Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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Chapter 10

Two Row Partitions and the Gaussian Polynomial

Let $n = ab$ with a, $b \in \mathbb{N}$ and take $\ell \in \mathbb{N}$ such that $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$. Let $\mathcal{P}_a^b(\ell)$ be the numbers of partitions of n having at most a parts each of size less that or equal to b, that is partitions of n fitting inside a $b \times a$ rectangle. Then $\mathcal{P}_a^b(\ell)$ is the co-efficient of q^ℓ in the Gaussian polynomial $\left[{a+b \atop b} \right]_q$, as in [1]. The Gaussian polynomial, $\left[{a+b \atop b} \right]_q$, is also called the Gaussian co-efficient or the generalized q-binomial coefficient.

Lemma 10.0.1. Take $n = ab$ and $\lambda = [n - \ell, \ell]$. Let $K = \mathcal{S}_b \wr \mathcal{S}_a$. The multiplicity of χ^λ in $1_K^{\mathcal{S}_n}$ equals $\mathcal{P}_a^b(\ell) - \mathcal{P}_a^b(\ell - 1)$.

Proof. Let $H = \mathcal{S}_{n-\ell} \times \mathcal{S}_\ell$ and $H' = \mathcal{S}_{n-\ell+1} \times \mathcal{S}_{\ell-1}$. Then $\chi^{[n-\ell,\ell]} = 1_H^{\mathcal{S}_n} - 1_{H'}^{\mathcal{S}_n}$ by the determinantal formula [14]. So $\langle 1_K^{\mathcal{S}_n}, \chi^{[n-\ell,\ell]} \rangle_{\mathcal{S}_n} = \langle 1_K^{\mathcal{S}_n}, 1_H^{\mathcal{S}_n} \rangle_{\mathcal{S}_n} - \langle 1_K^{\mathcal{S}_n}, 1_{H'}^{\mathcal{S}_n} \rangle_{\mathcal{S}_n}$. Hence it suffices to show $\langle 1_K^{\mathcal{S}_n}, 1_H^{\mathcal{S}_n} \rangle_{\mathcal{S}_n} = \mathcal{P}_a^b(\ell)$.

Now $\langle 1_K^{\mathcal{S}_n}, 1_H^{\mathcal{S}_n} \rangle_{\mathcal{S}_n}$ is the number of orbits of K acting on the cosets of H in $\mathcal{S}_n$ [11]. View the numbers 1 to n in blocks of size b, that is

$$|1, 2, \ldots, b|b + 1, \ldots 2b| \cdots |(a - 1)b + 1, \ldots ab|$$

The copies of H in $\mathcal{S}_n$ correspond to the different ways $\mathcal{S}_\ell$ sits in $\mathcal{S}_n$, that is subsets of $\{1, \ldots n\}$ of size ℓ. Given such a subset L (corresponding to a copy of H) it will be broken into a parts by intersection with the blocks above. Let μ_i be the size of the part of L in the ith block. Since K acts by $\mathcal{S}_b$ on each of the blocks, L is equivalent (under K) to a subset L' where the first μ_i numbers $\{i \cdot b+1, \dots i \cdot b+\mu_i\}$ are chosen from block i (starting with the 0^{th} block). Since K also has the wreath product action by $\mathcal{S}_a$ acting on the blocks, L' is equivalent to the subset L^*, where the blocks are reordered so the $\mu_i \geq \mu_{i+1}$. Hence L^* corresponds to a partition of the number ℓ into a parts of size at most b and every such partition corresponds to a copy of H in $\mathcal{S}_n$.

So every such partition is contained in some orbit of K on $\mathcal{S}_n/H$, and every orbit contains some such partition. Hence it suffices to shows that no two partitions are in the same orbit. Say $\mu = [\mu_0, \dots \mu_{a-1}]$ and $\nu = [\nu_0, \dots \nu_{a-1}]$ are partitions of ℓ where we allow $0 \leq \mu_i,\ \nu_i \leq b$. If μ and ν are in the same orbit, then there exists $g \in K$ such that $g \cdot \{i \cdot b+j | 0 \leq i \leq a-1, 1 \leq j \leq \mu_i\} = \{i \cdot b+j | 0 \leq i \leq a-1, 1 \leq j \leq \nu_i\}$. So $g(ib+j) = k_{i,j}b + c_{i,j}$. Since g moves complete blocks, we must have $k_{i,j} = k_{i,j'}$ for all $1 \leq j, j' \leq \mu_i$. As the action is injective, we must then have $c_{i,j} \neq c_{i,j'}$ for $j \neq j'$. Hence looking at the image, we have $\mu_i = |\{c_{i,j}\}| \leq \nu_{k_i}$.

Take $\mu \trianglerighteq \nu$, $\mu \neq \nu$. There exists i such that $\mu_i > \nu_i$ and $\mu_{i'} = \nu_{i'}$ for all $i' < i$. Then there are i such $\nu_{i'}$ with $\nu_{i'} \geq \mu_i$. But if $g \cdot \mu = \nu$ then $\mu_i = |\{c_{i,j}\}| \leq \nu_{k_i}$ implies there are at least $i+1$ such $\nu_{i'}$, which is a contradiction. Hence no orbit contains two such partitions, which finishes the proof. □

The ideas behind this proof are due to J. Saxl stemming from discussions of his paper [13].

Since the multiplicity of irreducibles in induced characters is non-negative [16], this lemma implies the well-known unimodality of the Gaussian coefficients [1]. Now $\mathcal{P}_a^b(\ell) = \mathcal{P}_b^a(\ell)$, since $\left[{a+b \atop b} \right]_q = \left[{a+b \atop a} \right]_q$ by taking conjugate partitions. Hence this lemma shows that Foulkes' Conjecture always holds for two row partitions, which is discussed in [14].

We can also interpret our results on the generalized Foulkes' Conjecture in terms of the Gaussian coefficient. From Theorem 1 we have:

Theorem 12. If $n = 2b = cd$, with $c, d \geq 2$, then for $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$,

$$\mathcal{P}_c^d(\ell) - \mathcal{P}_c^d(\ell-1) \geq \mathcal{P}_2^b(\ell) - \mathcal{P}_2^b(\ell-1)$$

Similarly, Theorem 3 gives:

Theorem 13. If $n = 3b = cd$, with $c, d \geq 3$, then for $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$,

$$\mathcal{P}_c^d(\ell) - \mathcal{P}_c^d(\ell - 1) \geq \mathcal{P}_3^b(\ell) - \mathcal{P}_3^b(\ell - 1)$$

Hence our results give insight into the relationship between the rates of growth of different Gaussian coefficients.

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