Generalized Foulkes’ Conjecture and Tableaux Construction

Thesis by

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Chapter 5

Proof of Theorem 1

Recall, Theorem 1 says that every irreducible occurring in $1_{S_n S_2}^S$ occurs in $1_{S_d S_c}^S$ with equal or greater multiplicity, where $n = 2b = cd$ and $b, c, d \geq 2$.

In Section 3.2 we proved Theorem 7, which showed that the irreducibles occurring in $1_{S_n S_2}^S$ were exactly those corresponding to partitions $\lambda = [n - s, s]$ for $s$ even and they occur with multiplicity one. (Since $n = 2b$ is even, it suffices to consider only the even values of $s$.) By Remark 2.2.8, to prove Theorem 1, it suffices to construct a non-zero tableau filled with $d$ copies of $c$ elements for each partition $[n - s, s]$, where $0 \leq s \leq \frac{n}{2}$, $s$ even and $n = cd$.

To do this we will construct some non-zero generic tableaux that when assembled via Theorem 8 will produce all the shapes and fillings needed. Since we are constructing generic tableaux for many partitions and fillings, we will not use a fixed $c$. However, we assume that every element listed in the body of the tableau occurs $d$ times, filling out the tail as needed. We apply weight-set counting to prove a tableau is non-zero. The tableaux we need are:

**Tableau $U_1$**

\[
U_1 = \begin{array}{ccc}
A & d-A & d-A \\
1 & 1 & 2 \\
2 & 1 & 2 \\
\end{array} \sim \begin{array}{c}
A \\
1 \\
2 \\
\end{array}
\]

\[\omega_2(U_1) = (0, A)\]
\[ \lambda = [2d - \lambda, \lambda] \]

For this first tableau, we listed \( U_1 \) both with and without the tail. Normally we will suppress the tail when writing these tableaux. \( U_1 \) is non-zero by Lemma 3.2.6 since \( \lambda \) is even. It is maximal since \((\lambda, 0)\) is the largest possible weight-set for this shape.

**Tableau \( U_2 \)**

\[
U_2 = \begin{array}{cccc}
A & A & B & B \\
1 & 3 & 1 & 3 \\
2 & 4 & 4 & 2
\end{array}
\quad A + B \leq d
\]
\[
\begin{array}{cc}
A, & B > 0
\end{array}
\]

\[ \omega_2 = (0, A+B, 0, A+B) \]

\[ \lambda = [4d - 2(A+B), 2(A+B)] \]

Examining the filling of \( U_2 \) and \( A, B > 0 \) we find the following constraints on any valid weight assignment: (Recall that \( U^* \) corresponds to a possible tableau \( \tau U_2 \).)

- If \( \omega_2(1|U^*) = 0 \) then \( \omega_2(2 \text{ and } 4|U^*) > 0 \).
- If \( \omega_2(2|U^*) = 0 \) then \( \omega_2(1 \text{ and } 3|U^*) > 0 \).
- We must have \( \omega_2(1 \text{ or } 2|U^*) > 0 \) and \( \omega_2(3 \text{ or } 4|U^*) > 0 \).

Since any valid weight assignment of \((0, A+B, 0, A+B)\) has exactly two zeros, the restrictions above show that \((1, 2, 3, 4)\) and \((2, 1, 4, 3)\) are the only valid weight assignments. These weights-sets correspond to applying \( \tau = (^A T \times ^A T \times ^B T \times ^B T \) and \( \tau = (^A T \times ^A T \times ^B T \times ^B T \times ^B T \times ^B T \times ^B T \times ^B T \) respectively. As both of these \( \tau \) have positive sign, \( q_{U_2} \neq 0 \). This tableau is maximal since every element \( x \) must have \( \omega_2(x) \leq A+B \).
Tableau $U_3$

\[
U_3 = \begin{array}{ccc}
A & B & B \\
1 & 1 & 2 \\
2 & 3 & 3 \\
\end{array} \\
A \text{ even} \\
A + B \leq d \\
B = \frac{d}{2} \\
d \text{ even}
\]

$\omega_2 = (0, A, d)$

$\lambda = [2d - A, A + d]$

To show $U_3$ is non-zero we will use weight-set counting on $\omega_2 = (0, A, d)$. There are two cases for which we need to determine weight assignments, $A + B < d$ and $A + B = d$.

When $A + B < d$, only the element 3 may be assigned a row two weight of $d$. So the distinct weight assignments are $(1, 2, 3)$ and $(2, 1, 3)$, which occur with $\tau = ()_T^A$ and $\tau = (12)_T^A \times ()_T^B \times ()_T^B$ respectively. Since $A$ is even, both $\tau$ have positive sign. Hence $U_3$ is non-zero.

If $A + B = d$, then $A = B = \frac{d}{2}$ and $d \equiv 0 \pmod{4}$. While every permutation corresponds to a distinct weight assignment, every weight assignment can only be obtained by having $\tau$ move complete column blocks. Since all of these blocks are even, $\tau$ is positive for every weight assignment and hence $U_3$ is non-zero. This tableau is maximal since $(d, A, 0)$ is the largest possible weight.

Tableau $V(d)$

\[
V = \frac{d}{1}
\]

$\omega_1(V) = (d)$

$\lambda = [d]$
This is just a single row with $d$ ones. Since there are no column permutations, this tableau is always non-zero. It is obviously maximal.

Having constructed these generic tableaux, we will use the notation $U_i(x)$ to denote the tableau $U_i$ with the parameter $\Lambda = x$ or $U_i(x,y)$ for $x = \Lambda$ and $y = B$ in $U_i$. We will use $fU_i$ to denote the join of $f$ copies of $U_i$. Note that these tableaux are all in maximal form.

For the proof of Theorem 1, the parity effects the construction process. To simplify notation, we define the $\ast$-function.

$$x^\ast = \begin{cases} 
  x & \text{if } x \text{ is even} \\
  x - 1 & \text{if } x \text{ is odd}
\end{cases}$$

We analyze $T$ by the parameters $r = n - 2s$ and $s$, where $\lambda = [r + s, s]$. For reference, we consider tableau of the following shape, with $r$ and $s$ even.

\[
T = \begin{array}{c}
\hline
\hline
s \\
\hline
r \\
\hline
\end{array}
\]

**Proof of Theorem 1.** To prove Theorem 1 we need to construct a non-zero tableau of shape $\lambda = [n - s, s]$ for $s \leq \frac{n}{2}$, with $s$ even and $n = cd$, $c, d \geq 2$. First we construct a general tableau that covers most $s$. Suppose $s \leq \frac{cd^*}{2}$. We know $s$ is even, so write $s = fd^* + e$, where $0 \leq e < d^*$, $e$ even. Since $s, d^*$, and $e$ are even, this is possible by the Euclidean algorithm.

Let $T = fU_1(d^*) \lor U_1(e)$. Note that the bound on $s$ guarantees that $2(f + 1) \leq c$ when $e > 0$, and $2f \leq c$ when $e = 0$. This insures that there are at most $c$ distinct elements in $T$. If there are fewer than $c$ elements in $T$ add all the remaining elements to the tail of $T$ by joining the appropriate number of $V(d)$’s. Suppressing the tail elements from the $U_1$’s and $V(d)$’s, $T$ looks like:
Theorem 8 shows $T$ is non-zero, provided the weight-sets are disjoint. Since the tableaux are in maximal form, the weights must be disjoint by Lemma 3.4.9. This covers the majority of the $s$. The remaining tableaux will be constructed according to the parity of $c$ and $d$.

Case I: $(c, d$ even$)$ In this case $\frac{c^*d^*}{2} = \frac{cd}{2}$, so $T$ constructed above covers all partitions.

Case II: $(d$ even, $c$ odd$)$ By the above construction, we have all tableaux with $s$ up to $\frac{(c-1)d}{2}$. Thus we only need those even partitions with $s = \frac{cd-k}{2}$ for $0 \leq k \leq d-2$, $k \equiv d \pmod{4}$. Take $T = \frac{c-3}{2}U_1(d) \cup U_3(\Lambda)$ for $0 \leq \Lambda \leq \frac{d}{2}$ with $\Lambda$ even. Then $s = \frac{c-3}{2}d + \Lambda + d = \frac{cd+d+2\Lambda}{2}$. Thus we have $k = d-2\Lambda$, which ranges over the correct parameters. Since $U_1$ and $U_3$ are in maximal form, Lemma 3.4.9 implies disjointness and Theorem 8 shows $T$ is non-zero.

Case III: $(c$ even, $d$ odd$)$ Since $r = n-2s = cd-2s$ we need $\lambda = [r+s, s]$ for $r \leq cd$ with $r \equiv cd \pmod{4}$. It suffices to construct a non-zero tableau for $r < 4d$. When $r \geq 4d$, let $r' = r-4dz$ with $r' < 4d$. Then if we construct a $\lambda' = [s+r', s]$ tableau $T'$ filled with $d$ copies of $c-4z$ elements, we get the needed tableau by $T = T' \cup 4zV(d)$. Hence we will take $r < 4d$.

When $c \equiv 0 \pmod{4}$ then $r \equiv 0 \pmod{4}$. Take $T = \frac{c-3}{4}U_2(d-1, 1) \cup U_2(d - \frac{r}{4} - 1, 1)$. This construction gives the shape $\frac{c-3}{4}[2d, 2d] + [2d + \frac{r}{4}, 2d - \frac{r}{4}] = [\frac{cd}{2} + \frac{r}{4}, \frac{cd}{2} - \frac{r}{4}]$ as desired. The parameters of these tableaux are positive unless $r = 4d-4$ since $r < 4d$, $r \equiv 0 \pmod{4}$, and $d \geq 2$. If $r = 4d-4$ then $d - \frac{r}{4} - 1 = 0$, so use $U_1(2) \cup 2V(d)$ instead of of $U_2(d - \frac{r}{4} - 1, 1)$.

For $c \equiv 2 \pmod{4}$ we will assume $r < 2d$. When $2d \leq r < 4d$ let $r' = r-2d$. Then
can construct a $\lambda = [r' + s, s]$ tableau $T'$ with $c \equiv 0 \pmod{4}$ and use $T = T' \lor 2V(d)$. Take $T = \frac{c-2}{4}U_2(d-1, 1) \lor U_1(\frac{2d-r}{2})$ with $V(d)$’s as needed. Note that $cd \equiv r \pmod{4}$ implies that $\frac{2d-r}{2}$ is even, while $r < 2d$ insures it is positive. So we get the shape $[\frac{cd}{2} + \frac{r}{2}, \frac{cd}{2} - \frac{r}{2}]$ as needed. Theorem 8 shows these $T$’s are non-zero provided the weight-sets are disjoint, which follows from maximality.

Note that since $cd = n$, $n$ even, then $c$ or $d$ is even. Thus we have constructed all cases.

Although it is not directly apparent from this construction, $c$ or $d$ even is often a necessary requirement for any non-zero two row tableau with $s$ even to exist. For instance, when $c = 3$ and $d = 7$, the shape $[11, 10]$ has $s$ even, but all tableaux are zero by Theorem 9.
Bibliography


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[22] David Wales, personal communication.