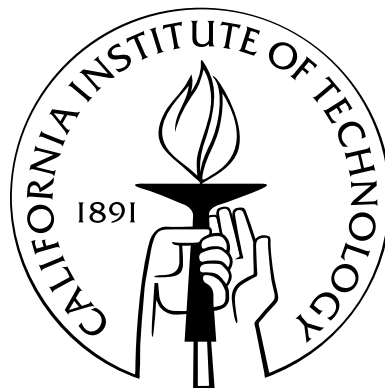


# Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

Rebecca Vessenes

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To God  
through whom all things, even this, are possible

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# Abstract

Foulkes conjectured that for  $n = ab$  and  $a \leq b$ , every irreducible module occurring as a constituent in  $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_n}$  occurs with greater or equal multiplicity in  $1_{\mathcal{S}_a \wr \mathcal{S}_b}^{\mathcal{S}_n}$ . We generalize part of this to say those irreducibles also occur in  $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ , where  $cd = n$  and  $c, d \geq a$ . We prove the generalized conjecture for  $a = 2$  and  $a = 3$ , by explicitly constructing the corresponding tableaux. We also prove the multiplicity constraint for certain cases. For these proofs we develop a theory of construction conditions for tableaux giving rise to  $\mathcal{S}_b \wr \mathcal{S}_a$  modules and in doing so, completely classify all such tableaux for  $a = 2$  and  $a = 3$ .

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