Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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Chapter 5

Proof of Theorem 1

Recall, Theorem 1 says that every irreducible occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with equal or greater multiplicity, where $n = 2b = cd$ and $b, c, d \geq 2$.

In Section 3.2 we proved Theorem 7, which showed that the irreducibles occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{\mathcal{S}_n}$ were exactly those corresponding to partitions $\lambda = [n-s, s]$ for s even and they occur with multiplicity one. (Since $n = 2b$ is even, it suffices to consider only the even values of s.) By Remark 2.2.8, to prove Theorem 1, it suffices to construct a non-zero tableau filled with d copies of c elements for each partition $[n-s, s]$, where $0 \leq s \leq \frac{n}{2}$, s even and $n = cd$.

To do this we will construct some non-zero generic tableaux that when assembled via Theorem 8 will produce all the shapes and fillings needed. Since we are constructing generic tableaux for many partitions and fillings, we will not use a fixed c. However, we assume that every element listed in the body of the tableau occurs d times, filling out the tail as needed. We apply weight-set counting to prove a tableau is non-zero. The tableaux we need are:

<u>Tableau U_1</u>

$$U_1 = \begin{array}{ccc} \text{A} & \text{d-A} & \text{d-A} \\ \hline 1 & 1 & 2 \\ 2 & & \end{array} \sim \begin{array}{c} \text{A} \\ \hline 1 \\ 2 \end{array} \qquad \begin{array}{l} \text{A even} \\ \text{A} \leq d \end{array}$$

$\omega_2(U_1) = (0, \text{A})$

$$\lambda = [2d - \text{A}, \text{A}]$$

For this first tableau, we listed U_1 both with and without the tail. Normally we will suppress the tail when writing these tableaux. U_1 is non-zero by Lemma 3.2.6 since A is even. It is maximal since $(\text{A}, 0)$ is the largest possible weight-set for this shape.

Tableau U_2

$$U_2 = \begin{array}{cccc} \text{A} & \text{A} & \text{B} & \text{B} \\ \hline 1 & 3 & 1 & 3 \\ 2 & 4 & 4 & 2 \end{array} \qquad \begin{array}{l} \text{A} + \text{B} \leq d \\ \text{A},\ \text{B} > 0 \end{array}$$

$$\omega_2 = (0, \text{A}{+}\text{B}, 0, \text{A}{+}\text{B})$$

$$\lambda = [4d - 2(\text{A}{+}\text{B}), 2(\text{A}{+}\text{B})]$$

Examining the filling of U_2 and A,B > 0 we find the following constraints on any valid weight assignment: (Recall that U^* corresponds to a possible tableau τU_2.)

- If $\omega_2(1|U^*) = 0$ then $\omega_2(2 \text{ and } 4|U^*) > 0$.
- If $\omega_2(2|U^*) = 0$ then $\omega_2(1 \text{ and } 3\ |U^*) > 0$.
- We must have $\omega_2(1 \text{ or } 2|U^*) > 0$ and $\omega_2(3 \text{ or } 4|U^*) > 0$.

Since any valid weight assignment of $(0, \text{A} + \text{B}, 0, \text{A} + \text{B})$ has exactly two zeros, the restrictions above show that $(1, 2, 3, 4)$ and $(2, 1, 4, 3)$ are the only valid weight assignments. These weights-sets correspond to applying $\tau = \overset{\text{A}}{()}_T \times \overset{\text{A}}{()}_T \times \overset{\text{B}}{()}_T \times \overset{\text{B}}{()}_T$ and $\tau = \overset{\text{A}}{(12)}_T \times \overset{\text{A}}{(12)}_T \times \overset{\text{B}}{(12)}_T \times \overset{\text{B}}{(12)}_T$ respectively. As both of these τ have positive sign, $\mathbf{q}_{U_2} \neq 0$. This tableau is maximal since every element x must have $\omega_2(x) \leq \text{A}{+}\text{B}$.

Tableau U_3

$$U_3 = \begin{array}{ccc} \mathrm{A} & \mathrm{B} & \mathrm{B} \\ \hline 1 & 1 & 2 \\ 2 & 3 & 3 \end{array} \qquad \begin{array}{l} \mathrm{A} \text{ even} \\ \mathrm{A} + \mathrm{B} \leq d \\ d \text{ even} \\ \mathrm{B} = \frac{d}{2} \end{array}$$

$$\omega_2 = (0, \mathrm{A}, d)$$

$$\lambda = [2d - \mathrm{A}, \mathrm{A} + d]$$

To show U_3 is non-zero we will use weight-set counting on $\omega_2 = (0, \mathrm{A}, d)$. There are two cases for which we need to determine weight assignments, $\mathrm{A} + \mathrm{B} < d$ and $\mathrm{A} + \mathrm{B} = d$.

When $\mathrm{A} + \mathrm{B} < d$, only the element 3 may be assigned a row two weight of d. So the distinct weight assignments are $(1, 2, 3)$ and $(2, 1, 3)$, which occur with $\tau = ()_T$ and $\tau = \overset{\mathrm{A}}{(12)}_T \times \overset{\mathrm{B}}{()}_t \times \overset{\mathrm{B}}{()}_T$ respectively. Since A is even, both τ have positive sign. Hence U_3 is non-zero.

If $\mathrm{A} + \mathrm{B} = d$, then $\mathrm{A} = \mathrm{B} = \frac{d}{2}$ and $d \equiv 0 \pmod 4$. While every permutation corresponds to a distinct weight assignment, every weight assignment can only be obtained by having τ move complete column blocks. Since all of these blocks are even, τ is positive for every weight assignment and hence U_3 is non-zero. This tableau is maximal since $(d, \mathrm{A}, 0)$ is the largest possible weight.

Tableau $V(d)$

$$V = \begin{array}{c} \mathrm{d} \\ \hline 1 \end{array}$$

$$\omega_1(V) = (d)$$

$$\lambda = [d]$$

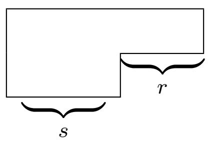
This is just a single row with d ones. Since there are no column permutations, this tableau is always non-zero. It is obviously maximal.

Having constructed these generic tableaux, we will use the notation $U_i(x)$ to denote the tableau U_i with the parameter $\mathrm{A} = x$ or $U_i(x, y)$ for $x = \mathrm{A}$ and $y = \mathrm{B}$ in U_i. We will use $\mathsf{f}U_i$ to denote the join of f copies of U_i. Note that these tableaux are all in maximal form.

For the proof of Theorem 1, the parity effects the construction process. To simplify notation, we define the *-function.

$$x^* = \begin{cases} x & x \text{ even} \\ x - 1 & x \text{ odd} \end{cases}$$

We analyze T by the parameters $r = n - 2s$ and s, where $\lambda = [r + s, s]$. For reference, we consider tableau of the following shape, with r and s even.

$T =$ 

Proof of Theorem 1. To prove Theorem 1 we need to construct a non-zero tableau of shape $\lambda = [n - s, s]$ for $s \leq \frac{n}{2}$, with s even and $n = cd$, $c, d \geq 2$. First we construct a general tableau that covers most s. Suppose $s \leq \frac{c^* d^*}{2}$. We know s is even, so write $s = \mathsf{f}d^* + e$, where $0 \leq e < d^*$, e even. Since s, d^*, and e are even, this is possible by the Euclidean algorithm.

Let $T = \mathsf{f}U_1(d^*) \vee U_1(e)$. Note that the bound on s guarantees that $2(\mathsf{f} + 1) \leq c$ when $e > 0$, and $2\mathsf{f} \leq c$ when $e = 0$. This insures that there are at most c distinct elements in T. If there are fewer than c elements in T add all the remaining elements to the tail of T by joining the appropriate number of $V(d)$'s. Suppressing the tail elements from the U_1's and $V(d)$'s, T looks like:

$$T = \begin{array}{ccccccccc} d^* & d^* & \cdots & d^* & v & d & \cdots & d \\ \hline 1 & 3 & \cdots & 2\mathsf{f}-1 & 2\mathsf{f}+1 & 2\mathsf{f}+3 & \cdots & c \\ 2 & 4 & \cdots & 2\mathsf{f} & 2\mathsf{f}+2 & & & \end{array}$$

Theorem 8 shows T is non-zero, provided the weight-sets are disjoint. Since the tableaux are in maximal form, the weights must be disjoint by Lemma 3.4.9. This covers the majority of the s. The remaining tableaux will be constructed according to the parity of c and d.

<u>Case I:</u> (c, d even) In this case $\frac{c^*d^*}{2} = \frac{cd}{2}$, so T constructed above covers all partitions.

<u>Case II:</u> (d even, c odd) By the above construction, we have all tableaux with s up to $\frac{(c-1)d}{2}$. Thus we only need those even partitions with $s = \frac{cd-k}{2}$ for $0 \leq k \leq d-2$, $k \equiv d \pmod 4$. Take $T = \frac{\mathsf{c}-3}{2}U_1(d) \vee U_3(\textsc{a})$ for $0 \leq \textsc{a} \leq \frac{d}{2}$ with $\textsc{a}$ even. Then $s = \frac{c-3}{2}d + \textsc{a} + d = \frac{cd-d+2\textsc{a}}{2}$. Thus we have $k = d - 2\textsc{a}$, which ranges over the correct parameters. Since U_1 and U_3 are in maximal form, Lemma 3.4.9 implies disjointness and Theorem 8 shows T is non-zero.

<u>Case III:</u> (c even, d odd) Since $r = n - 2s = cd - 2s$ s we need $\lambda = [r+s, s]$ for $r \leq cd$ with $r \equiv cd \pmod 4$. It suffices to construct a non-zero tableau for $r < 4d$. When $r \geq 4d$, let $r' = r - 4d\mathsf{z}$ with $r' < 4d$. Then if we construct a $\lambda' = [s + r', s]$ tableau T' filled with d copies of $c - 4\mathsf{z}$ elements, we get the needed tableau by $T = T' \vee 4\mathsf{z}V(d)$. Hence we will take $r < 4d$.

When $c \equiv 0 \pmod 4$ then $r \equiv 0 \pmod 4$. Take $T = \frac{c-4}{4}U_2(d-1,1) \vee U_2(d - \frac{r}{4} - 1, 1)$. This construction gives the shape $\frac{c-4}{4}[2d, 2d] + [2d + \frac{r}{2}, 2d - \frac{r}{2}] = [\frac{cd}{2} + \frac{r}{2}, \frac{cd}{2} - \frac{r}{2}]$ as desired. The parameters of these tableaux are positive unless $r = 4d - 4$ since $r < 4d$, $r \equiv 0 \pmod 4$, and $d \geq 2$. If $r = 4d - 4$ then $d - \frac{r}{4} - 1 = 0$, so use $U_1(2) \vee 2V(d)$ instead of of $U_2(d - \frac{r}{4} - 1, 1)$.

For $c \equiv 2 \pmod 4$ we will assume $r < 2d$. When $2d \leq r < 4d$ let $r' = r - 2d$. Then can construct a $\lambda' = [r'+s, s]$ tableau T' with $c \equiv 0 \pmod 4$ and use $T = T' \vee 2V(d)$. Take $T = \frac{c-2}{4}U_2(d-1,1) \vee U_1(\frac{2d-r}{2})$ with $V(d)$'s as needed. Note that $cd \equiv r \pmod 4$ implies that $\frac{2d-r}{2}$ is even, while $r < 2d$ insures it is positive. So we get the shape $[\frac{cd}{2} + \frac{r}{2}, \frac{cd}{2} - \frac{r}{2}]$ as needed. Theorem 8 shows these T's are non-zero provided the weight-sets are disjoint, which follows from maximality.

Note that since $cd = n$, n even, then c or d is even. Thus we have constructed all cases. □

Although it is not directly apparent from this construction, c or d even is often a necessary requirement for any non-zero two row tableau with s even to exist. For instance, when $c = 3$ and $d = 7$, the shape $[11, 10]$ has s even, but all tableaux are zero by Theorem 9.

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