Generalized Foulkes’ Conjecture and Tableaux Construction

Thesis by

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Chapter 10

Two Row Partitions and the Gaussian Polynomial

Let \( n = ab \) with \( a, b \in \mathbb{N} \) and take \( \ell \in \mathbb{N} \) such that \( 1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor \). Let \( \mathcal{P}_a^b(\ell) \) be the numbers of partitions of \( n \) having at most \( a \) parts each of size less that or equal to \( b \), that is partitions of \( n \) fitting inside a \( b \times a \) rectangle. Then \( \mathcal{P}_a^b(\ell) \) is the co-efficient of \( q^\ell \) in the Gaussian polynomial \( [a+b]_q^b \), as in [1]. The Gaussian polynomial, \( [a+b]_q^b \), is also called the Gaussian co-efficient or the generalized \( q \)-binomial coefficient.

Lemma 10.0.1. Take \( n = ab \) and \( \lambda = [n-\ell, \ell] \). Let \( K = S_b \wr S_a \). The multiplicity of \( \chi^\lambda \) in \( 1_{S_n}^S K \) equals \( \mathcal{P}_a^b(\ell) - \mathcal{P}_a^b(\ell-1) \).

Proof. Let \( H = S_{n-\ell} \times S_\ell \) and \( H' = S_{n-\ell+1} \times S_{\ell-1} \). Then \( \chi^{[n-\ell,\ell]} = 1_{S_n}^{S_H} - 1_{S_n}^{S_H'} \) by the determinantal formula [14]. So \( \langle 1_{S_H}^{S_n}, \chi^{[n-\ell,\ell]} \rangle_{S_n} = \langle 1_{S_H}^{S_n}, 1_{S_H}^{S_H} \rangle_{S_n} - \langle 1_{S_H}^{S_n}, 1_{S_H'}^{S_H} \rangle_{S_n} \). Hence it suffices to show \( \langle 1_{S_H}^{S_n}, 1_{S_H'}^{S_H} \rangle_{S_n} = \mathcal{P}_a^b(\ell) \). Now \( \langle 1_{S_H}^{S_n}, 1_{S_H}^{S_H} \rangle_{S_n} \) is the number of orbits of \( K \) acting on the cosets of \( H \) in \( S_n \) [11]. View the numbers 1 to \( n \) in blocks of size \( b \), that is

\[
|1, 2, \ldots, b|b+1, \ldots 2b| \cdots |(a-1)b+1, \ldots ab|
\]

The copies of \( H \) in \( S_n \) correspond to the different ways \( S_\ell \) sits in \( S_n \), that is subsets of \( \{1, \ldots n\} \) of size \( \ell \). Given such a subset \( L \) (corresponding to a copy of \( H \)) it will be broken into \( a \) parts by intersection with the blocks above. Let \( \mu_i \) be the size of the part of \( L \) in the \( i \)th block. Since \( K \) acts by \( S_b \) on each of the blocks, \( L \) is equivalent
(under $K$) to a subset $L'$ where the first $\mu_i$ numbers $\{i \cdot b + 1, \ldots i \cdot b + \mu_i\}$ are chosen from block $i$ (starting with the $0^{th}$ block). Since $K$ also has the wreath product action by $S_a$ acting on the blocks, $L'$ is equivalent to the subset $L^*$, where the blocks are reordered so the $\mu_i \geq \mu_{i+1}$. Hence $L^*$ corresponds to a partition of the number $\ell$ into $a$ parts of size at most $b$ and every such partition corresponds to a copy of $H$ in $S_n$.

So every such partition is contained in some orbit of $K$ on $S_n/H$, and every orbit contains some such partition. Hence it suffices to show that no two partitions are in the same orbit. Say $\mu = [\mu_0, \ldots \mu_{a-1}]$ and $\nu = [\nu_0, \ldots \nu_{a-1}]$ are partitions of $\ell$ where we allow $0 \leq \mu_i, \nu_i \leq b$. If $\mu$ and $\nu$ are in the same orbit, then there exists $g \in K$ such that $g \cdot \{i \cdot b + j | 0 \leq i \leq a-1, 1 \leq j \leq \mu_i\} = \{i \cdot b + j | 0 \leq i \leq a-1, 1 \leq j \leq \nu_i\}$. So $g(i b + j) = k_{i,j} b + c_{i,j}$. Since $g$ moves complete blocks, we must have $k_{i,j} = k_{i,j'}$ for all $1 \leq j, j' \leq \mu_i$. As the action is injective, we must then have $c_{i,j} \neq c_{i,j'}$ for $j \neq j'$. Hence looking at the image, we have $\mu_i = |\{c_{i,j}\}| \leq \nu_{k_i}$.

Take $\mu \succeq \nu$, $\mu \neq \nu$. There exists $i$ such that $\mu_i > \nu_i$ and $\mu_i' = \nu_i'$ for all $i' < i$. Then there are $i$ such $\nu_{i'}$ with $\nu_{i'} \geq \mu_i$. But if $g \cdot \mu = \nu$ then $\mu_i = |\{c_{i,j}\}| \leq \nu_{k_i}$ implies there are at least $i + 1$ such $\nu_{i'}$, which is a contradiction. Hence no orbit contains two such partitions, which finishes the proof.

The ideas behind this proof are due to J. Saxl stemming from discussions of his paper [13].

Since the multiplicity of irreducibles in induced characters is non-negative [16], this lemma implies the well-known unimodality of the Gaussian coefficients [1]. Now $P^a_b(\ell) = P^a(\ell)$, since $\left[\begin{smallmatrix} a+b \\ b \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right]_q$ by taking conjugate partitions. Hence this lemma shows that Foulkes’ Conjecture always holds for two row partitions, which is discussed in [14].

We can also interpret our results on the generalized Foulkes’ Conjecture in terms of the Gaussian coefficient. From Theorem 1 we have:

**Theorem 12.** If $n = 2b = cd$, with $c, d \geq 2$, then for $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$,

$$P^d_c(\ell) - P^d_c(\ell - 1) \geq P^b_c(\ell) - P^b(\ell - 1)$$
Similarly, Theorem 3 gives:

\textbf{Theorem 13.} If \( n = 3b = cd \), with \( c, d \geq 3 \), then for \( 1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor \),

\[ P^d_c(\ell) - P^d_c(\ell - 1) \geq P^b_3(\ell) - P^b_3(\ell - 1) \]

Hence our results give insight into the relationship between the rates of growth of different Gaussian coefficients.
Bibliography


[22] David Wales, personal communication.