Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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Chapter 11

The Alternating Character

Since Foulkes' Conjecture is based on the trivial character, it is natural to ask whether the ideas hold for the alternating character. The first question is, what we mean by the alternating character in terms of induced modules.

Consider, the 'alternating' character of the form $((-1)_{\mathcal{S}_{ab}} \downarrow_{\mathcal{S}_a \wr \mathcal{S}_b}) \uparrow^{\mathcal{S}_{ab}}$, that is the usual alternating character of $\mathcal{S}_{ab}$ restricted to the subgroup $\mathcal{S}_a \wr \mathcal{S}_b$, which is induced back up to $\mathcal{S}_{ab}$. A brief computer check of Foulkes' Conjecture using this character shows it holds for some small values of a and b. In fact, Foulkes' Conjecture is equivalent to the following conjecture using the alternating character.

Conjecture 3 (Foulkes' Conjecture for Alternating Characters). If $a \leq b$ then every irreducible character occurring in $((-1)_{\mathcal{S}_{ab}} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_a}) \uparrow^{\mathcal{S}_{ab}}$ occurs in $((-1)_{\mathcal{S}_{ab}} \downarrow_{\mathcal{S}_a \wr \mathcal{S}_b}) \uparrow^{\mathcal{S}_{ab}}$ with multiplicity greater than or equal to its multiplicity in $((-1)_{\mathcal{S}_{ab}} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_a}) \uparrow^{\mathcal{S}_{ab}}$.

Naturally, this conjecture also generalizes to:

Conjecture 4 (Generalized Foulkes' Conjecture for Alternating Characters). Given $n = ab$, $a \leq b$, if c, d are such that $cd = n$, and $c, d \geq a$, then every irreducible character occurring in $((-1)_{\mathcal{S}_n} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_a}) \uparrow^{\mathcal{S}_n}$ occurs in $((-1)_{\mathcal{S}_n} \downarrow_{\mathcal{S}_d \wr \mathcal{S}_c}) \uparrow^{\mathcal{S}_n}$ with multiplicity at least as large.

Showing the equivalences of Conjecture 1 or Conjecture 2, (Foulkes' Conjecture for trivial characters), and Conjectures 3 or 4 (Foulkes' Conjecture for alternating characters) is straightforward. We will assume Conjecture 1 (or Conjecture 2) holds and prove the alternating character version. The same argument shows the reverse equivalence.

Proof. First recall that if S is a subgroup of finite index in G, F an S-module and E a G-module over a field, then there is an isomorphism $\mathrm{Ind}_S^G(\mathrm{Res}_S(E) \otimes F) \simeq E \otimes \mathrm{Ind}_S^G(F)$. (See Chapter XVIII §7 of [16].)

Note that here we have used Ind for induction and Res for restriction of modules. Also, let $G = \mathcal{S}_n$, $S = \mathcal{S}_a \wr \mathcal{S}_b$ and $T = \mathcal{S}_b \wr \mathcal{S}_a$ or $\mathcal{S}_d \wr \mathcal{S}_c$ as appropriate. Let E be the G-module corresponding to the character (-1) on G. Since we are working over $\mathbb{C}$, we will use $\mathbb{C}$ to denote the trivial module over any group.

The characters we're comparing are $\chi_S = \mathrm{Ind}_S^G(\mathrm{Res}_S(E)) =(-1)_{\mathcal{S}_{ab}} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_a} \uparrow^{\mathcal{S}_{ab}}$ and $\chi_T = \mathrm{Ind}_T^G(\mathrm{Res}_T(E))$. Then $\chi_S \simeq \mathrm{Ind}_S^G(\mathrm{Res}_S(E) \otimes \mathbb{C}) \simeq E \otimes \mathrm{Ind}_S^G(\mathbb{C})$ by the isomorphism mentioned above. Similarly for χ_T. Switching notation back to characters, we get $\chi_S \simeq (-1)_G 1_S^G$ and $\chi_T \simeq (-1)_G 1_T^G$.

Since we've assumed Foulkes' Conjecture on trivial characters, we have $1_S^G = 1_T^G + \psi$ for some character ψ. Then $\chi_S = (-1)_G(1_T^G + \psi) = (-1)_T^G 1_T^G + (-1)_G \psi$. Hence $\chi_S \leq \chi_T$ as desired.

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Since we've proven Theorem 1 the argument above shows:

Theorem 14. If $2 \leq b$ then every irreducible character occurring in $((-1)_{\mathcal{S}_{2b}} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_2}) \uparrow^{\mathcal{S}_{2b}}$ occurs in $((-1)_{\mathcal{S}_{2b}} \downarrow_{\mathcal{S}_2 \wr \mathcal{S}_b}) \uparrow^{\mathcal{S}_{2b}}$ with multiplicity greater than or equal to its multiplicity in $((-1)_{\mathcal{S}_{2b}} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_2}) \uparrow^{\mathcal{S}_{2b}}$.

Similarly, Theorem 2 gives:

Theorem 15. Given $n = 3b$, $3 \leq b$, if c, d are such that $cd = n$, and $c, d \geq 3$, then every irreducible character occurring in $((-1)_{\mathcal{S}_n} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_3}) \uparrow^{\mathcal{S}_n}$ occurs in $((-1)_{\mathcal{S}_n} \downarrow_{\mathcal{S}_d \wr \mathcal{S}_c}) \uparrow^{\mathcal{S}_n}$.

While Theorem 3 shows:

Theorem 16. Let $n = 3b = cd$, with c, $d \geq 3$ and let $\lambda = [\lambda_1, \lambda_2]$ be a two row partition of n. Then every irreducible character χ^λ occurring in $((-1)_{\mathcal{S}_n} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_3}) \uparrow^{\mathcal{S}_n}$ occurs in $((-1)_{\mathcal{S}_n} \downarrow_{\mathcal{S}_d \wr \mathcal{S}_c}) \uparrow^{\mathcal{S}_n}$ with multiplicity at least as large.

Given the success of replacing the trivial character in Foulkes' Conjecture with this 'alternating' character, it is natural to investigate if other definitions of an alternating character yield similar results. One suggestion was to try $(-1)^{\mathcal{S}_{ab}}_{\mathcal{S}_a \wr \mathcal{A}_b} - (-1)^{\mathcal{S}_{ab}}_{\mathcal{S}_a \wr \mathcal{S}_b}$ for an induced alternating character in place of $1^{\mathcal{S}_{ab}}_{\mathcal{S}_a \wr \mathcal{S}_b}$ in Foulkes' Conjecture. Alas, a simple computer check via GAP [9] shows Foulkes' Conjecture for this character fails when $a = 3$ and $b = 4$. Other variations on this character, such as $(-1)^{\mathcal{S}_{ab}}_{\mathcal{A}_a \wr \mathcal{S}_b} - (-1)^{\mathcal{S}_{ab}}_{\mathcal{S}_a \wr \mathcal{S}_b}$ also fail at those values.

Chapter 12

Discussion of General Results

Theorems 1, 2, and 3 extend the current research on Foulkes' Conjecture. Although, the proof used combinatorial techniques on Young tableaux, the results correspondingly apply to areas such as Shur functions, Rational Homotopy Theory, and other means of interpreting Foulkes' Conjecture. In addition we may interpret the Foulkes' Conjecture using the alternating character via these theorems, which we discussed in Chapter 11.

While the construction of the tableaux themselves are cumbersome, the development of the theory illustrates new approaches to Young tableaux. These concepts could be carried forth in contexts involving tableaux other than its usage here. Although the main theorems are specific to the cases $a = 2$ and $a = 3$, some general results arise from this study.

The main theoretical techniques of this paper are that of weight-set counting in Theorem 4, the application of Theorem 8, and the use of maximality to show linear independence.

The theory and technique of weight-set counting developed in this paper can be implemented in general, as can the concept of maximal form. While we only used tableaux with three or fewer rows, the theoretical foundations of weight-set counting have been laid for tableaux of an arbitrary number of rows. Although the computations are impractical for a random tableau, the counting works smoothly for tableaux with suitable symmetries, particularly those tableaux in maximal form.

Moreover, the technique of weight-set counting is not dependent on a filling of content $[b^a]$. However, for non-uniform contents, one must watch carefully the action of $\mathcal{S}_a$; the weight-set counting may need to count all rows and the definition of maximality will need adjustment.

Similarly, the usage of Theorem 8 in constructing larger tableaux will also work for other contents and row quantities. The use of the Lemma 3.4.9, to show weight-set disjointness by maximality, however, has only been defined for three row partitions. It should be possible to generalize it for other partitions.

Tableau maximality is a very useful concept for showing weight-set disjointness and applying Theorem 8. It is vital in proving linear independence of tableaux. Linear independence through tableau maximality should allow more progress on issues such as multiplicity.

The methods of proving Theorem 2 could also apply to proving Conjecture 2 with other a's, not including multiplicities. Unfortunately, the computations are likely to be somewhat cumbersome, especially the establishments of non-zero shapes as done by Theorem 9. However, should those parameters be established through other techniques, the tableaux constructed for Theorem 2 should provide nearly all the needed shapes with three or fewer rows, thus reducing the work substantially. Moreover, the reduction procedures will also apply.

Specifically, given any $n = ab = cd$, if the shapes having multiplicity zero in $\mathcal{S}_b \wr \mathcal{S}_a$ are bounded, then to prove the generalized Foulkes' Conjecture for arbitrary c, $d \geq a$, we should only need to prove it for a limited number of c's. For instance, suppose a shape had multiplicity zero only if $\lambda_i - \lambda_{i+1} \leq f_i$. (For $n = 3b$, we had $f_1 = f_2 = 4$.) Assume d is even (the odd case, though more cumbersome should follows analogously). Then given a tableau, we can 'peel off' a column block of size d with the appropriate row length, for instance, $U_1(d)$ and $P_1(d)$ are the two and three row versions. We can repeat this process so long as $\lambda_i - \lambda_{i+1} > d + f_i$ (and there are at least as many elements as there are rows). Hence in the end, we need only construct a tableau with $\lambda_i - \lambda_{i+1} \leq d + f_i$. This tableau will need at most $\frac{1}{d}\sum_i i(d + f_i)$ elements, hence c will be bounded by this number. This should imply, given the f_i, if the generalized Foulkes' Conjecture is true for c up to some bound, it is true for all c. (Presumably, if these tableaux exists, we can find versions with maximal/disjoint weight sets as needed.) The existence of the f_i seems probable since Theorem 2 implies $f_1,\ f_2 \leq 4$ if $3|n$. It may be the case, as in Theorem 1 that the parity of λ_i strongly effects the multiplicity. However, since the 'peeling off' does not change the parity, this process should still go through.

In addition to this procedure, the investigations of Theorems 1 and 2 yield some general results. Take the character $1_{\mathcal{S}_a \wr \mathcal{S}_b}^{\mathcal{S}_{ab}}$ and consider the irreducible $\mathcal{S}^\lambda$ corresponding to $\lambda = [\lambda_1, \lambda_2]$. Then $\mathcal{S}^\lambda$ always has non-zero multiplicity whenever λ_1 and λ_2 are even. Moreover, this multiplicity is zero whenever $\lambda_2 = 1$ regardless of the choice of a and b. These theorems also have implications regarding the generalized Gaussian polynomial, as discussed in Chapter 10.

Finally, the techniques within the proof of Theorem 3 should extend beyond two row tableaux. Specifically, Theorem 2 can probably be strengthened to include multiplicities for all partitions. Such a result should follow the ideas of Theorem 3, though sufficient linearly independent three row tableaux for $c = 4$, 5, and 6 must first be established. However, weight-set maximality should be sufficient to demonstrate linear independence. In all, these results provide a strong foundation for those wishing to study the representation theory of wreath products of symmetric groups via tableaux.

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