Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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Chapter 2

Background

2.1 Tableaux

A *partition* $\lambda = [\lambda_1, \dots, \lambda_\ell]$ of a number n is an ordered tuple of positive integers such that $\sum \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1} > 0$; it is denoted by $\lambda \vdash n$. The *length* of λ is ℓ. A *Ferrers diagram* is the set $[\lambda] = \{(i,j) | 1 \leq i \leq \ell,\ 1 \leq j \leq \lambda_i\}$. We view $[\lambda]$ as a (left-justified) stack of boxes with row i having λ_i boxes.

A *tableau* of *shape* λ is a filling of the Ferrers diagram $[\lambda]$ with a set of elements, usually the positive integers. It is said to have *content* $\alpha = [\alpha_1, \dots, \alpha_k]$ if the integer i occurs exactly α_i times. A tableau is *semi-standard* if the entries are weakly increasing across the rows and strictly increasing down the columns. This notation is standard and further discussion can be found in [7] and [18].

Example 2.1.1. Consider the following tableaux:

$$P = \begin{array}{lll} 1 & 2 & 5 \\ 4 & 2 & \\ 3 & & \end{array} \qquad Q = \begin{array}{lll} 1 & 1 & 5 \\ 2 & 2 & \\ 3 & 3 & \end{array} \qquad R = \begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & 3 \end{array}$$

P is a $[3,2,1]$-tableau of content $\alpha = [1,2,1,1,1]$, while Q has shape $\lambda = [3,2,2]$ and content $\alpha = [2,2,2,0,1]$. Similarly, R has shape $[3,3]$ with content $[2^3] = [2,2,2]$. Both Q and R are semi-standard, but P is not.

2.2 Combinatorial Structures

There are two different permutation actions on tableaux: an action permuting the entry positions in the tableau and an action permuting the numbers filling the tableau. These actions commute with each other.

Let T be a λ-tableau filled with the numbers 1 to a, where $\lambda \vdash n$. The permutation of entry positions corresponds to an action of $\mathcal{S}_n$ on T. View the entry positions (i.e., boxes) of T as labelled 1 to n. Then $\sigma \in \mathcal{S}_n$ acts on T by permuting the entries in the positions moved by σ. To avoid confusion between entry positions and numbers in T, we will denote all entry positions with the subscript $_T$ when necessary. The permutation action of the numbers corresponds to their permutation by $\pi \in \mathcal{S}_a$.

Example 2.2.1. Take $n = 6$, $\lambda = [3, 2, 1]$, and $a = 3$. Consider $T = \begin{array}{lll} 3 & 3 & 1 \\ 1 & 2 & \\ 2 & & \end{array}$. In terms of entry positions, we label T as $\begin{array}{lll} 1_T & 2_T & 3_T \\ 4_T & 5_T & \\ 6_T & & \end{array}$. For $\sigma = (23)_T \in \mathcal{S}_6$, we have $\sigma T = \begin{array}{lll} 3 & 1 & 3 \\ 1 & 2 & \\ 2 & & \end{array}$, while the action of $\pi = (23) \in \mathcal{S}_3$ gives $\pi T = \begin{array}{lll} 2 & 2 & 1 \\ 1 & 3 & \\ 3 & & \end{array}$.

We generally restrict the action of $\mathcal{S}_n$ to two subgroups. Let R_T be the subgroup of $\mathcal{S}_n$ which set-wise fixes the rows of T, namely, the row permutations. Denote this action by σT for $\sigma \in R_T$. Let C_T be the subgroup of $\mathcal{S}_n$ which set-wise fixes the columns of T, namely, the column permutations. This action is denoted τT for $\tau \in C_T$. If λ' is the conjugate partition of λ, (i.e., the partition corresponding to column lengths) we have that $R_T \approx \mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_\ell}$ and $C_T \approx \mathcal{S}_{\lambda'_1} \times \cdots \times \mathcal{S}_{\lambda'_{\ell'}}$. Viewing the subgroups under these isomorphisms, we can label the entry positions by labelling each row (resp. column) with 1 to λ_i (resp. $\lambda_{i'}$). Under these labellings we write σ (resp. τ) as a direct product of the permutations for each row (resp. column).

Example 2.2.2. Let $T = \begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & \end{array}$, then for R_T we view T as labelled $\begin{array}{lll} 1_T & 2_T & 3_T \\ 1_T & 2_T & \end{array}$. Likewise we use the labelling $\begin{array}{lll} 1_T & 1_T & 1_T \\ 2_T & 2_T & \end{array}$ for C_T. Applying these actions to T, gives the following sets:

$$\{\sigma T \mid \sigma \in R_T\} = \left\{ \begin{matrix} 1\,1\,2 \\ 2\,3 \end{matrix}, \begin{matrix} 1\,2\,1 \\ 2\,3 \end{matrix}, \begin{matrix} 2\,1\,1 \\ 2\,3 \end{matrix}, \begin{matrix} 1\,1\,2 \\ 3\,2 \end{matrix}, \begin{matrix} 1\,2\,1 \\ 3\,2 \end{matrix}, \begin{matrix} 2\,1\,1 \\ 3\,2 \end{matrix} \right\}$$

with each element occurring twice.

$$\{\tau T \mid \tau \in C_T\} = \left\{ \begin{matrix} 1\,1\,2 \\ 2\,3 \end{matrix}, \begin{matrix} 2\,1\,2 \\ 1\,3 \end{matrix}, \begin{matrix} 1\,3\,2 \\ 2\,1 \end{matrix}, \begin{matrix} 2\,3\,2 \\ 1\,1 \end{matrix} \right\}$$

$$\{\pi T \mid \pi \in \mathcal{S}_a\} = \left\{ \begin{matrix} 1\,1\,2 \\ 2\,3 \end{matrix}, \begin{matrix} 2\,2\,1 \\ 1\,3 \end{matrix}, \begin{matrix} 3\,3\,2 \\ 2\,1 \end{matrix}, \begin{matrix} 1\,1\,3 \\ 3\,2 \end{matrix}, \begin{matrix} 2\,2\,3 \\ 3\,1 \end{matrix}, \begin{matrix} 3\,3\,1 \\ 1\,2 \end{matrix} \right\}$$

$$C_T = \{()_T \times ()_T \times ()_T,\ (12)_T \times ()_T \times ()_T,\ ()_T \times (12)_T \times ()_T,\ (12)_T \times (12)_T \times ()_T\}$$

Note that the actions of σ and τ do not commute, but π commutes with both σ and τ.

Given λ a partition of $n = ab$, let $\mathcal{W}^{\lambda,a}$ be the set of all tableaux of shape λ and content $[b^a] = [b, \dots, b]$, where the entries are 1 to a. Let $\mathcal{S}^{\lambda,a}$ be the set of all semi-standard tableaux in $\mathcal{W}^{\lambda,a}$. These sets and the following constructions were developed by Doran in [7]. Note that when $b = 1$, $\mathcal{S}^{\lambda}$ give rise to the Specht modules which are discussed extensively in [15] and [18]. The partitions of n index the Specht modules S^{λ}, which in turn correspond to precisely the irreducible modules of $\mathcal{S}_n$.

From $\mathcal{W}^{\lambda,a}$ we can construct the complex vector space $W^{\lambda,a}$ with the tableaux as a basis. The action of $\mathcal{S}_a$ on the tableaux give rise to a permutation representation. Inside $W^{\lambda,a}$ we construct the following objects.

Definition 2.2.3. Let $T \in \mathcal{W}^{\lambda,a}$. Let $\epsilon(\tau)$ be the sign of τ as a permutation. Inside $W^{\lambda,a}$ we have:

a) $\mathbf{e}_T = \sum_{\sigma \in R_T} \sum_{\tau \in C_T} \epsilon(\tau)\sigma\tau T$

b) $\mathbf{q}_T = \sum_{\pi\in\mathcal{S}_a}\sum_{\sigma\in R_T}\sum_{\tau\in C_T}\epsilon(\tau)\pi\sigma\tau T$

c) $\mathbf{m}_T = \sum_{\pi\in\mathcal{S}_a}\sum_{\sigma\in R_T}\pi\sigma T$

The tableaux are independent basis for $W^{\lambda,a}$, so for $T_1, T_2 \in \mathcal{W}^{\lambda,a}$, we have $T_1+T_1 = 2T_1$ but $T_1+T_2 = T_1+T_2$.

Example 2.2.4. Let $T = \begin{matrix}11\\22\end{matrix}$, then

$$\mathbf{e}_T = 4\times\left\{\begin{matrix}11\\22\end{matrix} + \begin{matrix}22\\11\end{matrix}\right\} - 2\left\{\begin{matrix}12\\21\end{matrix} + \begin{matrix}21\\12\end{matrix} + \begin{matrix}21\\21\end{matrix} + \begin{matrix}12\\12\end{matrix}\right\}$$

Here, symmetry gives $\mathbf{q}_T = 2\mathbf{e}_T$.

Definition 2.2.5. From $\mathbf{e}_T$ and $\mathbf{m}_T$ we can construct the following subspaces of $W^{\lambda,a}$:

a) $S^{\lambda,a} = \mathbb{C}[\mathbf{e}_T | T \in \mathcal{W}^{\lambda,a}]$

b) $M^{\lambda,a} = \mathbb{C}[\mathbf{m}_T | T \in \mathcal{W}^{\lambda,a}]$

These spaces are $\mathcal{S}_a$-modules. We have $\{\mathbf{e}_T | T \in \mathcal{S}^{\lambda,a}\}$, a basis for $S^{\lambda,a}$, and $\{\mathbf{m}_T | T \in \mathcal{S}^{\lambda,a}/\mathcal{S}_a\}$, a basis for $M^{\lambda,a}$. The set $\{\mathbf{q}_T | T \in \mathcal{W}^{\lambda,a}\}$ generates $S^{\lambda,a}\cap M^{\lambda,a}$, but does not form a basis. Background on these spaces and the proofs of the statements may be found in [7].

In [7], Doran uses Gay's result from [10]:

Lemma 2.2.6 (Gay's Result). The multiplicity of the irreducible module S^λ in $1_{\mathcal{S}_b\wr\mathcal{S}_a}^{\mathcal{S}_{ab}}$ equals the multiplicity of the trivial representation in $S^{\lambda,a}$.

From this, Doran reformulated Foulkes' Conjecture to:

Lemma 2.2.7. The dimension of $S^{\lambda,a}\cap M^{\lambda,a}$ equals the multiplicity of the irreducible S^λ in $1_{\mathcal{S}_b\wr\mathcal{S}_a}^{\mathcal{S}_{ab}}$.

A proof of this lemma in terms of $\mathbf{q}_T$'s is presented in Appendix A. From this, Foulkes' Conjecture is equivalent to proving $\dim(S^{\lambda,a}\cap M^{\lambda,a}) \le \dim(S^{\lambda,b}\cap M^{\lambda,b})$ for $a \le b$ and all $\lambda \vdash n$. Proving Conjecture 2 is equivalent to showing that for all $\lambda \vdash n$, $\dim(S^{\lambda,a}\cap M^{\lambda,a}) \le \dim(S^{\lambda,c}\cap M^{\lambda,c})$ when $ab = n = cd$ with $c, d \ge a$.

Remark 2.2.8. In terms of tableaux, proving Theorem 1 is equivalent to exhibiting m_λ non-zero linearly independent $\mathbf{q}_T$, where T has shape λ and content $[d^c]$, with m_λ the multiplicity of S^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{\mathcal{S}_{2b}}$. Theorem 2 is provable by exhibiting a non-zero $\mathbf{q}_T$ with T having shape λ and content $[d^c]$ for all λ such that the multiplicity of S^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$ is non-zero. For Theorem 3 we want m_λ linearly independent tableaux T with content $[d^c]$ such that $\mathbf{q}_T$ is non-zero, where $\lambda = [\lambda_1, \lambda_2]$ and m_λ is the multiplicity of S^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$.

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