Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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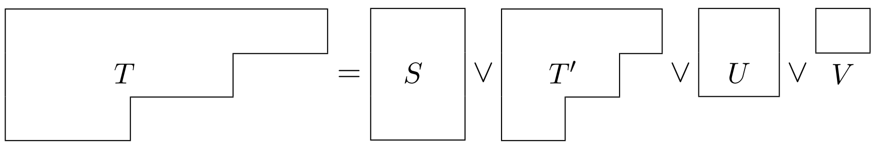
Chapter 6

Proof of Theorem 2

From Remark 2.2.8, to prove Theorem 2, it suffices to construct non-zero tableaux filled with d copies of c elements for all required partitions of $n = cd$, $c, d \geq 3$. These partitions were determined in Theorem 10.

Our approach is similar to the proof of Theorem 1 in Chapter 5. Using some generic non-zero tableaux (like U_i and V in Chapter 5) with c elements, we join them together by Theorem 8 to form a tableau of the appropriate shape and filling. However, unlike in Chapter 5, a large number of generic tableaux are needed. Since the cataloging of non-zero tableaux is quite tedious, we post-pone the construction until Chapter 7. Namely, our proof here will presuppose the construction of all tableaux of the required shapes for $c \leq 8$.

The general idea is to write a tableau T as follows:



for an appropriate T', where S, U, and V generic constructions based on the parity of d. This reduces the construction of T to a construction of T' where the shape parameters, (r, s, and t), of T' are small. Thus we only need to construct tableaux for a limited number of cases corresponding to small shapes. The tableaux S, U, and V are based on the following non-zero maximal tableaux: (Here U_1, U_2 and V occurred in Chapter 5.)

$$S_0 = P_1(d) = \begin{array}{c} d \\ \hline 1 \\ 2 \\ 3 \end{array} \qquad d \text{ even}, \qquad S_1 = \begin{array}{cccccc} \text{A} & \text{A} & \text{B} & \text{B} & \text{B} & \text{B} \\ \hline 5 & 6 & 6 & 6 & 5 & 5 \\ 4 & 3 & 3 & 4 & 3 & 4 \\ 1 & 2 & 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} \text{A} = \frac{d-\mathsf{x}}{3} + \mathsf{x} \\ \text{B} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3, \\ \mathsf{x} \in \{0,1,2\} \end{array}$$

$$U_1(d) = \begin{array}{c} d \\ \hline 1 \\ 2 \end{array} \qquad d \text{ even}, \qquad U_2 = \begin{array}{cccc} \text{A} & \text{A} & \text{B} & \text{B} \\ \hline 1 & 3 & 1 & 3 \\ 2 & 4 & 4 & 2 \end{array} \qquad \begin{array}{l} \text{A} = d - 1 \\ \text{B} = 1 \end{array}$$

$$V = \begin{array}{c} d \\ \hline 1 \end{array}$$

Let $\lambda = [r+s+t, s+t, t]$. We can write $T = S \vee T' \vee U \vee V$, for appropriate T' provided T' is maximail. Then T' will be filled with d copies of c' elements, for some $c' < c$, which will eventually allow us to reduce to $c \leq 8$. If T' is non-zero and maximal then by Lemma 3.4.9 and the Theorem 8 $\mathbf{q}_T \neq 0$ as desired. For simplicity, we will base our construction on the parity of d.

6.1 Case: d even

To see how to write T as $T = S \vee U \vee V \vee T'$ for an appropriate T' we first discuss the individual reductions allowing us to write $T = S \vee T'$, $T = U \vee T'$, or $T = V \vee T'$. Then successive applications of these reductions yield our desired decomposition. An analysis of these reductions also computes the resulting bounds on the shape of T'. An example application follows the reductions listed below. The reader may wish to refer to Example 6.1.1 while reading these reductions.

Reduction 1: Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take f to be the maximum integer such that $\mathsf{f}d \leq \lambda_3$ and $c - 3\mathsf{f} \geq 3$. Let $S = \mathsf{f}P_1(d)$ be the join of f copies of $P_1(d)$. Then by Theorem 8, we may write $T = S \vee T'$ for T' a $\lambda' = (\lambda_1 - d\mathsf{f}, \lambda_2 - d\mathsf{f}, \lambda_3 - d\mathsf{f})$-tableau filled with d copies of $c' = c - 3\mathsf{f}$ elements, provided the weight-sets of S and T' are disjoint. The choice of f means that in T', $t' = \lambda_3' = \lambda_3 - d\mathsf{f} < d$ or $c' = c - 3\mathsf{f} < 6$. Thus we need only consider tableaux with $t = \lambda_3 < d$ or $c < 6$. The $c < 6$ condition coresponds to the requirement $c - 3\mathsf{f} \geq 3$. We need this requirement so that there are at least three elements available with which to fill the remaining tableau, T'.

Reduction 2: Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take g to be the maximum integer such that $\mathsf{g}d \leq \lambda_2 - \lambda_3 = s$ and $c - 2\mathsf{g} \geq 3$. Let $U = \mathsf{g}U_1(d)$ be the join of g copies of $U_1(d)$. Then by Theorem 8, we may write $T = U \vee T'$ for T' a $\lambda' = (\lambda_1 - d\mathsf{g}, \lambda_2 - d\mathsf{g}, \lambda_3)$-tableau filled with d copies of $c' = c - 2\mathsf{g}$ elements, provided the weight-sets of U and T' are disjoint. The choice of g means that in T', $s' = \lambda_2' - \lambda_3' = \lambda_2 - d\mathsf{g} - \lambda'\_3 < d$ or $c' = c - 2\mathsf{g} < 5$. However, we will need the existence of a non-zero T' in the specified shape. As was shown in Theorem 9, this is not always the case for some s. Specifically, when $s < 5$ non-zero tableaux do not exist for certain shapes when $c = 3$. (Consider $\lambda = [6 + d, 2 + d, 1] = [9, 5, 1]$ with $d = 3$ and $c = 5$. Applying Reduction 2 yields $\lambda' = [5, 2, 1]$ with $c = 3$. All such tableaux are zero by Theorem 10 since $s = 1$.) To account for this, we modify the construction above to use $\mathsf{g} - 1$ copies of $U_1(d)$ when $\mathsf{g} > 0$ and $s' < 5$. In such a case, the modified T' now has $s' < d + 5$. Thus we need only consider arbitrary tableaux with $s < d + 5$ or $c < 5$.

Reduction 3: Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take h to be the maximum integer such that $\mathsf{h}d \leq \lambda_1 - \lambda_2 = r$ and $c - \mathsf{h} \geq 3$. Let $V = \mathsf{h}V_1(d)$ be the join of h copies of $V_1(d)$. Then by Theorem 8, we may write $T = V \vee T'$ for T' a $\lambda' = (\lambda_1 - d\mathsf{h}, \lambda_2, \lambda_3)$-tableau filled with d copies of $c' = c - \mathsf{h}$ elements, provided the weight-sets of V and T' are disjoint. The choice of h means that in T', $r' = \lambda_1' - \lambda_2' = \lambda_1 - d\mathsf{h} - \lambda_2 < d$ or $c' = c - \mathsf{h} < 4$. However, we will need the existence of non-zero T' in the specified shape. As was shown in Theorem 9, this is not always the case for some r. Specifically, when $r < 5$ non-zero tableaux do not exist for certain shapes when $c = 3$. To account for this, we modify the construction above to use $\mathsf{h} - 1$ copies of $U_1(d)$ when $\mathsf{h} > 0$ and $r' < 5$. In that case, the modified T' now has $r' < d+5$. Thus we need only consider arbitrary tableaux with $r < d+5$ or $c < 4$.

Conclusion: When d is even, we can apply these reductions successively. Take an arbitrary λ-tableau T filled with d copies of c elements and assume $c \geq 6$. We use $T^{(i)}$ to represent the appropriate T' obtained in these reductions. By Reduction 1, $T = S \vee T^{(1)}$, where $S = \mathsf{f}P_1(d)$ and $T^{(1)}$has $t = \lambda_3(T^{(1)}) < d$ and is filled with $c^{(1)} = c - 3\mathsf{f}$ elements.

Now, if $c^{(1)} \geq 6$ apply Reduction 2 to $T^{(1)}$. Since $c^{(1)} \geq 6$, then by Reduction 2, write $T^{(1)} = U \vee T^{(2)}$ where $U = \mathsf{g}U_1(d)$ and $T^{(2)}$ has $t < d$ (since $T^{(1)}$ does) and $s < d+5$. Here $T^{(2)}$ is filled with $c^{(2)} = c^{(1)} - 2\mathsf{g}$ elements.

Finally if $c^{(2)} \geq 6$ apply Reduction 3. This gives $T^{(2)} = V \vee T^{(3)}$, where $V = \mathsf{h}V_1(d)$ and $T^{(3)}$ has $t < d$, $s < d+5$, and $r < d+5$. Here $T^{(3)}$ is filled with $c^{(3)} = c^{(2)} - \mathsf{h}$ elements.

Hence $T = S \vee U \vee V \vee T^{(i)}$ where either $T^{(i)}$ is filled with fewer than 6 elements, or $T^{(i)}$ has $t < d$, $s < d+5$, and $r < d+5$. In the second case, $T^{(i)}$ must be filled with $3t + 2s + r = cd$ elements. This is less than or equal to $3(d-1) + 2(d+4) + (d+4) = 6d + 9 \leq 8d$ if $d > 4$. (If $d = 4$ we have $6d + 8 \leq 8d$ and it's not possible to have $6d + 9 = 9d$ when $d = 4$. For $d = 3$ additional reductions apply.) Hence we only need those tableaux with $c \leq 8$. Moreover, if r or $s < 5$ in $T^{(i)}$, then r or $s < 5$ in T, because the reductions do not reduce r or s to less than 5. Hence $T^{(i)} = T'$ has a shape occurring in Theorem 10 since all partitions of n with r and $s \geq 5$ are needed.

This reduction uses Theorem 8. Our usage only requires verification that the weight-sets are disjoint. However, the tableaux S, U, and V are in maximal form. Hence for appropriately chosen tableaux (i.e., ones in maximal form), an application of Lemma 3.4.9 can easily prove weight-set disjointness.

Example 6.1.1. To see how this reduction works, let us consider a specific shape, $\lambda = [9d-2, 5d, d+2]$ where $d \geq 6$, d even and $c = 15$. This shape has $t = d+2$, $s = 4d-2$, and $r = 4d-2$. First we apply Reduction 1, which joins $P_1(d)$ in order to have $t < d$.

$$[9d-2, 5d, d+2] = P_1(d) \vee [8d-2, 4d, 2]$$

Then we apply Reduction 2 to the shape $[8d-2, 4d, 2]$, which has $s = 4d-2$ to reduce to $s < d+5$ by joining three copies of $U_1(d)$.

$$[8d-2, 4d, 2] = 3U_1(d) \vee [5d-2, d, 2]$$

Applying Reduction 3 to shape $[5d-2, d, 2]$, which has $r = 4d-2$ we normally want to reduce r to be between 5 and $d+5$. Here we won't necessarily reduce r fully, so that the resulting tableau will be familiar. Instead we will reduce to $r = 2d-2$ (which may be reduced further depending on d) by joining two copies of $V(d)$.

$$[5d-2, d, 2] = 2V(d) \vee [3d-2, d, 2]$$

Hence, when we combine all these reductions, we get

$$[9d-2, 5d, d+2] = P_1(d) \vee 3U_1(d) \vee 2V(d) \vee [3d-2, d, 2]$$

A non-zero tableau of shape $[3d-2, d, 2]$ is Q^* of Example 3.2.7 with $\text{A} = 1$, $\text{B} = 1$, $\text{C} = d-2$. Therefore, writing

$$T = P_1(d) \vee Q^*(1, 1, d-2) \vee 3U_1(d) \vee 2V(d)$$

and omitting the extra tail of Q^* we have

$$T = \begin{array}{ccccccccccc} \text{d} & 1 & 1 & 1 & 1 & \text{d-2} & \text{d} & \text{d} & \text{d} & \text{d} & \text{d} \\ \hline 1 & 5 & 5 & 5 & 6 & 4 & 8 & 10 & 12 & 14 & 15 \\ 2 & 4 & 7 & 7 & 4 & 7 & 9 & 11 & 13 & & \\ 3 & 6 & 6 & & & & & & & & \end{array}$$

As Q^* is in maximal form, $\mathbf{q}_T \neq 0$.

6.2 Case: d odd

When d is odd, we proceed exactly as in the even case, except the tableaux we use are slightly different. Namely, we use S_1 instead of P_1 and U_2 instead of U_1. These adjustments are necessary for Reductions 1 and 2 since $P_1(d)$ and $U_1(d)$ are zero for d odd. Reduction 3 remains unchanged however. For completeness, we rewrite these reductions in terms of d odd. However, these reductions alone are not enough to reduce to $c \leq 8$. So after these reductions, we apply a few more in order to reduce the size of tableaux we need to consider.

Reduction $1'$**:** Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take w to be the maximum integer such that $\mathsf{w} \cdot 2d \leq \lambda_3$ and $c - 6\mathsf{w} \geq 3$. Let $S = \mathsf{w}S_1(d)$ be the join of w copies of $S_1(d)$. Then by Theorem 8, we may write $T = S \vee T'$ for T' a $\lambda' = (\lambda_1 - 2d \cdot \mathsf{w}, \lambda_2 - 2d \cdot \mathsf{w}, \lambda_3 - 2d \cdot \mathsf{w})$-tableau filled with d copies of $c' = c - 6\mathsf{w}$ elements, provided the weight-sets of S and T' are disjoint. The choice of w means that in T', $t' = \lambda'_3 = \lambda_3 - 2d \cdot \mathsf{w} < 2d$ or $c' = c - 6\mathsf{w} < 9$. Thus we need only consider tableaux with $t = \lambda_3 < 2d$ or $c < 9$.

Reduction $2'$**:** Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take v to be the maximum integer such that $\mathsf{v} \cdot 2d \leq \lambda_2 - \lambda_3 = s$ and $c - 4\mathsf{v} \geq 3$. Let $U = \mathsf{v}U_2(d)$ be the join of v copies of $U_2(d)$. Then by Theorem 8, we may write $T = U \vee T'$ for T' a $\lambda' = (\lambda_1 - 2d \cdot \mathsf{v}, \lambda_2 - 2d \cdot \mathsf{v}, \lambda_3)$-tableau filled with d copies of $c' = c - 4\mathsf{v}$ elements, provided the weight-sets of U and T' are disjoint. The choice of v means that in T', $s' = \lambda'_2 - \lambda'_3 = \lambda_2 - d \cdot \mathsf{v} - \lambda'_3 < 2d$ or $c' = c - 4\mathsf{v} < 8$. As in the even case, to account for the shapes $s < 5$, we modify this reduction to use $\mathsf{v} - 1$ copies of $U_1(d)$ when $\mathsf{v} > 0$ and $s' < 5$. Then the modified T' now has $s' < 2d + 5$. Thus we need only consider arbitrary tableaux with $s < 2d + 5$ or $c < 8$.

Summary: The same argument as in the even case works for the d odd cases, though the numbers are adjusted slightly. Take an arbitrary λ-tableau T with filled with d copies of c elements, but this time assume $c \geq 9$. Then by applications of

Reductions 1', 2' and 3, $T = S \vee U \vee V \vee T^{(i)}$ where either $T^{(i)}$ is filled with fewer than 9 elements, or $T^{(i)}$ has $t < 2d$, $s < 2d+5$, and $r < d+5$. In the second case, $T^{(i)}$ must be filled with $3t+2s+r$ elements, which is less than or equal to $3(2d-1)+2(2d+4)+(d+4) = 11d+9$ as $d \geq 3$. Moreover, if r or $s < 5$ in $T^{(i)}$, then r or $s < 5$ in T. Hence $T^{(i)}$ has a required shape of Theorem 10. However, we wish to have $T^{(i)}$ fillable with $c \leq 8$. To do this we have additional reduction techniques. However, these techniques are very sensitive to the parameters in $T^{(i)}$, so we will categorize them by such. The additional non-zero maximal tableaux we use are

$$U_1(d-1) = \begin{array}{c} d-1 \\ \hline 1 \\ 2 \end{array} \qquad d \text{ odd} \qquad \omega_2 = (0, d-1)$$

$$P_1(d-1) = \begin{array}{c} d-1 \\ \hline 1 \\ 2 \\ 3 \end{array} \qquad d \text{ odd} \qquad \omega_{2,3} = \left(\begin{smallmatrix} 0 & d-1 & 0 \\ 0 & 0 & d-1 \end{smallmatrix}\right)$$

$$P_4(d-2,1,1) = \begin{array}{cccc} \multicolumn{4}{c}{d-2} \\ \hline 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & 2 \\ 3 & & & \end{array} \qquad d \text{ odd} \qquad \omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 1 \\ 0 & 0 & d-2 \end{smallmatrix}\right)$$

Start with a tableau T where $t \leq 2d-1$, $s \leq d+4$, $s \neq 1$, $r \leq d+4$, $r \neq 1$, d odd and $s+t$ even if r or s in $\{0,2,4\}$. (These are the partitions required by Theorem 10 after the previous reductions have been applied.) First consider those tableaux with $r \geq 10$, which implies $d \geq 6$.

Case A: Assume $r \geq 10$, $s < d+4$, $t < d-1$. Then $3t+2s+r \leq 3(d-2) + 2(d+3)+d+4 = 6d+4 \leq 8d$. Hence this case is covered by $c \leq 8$.

Case B: Assume $r \geq 10$, $s \geq d+4$, $t \geq d-1$. Write $T = P_1(d-1) \vee U_1(d-1) \vee T'$. If (r,s,t) are the parameters of T, then T' has parameters $(r', s', t') = (r-5, s-(d+1), t-(d-1))$. Thus $5 \leq r' \leq d-1$, $5 \leq s' \leq d+5$, and $0 \leq t' \leq d$. Then

$3t + 2s + r \leq 3d + 2(d + 5) + d - 1 = 6d + 9 \leq 8d$. Note that no exceptional r or s cases occur in T'. Hence this case is covered by $c \leq 8$.

Case C: Assume $r \geq 10$, $s < d+4$, $t \geq d-1$. Write $T = P_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r-3, s, t-(d-1))$. Thus $7 \leq r' \leq d+1$, $s' \leq d+3$, and $0 \leq t' \leq d$. Then $3t+2s+r \leq 3d+2(d+3)+d+1 = 6d + 7 \leq 8d$. Note that no exceptional r or s cases occur in T'. Hence this case is covered by $c \leq 8$.

Case D: Assume $r \geq 10$, $s \geq d+4$, $t < d-1$. Write $T = U_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r - 2, s - (d - 1), t)$. Thus $8 \leq r' \leq d + 2$, $s' \leq d + 5$, and $0 \leq t' \leq d - 1$. Then $3t + 2s + r \leq 3(d-1) + 2(d+5) + d+2 = 6d+9 \leq 8d$. Note that no exceptional r or s cases occur in T'. Hence this case is covered by $c \leq 8$.

For $r < 10$ the arguments depend more on the values of r, but the general idea is the same.

Case E: Assume $r < 10$, $s < d + 4$, $t < d - 1$. Then $3t + 2s + r \leq 3(d - 1) + 2(d + 3) + 9 = 5d + 12 \leq 8d$ for $d \geq 4$. When $d = 3$, $c = 9$ is a possibility. Now, when $d = 3$, we have $r \leq d + 4 = 7$, $s \leq d + 3 = 6$, and $t \leq d - 2 = 1$. So here $3t + 2s + r \leq 3 + 2 \cdot 6 + 7 = 22 < 8d$. Hence $c \leq 8$ tableaux will suffice to cover this case. Note that the $s + t$ parity is preserved in the exceptional r and s cases.

Case F: Assume $r < 10$, $s \geq d + 4$, $t < d - 1$. If $r = 9, 8, 7, 5$ then write $T = U_1(d - 1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r - 2, s - (d - 1), t)$. Then $r' \in \{7, 6, 5, 3\}$, $s' \leq d+5$, and $0 \leq t' \leq d-1$. Then $3t' + 2s' + r' \leq 3(d - 2) + 2(d + 5) + 7 = 5d + 11 \leq 8d$ for $d > 3$. When $d = 3$, then $s' \leq 8$, $t' \leq 1$. We have $3t' + 2s' + r' \leq 3 + 16 + 7 = 26 < 9d$. Hence $c \leq 8$ tableaux will suffice to cover these case.

If $r = 2, 4$, or 6, then write $T = U_1(d - 1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r-2, s-(d-1), t)$. Thus $r' \in \{4, 2, 0\}$, $s' \leq d+5$, and $0 \leq t' \leq d-1$. Then $3t' + 2s' + r' \leq 3(d-2) + 2(d+5) + 4 = 5d + 8 \leq 8d$. Hence $c \leq 8$ tableaux will suffice to cover these cases if the tableau exists.

Note that $U_1(d-1)$ preserves the parity of $s+t$, so if $s+t$ are even (always the case when $r = 2$ or 4), we will have $s' + t'$ even which is necessary for $r' = 0$, 2, or 4. Hence this construction works except when $r = 6$ and $s+t$ odd. Then $3t + 2s + r \leq 3(d-2) + 2(2d+4) + 6 = 7d + 8 \leq 8d$ for $d \geq 8$ otherwise it is less than $9d$ unless $d = 3$. If $3t + 2s + 6 = 9d$, then t is odd so write $t = d - 2k$ with $k \geq 1$. Then we have $s = 3d + 3k - 3$. Since $s \leq 2d + 4$, we have $d + 3k \leq 7$, so $d = 3, k = 1$ is the only solution. This corresponds to $t = 1$, $s = 9$, $r = 6$. Since $s + t$ even, this case has already been done. For $d = 3$ we also need to consider $c = 10$. However, $3 + 20 + 6 < 10d$ so no such partition will occur.

For $r = 0$ we have $s+t$ even. Then $3t+2s+r \leq 3(d-2)+2(2d+4) = 7d+2 \leq 8d$, so this case is covered by $c \leq 8$ tableaux.

For $r = 3$ we $3t + 2s + r \leq 3(d-2) + 2(2d+4) + 3 = 7d + 5 \leq 8d$ for $d \geq 5$. When $d = 3$ we have $3 + 20 + 3 < 9d$, so such a partition does not occur. Hence $r = 3$ is covered by $c \leq 8$.

Case G: Assume $r < 10$, $s < d + 4$, $t \geq d - 1$. If $r = 9, 8, 6$ then write $T = P_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r - 3, s, t - (d-1))$. Thus $r' \in \{6, 5, 3\}$, $s' \leq d + 3$, and $0 \leq t' \leq d$. Then $3t' + 2s' + r' \leq 3d + 2(d+3) + 6 = 5d + 12 \leq 8d$ for $d \geq 5$. When $d = 3$, then $s' \leq 6$, $t' \leq 3$. Hence we have $3t' + 2s' + r' \leq 9 + 12 + 6 = 27 = 9d$, so the only solution is $t' = 3$, $s' = 6$, $r' = 6$. This we can further reduce by writing $T' = V(d) \vee T''$ where T'' is a $c = 8$ tableau of parameters $t = 3$, $s = 6$, and $r = 3$. Hence $c \leq 8$ tableaux will suffice.

If $r = 7, 5$, or 3 and $s + t$ even then write $T = P_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r - 3, s, t - (d-1))$ and the parity of $s' + t'$ is preserved. Thus $r' \in \{0, 2, 4\}$, $s' \leq d + 3$, and $0 \leq t' \leq d$. Then $3t' + 2s' + r' \leq 3d + 2(d+3) + 4 = 5d + 10 \leq 8d$ for $d \geq 5$. For $d = 3$ we have

$3t' + 2s' + r' \leq 9 + 12 + 4 = 25 < 9d$, so this does not occur. Hence the $c \geq 8$ tableaux will suffice.

Consider $r = 0, 2, 4$ with $s + t$ even, or $r = 3, 5, 7$ with $s + t$ odd. If $s \geq 8$ write $T = P_4(d-2, 1, 1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r, s-3, t-(d-2))$ and the parity of $s'+t'$ is preserved. Thus , $5 \leq s' \leq d$, and $0 \leq t' \leq d+1$. Then $3t' + 2s' + r' \leq 3(d+1) + 2(d) + 7 = 5d + 10 \leq 8d$ for $d \geq 5$. For $d = 3$ we have $3t' + 2s' + r' \leq 12 + 6 + 7 = 25 < 9d$, so this does not occur. Hence the $c \geq 8$ tableaux will suffice.

For $r = 3, 5$, or 7, $s+t$ odd, and $s \leq 7$, we have $3t + 3s + r \leq 3(2d-1) + 2 \cdot 7 + 7 = 6d + 18 \leq 8d$ for $d \geq 9$. If $3t + 2s + r = 9d$, then since r is odd, we have $t = 2d - 2k$ for $k \geq 1$. This implies $s = d + 3k + \frac{d-r}{2}$. Since $s \leq 7$, the only possible solutions are $(r, s, t) = (7, 4, 4)$, $(5, 5, 4)$, $(3, 6, 4)$, $(7, 7, 2)$ when $d = 3$ and $(7, 4, 8)$ when $d = 5$. For those with $s + t$ even, the case has already been done. For $(r, s, t) = (7, 7, 2)$ write $T = U_1(2) \vee T'$ where T' has parameters $(5, 5, 2)$ and is a $c = 7$ tableau. Hence we still need $(r, s, t) = (5, 5, 4)$ for $d = 3$. But this is $U_1(2) \vee T'$, where T' is a $c = 7$ tableau of parameters $(3, 3, 4)$.

We also need to consider those partitions with $c = 10$. Then we have $3t + 2s + r = 10d$ which implies $t = 2d - 2k - 1$ as t is odd. So $s = 2d + 3k + 1 - \frac{r-1}{2}$. The only solutions with $s \leq 7$ are $(r, s, t) = (3, 6, 5)$, $(5, 5, 5)$, $(7, 4, 5)$, and $(7, 7, 3)$ all with $d = 3$. The cases with $s + t$ even have been done already. For $(r, s, t) = (3, 6, 5)$, use $T = P_4(1, 1, 1) \vee T'$ where T' is a $c = 7$ tableau with $(r', s,', t') = (3, 3, 4)$. For $(7, 4, 5)$ we have $s = 4$ and $s + t$ odd, so this case partition is not needed.

When $d = 3$ we also may have $c = 11$ or $c = 12$. Proceeding as above, the only solutions are $(r, s, t) = (7, 7, 4)$ and $(7, 7, 5)$. The second case has $s+t$ even and hence is not needed. The first case can be reduced to $T = U_1(2) \vee T'$, where T' is a $c = 9$ case with parameters $(5, 5, 4)$. But this is $U_1(2) \vee T'$, where T' is a $c = 7$ tableau of parameters $(3, 3, 4)$.

If $r = 0, 2, 4$, $s + t$ even, and $s \leq 7$, we have $3t + 3s + r \leq 3(2d-1) + 2 \cdot 7 + 4 = 6d + 15 \leq 8d$ for $d \geq 9$. For $c = 9$, we have $t = 2d - 2k - 1$ as t is odd. Then $s = d + 3k + 1 + \frac{d+1-r}{2}$. Since $s \leq 7$ the only solutions are $(r, s, t) = (0, 6, 5)$, $(2, 5, 5)$, $(4, 4, 5)$, and $(4,7,3)$ with $d=3$ and $(4,7,9)$. Note that only those with $s+t$ even are needed. For these cases write $T=U_1(d-1)\vee T'$, where T' is a $c'=7$ tableau with parameters $(r-2, s-(d-1), t)$. Since this preserves the parity of $s+t$ and does not cause $r', s'=1$, the tableau exists.

For $d=3$ we may also have $c=10$ or 11. Proceeding as above, the only solutions are $(r,s,t)=(4,7,4)$ which has $s+t$ odd, hence it is not need, and $(4,7,5)$ which is $U_1(2)\vee U_1(2)\vee T'$ where T' is a $c=7$ tableau with $(r,s,t)=(0,3,5)$.

Case H: Assume $r<10$, $s\geq d+4$, $t\geq d-1$. If $r=8$ or $r=9,7,5$ and $s+t$ even then write $T=P_1(d-1)\vee U_1(d-1)\vee T'$ where T' has parameters $(r',s',t')=(r-5, s-(d-1), t-(d-1))$. Thus $r'\leq 4$, $s'\leq d+5$, and $t'\leq d$. Then $3t'+2s'+r'\leq 3d+2(d+5)+4=5d+14\leq 8d$ for $d>3$. When $d=3$ we can have $3t'+2s'+r'=9d$ only for $(r',s',t')=(2,8,3)$ or $(4,7,3)$. But $s+t$ even means only $(4,7,3)$ is needed. This is $U_1(d-1)\vee T''$ where T'' is a $c=7$ tableau with $(r,s,t)=(2,5,3)$. Hence we've reduced to $c\leq 8$ cases.

For $r=9,7,5$ with $s+t$ odd, write $T=P_4(d-2,1,1)\vee U_1(d-1)\vee T'$ where T' has parameters $(r',s',t')=(r-2, s-(d-1)-3, t-(d-2))$. Hence $r'\leq 7$, $2\leq s'\leq d+2$, and $t'\leq d+1$. We will have $s'\notin\{0,1,2,4\}$ provided $s\neq d+4$ or $d+6$. Then $3t'+2s'+r'\leq 3(d+1)+2(d+2)+7=5d+14\leq 8d$ for $d>3$. When $s=d+4$ or $d+6$ write $T=P_4(d-2,1,1)\vee T'$ where T' has parameters $(r',s',t')=(r, s-3, t-(d-2))$. Then $3t'+2s'+r\leq 3(d+1)+2(d+3)+9=5d+18\leq 8d$ for $d\geq 5$.

For $d=5$, $s'=d+3$ or $d+1$ there are no partitions with $t'\leq d+1$. Now consider $d=3$ with $s'\leq d+3$, $t'\leq d+1$. Since $r\leq d+4$ we have $r=3,5$, or 7. Given these parameters, solving $3t'+2s'+r'=3c$ for $c\geq 9$, the only solutions are $(r',s',t')=(3,6,4)$, $(5,5,4)$, and $(7,4,4)$. Since $s+t$ odd the only partition we need to construct is $(r',s',t')=(5,5,4)$ with $c=9$. But this is $U_1(2)\vee T'$ where T' is a $c=7$ tableau of parameters $(3,3,4)$.

For $r=2,4,6$ with $s+t$ even, write $T=P_4(d-2,1,1)\vee U_1(d-1)\vee T'$ where T' has parameters $(r',s',t')=(r-2, s-(d-1)-3, t-(d-2))$. Thus $r'\leq 4$, $2\leq s'\leq d+2$, and $t'\leq d+1$. Since $s+t$ even, we need only that $s'\neq 1$. Then

$3t' + 2s' + r' \leq 3(d+1) + 2(d+2) + 4 = 5d + 11 \leq 8d$ for $d > 3$. For $d = 3$, $d \cdot 4 + 2 \cdot 5 + 4 = 26 < 9d$ so that case is not needed.

For $r = 6$, $s+t$ odd write $T = P_1(d-1) \vee T'$, where T' has parameters $(r', s', t') = (3, s, t-(d-1))$. Thus $3t' + 2s' + r' \leq 3d + 2(2d+4) + 3 = 7d + 11 \leq 8d$ for $d \geq 11$. Solving $3t'+2s'+r' = 9d$ given the parameters, the only solution is $(r', s', t') = (3, 9, 2)$. This is $P_4(1,1,1) \vee T''$ where T'' is a $c = 6$ tableau with parameters $(3, 6, 1)$. When $c = 10$, we also have $(3, 9, 3)$ as a solution, but this does not have $s + t$ odd.

For $r = 3$, write $T = P_4(d-2, 1, 1) \vee T'$ where T' has parameters $(r', s', t') = (3, s-3, t-(d-2))$. Thus $3t' + 2s' + r' \leq 3(d+1) + 2(2d+1) + 3 = 7d + 8 \leq 8d$ for $d > 5$. (The $c = 8$ constructions will hold unless $s - 3 \in \{0, 1, 2, 4\}$, but since $s \geq d + 4$ this can only occur with $s = d + 1$, $d = 3$. Then our original tableau will have $t \leq 2d - 1$, $s = d + 1$, $r = 3$ which is satisfiable by a $c \leq 8$ tableau.) Solving $3t' + 2s' + r' = 9d$ given the parameters, the only solutions are $(r', s', t') = (3, 6, 4)$, $(3, 9, 2)$, and $(3, 12, 6)$ with $d = 5$. The $(3, 6, 4)$ case is $P_4(1, 1, 1) \vee T''$ where T'' is a $c = 6$ tableau with parameters $(3, 3, 3)$. The $(3, 9, 2)$ case is $P_4(1, 1, 1) \vee T''$ where T'' is a $c = 6$ tableau with parameters $(3, 6, 1)$. When $d = 5$, the $(3, 12, 6)$ case is $P_4(3, 1, 1) \vee T''$ where T'' is a $c = 6$ tableau with parameters $(3, 9, 3)$. There are no tableaux with $c > 9$.

For $r = 0$ we have $s+t$ even. Write $T = P_4(d-2, 1, 1) \vee T'$ where T' has parameters $(r', s', t') = (3, s-3, t-(d-2))$. So $3t' + 2s' + r' \leq 3(d+1) + 2(2d+1) = 7d + 5 \leq 8d$ for $d > 3$. As P_4 preserves parity and $s + t$ even, the $c = 8$ constructions will work provided $s' \neq 1$. But $s \geq d + 4$ forces $s' \geq d + 1 \geq 4$. When $d = 3$ we have $3 \cdot 4 + 2 \cdot 7 = 26 < 9d$, hence all cases are covered.

Conclusion: When d is odd, we need all tableaux with $c \leq 8$. The basic reduction requires those tableaux to be disjoint from multiple copies of S_1, $U_2(d-1, 1)$ and $V(d)$. The further reductions also require the tableau to be disjoint from $P_1(d-1)$, $U_1(d-1)$, $P_1(d-1) \vee U_1(d-1)$, $P_4(d-2, 1, 1)$, and $U_1(d-1) \vee P_4(d-2, 1, 1)$.

For d even we need all tableaux with $c \leq 6$, along with those tableaux having $t \leq d - 1$, $s \leq d + 4$, $r \leq d + 4$ when $c = 7$ or 8. These tableaux need to be disjoint from $P_1(d)$, $U_1(d)$, and $V(d)$. These tableaux will be listed in Chapter 7. In Chapter 8, we verify that all necessary tableaux have been produced.

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