaluate compactness measures within their models.[31]

Most comparisons of compactness measures have been informal: Frolov (1974) comments on a number of compactness measures used by geographers. Young (1988) shows how a number of compactness criteria can produce counterintuitive results.

Much research into compactness has focused on creating compactness standards, mostly *ad hoc*. Table 2-1 shows many of these standards. Empirical research in this area has been limited primarily to the measurement of particular districts or plans. No one, previously, has compared a large sample of standards over a wide range of district plans. Instead, the majority of studies selectively apply chosen measures of compactness to actual and proposed district plans (Hofeller and Grofman 1990; Niemi and Wilkerson 1990; Pildes and Niemi 1993; Reock 1961; Schwartzberg 1966). Two studies compare a

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[31] Sherstyuk (Sherstyuk 1993) shows that both population equality and substantive contiguity (i.e., excluding “telephone line” style contiguity) limit the opportunity for manipulation. She concludes in general that the addition of any substantive redistricting criteron tends to make gerrymandering more difficult but that the effects and neutrality of such criteria as compactness will depend on population distribution, and may conflict with redistricting goals.

variety of compactness measures using empirical data: Flaherty and Crumplin (Flaherty and Crumplin 1992) use several measures of compactness to measure proposed provincial districts in British Columbia; they recommend two particular area measurements, but do not explicitly compare the consistency of different measures. In one of the few studies to examine the consistency of compactness measures, Niemi et al. (1991) evaluate Congressional district plans for the states of California and Colorado, and state house plans from Indiana, Rhode Island, and New York. They calculate the correlation among selected measures on these plans and find varying levels of consistency among measures, concluding that measurements are most useful when several standards are used simultaneously to compare different plans for the same state.

	Length v. Width	*Earliest Use*
LW_1	W/L: where L is longest diameter and W is the maximum diameter perpendicular to L	(Harris 1964)
LW_2	W/L: from circumscribing rectangle with minimum perimeter	(Niemi, et al. 1991)
LW_3	1/(W/L): rectangle enclosing district and touching it on all four sides for which ratio of length to width is maximum	(Niemi, et al. 1991) modification of (Young 1988)
LW_4	W/L, where L is longest axis and W and L are that of a rectangle enclosing district and touching it on all four sides	(Niemi, et al. 1991)
LW_5	L–W where L and W are measured on north-south and east-west axes, respectively	(Eig and Seitzinger 1981)
LW_6	diameter of inscribed circle/diameter of circumscribed circle	(Frolov 1974)
LW_7	minimum shape diameter/maximum shape diameter	(Flaherty and Crumplin 1992)
	Measurements Based on Area	
A_1	The ratio of the district area to area of minimum circumscribing circle	(Frolov 1974)
A_2	The ratio of district area to the area of the minimum circumscribing hexagon[32]	(Geisler 1985), cited in (Niemi and Wilkerson 1990)
A_3	The ratio of district area to the area of the minimum convex shape that completely contains the district	(Niemi, et al. 1991)
A_4	The ratio of district area to area of the circle with diameter equal to the districts' longest axis	(Gibbs 1961)
A_5	The area of the inscribed circle/area of circumscribed circle	(Flaherty and Crumplin 1992)
A_6	The area of the inscribed circle/area of shape	(Ehrenburg 1892) cited in (Frolov 1974)
A_7	(area of intersection of the shape and circle of equal area)/(area of the union of the shape and the circle of equal area)	(Lee and Sallee 1970)
	Measurements Based on Perimeter/Area Ratios	
PA_1	The ratio of district area to the area of circle with same perimeter	(Cox 1927) cited in (Niemi, et al. 1991)
PA_2	$1\text{-}PA_1^{(1/2)}$	(Attneave and Arnoult 1936) cited in (Niemi, et al. 1991)
PA_3	The ratio of perimeter of the district to the perimeter of a circle with equal area	(Nagel 1835) cited in (Frolov 1974)
PA_4	The perimeter of a district as a percentage of the minimum perimeter enclosing that area (=100(PA_3))	(Pounds 1972)
PA_5	A/0.282P	(Flaherty and Crumplin 1992)

[32] When I analyze this measure, I assume that these hexagons must be regular.

PA_6	$A/(0.282P)^2$	(Flaherty and Crumplin 1992)
	Other Shape Measures	
OS_1	The moment of inertia — the variance of distances from all points in the district to the district's areal center of gravity, normalized. Where A is the area of the shape, r is the distance from the center and D is the set of points in the shape this is $\frac{A}{\sqrt{2\iint_D r^2 dD}}$.	(Boyce and Clark 1964)
OS_2	The average distance from the district's areal center to the point on district perimeter reached by a set of equally spaced lines	(Boyce and Clark 1964)
OS_3	(radius of circle having same area as shape)/(radius of circumscribing circle)	(Flaherty and Crumplin 1992)
OS_4	(N-R)/(N+R) where N,R is # of (non)reflexive interior angles (respectively)	(Taylor 1973)

Table 2-1. Shape based measures of compactness for districts.

Although we now have a large number of compactness standards to choose from, and we know how some districting plans measure up under a few of these, we still do not know what these measurements mean. What are compactness criteria measuring — do they measure gerrymandering? Are these measures consistent with each other — does it matter, really, which one we use? How effective are compactness standards at preventing gerrymandering, which measures should we use, and how should particular minimum compactness levels be set? Beyond preventing manipulation, are compactness standards neutral? What other effects could compactness standards have on politics?

In the next section, I address the first two questions. I use an an axiomatic analysis to test the consistency of existing compactness measures against our intuitions about how gerrymandering is performed in practice. I then develop a methodology to answer the third question by quantifying the strictness of each compactness standard.[33]

2.3. An axiomatic examination of compactness criteria

Recent surveys of compactness criteria list 36 different measurement formulas.[34] While there are a plethora of different measurements, and many assertions as to their effectiveness, only *ad hoc* criteria are used to distinguish between them.[35]

We can learn more about compactness measures by examining their formal properties. In this section I use an axiomatic approach to analyze the consistency of different compactness criteria. Then, in the sections that follow, I will extend this formal analysis with an exhaustive analysis of small districts.

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[33] This methodology identifies effective standards and leaves open the question of whether compactness standards are politically neutral. In Chapter 4, I show that they are not.

[34] See Niemi, et al., (1991) for the most comprehensive listing. (See also Flaherty and Crumplin 1992; Frolov 1974; Young 1988 for alternative treatments.)

[35] For an isolated exception see Blair & Bliss (1967), a largely overlooked, but more formal approach.

This section proceeds as follows. First, I will use a hypothetical example to introduce the issues surrounding geographical district manipulation. Second, I describe three commonly recognized techniques for manipulating district maps, and offer a set of principles that attempt to capture these different types of manipulation. Third, I show how we can use these axioms to eliminate a majority of the standards found in the literature as inconsistent or nonsensical.

2.3.1. A Hypothetical Example

As an introduction to redistricting with compactness standards, consider a hypothetical square state. This square state is inhabited by two parties with distinct policy preferences, the “Republicans” and the “Democrats.” Members of these factions live in each block.

The political structure of this hypothetical state is as simple as its population. The state is divided into four districts, each of which is composed of some number of indivisible blocks, and from each of which a member of the legislature is elected. When one party outnumbers another in a district, a candidate from that party is elected (Figure 2-1).[36]

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[36] Here I am assuming that ties are decided by a coin toss, everyone votes, and that everyone votes according to their party identification. While these assumptions simplifies reality, it is reassuring that one can predict nearly 90% of contemporary California elections by using only the partisan registration percentages in each district (Kousser

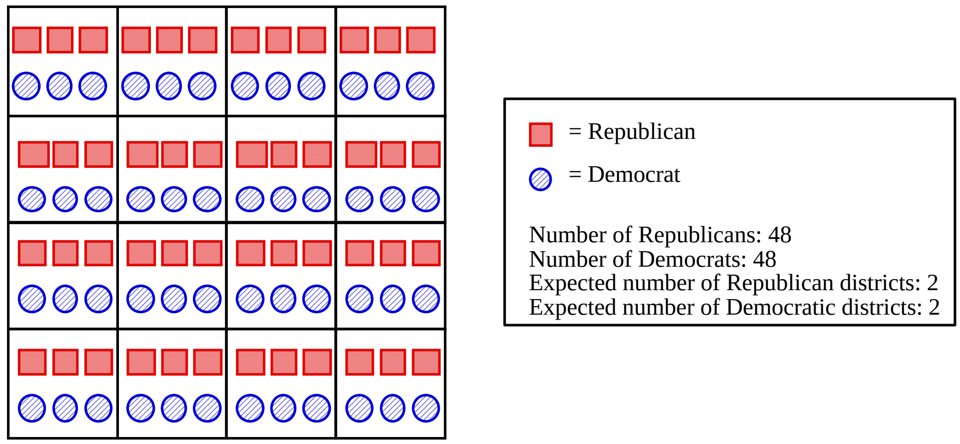

Figure 2-1. Hypothetical state with uniform population distribution.

Consider the situation above, where the population of each group is uniformly distributed across the state. In this most unlikely case, which is illustrated in Figure 2-1, redistricting rules do not matter. No matter which population blocks we use for each district, there will always be, on average, two members from each party in the legislature.

However, if the population distribution is not the same for every block, the situation may be much different: The particular districting plan that the legislature uses and the rules that govern the creation of districting plans in general may strongly influence the composition of the legislature (Figure 2-2).

Rules that constrain a legislature's actions do not necessarily constrain legislative outcomes. For example, if the legislature is required only to draw districts that are contiguous and equal in population, it will still be able to choose between ones that give

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1995).

an expected majority of seats to either party (Figure 2-2: A,B). Whether these districting rules limit outcomes depends upon how the voting population is geographically distributed.

What happens when compactness is added to the list of district requirements? In fact, the legislature's ability to affect elections depends greatly upon how we measure compactness. Suppose we use Theobald's measure (Section 2.3.2) and define compactness to be the maximum difference between an individual district area and the average area of all districts.[37] In this case, the legislature is not additionally constrained (Figure 2-2: A,B), because every plan that meets the equal population standard will also meet the compactness standard.[38]

But suppose, on the other hand, we use the state of Colorado's definition of compactness, and equate the compactness of a plan with the sum of all the perimeters of its districts. Low numbers are more compact. This compactness measure leads inevitably

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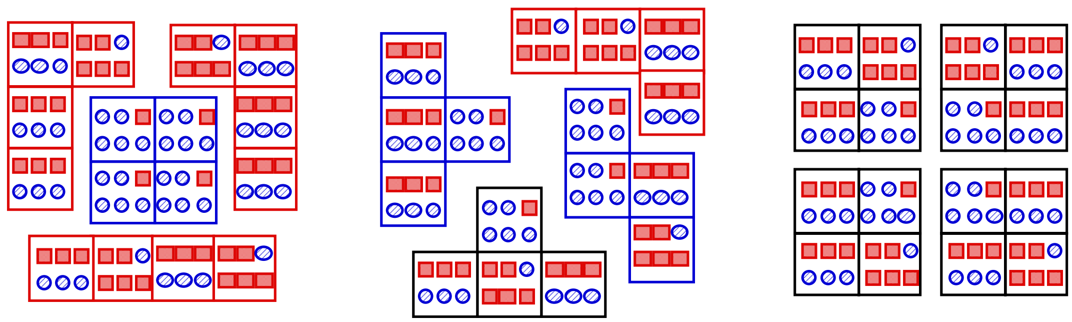
[37] Formally, if we take a set of districts, number them from 1..*N*, refer to their individual area's as $A_{i,}$, and to the mean of all district areas as $\bar{A}$, then the compactness score of a plan is $\max_i \left| A_i - \bar{A} \right|$. Under this measure a perfectly compact plan has a score of zero.

[38] This is a result of the uniform population density in this state — each population bloc contains both a uniform amount of population and has a uniform area.

to plan C in Figure 2. Plan C is more compact than any other possible plan, even if we discard requirements of contiguity and population equality.[39] In this particular case, a compactness standard gives each party an equal chance of controlling the legislature.



Plan A Plan B Plan C

Figure 2-2. Possible redistricting plans under different rules. All three plans meet contiguity, equal population standards, and the first compactness standard defined above. And plans use the same population map. Only Plan C meets the second compactness standard.

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[39] Many compactness measures find plan C to be uniquely and optimally compact, including comparison to ideal district shape (circle, square, hexagon), length to width ratio, population dispersion and perimeter/area ratios. The intuition behind this is that a square is the most "regular" shape that can be created using these population units, and that only one plan allows all districts to be squares. I have verified that this is, indeed, the optimal plan through an exhaustive analysis.

This example might seem to imply that compactness rules decrease a plan's partisan bias. Consider, however, the result of the same rules when they are applied to a different distribution of population. In Figure 2-3, our square state contains the same number of Republicans and Democrats as in Figure 2-2, but their locations have changed. In this case, our previously "fair" compactness rule ensures that the Republicans will control the legislature. The compactness rule still limits gerrymandering, in the sense that it makes it impossible to manipulate district lines; however, the rule has a clearly disproportionate effect on different parties. In fact, if someone who knew the population distribution had suggested such a compactness rule, we would have a strong reason to suspect them of partisanship.

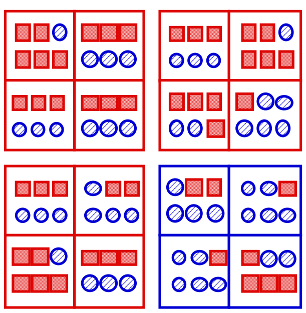

Figure 2-3. Compact plan for another population distribution.

This example illustrates four claims that I will pursue in the rest of this section: First, compactness and other rules governing the shapes of districts may have a powerful effect on the composition of a legislature. Second, rules *can* limit the possibility of manipulation, but some may be stronger than others. Third, the effects that rules have interact with the way in which populations are distributed. Fourth, the effects of a compactness standard may depend very much on how we measure compactness.

It is important to realize that these criteria neither measure nor constrain electoral manipulation directly — they say nothing about the electoral results that can be expected from a particular set of districts. Instead they are proxies that attempt to reflect the ways that gerrymanderers distort district shape to manipulate elections.

2.3.2. Manipulating The Shape Of Individual Districts — Six Axioms

Since most measures of compactness have concentrated on the geographical manipulation individual districts, I will address these measures first. Later, in Section 4.5, I will discuss measures that are based upon the compactness of an entire districting plan and measures that are based upon population dispersion instead of geography.

Most compactness measures claim to describe the *shape* of a district. Therefore we should require that any index of compactness give the same score to two districts that have the same shape.[40] Blair and Bliss (1967) suggest that two objects should be said to have the same shape if we can make them identical through translation, rotation and uniform scaling.[41] I adopt this characterization (Figure 2-4).

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[40] For district measures that capture population distribution, we would require the measurement to produce identical responses for identical shapes and distributions of population, rather than identical geography (Section 2.3.3).

[41] The definitions of shape that are used by Blair and Bliss (1967) differ slightly, but the axioms are similar. In addition to the three general properties, they claim that a circle

In general, three types of shape distortion and manipulation have been recognized: dispersion, dissection, and indentation (Blair and Bliss 1967; Flaherty and Crumplin 1992; Frolov 1974).[42] While there is no consensus on how these concepts should be precisely measured, it is easy to describe each intuitively. *Dispersion* reflects the symmetry of a shape around its center — a circle is evenly dispersed, whereas a ellipse is less evenly dispersed. *Dissection* reflects discontinuity in the distribution of points across the convex hull of a shape — shapes with holes cut out of them are highly dissected. *Indentation* reflects the smoothness of the perimeter of a shape — most coastlines are examples of indented shapes (Figure 2-4).

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should be judged to be maximally compact under any reasonable index. I leave this out, as it is an implication of the axiom's I suggest later.

[42] While these authors refer to a shared set of concepts, their terminology sometimes varies. For example, where Flaherty & Crumplin (1992) refer to *compactness,* I refer to *dispersion* in order to distinguish this concept from the more inclusive meaning of the term in general use. Niemi et al. (1991) points out the importance of population.

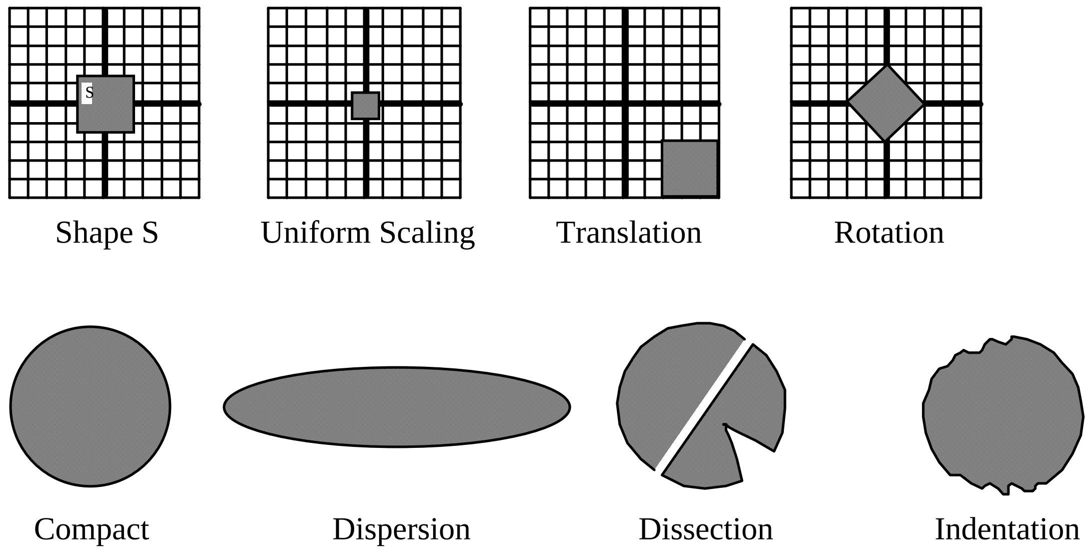

Figure 2-4. Transformation of shapes.

Suppose that we were to place legal limits on the amount of indentation, dispersion and dissection allowed in each district. How would this affect gerrymandering? As we made these limits more stringent, the set of plans from which a gerrymanderer could choose would shrink. In a very simple world, indentation, dispersion, and dissection might closely reflect the ability of a gerrymanderer to pick and choose district plans to her liking.

For example, imagine that the voting patterns within each district are completely predictable, that all voting data is known with certainty, and that people are evenly distributed over each square mile of our hypothetical state. Also imagine that there are only two parties, that they have an equal number of loyal voters, and that these voters are uniformly randomly distributed across space. In this case, the amount of indentation (or dispersion or dissection) that district planners are allowed to use when they create a districting plan directly limits the set of districts from which planners can choose, and will increase the *ex-ante* probability of their being able to choose a winning plan for their party.

If, as the literature indicates, these three types of shape distortion are good proxies for geographical manipulation, then acceptable measures of geographic compactness should capture at least one, if not more, of these principles. In the remainder of this section, I formalize these principles.

First, we will need some definitions:

Let a shape $S = \{s_1, \ldots, s_i\}$ be a finite, nonempty set of simple, continuous, closed, nonoverlapping subsets of the plane where $Area(s_i \cap s_j) = Perimeter(s_i \cap s_j) = 0, \forall i \neq j$.

Let $P: S \rightarrow \Re_+$ be the length of the perimeter of the shape, and let $A: S \rightarrow \Re_+$ be the area of the shape.

Let a compactness measure *C*, be a function $C: S \rightarrow \Re$.

Using these definitions, we can now formally define what it means for a compact measure to capture shape:

1. *Scale independence*: if two shapes differ only in scale, then they should be equally compact.

Formally, $\forall \alpha \in (0,1] \Rightarrow C(\alpha S) = C(S)$.

2. *Rotation independence*: if S_1, S_2 are two shapes which differ only in rotation around the origin, they should be equally compact. Formally, if θ is an angle, and then

$$S' = \{p' | p \in S, p'_x = p_x \cos\theta - p_y \sin\theta, p'_y = p_x \sin\theta - p_y \cos\theta\} \Rightarrow C(S) = C(S')$$

.

3. *Translation independence*: if S_1, S_2 are two shapes which differ only in position, they should be equally compact. Formally, $\theta \in \Re^2 \Rightarrow C(S+\theta) = C(S)$.

A compactness measure must not violate any of these three principles. It would be strange indeed if we could change a district's shape simply by uniformly scaling, rotating, or moving the map upon which it is drawn. If a compactness measure does not meet these basic standards,[43] political actors would be able to manipulate the compactness of their districts simply by manipulating the maps upon which they are drawn.

In the next three principles, I capture the concepts of dispersion, dissection, and indentation. First, let us take dispersion. Compactness measures that claim to capture dispersion are usually based on the ratio of a shape's perimeter to its area. These

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[43] Remember that these measures are based upon geography alone. Violations of principles 1-3, as stated here, might be quite reasonable if we were measuring population: For example, moving a square district to a different part of the map could completely change the population distribution within that district. Fortunately, we can both preserve these principles and reflect population distributions — see Section 2.3.3.

measures work well for convex shapes, but can confuse indentation and dispersion for nonconvex shapes (Figure 2-5).

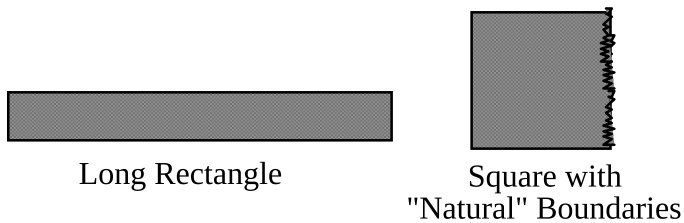

Figure 2-5. The Perimeter/Area ratio fails to capture dispersion. The P/A of the figure on the left is less than that of the one on the right.

For example, the shapes in Figure 2-5 have equal area but the perimeter of the "long rectangle" below is much less than the "coastal" square to its right. Measures based on the perimeter/area ratio judge the square to be more dispersed, whereas, intuitively we can see that it is really less dispersed, but more indented.

By measuring the perimeter of the convex hull of the shape, we can avoid this confusion. Intuitively, the convex hull, which we will refer to as "CO," smoothes out the bumps in the shape and allows us to look at its broader outline.

4. *Minimal dispersion*: A compactness measure reflects the principle of dispersion if, for all shapes S_1, S_2, if S_1 and S_2 are of equal area, and the perimeter of the convex hull of S_1 is larger, S_1 is less compact:

Formally, $A(S_1) = A(S_2) \& P(CO(S_1)) > P(CO(S_2)) \Rightarrow C(S_2) > C(S_1)$.

We can also use the convex hull to compare two shapes that have the same general outlines, so as to see which is relatively more dissected or indented:

5. *Minimal dissection*: Let *CO(S)* be the convex hull of shape *S*. If S_1 and S_2 are any two shapes with identical convex hulls, and S_1 has a strictly smaller area, then S_1 should be judged less compact:

Formally, if $CO(S_1) = CO(S_2)$, and $A(S_1) < A(S_2) \Rightarrow C(S_2) > C(S_1)$.

6. *Minimal indentation*: If S_1 and S_2 have identical convex hulls and S_1 has a strictly larger perimeter/area ratio, S_1 should be judged less compact.

Formally, if $CO(S_1) = CO(S_2)$, then $\frac{P(S_1)}{A(S_1)} > \frac{P(S_2)}{A(S_2)} \Rightarrow C(S_1) < C(S_2)$.[44]

In addition to capturing recognized methods of manipulation, any compactness measure that satisfies any of these axioms will have two other nice properties. Contiguity is usually assumed, *ad hoc*, to limit manipulation;[45] similarly, a circle is often assumed to

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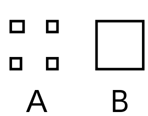
[44] One alternative to 6 that might be offered is 6': $CO(S_1) = CO(S_2)$, then $P(S_1) > P(S_2) \Rightarrow C(S_1) < C(S_2)$.

Since 6' is analogous to 5, it seems natural, at first glance, but leads to surprising conclusions involving noncontiguous districts.

For example, under 6', shape A is more compact than shape B:



[45] Though see Sherstyuk 1993 for a formal approach to contiguity.

be maximally compact. Under the six axioms above, we need not assume these properties, but can derive them (See the Appendix for proofs):

Result 1. If a compactness measure satisfies axioms 1–4 and either axiom 5 or axiom 6, then it has the following properties:

Contiguity: For any given perimeter or area, the maximally compact shape is contiguous. If C satisfies axiom 5, this is true for any given convex hull as well.

Circle Compactness: A circle is the most compact shape.

Most research on compactness assumes that it can be measured on a single unidimensional scale. Like Flaherty and Crumplin (1992) and Niemi, et al. (1991), I find this assumption to be incorrect. Many measures seem simply to be measuring a different aspect of compactness — there is more than one way to manipulate a shape.

Result 2. It is impossible for a single index of compactness to meet axioms 5 and 6 simultaneously (Figure 2-6). Axiom's five and six can contradict: Under Axiom 5, C(B)>C(A), but under Axiom 6 C(A)>C(B).[46] (See Figure 2-6.)

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[46] I could reformulate axiom 6 avoid this conflict. For example, let us define axiom 6' as follows:

$A(S_1) = A(S_2)$, then $\frac{P(S_1)}{A(S_1)} > \frac{P(S_2)}{A(S_2)} \Rightarrow C(S_1) < C(S_2)$. But this seems to blur the distinction between indentation and dispersion.

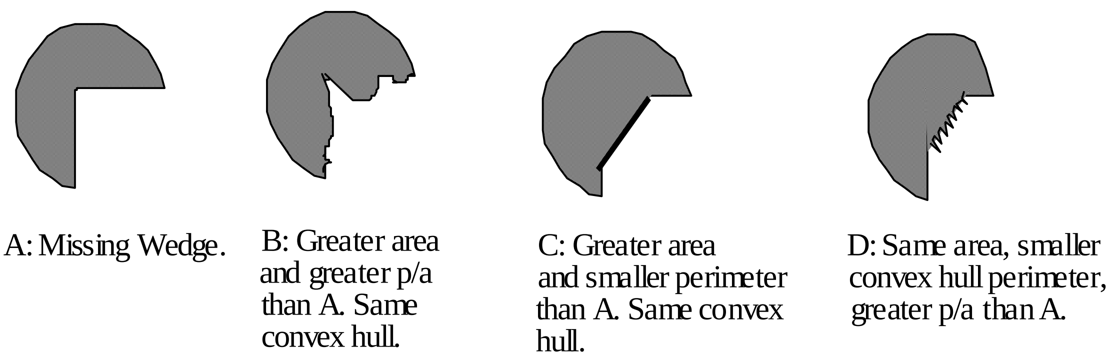

Figure 2-6. Shapes *A* and *B* have identical convex hulls, P(A) > P(B), and A(A) > A(B). Under axiom 5, *A* should be more compact, while *B* should be more compact under axiom 6.

What does Result 2 mean? There is significant debate over whether compactness standards are measuring the same things. This result shows that differences among compactness measures exist and are, to an extent, unavoidable.[47]

[47] As I discussed at the beginning of this section, geographical methods of manipulating districts remain proxies for the end goal of gerrymandering — influencing the results of an election. While in some ways more direct, this characterization, less general because it requires assumptions about underlying population distributions, optimization methods for creating compact plans, and electoral goals. First, gerrymanderer's may have different competing goals for electoral results — incumbent protection and partisanship conflict (Owen and Grofman 1988). Second, optimal compact gerrymanders are very difficult to create, and in practice the effect of a compactness

These principles set bounds on a reasonable compactness standard — if a compactness measure contradicts all three of principles of compactness (or any of the shape principles), we should suspect it of measuring something other than geographic compactness. Table 2-2 summarizes the results of applying these axioms. (See this chapter's Appendix for proofs of these results.) In it I list each measure and the axioms that it violates.

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standard will vary with the particular methods used for creating districts. (See Chapter five.) Third, the extent to which a particular compactness standard constrains a (partisan) gerrymanderer's ability to gain seats depends not only on the compactness measure, but on the spatial distribution of (partisan) voters. (See Chapter 4.)

Measure	*Axiom 1*	*Axiom 2*	*Axiom 3*	*Axiom 4*	*Axiom 5*	*Axiom 6*
LW_1				**V**	**V**	**V**
LW_2				**V**	**V**	**V**
LW_3				**V**	**V**	**V**
LW_4				**V**	**V**	**V**
LW_5		**V**		**V**	**V**	**V**
LW_6				**V**	**V**	**V**
LW_7				**V**	**V**	**V**
A_1				**V**		**V**
A_2				**V**		**V**
A_3				**V**		**V**
A_4				**V**	**V**	**V**
A_5				**V**	**V**	**V**
A_6				**V**	**V**	**V**
A_7				**V**	**V**	**V**
PA_1	**V***			**V**	**V**	
PA_2	**V***			**V**	**V**	
PA_3	**V***			**V**	**V**	
PA_4	**V***			**V**	**V**	
PA_5	**V**			**V**	**V**	
PA_6	**V***			**V**	**V**	
OS_1				**V**	**V**	**V**
OS_2		**V**		**V**	**V**	**V**
OS_3				**V**		**V**
OS_4				**V**	**V**	**V**

Table 2-2. Violations of the measurement axioms by compactness measure are marked by a 'V' in the cell.

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* In these cases the measure is sensitive to the scale of the measuring unit used to measure the district boundaries, not to the scale of the map upon which the district boundaries are represented.

The results in this table enable us, in two ways, to trim[48] the set of compactness standards that courts and political scientists should consider adopting: First, although most of the compactness measures meet our first three axioms, eight measures violate, in their standard form, these basic axioms for measuring shape. Three measures, LW_5, OS_2 and PA_5, unequivocally violate these axioms, and so should be rejected. Five others, the remaining PA measures, in their current form, violate axiom 1, but they can be saved we are careful to measure the boundaries of all districts with the same precision.

Second, 13 compactness measures violate all 3 axioms of compactness. In other words, they do not comport with the commonly held intuitions about how gerrymandering is accomplished geographically. They should be rejected in the absence of compelling theoretical or empirical evidence that these measures are, in fact, measuring other aspects of gerrymandering.

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[48] In addition, the ubiquitous violation of Axiom 4 points to a way in which our measurements of compactness can become more complete: by using at least one measurement that captures indentation. In Appendix 1 I create, for example, one such a measure.

2.3.3. Extension 1: Capturing Population Distribution.

District lines affect elections only when these lines affect how we assign voters to districts: A meandering district line in a dense urban area may indicate political manipulation, but the same line, when found following a river in a sparsely populated rural area, may be devoid of political content. Measures of population compactness attempt to capture this distinction.

If the population density in a state is uniform, the first three of these population measures are equivalent to the geographical measures A3, A1, and OS1, respectively.[49] In general, the difference between these measures and their geographically-based counterparts will depend on how people are distributed in a state (Table 2-3).

---

[49] Population measure four is designed as a computationally simpler approximation of POP_3 (Papayanopoulos 1973). Given that computers are now sufficiently powerful to calculate good approximations of POP_3 directly, this may no longer be necessary.

Also note that POP_1, like its shape counterpart A_3, violates the modified axiom 4 even where population density is uniform.

	Population measures	
POP_1	ratio of district population to the population of the minimum convex shape that completely contains the district	(Niemi, et al. 1991)
POP_2	ratio of the district population to the population in the minimum circumscribing circle	(Niemi, et al. 1991)
POP_3	population moment of inertia, normalized	(Weaver and Hess 1963)
POP_4	sum of all pair-wise distances between centers of subunits of legislative population, weighted by subunit population	(Papayanopoulos 1973)
	Plan Compactness Measures	
PL_1	The sum of the district perimeters	(Adams 1977)
PL_2	The maximum absolute deviation from the average district area	(Theobald 1970) cited in (Niemi, et al. 1991)

Table 2-3. Measures of compactness that evaluate properties other than district shape.

Niemi, et al., (1991) argue reasonably that the compactness of a district should not change when we include unpopulated parcels of land. We can go farther, however, in making measures sensitive to population. It is reasonable to expect that the more people we are allowed to shift from district to district, the larger the potential for political manipulation; therefore, compactness should reflect not only the presence of people, but their numbers.

Fortunately, there is an easy way to transform a compactness measure that evaluates only geography into a compactness measure that it is sensitive to population, as well. We can make this conversion not by changing the measure itself, but by changing the map to which we apply the compactness criteria. By using a map where population and area are made equivalent, we can measure population compactness with any of our geographical compactness measures.

Tobler (1973) shows how to generate this type of map.[50] Compactness measurements made on these maps automatically reflect manipulation of population. For example, if we examine a boundary line that follows an unpopulated river bed, it will seem highly indented on a conventional map, while it will show no indentation at all on the transformed map. On the other hand, if our boundary line is in the middle of a densely populated urban area, any irregularities will become magnified on the population map. If we measure compactness using these maps, we truly look at people, not acres.

2.3.4. Extension 2: Measuring The Compactness Of Plans.

Since districts share borders, the shapes of districts in a plan are interdependent. Most compactness measures, however, examine districts in isolation from the plan in which they are embedded. Even when researchers are forced to define some measure of plan compactness, instead of evaluating the plan as a whole, they often simply measure the compactness of each district and then use the mean or minimum of these district-based scores.

Measures of plan compactness should be sensitive to improvements in districts. At the least, a plan measure should reward making one district more compact, if other districts are not made any worse. Otherwise, the compactness of an entire plan may be

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[50] Tobler created this transformation as a method for drawing equal population districts, not for creating compact districts per se. He does suggest however, that a hexagon could be used as an appropriate “compact” shape.

determined by a single district, which is unlikely to reflect adequately the degree of electoral manipulation in the whole plan. Researchers have created only two measures that take the entire plan as the basic unit for which compactness is measured (Table 2-3).

Axiom 7 states this criterion more formally. First we need a few definitions. Let plans P_1,P_2 be sets of shapes. We are given a measure C of shape compactness, satisfying axioms 1–3, and 4, 5 or 6. Define $CP: P \rightarrow \Re$ to be a measure of plan compactness.

Axiom 7 (weak Pareto comparison): If every district in plan 1 is at least as good as every district in plan 2 and one district is better, then plan 1 is more compact. Formally, let f be a bijection mapping each element of P_1 to a single element of P_2:

$$\exists f, s.t.\ C(S_i) \geq C(f(S_i)) \forall S_i \in P_1, \text{and } \exists S_j \in P_i\ \ s.t.\ C(S_j) \geq C(f(S_j)) \Rightarrow CP(P_1) > CP(P_2)$$

If a district-compactness measure, C, satisfies axioms 1–3 and 4, 5, or 6 then the mean district compactness, $PC_{mean}(P) = \frac{\sum_{S_i \in P} C(S_i)}{\# P}$, satisfies axiom 7. Measuring the compactness of a plan based on the minimum district compactness, $PC_{\min}(S) = \min_{\forall S_i \in P}(C(S_i))$, violates axiom 7 (Figure 2-7). In addition, both PL_1 and PL_2 violate scale invariance, although if we wish only to compare two plans that are drawn upon the same map, this is not a serious defect. Even so, PL_2 is unsatisfactory because it violates axioms 4, 5, and 6[51] (Figure 2-7).

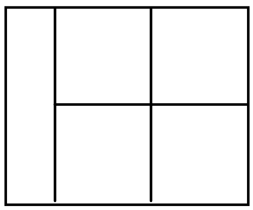
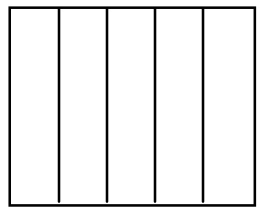
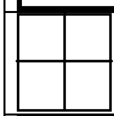
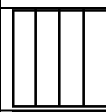
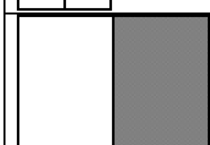
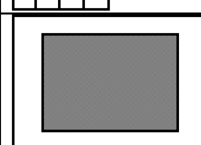
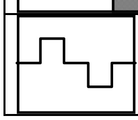
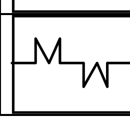
More Compact	*Less Compact*	*Measures Not Classifying These Plans Correctly (Axiom Violated)*
		Any plan measure based on the score of the minimally compact district. (Axiom 7)
		PL_1,PL_2 (Axiom 1)
		PL_2 (Axiom 4)
		PL_2 (Axiom 5)
		PL_2 (Axiom 6)

Figure 2-7. Violations of Axioms 1, 4, 5, 6, and 7 by measures of plan compactness.

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[51] Note that on PL_2, when applied to a uniform population map such as we described in the previous section, is equivalent to the equal-population standard.

2.4. Evaluating The Consistency Of Compactness Measures With Small Cases

The analysis in the previous section showed that the worst of the individual compactness measures fail even to measure shape adequately, and that the best of them capture only limited aspects of geographical manipulation. In this section I use an exhaustive analysis of small cases to quantify the amount of agreement among various compactness measures and to quantify how sensitive these measures are to manipulation of district shape.

2.4.1. Generating Districts And Plans

If two compactness measures produce the same rankings over all sets of districts, they are identical for all practical purposes. While few compactness measures are completely identical in this way, we can use several straightforward statistics to analyze the agreement between rankings over a given set of districts.

How do I choose an unbiased set of "test" districts for our comparisons? I use the exhaustive set, the set of all districts that can be created on a given map with population blocks of unit size. The choice to use an exhaustive set allows me to avoid bias in the selection of particular districts; but because this set grows very quickly as the number of census blocks in a district increases, it limits the size of the districts that can be examined.

I start by creating a small artificial district map that consists of a rectangle of population blocks (similar to the examples in Section 2.3.1). I use combinations of these blocks to form individual districts. I then create an exhaustive compactness *ranking* by using one compactness criterion to rank all the possible districts that can be created on that map. Finally, I compare the rankings produced by different compactness rankings to determine the similarity among measures.

Generating district plans is a bit more complicated than generating individual districts. We can characterize redistricting mathematically as a partitioning problem.[52] Imagine that each state in the U.S. is composed of indivisible population units,[53] in this case creating a plan is equivalent to partitioning these units. If we care about what a plan looks like, then we can add a value function to our partitioning that incorporates such criteria as contiguity, compactness, and population equality. I create two sets of plans for

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[52]A partition divides a set into component groups which are exhaustive and exclusive. More formally:

For any set $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$, a *partition* is defined as
a set of sets $\mathbf{Y} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ $s.t.$
(1) $\forall x_i \in \mathbf{x},\ \exists \mathbf{y}_j \in \mathbf{Y}, s.t.\ x_i \in \mathbf{y}_j$
(2) $\forall i, \forall j \neq i, \mathbf{y}_j \cap y_i = \varnothing$

See Stanton (1986) for an overview of algorithms to create exhaustive lists of partitions.

[53] Census blocs for redistricting purposes may often be considered to be practically indivisible.

each map: The first set is exhaustive; it includes all possible redistricting plans. The second set is more selective; it contains all the plans that meet the constraint of population equality. After creating these plans, I then use them to create exhaustive rankings for the plan compactness measures, just as I described using exhaustive sets of districts.

Even when we use maps that contain a small number of population units, we can create a surprisingly large number of distinct district plans. For example, if we want to produce all possible districts from an *n* by *m* rectangle of population blocks, the number of districts, *d*, that we can create is represented by the function $d = 2^{(n)(m)}$. Not only is this large, but it grows exponentially as we increase the number of population blocks in our map. The number of plans in an exhaustive set can grow even faster than the number of districts. If we have *n* by *m* population blocks and want to create *r* districts, we can create $S((nm),r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i\left(\frac{r!}{(r-i)!\,i!}\right)(r-i)^{nm}$ plans.[54] If, however, each district in a plan has exactly the same number of blocks, *k*, then the number of plans we need to create is a bit smaller: $\frac{(nm)!}{r!(k!)^r}$.

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[54] *S* is known as a "Stirling Number of the Second Kind." See Even (1973) for an introduction.

2.4.2. Results From The Small Case Analysis

Since the length of our list of exhaustive rankings tends to grow exponentially as we add population units, we can only use this technique on relatively small maps. I examine all rectangular maps that measure 4 by 4 or smaller (2x2,2x3,3x3,2x4,3x4,4x4). Even though these sizes are small, the number of plans we can generate from them is large — up to 90,000 different plans can be generated from the 4 by 4 grid.

The results from this exhaustive analysis reinforce our previous theoretical analysis: many district and plan compactness measures judge districts quite differently. Furthermore, some measures are much more sensitive to the manipulation of district lines.

<u>Measures Of District Compactness Are Inconsistent.</u>

We first turn to measures of district compactness. For this part of the analysis, I selected seven district measures that either satisfied a large number of the six axiomatic criteria or have received particular attention in the literature: measures A_1, OS_1, PA_6, LW_5, PA_3, PA_5, and A_7, as defined previously. I used these seven different compactness measures to rank all the districts that could be created for each map.

Box-plots[55] allow us to compare the distribution of compactness scores when each of these compactness measures is applied to an identical set of districts. These distributions

[55] Box plots are commonly used to compare distributions. In these plots, the top and

of compactness scores have two striking features: First, district scores are concentrated in a narrow range, and, second, there are few extremely compact districts. (Figure 2-8).

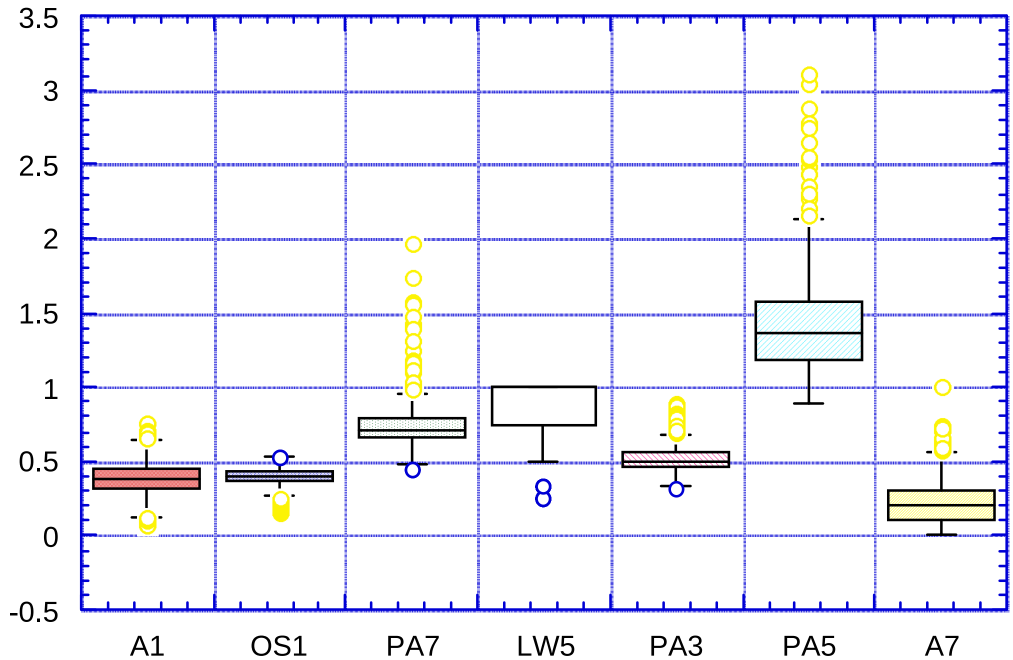

Figure 2-8. Box -plots of district compactness scores for all districts (the exhaustive set) on a 3x4 map.

How can we use these observations to design better redistricting regulations? Some researchers have proposed that we require all districts to meet a specified minimum level of compactness, while others would use compactness scores only as a relative measure to

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bottom of the box correspond to the 25th and 75th percentiles of the variable, while the whisker lines extend beyond the box by one and one-half times the interquartile range (so that approximately 99 percent of normally distributed data will lie within them.) The median is identified by a horizontal line, and outliers are identified by the small circles.

make comparisons between districts. My results suggest that compactness scores are more useful as relative measures than as absolute measures. Since the distribution of compactness scores is so narrow, it will be very difficult to set a compactness limit that is restrictive without being draconian.

Although the shapes of the distributions of compactness rankings were similar across compactness measures, there were two striking differences among the rankings themselves. First, all compactness measures were not equally strict when judging differences between districts, in fact, some measures ignored all but the grossest differences. One way we can measure this sensitivity is by examining the number of different classes of equivalent scores assigned within each ranking, or *equivalence classes*. The greater the number of equivalence classes, the more sensitive is the measure to district shape.[56]

Table 2-4 shows the number of equivalence classes that each compactness measure produced when it ranked all the districts for a map of given size. In the final column,

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[56] An alternative way of characterizing the strictness of plans is to compare how difficult it is to generate compact plans under each standard, and to what extent this constrains gerrymandering for given electoral goals. (See Note 119, below.) In Chapter 6, I analyze three compactness standards in this way. As expected, the compactness measure which generated few equivalence classes in this study was also difficult under this alternative characterization of strictness.

notice the extreme contrast between LW_5 and OS_1. LW_5, which is one of the few compactness measures that has been put into law, is extremely insensitive — all 65,000 are assigned one of only 6 distinct scores. Other things equal, we should prefer measures that are more sensitive to district shape to those that are less sensitive, because compactness criteria that capture only the most dramatic differences in district shape are unlikely to strongly restrict gerrymandering.

	2x2 (11)	*3x2* (57)	*2x4* (247)	*3x3* (502)	*3x4* (4083)	*4x4* (65519)
A1	4	11	19	19	39	55
OS1	4	17	45	57	299	953
PA6	2	8	16	17	41	56
LW5	2	4	5	4	6	6
PA3	4	11	20	24	48	88
PA5	4	8	14	17	36	63
A7	3	2	11	15	25	47

Table 2-4. Equivalence classes produced by different measures of compactness. Grid size (number of shapes) is shown in columns.

A second striking difference among compactness measures is the order in which they rank particular districts — many measures do not seem to be measuring the same thing. Table 2-5 shows the concordance[57] among cardinal compactness scores of all districts on

[57] Compactness is for most purposes a relative measure, not an absolute measure. When the courts compare two districts, they will not ask “How do these districts score?” but “Which district is more compact?” To evaluate the similarity of relative compactness

the 3 by 4 map. If these different measures ranked districts in the same way, then all of these concordances should be close to one; many measures seem to have little relationship to each other.

	A1	*OS1*	*PA6*	*LW5*	*PA3*	*PA5*
OS1	0.67 (0.83)					
PA6	-0.25 (-0.39)	-0.19 (-0.39)				
LW5	0.14 (0.24)	0.09 (0.11)	-0.20 (-0.20)			
PA3	0.29 (0.48)	0.25 (0.36)	0.51 (0.60)	-0.38 (0)		
PA5	0.58 (0.73)	0.74 (0.86)	0.07 (0)	0.34 (0)	0.47 (0.71)	
A7	0.43 (0.56)	0.58 (0.74)	0.02 (0)	0.33 (0)	0.34 (0.48)	0.56 (0.72)

Table 2-5. Degree of agreement (Kendall's τ) in pair-wise comparisons for the 3x4 case. (Correlations are reported in parentheses.)

Although compactness measures disagree over how districts should be ranked, if compactness measured agreed about which districts were "best" — at the top of the rankings — other disagreements might be less important. I investigated the possibility of this sort of agreement in two ways: First I recomputed Table 2-5 using only the top 10

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judgments between each pair of measures, I use Kendall's τ_β. For each pair of measures, I count the number of times where both measures agree that one district is more compact than the other, C ("concurrences"), the number of strict disagreements, *D*, and the number of unilateral ties T_X & T_Y. I then compute Kendall's τ_β using the following formula[57]:

$$\tau_\beta = \frac{C-D}{\sqrt{(C+D+T_x)(C+D+T_y)}}.$$

percent of district rankings, but compactness measures continued to disagree.[58] Second, I calculated the similarity between the ten districts chosen as *most* compact by each measure (Table 2-6).[59] Still, the differences between measures remain: compactness measures disagree over good districts as much as they disagree over bad districts.

	A1	*OS1*	*PA6*	*LW5*	*PA3*	*PA5*
OS1	0.59					
PA6	0.10	0.15				
LW5	0.20	0.23	0.14			
PA3	0.45	0.50	0.10	0.18		
PA5	0.59	0.82	0.15	0.23	0.81	
A7	0.44	0.45	0.10	0.16	0.46	0.40

Table 2-6. Similarity between top ranked shapes.

Population Measures And Consistency.

Each of the seven measures that I tested in the previous sections looks only at geography. Would compactness measures that are based on population be more consistent? Only two population-based measures meet our previously discussed

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[58] To limit comparisons in this way, one must choose a particular measure to select the top 10 percent of districts, or repeat the process for each measure. I chose the latter approach as more thorough, but omit the seven resulting tables to save space.

[59] Similarities between districts are computed with the following formula $\frac{\#blocks(S_1 \cap S_2)}{\#blocks(S_1 \cup S_2)}$. This is based on Lee and Sallee (1970) although I have adapted it to the discrete case.

requirements, and I examined both of these. So as to see how these measures would be affected by differences in population distribution, I assigned a random population weight (a discrete uniform distribution) to each population block in the model. I then examined these measures using the same techniques that I used for the previous analyses.

Range of Population Distribution	τ*(AC,PAC)*	τ*(MI,PMI)*	τ*(PMI,PAC)*	*Equivalence Classes (PAC)*	*Equivalence Classes (PMI)*
(0,1)	0.59	0.50	0.74	19	121
(0,2)	0.56	0.58	0.77	23	192
(0,3)	0.56	0.42	0.73	69	290
(0,4)	0.59	0.58	0.77	113	340
(0,5)	0.81	0.81	0.54	123	340

Table 2-7. Degree of agreement between rankings, and equivalence classes, for population based measures of district compactness. Each row records results using different parameters for the random distribution of population. The first three columns show the index of concordance between pairs of measures (3 by 4 map).

In Table 2-7, I compare rankings between the population-based measures and between these measures and their geographical counterparts. Because the model assigned random weights to population blocks in each run, the results varied somewhat from run to run, but the patterns in the data remained consistent: Population based measures are no more consistent with each other than are their geographical counterparts.[60]

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[60] Population measures are, however, unsurprisingly, more closely related to other population measures than to other geographical measures.

How Effective Is Mandatory Plan Compactness?

When we use compactness measures to rank entire district plans rather than single districts, we continue to see inconsistencies among different measures. I used eight different measures of plan compactness: the two measures specifically designed for plans (PL1,PL2), and the six measures based upon the average and the minimum of individual district scores (A1,OS1, and PA6). I used these measures to evaluate exhaustive sets of both *balanced* and *unbalanced* plans: Plans are *balanced* when each district has an equal area, and they are *unbalanced* when they may contain districts with unequal (but nonzero) area.[61] The distributions of compactness scores for balanced and unbalanced plans in a typical case are shown below: (Figure 2-9)

[61] Note that PL2 is zero for all balanced plans.

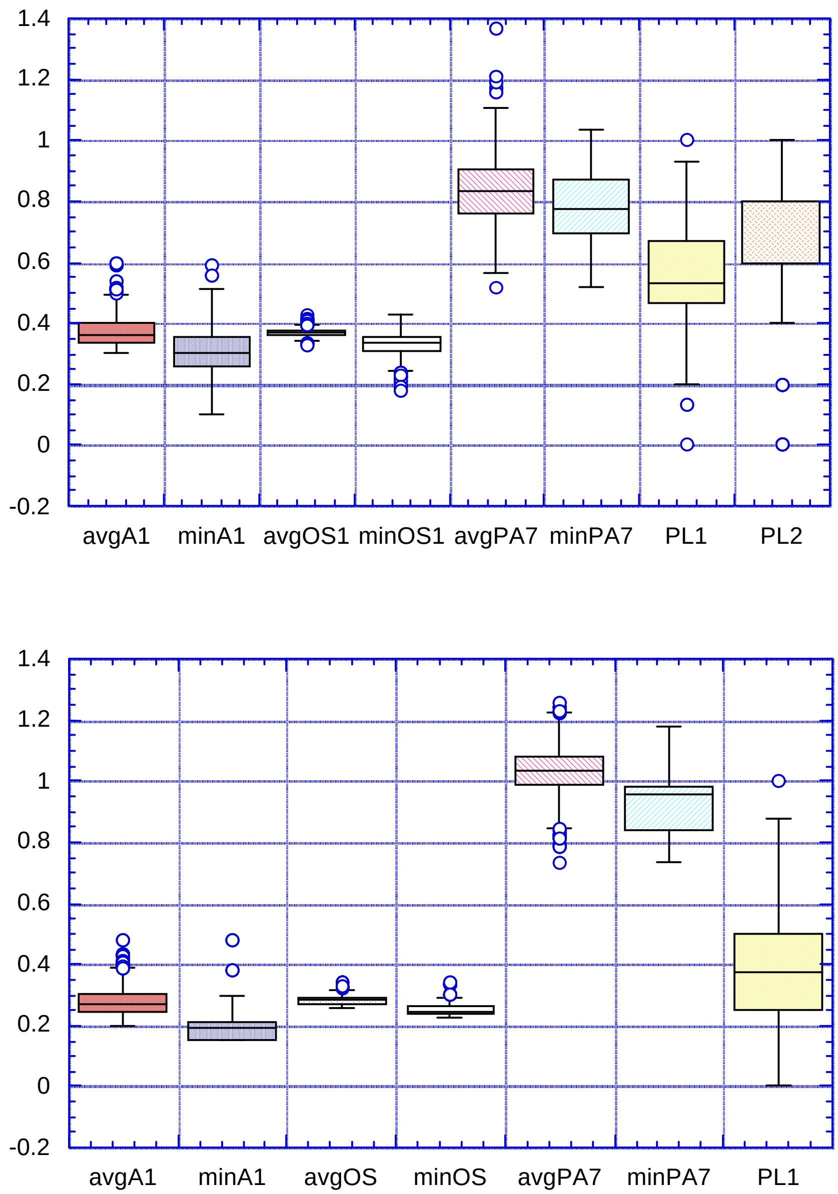

Figure 2-9. Box plots of plan compactness for a 3x4 grid, partitioned into two districts. "Avg" indicates the average district score using a given measure, while "min" indicates the score of the minimally compact district under that measure.

Measures PL1 and PL2 are normalized to (0,1). Top: distribution of balanced plans. Bottom: distributions of unbalanced plans.

As in our examination of individual districts, similarities among the distributions of compactness scores belie differences in the ways that each compactness measure ranked plans. If anything, plan measures disagreed more frequently over how to evaluate plans than district measures differed on the rankings of individual districts (Table 2-8):

	Avg. A1	*Min. A1*	*Avg. OS1*	*Min. OS1*	*Avg. PA6*	*Min. PA6*	*PL1*
Min. A1	0.47						
Avg. OS1	0.40	0.45					
Min. OS1	0.19	0.54	0.49				
Avg. PA6	-0.17	-0.28	0.05	-0.17			
Min. PA6	-0.13	-0.20	0.14	-0.07	0.75		
PL1	-0.13	0.09	-0.22	0.12	-0.76	-0.68	
PL2	0.01	-0.21	-0.22	-0.56	0.25	0.06	-0.30

Table 2-8. Similarities in plan compactness rankings (Kendall's τ_β) or the 3x4 case with 2 districts.

Table 2-9 shows the number of equivalence classes created by each plan compactness measure for a selected map. The number of classes varies between plans measures, supporting our previous conclusions that some measures are much more sensitive than others.

Grid Size (# of plans)	*districts per plan*	*Avg. A1*	*Min. A1*	*Avg. OS1*	*Min. OS1*	*Avg. PA6*	*Min. PA6*	*PL1*	*PL2a*
2x4 (35)	2	7	4	13	13	12	9	7	n/a
2x4 (126)	2	16	15	33	26	28	11	8	4
3x3 (280)	3	14	3	28	7	28	5	7	n/a
3x3 (3024)	3	93	16	309	13	162	12	9	5
3x4 (462)	2	12	4	66	48	61	20	12	n/a
3x4 (2046)	2	61	29	296	142	157	25	15	6
3x4 (5775)	3	60	5	570	30	438	17	12	n/a
3x4 (88534)	3	689	31	7960	74	2886	32	14	7
3x4 (15400)	4	152	7	571	12	715	9	9	n/a

Table 2-9. Equivalence classes for plan based measures of compactness. Balanced plans are indicated by *italics*.

2.5. Discussion

In this chapter, I have answered the question “Are compactness measures consistent?” and started to answer the question of “Are compactness measures effective?” Many advocates of compactness assume that the choice of a particular compactness measure is relatively unimportant. My research shows this assumption to be false: The worst compactness measures, such as raw ratios of perimeter to area and length to width, fail to capture any of the common intuitions about how geographical gerrymandering works. The best compactness measures can capture only limited aspects of geographical manipulation — gerrymandering is multifaceted, and no single one-dimensional index suffices to capture all aspects of it.

Compactness standards are only proxies for electoral manipulation — no author has based their compactness measure on an explicit theory of the electoral effects of district lines. Most measures claim to flag suspect district shapes, shapes that may indicate undue manipulation of district lines. Previous examinations of compactness have been hindered by the absence of a set of reasonable minimal criteria for compactness measures, resulting in a multiplication of measures of questionable value. Although no single perfect measure of compactness exists, by developing a set of minimal standards for compactness measures, I have been able to eliminate many measures that fail to comport with common intuitions about gerrymandering and common understandings about measuring ìshapeî. And I have also been able to develop corrections for commonly used, but flawed, measures of compactness.

Appendix: Proofs

We can simplify the analysis by recognizing that some measures[62] will produce identical rankings over all shapes, and will be indistinguishable under axioms 1–6.[63]

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[62] In particular, the following sets are clearly identical: (LW_3, LW_4), (LW_6,A_5), (A_1,OS_3), (PA_1,PA_2,PA_6), and (PA_3,PA_4).

[63] While one measure might be simpler than another, in practice, to compute, I ignore this distinction.

Measures OS_3 and A_1, A_2 and A_3 clearly satisfy axioms 1–3 and 5 thus meeting the axiomatic criteria. However, most of the other measurements violate at least one of the first three, or all of the latter three axioms, raising doubts as to their consistency.

Many of these indices violate at least one of the first three “shape” axioms:

- Measure PA_5 violate axiom 1. Convex districts of exactly the same shape, but different sizes may be assigned different values.[64] All of the perimeter/area measures, PA_1—PA_6, are subject to a more subtle violation of scale invariance in practice, which has not been previously recognized. If districts have natural boundaries, these measures can be affected by precisely how we measure district lines, for districts will seem to be less compact when seen on a map which has a fine scale than on a map with a larger scale.[65] For comparisons to be consistent, we must use the same precision to measure all district lines.

[64] Flaherty and Crumplin (1992) note that, in general, that perimeter/area measures are not scale invariant.

[65] Suppose you were trying to measure the length of a section of California shoreline, perhaps the section between San Francisco and Los Angeles. If you used a coarse approximation, perhaps by measuring the length of Route 1, which runs along the shore nearby, you would guess that the shoreline is several hundred miles long. If you tried to make more precise measurements by walking along the beach, your path might expand to several thousands of miles. Finer measurements will reveal the shore to be of ever-

- For any finite number of sample points, chosen at fixed positions along the edge of the shape, OS_2 violates axiom 2, because rotating a shape may change the choice of sample points, and hence the compactness measurement (Young 1988).[66] Measure LW_5 also, by its definition, fails axiom 2.

Most compactness indices reflect at least one principle of shape manipulation, but not others. In most cases, these measures obviously satisfy one shape axiom, but violate others. I demonstrate these violations by producing shapes that are misclassified by particular measures.

- Measures A_1, A_2, A_3, and OS_3 clearly satisfy axiom 5,[67] although they violate axioms 4 and 6 (Figure 2-10, Figure 2-11).[68]

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increasing length.

[66] Young (1988) does not use an axiomatic characterization, but his example can easily be applied to show the violation. He gives an example where changing the particular sample points used to measure a shape changes its compactness. If, instead, we consider the sample points to be fixed in orientation (e.g., one point sampled at "12-0'clock," "1-O'clock," etc.), and instead rotate the shape, Young's example shows that OS_2 is not rotation invariant. We can fix this rotation invariance by using a rotation-invariant reference point for our samples, but this would simply force the measure to be fail axiom 3.

[67] If two shapes have the same convex hull, the radius and area of the circumscribing

- Measure OS_1 violates axioms 4, 5, 6 (Figure 2-10, Figure 2-12, Figure 2-11).[69]

- While the perimeter-area measures violate scale invariance, we can normalize them to correct this defect, and, at the same time, satisfy axiom 7. One way to correct these violations is to normalize the shape being measured by the area of its convex hull. I define a new measure PA_7 to be: $\frac{P\left(\frac{1}{\alpha}S\right)}{A\left(\frac{1}{\alpha}S\right)}$, where $A(Co(S))=\alpha$/ PA_7 satisfies axioms 1–3, and 6. Similar transformations could be used for other measures. All of the perimeter-area measures can violate axiom 5 (Figure 2-10).

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circle, hexagon or convex figure will be the same. Hence if shape A has the same convex hull as B yet B has a greater area, B will be ranked higher under these measures — satisfying axiom 5.

[68] Under the A_3 measure any convex shape is perfectly compact - contradicting axiom 4. Measure A_2 is created under the assumption than the most compact figure is a hexagon, which leads to a similar violation of axiom 4. While A_1 does agree with axiom 4's implication that the most compact shape is a circle, it can violate the axiom in less obvious ways, as illustrated below.

[69] Blair and Bliss (1967) show that OS_1 satisfies axioms 1–3. They also show that under OS_1 the most compact shape is a circle. While OS_1 fails axioms 1–3, it does seem to be capturing legitimate aspects of shape manipulation; rather than focussing on dissection or dispersion alone, it may be capturing a combination of both.

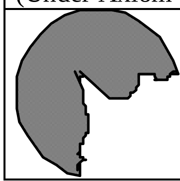
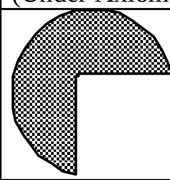
More Compact (Under Axiom 5)	Less Compact (Under Axiom5)	Measures Not Classifying These Correctly
		$PA_1 – PA_6$

Figure 2-10. Violations of axiom 5. Shapes on the left have the same convex hull, but greater area hulls than those on the right.

A number of compactness indices violate all three principles:

- All measures listed violate axiom 4 (Figure 2-11). We can, however, create a measure that satisfies axiom 4, as well as axioms 1-3, $OS_5 = \frac{P(CO(s_{norm}))}{A(S_{norm})}$.[70]

- All with the exceptions of PA_1–PA_6 and OS_2 violate axiom 6, because changes in perimeter that do not affect convex hull, area and shape diameters are ignored.

- Measures LW_1–LW_7, A_4–A_7 and OS_4 violate all three compactness axioms (Figure 2-12, Figure 2-11).

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70 To avoid violating axiom 1, we uniformly scale the shape so that it has unit area, producing S_{norm}.

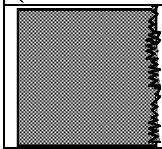
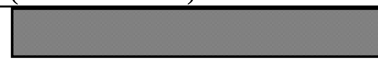
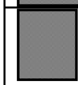
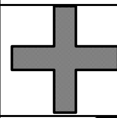
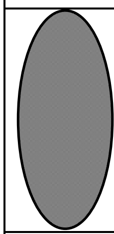
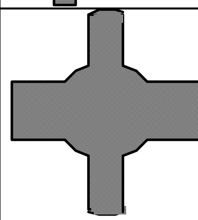
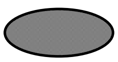
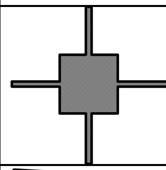
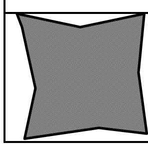
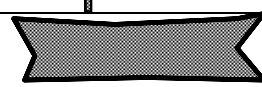
More Compact (Under Axiom 4)	Less Compact (Under Axiom 4)	Measures Not Classifying These Correctly
		PA_1–PA_6
		LW_1–LW_5
		LW_6,LW_7, A_1,A_2, OS_3,A_4–A_7
		OS_1,OS_2[71]
		OS_4,A_3

Figure 2-11. Violations of axiom 4. Shapes on the left have the same area, but smaller convex hulls than those on the right.

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[71] Remember that this measure uses a sample of points on the perimeter: This particular example works if we take our sample points at the compass points; it is easy to create other examples for other specified sampling methods.

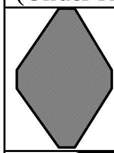
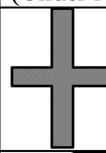
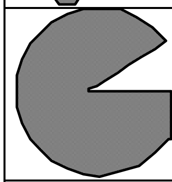
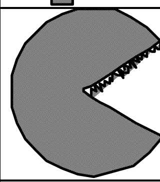
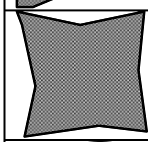
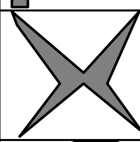
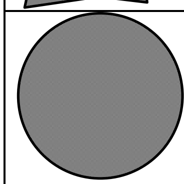
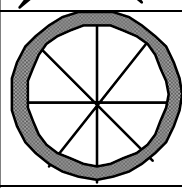
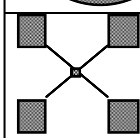
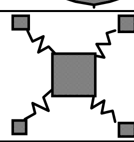
More Compact (Under Axioms 5,6)	Less Compact (Under Axioms 5,6)	Measures Not Classifying These Correctly
		LW_1–LW_5
		LW_6–LW_7, A_4–A_6
		A_7
		OS_4
		OS_2[72]
		OS_1

Figure 2-12. Violations of axioms 5 and 6. Shapes on the left have the same convex hulls, greater area and a smaller perimeter/area ratio than those on the right.

For any compactness measure C, satisfying axioms 1–4 and either 5 or 6, the two properties hold.

[72] This example is based on the arguments in Young (1988).

Property 1 (Contiguity): For any given perimeter or area, the maximally compact shape is contiguous. If C satisfies axiom 6, this is true for any given convex hull as well.

Property 2: A circle is the most compact district.

The shape that uniquely minimizes the perimeter for any given area is a circle.[73] Thus any compactness measure satisfying axiom 4 will judge a circle to be most compact, and property 1 is shown.

This fact can also be used to show part of property 1. Because a circle is contiguous, and a circle is the most compact shape, the most compact shape for any given perimeter or area is contiguous.

This leaves me to show that under axiom 6 the most compact shape with a given convex hull is contiguous. I do this by showing that for any given convex hull *X*, the shape s.t. $S^*=CO(S^*)=X$ is most compact. Because this shape is contiguous, property 2 follows.

C satisfies axiom 6.

[73] This is a well known isoperimetric inequality. For a compendium with original sources, see (Mitrinovic, Pecaric and Volenec 1989).

Define: A shape S is *discontiguous* if $\exists s_i \in S, s.t.\ s_i \cap \left(\bigcup_{s_{j \neq i} \in S} s_i \right) = \varnothing$. Let S^- $= S \cap CO(S)$. A shape S is *measurably* discontigous if it discontiguous and *A(S⁻)•0,P(S⁻)•0.*

If *CO(S)=X*, *S•S**, *S* is measurably discontiguous, then A(S)<A(S*). Hence C(S*)>C(S) by axiom 6. Q.E.D. I conjecture without proof that *S** minimizes the perimeter/area ratio for all shapes with convex hull *X*.