Generalized Foulkes’ Conjecture and Tableaux Construction

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Chapter 2

Background

2.1 Tableaux

A partition \( \lambda = [\lambda_1, \ldots, \lambda_\ell] \) of a number \( n \) is an ordered tuple of positive integers such that \( \sum \lambda_i = n \) and \( \lambda_i \geq \lambda_{i+1} > 0 \); it is denoted by \( \lambda \vdash n \). The length of \( \lambda \) is \( \ell \).

A Ferrers diagram is the set \([\lambda] = \{(i, j) \mid 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}\). We view \([\lambda]\) as a (left-justified) stack of boxes with row \( i \) having \( \lambda_i \) boxes.

A tableau of shape \( \lambda \) is a filling of the Ferrers diagram \([\lambda]\) with a set of elements, usually the positive integers. It is said to have content \( \alpha = [\alpha_1, \ldots, \alpha_k] \) if the integer \( i \) occurs exactly \( \alpha_i \) times. A tableau is semi-standard if the entries are weakly increasing across the rows and strictly increasing down the columns. This notation is standard and further discussion can be found in [7] and [18].

Example 2.1.1. Consider the following tableaux:

\[
\begin{align*}
P &= 1 & 2 & 5 \\
& 4 & 2 \\
& 3 \\
Q &= 1 & 1 & 5 \\
& 2 & 2 \\
& 3 & 3 \\
R &= 1 & 1 & 2 \\
& 2 & 3 & 3
\end{align*}
\]

\( P \) is a \([3, 2, 1] \)-tableau of content \( \alpha = [1, 2, 1, 1, 1] \), while \( Q \) has shape \( \lambda = [3, 2, 2] \) and content \( \alpha = [2, 2, 2, 0, 1] \). Similarly, \( R \) has shape \([3, 3]\) with content \([2^2] = [2, 2, 2]\). Both \( Q \) and \( R \) are semi-standard, but \( P \) is not.
2.2 Combinatorial Structures

There are two different permutation actions on tableaux: an action permuting the entry positions in the tableau and an action permuting the numbers filling the tableau. These actions commute with each other.

Let $T$ be a $\lambda$-tableau filled with the numbers 1 to $a$, where $\lambda \vdash n$. The permutation of entry positions corresponds to an action of $S_n$ on $T$. View the entry positions (i.e., boxes) of $T$ as labelled 1 to $n$. Then $\sigma \in S_n$ acts on $T$ by permuting the entries in the positions moved by $\sigma$. To avoid confusion between entry positions and numbers in $T$, we will denote all entry positions with the subscript $T$ when necessary. The permutation action of the numbers corresponds to their permutation by $\pi \in S_a$.

Example 2.2.1. Take $n = 6$, $\lambda = [3,2,1]$, and $a = 3$. Consider $T = \begin{array}{ccc}3 & 3 & 1 \\ 1 & 2 & \\ 2\end{array}$. In terms of entry positions, we label $T$ as $\begin{array}{ccc}1_T & 2_T & 3_T \\ 4_T & 5_T & \\ 6_T\end{array}$. For $\sigma = (23)_T \in S_6$, we have $\sigma T = \begin{array}{ccc}3 & 1 & 3 \\ 1_2 & & \\ 2\end{array}$, while the action of $\pi = (23) \in S_3$ gives $\pi T = \begin{array}{ccc}2 & 2 & 1 \\ 1 & 3 & \\ 3\end{array}$.

We generally restrict the action of $S_n$ to two subgroups. Let $R_T$ be the subgroup of $S_n$ which set-wise fixes the rows of $T$, namely, the row permutations. Denote this action by $\sigma T$ for $\sigma \in R_T$. Let $C_T$ be the subgroup of $S_n$ which set-wise fixes the columns of $T$, namely, the column permutations. This action is denoted $\tau T$ for $\tau \in C_T$. If $\lambda'$ is the conjugate partition of $\lambda$, (i.e., the partition corresponding to column lengths) we have that $R_T \approx S_{\lambda_1} \times \cdots \times S_{\lambda_i}$ and $C_T \approx S_{\lambda'_1} \times \cdots \times S_{\lambda'_\ell}$. Viewing the subgroups under these isomorphisms, we can label the entry positions by labelling each row (resp. column) with 1 to $\lambda_i$ (resp. $\lambda'_i$). Under these labellings we write $\sigma$ (resp. $\tau$) as a direct product of the permutations for each row (resp. column).

Example 2.2.2. Let $T = \begin{array}{cc}1 & 1 \\ 2 & 3\end{array}$, then for $R_T$ we view $T$ as labelled $\begin{array}{ccc}1_T & 2_T & 3_T \\ 1_T & 2_T & \end{array}$. Likewise we use the labelling $\begin{array}{ccc}1_T & 1_T & 1_T \\ 2_T & 2_T & \end{array}$ for $C_T$. Applying these actions to $T$, gives the following sets:
\{\sigma T \mid \sigma \in R_T\} = \left\{ \begin{array}{cccccc} 1 & 1 & 2 & 1 & 2 & 1 \\ 2 & 3 & 2 & 3 & 3 & 2 \end{array} \right\}

with each element occurring twice.

\{\tau T \mid \tau \in C_T\} = \left\{ \begin{array}{cccccc} 1 & 1 & 2 & 2 & 1 & 2 \\ 2 & 3 & 1 & 3 & 2 & 1 \\ 1 & 3 & 2 & 1 & 1 \end{array} \right\}

\{\pi T \mid \pi \in S_a\} = \left\{ \begin{array}{cccccc} 1 & 1 & 2 & 2 & 1 & 3 \\ 2 & 3 & 1 & 3 & 2 & 1 \\ 2 & 3 & 3 & 2 & 3 & 1 \end{array} \right\}

\quad C_T = \{() \times () \times ()_T, (12) \times () \times ()_T, () \times (12) \times ()_T, (12) \times (12) \times ()_T\}

Note that the actions of \(\sigma\) and \(\tau\) do not commute, but \(\pi\) commutes with both \(\sigma\) and \(\tau\).

Given \(\lambda\) a partition of \(n = ab\), let \(W^{\lambda,a}\) be the set of all tableaux of shape \(\lambda\) and content \([b^a] = [b, \ldots, b]\), where the entries are 1 to \(a\). Let \(S^{\lambda,a}\) be the set of all semi-standard tableaux in \(W^{\lambda,a}\). These sets and the following constructions were developed by Doran in [7]. Note that when \(b = 1\), \(S^\lambda\) give rise to the Specht modules which are discussed extensively in [15] and [18]. The partitions of \(n\) index the Specht modules \(S^\lambda\), which in turn correspond to precisely the irreducible modules of \(S_n\).

From \(W^{\lambda,a}\) we can construct the complex vector space \(W^{\lambda,a}\) with the tableaux as a basis. The action of \(S_n\) on the tableaux give rise to a permutation representation. Inside \(W^{\lambda,a}\) we construct the following objects.

**Definition 2.2.3.** Let \(T \in W^{\lambda,a}\). Let \(\epsilon(\tau)\) be the sign of \(\tau\) as a permutation. Inside \(W^{\lambda,a}\) we have:

\[ e_T = \sum_{\sigma \in R_T} \sum_{\tau \in C_T} \epsilon(\tau) \sigma T \]
b) $q_T = \sum_{\pi \in S} \sum_{\sigma \in R} \sum_{\tau \in C} \epsilon(\tau) \pi \sigma \tau T$

c) $m_T = \sum_{\pi \in S} \sum_{\sigma \in R} \pi \sigma T$

The tableaux are independent basis for $W^{\lambda,a}$, so for $T_1, T_2 \in W^{\lambda,a}$, we have $T_1 + T_1 = 2T_1$ but $T_1 + T_2 = T_1 + T_2$.

Example 2.2.4. Let $T = \begin{pmatrix} 11 \\ 22 \end{pmatrix}$, then

$$e_T = 4 \times \left\{ \begin{pmatrix} 11 \\ 22 \end{pmatrix} + \begin{pmatrix} 22 \\ 11 \end{pmatrix} \right\} - 2 \left\{ \begin{pmatrix} 12 \\ 21 \end{pmatrix} + \begin{pmatrix} 21 \\ 12 \end{pmatrix} + \begin{pmatrix} 12 \\ 12 \end{pmatrix} \right\}$$

Here, symmetry gives $q_T = 2e_T$.

Definition 2.2.5. From $e_T$ and $m_T$ we can construct the following subspaces of $W^{\lambda,a}$:

a) $S^{\lambda,a} = \mathbb{C}[e_T | T \in W^{\lambda,a}]$

b) $M^{\lambda,a} = \mathbb{C}[m_T | T \in W^{\lambda,a}]$

These spaces are $S_a$-modules. We have $\{e_T | T \in S^{\lambda,a}\}$, a basis for $S^{\lambda,a}$, and $\{m_T | T \in S^{\lambda,a}/S_a\}$, a basis for $M^{\lambda,a}$. The set $\{q_T | T \in W^{\lambda,a}\}$ generates $S^{\lambda,a} \cap M^{\lambda,a}$, but does not form a basis. Background on these spaces and the proofs of the statements may be found in [7].

In [7], Doran uses Gay’s result from [10]:

Lemma 2.2.6 (Gay’s Result). The multiplicity of the irreducible module $S^\lambda$ in $1_S^{\lambda,b}S_c$ equals the multiplicity of the trivial representation in $S^{\lambda,a}$.

From this, Doran reformulated Foulkes’ Conjecture to:

Lemma 2.2.7. The dimension of $S^{\lambda,a} \cap M^{\lambda,a}$ equals the multiplicity of the irreducible $S^\lambda$ in $1_S^{\lambda,b}S_c$.

A proof of this lemma in terms of $q_T$ ’s is presented in Appendix A. From this, Foulkes’ Conjecture is equivalent to proving $\dim(S^{\lambda,a} \cap M^{\lambda,a}) \leq \dim(S^{\lambda,b} \cap M^{\lambda,b})$ for $a \leq b$ and all $\lambda \vdash n$. Proving Conjecture 2 is equivalent to showing that for all $\lambda \vdash n$, $\dim(S^{\lambda,a} \cap M^{\lambda,a}) \leq \dim(S^{\lambda,c} \cap M^{\lambda,c})$ when $ab = n = cd$ with $c, d \geq a$. 
Remark 2.2.8. In terms of tableaux, proving Theorem 1 is equivalent to exhibiting $m_{\lambda}$ non-zero linearly independent $q_{T}$, where $T$ has shape $\lambda$ and content $[d^c]$, with $m_{\lambda}$ the multiplicity of $S^{\lambda}$ in $1_{S_0 S_2}$. Theorem 2 is provable by exhibiting a non-zero $q_{T}$ with $T$ having shape $\lambda$ and content $[d^c]$ for all $\lambda$ such that the multiplicity of $S^{\lambda}$ in $1_{S_0 S_3}$ is non-zero. For Theorem 3 we want $m_{\lambda}$ linearly independent tableaux $T$ with content $[d^c]$ such that $q_{T}$ is non-zero, where $\lambda = [\lambda_1, \lambda_2]$ and $m_{\lambda}$ is the multiplicity of $S^{\lambda}$ in $1_{S_0 S_3}$. 
Bibliography


198


[22] David Wales, personal communication.