Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

Rebecca Vessenes

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To God

through whom all things, even this, are possible

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Abstract

Foulkes conjectured that for $n = ab$ and $a \leq b$, every irreducible module occurring as a constituent in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_n}$ occurs with greater or equal multiplicity in $1_{\mathcal{S}_a \wr \mathcal{S}_b}^{\mathcal{S}_n}$. We generalize part of this to say those irreducibles also occur in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$, where $cd = n$ and $c, d \geq a$. We prove the generalized conjecture for $a = 2$ and $a = 3$, by explicitly constructing the corresponding tableaux. We also prove the multiplicity constraint for certain cases. For these proofs we develop a theory of construction conditions for tableaux giving rise to $\mathcal{S}_b \wr \mathcal{S}_a$ modules and in doing so, completely classify all such tableaux for $a = 2$ and $a = 3$.

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