Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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Chapter 4

The Tableaux of $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$

In this chapter we completely classify and discuss those tableaux occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$. To begin with, we determine exactly which tableaux associated with this space have $\mathbf{q}_T \neq 0$. This is done in Section 4.1. In Section 4.2, we use this information and the results from Thrall's paper, [20], to determine precisely which partitions occur. Finally in Section 4.3, we construct a complete tableau basis for each the tableaux space $S^{\lambda,3} \cap M^{\lambda,3}$ associated to the irreducibles of $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$

4.1 Classification of $\mathbf{q}_T \neq 0$, for T filled with 1, 2, 3

Let T be a λ-tableau filled with b copies of the numbers 1, 2, and 3. By Lemma 3.1.5, T has at most three rows. By Remark 3.0.9 entry permutations, column permutations and column exchanges do not change whether $\mathbf{q}_T$ is non-zero. Hence we may take T to be in the following the form:

$$T = \begin{array}{ccccccc} \scriptstyle\mathrm{K} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{N} & \scriptstyle\mathrm{O} & \scriptstyle\mathrm{P} & \scriptstyle\mathrm{Q} \\ \hline 1 & 1 & 1 & 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 3 & & & \\ 3 & & & & & & \end{array} \qquad \begin{cases} \mathrm{K} + \mathrm{L} + \mathrm{M} + \mathrm{O} = b \\ \mathrm{K} + \mathrm{L} + \mathrm{N} + \mathrm{P} = b \\ \mathrm{K} + \mathrm{M} + \mathrm{N} + \mathrm{Q} = b \\ \mathrm{L} \geq \mathrm{M} \geq \mathrm{N} \geq 0 \end{cases}$$

However, since we know there are exactly b copies of every number in T, we may omit the tail and simply write T as $T = \begin{array}{cccc} \mathrm{K} & \mathrm{L} & \mathrm{M} & \mathrm{N} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$, retaining the condition $\mathrm{L} \geq \mathrm{M} \geq \mathrm{N} \geq 0$ and assuming the tail.

Theorem 9. With T as described above,

$$\mathbf{q}_T = 0 \iff \begin{cases} \mathrm{K}+\mathrm{L} \text{ odd} & \mathrm{L} > \mathrm{M} = \mathrm{N} \geq 0 \\ \mathrm{K}+\mathrm{N} \text{ odd} & \mathrm{L} = \mathrm{M} \geq \mathrm{N} \geq 0 \\ \mathrm{K}+\mathrm{M} \text{ even} & \mathrm{L} = \mathrm{M}+2,\ \mathrm{N} = \mathrm{M}\text{-}1 \geq 0 \\ \mathrm{K}+\mathrm{L} \text{ even} & \mathrm{L} = \mathrm{M}+1,\ \mathrm{M}{=}\mathrm{N} \geq 0 \end{cases}$$

Moreover, when $\mathbf{q}_T \neq 0$, it is non-zero by weight-set counting.

Proof. $\Longleftarrow$ Using Remark 3.0.9, to show $\mathbf{q}_T = 0$ it suffices to exhibit $\pi \in \mathcal{S}_3$, $\tau \in C_T$, $\epsilon(\tau) = -1$, such that $\pi\tau T = T$ up to an exchange of columns.

For $\mathrm{L} > \mathrm{M} = \mathrm{N}$, $\mathrm{L}+\mathrm{K}$ odd, take $\tau = \overset{\mathrm{K}}{(12)}_T \times \overset{\mathrm{L}}{(12)}_T \times \overset{\mathrm{M}}{()}_T \times \overset{\mathrm{N}}{()}_T$ and $\pi = (12)$. So $\epsilon(\tau) = (-1)^{\mathrm{K}+\mathrm{L}} = -1$. Then $\tau T = \begin{array}{cccc} \mathrm{K} & \mathrm{L} & \mathrm{M} & \mathrm{N} \\ \hline 2 & 2 & 1 & 2 \\ 1 & 1 & 3 & 3 \\ 3 & & & \end{array}$ and $\pi\tau T = \begin{array}{cccc} \mathrm{K} & \mathrm{L} & \mathrm{M} & \mathrm{N} \\ \hline 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$. Since $\mathrm{M}{=}\ \mathrm{N}$, exchanging columns gives T.

For $\mathrm{L} = \mathrm{M} \geq \mathrm{N} \geq 0$, $\mathrm{K}+\mathrm{N}$ odd use $\pi = (23)$, $\tau = \overset{\mathrm{K}}{(23)}_T \times \overset{\mathrm{L}}{()}_T \times \overset{\mathrm{M}}{()}_T \times \overset{\mathrm{N}}{(12)}_T$ and interchange columns L and M.

Now consider when $\mathrm{L} = \mathrm{M}{+}1$, $\mathrm{M} = \mathrm{N}$ with $\mathrm{K}+\mathrm{L}$ even. Then $T = \begin{array}{ccccc} \mathrm{K} & & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ \hline 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & & & & \end{array}$.

Write $T = T^* \vee T_1$ where $T^* = \begin{array}{cccc} \mathrm{K} & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$ and $T_1 = \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$. Let $T_2 = \begin{array}{cc} 2 & 3 \\ 1 & \end{array}$. Take $\sigma' \in R_{T_i}$ so $\sigma' T_2 = \begin{array}{cc} 3 & 2 \\ 1 & \end{array}$. Let $\pi' = (123) \in \mathcal{S}_3$ and $\tau^* = \overset{\mathrm{K}}{(132)} \times \overset{\mathrm{M}}{()} \times \overset{\mathrm{M}}{(12)} \times \overset{\mathrm{M}}{(12)} \in C_{T^*}$.

Then via reordering the columns by $\tilde{\sigma}$, we have $\tilde{\sigma}\tau^*\pi'\sigma'(T^* \vee T_2) = T^* \vee T_1$. Note that $\epsilon(\tau^*) = 1$ and $\tilde{\sigma}$, σ', π' commute with each other and all $\tau \in C_{T^*}$. Then

$$\begin{aligned}
\mathbf{q}_T &= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_T} \sigma\pi\,\epsilon(\tau)\tau T \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_T} \sigma\pi\,\epsilon(\tau)\tau(T^*\vee T_1) \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3} \sigma\pi[(\sum_{\tau\in C_{T^*}} \epsilon(\tau)\tau T^*) \vee (\sum_{\tau'\in C_{T_1}} \epsilon(\tau')\tau' T_1)] \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3} \sigma\pi[(\sum_{\tau\in C_{T^*}} \epsilon(\tau)\tau T^*) \vee (T_1 - T_2)] \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau(T^*\vee T_1 - T^*\vee T_2) \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_1 - \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau\tilde{\sigma}\tau^*\pi'\sigma' T^*\vee T_2 - \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\,\epsilon(\tau^*)\tau T^*\vee T_2 - \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 - \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 \\
&= 0
\end{aligned}$$

Note that where appropriate we commute $\tilde{\sigma}$, σ', and π' to combine with σ and π and reparameterize. The $\epsilon(\tau^*)$ factor arises from the reparamterization of $\tau\tau^*$.

Now consider when L $=$ M$+2$, N $=$ M-1 with K $+$ M even. So $T = \begin{array}{cccccc} \text{K} & \text{M-1} & & \text{M-1} & \text{M-1} \\ \hline 1 & 1 & 1\,1\,1\,1 & 1 & 2 \\ 2 & 2 & 2\,2\,2\,3 & 3 & 3 \\ 3 & & & & \end{array}$. Write $T = T^* \vee T_1$ where $T^* = \begin{array}{cccc} \text{K} & \text{M-1} & \text{M-1} & \text{M-1} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$ and $T_1 = \begin{array}{l} 1\,1\,1\,1\,2\,3\,3\,3 \\ 2\,2\,2\,3 \end{array}$.

We will use the idea of the previous case to show $\mathbf{q}_T = 0$. However, we need to sum over all $\epsilon(\tau)\tau T_1$ with τ in C_{T_1}. These tableaux have different symmetry relations with each other, so we will list all the $\epsilon(\tau)\tau T$ and their relations. Note that the tail is omitted for readability.

$$T_1 = \begin{matrix} 1\,1\,1\,1 \\ 2\,2\,2\,3 \end{matrix} \qquad T_2 = -\begin{matrix} 2\,1\,1\,1 \\ 1\,2\,2\,3 \end{matrix} \qquad T_3 = -\begin{matrix} 1\,2\,1\,1 \\ 2\,1\,2\,3 \end{matrix} \qquad T_4 = -\begin{matrix} 1\,1\,2\,1 \\ 2\,2\,1\,3 \end{matrix}$$

$$T_5 = -\begin{matrix} 1\,1\,1\,3 \\ 2\,2\,2\,1 \end{matrix} \qquad T_6 = \begin{matrix} 2\,2\,1\,1 \\ 1\,1\,2\,3 \end{matrix} \qquad T_7 = \begin{matrix} 2\,1\,2\,1 \\ 1\,2\,1\,3 \end{matrix} \qquad T_8 = \begin{matrix} 2\,1\,1\,3 \\ 1\,2\,2\,1 \end{matrix}$$

$$T_9 = \begin{matrix} 1\,2\,2\,1 \\ 2\,1\,1\,3 \end{matrix} \qquad T_{10} = \begin{matrix} 1\,2\,1\,3 \\ 2\,1\,2\,1 \end{matrix} \qquad T_{11} = \begin{matrix} 1\,1\,2\,3 \\ 2\,2\,1\,1 \end{matrix} \qquad T_{12} = -\begin{matrix} 1\,2\,2\,3 \\ 2\,1\,1\,1 \end{matrix}$$

$$T_{13} = -\begin{matrix} 2\,1\,2\,3 \\ 1\,2\,1\,1 \end{matrix} \qquad T_{14} = -\begin{matrix} 2\,2\,1\,3 \\ 1\,1\,2\,1 \end{matrix} \qquad T_{15} = -\begin{matrix} 2\,2\,2\,1 \\ 1\,1\,1\,3 \end{matrix} \qquad T_{16} = \begin{matrix} 2\,2\,2\,3 \\ 1\,1\,1\,1 \end{matrix}$$

For appropriate $\sigma' \in R_{T_1}$ we have the following relations. All permutation listed are from $\mathcal{S}_3$.

$$T_1 = -(123)\sigma' T_{14} \quad T_2 = -(123)\sigma' T_9 \quad T_3 = -(123)\sigma' T_6 \quad T_4 = -(123)\sigma' T_7$$

$$T_5 = (12)\sigma' T_{13} \qquad T_8 = (12)\sigma' T_8 \qquad T_{10} = (12)\sigma' T_{10} \qquad T_{11} = (12)\sigma' T_{11}$$

$$T_{12} = (23)\sigma' T_{15} \qquad T_{16} = (23)\sigma' T_{16}$$

We also have some relations on T^*, namely that for $\pi^* \in \mathcal{S}_3$ a transposition, then $\pi^* T^* = \tau_{\pi^*} T^*$ for $\tau_{\pi^*} \in C_{T^*}$ with $\epsilon(\tau_{\pi^*}) = (-1)^{\mathrm{K}+\mathrm{M}-1}$. Also $(132)T^* = \tau_{(132)} T^*$ for $\tau_{(132)} \in C_{T^*}$ with $\epsilon(\tau_{(132)}) = (-1)^{2(\mathrm{M}-1)} = 1$. Then

$$\begin{aligned}
&\sum_{\sigma \in R_T} \sum_{\pi \in \mathcal{S}_3} \sum_{\tau \in C_{T^*}} \pi\sigma\, \epsilon(\tau)\tau(T^* \vee (\pi^*)^{-1}\sigma' T_i) \\
&= \sum_{\sigma \in R_T} \sum_{\pi \in \mathcal{S}_3} \sum_{\tau \in C_{T^*}} \pi\sigma\, \epsilon(\tau)\tau(\pi^* T^* \vee \sigma' T_i) \\
&= \sum_{\sigma \in R_T} \sum_{\pi \in \mathcal{S}_3} \sum_{\tau \in C_{T^*}} \pi\sigma\, \epsilon(\tau)\, \epsilon(\tau_{\pi^*})\tau(T^* \vee \sigma' T_i) \\
&= \sum_{\sigma \in R_T} \sum_{\pi \in \mathcal{S}_3} \sum_{\tau \in C_{T^*}} \pi\sigma\, \epsilon(\tau)\, \epsilon(\tau_{\pi^*})\tau(T^* \vee T_i)
\end{aligned}$$

Hence if $T_i = -(123)\sigma' T_j$ or $T_i = \pi^* \sigma' T_j$ for π^* a transposition, then $\sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \pi\sigma\, \epsilon(\tau)\tau[(T^* \vee T_i) + (T^* \vee T_j)] = 0$. Also, if $T_i = \pi^*\sigma' T_j$, then $\sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \pi\sigma\, \epsilon(\tau)\tau(T^* \vee T_i) = 0$. So using the cancellations above, we have

$$\begin{aligned}
\mathbf{q}_T &= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_T} \pi\sigma\, \epsilon(\tau)\tau(T^* \vee T_1) \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3} \pi\sigma[(\sum_{\tau\in C_{T^*}} \epsilon(\tau)\tau T^*) \vee (\sum_{\tau'\in C_{T_1}} \epsilon(\tau')\tau T_1)] \\
&= \sum_{i=1}^{16}\sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \pi\sigma\, \epsilon(\tau)\tau(T^* \vee T_i) \\
&= 0
\end{aligned}$$

$\Longrightarrow$ To prove these are the only non-zero cases, we will use weight-set counting of Theorem 4 to show $\mathbf{q}_T \neq 0$ in the remaining cases. Given a weight, for every weight assignment $\pi \in \mathcal{S}_3$ we will count (with sign) the number of $\tau \in C_T$ such that $\pi\omega(T) = \omega(\tau T)$ and show the sum of these numbers is non-zero. In some cases it may be necessary to use $\omega(\tau' T)$ instead of $\omega(T)$ to show the weight-sum is non-zero. Since applying τ' affects only the sign of $\mathbf{q}_T$, this will not change our result.

Specifically we wish to show if:

$$\begin{array}{lll}
\text{K}+\text{L even} & \text{L} > \text{M} = \text{N} \geq 0 & \\
\text{K}+\text{N even} & \text{L} = \text{M} \geq \text{N} \geq 0 & \text{then } \mathbf{q}_T \neq 0 \\
\text{L} > \text{M} > \text{N} & &
\end{array}$$

unless L=M+1, M=N, K+L even, or L=M+1, M-1=N, K+M even.

Note that if $\text{K} = \text{L} = \text{M} = \text{N} = 0$ then T has only one row and $C_T = 1$. In this case $\mathbf{q}_{\sigma T} = \mathbf{q}_T$, so there is exactly one distinct T. Since $\text{K} + \text{N} = 0$, so the statement holds. Similarly, if $\text{L} = \text{M} = \text{N} = 0$, applying Lemma 3.2.6 gives K even. Hence we may assume $\text{L} > 0$.

We will use the first weight-set to illustrate the technique and notation of weight-set counting. This method of argument will be used extensively throughout the rest of the paper. We will list our weight-counting in a table of the following form:

$\omega_i = (\mathrm{L}, \mathrm{M}, \mathrm{N})$	Tableau	#	ϵ	τ	bound
(x, y, z)	T'	j	$(-1)^{\epsilon}$	$\overset{\mathrm{L}}{\tau_1} \times \overset{\mathrm{M}}{\tau_2} \times \overset{\mathrm{N}}{\tau_3}$	M=N

The column headings are: the weight being used, the form of the tableau, the number of τ corresponding to this weight, the sign of the τ, the form of τ, and any bounds required. The subsequent lines correspond to different weight assignments in line i. By (x, y, z), we mean $\omega_i(x, y, z) = (\mathrm{L}, \mathrm{M}, \mathrm{N})$. When M=N, there are j distinct τ such that $\tau = \overset{\mathrm{L}}{\tau_1} \times \overset{\mathrm{M}}{\tau_2} \times \overset{\mathrm{N}}{\tau_3}$. All the τ have sign $(-1)^{\epsilon}$ and τT is of the form T'.

The following is a standard formula that we will use in computing these weight sums. Its proof is a straightforward inductive application of Pascal's Identity. For notation purposes, we take $\binom{\mathsf{a}}{\mathsf{b}} = 0$ for $\mathsf{b} > \mathsf{a}$ or $\mathsf{b} < 0$. We also use the convention $\binom{0}{0} = 1$.

Lemma 4.1.1. $\binom{\mathsf{a}+\mathsf{h}}{\mathsf{b}} - \binom{\mathsf{a}}{\mathsf{b}} = \sum_{i=1}^{\mathsf{h}} \binom{\mathsf{a}+\mathsf{h}-i}{\mathsf{b}-1}$

Consider the cases where $\mathrm{K} \geq 0$ and at least $\mathrm{L} > 0$. We will apply weight-set counting to rows two and three. For the most part, the table should be self explanatory, though we will discuss the first weight-set table for clarity.

We'll start with $T = \begin{array}{cccc} \scriptstyle\mathrm{K} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{N} \\ \hline 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & 2 \\ 3 & & & \end{array}$, $\quad \omega_{2,3} = \left(\begin{smallmatrix} 0 & \mathrm{K}+\mathrm{L}+\mathrm{N} & \mathrm{M} \\ 0 & 0 & \mathrm{K} \end{smallmatrix} \right)$

$\omega_{2,3}$	Tableau	#	ϵ	τ	bound
$(1,2,3)$	$T = \begin{array}{cccc} \scriptstyle\mathrm{K} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{N} \\ \hline 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & 2 \\ 3 \end{array}$	1	$(-1)^0$	$\overset{\mathrm{K}}{()}_T \times \overset{\mathrm{L}}{()}_T \times \overset{\mathrm{M}}{()}_T \times \overset{\mathrm{N}}{()}_T$	
$(1,3,2)$	$T = \begin{array}{cccc} \scriptstyle\mathrm{K} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{N} \\ \hline 1 & 1 & 1 & 2 \\ 3 & 2 & 3 & 3 \\ 2 \end{array}$	1	$(-1)^{\mathrm{K}+\mathrm{N}}$	$\overset{\mathrm{K}}{(23)}_T \times \overset{\mathrm{L}}{()}_T \times \overset{\mathrm{M}}{()}_T \times \overset{\mathrm{N}}{(12)}_T$	$\mathrm{L} = \mathrm{M}$
$(2,1,3)$	$T = \begin{array}{cccc} \scriptstyle\mathrm{K} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{N} \\ \hline 2 & 2 & 31 & 2 \\ 1 & 1 & 13 & 3 \\ 3 \end{array}$	$\binom{\mathrm{M}}{\mathrm{N}}$	$(-1)^{\mathrm{K}+\mathrm{L}}$	$\overset{\mathrm{K}}{(12)}_T \times \overset{\mathrm{L}}{(12)}_T \times \overset{\mathrm{N}}{(12)}_T \times \overset{\mathrm{N}}{(12)}_T$	
$(2,3,1)$	$T = \begin{array}{cccc} \scriptstyle\mathrm{K} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{N} \\ \hline 2 & 2 & 1 & 2 \\ 3 & 1 & 3 & 3 \\ 1 \end{array}$	1	$(-1)^{\mathrm{L}+\mathrm{N}}$	$\overset{\mathrm{K}}{(123)}_T \times \overset{\mathrm{L}}{(12)}_T \times \overset{\mathrm{M}}{()}_T \times \overset{\mathrm{N}}{(12)}_T$	$\mathrm{L} = \mathrm{M}$
$(3,1,2)$	$T = \begin{array}{cccc} \scriptstyle\mathrm{K} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{N} \\ \hline 3 & 21 & 3 & 3 \\ 1 & 12 & 1 & 2 \\ 2 \end{array}$	$\binom{\mathrm{L}}{\mathrm{M}-\mathrm{N}}$	$(-1)^{\mathrm{L}+\mathrm{N}}$	$\overset{\mathrm{K}}{(132)}_T \times \overset{\mathrm{L}+\mathrm{N}-\mathrm{M}}{(12)}_T \times \overset{\mathrm{M}}{(12)}_T \times \overset{\mathrm{N}}{()}_T$	
$(3,2,1)$	$T = \begin{array}{cccc} \scriptstyle\mathrm{K} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{N} \\ \hline 3 & 1 & 3 & 3 \\ 2 & 2 & 1 & 2 \\ 1 \end{array}$	1	$(-1)^{\mathrm{K}+\mathrm{M}}$	$\overset{\mathrm{K}}{(13)}_T \times \overset{\mathrm{L}}{()}_T \times \overset{\mathrm{M}}{(12)}_T \times \overset{\mathrm{N}}{()}_T$	

To understand how these τ are obtained, first apply the permutations needed to have $\omega(x) = 0$, for the appropriate x. Additionally, apply necessary permutations so that row three of T has the correct weight. Once this is done, there will only be one column block whose permutations have not been specified. Apply the number of permutations needed to get the correct weight.

For the first line of this table, we see that T has the desired weight and any column permutations will change this. Thus there is exactly one τ and it is positive.

In line two, we have $\omega_3(2) = \mathrm{K}$ and $\omega_{2,3}(1) = 0$. Hence we must apply $(23)_T$ to column block K. Columns $\frac{1}{2}$ and $\frac{1}{3}$ cannot move since $\omega_{2,3}(1) = 0$. This gives L 2's in row two, so we must have L=M and apply $(12)_T$ to block N. When L=M, this completely determines τ, and it has sign $(-1)^{\mathrm{K}+\mathrm{N}}$. No such τ exists for $\mathrm{L} > \mathrm{M}$.

Line three counts $\omega_{2,3}(2,1,3)$. Since there can be no 2's in either rows two or three, τ must contain $\overset{\mathrm{K}}{(12)}_T \times \overset{\mathrm{L}}{(12)}_T \times \overset{\mathrm{N}}{(12)}_T$. As row three already contains K 3's, column block K needs no other permutation. Hence column $\genfrac{}{}{0pt}{}{1}{3}$ is the only column where τ has not yet been determined. As it stands, we already have K+L 1's in row two, hence only N more are required. Thus we need to apply $\overset{\mathrm{N}}{(13)}_T$ to $\genfrac{}{}{0pt}{}{1}{3}$. There are $\binom{\mathrm{M}}{\mathrm{N}}$ ways to choose which N columns move within the M block. Hence we get $\binom{\mathrm{M}}{\mathrm{N}}$ distinct τ of the form described.

In line four we apply $(132)_T$ to block K in order to have $\omega_3(1) = \mathrm{K}$ and $\omega_{2,3}(2) = 0$. Additionally, $\omega_{2,3}(2) = 0$ means we must apply $(12)_T$ to block L and $(12)_T$ to block N. This gives L 1's in row two, so we must have L=M and leave block M unchanged. In this case there is one such τ; it has sign $(-1)^{\mathrm{L+N}}$.

Line five is similar to line three. From the constraints $\omega_3(1) = \mathrm{K}$ and $\omega_{2,3}(3) = 0$ we have that τ contains $\overset{\mathrm{K}}{(123)}_T \times \overset{\mathrm{M}}{(12)}_T \times \overset{\mathrm{N}}{()}_T$. In order to have the correct weight for row two, we need L+N-M more 2's. There are $\binom{\mathrm{L}}{\mathrm{M\text{-}N}}$ ways to choose $\overset{\mathrm{L+N\text{-}M}}{(12)}_T$ from block L and all these τ have sign $(-1)^{\mathrm{L+N}}$.

The last line has a similar argument to line one. We need only apply $\tau = \overset{\mathrm{K}}{(13)}_T \times \overset{\mathrm{M}}{(12)}_T$ to get the correct weight and this is the only possible τ.

From this table we get the following sums.

$$1 + (-1)^{\mathrm{K+L}}\binom{\mathrm{M}}{\mathrm{M-N}} + (-1)^{\mathrm{L+N}}\binom{\mathrm{L}}{\mathrm{M-N}} + (-1)^{\mathrm{K+M}} \qquad \mathrm{L} \neq \mathrm{M} \qquad (4.1.1)$$

$$1 + (-1)^{\mathrm{K+N}} + (-1)^{\mathrm{K+L}}\binom{\mathrm{L}}{\mathrm{N}} + (-1)^{\mathrm{L+N}} + (-1)^{\mathrm{L+N}}\binom{\mathrm{L}}{\mathrm{N}} + (-1)^{\mathrm{K+L}} \qquad \mathrm{L} = \mathrm{M} \qquad (4.1.2)$$

<u>Case I:</u> ($\mathrm{L} > \mathrm{M} > \mathrm{N}$). Here, (4.1.1) equals zero only if $|\binom{\mathrm{L}}{\mathrm{M-N}} \pm \binom{\mathrm{M}}{\mathrm{M-N}}| = 0$ or 2. For it to equal 0, we must have $\mathrm{M} = \mathrm{N}$ or $\mathrm{L} = \mathrm{M}$.

To have $|\binom{\mathrm{L}}{\mathrm{M-N}} - \binom{\mathrm{M}}{\mathrm{M-N}}| = 2$, and (4.1.1) equal to zero, we must have $\mathrm{K} + \mathrm{M}$ even and $\mathrm{K} + \mathrm{N}$ odd. Applying Lemma 4.1.1, we get $\mathrm{L} = \mathrm{M} + 1$ or $\mathrm{L} = \mathrm{M} + 2$.

If $\mathrm{L} = \mathrm{M} + 1$, then (4.1.1) becomes $1 - \binom{\mathrm{M}}{\mathrm{M-N}} + \binom{\mathrm{M+1}}{\mathrm{M-N}} + 1$. Now $\binom{\mathrm{M+1}}{\mathrm{M-N}} - \binom{\mathrm{M}}{\mathrm{M-N}} = 2$ only if $\binom{\mathrm{M}}{\mathrm{M-N-1}} = 2$, that is $\mathrm{M} = 2$ and $\mathrm{N} = 0$. This contradicts $\mathrm{K} + \mathrm{M}$ even and $\mathrm{K} + \mathrm{N}$ odd. Hence (4.1.1) is non-zero for $\mathrm{L} = \mathrm{M} + 1$.

If $\mathrm{L} = \mathrm{M} + 2$ and $\mathrm{K} + \mathrm{M}$ even, (4.1.1) becomes $1 - \binom{\mathrm{M}}{\mathrm{M-N}} + (-1)^{\mathrm{M+N}}\binom{\mathrm{M+2}}{\mathrm{M-N}} + 1$.

For this expression to be zero we must have $\mathrm{N} = \mathrm{M} - 1$, in which case we've already shown $\mathbf{q}_T = 0$. Thus (4.1.1) is non-zero unless $\mathrm{L} = \mathrm{M} + 2$ and $\mathrm{N} = \mathrm{M} - 1$.

Finally, $\binom{\mathrm{L}}{\mathrm{M}-\mathrm{N}} + \binom{\mathrm{M}}{\mathrm{M}-\mathrm{N}} = 2$ only when $\mathrm{M} = \mathrm{N}$.

<u>Case II:</u> $(\mathrm{L} = \mathrm{M} \geq \mathrm{N}, \mathrm{L} > 0)$. We need only show the expression (4.1.2) is non-zero for $\mathrm{K}+\mathrm{N}$ even. From this, (4.1.2) becomes $2+(-1)^{\mathrm{K}+\mathrm{L}}(1+\binom{\mathrm{L}}{\mathrm{N}})+(-1)^{\mathrm{L}+\mathrm{N}}(1+\binom{\mathrm{L}}{\mathrm{N}})$. The parity of K and N is the same, so we reduce to determining when $1+(-1)^{\mathrm{K}+\mathrm{L}}((1+\binom{\mathrm{L}}{\mathrm{N}}) = 0$. Since $\binom{\mathrm{L}}{\mathrm{N}} > 0$, this cannot occur.

<u>Case III:</u> $(\mathrm{L} > \mathrm{M} = \mathrm{N})$. We want a non-zero weight sum for $\mathrm{K}+\mathrm{L}$ even. Under these conditions, expression (4.1.1) becomes $1 + 1 + (-1)^{\mathrm{L}+\mathrm{N}} + (-1)^{\mathrm{K}+\mathrm{N}}$. This is non-zero unless $\mathrm{K} + \mathrm{N}$ is odd.

It remains to show $\mathbf{q}_T \neq 0$ for $\mathrm{L} > \mathrm{N} = \mathrm{M}$, $\mathrm{K} + \mathrm{L}$ even, $\mathrm{K} + \mathrm{M}$ odd.

For this, consider the following weight-set counting on $T = \begin{array}{cccc} \scriptstyle\mathrm{K} & \scriptstyle\mathrm{L} & \scriptstyle\mathrm{M} & \scriptstyle\mathrm{N} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$ with $\omega_{2,3} = \begin{pmatrix} \mathrm{K}+\mathrm{L}+\mathrm{M}\text{-}1 & 0 & \mathrm{N}+1 \\ 0 & 0 & \mathrm{K} \end{pmatrix}$. Note that this weight-set is not the weight of T. We are counting which permutations τ will correspond to a weight assignment π, where $\omega_{2,3} = \pi\omega_{2,3}(\tau T)$.

$\omega_{2,3}$	Tableau	#	ϵ	τ	Bound
$(1,2,3)$	$T = \begin{array}{cccc} \scriptstyle K & \scriptstyle L & \scriptstyle M & \scriptstyle N \\ \hline 2 & 2 & 31 & 2 \\ 1 & 1 & 13 & 3 \\ 3 & & & \end{array}$	$\binom{M}{M-1}$	$(-1)^{K+L+M-1}$	$\overset{K}{(12)}_T \times \overset{L}{(12)}_T \times \overset{M-1}{(12)}_T \times \overset{N}{()}_T$	$M \geq 1$
$(1,3,2)$	$T = \begin{array}{cccc} \scriptstyle K & \scriptstyle L & \scriptstyle M & \scriptstyle N \\ \hline 3 & 21 & 3 & 3 \\ 2 & 12 & 1 & 2 \\ 1 & & & \end{array}$	$\binom{L}{L-1}$	$(-1)^{M+N+L-1}$	$\overset{K}{(132)}_T \times \overset{L-1}{(12)}_T \times \overset{M}{(12)}_T \times \overset{N}{(12)}_T$	$L \geq 1$
$(2,1,3)$	$T = \begin{array}{cccc} \scriptstyle K & \scriptstyle L & \scriptstyle M & \scriptstyle N \\ \hline 1 & 1 & 1 & 32 \\ 2 & 2 & 3 & 23 \\ 3 & & & \end{array}$	$\binom{N}{M-1}$	$(-1)^{M-1}$	$\overset{K}{()}_T \times \overset{L}{()}_T \times \overset{M}{()}_T \times \overset{M-1}{(12)}_T$	$M \geq 1$ $M-1 \leq N$
$(2,3,1)$	$T = \begin{array}{cccc} \scriptstyle K & \scriptstyle L & \scriptstyle M & \scriptstyle N \\ \hline 3 & 21 & 3 & 2 \\ 2 & 12 & 1 & 3 \\ 1 & & & \end{array}$	$\binom{L}{N+1-M}$	$(-1)^{K+1}$	$\overset{K}{(13)}_T \times \overset{N+1-M}{(12)}_T \times \overset{M}{(12)}_T \times \overset{N}{(12)}_T$	$M \leq N+1$ $N+1-M \leq L$
$(3,1,2)$	$T = \begin{array}{cccc} \scriptstyle K & \scriptstyle L & \scriptstyle M & \scriptstyle N \\ \hline 1 & 1 & 1 & 32 \\ 3 & 2 & 3 & 23 \\ 2 & & & \end{array}$	$\binom{N}{L-1}$	$(-1)^{K+N+L+1}$	$\overset{K}{(23)}_T \times \overset{L}{()}_T \times \overset{M}{()}_T \times \overset{N+1-L}{(12)}_T$	$L \geq 1$ $L-1 \leq N$
$(3,2,1)$	$T = \begin{array}{cccc} \scriptstyle K & \scriptstyle L & \scriptstyle M & \scriptstyle N \\ \hline 2 & 2 & 31 & 2 \\ 3 & 1 & 13 & 3 \\ 1 & & & \end{array}$	$\binom{M}{N+1-L}$	$(-1)^{N+1}$	$\overset{K}{(123)}_T \times \overset{L}{(12)}_T \times \overset{N+1-L}{(12)}_T \times \overset{N}{()}_T$	$L \leq N+1$ $N+1-L \leq M$

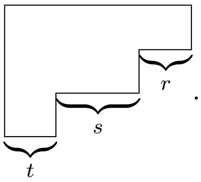
Since we previously dealt with the $L = N+1$ case, the last two lines of the table do not contribute. Hence the table gives the weight sum:

$$(-1)^{M-1}M + (-1)^{L-1}L + (-1)^{M-1}M + (-1)^{L+1}L \qquad L \neq N+1,\ M \neq 0 \tag{4.1.3}$$

Consider $L > M = N$, $K + L$ even, $K + N$ odd. For $L \neq N+1$, $M \neq 0$, (4.1.3) is $2((-1)^{M-1}M + (-1)^{L-1}L)$, which is non-zero as $L \neq M$. If $M = 0$, we must have K and L odd. This makes (4.1.3) $L + L$ which is not equal to zero since $L \neq 0$. For $L = N+1$ we've already shown $\mathbf{q}_T = 0$. Thus we determined all the non-zero tableaux of $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$. $\square$

4.2 The Irreducibles Partitions of $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$

For Theorem 2, we need to know which irreducibles occur (i.e., have non-zero multiplicity) in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$. We call a shape (or partition) non-zero if the multiplicity of the corresponding irreducible in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$ is non-zero. By Lemma 2.2.7, the non-zero partitions are those partitions where $\dim(S^{\lambda,3} \cap M^{\lambda,3}) > 0$. Since $\mathbf{q}_T$ generates $S^{\lambda,a} \cap M^{\lambda,a}$ we need only determine from Theorem 9 which partitions have non-zero tableaux. These partitions and their multiplicities were completely determined by Thrall in [20]. We will first derive them from Theorem 9 and then confirm it with Thrall's result.

We will only consider shapes $[\lambda]$ which are partitions of $3b$. Consider $T =$ 

. This labeling will be useful in our later constructions, so we will derive the non-zero shapes in terms of it. When T is of the form $\begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$, we have $t = \text{K}$, $s = \text{L}+\text{M}+\text{N}$ and the tail $r = 3b - 2s - 3t$. Since 1, 2, and 3 all occur $\text{K} = t$ times in the part of T above t, we will sometimes replace b with $b' = b - \text{K}$ when considering the multiplicity of elements in columns r and s, since the subtableau formed by columns r, s and the tail will have the elements 1, 2, and 3 occurring b' times each. In partition notation we have $\lambda = [r + s + t, s + t, t]$.

4.2.1 Non-Zero Partitions from Theorem 9

Definition 4.2.1. For our purposes, we call a partition (or shape) λ of n *required* if there is a non-zero λ-tableau T filled with $\frac{n}{3}$ copies of the elements 1, 2, and 3. These are precisely the tableaux determined in Theorem 9. Specifically, a required shape is one for which we must construct an appropriately filled non-zero tableau in order to prove Theorems 1 and 2. These shapes are explicitly determined in Theorem 10.

To determine the non-zero $[\lambda] = [r + s + t, s + t, t]$, we analyze the required partitions that correspond to non-zero tableaux in Theorem 9. We find that:

Theorem 10. The only partitions $[r+s+t, s+t, t]$ of $n = 3b$ which do not occur in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ are those with s or $r = 1$ as well as those having $s+t$ odd and s or $r \in \{0, 2, 4\}$. Equivalently, a partition is non-zero if, for r, $s \neq 1$, when r or s is in $\{0, 2, 4\}$, then $s + t$ is even.

Proof. For a given partition $\lambda = [r + s + t, s + t, t]$, we need only find $(\text{L}, \text{M}, \text{N})$ with $\text{L} + \text{N} + \text{N} = s$, $\text{K} = t$, and the conditions of Theorem 9 satisfied to show $\mathbf{q}_T \neq 0$. This shows that $\mathcal{S}^{\lambda}$ must occur in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$. Since all values of $t = \text{K}$ can occur when $\text{L} > \text{M} > \text{N}$, (if $\text{L} \neq \text{M} + 2$), we consider tableaux of this form. Given T, the elements 1, 2 and 3 will occur $b' = b - t$ times in the remaining columns L, M, N, and the tail of T. For each t we need to determine which s for $0 \leq s \leq \frac{b'}{2}$ yield required non-zero partitions. To do so we will take the following parameterizations of $(\text{L}, \text{M}, \text{N})$ and determine the corresponding s.

$$
\begin{aligned}
(\text{L}, \text{M}, \text{N}) &= (i+2, i+1, i), \text{ we get} & s &= 3i + 3 \text{ for} & 0 \leq i &\leq \frac{b'-3}{2} \\
&= (i+3, i+2, i) & s &= 3i + 5 & 0 \leq i &\leq \frac{b'-5}{2} \\
&= (i+4, i+3, i) & s &= 3i + 7 & 0 \leq i &\leq \frac{b'-7}{2}
\end{aligned}
$$

For a given parameterization of $(\text{L}, \text{M}, \text{N})$ by i, we have $s = \text{L} + \text{M} + \text{N}$. Since there are at most b' 1's in the r and s sections, we must have $\text{L} + \text{M}] \leq b'$ which gives the upper bound on i.

These parameterizations of $(\text{L}, \text{M}, \text{N})$ are non-zero by Theorem 9 since $\text{L} > \text{M} > \text{N}$ and $\text{L} \neq \text{M} + 2$. Moreover, the parameterizations cover all equivalence classes of s (mod 3). Hence this tells us that all partitions with $s \geq 5$ or $s = 3$ are non-zero, leaving aside the upper bound on s for now. When $s = 4$, the possibilities for $(\text{L}, \text{M}, \text{N})$ are $(4, 0, 0)$, $(2, 2, 0)$, and $(2, 1, 1)$ which are non-zero only when $s+t$, (i.e., K) is even. For $(3, 1, 0)$ the tableau is always zero since $\text{L} = \text{M}+2$, $\text{N} = \text{M}-1$. For $s = 2$ the only possibilities are $(2, 0, 0)$ and $(1, 1, 0)$. The non-zero conditions of Theorem 9 require $\text{K} = t$, and hence $s + t$, to be even in both cases. Similarly for $s = 0$, we must have K even for T to be non-zero, as shown in Lemma 3.2.6. For $s = 1$, the only possibility is $(\mathrm{L}, \mathrm{M}, \mathrm{N}) = (1, 0, 0)$, and such a tableau is always zero.

Determining the upper bounds on s corresponds to determining lower bounds on r. We will similarly give parameterizations of $(\mathrm{L}, \mathrm{M}, \mathrm{N})$ which will cover all equivalence classes of $r \pmod 3$. Using the equation $3b' - 2s = r$ we get the corresponding equations and lower bounds.

$$\text{For } s = 3i+3, \qquad i \le \frac{b'-3}{2} \quad \text{then} \qquad r \ge 3,\ r \equiv 0 \pmod 3 \tag{4.2.1}$$

$$s = 3i+5, \qquad i \le \frac{b'-5}{2} \qquad r \ge 5,\ r \equiv 2 \pmod 3 \tag{4.2.2}$$

$$s = 3i+7, \qquad i \le \frac{b'-7}{2} \qquad r \ge 7,\ r \equiv 1 \pmod 3 \tag{4.2.3}$$

This is the parameterization table for s rewritten in terms of r. So for $r \ge 5$, all partitions are non-zero, provided the conditions on s are met.

If $r = 0$, we must have $\mathrm{L} = \mathrm{M} = \mathrm{N} = \frac{b'}{2}$. By Theorem 9, this shape is non-zero only if $\mathrm{K}+\mathrm{L}$ is even. Thus only the shapes with $s+t = \mathrm{K}+\mathrm{L}+\mathrm{M}+\mathrm{N}$ even are non-zero. For $r = 1$ we get the constraints $\mathrm{L} + \mathrm{M} = \mathrm{L} + \mathrm{N} = b'$ and $\mathrm{M} + \mathrm{N} = b' - 1$. This means $\mathrm{M} = \mathrm{N}$, and $\mathrm{L}{=}\mathrm{M}{+}1$. These tableaux are always zero.

When $r = 2$, we must have either $\mathrm{L} = \mathrm{M}$ or $\mathrm{M} = \mathrm{N}$. These shapes will be non-zero if $\mathrm{K}+\mathrm{N}$ or $\mathrm{K}+\mathrm{L}$ is even respectively. In either case, $s + t$ is even.

For $r = 4$, we must have either $\mathrm{M} = \mathrm{N}$, $\mathrm{L} = \mathrm{M}$, or $(\mathrm{L}, \mathrm{M}, \mathrm{N}) = (\frac{b'+2}{2}, \frac{b'-2}{2}, \frac{b'-4}{2})$. In the first two cases, the non-zero conditions of Theorem 9 force $s + t$ to be even. In the last case, the tableau is always zero.

If $r = 3$ or 5 then $i = \frac{b'-3}{2}$ or $i = \frac{b'-5}{2}$ is an integer for b' odd so the parameterization listed works. For b' even, these partitions are not required since $3b' - 2s = r$, would make r even.

In addition, Theorem 9 shows the remaining partitions are zero.

$\square$

4.2.2 Partition Multiplicities according to Thrall

In [20], Thrall determines the partitions occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$ with multiplicity, which he calls $f(\lambda)$. If $\lambda = [\lambda_1, \lambda_2, \lambda_3]$, he gives the following method to compute $f(\lambda)$:

> To the minimum of $1 + \lambda_1 - \lambda_2$ and $1 + \lambda_2 - \lambda_3$, add whichever of -2, 0, $+2$ will give a result divisible by 3. If this result is even, divide by 6 to get $f(\lambda)$. If the result is odd, add or subtract 3 according to λ_2 being even or odd and then divide by 6 to get $f(\lambda)$.

Letting $\lambda = [r+s+t, s+t, t]$ we have $\min(1+\lambda_1-\lambda_2, 1+\lambda_2-\lambda_3) = 1+\min(r, s)$. Then $f(\lambda) = \frac{1+\min(r,s)+x+y}{6}$ where $x \in \{-2, 0, 2\}$ such that $1 + \min(r, s) + x \equiv 0 \pmod 3$, $y \in \{-3, 0, 3\}$ such that $1 + \min(r, s) + x + y \equiv 0 \pmod 6$, and $y \geq 0$ if $s + t$ even and $y \leq 0$ if $s + t$ odd.

Theorem 11. Writing $s = 6k + j$, $r = 6h + i$ with $0 \leq i, j \leq 5$, then by [20], the multiplicity of $\mathcal{S}^\lambda$ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$ is $f(\lambda)$, where

$$f(\lambda) = \begin{cases} k & s \leq r \quad i = 0,\ 2,\ 4, \quad s+t \text{ odd} \\ k & s \leq r \quad i = 1 \\ k+1 & s \leq r \quad i = 0,\ 2,\ 4, \quad s+t \text{ even} \\ k+1 & s \leq r \quad i = 3,\ 5 \\ h & r < s \quad i = 0,\ 2,\ 4, \quad s+t \text{ odd} \\ h & r < s \quad i = 1 \\ h+1 & r < s \quad i = 0,\ 2,\ 4, \quad s+t \text{ even} \\ h+1 & r < s \quad i = 3,\ 5 \end{cases}$$

Hence $f(\lambda) \neq 0$ for $r, s \neq 1$, provided $s + t$ is even when r or s is in $\{0, 2, 4\}$. This agrees with our results in Section 4.1.

4.3 Construction of Basis Tableaux for $c = 3$

Recall that the space $S^{\lambda,c} \cap M^{\lambda,c}$ is spanned by $\{\mathbf{q}_T\}$ where the T are λ-tableaux filled with the numbers 1 to c. By Lemma 2.2.7, $\dim(S^{\lambda,c} \cap M^{\lambda,c})$ equals the multiplicity of $\mathcal{S}^\lambda$ in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$. Given a partition λ of $n = 3b$, we want a set of tableaux $\{\mathcal{B}_p\}$ such that $\{\mathbf{q}_{\mathcal{B}_p}\}$ is linearly independent and that $|\{\mathcal{B}_p\}|$ is the multiplicity of the irreducible corresponding to λ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$. We call these $\mathcal{B}_p$ the basis tableaux for $c = 3$. These tableaux will be used in Chapter 9 for the proof of Theorem 3. We will build these tableaux from the following components:

$$\mathcal{M}_1 = \begin{array}{ccc} 1 & 2 & 3 \end{array} \qquad \mathcal{M}_2 = \begin{array}{cccccc} 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{array}$$

$$\mathcal{M}_3 = \begin{array}{cc} 3 & 3 \\ 1 & 1 \\ 2 & 2 \end{array} \qquad \mathcal{M}_4 = \begin{array}{cccc} 3 & 3 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & & & \end{array}$$

$$\mathcal{N}_{\text{A}} = \begin{array}{cccc} \text{A-1} & & & \text{A-1} \\ \hline 2 & 3 & 2 & 3 \\ 1 & 1 & & \end{array} \qquad 1 < \text{A} \leq \tfrac{n}{3} = b$$

In constructing these basis tableaux, we want tableaux filled with only the numbers 1, 2, and 3. We will use $\veebar$ to denote the joining of tableaux without renumbering them. For example, $\mathcal{M}_2 \veebar \mathcal{M}_1 = \begin{array}{ccccccccc} 2 & 2 & 3 & 3 & 3 & 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 & & & \end{array}$.

Let $\lambda = [r + s + t, s + t, t]$. When t is even, write $s = 6k + j$ and $r = 6h + i$, with $0 \leq i, j \leq 5$. Let $g = min(k, h)$. Since λ is a partition of $n = 3b$ we have $3t + 2s + r = 3b$. Hence $2j + i \equiv 0 \pmod 3$. Let $\delta = \frac{i-j}{3}$. (So $\delta = 0$ for $i = j$, $\delta = 1$ for $i = j + 3$, and $\delta = -1$ for $i = j - 3$.) When t is odd, we proceed as above, except let $s - 3 = 6k' + j'$, $\delta = \frac{i-j'}{3}$, and $g = min(k', h)$.

For $p = 1, 2, \ldots, g$, we define the basis tableaux:

$$\mathcal{B}_p = \tfrac{t}{2}\mathcal{M}_3 \veebar \mathcal{N}_{6p+j} \veebar (k-p)\mathcal{M}_2 \veebar (2h - 2p + \delta)\mathcal{M}_1 \qquad (t \text{ even})$$

$$\mathcal{B}_p = \tfrac{t-1}{2}\mathcal{M}_3 \veebar \mathcal{M}_4 \veebar \mathcal{N}_{6p+j'} \veebar (k'-p)\mathcal{M}_2 \veebar (2h - 2p + \delta)\mathcal{M}_1 \qquad (t \text{ odd})$$

Additionally, for j, $j' \neq 1$ we have the tableau $\mathcal{B}_0$, which is $\mathcal{B}_p$ with $p = 0$ under certain conditions.

$$
\begin{aligned}
\mathcal{B}_0 &= \tfrac{t}{2}\mathcal{M}_3 \veebar \mathcal{N}_j \veebar k\mathcal{M}_2 \veebar (2h+\delta)\mathcal{M}_1, && j > 1 && (t \text{ even})\\
\mathcal{B}_0 &= \tfrac{t}{2}\mathcal{M}_3 \veebar k\mathcal{M}_2 \veebar (2h+\delta)\mathcal{M}_1 && j = 0 && (t \text{ even})\\
\mathcal{B}_0 &= \tfrac{t-1}{2}\mathcal{M}_3 \veebar \mathcal{M}_4 \veebar \mathcal{N}_{j'} \veebar k'\mathcal{M}_2 \veebar (2h+\delta)\mathcal{M}_1, && j' > 1 && (t \text{ odd})\\
\mathcal{B}_0 &= \tfrac{t-1}{2}\mathcal{M}_3 \vee \mathcal{M}_4 \veebar k'\mathcal{M}_2 \veebar (2h+\delta)\mathcal{M}_1 && j' = 0 && (t \text{ odd})
\end{aligned}
$$

Note that if $\delta = -1$, then $\mathcal{B}_0$ exists only for $h \geq 1$ and $\mathcal{B}_g$ exists only for $g < h$.

To demonstrate that the $\{\mathcal{B}_p\}$ is a basis, we need to verify that they:

- Have the correct shape,
- Are non-zero and maximal,
- Are linearly independent,
- Span the space.

Shape: First consider the shape of these tableaux. We need to show these tableaux have shape $\lambda = [r+s+t, s+t, t]$. For t even:

$$
\begin{aligned}
\lambda_3(\mathcal{B}_p) &= \tfrac{t}{2}\lambda_3(\mathcal{M}_3) + \lambda_3(\mathcal{N}_{6p+j}) + (k-p)\lambda_3(\mathcal{M}_2) + (2h-2p+\delta)\lambda_3(\mathcal{M}_1)\\
&= \tfrac{t}{2} * 2 + (k-p) * 0 + (2h-2p+\delta) * 0\\
&= t
\end{aligned}
$$

$$
\begin{aligned}
\lambda_2(\mathcal{B}_p) =& \tfrac{t}{2}\lambda_2(\mathcal{M}_3) + \lambda_2(\mathcal{N}_{6p+j}) + (k-p)\lambda_2(\mathcal{M}_2) + (2h-2p+\delta)\lambda_2(\mathcal{M}_1)\\
&= t + 6p + j + (k-p) * 6 + (2h-2p+\delta) * 0\\
&= t + 6k + j\\
&= t + s
\end{aligned}
$$

$$\begin{aligned}
\lambda_1(\mathcal{B}_p) &= \tfrac{t}{2}\lambda_1(\mathcal{M}_3) + \lambda_1(\mathcal{N}_{6p+j}) + (k-p)\lambda_1(\mathcal{M}_2) + (2h-2p+\delta)\lambda_1(\mathcal{M}_1)\\
&= t + (6p+j)*2 + (k-p)*6 + (2h-2p+\delta)*3\\
&= t + 6k + j + 6h + j + 3\delta\\
&= t + s + 6h + j + 3\delta\\
&= t + s + 6h + i\\
&= t + s + r
\end{aligned}$$

When t is odd we have:

$$\begin{aligned}
\lambda_3(\mathcal{B}_p) &= \tfrac{t-1}{2}\lambda_3(\mathcal{M}_3) + \lambda_3(\mathcal{M}_4) + \lambda_3(\mathcal{N}_{6p+j'}) + (k'-p)\lambda_3(\mathcal{M}_2) + (2h-2p+\delta)\lambda_3(\mathcal{M}_1)\\
&= \tfrac{t-1}{2} * 2 + 1 + (k'-p)*0 + (2h-2p+\delta)*0\\
&= t
\end{aligned}$$

$$\begin{aligned}
\lambda_2(\mathcal{B}_p) &= \tfrac{t-1}{2}\lambda_2(\mathcal{M}_3) + \lambda_2(\mathcal{M}_4) + \lambda_2(\mathcal{N}_{6p+j'}) + (k'-p)\lambda_2(\mathcal{M}_2) + (2h-2p+\delta)\lambda_2(\mathcal{M}_1)\\
&= t - 1 + 4 + 6p + j + (k-p)*6 + (2h-2p+\delta)*0\\
&= t + 3 + 6k' + j'\\
&= t + s
\end{aligned}$$

$$\begin{aligned}
\lambda_1(\mathcal{B}_p) &= \tfrac{t}{2}\lambda_1(\mathcal{M}_3) + \lambda_1(\mathcal{M}_4) + \lambda_1(\mathcal{N}_{6p+j'}) + (k'-p)\lambda_1(\mathcal{M}_2) + (2h-2p+\delta)\lambda_1(\mathcal{M}_1)\\
&= t - 1 + 4 + (6p+j')*2 + (k-p)*6 + (2h-2p+\delta)*3\\
&= t + 3 + 6k' + j' + 6h + j' + 3\delta\\
&= t + s + 6h + j' + 3\delta\\
&= t + s + 6h + i\\
&= t + s + r
\end{aligned}$$

A similar computation works for the shape of $\mathcal{B}_0$. Hence these tableaux have the correct shape. Moreover, within each component, the same number of 1's, 2's and 3's were used. Hence the $\mathcal{B}_p$ have the correct number of 1's 2's and 3's.

Maximality: When t is even, a generic basis element (with the tail suppressed) looks like:

$$\mathcal{B}_p = \begin{array}{cccc} \text{t} & \text{A} & \text{B} & \text{C} \\ \hline 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \qquad \begin{array}{l} \text{A} = 2(k-p)+6p+j-1 \\ \text{B} = 2(k-p)+1 \\ \text{C} = 2(k-p) \end{array}$$

Then $\omega_{2,3}(\mathcal{B}_p) = \left(\begin{smallmatrix} t+6p+j+4(k-p) & 2(k-p) & 0 \\ 0 & t & 0 \end{smallmatrix}\right)$. Since $\text{B} > \text{C}$ and $\text{A} > \text{B}$ (for $p \geq 1$), this weight is maximal. $\mathcal{B}_p$ is also non-zero, since the only other possible weight assignment is $\left(\begin{smallmatrix} t+6p+j+4(k-p) & 0 & 2(k-p) \\ 0 & 0 & t \end{smallmatrix}\right)$. This has sign $(-1)^{t+2(k-p)} = 1$ as t is even.

When t is even, the tableau $\mathcal{B}_0$ is the same as $\mathcal{B}_p$ with $p = 0$ when $j > 1$. In this case $\text{A} > \text{B}$ and the above argument holds. There is no $\mathcal{B}_0$ for $j = 1$. When $j = 0$, we have (suppressing the tail):

$$\mathcal{B}_0 = \begin{array}{cccc} \text{t} & \text{A} & \text{A} & \text{A} \\ \hline 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \qquad \text{A} = 2k$$

Then $\omega_{2,3}(\mathcal{B}_0) = \left(\begin{smallmatrix} t+4k & 2k & 0 \\ 0 & t & 0 \end{smallmatrix}\right)$. This weight is clearly maximal. Although there are many different weight assignments possible for $\mathcal{B}_0$, all weight assignments are positive since each column block is even. Hence $\mathcal{B}_0$ is non-zero.

When t is odd, a generic basis element (with the tail suppressed) looks like:

$$\mathcal{B}_p = \begin{array}{cccc} \text{t} & \text{A} & \text{B} & \text{C} \\ \hline 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \qquad \begin{array}{l} \text{A} = 2(k'-p)+6p+j' \\ \text{B} = 2(k'-p)+2 \\ \text{C} = 2(k'-p)+1 \end{array}$$

Then $\omega_{2,3}(\mathcal{B}_p) = \left(\begin{smallmatrix} t+6p+j'+4(k'-p)+2 & 2(k'-p)+1 & 0 \\ 0 & t & 0 \end{smallmatrix}\right)$. Since $\text{B} > \text{C}$ and $\text{A} \geq \text{B}$ (for $p \geq 1$), this weight is maximal. $\mathcal{B}_p$ is also non-zero since the only other possible weight assignment is $\left(\begin{smallmatrix} t+6p+j'+4(k'-p)+2 & 0 & 2(k'-p)+1 \\ 0 & 0 & t \end{smallmatrix}\right)$. This has sign $(-1)^{t+2(k'-p)+1} = 1$ as t is odd.

When t is odd, the tableau $\mathcal{B}_0$ is the same as $\mathcal{B}_p$ with $p = 0$ when $j' > 1$. In this case $\textsc{a} \geq \textsc{b}$ and the above argument holds. There is no $\mathcal{B}_0$ for $j' = 1$ (since $\mathbf{q}_{\mathcal{B}_0}$ is zero when $j' = 1$). When $j' = 0$, we have (suppressing the tail):

$$\mathcal{B}_0 = \begin{array}{cccc} \mathrm{t} & \textsc{a} & \textsc{a} & \textsc{a} \\ \hline 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \qquad \textsc{a} = 2k' + 1$$

Then $\omega_{2,3}(\mathcal{B}_0) = \left(\begin{smallmatrix} t+4k'+2 & 2k'+1 & 0 \\ 0 & t & 0 \end{smallmatrix}\right)$. This weight is clearly maximal. Although there are many different weight assignments possible for $\mathcal{B}_0$, These weight assignments always move exactly two or four column blocks. Since the size of the column blocks is odd, all weight assignments are positive. Hence $\mathcal{B}_0$ is non-zero.

Linear Independence: To show the tableaux $\mathcal{B}_p$ are linearly independent, by Lemma 3.4.12 it suffices to show their max weights are distinct. First consider t even. Say $\mathsf{w}_{2,3}(\mathcal{B}_p) = \mathsf{w}_{2,3}(\mathcal{B}_{p'})$, with $p < p'$. Then we must have $t + 6p + j + 4(k - p) = 2(k - p')$, which forces $k = p = j = t = 0$. Then our partition is just $[n]$, so only one tableau is needed. If $\mathsf{w}_{2,3}(\mathcal{B}_p) = \mathsf{w}_{2,3}(\mathcal{B}_0)$ with $p > 0$, then we get $t + 4k = 2(k - p)$ which implies $t = k = p = 0$. Hence the tableaux are linearly independent. When t is odd, the max weights must be distinct, since an argument similar to the one above shows $t = 0$ which is not possible.

Span: Since the tableaux $\mathcal{B}_p$ are linearly independent they will span the space $S^{\lambda,3} \cap M^{\lambda,3}$ if $|\{\mathcal{B}_p\}| = m_\lambda$, where $m_\lambda = f(\lambda)$ as determined by [20]. (We listed $f(\lambda)$ explicitly in Theorem 11.)

First consider the case of t even. Given $\lambda = [r + s + t+, s + t, t]$ with t even, then m_λ depends on the relative sizes of r and s. For $s \leq r$, we need $k + 1$ tableaux for $j \neq 1$ and k tableaux for $j = 1$. When $s \leq r$, we have $g = k$. Thus we get k different $\mathcal{B}_p$'s and when $j \neq 1$ we have $\mathcal{B}_0$ as well. The restriction on the tableaux when $\delta = -1$ occurs only when $g = h$. However, then $h = k$ and $i = j - 3$ which contradicts $s \leq r$, so this case does not occur here. Hence we have a full set of basis tableaux.

If $r < s$, we have $g = h$. How many tableaux we have depends on δ. Note that $\mathcal{B}_h$ exists only for $\delta \neq -1$, and $\mathcal{B}_0$ requires $h > 0$ for $\delta = -1$. The number of tableaux needed according to Theorem 11 also depends on δ.

Since $s \equiv j \pmod 2$ and $3\delta = i - j$, when $\delta = 0$ we have $i \equiv s \pmod 2$. Then s is even for $i = 0$, 2, and 4, hence by [20] we need we need $h + 1$ tableaux for $i \neq 1$ and h tableaux for $i = 1$. Since $\delta \neq -1$, there are h distinct $\mathcal{B}_p$. We also have $\mathcal{B}_0$ when $i \neq 1$ since $i = j$. Hence we have a full set of basis tableaux.

If $\delta = 1$, all the tableaux described when $\delta = 0$ occur. Since $\delta = 1$, we have $i = j + 3$ which implies $i = 3, 4$ or 5. For $i = 4$ we need need h tableaux since s is odd, while for $i = 3, 5$ we need $h + 1$ tableaux. When $i = 4$ we have h different $\mathcal{B}_p$ (though no $\mathcal{B}_0$ since $j = 1$). When $i = 3, 5$ then $j = 0, 2$. Hence $\mathcal{B}_0$ exists and we obtain the complete set of basis tableaux.

For $\delta = -1$ we have $i = j - 3$, so $i = 0, 1, 2$. We need exactly h tableaux since either $i = 1$ or s is odd. However, we no longer have $\mathcal{B}_h$, so we get only $h-1$ tableaux from the $\mathcal{B}_p$. In addition, when $h \geq 1$ we have $\mathcal{B}_0$ since $j = i + 3 \geq 3$ and so $j > 1$. Hence we have h tableaux for $h \geq 1$. If $h = 0$ then no tableaux are needed since either $r = 1$ or $r = 0$ or $r = 2$ with s odd. Thus the correct number of tableaux is given.

Now consider the case when t is odd. When $s \leq r$ we need k tableaux for $j = 0$, 1, 2, or 4 and $k + 1$ tableaux for $j = 3$ or 5. If $j = 3$ or 5, then $k' = k = g$ so there are k tableaux $\mathcal{B}_p$, in addition to the tableau $\mathcal{B}_0$ (since $j' = j - 3 \neq 1$). If $j < 3$ then $g = k' = k - 1$, so there are $k - 1$ tableaux $\mathcal{B}_p$ in addition to the tableau $\mathcal{B}_0$. For $j = 4$, we have k tableaux $\mathcal{B}_p$, since $g = k' = k$. However since $j' = 1$, $\mathcal{B}_0$ does not exists. The restriction on the tableaux when $\delta = -1$ occurs only when $g = h$. However, then $h = k' = k$ and $i = j - 3$ which contradicts $s \leq r$, so this case does not occur here. Hence we have a full set of basis tableaux.

When $r < s$ we need h tableaux when $i = 1$. For $i = 0, 2$, and 4 we need h tableaux when s is even and $h + 1$ tableaux when s is odd. For $i = 3$ or 5 we need $h + 1$ tableaux. Note that $s \not\equiv j' \pmod 2$.

If $\delta = 0$ then $i = j'$. We have h tableaux $\mathcal{B}_p$, along with $\mathcal{B}_0$ for $j' \neq 1$. Hence we have h tableaux when $i = 1$, and $h+1$ tableaux otherwise. Since s is odd when $i = 0$, 2, or 4, this is the correct number.

If $\delta = 1$ then $i = j' + 3$, so $i = 3$, 4 or 5. If $i = 3$ or 5, then $j' \neq 1$ and there are $h+1$ tableaux as desired. If $i = 4$ then $j' = 1$, hence there are only h tableaux $\mathcal{B}_p$. However, only h tableaux are needed here since s is even.

If $\delta = -1$ then $i = j' - 3$, so $i = 0$, 1 or 2. Since s is even when $i \neq 1$, only h tableaux are required in this case. However, we no longer have $\mathcal{B}_h$, so there are $h - 1$ tableaux $\mathcal{B}_p$. In addition, when $h \geq 1$, we have $\mathcal{B}_0$, since $j' > 1$, so the correct number of tableaux are obtained. If $h = 0$ then no tableaux are needed since either $r = 1$ or $r = 0$ or $r = 2$ with $s+t$ odd. Thus the correct number of tableaux is given.

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