Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

Rebecca Vessenes

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

2004
(Submitted May 27, 2004)

Chapter 3

Theory of Tableaux Construction

Throughout this and subsequent chapters we will use T to represent an arbitrary tableau, σ an element of R_T, τ an element of C_T, and π an element of $\mathcal{S}_a$ for T filled with 1 to a.

Remark 3.0.9.

a) $\mathbf{e}_{\tau T} = \epsilon(\tau)\mathbf{e}_T$

b) $\mathbf{e}_{\pi T} = \pi\mathbf{e}_T$

c) $\mathbf{q}_{\pi T} = \pi\mathbf{q}_T = \mathbf{q}_T$

d) $\mathbf{q}_T = \Sigma_{\pi\in\mathcal{S}_a}\pi\mathbf{e}_T$

e) $\mathbf{q}_{\tau T} = \epsilon(\tau)\mathbf{q}_T$

These are standard computations, which are discussed in [7] and [18]. This remark shows that we can ignore the effects of permuting entries when constructing the tableaux. Also, we may order the columns however we choose at the cost of a sign.

3.1 Filling Tableaux

Definition 3.1.1. In T, the *weight* of a number x in row i, denoted $\omega_i(x)$, is the number of times x occurs in row i of T. When T is not clear from context, we write $\omega_i(x|T)$ in place of $\omega_i(x)$. We extend this so that $\omega_i(x_1,\ldots,x_j|T) = (\omega_i(x_1|T),\ldots,\omega_i(x_j|T))$.

Implicitly, we take $\omega_i(T) = (\omega_i(1), \ldots, \omega_i(a))$, which is called the *row-weight* of row i of T. Similarly, $\omega(x_j) = \begin{pmatrix} \omega_1(x_j) \\ \vdots \\ \omega_\ell(x_j) \end{pmatrix}$ is the *weight vector* of x_j of T. Hence $\omega(T)$ is the matrix corresponding to $\omega_i(j|T)$. Note that row permutations do not effect weight, so $\omega(\sigma T) = \omega(T)$.

Example 3.1.2. $T = \begin{matrix} 1\ 2\ 2 \\ 3\ 4\ 4 \\ 5\ 5 \end{matrix}$. We have $\omega(T) = \left(\begin{smallmatrix} 1&2&0&0&0 \\ 0&0&1&2&0 \\ 0&0&0&0&2 \end{smallmatrix}\right)$. From this we can read that $\omega_2(3) = 1$.

Weights for a tableau are only comparable with tableaux of the same shape and content. Recall that $\mathcal{W}^{\lambda,a}$ is the set of all λ-tableaux with content $[b^a]$. Let N^a be the set of all a-tuples w with non-negative integer entries and $N^{a,\ell}$ those $\ell \times a$ matrices. Then we can view the row-weight function as a linear operator $\omega_i : \mathbb{Z}[T|T \in \mathcal{W}^{\lambda,a}] \to \mathbb{Z}[w|w \in N^a]$, where $\{w|w \in N^a\}$ is a $\mathbb{Z}$-basis, or correspondingly, $\omega : \mathbb{Z}[T|T \in \mathcal{W}^{\lambda,a}] \to \mathbb{Z}[w|w \in N^{a,\ell}]$, with $\ell = \ell(\lambda)$.

This means we treat weights like linearly independent basis in $\mathbb{Z}[w|w \in N^{a,\ell}]$. Hence $\omega(T_1 + aT_2) = \omega(T_1) + a\omega(T_2)$. If $T_1 = \begin{matrix} 1\ 1 \\ 2 \end{matrix}$ and $T_2 = \begin{matrix} 2\ 1 \\ 1 \end{matrix}$ then $\omega_1(T_1) = (2,0)$ and $\omega_1(T_2) = (1,1)$. So $\omega_1(T_1 + T_2) = (2,0) + (1,1)$ and $\omega_1(T_1 + T_1) = (2,0) + (2,0) = 2 \cdot (2,0)$. For convenience, we take $\omega(0) = 0$.

Definition 3.1.3. Let $rowsum_T(i)$ be the sum of all the entries in row i of T, $\sum_j \omega_i(j)j$.

Lemma 3.1.4. $\mathbf{e}_T = 0$ iff T has a repeat entry in a column.

Proof. $\Rightarrow$ It suffices to show that if T has no repetitions within a column, then $\mathbf{e}_T \neq 0$. By Remark 3.0.9 column permutations only change the sign of $\mathbf{e}_T$, so without loss of generality, we may assume the columns of T are strictly increasing. If $\mathbf{e}_T = 0$ then there exists $\sigma \in R_T$, $\tau \in C_T$ such that $\epsilon(\tau) = -1$ and $\sigma\tau T = T$, and we say that T cancels in the summation. Since $\sigma\tau T = T$, we must have $\omega(\sigma\tau T) = \omega(T)$ and so $\omega(\tau T) = \omega(T)$. This implies that $rowsum_T = rowsum_{\tau T}$ for all rows.

Let r be the first row in which τ moves an entry of T. Let $\alpha_{i_1}, \ldots, \alpha_{i_k}$ be the entries of row r moved by τ. Say τ moves β_j to α_{i_j}. Since r is the first row moved by τ, $\beta_j > \alpha_{i_j}$. Then $rowsum_T(r) = \sum_{j=1}^{k} \alpha_{i_j} + \sum_{i \neq i_j} \alpha_i < \sum_{j=1}^{k} \beta_j + \sum_{i \neq i_j} \alpha_i = rowsum_{\tau T}(r)$. Contradiction. Therefore $\mathbf{e}_T \neq 0$.

$\Leftarrow$ If T has a repeated entry in a column, then there exists a transposition $\tau \in C_T$ such that $\tau T = T$. Hence $\mathbf{e}_T = \mathbf{e}_{\tau T} = \epsilon(\tau)\mathbf{e}_T = -\mathbf{e}_T$ and so $\mathbf{e}_T = 0$.

□

Knowing that any tableau with repeated numbers in a column makes $\mathbf{e}_T = 0$ is very useful for our construction of non-zero tableaux. We summarize this fact as follows:

Lemma 3.1.5. Any tableau T filled with the numbers 1 to a having more than a rows will have $\mathbf{e}_T = 0$ and $\mathbf{q}_T = 0$, as will any T having repetitions within a column.

Hence every tableau T filled with the numbers 1, 2, and 3, with $\mathbf{q}_T \neq 0$, will have at most three rows and all column entries will be distinct. Due to this, from now on we will assume all tableaux have distinct column entries.

Notation: Many of the tableaux we construct will have multiple identical columns. We call a group of such columns, a *column block*. For both clarity and space we denote a column block by one copy of the column with the number of repetitions listed above. If the number of column copies is omitted, it is assumed to be one. For example, $T = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}$ would be denoted by $T = \begin{array}{c} 3 \\ \hline 1 \\ 2 \end{array}$, while $T = \begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$ has K copies of $\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$ and L copies of $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$, M copies of $\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$, and N copies of $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$. We call the columns of T having only one entry the *tail* of T. When specifying T by column blocks, we may omit the tail, provided the content of T is known. The rest of T is called the *body*. For instance, if $T = \begin{array}{ccccc} 2 & & & & \\ \hline 1 & 1 & 2 & 2 & 3 \\ 3 & 2 & 4 & & \\ 4 & & & & \end{array}$ and we know that T has content

$[3,3,3,3]$, we can just list the body, $T = \begin{array}{l} \;\;2 \\ \hline 1\;1\;2 \\ 3\;2\;4 \\ 4 \end{array}$ instead. It is assumed that any entries not specified are contained in the tail.

We also use this abbreviated notation when describing elements of C_T. We write $\tau \in C_T$ as a direct product of permutations on the column blocks. Since the only permutation possible on the tail is the identity, we omit the permutations corresponding to the tail. Hence for T listed above, τ is of the form $\overset{2}{\tau_1} \times \tau_2 \times \tau_3$. We write $\overset{\kappa}{\tau_i}$ if the same permutation τ_i is to be applied to κ columns within a column block. When κ is less than the size of the column block, we understand $\overset{\kappa}{\tau_i}$ to mean that τ_i is applied to all κ of the columns (determined by context) and the identity permutation is applied to the remaining columns within the block. For instance, on the previous T, there are two permutations of the form $\overset{1}{(13)_T} \times ()_T \times (12)_T$, which produce $\begin{array}{l} 4\;1\;1\;4 \\ 3\;3\;2\;2 \\ 1\;4 \end{array}$ and $\begin{array}{l} 1\;4\;1\;4 \\ 3\;3\;2\;2 \\ 4\;1 \end{array}$.

3.2 Showing Tableaux are Non-Zero

Definition 3.2.1. A tableau T is said to be *non-zero* if $\mathbf{q}_T \neq 0$. Two tableaux are said to be *distinct* if $\mathbf{q}_{T_1} \neq \pm\mathbf{q}_{T_2}$, otherwise T_1 and T_2 are said to be equivalent.

Since $\mathbf{q}_T$ involves many summands, showing $\mathbf{q}_T \neq 0$ by direct summation is not practical. Instead, we use a technique called *weight-set counting*. Weight-set counting involves summing only those tableaux with a given weight; if that sum is non-zero, the entire $\mathbf{q}_T$ summation must be non zero.

Definition 3.2.2. Given a tableau T, let $\omega(T) = (\overline{x}_1, \ldots, \overline{x}_a)$, where $\overline{x}_j$ is the weight vector $\omega(j|T) = \overline{x}_j$ of the element j in T. A *weight assignment* of T is a pairing between the set of elements of T with the multiset of weight vectors of $\omega(T)$. We denote the pairing of the weight vector $\overline{x}_j$ with the element k_j by $\omega(k_j|T^*) = \overline{x}_j$. (The T^* represents a possible tableau which has the weight of k_j being $\overline{x}_j$.) Note that if the vectors $\overline{x}_j$ and $\overline{x}_{j'}$ are equal, the weight assignment pairing k_j with $\overline{x}_j$ is the same as the weight assignment pairing k_j with $\overline{x}_{j'}$. What matters in a weight assignment is the vector paired with each element, not how we label the vectors. We usually indicate a weight assignment by writing $(k_1, \dots, k_a)$, by which we mean $\omega(k_1, \dots, k_a|T^*) = (\overline{x}_1, \dots, \overline{x}_a) = \omega(T)$.

Given a permutation π we can create a weight assignment by assigning the element $\pi(k)$ to $\overline{x}_k$, since $\omega(\pi(k)|\pi T) = \omega(k|T) = \overline{x}_k$. Similarly, given such pairing $(k_1, \dots, k_a)$ we can construct a permutation π by taking $\pi = (k_1, \dots, k_a)$ in one-line notation.

For example, let $T = \begin{array}{lll} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & & \end{array}$. Then $\omega(T) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. The weight assignment $(3, 1, 2)$ means $\omega(3, 1, 2|T^*) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ for some tableau T^*. This weight assignment corresponds to the weight permutation $\pi = (132)$ in cycle notation (from left to right).

Note, however, that such a listing $(k_1, \dots, k_a)$ of a weight assignment is not necessarily unique. For instance, if $\overline{x}_1 = \overline{x}_2$ then $(k_1, k_2, k_3, \dots k_a)$ and $(k_2, k_1, k_3, \dots k_a)$ represent the same weight assignment (pairing) but give rise to different permutations. The numbers of such permutations corresponding to the same weight assignment depends only on the vector symmetries of $\omega(T)$. This number is denoted $s(\omega(T))$.

For example, if $T = \begin{array}{lllll} 1 & 1 & 2 & 2 & 3 \end{array}$, then $\omega_T = (2, 2, 1)$. There are three distinct weight assignments of T corresponding to which of the three elements is assigned a weight of 1. Since there are two permutations arising from such an assignment (for instance, $\omega(213|T) = (2, 2, 1)$ as well) we have $s(\omega(T)) = 2$.

Definition 3.2.3. A weight assignment $(k_1, \dots, k_a)$ is *valid* for T if there exists $\tau \in C_T$ such that $\omega(k_1, \dots, k_a|\tau T) = \omega(T)$, i.e., $T^* = \tau T$. If this happens we say τ is valid for $(k_1, \dots, k_a)$, otherwise τ is *invalid.* Given a valid τ we say τ is *positive* if $\epsilon(\tau) = 1$ and *negative* if $\epsilon(\tau) = -1$.

Example 3.2.4. Let $T = \begin{array}{lllllll} 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 3 & 1 & 1 & & & & \\ 4 & 4 & & & & & \end{array}$. We have $\omega(T) = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Recall that for C_T we use the labelling $\begin{array}{lll} 1_T & 1_T & 1_T \\ 2_T & 2_T & 2_T \\ 3_T & 3_T & \end{array}$ (the tail is omitted since all column permutations on it are trivial). Now $(2,3,1,4)$ is a valid weight assignment since $\tau = (12)_T \times (12)_T \times (12)_T \in C_T$ has $\tau T = \begin{matrix} 3\,1\,1\,2\,3\,3\,4 \\ 1\,2\,2 \\ 4\,4 \end{matrix}$ with $\omega(\tau T) = \left(\begin{smallmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{smallmatrix}\right)$, So $\omega(2,3,1,4|\tau T) = \omega(T)$. The weight assignment, $(2,3,1,4)$ corresponds to the permutation $\pi = (123)$ in cycle notation, meaning $\omega(1,2,3,4|\tau T) = \omega(1,2,3,4|\pi T)$

However, $(1,4,3,2)$ is not a valid weight assignment since then we must have $\omega_3(2|\tau T) = 2$ for some τ, but there is no column permutation that will put two 2's in the third row. We will make frequent use of weight assignments in order to determine when T is non-zero.

Definition 3.2.5. Given T, consider the following functions:

- $\mathcal{P}(\pi(T))$ = the number of $\tau \in C_T$ such that $\epsilon(\tau) = 1$ and $\omega(\pi^{-1}(1), \ldots, \pi^{-1}(a)|\tau T) = \omega(T)$.
- $\mathcal{N}(\pi(T))$ = the number of $\tau \in C_T$ such that $\epsilon(\tau) = -1$ and $\omega(\pi^{-1}(1), \ldots, \pi^{-1}(a)|\tau T) = \omega(T)$.
- $\mathcal{P}(T) = \sum_\pi \mathcal{P}(\pi(T))$, where π correspond to distinct weight assignments of $\omega(T)$
- $\mathcal{N}(T) = \sum_\pi \mathcal{N}(\pi(T))$, where π correspond to distinct weight assignments of $\omega(T)$

Theorem 4. (Weight-set Counting) If $\mathbf{q}_T = 0$ then $\mathcal{P}(T) = \mathcal{N}(T)$.

Proof. Let $\mathcal{D}$ be the set permutations corresponding to distinct weight assignments of T.

$$\mathbf{q}_T = 0 \tag{3.2.1}$$

$$\Rightarrow \sum_\pi \sum_\sigma \sum_\tau \epsilon(\tau)\omega(\sigma\pi\tau T) = 0 \tag{3.2.2}$$

$$\Rightarrow \sum_\pi \sum_\tau \epsilon(\tau)\omega(\pi\tau T) = 0 \tag{3.2.3}$$

$$\Rightarrow \sum_\pi \sum_\tau \epsilon(\tau)\omega(\pi\tau T) = 0 \text{ s.t. } \omega(1, \ldots, a|\pi\tau T) = \omega(T) \tag{3.2.4}$$

$$\Rightarrow \sum_{\pi} \sum_{\tau} \epsilon(\tau) = 0 \text{ s.t. } \omega(1, \ldots, a|\pi\tau T) = \omega(T) \tag{3.2.5}$$

$$\Rightarrow \sum_{\pi} \sum_{\tau} \epsilon(\tau) = 0 \text{ s.t. } \omega(\pi^{-1}(1), \ldots, \pi^{-1}(a)|\tau T) = \omega(T) \tag{3.2.6}$$

$$\Rightarrow s(\omega(T)) \sum_{\pi \in \mathcal{D}} (\mathcal{P}(\pi T) - \mathcal{N}(\pi T)) = 0 \tag{3.2.7}$$

$$\Rightarrow s(\omega(T))(\mathcal{P}(T) - \mathcal{N}(T)) \tag{3.2.8}$$

$$\Rightarrow \mathcal{P}(T) - \mathcal{N}(T) = 0 \tag{3.2.9}$$

If $\mathbf{q}_T = 0$, taking the weight of both sides gives Eq. (3.2.2). Since row permutations do not effect weights, we can reduce to Eq. (3.2.3). As distinct weights can not cancel, we can consider only those tableaux with the same weight as $\omega(T)$, hence we must have Eq. (3.2.4). Since we are only summing over tableaux of a fixed weight we may drop the weight from the sum and simply add the sign of τ, for Eq. (3.2.5). By definition of weight assignments, $\omega(1, \ldots, a|\pi\tau T) = \omega(\pi^{-1}(1), \ldots, \pi^{-1}(a)|\tau T)$ hence we get Eq. (3.2.6). Restricting the sum to distinct weight-sets gives the factor of $s(\omega(T))$, and if we split over the sign of τ' we get Eq. (3.2.7), which by definition is the same as Eq. (3.2.8). Factoring out $s(\omega(T))$ yields Eq. (3.2.9). Thus $\mathcal{P}(T) = \mathcal{N}(T)$. $\square$

We can also write Theorem 4 as:

Corollary 5. Let $\mathcal{A} = \{\tau|\omega(\tau T) = \omega(\pi T) \text{ for some } \pi \in \mathcal{S}_a\}$. If $\sum_{\tau \in \mathcal{A}} \epsilon(\tau) \neq 0$, then $\mathbf{q}_T \neq 0$ by weight-set counting on $\omega(T)$.

An easy application of this theorem is the following useful lemma.

Lemma 3.2.6. If T is a tableau consisting of a single column block (and an arbitrary tail), for instance, $T = \frac{\overset{\kappa}{}}{\begin{smallmatrix}1\\2\end{smallmatrix}}$, then $\mathbf{q}_T \neq 0$ iff κ even.

Proof. We have $\mathsf{w} = \omega(T) = \left(\begin{smallmatrix}\kappa & 0\\ 0 & \kappa\end{smallmatrix}\right)$. So there are two distinct weight assignments for T, namely, $\omega(1, 2|T^*) = \mathsf{w}$ and $\omega(2, 1|T^*) = \mathsf{w}$. We wish to determine for which τ does $T^* = \tau T$ satisfy one of these equations. For $\omega(1, 2|\tau T) = \mathsf{w}$, we must have κ 2's in the second row, hence none of the columns of T may move. The only τ satisfying this is $\tau = \overset{\kappa}{()}_T$, which is positive. For $\omega(2,1|\tau T) = \mathsf{w}$, we must have κ 1's in the second row, thus τ must exchange row 1 and row 2 for every column in T. Thus $\tau = \overset{\kappa}{(12)}_T$ and $\epsilon(\tau) = (-1)^{\kappa}$. Hence when κ is even, we have $\mathcal{P}(T) = 2$ and $\mathcal{N}(T) = 0$, so $\mathbf{q}_T$ is non-zero. The same idea applies for T having more than two rows.

When κ is odd, however, we have $\mathcal{P}(T) = \mathcal{N}(T)$ so the Theorem 4 does not apply. Instead, let $\tau = \overset{\kappa}{(12)}_T$ and let $\pi \in \mathcal{S}_a$ be the corresponding entry transposition (in our example $\pi = (12)$). Then $\pi\tau T = T$ and $\epsilon(\tau) = -1$. Thus $\mathbf{q}_{\pi\tau t} = \pi\mathbf{q}_{\tau T} = \mathbf{q}_{\tau T} = \epsilon(\tau)\mathbf{q}_T = -\mathbf{q}_T$. So $\mathbf{q}_T = 0$. □

It is also true that if $\mathbf{q}_T \neq 0$, then it is non-zero by weight-set counting on some weight $\omega(\tau T)$. That is we can't have $\mathcal{P}(\tau T) = \mathcal{N}(\tau T)$ for all $\tau \in C_T$ and still have $\mathbf{q}_T \neq 0$.

Theorem 6. If $\mathbf{q}_T \neq 0$, then it is non-zero by weight set counting on τT for some $\tau \in C_T$.

Proof. Recall that $\mathbf{q}_T = \sum_\pi \sum_\sigma \sum_\tau \pi\sigma\, \epsilon(\tau)\tau T$. Since a tableau may be written in multiple ways (such as $T' = \pi\sigma\tau T = \pi'\sigma'\tau' T$), we need to be careful of how we denote a tableau. Consider the terms of $\mathbf{q}_T$ partitioned in to classes $\{\tau T\} = \{\pi\sigma\tau T | \pi \in \mathcal{S}_a,\ \sigma \in R_T\}$, using $\{\tau T\}$ as the class representative. Note that all tableaux in a given class have the same sign and generic weight, i.e., the same weight modulo the action of $\mathcal{S}_a$. Hence if $\tau T = \pi'\sigma'\tau' T$ for some π', σ', τ', with $\epsilon(\tau\tau') = -1$ then $\pi\sigma\tau T = \pi\pi'\sigma\sigma'\tau' T$, so the classes are equal set wise, but of opposite sign. This holds for any element of the class, not just τT.

This means we may view the tableaux which cancel in $\mathbf{q}_T$ as a matching between equal classes of opposite sign. So if $\mathbf{q}_T \neq 0$, there is a set of tableaux $\mathcal{T}$ which are not canceled in $\mathbf{q}_T$. Now for any $\tau T \in \mathcal{T}$ either the weight of τT does not cancel, (that is $\omega(\tau T) \neq \overline{\pi}\omega(\pi'\sigma'\tau' T)$ for all $\overline{\pi} \in \mathcal{S}_a$ and all $\pi'\sigma'\tau' T \in \mathcal{T}$ with $\epsilon(\tau\tau') = -1$) or the weight cancels with some $\pi'\sigma'\tau' T$ having $\epsilon(\tau\tau') = -1$ (that is $\omega(\tau T) = \overline{\pi}\omega(\pi'\sigma'\tau' T)$ for some $\pi'\sigma'\tau' T \in \mathcal{T}$). If the weight does not cancel for some τT, the $\mathbf{q}_T \neq 0$ by weight set counting on τT. Assume all the weights do cancel. Then $\omega(\tau T) = \overline{\pi}\omega(\pi'\sigma'\tau' T)$, so for every row, τT and $\overline{\pi}\pi'\sigma'\tau' T$ have the same number of each symbol. Hence there exists $\overline{\sigma} \in R_T$ such that $\tau T = \overline{\pi}\pi'\overline{\sigma}\sigma'\tau' T$. But then, since $\epsilon(\tau\tau') = -1$, the classes $\{\tau T\}$ and $\{\tau' T\}$ will cancel in $\mathbf{q}_T$ contradicting τT, $\pi'\sigma'\tau' T \in \mathfrak{T}$. Hence $\mathbf{q}_T \neq 0$ by weight set counting on some τT. □

We will not need Theorem 6 for our results; it is included for theoretical interest and completeness. Theorem 4 is used quite heavily, however. For instance, it allows use to directly establish the multiplicities of the irreducible characters in $1^{\mathcal{S}_{2b}}_{\mathcal{S}_b \wr \mathcal{S}_2}$.

Theorem 7. The only irreducible characters occurring in $1^{2b}_{\mathcal{S}_b \wr \mathcal{S}_2}$ are those corresponding to all partitions $\lambda = [\lambda_1, \lambda_2]$ of $2b$ where λ_i is even. Moreover, these characters occur with multiplicity 1.

Proof. By Remark 2.2.8 we need only consider those shapes with $\mathbf{q}_T$ non-zero. By Lemma 3.0.9 and Lemma 3.1.5 all distinct non-zero tableaux filled with b 1's and b 2's must be equivalent to $T = \begin{array}{c} \kappa \\ \hline 1 \\ 2 \end{array}$, (not including tail). Hence there is at most one distinct non-zero tableau for any shape λ. By Lemma 3.2.6, when $\lambda = [2b-\kappa, \kappa]$ then $\mathbf{q}_T \neq 0$ iff κ is even. Since $\langle \mathbf{q}_T \rangle = S^{\lambda,2} \cap M^{\lambda,2}$ we must have $\dim(S^{\lambda,2} \cap M^{\lambda,2}) = 1$ if κ is even and zero otherwise.

This is a well-known result, appearing in [13] and [17].

□

The weight-set counting of Theorem 4 is useful for much more complicated tableaux as well. To illustrate the general usage of the theorem, we list here a slightly more involved example.

Example 3.2.7. To see directly how weight-set counting works, consider the following example. The tableau Q^* is listed below using the column block notation, with the conditions on the block size listed to the right. Underneath the tableau we list the weight and shape of the tableau.

$$Q^* = \begin{array}{ccccc} \text{A} & \text{A} & \text{B} & \text{B} & \text{C} \\ \hline 2 & 2 & 2 & 3 & 1 \\ 1 & 4 & 4 & 1 & 4 \\ 3 & 3 & & & \end{array} \qquad \begin{array}{l} \text{A} + \text{B} + \text{C} = d \\ 2\text{A} + \text{B} < d \\ \text{C even} \\ \text{A}, \text{B} > 0 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} \text{A}+\text{B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{smallmatrix} \right)$$

$$\lambda = [3d - 2\text{A} - \text{B}, d + \text{A}+\text{B}, 2\text{A}]$$

We want to show that $\mathbf{q}_{Q^*} \neq 0$, which we do by showing $\mathcal{P}(Q^*) - \mathcal{N}(Q^*) \neq 0$ and applying Theorem 4. Now $\mathcal{P}(Q^*)$ is the number of $\tau \in C_{Q^*}$ with $\epsilon(\tau) = 1$ such that $\omega(i_1, i_2, i_3, i_4|\tau Q^*) = \omega(Q^*)$ for some distinct weight assignment (i_1, i_2, i_3, i_4). (Equivalently, τ is such that $\{\omega(i|\tau Q^*)\|i = 1, 2, 3, 4\} = \{\omega(i|Q^*)\|i = 1, 2, 3, 4\}$.) Similarly, $\mathcal{N}(Q^*)$ is those with $\epsilon(\tau) = -1$. The easiest way to count these τ is to use weight assignments. First we determine which weight assignments might be possible using some general properties of the tableau. Then we count how many τ correspond to each weight assignment (i.e., for which τ is the weight assignment valid) and determine $\epsilon(\tau)$. Finally, we add this signed sum to determine $\mathcal{P}(Q^*) - \mathcal{N}(Q^*)$.

First we want to determine which weight assignments are possible for Q^*. That is, determine for which 4-tuples $\mathbf{x} = (i_1, i_2, i_3, i_4)$ there might exist $\tau \in C_{Q^*}$ such that $\omega_{2,3}(\mathbf{x}|\tau Q^*) = \left(\begin{smallmatrix} \text{A}+\text{B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{smallmatrix} \right)$. Let $\mathbf{w} = \left(\begin{smallmatrix} \text{A}+\text{B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{smallmatrix} \right)$. Simply looking at Q^*, there are a few restrictions on what $\mathbf{x}$ can be.

Notice the body contains d copies of the elements 1 and 4, but fewer than d copies of 2 and 3 since $2\text{A}+\text{B} < d$. Also note that the body of τQ^* contains the same elements as the body of Q^*. This implies that not all elements can have d copies in row two of τQ^* for some τ. If $\omega_{2,3}(\mathbf{x}|\tau Q^*) = \mathbf{w}$ then either $\omega_2(1|\tau Q^*) = d$ or $\omega_2(4|\tau Q^*) = d$; namely, only the elements 1 and 4 may have $\omega_2(i|\tau Q^*) = d$. Hence any valid weight assignment must have $i_4 = 1$ or 4. If $\omega_2(1|\tau Q^*) = d$, then there is only one other non-zero weight to assign in row two. As $\text{B} > 0$ the remaining columns (the second A block and the first B block) must have the same element in row two, namely, 2 or

4. That means we must have either $\omega_2(2|\tau Q^*) = \text{A}+\text{B}$ or $\omega_2(4|\tau Q^*) = \text{A}+\text{B}$, so the weight assignment must have $i_1 = 2$ or 4. Similarly, if $\omega_2(4|\tau Q^*) = d$, then since $\text{B} > 0$ we must have either $\omega_2(1|\tau Q^*) = \text{A}+\text{B}$ or $\omega_2(3|\tau Q^*) = \text{A}+\text{B}$, that is $i_1 = 1$ or 3.

We also consider which elements j may have $\omega_3(j|\tau Q^*) = 2\text{A}$. We find that only with $j = 2$ or 3 may this occur since these are the only elements for which both A blocks will be the same in row three. (That is if $\omega_{2,3}(\tau Q^*) = \mathbf{w}$ then either $\omega_3(2|\tau Q^*) = 2\text{A}$ or $\omega_3(3|\tau Q^*) = 2\text{A}$, so any valid weight assignment has $i_3 = 2$ or 3.)

There are six distinct weight assignments $\mathbf{x} = (i_1, i_2, i_3, i_4)$ meeting these conditions: $(1, 2, 3, 4)$, $(1, 3, 2, 4)$, $(4, 3, 2, 1)$, $(3, 1, 2, 4)$, $(4, 2, 3, 1)$, and $(2, 4, 3, 1)$. In the table below, for each weight assignment we list for what type of tableau τQ^* it is valid, the form τ used, the number of such τ, and the sign of τ. This is an easy way to summarize the counting of τ and their signs. (We will omit the subscripts $_{Q^*}$ when writing τ for easy reading. Remember that τ is labeled by the *entry positions* of Q^* and not the elements.)

$\omega_{2,3}(\mathbf{x}\mid\tau Q^*)$ $= \left(\begin{smallmatrix} \text{A+B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{smallmatrix}\right)$	τQ^*	τ	#	$\epsilon(\tau)$
$\mathbf{x} = (1,2,3,4)$	A A B B C 2 2 2 3 1 1 4 4 1 4 3 3	$\overset{\text{A}}{()} \times \overset{\text{A}}{()} \times \overset{\text{B}}{()} \times \overset{\text{B}}{()} \times \overset{\text{C}}{()}$	1	$(-1)^0$
$\mathbf{x} = (1,3,2,4)$	A A B B C 3 3 2 3 1 1 4 4 1 4 2 2	$\overset{\text{A}}{(13)} \times \overset{\text{A}}{(13)} \times \overset{\text{B}}{()} \times \overset{\text{B}}{()} \times \overset{\text{C}}{()}$	1	$(-1)^{2\text{A}}$
$\mathbf{x} = (3,1,2,4)$	A A B B C 1 3 2 1 1 3 4 4 3 4 2 2	$\overset{\text{A}}{(132)} \times \overset{\text{A}}{(13)} \times \overset{\text{B}}{()} \times \overset{\text{B}}{(12)} \times \overset{\text{C}}{()}$	1	$(-1)^{\text{A+B}}$
$\mathbf{x} = (4,2,3,1)$	A A B B C 2 2 2 3 4 1 4 4 1 1 3 3	$\overset{\text{A}}{()} \times \overset{\text{A}}{()} \times \overset{\text{B}}{()} \times \overset{\text{B}}{()} \times \overset{\text{C}}{(12)}$	1	$(-1)^{\text{C}}$
$\mathbf{x} = (4,3,2,1)$	A A B B C 3 3 2 3 4 1 4 4 1 1 2 2	$\overset{\text{A}}{(13)} \times \overset{\text{A}}{(13)} \times \overset{\text{B}}{()} \times \overset{\text{B}}{()} \times \overset{\text{C}}{(12)}$	1	$(-1)^{2\text{A+C}}$
$\mathbf{x} = (2,4,3,1)$	A A B B C 2 4 4 3 4 1 2 2 1 1 3 3	$\overset{\text{A}}{()} \times \overset{\text{A}}{(12)} \times \overset{\text{B}}{(12)} \times \overset{\text{B}}{()} \times \overset{\text{C}}{(12)}$	1	$(-1)^{\text{A+B+C}}$

To see how we obtain such a table, consider the last row. We want a tableau τQ^* such that $\omega_{2,3}(2,4,3,1|\tau Q^*) = \mathbf{w}$. This means that $\omega_3(3|\tau Q^*) = 2\text{A}$ so τ cannot move any entries in row 3. We also have $\omega_2(4|\tau Q^*) = 0$, so examining Q^*, we know that τ acts non-trivially on the second column block A, the first column block B, and column block C, namely, $\tau = \overset{\text{A}}{*} \times \overset{\text{A}}{(12)}_T \times \overset{\text{B}}{(12)}_T \times \overset{\text{B}}{*} \times \overset{\text{C}}{(12)}_T$. If τ were to act non-trivially on the first column block A or the second column block B, the number of 1's in row two of τQ^* would decrease. Since we must have $\omega_2(1|\tau Q^*) = d$, this cannot happen. Hence $\tau = \overset{\text{A}}{()}_T \times \overset{\text{A}}{(12)}_T \times \overset{\text{B}}{(12)}_T \times \overset{\text{B}}{()}_T \times \overset{\text{C}}{(12)}_T$ and has been completely determined for us. Then we have $\omega_{2,3}(2,4,3,1|\tau Q^*) = \mathbf{w}$, so such a τ exists and is unique. For reference, τ and τQ^* are listed. Once τ has been determined, computing $\epsilon(\tau) = (-1)^{\text{A+B+C}}$ is straightforward.

Finally, to compute the weight sum for $\mathbf{w}$, we sum the product of the number of τ with the sign of τ, that is $\# \cdot \epsilon(\tau)$. Here the sum is $1+1+(-1)^{\text{A+B}}+1+1+(-1)^{\text{A+B}}$. This sum is between 2 and 6, depending on the parity of A and B. Since it is non-zero in all cases, Theorem 4 shows $\mathbf{q}_{Q^*} \neq 0$.

3.3 Joining Tableaux

Definition 3.3.1. The *join* of two tableaux, U and V, denoted $U \vee V$ is a way of combining tableaux together. If the entries of U and V are not disjoint, renumber V so that they are. For instance, if U contains the numbers 1 to n and V contains the numbers 1 to m we first renumber V with the numbers $n+1$ to $n+m$. Then concatenate the tableaux and sort the columns by length. Note that entries of every column remain fixed, only the order of the columns change.

Example 3.3.2. $U = \begin{array}{lll} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & \end{array}$ and $V = \begin{array}{lll} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & & \end{array}$ We renumber V to get $V = \begin{array}{lll} 5 & 6 & 7 \\ 6 & 8 & 8 \\ 7 & & \end{array}$.

Concatenating gives $\begin{array}{llllll} 1 & 1 & 3 & 5 & 6 & 7 \\ 2 & 2 & 4 & 6 & 8 & 8 \\ 3 & 4 & & 7 & & \end{array}$. When sorted we get $T = U \vee V = \begin{array}{llllll} 1 & 1 & 5 & 3 & 6 & 7 \\ 2 & 2 & 6 & 4 & 8 & 8 \\ 3 & 4 & 7 & & & \end{array}$.

Note that since applying permutations of $\mathcal{S}_a$ to a tableau has no effect on $\mathbf{q}_T$ and $\mathbf{m}_T$, the renumbering of a tableau is irrelevant. Also, any $\sigma \in R_T$ that only interchanges columns will commute with all $\tau \in C_T$. Hence column sorting has no effect of $\mathbf{e}_T$, $\mathbf{q}_T$, and $\mathbf{m}_T$, since there is no sign change for row permutations. This join operation also joins the weight-sets, namely, $\omega_3(U \vee V) = \omega_3(U), \omega_3(V) = (0,0,1,1,0,0,1,0)$.

Definition 3.3.3. Let $T = U \vee V$ for tableaux U filled with 1 to m and V filled with $m+1$ to a. The weights, $\omega(U)$ and $\omega(V)$ are *disjoint* (equivalently, $\omega(T)$ *splits* over U and V), if every valid weight assignment of $\omega(T)$, can be obtained from concatenating valid weight assignments of $\omega(U)$ and $\omega(V)$.

Say $\omega(1,\ldots,m|U) = (\overline{x}_1,\ldots,\overline{x}_m)$ where the $\overline{x}_i$ are weight vectors and $\omega(m+1,\ldots,a|V) = (\overline{x}_{m+1},\ldots,\overline{x}_a)$. So $\omega(1,\ldots,a|T) = (\overline{x}_1,\ldots,\overline{x}_a)$. Consider a valid weight assignment of T assigning to the element j the weight vector $\overline{x}_{k_j}$. Then $\omega(1,\ldots,a|\tau T) = (\overline{x}_{k_1},\ldots,\overline{x}_{k_a})$ for some $\tau \in C_T$. This restricts to a valid weight assignment of $\tau_{|_U} U$ by considering only the elements 1 to m. This restriction is unique because a weight assignment is defined by the vector-element pairing, not the label assigned to the vector. If this restriction corresponds to a weight assignment of $\omega(U)$ (i.e., the weights assigned to elements 1 to m are the same as the weights of $\omega(U)$ as vectors) then the weight assignment of T arose from valid weight assignments of U. Similarly for V.

By restricting to a weight assignment of $\omega(U)$ we mean $\{\overline{x}_{k_i}|i = 1\ldots m\} = \{\overline{x}_i|i = 1\ldots m\}$, i.e., the weights assigned to U are equivalent to those of $\omega(U)$. If this result is a weight assignment of $\omega(U)$, it is valid for the tableau $\tau_{|_U} U$. If this is true for all valid weight assignments of T, then $\omega(T)$ splits and the weights of U and V are disjoint.

Example 3.3.4. Consider $U = \begin{matrix} 1\,1 \\ 2\,2 \end{matrix}$ and $V = \begin{matrix} 3\,3\,3\,3 \\ 4\,4\,4\,4 \end{matrix}$. So $T = \begin{matrix} 1\,1\,3\,3\,3\,3 \\ 2\,2\,4\,4\,4\,4 \end{matrix}$ and $\omega_2(1,2,3,4|T) = (0,2,0,4)$. There are four valid weight assignments possible for T: (Recall that T^* represents any possible tableau τT.)

$$\omega_2(1,2,3,4|T^*) = (0,2,0,4)$$
$$\omega_2(1,2,3,4|T^*) = (2,0,0,4)$$
$$\omega_2(1,2,3,4|T^*) = (0,2,4,0)$$
$$\omega_2(1,2,3,4|T^*) = (2,0,4,0)$$

When we restrict these weight assignments to U we get two possible assignments for U, $\omega_2(1,2|U^*) = (0,2)$ and $\omega_2(1,2|U^*) = (2,0)$. Since the original weight of U is $\omega_2(1,2|U) = (0,2)$, both of these assignments are assignments of $(0,2)$ and both are valid for U (simply take $\tau_U = ()$ and $\tau_U = (12)_T \times (12)_T$). A similar argument holds for V. Thus the $\omega(T)$ splits over U and V.

However, weight-set disjointness is highly dependent on the filling of T. Consider instead, $U = \begin{matrix} 1\,1 \\ 2\,2 \end{matrix}$ and $V = \begin{matrix} 4\,3 \\ 3\,4 \end{matrix}$. So $T = \begin{matrix} 1\,1\,4\,3 \\ 2\,2\,3\,4 \end{matrix}$ and $\omega_2(1,2,3,4|T) = (0,2,1,1)$.

Then $\omega_2(1,2,3,4|T^*) = (1,1,2,0)$ is a valid weight assignment for T by $T^* = \tau T$ with $\tau = (12)_T \times ()_T \times ()_T \times (12)_T$. However, $\omega_2(1,2|U^*) = (1,1)$ it is not a valid weight assignment of $\omega_2(U) = (2,0)$, even though there exists τ such that $\omega_2(\tau U) = (1,1)$. Hence the weights are not disjoint.

Although this definition of disjointness is a bit involved, in Section 3.4 we will give a sufficient (but not necessary) condition on the tableau which is easier to check. However, we use disjointness here to obtain the full generality of Theorem 8, which is one of the fundamental tools we use to construct non-zero tableaux.

Theorem 8. Let U and V be tableaux such that

- $elements(U) = \{1, \ldots m\}$ and $elements(V) = \{m+1, \ldots a\}$ (renumber if necessary)
- The weights $\omega(1,\ldots,m|U)$, $\omega(m+1,\ldots,a|V)$ are such that $\mathbf{q}_U$ and $\mathbf{q}_V$ are non-zero by weight-set counting on ω.
- The weight assignments corresponding to $\omega(1,\ldots,m|U)$ and $\omega(m+1,\ldots,a|V)$ are disjoint.

Then for $T = U \vee V$, we have $\mathbf{q}_T \neq 0$ by weight-set counting on $\omega(T)$.

Proof. By weight-set counting on U and V we have $\mathcal{P}(U) - \mathcal{N}(U) \neq 0$ and $\mathcal{P}(V) - \mathcal{N}(V) \neq 0$. By Theorem 4, showing $\mathcal{P}(T) - \mathcal{N}(T) \neq 0$ implies $\mathbf{q}_T \neq 0$. Thus it suffices to show $\mathcal{P}(T) - \mathcal{N}(T) = (\mathcal{P}(U) - \mathcal{N}(U))(\mathcal{P}(V) - \mathcal{N}(V))$ or equivalently $\mathcal{P}(T) = \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$ and $\mathcal{N}(T) = \mathcal{N}(U)\mathcal{P}(V) + \mathcal{P}(U)\mathcal{N}(V)$. We will show $\mathcal{P}(T) = \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$. The $\mathcal{N}(T)$ claim follows similarly.

Consider the weight assignment $(k_1,\ldots,k_a)$ of T. If $\omega(k_1,\ldots,k_a|\tau T) = \omega(1,\ldots,a|T)$ with $\epsilon(\tau) = 1$ (i.e., τ is positive for $(k_1,\ldots,k_a)$) then it is counted in $\mathcal{P}(T)$. Since the weight splits, we have $\omega(k_1,\ldots,k_a|\tau T) = \omega(k_1,\ldots k_m|\tau_{|U} U)\omega(k_{m+1},\ldots k_a|\tau_{|V} V) = \omega(1,\ldots,m|U)\omega(m+1,\ldots,a|V) = \omega(1,\ldots,a|T)$ with $\epsilon(\tau) = \epsilon(\tau_{|U})\dot{\epsilon}(\tau_{|V}) = 1$. Hence either $\epsilon(\tau_{|U}) = \epsilon(\tau_{|V}) = 1$ or $\epsilon(\tau_{|U}) = \epsilon(\tau_{|V}) = -1$. If $\epsilon(\tau_{|U}) = 1$ then since $\omega(k_1,\ldots k_m|\tau_{|U} U) = \omega(U)$ is a valid weight assignment (because the weights are disjoint), it is counted in $\mathcal{P}(U)$. Similarly for the $\mathcal{P}(V)$ cases. The $\epsilon(\tau_{|_U}) = -1$ cases are counted in $\mathcal{N}(U)$. Thus $\mathcal{P}(T) \leq \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$.

Now any weight assignments $\omega(k_1, \ldots, k_a|\tau_{|_U} U) = \omega(1, \ldots, a|U)$ in $\mathcal{P}(U)$ and $\omega(k_{m+1}, \ldots, a|\tau_{|_V} V) = \omega(m+1, \ldots, a|V)$ in $\mathcal{P}(V)$ must also correspond to the valid weight assignment $\omega(k_1, \ldots, k_a|\tau T) = \omega(1, \ldots, a|T)$ with $\tau = \tau_{|_U} \times \tau_{|_V}$. Moreover $\epsilon(\tau) = \epsilon(\tau_{|_U})\dot{\epsilon}(\tau_{|_V}) = 1$. So this weight assignment is in $\mathcal{P}(T)$. Similarly for weight assignments in $\mathcal{N}(U)$ and $\mathcal{N}(V)$. Hence $\mathcal{P}(T) \geq \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$. Thus we have $\mathcal{P}(T) = \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$ as desired. □

Theorem 8 allows us to construct a non-zero tableau from smaller non-zero tableaux. The main difficulty in applying this theorem is showing that the weight-sets are disjoint. To deal with this, we develop an idea of maximality of weights which is sufficient for weight-set disjointness.

3.4 Maximal Weights

Since the action of $\mathcal{S}_a$ on a tableau T does not change the resulting $\mathbf{q}_T$, we generalize the definition of tableau weight to account for this. We put an order on this generic weight, thus defining maximal weights.

Definition 3.4.1. Given a tableau T, the *generic form* of $\omega_i(T)$ is $\mathsf{w}_i(T) = \omega_i(\pi T) = (x_1, \ldots, x_a)$ for any $\pi \in \mathcal{S}_a$ such that $x_j \geq x_{j+1}$ for all j. In essence, $\mathsf{w}_i(T)$ is the weights of $\omega_i(T)$ listed in decreasing order. This definition works for any row of T.

We define the generic form of a weight on the entire tableau (assuming T has at most three rows) by, $\mathsf{w}(T) = \omega_{2,3}(\pi T) = \left(\begin{smallmatrix} x_1 & \cdots & x_a \\ y_1 & \cdots & y_a \end{smallmatrix} \right)$ for any $\pi \in \mathcal{S}_a$ such that $y_j \geq y_{j+1}$ for all j and if $y_j = y_{j+1}$ then $x_j \geq x_{j+1}$. We consider only the weight vectors of the second and third rows and list the vectors so that the row three weights are decreasing. If two vectors have the same weight for row three, we list the vector with the larger weight in row two first.

For instance, if $T = \begin{matrix} 2\,2\,1 \\ 1\,1\,2 \\ 3\,3\,3 \end{matrix}$ then $\omega_{2,3}(T) = \left(\begin{smallmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right)$. Hence $\mathsf{w}(T) = \left(\begin{smallmatrix} 0 & 2 & 1 \\ 3 & 0 & 0 \end{smallmatrix}\right)$.

Definition 3.4.2. We put an order on generic weights by $\mathsf{w}_i(T_1) > \mathsf{w}_i(T_2)$ if when $\mathsf{w}_i(T_1) = (x_1, \dots x_a)$, $\mathsf{w}_i(T_2) = (v_1, \dots v_a)$, there exists $k \geq 1$ such that $x_j = v_j$ for $j < k$ and $x_k > v_k$. We say $\mathsf{w}(T_1) > \mathsf{w}(T_2)$ if

1. $\mathsf{w}_3(T_1) > \mathsf{w}_3(T_2)$ or
2. $\mathsf{w}_3(T_1) = \mathsf{w}_3(T_2)$ and $\mathsf{w}_2(T_1) > \mathsf{w}_2(T_2)$ or
3. $\mathsf{w}_3(T_1) = \mathsf{w}_3(T_2)$, $\mathsf{w}_2(T_1) = \mathsf{w}_2(T_2)$, and if we have $\mathsf{w}(T_1) = \left(\begin{smallmatrix} y_1 & \cdots & y_a \\ x_1 & \cdots & x_a \end{smallmatrix}\right)$ and $\mathsf{w}(T_2) = \left(\begin{smallmatrix} z_1 & \cdots & z_a \\ x_1 & \cdots & x_a \end{smallmatrix}\right)$ then there exists $k \geq 1$ such that $y_j = z_j$ for $j < k$ and $y_k > z_k$.

We also apply this ordering to sets of weight vectors, by associating to each set the generic weight vector formed by concatenating the given weights in order. So to the set $A = \{\binom{0}{2}, \binom{1}{1}\}$ we associate the weight $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 1 \end{smallmatrix}\right)$.

Definition 3.4.3. We define the *maximum generic weight* of row i of T to be the maximum with respect to $>$ of $\{\mathsf{w}_i(\tau T) | \tau \in C_T\}$, where w is the generic weight defined above. Similarly the *maximum generic weight* of T is the maximum with respect to $>$ of $\{\mathsf{w}(\tau T) | \tau \in C_T\}$. Note that the maximum generic weight of T is based only on the weights of rows two and three. As such we ignore the weight of the first row.

Example 3.4.4. Let $T = \begin{matrix} 1\,2 \\ 2\,1 \end{matrix}$. Then the generic weights $\mathsf{w}_2(\tau T)$ are $(1,1)$ and $(2,0)$, with $(2,0)$ (the generic weight of $\tau T = \begin{matrix} 1\,1 \\ 2\,2 \end{matrix}$ or $\begin{matrix} 2\,2 \\ 1\,1 \end{matrix}$) as maximum. Here we've suppressed writing $\mathsf{w}_3(T)$ since T has only two rows.

If $T = \begin{matrix} 1\,1\,4\,4\,4 \\ 2\,2\,3\,3\,3 \\ 3\,3 \end{matrix}$ then the maximum generic weight of T (of the second and third rows) is $\left(\begin{smallmatrix} 3 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{smallmatrix}\right)$ which is $\omega_{2,3}(T)$ in generic form, $\mathsf{w}(T)$.

Definition 3.4.5. Let T be a tableau having three or fewer rows. Let w^{m} be the maximum generic weight of T. We say w^{m} is the *max weight* for T if w^{m} occurs in $\mathbf{q}_T$. That is $\mathbf{q}_T \neq 0$ by weight-set counting on w^{m}.

Unlike the maximum weight, the max weight of a tableau may not exist since the weight may not occur in $\mathbf{q}_T$. For instance, consider $T = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{matrix}$. We know $\mathbf{q}_T = 0$ by Lemma 3.2.6, yet T has $\left(\begin{smallmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \end{smallmatrix}\right)$ as its maximum weight.

The max weight for T is always the maximum weight for row three of T, but it need not be the maximum weight for row two of T. Consider $T = \begin{matrix} 5 & 2 & 3 & 3 \\ 1 & 4 & 4 & 1 \\ 3 & 3 & & \end{matrix}$. The maximum generic weight of T is $\left(\begin{smallmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \end{smallmatrix}\right)$, but the maximum generic weight of row two of T is $(4, 0, 0, 0)$ which is not the generic form of $(0, 2, 2, 0)$.

Definition 3.4.6. If w^{m} is the max weight for T, we say T is in *maximal form* provided $\omega_{2,3}(\pi T) = w^{\mathsf{m}}$ for some $\pi \in \mathcal{S}_a$. This only requires that some permutation of the weight vectors of $\omega_{2,3}(T)$ be equal to the max weight of T.

While the max weight may not exist for all tableaux, it is easy to show weight-set disjointness for those tableaux which are in maximal form. In order to prove this, we use the following lemmas regarding our ordering.

Lemma 3.4.7. Given two weights, W_1 and W_2, of the same length, let $C_k = \{\binom{x}{y} | \binom{x}{y} \in W_k\}$, $k = 1, 2$ be the multisets of weight vectors in each of these weights. Let $A = C_1 \setminus (C_1 \cap C_2)$, the weight vectors of W_1 not in W_2. Similarly, let $B = C_2 \setminus (C_1 \cap C_2)$. If $W_1 \geq W_2$, then $A \geq B$.

Proof. Without loss of generality, assume W_i is written in maximal form (i.e., is equal to its generic weight). If $W_1 = W_2$ then $A = B = \varnothing$, so the result holds trivially. So suppose $W_1 > W_2$. Then either W_1 differs from W_2 at some place in the third row, or the third rows are equal and they differ at some place in the second row (at least vectorwise).

Define $A^y = \{\binom{x_i}{y_i} | \binom{x_i}{y_i} \in A,\, y_i = y\}$ and $B^v = \{\binom{u_i}{v_i} | \binom{u_i}{v_i} \in B,\, v_i = v\}$. Defining C_1^y and C_2^v similarly, we have $A^y = C_1^y \setminus (C_1^y \cap C_2^y)$ and $B^v = C_2^v \setminus (C_1^v \cap C_2^v)$.

Let $W_1 = \left(\begin{smallmatrix} x_1 & \cdots & x_m \\ y_1 & \cdots & y_m \end{smallmatrix}\right)$ and $W_2 = \left(\begin{smallmatrix} u_1 & \cdots & u_m \\ v_1 & \cdots & v_m \end{smallmatrix}\right)$. If W_1 differs from W_2 in the third row, then there exists j such that $y_j > v_j$ and $y_i = v_i$ for all $i < j$. Hence $|C_1^{y_i}| = |C_2^{v_i}| = |C_2^{y_i}|$ for $y_i > y_j$, so $|A^{y_i}| = |B^{v_i}| = |B^{y_i}|$. Then to show $A > B$ it suffices to show

$|A^{y_j}| > |B^{y_j}|$. But since $y_j > v_j$, we have $|C_1^{y_j}| > |C_2^{y_j}|$. Thus $\mathsf{w}_3(A) > \mathsf{w}_3(B)$ and the result follows. Note that when the third rows are equal, this argument shows $\mathsf{w}_3(A) = \mathsf{w}_3(B)$.

If W_1 and W_2 are equal in the third row but the generic weights of their second rows differ, we can apply the same argument as above, where A^x, B^x, C_i^x are the appropriate sets of weight vectors with the second row weight equal to x. This shows $\mathsf{w}_2(A) > \mathsf{w}_2(B)$. Since we've already have $\mathsf{w}_3(A) = \mathsf{w}_3(B)$, the result follows.

If W_1 and W_2 have the same generic weights in the second and third rows, then by above we know the generic weights of rows two and three of A and B are the same. By definition, $A \cap B = \varnothing$, so the second row of the first vectors in A and B are different. Now W_1 and W_2 agree in the third row, so the first vectors where they differ in the second row must be the first vector in A and B respectively. Since $W_1 > W_2$ we have the large vector occurring in W_1 and hence in A. Thus $A > B$. □

The following lemma is an obvious property of the ordering, but is included due to the non-standard ordering used.

Lemma 3.4.8. Given sets A and B of weight vectors, if $A \geq B$ and $B \geq A$ then $A = B$.

Proof. View A and B as generic weight vectors W_A and W_B. Let $W_A = \left(\begin{smallmatrix} x_1 & \cdots & x_m \\ y_1 & \cdots & y_m \end{smallmatrix}\right)$ and $W_B = \left(\begin{smallmatrix} u_1 & \cdots & u_m \\ v_1 & \cdots & v_m \end{smallmatrix}\right)$. Since $A \geq B$ we have $(y_1, \ldots, y_m) \geq (v_1, \ldots, v_m)$. So either the rows are equal or there exists j such that $y_j > v_j$ and $y_i = v_i$ for all $i < j$. As $B \geq A$ we would similarly get $v_j \geq y_j$, which is a contradiction. Hence $\mathsf{w}_3(A) = \mathsf{w}_3(B)$. A similar argument shows $\mathsf{w}_2(A) = \mathsf{w}_2(B)$.

If $A \neq B$, let j be the first place where they differ. Then $A \geq B$ implies $\binom{x_j}{y_j} > \binom{u_j}{v_j}$. Since $y_j = v_j$, this means $x_j > u_j$. But the same argument on $B \geq A$ implies $u_j > x_j$, which is a contradiction. Thus $A = B$. □

Now using these lemmas we can show that the max weights of tableaux are necessarily disjoint.

Lemma 3.4.9. If U and V are tableaux in maximal form, with U containing b copies of the elements 1 to m and V containing b copies of the elements $m+1$ to a (after renumbering as necessary), then $\omega(U)$ and $\omega(V)$ are disjoint.

Proof. Let U and V be in maximal form. Let $\omega_{2,3}(U \vee V) = \left(\begin{smallmatrix} x_1 & \cdots & x_m & x_{m+1} & \cdots & x_a \\ y_1 & \cdots & y_m & y_{m+1} & \cdots & y_a \end{smallmatrix}\right)$. To show that U and V are disjoint, we need to show that any valid weight assignment of $U \vee V$ restricts to a valid weight assignment of U and V. Let $\left(\begin{smallmatrix} x_{k_1} & \cdots & x_{k_m} & x_{k_{m+1}} & \cdots & x_{k_a} \\ y_{k_1} & \cdots & y_{k_m} & y_{k_{m+1}} & \cdots & y_{k_a} \end{smallmatrix}\right)$ be a valid weight assignment of $U \vee V$. That means there exists τ such that $\omega_{2,3}(\tau[U \vee V]) = \left(\begin{smallmatrix} x_{k_1} & \cdots & x_{k_m} & x_{k_{m+1}} & \cdots & x_{k_a} \\ y_{k_1} & \cdots & y_{k_m} & y_{k_{m+1}} & \cdots & y_{k_a} \end{smallmatrix}\right)$. We want to show it restricts to a valid weight assignment of U, that is $\omega_{2,3}(\tau_{|_U} U) = \left(\begin{smallmatrix} x_{k_1} & \cdots & x_{k_m} \\ y_{k_1} & \cdots & y_{k_m} \end{smallmatrix}\right) = \omega_{2,3}(\pi U)$ for some $\pi \in \mathcal{S}_m$. This is equivalent to showing $\{\binom{x_i}{y_i} | i = 1 \dots m\} = \{\binom{x_{k_i}}{y_{k_i}} | i = 1 \dots m\}$.

Let $C = \{\binom{x_i}{y_i} | i = 1 \dots m\}$ be set of the weight vectors of $\omega_{2,3}(U)$ and $D = \{\binom{x_{k_i}}{y_{k_i}} | i = 1 \dots m\}$ the set of weight vectors of $\omega_{2,3}(\tau_{|_U} U)$. Define $A = C \setminus (C \cap D)$ and $B = D \setminus (C \cap D)$. So A consists of those weight vectors of U assigned to V which are distinct from the weight vectors of V assigned to U under this weight assignment. That is, the vectors in A occur in $\omega_{2,3}(U)$ but not in $\omega_{2,3}(\tau_{|_U} U)$. The set B is the weight vectors are those vectors coming from $\omega_{2,3}(\tau_{|U} U)$ which are not in $\omega_{2,3}(U)$. To prove disjointness, we need to show that $C = D$, which is equivalent to showing $A = B = \varnothing$.

Now U is in maximal form, so $\omega_{2,3}(U) \geq \omega_{2,3}(\tau' U)$ for all τ'. Hence $\omega_{2,3}(U) \geq \omega_{2,3}(\tau_{|_U} U)$. So by Lemma 3.4.7, we have $A \geq B$. But we can also view A as the weight vectors of U in $\omega_{2,3}(\tau_{|_V} V)$ which are not in $\omega_{2,3}(V)$. Similarly B is the set of vectors from $\omega_{2,3}(V)$ which are not in $\omega_{2,3}(\tau_{|_V} V)$. Since V is also in maximal form, $\omega_{2,3}(V) \geq \omega_{2,3}(\tau_{|_V} V)$. Hence by Lemma 3.4.7, we have $B \geq A$. Thus Lemma 3.4.8 shows $A = B$. However, $A \cap B = \varnothing$ by definition, so $A = B = \varnothing$. Thus the weights are disjoint. $\square$

Example 3.4.10. Suppose U is a tableau in maximal form such that $\omega_{2,3}(1,2,3,4|U) = \left(\begin{smallmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{smallmatrix}\right)$ and V is a tableau in maximal form with $\omega_{2,3}(5,6,7,8,9,10,11|V) = \left(\begin{smallmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$. Suppose these weights were not disjoint. That means we must be able to assign some weight $\binom{x}{y}$ of U to V and some weight

$\binom{x'}{y'}$ of V to U.

First consider the vector $\binom{1}{2}$ of U (i.e., the vector with the largest weight in row three). Since V is in maximal form, we know that there can be at most one copies of any element in row three of τV for any τ. Since $2 > 1$, this vector cannot be assigned to V. Similarly, once we know that $\binom{1}{2}$ remains a weight of U, the largest row three weight we can assign to U is 0. Hence $\binom{1}{1}$ remains with V.

Now consider $\binom{3}{0}$ of U. Having V in maximal form means that when $\binom{1}{1}$ are assigned to V, a vector $\binom{*}{0}$ assigned to V must have $* \leq 2$. Thus $\binom{3}{0}$ is assigned to U. Therefore the only vectors of U and V that can be assigned to each other are the $\binom{0}{0}$ vectors. However, since a weight assignment is based only on the vector and not its label, this is the same as a weight assignment arising from U and V. Hence the weights are disjoint.

Lemma 3.4.9 shows that if U and V are in maximal form, $\mathbf{q}_{U \vee V} \neq 0$ by Theorem 8. We will apply Theorem 8 repeatedly when constructing tableaux. As such, we want $U \vee V$ to be in maximal form whenever U and V are.

Lemma 3.4.11. If T_1 and T_2 are maximal tableaux filled with different elements, then $T_1 \vee T_2$ is maximal.

Proof. The tableaux have no elements in common so the weight splits over the join. Hence $\omega(\tau[T_1 \vee T_2]) = \omega(\tau_1 T_1 \vee \tau_2 T_2) = \omega(\tau_1 T_1) \vee \omega(\tau_2 T_2)$. Since each tableau weight was maximal, so too is their join. $\square$

Hence maximality is preserved under the join operation. Through this join operation, we will construct collections of tableaux. To show these tableaux are linearly independent (over $\mathbb{C}$), we can simply compare max weights.

Lemma 3.4.12. Let $\{\mathcal{B}_p\}$ be a set of tableaux in maximal form. If the max weights of these tableaux are distinct, then $\{\mathbf{q}_{\mathcal{B}_p}\}$ is linearly independent.

Proof. Assume $\{\mathbf{q}_{\mathcal{B}_p}\}$ is not linearly independent. Let $\mathcal{B}_k$ be the tableau with the largest weight such that $\mathbf{q}_{\mathcal{B}_k}$ is not linearly independent from the rest of $\{\mathbf{q}_{\mathcal{B}_p}\}$. Write

$\mathbf{q}_{\mathcal{B}_k} = \sum a_p \mathbf{q}_{\mathcal{B}_p}$. Then $\omega(\mathbf{q}_{\mathcal{B}_k}) = \sum a_p \omega(\mathbf{q}_{\mathcal{B}_p})$. Since $\mathcal{B}_k$ is in maximal form, $\omega(\mathcal{B}_k)$ occurs with non-zero coefficient in $\omega(\mathbf{q}_{\mathcal{B}_k})$. Hence $\omega(\mathcal{B}_k)$ must occur with non-zero coefficient in $\omega(\mathbf{q}_{\mathcal{B}_p})$ for some p. However, $\mathcal{B}_k$ was chosen such that $\omega(\mathcal{B}_k) \geq \omega(\mathcal{B}_p)$. By hypothesis these weights are distinct, so the inequality is strict. But the $\mathcal{B}_p$ are in maximal form, so $\omega(\mathcal{B}_p)$ is the largest weight occurring in $\mathbf{q}_{\mathcal{B}_p}$. Hence $\omega(\mathcal{B}_k)$ does not occurring in $\sum a_p \omega(\mathbf{q}_{\mathcal{B}_p})$, contradicting the linear dependence. □

We will use Lemma 3.4.12 heavily in Chapter 9 to prove Theorem 3. To make use of this lemma, we need to have distinctness of max weights. When the tableaux are formed via the join operation, we can sometimes simplify the proof of max weight distinctness via the following lemma:

Lemma 3.4.13. Suppose we have the two row tableaux T_1, T_2, T_3, and T_4, were the non-zero max weights are as follows: $\omega(T_1) = (A, B)$, $\omega(T_2) = (C, D)$, $\omega(T_3) = (a, b)$, and $\omega(T_4) = (c, d)$. Assume $\omega(T_1) \neq \omega(T_3)$, $\omega(T_2) \neq \omega(T_4)$, but $\lambda_2(T_1) = \lambda_2(T_3)$ and $\lambda_2(T_2) = \lambda_2(T_4)$. If $\lambda_2(T_1) \neq \lambda_2(T_2)$, then $\omega(T_1 \vee T_2) \neq \omega(T_3 \vee T_4)$.

Proof. Since $\lambda_2(T_1) \neq \lambda_2(T_2)$ and $\omega(T_1) \neq \omega(T_3)$, assume $A+B > C+D$ and $A > a$. If $\omega(T_1 \vee T_2) = \omega(T_3 \vee T_4)$, either $A = c$ or $A = d$ since $a \geq b$ by maximality of $\omega(T_3)$. Consider $A = c$, then $B > d$ and $B \neq c$ because $C + D = c + d$. Then $A > a$ and $A + B = a + b$, implies $B < b \leq a$. Hence there is no weight equal to B, so this cannot occur. Similarly if $A = d$ then $c < B$ and hence there is no weight equal to B. Thus the weights are distinct. □

Although Lemma 3.4.13 applies directly to the join of only two tableaux, it may often be applied in a broader context. Namely, if many of the tableaux being joined are the same, the question of distinct weights reduces to looking only at the weights of those tableaux which differ. This approach will be used and discussed in Chapter 9.

Bibliography

[1] George E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original. MR 99c:11126

[2] S. C. Black and R. J. List, *A note on plethysm*, European J. Combin. **10** (1989), no. 1, 111–112. MR 89m:20011

[3] Emmanuel Briand, *Polynômes multisymétriques*, Ph. D. dissertation, University Rennes I, Rennes, France, October 2002.

[4] Michel Brion, *Stable properties of plethysm: on two conjectures of Foulkes*, Manuscripta Math. **80** (1993), no. 4, 347–371. MR 95c:20056

[5] C. Coker, *A problem related to Foulkes's conjecture*, Graphs Combin. **9** (1993), no. 2, 117–134. MR 94g:20019

[6] Suzie C. Dent and Johannes Siemons, *On a conjecture of Foulkes*, J. Algebra **226** (2000), no. 1, 236–249. MR 2001f:20026

[7] William F. Doran, IV, *On Foulkes' conjecture*, J. Pure Appl. Algebra **130** (1998), no. 1, 85–98. MR 99h:20014

[8] H. O. Foulkes, *Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form*, J. London Math. Soc. **25** (1950), 205–209. MR 12,236e

[9] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.3*, 2002, (`http://www.gap-system.org`).

[10] David A. Gay, *Characters of the Weyl group of $SU(n)$ on zero weight spaces and centralizers of permutation representations*, Rocky Mountain J. Math. **6** (1976), no. 3, 449–455. MR 54 #2886

[11] Larry C. Grove, *Groups and characters*, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1997, A Wiley-Interscience Publication. MR 98e:20012

[12] Roger Howe, $(\mathrm{GL}_n, \mathrm{GL}_m)$*-duality and symmetric plethysm*, Proc. Indian Acad. Sci. Math. Sci. **97** (1987), no. 1-3, 85–109 (1988). MR 90b:22020

[13] N. F. J. Inglis, R. W. Richardson, and J. Saxl, *An explicit model for the complex representations of S_n*, Arch. Math. (Basel) **54** (1990), no. 3, 258–259. MR 91d:20017

[14] G. James and A. Kerber, *Representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley, Reading, MA, 1981.

[15] G. D. James, *The representation theory of the symmetric group*, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978.

[16] Serge Lang, *Algebra*, 3 ed., Addison Wesley, Reading Massachusetts, 1999.

[17] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications. MR 96h:05207

[18] Bruce E. Sagan, *The symmetric group*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991, Representations, combinatorial algorithms, and symmetric functions. MR 93f:05102

[19] Richard P. Stanley, *Positivity problems and conjectures in algebraic combinatorics*, Mathematics: Frontiers and Perspectives (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.), American Mathematical Society, Providence, RI, 2000, pp. 295–319.

[20] R. M. Thrall, *On symmetrized Kronecker powers and the structure of the free Lie ring*, Amer. J. Math. **64** (1942), 371–388. MR 3,262d

[21] Rebecca Vessenes, *Foulkes' conjecture and tableaux construction*, J. Albegra (2004), forthcoming.

[22] David Wales, personal communication.

[23] Jie Wu, *Foulkes conjecture in representation theory and its relations in rational homotopy theory*, `http://www.math.nus.edu.sg/~matwujie/Foulkes.pdf`.