Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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Chapter 1

Introduction and Statement of Main Results

Foulkes' Conjecture is an outstanding problem in the areas of plethysms, rational homotopy theory, multisymmetric functions, and representation theory of symmetric groups. In representation terms, Foulkes' Conjecture deals with induced permutation characters of wreath products of symmetric groups. The wreath product of symmetric groups $\mathcal{S}_a$ and $\mathcal{S}_b$, denoted $\mathcal{S}_a \wr \mathcal{S}_b$, is the normalizer of the Young subgroup $\mathcal{S}_a \times \cdots \times \mathcal{S}_a$ (b times) in $\mathcal{S}_{ab}$. Let 1_H be the trivial representation of a group H and 1_H^G the induced representation from H to G. From this we can state Foulkes' Conjecture:

Conjecture 1. (Foulkes' Conjecture) If $a \leq b$ then every irreducible character occurring as a constituent in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_{ab}}$ occurs in $1_{\mathcal{S}_a \wr \mathcal{S}_b}^{\mathcal{S}_{ab}}$ with multiplicity greater than or equal to its multiplicity in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_{ab}}$.

Foulkes made this conjecture in [8] from his work on plethysms. The $a = 2$ case was contained in Thrall's 1942 work on symmetrized Kronecker powers, [20], and proved again in [14] by James and Kerber using Gaussian coefficients. Coker also gives a proof for $a = 2$ using eigenvalues in [5], while Doran used transition matrices in [7]. The $a = 3$ case was proven by Dent and Siemons using mappings of unordered partitions [6]. In [2], Black and List formulated Foulkes' Conjecture in terms of matrix incidences and Wu has rephrase it in terms of rational homotopy theory in [23]. Howe, in [12], used a plethystic approach to interpret Foulkes' Conjecture via canonical morphisms between symmetric power modules. Using this, Brion, [4], showed the conjecture holds for b sufficiently large with respect to a and Briand [3] proved Foulkes' Conjecture for $a = 4$. Also using symmetric powers and plethysms, Stanley, [19], places Foulkes' Conjecture inside a larger body of open positivity conjectures in Algebraic Combinatorics. With a more combinatorial approach, Doran gave additional formulations in [7] using tableaux spaces. Doran also suggested generalizing Foulkes' Conjecture to:

Conjecture 2. (Generalized Foulkes' Conjecture) Given $n = ab$, $a \leq b$, if c, d are such that $cd = n$, and $c, d \geq a$, then every irreducible character occurring as a constituent in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with multiplicity at least as large.

For $c = a$ and $d = b$ this becomes the standard Foulkes' Conjecture. Note that $c, d \geq a$ is necessary. This is easily verified by using GAP, [9], which shows that some irreducibles in $1_{\mathcal{S}_4 \wr \mathcal{S}_3}^{\mathcal{S}_{12}}$ do not occur in $1_{\mathcal{S}_6 \wr \mathcal{S}_2}^{\mathcal{S}_{12}}$. Conjecture 2 holds for small n, (less than 28), by computer verification also using GAP, [9]. In Chapter 5 we will prove it holds for $a = 2$ by construction. Namely, we will show:

Theorem 1. Given $b \geq 2$, let $n = 2b$. If c, d are such that $cd = n$, and $c, d \geq 2$, then every irreducible occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with equal or larger multiplicity.

We can also prove the following variation on Conjecture 2 for $a = 3$. The bulk of the proof is discussed in Chapter 6 with supporting details in Chapters 7 and 8.

Theorem 2. Let $n = 3b = cd$, with $c, d \geq 3$. Then every irreducible character occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$.

This theorem can be strengthened when the irreducibles involved correspond to two row partitions of n. We prove this version in Chapter 9.

Theorem 3. Let $n = 3b = cd$, with $c, d \geq 3$ and let $\lambda = [\lambda_1, \lambda_2]$ be a two row partition of n. Then every irreducible character χ^λ occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with multiplicity at least as large.

In Chapter 2 we describe the necessary concepts to approach the Foulkes' Conjecture combinatorially using tableaux. Much of this framework was developed by

Doran in [7]. We develop the theory and techniques behind constructing appropriate tableaux in Chapter 3. In Chapter 4 we completely classify and discuss all tableaux that occur in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$. The proof of Theorem 1 is given in Chapter 5. We prove Theorem 2 in Chapter 6, though the required tableau constructions are postponed until Chapter 7. In Chapter 8 we show that these tableaux suffice to cover all necessary cases. We prove Theorem 3 in Chapter 9. The corresponding results for the alternating character is given in Chapter 11. Theorems 1 and 3 also can be interpreted in terms of the Gaussian coefficient; this is discussed in Chapter 10. Further implications of all these results are listed in Chapter 12. Some of these results appear in a forthcoming article in the Journal of Algebra, [21], namely, Chapters 1, 2, 3, 5, and portions of Chapters 4 and 6.

Bibliography

[1] George E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original. MR 99c:11126

[2] S. C. Black and R. J. List, *A note on plethysm*, European J. Combin. **10** (1989), no. 1, 111–112. MR 89m:20011

[3] Emmanuel Briand, *Polynômes multisymétriques*, Ph. D. dissertation, University Rennes I, Rennes, France, October 2002.

[4] Michel Brion, *Stable properties of plethysm: on two conjectures of Foulkes*, Manuscripta Math. **80** (1993), no. 4, 347–371. MR 95c:20056

[5] C. Coker, *A problem related to Foulkes's conjecture*, Graphs Combin. **9** (1993), no. 2, 117–134. MR 94g:20019

[6] Suzie C. Dent and Johannes Siemons, *On a conjecture of Foulkes*, J. Algebra **226** (2000), no. 1, 236–249. MR 2001f:20026

[7] William F. Doran, IV, *On Foulkes' conjecture*, J. Pure Appl. Algebra **130** (1998), no. 1, 85–98. MR 99h:20014

[8] H. O. Foulkes, *Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form*, J. London Math. Soc. **25** (1950), 205–209. MR 12,236e

[9] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.3*, 2002, (`http://www.gap-system.org`).

[10] David A. Gay, *Characters of the Weyl group of $SU(n)$ on zero weight spaces and centralizers of permutation representations*, Rocky Mountain J. Math. **6** (1976), no. 3, 449–455. MR 54 #2886

[11] Larry C. Grove, *Groups and characters*, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1997, A Wiley-Interscience Publication. MR 98e:20012

[12] Roger Howe, $(\mathrm{GL}_n, \mathrm{GL}_m)$*-duality and symmetric plethysm*, Proc. Indian Acad. Sci. Math. Sci. **97** (1987), no. 1-3, 85–109 (1988). MR 90b:22020

[13] N. F. J. Inglis, R. W. Richardson, and J. Saxl, *An explicit model for the complex representations of S_n*, Arch. Math. (Basel) **54** (1990), no. 3, 258–259. MR 91d:20017

[14] G. James and A. Kerber, *Representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley, Reading, MA, 1981.

[15] G. D. James, *The representation theory of the symmetric group*, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978.

[16] Serge Lang, *Algebra*, 3 ed., Addison Wesley, Reading Massachusetts, 1999.

[17] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications. MR 96h:05207

[18] Bruce E. Sagan, *The symmetric group*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991, Representations, combinatorial algorithms, and symmetric functions. MR 93f:05102

[19] Richard P. Stanley, *Positivity problems and conjectures in algebraic combinatorics*, Mathematics: Frontiers and Perspectives (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.), American Mathematical Society, Providence, RI, 2000, pp. 295–319.

[20] R. M. Thrall, *On symmetrized Kronecker powers and the structure of the free Lie ring*, Amer. J. Math. **64** (1942), 371–388. MR 3,262d

[21] Rebecca Vessenes, *Foulkes' conjecture and tableaux construction*, J. Albegra (2004), forthcoming.

[22] David Wales, personal communication.

[23] Jie Wu, *Foulkes conjecture in representation theory and its relations in rational homotopy theory*, `http://www.math.nus.edu.sg/~matwujie/Foulkes.pdf`.