Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

Rebecca Vessenes

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To God

through whom all things, even this, are possible

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Abstract

Foulkes conjectured that for $n = ab$ and $a \leq b$, every irreducible module occurring as a constituent in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_n}$ occurs with greater or equal multiplicity in $1_{\mathcal{S}_a \wr \mathcal{S}_b}^{\mathcal{S}_n}$. We generalize part of this to say those irreducibles also occur in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$, where $cd = n$ and $c, d \geq a$. We prove the generalized conjecture for $a = 2$ and $a = 3$, by explicitly constructing the corresponding tableaux. We also prove the multiplicity constraint for certain cases. For these proofs we develop a theory of construction conditions for tableaux giving rise to $\mathcal{S}_b \wr \mathcal{S}_a$ modules and in doing so, completely classify all such tableaux for $a = 2$ and $a = 3$.

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Chapter 1

Introduction and Statement of Main Results

Foulkes' Conjecture is an outstanding problem in the areas of plethysms, rational homotopy theory, multisymmetric functions, and representation theory of symmetric groups. In representation terms, Foulkes' Conjecture deals with induced permutation characters of wreath products of symmetric groups. The wreath product of symmetric groups $\mathcal{S}_a$ and $\mathcal{S}_b$, denoted $\mathcal{S}_a \wr \mathcal{S}_b$, is the normalizer of the Young subgroup $\mathcal{S}_a \times \cdots \times \mathcal{S}_a$ (b times) in $\mathcal{S}_{ab}$. Let 1_H be the trivial representation of a group H and 1_H^G the induced representation from H to G. From this we can state Foulkes' Conjecture:

Conjecture 1. (Foulkes' Conjecture) If $a \leq b$ then every irreducible character occurring as a constituent in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_{ab}}$ occurs in $1_{\mathcal{S}_a \wr \mathcal{S}_b}^{\mathcal{S}_{ab}}$ with multiplicity greater than or equal to its multiplicity in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_{ab}}$.

Foulkes made this conjecture in [8] from his work on plethysms. The $a = 2$ case was contained in Thrall's 1942 work on symmetrized Kronecker powers, [20], and proved again in [14] by James and Kerber using Gaussian coefficients. Coker also gives a proof for $a = 2$ using eigenvalues in [5], while Doran used transition matrices in [7]. The $a = 3$ case was proven by Dent and Siemons using mappings of unordered partitions [6]. In [2], Black and List formulated Foulkes' Conjecture in terms of matrix incidences and Wu has rephrase it in terms of rational homotopy theory in [23]. Howe, in [12], used a plethystic approach to interpret Foulkes' Conjecture via canonical morphisms between symmetric power modules. Using this, Brion, [4], showed the conjecture holds for b sufficiently large with respect to a and Briand [3] proved Foulkes' Conjecture for $a = 4$. Also using symmetric powers and plethysms, Stanley, [19], places Foulkes' Conjecture inside a larger body of open positivity conjectures in Algebraic Combinatorics. With a more combinatorial approach, Doran gave additional formulations in [7] using tableaux spaces. Doran also suggested generalizing Foulkes' Conjecture to:

Conjecture 2. (Generalized Foulkes' Conjecture) Given $n = ab$, $a \leq b$, if c, d are such that $cd = n$, and $c, d \geq a$, then every irreducible character occurring as a constituent in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with multiplicity at least as large.

For $c = a$ and $d = b$ this becomes the standard Foulkes' Conjecture. Note that $c, d \geq a$ is necessary. This is easily verified by using GAP, [9], which shows that some irreducibles in $1_{\mathcal{S}_4 \wr \mathcal{S}_3}^{\mathcal{S}_{12}}$ do not occur in $1_{\mathcal{S}_6 \wr \mathcal{S}_2}^{\mathcal{S}_{12}}$. Conjecture 2 holds for small n, (less than 28), by computer verification also using GAP, [9]. In Chapter 5 we will prove it holds for $a = 2$ by construction. Namely, we will show:

Theorem 1. Given $b \geq 2$, let $n = 2b$. If c, d are such that $cd = n$, and $c, d \geq 2$, then every irreducible occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with equal or larger multiplicity.

We can also prove the following variation on Conjecture 2 for $a = 3$. The bulk of the proof is discussed in Chapter 6 with supporting details in Chapters 7 and 8.

Theorem 2. Let $n = 3b = cd$, with $c, d \geq 3$. Then every irreducible character occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$.

This theorem can be strengthened when the irreducibles involved correspond to two row partitions of n. We prove this version in Chapter 9.

Theorem 3. Let $n = 3b = cd$, with $c, d \geq 3$ and let $\lambda = [\lambda_1, \lambda_2]$ be a two row partition of n. Then every irreducible character χ^λ occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with multiplicity at least as large.

In Chapter 2 we describe the necessary concepts to approach the Foulkes' Conjecture combinatorially using tableaux. Much of this framework was developed by

Doran in [7]. We develop the theory and techniques behind constructing appropriate tableaux in Chapter 3. In Chapter 4 we completely classify and discuss all tableaux that occur in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$. The proof of Theorem 1 is given in Chapter 5. We prove Theorem 2 in Chapter 6, though the required tableau constructions are postponed until Chapter 7. In Chapter 8 we show that these tableaux suffice to cover all necessary cases. We prove Theorem 3 in Chapter 9. The corresponding results for the alternating character is given in Chapter 11. Theorems 1 and 3 also can be interpreted in terms of the Gaussian coefficient; this is discussed in Chapter 10. Further implications of all these results are listed in Chapter 12. Some of these results appear in a forthcoming article in the Journal of Algebra, [21], namely, Chapters 1, 2, 3, 5, and portions of Chapters 4 and 6.

Chapter 2

Background

2.1 Tableaux

A *partition* $\lambda = [\lambda_1, \ldots, \lambda_\ell]$ of a number n is an ordered tuple of positive integers such that $\sum \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1} > 0$; it is denoted by $\lambda \vdash n$. The *length* of λ is ℓ. A *Ferrers diagram* is the set $[\lambda] = \{(i,j) | 1 \leq i \leq \ell,\ 1 \leq j \leq \lambda_i\}$. We view $[\lambda]$ as a (left-justified) stack of boxes with row i having λ_i boxes.

A *tableau* of *shape* λ is a filling of the Ferrers diagram $[\lambda]$ with a set of elements, usually the positive integers. It is said to have *content* $\alpha = [\alpha_1, \ldots, \alpha_k]$ if the integer i occurs exactly α_i times. A tableau is *semi-standard* if the entries are weakly increasing across the rows and strictly increasing down the columns. This notation is standard and further discussion can be found in [7] and [18].

Example 2.1.1. Consider the following tableaux:

$$P = \begin{array}{lll} 1 & 2 & 5 \\ 4 & 2 & \\ 3 & & \end{array} \qquad Q = \begin{array}{lll} 1 & 1 & 5 \\ 2 & 2 & \\ 3 & 3 & \end{array} \qquad R = \begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & 3 \end{array}$$

P is a $[3,2,1]$-tableau of content $\alpha = [1,2,1,1,1]$, while Q has shape $\lambda = [3,2,2]$ and content $\alpha = [2,2,2,0,1]$. Similarly, R has shape $[3,3]$ with content $[2^3] = [2,2,2]$. Both Q and R are semi-standard, but P is not.

2.2 Combinatorial Structures

There are two different permutation actions on tableaux: an action permuting the entry positions in the tableau and an action permuting the numbers filling the tableau. These actions commute with each other.

Let T be a λ-tableau filled with the numbers 1 to a, where $\lambda \vdash n$. The permutation of entry positions corresponds to an action of $\mathcal{S}_n$ on T. View the entry positions (i.e., boxes) of T as labelled 1 to n. Then $\sigma \in \mathcal{S}_n$ acts on T by permuting the entries in the positions moved by σ. To avoid confusion between entry positions and numbers in T, we will denote all entry positions with the subscript $_T$ when necessary. The permutation action of the numbers corresponds to their permutation by $\pi \in \mathcal{S}_a$.

Example 2.2.1. Take $n = 6$, $\lambda = [3, 2, 1]$, and $a = 3$. Consider $T = \begin{array}{lll} 3 & 3 & 1 \\ 1 & 2 & \\ 2 & & \end{array}$. In terms of entry positions, we label T as $\begin{array}{lll} 1_T & 2_T & 3_T \\ 4_T & 5_T & \\ 6_T & & \end{array}$. For $\sigma = (23)_T \in \mathcal{S}_6$, we have $\sigma T = \begin{array}{lll} 3 & 1 & 3 \\ 1 & 2 & \\ 2 & & \end{array}$, while the action of $\pi = (23) \in \mathcal{S}_3$ gives $\pi T = \begin{array}{lll} 2 & 2 & 1 \\ 1 & 3 & \\ 3 & & \end{array}$.

We generally restrict the action of $\mathcal{S}_n$ to two subgroups. Let R_T be the subgroup of $\mathcal{S}_n$ which set-wise fixes the rows of T, namely, the row permutations. Denote this action by σT for $\sigma \in R_T$. Let C_T be the subgroup of $\mathcal{S}_n$ which set-wise fixes the columns of T, namely, the column permutations. This action is denoted τT for $\tau \in C_T$. If λ' is the conjugate partition of λ, (i.e., the partition corresponding to column lengths) we have that $R_T \approx \mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_\ell}$ and $C_T \approx \mathcal{S}_{\lambda'_1} \times \cdots \times \mathcal{S}_{\lambda'_{\ell'}}$. Viewing the subgroups under these isomorphisms, we can label the entry positions by labelling each row (resp. column) with 1 to λ_i (resp. $\lambda_{i'}$). Under these labellings we write σ (resp. τ) as a direct product of the permutations for each row (resp. column).

Example 2.2.2. Let $T = \begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & \end{array}$, then for R_T we view T as labelled $\begin{array}{lll} 1_T & 2_T & 3_T \\ 1_T & 2_T & \end{array}$. Likewise we use the labelling $\begin{array}{lll} 1_T & 1_T & 1_T \\ 2_T & 2_T & \end{array}$ for C_T. Applying these actions to T, gives the following sets:

$$\{\sigma T \,|\, \sigma \in R_T\} = \left\{ \begin{array}{l}1\,1\,2\\2\,3\end{array}, \begin{array}{l}1\,2\,1\\2\,3\end{array}, \begin{array}{l}2\,1\,1\\2\,3\end{array}, \begin{array}{l}1\,1\,2\\3\,2\end{array}, \begin{array}{l}1\,2\,1\\3\,2\end{array}, \begin{array}{l}2\,1\,1\\3\,2\end{array} \right\}$$

with each element occurring twice.

$$\{\tau T \,|\, \tau \in C_T\} = \left\{ \begin{array}{l}1\,1\,2\\2\,3\end{array}, \begin{array}{l}2\,1\,2\\1\,3\end{array}, \begin{array}{l}1\,3\,2\\2\,1\end{array}, \begin{array}{l}2\,3\,2\\1\,1\end{array} \right\}$$

$$\{\pi T \,|\, \pi \in \mathcal{S}_a\} = \left\{ \begin{array}{l}1\,1\,2\\2\,3\end{array}, \begin{array}{l}2\,2\,1\\1\,3\end{array}, \begin{array}{l}3\,3\,2\\2\,1\end{array}, \begin{array}{l}1\,1\,3\\3\,2\end{array}, \begin{array}{l}2\,2\,3\\3\,1\end{array}, \begin{array}{l}3\,3\,1\\1\,2\end{array} \right\}$$

$$C_T = \{()_T \times ()_T \times ()_T,\ (12)_T \times ()_T \times ()_T,\ ()_T \times (12)_T \times ()_T,\ (12)_T \times (12)_T \times ()_T\}$$

Note that the actions of σ and τ do not commute, but π commutes with both σ and τ.

Given λ a partition of $n = ab$, let $\mathcal{W}^{\lambda,a}$ be the set of all tableaux of shape λ and content $[b^a] = [b, \ldots, b]$, where the entries are 1 to a. Let $\mathcal{S}^{\lambda,a}$ be the set of all semi-standard tableaux in $\mathcal{W}^{\lambda,a}$. These sets and the following constructions were developed by Doran in [7]. Note that when $b = 1$, $\mathcal{S}^{\lambda}$ give rise to the Specht modules which are discussed extensively in [15] and [18]. The partitions of n index the Specht modules S^{λ}, which in turn correspond to precisely the irreducible modules of $\mathcal{S}_n$.

From $\mathcal{W}^{\lambda,a}$ we can construct the complex vector space $W^{\lambda,a}$ with the tableaux as a basis. The action of $\mathcal{S}_a$ on the tableaux give rise to a permutation representation. Inside $W^{\lambda,a}$ we construct the following objects.

Definition 2.2.3. Let $T \in \mathcal{W}^{\lambda,a}$. Let $\epsilon(\tau)$ be the sign of τ as a permutation. Inside $W^{\lambda,a}$ we have:

a) $\mathbf{e}_T = \sum_{\sigma\in R_T} \sum_{\tau\in C_T} \epsilon(\tau)\sigma\tau T$

b) $\mathbf{q}_T = \sum_{\pi \in \mathcal{S}_a} \sum_{\sigma \in R_T} \sum_{\tau \in C_T} \epsilon(\tau)\pi\sigma\tau T$

c) $\mathbf{m}_T = \sum_{\pi \in \mathcal{S}_a} \sum_{\sigma \in R_T} \pi\sigma T$

The tableaux are independent basis for $W^{\lambda,a}$, so for $T_1, T_2 \in \mathcal{W}^{\lambda,a}$, we have $T_1 + T_1 = 2T_1$ but $T_1 + T_2 = T_1 + T_2$.

Example 2.2.4. Let $T = \begin{matrix} 11 \\ 22 \end{matrix}$, then

$$\mathbf{e}_T = 4 \times \left\{ \begin{matrix} 11 \\ 22 \end{matrix} + \begin{matrix} 22 \\ 11 \end{matrix} \right\} - 2 \left\{ \begin{matrix} 12 \\ 21 \end{matrix} + \begin{matrix} 21 \\ 12 \end{matrix} + \begin{matrix} 21 \\ 21 \end{matrix} + \begin{matrix} 12 \\ 12 \end{matrix} \right\}$$

Here, symmetry gives $\mathbf{q}_T = 2\mathbf{e}_T$.

Definition 2.2.5. From $\mathbf{e}_T$ and $\mathbf{m}_T$ we can construct the following subspaces of $W^{\lambda,a}$:

a) $S^{\lambda,a} = \mathbb{C}[\mathbf{e}_T | T \in \mathcal{W}^{\lambda,a}]$

b) $M^{\lambda,a} = \mathbb{C}[\mathbf{m}_T | T \in \mathcal{W}^{\lambda,a}]$

These spaces are $\mathcal{S}_a$-modules. We have $\{\mathbf{e}_T | T \in \mathcal{S}^{\lambda,a}\}$, a basis for $S^{\lambda,a}$, and $\{\mathbf{m}_T | T \in \mathcal{S}^{\lambda,a}/\mathcal{S}_a\}$, a basis for $M^{\lambda,a}$. The set $\{\mathbf{q}_T | T \in \mathcal{W}^{\lambda,a}\}$ generates $S^{\lambda,a} \cap M^{\lambda,a}$, but does not form a basis. Background on these spaces and the proofs of the statements may be found in [7].

In [7], Doran uses Gay's result from [10]:

Lemma 2.2.6 (Gay's Result). The multiplicity of the irreducible module S^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_{ab}}$ equals the multiplicity of the trivial representation in $S^{\lambda,a}$.

From this, Doran reformulated Foulkes' Conjecture to:

Lemma 2.2.7. The dimension of $S^{\lambda,a} \cap M^{\lambda,a}$ equals the multiplicity of the irreducible S^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_{ab}}$.

A proof of this lemma in terms of $\mathbf{q}_T$'s is presented in Appendix A. From this, Foulkes' Conjecture is equivalent to proving $\dim(S^{\lambda,a} \cap M^{\lambda,a}) \leq \dim(S^{\lambda,b} \cap M^{\lambda,b})$ for $a \leq b$ and all $\lambda \vdash n$. Proving Conjecture 2 is equivalent to showing that for all $\lambda \vdash n$, $\dim(S^{\lambda,a} \cap M^{\lambda,a}) \leq \dim(S^{\lambda,c} \cap M^{\lambda,c})$ when $ab = n = cd$ with $c, d \geq a$.

Remark 2.2.8. In terms of tableaux, proving Theorem 1 is equivalent to exhibiting m_λ non-zero linearly independent $\mathbf{q}_T$, where T has shape λ and content $[d^c]$, with m_λ the multiplicity of S^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{\mathcal{S}_{2b}}$. Theorem 2 is provable by exhibiting a non-zero $\mathbf{q}_T$ with T having shape λ and content $[d^c]$ for all λ such that the multiplicity of S^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$ is non-zero. For Theorem 3 we want m_λ linearly independent tableaux T with content $[d^c]$ such that $\mathbf{q}_T$ is non-zero, where $\lambda = [\lambda_1, \lambda_2]$ and m_λ is the multiplicity of S^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$.

Chapter 3

Theory of Tableaux Construction

Throughout this and subsequent chapters we will use T to represent an arbitrary tableau, σ an element of R_T, τ an element of C_T, and π an element of $\mathcal{S}_a$ for T filled with 1 to a.

Remark 3.0.9.

a) $\mathbf{e}_{\tau T} = \epsilon(\tau)\mathbf{e}_T$

b) $\mathbf{e}_{\pi T} = \pi\mathbf{e}_T$

c) $\mathbf{q}_{\pi T} = \pi\mathbf{q}_T = \mathbf{q}_T$

d) $\mathbf{q}_T = \Sigma_{\pi\in\mathcal{S}_a}\pi\mathbf{e}_T$

e) $\mathbf{q}_{\tau T} = \epsilon(\tau)\mathbf{q}_T$

These are standard computations, which are discussed in [7] and [18]. This remark shows that we can ignore the effects of permuting entries when constructing the tableaux. Also, we may order the columns however we choose at the cost of a sign.

3.1 Filling Tableaux

Definition 3.1.1. In T, the *weight* of a number x in row i, denoted $\omega_i(x)$, is the number of times x occurs in row i of T. When T is not clear from context, we write $\omega_i(x|T)$ in place of $\omega_i(x)$. We extend this so that $\omega_i(x_1,\ldots,x_j|T) = (\omega_i(x_1|T),\ldots,\omega_i(x_j|T))$.

Implicitly, we take $\omega_i(T) = (\omega_i(1), \ldots, \omega_i(a))$, which is called the *row-weight* of row i of T. Similarly, $\omega(x_j) = \begin{pmatrix} \omega_1(x_j) \\ \vdots \\ \omega_\ell(x_j) \end{pmatrix}$ is the *weight vector* of x_j of T. Hence $\omega(T)$ is the matrix corresponding to $\omega_i(j|T)$. Note that row permutations do not effect weight, so $\omega(\sigma T) = \omega(T)$.

Example 3.1.2. $T = \begin{matrix} 1\,2\,2 \\ 3\,4\,4 \\ 5\,5 \end{matrix}$. We have $\omega(T) = \begin{pmatrix} 1&2&0&0&0 \\ 0&0&1&2&0 \\ 0&0&0&0&2 \end{pmatrix}$. From this we can read that $\omega_2(3) = 1$.

Weights for a tableau are only comparable with tableaux of the same shape and content. Recall that $\mathcal{W}^{\lambda,a}$ is the set of all λ-tableaux with content $[b^a]$. Let N^a be the set of all a-tuples w with non-negative integer entries and $N^{a,\ell}$ those $\ell \times a$ matrices. Then we can view the row-weight function as a linear operator $\omega_i : \mathbb{Z}[T|T \in \mathcal{W}^{\lambda,a}] \to \mathbb{Z}[w|w \in N^a]$, where $\{w|w \in N^a\}$ is a $\mathbb{Z}$-basis, or correspondingly, $\omega : \mathbb{Z}[T|T \in \mathcal{W}^{\lambda,a}] \to \mathbb{Z}[w|w \in N^{a,\ell}]$, with $\ell = \ell(\lambda)$.

This means we treat weights like linearly independent basis in $\mathbb{Z}[w|w \in N^{a,\ell}]$. Hence $\omega(T_1 + aT_2) = \omega(T_1) + a\omega(T_2)$. If $T_1 = \begin{matrix} 1\,1 \\ 2 \end{matrix}$ and $T_2 = \begin{matrix} 2\,1 \\ 1 \end{matrix}$ then $\omega_1(T_1) = (2, 0)$ and $\omega_1(T_2) = (1, 1)$. So $\omega_1(T_1 + T_2) = (2, 0) + (1, 1)$ and $\omega_1(T_1 + T_1) = (2, 0) + (2, 0) = 2 \cdot (2, 0)$. For convenience, we take $\omega(0) = 0$.

Definition 3.1.3. Let $rowsum_T(i)$ be the sum of all the entries in row i of T, $\sum_j \omega_i(j)j$.

Lemma 3.1.4. $\mathbf{e}_T = 0$ iff T has a repeat entry in a column.

Proof. $\Rightarrow$ It suffices to show that if T has no repetitions within a column, then $\mathbf{e}_T \neq 0$. By Remark 3.0.9 column permutations only change the sign of $\mathbf{e}_T$, so without loss of generality, we may assume the columns of T are strictly increasing. If $\mathbf{e}_T = 0$ then there exists $\sigma \in R_T$, $\tau \in C_T$ such that $\epsilon(\tau) = -1$ and $\sigma\tau T = T$, and we say that T cancels in the summation. Since $\sigma\tau T = T$, we must have $\omega(\sigma\tau T) = \omega(T)$ and so $\omega(\tau T) = \omega(T)$. This implies that $rowsum_T = rowsum_{\tau T}$ for all rows.

Let r be the first row in which τ moves an entry of T. Let $\alpha_{i_1}, \ldots, \alpha_{i_k}$ be the entries of row r moved by τ. Say τ moves β_j to α_{i_j}. Since r is the first row moved by τ, $\beta_j > \alpha_{i_j}$. Then $rowsum_T(r) = \sum_{j=1}^k \alpha_{i_j} + \sum_{i \neq i_j} \alpha_i < \sum_{j=1}^k \beta_j + \sum_{i \neq i_j} \alpha_i = rowsum_{\tau T}(r)$. Contradiction. Therefore $\mathbf{e}_T \neq 0$.

$\Leftarrow$ If T has a repeated entry in a column, then there exists a transposition $\tau \in C_T$ such that $\tau T = T$. Hence $\mathbf{e}_T = \mathbf{e}_{\tau T} = \epsilon(\tau)\mathbf{e}_T = -\mathbf{e}_T$ and so $\mathbf{e}_T = 0$.

□

Knowing that any tableau with repeated numbers in a column makes $\mathbf{e}_T = 0$ is very useful for our construction of non-zero tableaux. We summarize this fact as follows:

Lemma 3.1.5. Any tableau T filled with the numbers 1 to a having more than a rows will have $\mathbf{e}_T = 0$ and $\mathbf{q}_T = 0$, as will any T having repetitions within a column.

Hence every tableau T filled with the numbers 1, 2, and 3, with $\mathbf{q}_T \neq 0$, will have at most three rows and all column entries will be distinct. Due to this, from now on we will assume all tableaux have distinct column entries.

Notation: Many of the tableaux we construct will have multiple identical columns. We call a group of such columns, a *column block*. For both clarity and space we denote a column block by one copy of the column with the number of repetitions listed above. If the number of column copies is omitted, it is assumed to be one. For example, $T = \begin{matrix} 1\,1\,1 \\ 2\,2\,2 \end{matrix}$ would be denoted by $T = \begin{array}{c} \scriptstyle 3 \\ \hline 1 \\ 2 \end{array}$, while $T = \begin{array}{cccc} \scriptstyle K & \scriptstyle L & \scriptstyle M & \scriptstyle N \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$

has K copies of $\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$ and L copies of $\frac{1}{2}$, M copies of $\frac{1}{3}$, and N copies of $\frac{2}{3}$. We call the columns of T having only one entry the *tail* of T. When specifying T by column blocks, we may omit the tail, provided the content of T is known. The rest of T is called the *body*. For instance, if $T = \begin{array}{ccccc} 2 & & & & \\ \hline 1 & 1 & 2 & 2 & 3 \\ 3 & 2 & 4 & & \\ 4 & & & & \end{array}$ and we know that T has content

$[3,3,3,3]$, we can just list the body, $T = \begin{array}{c} 2 \\ \hline 1\ 1\ 2 \\ 3\ 2\ 4 \\ 4 \end{array}$ instead. It is assumed that any entries not specified are contained in the tail.

We also use this abbreviated notation when describing elements of C_T. We write $\tau \in C_T$ as a direct product of permutations on the column blocks. Since the only permutation possible on the tail is the identity, we omit the permutations corresponding to the tail. Hence for T listed above, τ is of the form $\overset{2}{\tau_1} \times \tau_2 \times \tau_3$. We write $\overset{\kappa}{\tau_i}$ if the same permutation τ_i is to be applied to κ columns within a column block. When κ is less than the size of the column block, we understand $\overset{\kappa}{\tau_i}$ to mean that τ_i is applied to all κ of the columns (determined by context) and the identity permutation is applied to the remaining columns within the block. For instance, on the previous T, there are two permutations of the form $\overset{1}{(13)_T} \times ()_T \times (12)_T$, which produce $\begin{array}{l} 4\ 1\ 1\ 4 \\ 3\ 3\ 2\ 2 \\ 1\ 4 \end{array}$ and $\begin{array}{l} 1\ 4\ 1\ 4 \\ 3\ 3\ 2\ 2 \\ 4\ 1 \end{array}$.

3.2 Showing Tableaux are Non-Zero

Definition 3.2.1. A tableau T is said to be *non-zero* if $\mathbf{q}_T \neq 0$. Two tableaux are said to be *distinct* if $\mathbf{q}_{T_1} \neq \pm\mathbf{q}_{T_2}$, otherwise T_1 and T_2 are said to be equivalent.

Since $\mathbf{q}_T$ involves many summands, showing $\mathbf{q}_T \neq 0$ by direct summation is not practical. Instead, we use a technique called *weight-set counting*. Weight-set counting involves summing only those tableaux with a given weight; if that sum is non-zero, the entire $\mathbf{q}_T$ summation must be non zero.

Definition 3.2.2. Given a tableau T, let $\omega(T) = (\overline{x}_1, \ldots, \overline{x}_a)$, where $\overline{x}_j$ is the weight vector $\omega(j|T) = \overline{x}_j$ of the element j in T. A *weight assignment* of T is a pairing between the set of elements of T with the multiset of weight vectors of $\omega(T)$. We denote the pairing of the weight vector $\overline{x}_j$ with the element k_j by $\omega(k_j|T^*) = \overline{x}_j$. (The T^* represents a possible tableau which has the weight of k_j being $\overline{x}_j$.) Note that if the vectors $\overline{x}_j$ and $\overline{x}_{j'}$ are equal, the weight assignment pairing k_j with $\overline{x}_j$ is the same as the weight assignment pairing k_j with $\overline{x}_{j'}$. What matters in a weight assignment is the vector paired with each element, not how we label the vectors. We usually indicate a weight assignment by writing $(k_1, \ldots, k_a)$, by which we mean $\omega(k_1, \ldots, k_a|T^*) = (\overline{x}_1, \ldots, \overline{x}_a) = \omega(T)$.

Given a permutation π we can create a weight assignment by assigning the element $\pi(k)$ to $\overline{x}_k$, since $\omega(\pi(k)|\pi T) = \omega(k|T) = \overline{x}_k$. Similarly, given such pairing $(k_1, \ldots, k_a)$ we can construct a permutation π by taking $\pi = (k_1, \ldots, k_a)$ in one-line notation.

For example, let $T = \begin{array}{lll} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & & \end{array}$. Then $\omega(T) = \left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. The weight assignment $(3, 1, 2)$ means $\omega(3, 1, 2|T^*) = \left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ for some tableau T^*. This weight assignment corresponds to the weight permutation $\pi = (132)$ in cycle notation (from left to right).

Note, however, that such a listing $(k_1, \ldots, k_a)$ of a weight assignment is not necessarily unique. For instance, if $\overline{x}_1 = \overline{x}_2$ then $(k_1, k_2, k_3, \ldots k_a)$ and $(k_2, k_1, k_3, \ldots k_a)$ represent the same weight assignment (pairing) but give rise to different permutations. The numbers of such permutations corresponding to the same weight assignment depends only on the vector symmetries of $\omega(T)$. This number is denoted $s(\omega(T))$.

For example, if $T = \begin{array}{lllll} 1 & 1 & 2 & 2 & 3 \end{array}$, then $\omega_T = (2, 2, 1)$. There are three distinct weight assignments of T corresponding to which of the three elements is assigned a weight of 1. Since there are two permutations arising from such an assignment (for instance, $\omega(213|T) = (2, 2, 1)$ as well) we have $s(\omega(T)) = 2$.

Definition 3.2.3. A weight assignment $(k_1, \ldots, k_a)$ is *valid* for T if there exists $\tau \in C_T$ such that $\omega(k_1, \ldots, k_a|\tau T) = \omega(T)$, i.e., $T^* = \tau T$. If this happens we say τ is valid for $(k_1, \ldots, k_a)$, otherwise τ is *invalid*. Given a valid τ we say τ is *positive* if $\epsilon(\tau) = 1$ and *negative* if $\epsilon(\tau) = -1$.

Example 3.2.4. Let $T = \begin{array}{lllllll} 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 3 & 1 & 1 & & & & \\ 4 & 4 & & & & & \end{array}$. We have $\omega(T) = \left(\begin{smallmatrix} 1 & 3 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{smallmatrix}\right)$. Recall that for C_T we use the labelling $\begin{array}{lll} 1_T & 1_T & 1_T \\ 2_T & 2_T & 2_T \\ 3_T & 3_T & \end{array}$ (the tail is omitted since all column permutations on it are trivial). Now $(2,3,1,4)$ is a valid weight assignment since $\tau = (12)_T \times (12)_T \times (12)_T \in C_T$ has $\tau T = \begin{array}{lllllll} 3 & 1 & 1 & 2 & 3 & 3 & 4 \\ 1 & 2 & 2 & & & & \\ 4 & 4 & & & & & \end{array}$ with $\omega(\tau T) = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, So $\omega(2,3,1,4|\tau T) = \omega(T)$. The weight assignment, $(2,3,1,4)$ corresponds to the permutation $\pi = (123)$ in cycle notation, meaning $\omega(1,2,3,4|\tau T) = \omega(1,2,3,4|\pi T)$

However, $(1,4,3,2)$ is not a valid weight assignment since then we must have $\omega_3(2|\tau T) = 2$ for some τ, but there is no column permutation that will put two 2's in the third row. We will make frequent use of weight assignments in order to determine when T is non-zero.

Definition 3.2.5. Given T, consider the following functions:

- $\mathcal{P}(\pi(T))$ = the number of $\tau \in C_T$ such that $\epsilon(\tau) = 1$ and $\omega(\pi^{-1}(1), \ldots, \pi^{-1}(a)|\tau T) = \omega(T)$.

- $\mathcal{N}(\pi(T))$ = the number of $\tau \in C_T$ such that $\epsilon(\tau) = -1$ and $\omega(\pi^{-1}(1), \ldots, \pi^{-1}(a)|\tau T) = \omega(T)$.

- $\mathcal{P}(T) = \sum_\pi \mathcal{P}(\pi(T))$, where π correspond to distinct weight assignments of $\omega(T)$

- $\mathcal{N}(T) = \sum_\pi \mathcal{N}(\pi(T))$, where π correspond to distinct weight assignments of $\omega(T)$

Theorem 4. (Weight-set Counting) If $\mathbf{q}_T = 0$ then $\mathcal{P}(T) = \mathcal{N}(T)$.

Proof. Let $\mathcal{D}$ be the set permutations corresponding to distinct weight assignments of T.

$$\mathbf{q}_T = 0 \tag{3.2.1}$$

$$\Rightarrow \sum_\pi \sum_\sigma \sum_\tau \epsilon(\tau)\omega(\sigma\pi\tau T) = 0 \tag{3.2.2}$$

$$\Rightarrow \sum_\pi \sum_\tau \epsilon(\tau)\omega(\pi\tau T) = 0 \tag{3.2.3}$$

$$\Rightarrow \sum_\pi \sum_\tau \epsilon(\tau)\omega(\pi\tau T) = 0 \text{ s.t. } \omega(1, \ldots, a|\pi\tau T) = \omega(T) \tag{3.2.4}$$

$$\Rightarrow \sum_{\pi} \sum_{\tau} \epsilon(\tau) = 0 \text{ s.t. } \omega(1,\ldots,a|\pi\tau T) = \omega(T) \tag{3.2.5}$$

$$\Rightarrow \sum_{\pi} \sum_{\tau} \epsilon(\tau) = 0 \text{ s.t. } \omega(\pi^{-1}(1),\ldots,\pi^{-1}(a)|\tau T) = \omega(T) \tag{3.2.6}$$

$$\Rightarrow s(\omega(T)) \sum_{\pi \in \mathcal{D}} (\mathcal{P}(\pi T) - \mathcal{N}(\pi T)) = 0 \tag{3.2.7}$$

$$\Rightarrow s(\omega(T))(\mathcal{P}(T) - \mathcal{N}(T)) \tag{3.2.8}$$

$$\Rightarrow \mathcal{P}(T) - \mathcal{N}(T) = 0 \tag{3.2.9}$$

If $\mathbf{q}_T = 0$, taking the weight of both sides gives Eq. (3.2.2). Since row permutations do not effect weights, we can reduce to Eq. (3.2.3). As distinct weights can not cancel, we can consider only those tableaux with the same weight as $\omega(T)$, hence we must have Eq. (3.2.4). Since we are only summing over tableaux of a fixed weight we may drop the weight from the sum and simply add the sign of τ, for Eq. (3.2.5). By definition of weight assignments, $\omega(1,\ldots,a|\pi\tau T) = \omega(\pi^{-1}(1),\ldots,\pi^{-1}(a)|\tau T)$ hence we get Eq. (3.2.6). Restricting the sum to distinct weight-sets gives the factor of $s(\omega(T))$, and if we split over the sign of τ' we get Eq. (3.2.7), which by definition is the same as Eq. (3.2.8). Factoring out $s(\omega(T))$ yields Eq. (3.2.9). Thus $\mathcal{P}(T) = \mathcal{N}(T)$. □

We can also write Theorem 4 as:

Corollary 5. Let $\mathcal{A} = \{\tau|\omega(\tau T) = \omega(\pi T)$ for some $\pi \in \mathcal{S}_a\}$. If $\sum_{\tau\in\mathcal{A}} \epsilon(\tau) \neq 0$, then $\mathbf{q}_T \neq 0$ by weight-set counting on $\omega(T)$.

An easy application of this theorem is the following useful lemma.

Lemma 3.2.6. If T is a tableau consisting of a single column block (and an arbitrary tail), for instance, $T = \frac{\overset{\kappa}{}}{\begin{matrix}1\\2\end{matrix}}$, then $\mathbf{q}_T \neq 0$ iff κ even.

Proof. We have $\mathsf{w} = \omega(T) = \left(\begin{smallmatrix}\kappa & 0\\ 0 & \kappa\end{smallmatrix}\right)$. So there are two distinct weight assignments for T, namely, $\omega(1,2|T^*) = \mathsf{w}$ and $\omega(2,1|T^*) = \mathsf{w}$. We wish to determine for which τ does $T^* = \tau T$ satisfy one of these equations. For $\omega(1,2|\tau T) = \mathsf{w}$, we must have κ 2's in the second row, hence none of the columns of T may move. The only τ satisfying this is $\tau = \overset{\kappa}{()}_T$, which is positive. For $\omega(2,1|\tau T) = \mathsf{w}$, we must have κ 1's in the second row, thus τ must exchange row 1 and row 2 for every column in T. Thus $\tau = \overset{\kappa}{(12)}_T$ and $\epsilon(\tau) = (-1)^{\kappa}$. Hence when κ is even, we have $\mathcal{P}(T) = 2$ and $\mathcal{N}(T) = 0$, so $\mathbf{q}_T$ is non-zero. The same idea applies for T having more than two rows.

When κ is odd, however, we have $\mathcal{P}(T) = \mathcal{N}(T)$ so the Theorem 4 does not apply. Instead, let $\tau = \overset{\kappa}{(12)}_T$ and let $\pi \in \mathcal{S}_a$ be the corresponding entry transposition (in our example $\pi = (12)$). Then $\pi\tau T = T$ and $\epsilon(\tau) = -1$. Thus $\mathbf{q}_{\pi\tau t} = \pi \mathbf{q}_{\tau T} = \mathbf{q}_{\tau T} = \epsilon(\tau)\mathbf{q}_T = -\mathbf{q}_T$. So $\mathbf{q}_T = 0$. □

It is also true that if $\mathbf{q}_T \neq 0$, then it is non-zero by weight-set counting on some weight $\omega(\tau T)$. That is we can't have $\mathcal{P}(\tau T) = \mathcal{N}(\tau T)$ for all $\tau \in C_T$ and still have $\mathbf{q}_T \neq 0$.

Theorem 6. If $\mathbf{q}_T \neq 0$, then it is non-zero by weight set counting on τT for some $\tau \in C_T$.

Proof. Recall that $\mathbf{q}_T = \sum_\pi \sum_\sigma \sum_\tau \pi\sigma\, \epsilon(\tau)\tau T$. Since a tableau may be written in multiple ways (such as $T' = \pi\sigma\tau T = \pi'\sigma'\tau' T$), we need to be careful of how we denote a tableau. Consider the terms of $\mathbf{q}_T$ partitioned in to classes $\{\tau T\} = \{\pi\sigma\tau T | \pi \in \mathcal{S}_a,\ \sigma \in R_T\}$, using $\{\tau T\}$ as the class representative. Note that all tableaux in a given class have the same sign and generic weight, i.e., the same weight modulo the action of $\mathcal{S}_a$. Hence if $\tau T = \pi'\sigma'\tau' T$ for some π', σ', τ', with $\epsilon(\tau\tau') = -1$ then $\pi\sigma\tau T = \pi\pi'\sigma\sigma'\tau' T$, so the classes are equal set wise, but of opposite sign. This holds for any element of the class, not just τT.

This means we may view the tableaux which cancel in $\mathbf{q}_T$ as a matching between equal classes of opposite sign. So if $\mathbf{q}_T \neq 0$, there is a set of tableaux $\mathcal{T}$ which are not canceled in $\mathbf{q}_T$. Now for any $\tau T \in \mathcal{T}$ either the weight of τT does not cancel, (that is $\omega(\tau T) \neq \overline{\pi}\omega(\pi'\sigma'\tau' T)$ for all $\overline{\pi} \in \mathcal{S}_a$ and all $\pi'\sigma'\tau' T \in \mathcal{T}$ with $\epsilon(\tau\tau') = -1$) or the weight cancels with some $\pi'\sigma'\tau' T$ having $\epsilon(\tau\tau') = -1$ (that is $\omega(\tau T) = \overline{\pi}\omega(\pi'\sigma'\tau' T)$ for some $\pi'\sigma'\tau' T \in \mathcal{T}$). If the weight does not cancel for some τT, the $\mathbf{q}_T \neq 0$ by weight set counting on τT. Assume all the weights do cancel. Then $\omega(\tau T) = \overline{\pi}\omega(\pi'\sigma'\tau' T)$, so for every row, τT and $\overline{\pi}\pi'\sigma'\tau' T$ have the same number of each symbol. Hence there exists $\overline{\sigma} \in R_T$ such that $\tau T = \overline{\pi}\pi'\overline{\sigma}\sigma'\tau' T$. But then, since $\epsilon(\tau\tau') = -1$, the classes $\{\tau T\}$ and $\{\tau' T\}$ will cancel in $\mathbf{q}_T$ contradicting τT, $\pi'\sigma'\tau' T \in \mathfrak{T}$. Hence $\mathbf{q}_T \neq 0$ by weight set counting on some τT. □

We will not need Theorem 6 for our results; it is included for theoretical interest and completeness. Theorem 4 is used quite heavily, however. For instance, it allows use to directly establish the multiplicities of the irreducible characters in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{\mathcal{S}_{2b}}$.

Theorem 7. The only irreducible characters occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{2b}$ are those corresponding to all partitions $\lambda = [\lambda_1, \lambda_2]$ of $2b$ where λ_i is even. Moreover, these characters occur with multiplicity 1.

Proof. By Remark 2.2.8 we need only consider those shapes with $\mathbf{q}_T$ non-zero. By Lemma 3.0.9 and Lemma 3.1.5 all distinct non-zero tableaux filled with b 1's and b 2's must be equivalent to $T = \begin{array}{c} \kappa \\ \hline 1 \\ 2 \end{array}$, (not including tail). Hence there is at most one distinct non-zero tableau for any shape λ. By Lemma 3.2.6, when $\lambda = [2b - \kappa, \kappa]$ then $\mathbf{q}_T \neq 0$ iff κ is even. Since $\langle \mathbf{q}_T \rangle = S^{\lambda,2} \cap M^{\lambda,2}$ we must have $\dim(S^{\lambda,2} \cap M^{\lambda,2}) = 1$ if κ is even and zero otherwise.

This is a well-known result, appearing in [13] and [17].

□

The weight-set counting of Theorem 4 is useful for much more complicated tableaux as well. To illustrate the general usage of the theorem, we list here a slightly more involved example.

Example 3.2.7. To see directly how weight-set counting works, consider the following example. The tableau Q^* is listed below using the column block notation, with the conditions on the block size listed to the right. Underneath the tableau we list the weight and shape of the tableau.

$$Q^* = \begin{array}{ccccc} \text{A} & \text{A} & \text{B} & \text{B} & \text{C} \\ \hline 2 & 2 & 2 & 3 & 1 \\ 1 & 4 & 4 & 1 & 4 \\ 3 & 3 & & & \end{array} \qquad \begin{array}{l} \text{A} + \text{B} + \text{C} = d \\ 2\text{A} + \text{B} < d \\ \text{C even} \\ \text{A}, \text{B} > 0 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} \text{A+B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{smallmatrix} \right)$$

$$\lambda = [3d - 2\text{A} - \text{B}, d + \text{A+B}, 2\text{A}]$$

We want to show that $\mathbf{q}_{Q^*} \neq 0$, which we do by showing $\mathcal{P}(Q^*) - \mathcal{N}(Q^*) \neq 0$ and applying Theorem 4. Now $\mathcal{P}(Q^*)$ is the number of $\tau \in C_{Q^*}$ with $\epsilon(\tau) = 1$ such that $\omega(i_1, i_2, i_3, i_4|\tau Q^*) = \omega(Q^*)$ for some distinct weight assignment (i_1, i_2, i_3, i_4). (Equivalently, τ is such that $\{\omega(i|\tau Q^*) \| i = 1, 2, 3, 4\} = \{\omega(i|Q^*) \| i = 1, 2, 3, 4\}$.) Similarly, $\mathcal{N}(Q^*)$ is those with $\epsilon(\tau) = -1$. The easiest way to count these τ is to use weight assignments. First we determine which weight assignments might be possible using some general properties of the tableau. Then we count how many τ correspond to each weight assignment (i.e., for which τ is the weight assignment valid) and determine $\epsilon(\tau)$. Finally, we add this signed sum to determine $\mathcal{P}(Q^*) - \mathcal{N}(Q^*)$.

First we want to determine which weight assignments are possible for Q^*. That is, determine for which 4-tuples $\mathbf{x} = (i_1, i_2, i_3, i_4)$ there might exist $\tau \in C_{Q^*}$ such that $\omega_{2,3}(\mathbf{x}|\tau Q^*) = \left(\begin{smallmatrix} \text{A+B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{smallmatrix} \right)$. Let $\mathbf{w} = \left(\begin{smallmatrix} \text{A+B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{smallmatrix} \right)$. Simply looking at Q^*, there are a few restrictions on what $\mathbf{x}$ can be.

Notice the body contains d copies of the elements 1 and 4, but fewer than d copies of 2 and 3 since $2\text{A}+\text{B} < d$. Also note that the body of τQ^* contains the same elements as the body of Q^*. This implies that not all elements can have d copies in row two of τQ^* for some τ. If $\omega_{2,3}(\mathbf{x}|\tau Q^*) = \mathbf{w}$ then either $\omega_2(1|\tau Q^*) = d$ or $\omega_2(4|\tau Q^*) = d$; namely, only the elements 1 and 4 may have $\omega_2(i|\tau Q^*) = d$. Hence any valid weight assignment must have $i_4 = 1$ or 4. If $\omega_2(1|\tau Q^*) = d$, then there is only one other non-zero weight to assign in row two. As $\text{B} > 0$ the remaining columns (the second A block and the first B block) must have the same element in row two, namely, 2 or

4. That means we must have either $\omega_2(2|\tau Q^*) = \text{A+B}$ or $\omega_2(4|\tau Q^*) = \text{A+B}$, so the weight assignment must have $i_1 = 2$ or 4. Similarly, if $\omega_2(4|\tau Q^*) = d$, then since $\text{B} > 0$ we must have either $\omega_2(1|\tau Q^*) = \text{A+B}$ or $\omega_2(3|\tau Q^*) = \text{A+B}$, that is $i_1 = 1$ or 3.

We also consider which elements j may have $\omega_3(j|\tau Q^*) = 2\text{A}$. We find that only with $j = 2$ or 3 may this occur since these are the only elements for which both A blocks will be the same in row three. (That is if $\omega_{2,3}(\tau Q^*) = \mathbf{w}$ then either $\omega_3(2|\tau Q^*) = 2\text{A}$ or $\omega_3(3|\tau Q^*) = 2\text{A}$, so any valid weight assignment has $i_3 = 2$ or 3.)

There are six distinct weight assignments $\mathbf{x} = (i_1, i_2, i_3, i_4)$ meeting these conditions: $(1, 2, 3, 4)$, $(1, 3, 2, 4)$, $(4, 3, 2, 1)$, $(3, 1, 2, 4)$, $(4, 2, 3, 1)$, and $(2, 4, 3, 1)$. In the table below, for each weight assignment we list for what type of tableau τQ^* it is valid, the form τ used, the number of such τ, and the sign of τ. This is an easy way to summarize the counting of τ and their signs. (We will omit the subscripts $_{Q^*}$ when writing τ for easy reading. Remember that τ is labeled by the *entry positions* of Q^* and not the elements.)

$\omega_{2,3}(\mathbf{x}\vert\tau Q^*)$ $= \begin{pmatrix} \text{A+B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{pmatrix}$	τQ^*	τ	#	$\epsilon(\tau)$
$\mathbf{x} = (1,2,3,4)$	A A B B C 2 2 2 3 1 1 4 4 1 4 3 3	$\overset{\text{A}}{()} \times \overset{\text{A}}{()} \times \overset{\text{B}}{()} \times \overset{\text{B}}{()} \times \overset{\text{C}}{()}$	1	$(-1)^0$
$\mathbf{x} = (1,3,2,4)$	A A B B C 3 3 2 3 1 1 4 4 1 4 2 2	$\overset{\text{A}}{(13)} \times \overset{\text{A}}{(13)} \times \overset{\text{B}}{()} \times \overset{\text{B}}{()} \times \overset{\text{C}}{()}$	1	$(-1)^{2\text{A}}$
$\mathbf{x} = (3,1,2,4)$	A A B B C 1 3 2 1 1 3 4 4 3 4 2 2	$\overset{\text{A}}{(132)} \times \overset{\text{A}}{(13)} \times \overset{\text{B}}{()} \times \overset{\text{B}}{(12)} \times \overset{\text{C}}{()}$	1	$(-1)^{\text{A+B}}$
$\mathbf{x} = (4,2,3,1)$	A A B B C 2 2 2 3 4 1 4 4 1 1 3 3	$\overset{\text{A}}{()} \times \overset{\text{A}}{()} \times \overset{\text{B}}{()} \times \overset{\text{B}}{()} \times \overset{\text{C}}{(12)}$	1	$(-1)^{\text{C}}$
$\mathbf{x} = (4,3,2,1)$	A A B B C 3 3 2 3 4 1 4 4 1 1 2 2	$\overset{\text{A}}{(13)} \times \overset{\text{A}}{(13)} \times \overset{\text{B}}{()} \times \overset{\text{B}}{()} \times \overset{\text{C}}{(12)}$	1	$(-1)^{2\text{A+C}}$
$\mathbf{x} = (2,4,3,1)$	A A B B C 2 4 4 3 4 1 2 2 1 1 3 3	$\overset{\text{A}}{()} \times \overset{\text{A}}{(12)} \times \overset{\text{B}}{(12)} \times \overset{\text{B}}{()} \times \overset{\text{C}}{(12)}$	1	$(-1)^{\text{A+B+C}}$

To see how we obtain such a table, consider the last row. We want a tableau τQ^* such that $\omega_{2,3}(2,4,3,1|\tau Q^*) = \mathbf{w}$. This means that $\omega_3(3|\tau Q^*) = 2\text{A}$ so τ cannot move any entries in row 3. We also have $\omega_2(4|\tau Q^*) = 0$, so examining Q^*, we know that τ acts non-trivially on the second column block A, the first column block B, and column block C, namely, $\tau = \overset{\text{A}}{*} \times \overset{\text{A}}{(12)}_T \times \overset{\text{B}}{(12)}_T \times \overset{\text{B}}{*} \times \overset{\text{C}}{(12)}_T$. If τ were to act non-trivially on the first column block A or the second column block B, the number of 1's in row two of τQ^* would decrease. Since we must have $\omega_2(1|\tau Q^*) = d$, this cannot happen. Hence $\tau = \overset{\text{A}}{()}_T \times \overset{\text{A}}{(12)}_T \times \overset{\text{B}}{(12)}_T \times \overset{\text{B}}{()}_T \times \overset{\text{C}}{(12)}_T$ and has been completely determined for us. Then we have $\omega_{2,3}(2,4,3,1|\tau Q^*) = \mathbf{w}$, so such a τ exists and is unique. For reference, τ and τQ^* are listed. Once τ has been determined, computing $\epsilon(\tau) = (-1)^{\text{A+B+C}}$ is straightforward.

Finally, to compute the weight sum for $\mathbf{w}$, we sum the product of the number of τ with the sign of τ, that is $\# \cdot \epsilon(\tau)$. Here the sum is $1+1+(-1)^{\mathrm{A+B}}+1+1+(-1)^{\mathrm{A+B}}$. This sum is between 2 and 6, depending on the parity of A and B. Since it is non-zero in all cases, Theorem 4 shows $\mathbf{q}_{Q^*} \neq 0$.

3.3 Joining Tableaux

Definition 3.3.1. The *join* of two tableaux, U and V, denoted $U \vee V$ is a way of combining tableaux together. If the entries of U and V are not disjoint, renumber V so that they are. For instance, if U contains the numbers 1 to n and V contains the numbers 1 to m we first renumber V with the numbers $n+1$ to $n+m$. Then concatenate the tableaux and sort the columns by length. Note that entries of every column remain fixed, only the order of the columns change.

Example 3.3.2. $U = \begin{matrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & \end{matrix}$ and $V = \begin{matrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & & \end{matrix}$ We renumber V to get $V = \begin{matrix} 5 & 6 & 7 \\ 6 & 8 & 8 \\ 7 & & \end{matrix}$.

Concatenating gives $\begin{matrix} 1 & 1 & 3 & 5 & 6 & 7 \\ 2 & 2 & 4 & 6 & 8 & 8 \\ 3 & 4 & & 7 & & \end{matrix}$. When sorted we get $T = U \vee V = \begin{matrix} 1 & 1 & 5 & 3 & 6 & 7 \\ 2 & 2 & 6 & 4 & 8 & 8 \\ 3 & 4 & 7 & & & \end{matrix}$.

Note that since applying permutations of $\mathcal{S}_a$ to a tableau has no effect on $\mathbf{q}_T$ and $\mathbf{m}_T$, the renumbering of a tableau is irrelevant. Also, any $\sigma \in R_T$ that only interchanges columns will commute with all $\tau \in C_T$. Hence column sorting has no effect of $\mathbf{e}_T$, $\mathbf{q}_T$, and $\mathbf{m}_T$, since there is no sign change for row permutations. This join operation also joins the weight-sets, namely, $\omega_3(U \vee V) = \omega_3(U), \omega_3(V) = (0,0,1,1,0,0,1,0)$.

Definition 3.3.3. Let $T = U \vee V$ for tableaux U filled with 1 to m and V filled with $m+1$ to a. The weights, $\omega(U)$ and $\omega(V)$ are *disjoint* (equivalently, $\omega(T)$ *splits* over U and V), if every valid weight assignment of $\omega(T)$, can be obtained from concatenating valid weight assignments of $\omega(U)$ and $\omega(V)$.

Say $\omega(1,\ldots,m|U) = (\overline{x}_1,\ldots,\overline{x}_m)$ where the $\overline{x}_i$ are weight vectors and $\omega(m+1,\ldots,a|V) = (\overline{x}_{m+1},\ldots,\overline{x}_a)$. So $\omega(1,\ldots,a|T) = (\overline{x}_1,\ldots,\overline{x}_a)$. Consider a valid weight assignment of T assigning to the element j the weight vector $\overline{x}_{k_j}$. Then $\omega(1,\ldots,a|\tau T) = (\overline{x}_{k_1},\ldots,\overline{x}_{k_a})$ for some $\tau \in C_T$. This restricts to a valid weight assignment of $\tau_{|_U} U$ by considering only the elements 1 to m. This restriction is unique because a weight assignment is defined by the vector-element pairing, not the label assigned to the vector. If this restriction corresponds to a weight assignment of $\omega(U)$ (i.e., the weights assigned to elements 1 to m are the same as the weights of $\omega(U)$ as vectors) then the weight assignment of T arose from valid weight assignments of U. Similarly for V.

By restricting to a weight assignment of $\omega(U)$ we mean $\{\overline{x}_{k_i}|i = 1\ldots m\} = \{\overline{x}_i|i = 1\ldots m\}$, i.e., the weights assigned to U are equivalent to those of $\omega(U)$. If this result is a weight assignment of $\omega(U)$, it is valid for the tableau $\tau_{|_U} U$. If this is true for all valid weight assignments of T, then $\omega(T)$ splits and the weights of U and V are disjoint.

Example 3.3.4. Consider $U = \begin{matrix} 1\,1 \\ 2\,2 \end{matrix}$ and $V = \begin{matrix} 3\,3\,3\,3 \\ 4\,4\,4\,4 \end{matrix}$. So $T = \begin{matrix} 1\,1\,3\,3\,3\,3 \\ 2\,2\,4\,4\,4\,4 \end{matrix}$ and $\omega_2(1,2,3,4|T) = (0,2,0,4)$. There are four valid weight assignments possible for T: (Recall that T^* represents any possible tableau τT.)

$$\omega_2(1,2,3,4|T^*) = (0,2,0,4)$$
$$\omega_2(1,2,3,4|T^*) = (2,0,0,4)$$
$$\omega_2(1,2,3,4|T^*) = (0,2,4,0)$$
$$\omega_2(1,2,3,4|T^*) = (2,0,4,0)$$

When we restrict these weight assignments to U we get two possible assignments for U, $\omega_2(1,2|U^*) = (0,2)$ and $\omega_2(1,2|U^*) = (2,0)$. Since the original weight of U is $\omega_2(1,2|U) = (0,2)$, both of these assignments are assignments of $(0,2)$ and both are valid for U (simply take $\tau_U = ()$ and $\tau_U = (12)_T \times (12)_T$). A similar argument holds for V. Thus the $\omega(T)$ splits over U and V.

However, weight-set disjointness is highly dependent on the filling of T. Consider instead, $U = \begin{matrix} 1\,1 \\ 2\,2 \end{matrix}$ and $V = \begin{matrix} 4\,3 \\ 3\,4 \end{matrix}$. So $T = \begin{matrix} 1\,1\,4\,3 \\ 2\,2\,3\,4 \end{matrix}$ and $\omega_2(1,2,3,4|T) = (0,2,1,1)$.

Then $\omega_2(1,2,3,4|T^*) = (1,1,2,0)$ is a valid weight assignment for T by $T^* = \tau T$ with $\tau = (12)_T \times ()_T \times ()_T \times (12)_T$. However, $\omega_2(1,2|U^*) = (1,1)$ it is not a valid weight assignment of $\omega_2(U) = (2,0)$, even though there exists τ such that $\omega_2(\tau U) = (1,1)$. Hence the weights are not disjoint.

Although this definition of disjointness is a bit involved, in Section 3.4 we will give a sufficient (but not necessary) condition on the tableau which is easier to check. However, we use disjointness here to obtain the full generality of Theorem 8, which is one of the fundamental tools we use to construct non-zero tableaux.

Theorem 8. Let U and V be tableaux such that

- *elements*$(U) = \{1, \ldots m\}$ and *elements*$(V) = \{m+1, \ldots a\}$ (renumber if necessary)
- The weights $\omega(1, \ldots, m|U)$, $\omega(m+1, \ldots, a|V)$ are such that $\mathbf{q}_U$ and $\mathbf{q}_V$ are non-zero by weight-set counting on ω.
- The weight assignments corresponding to $\omega(1, \ldots, m|U)$ and $\omega(m+1, \ldots, a|V)$ are disjoint.

Then for $T = U \vee V$, we have $\mathbf{q}_T \neq 0$ by weight-set counting on $\omega(T)$.

Proof. By weight-set counting on U and V we have $\mathcal{P}(U) - \mathcal{N}(U) \neq 0$ and $\mathcal{P}(V) - \mathcal{N}(V) \neq 0$. By Theorem 4, showing $\mathcal{P}(T) - \mathcal{N}(T) \neq 0$ implies $\mathbf{q}_T \neq 0$. Thus it suffices to show $\mathcal{P}(T) - \mathcal{N}(T) = (\mathcal{P}(U) - \mathcal{N}(U))(\mathcal{P}(V) - \mathcal{N}(V))$ or equivalently $\mathcal{P}(T) = \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$ and $\mathcal{N}(T) = \mathcal{N}(U)\mathcal{P}(V) + \mathcal{P}(U)\mathcal{N}(V)$. We will show $\mathcal{P}(T) = \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$. The $\mathcal{N}(T)$ claim follows similarly.

Consider the weight assignment $(k_1, \ldots, k_a)$ of T. If $\omega(k_1, \ldots, k_a|\tau T) = \omega(1, \ldots, a|T)$ with $\epsilon(\tau) = 1$ (i.e., τ is positive for $(k_1, \ldots, k_a)$) then it is counted in $\mathcal{P}(T)$. Since the weight splits, we have $\omega(k_1, \ldots, k_a|\tau T) = \omega(k_1, \ldots k_m|\tau_{|U} U)\omega(k_{m+1}, \ldots k_a|\tau_{|V} V) = \omega(1, \ldots, m|U)\omega(m+1, \ldots, a|V) = \omega(1, \ldots, a|T)$ with $\epsilon(\tau) = \epsilon(\tau_{|U})\dot{\epsilon}(\tau_{|V}) = 1$. Hence either $\epsilon(\tau_{|U}) = \epsilon(\tau_{|V}) = 1$ or $\epsilon(\tau_{|U}) = \epsilon(\tau_{|V}) = -1$. If $\epsilon(\tau_{|U}) = 1$ then since $\omega(k_1, \ldots k_m|\tau_{|U} U) = \omega(U)$ is a valid weight assignment (because the weights are disjoint), it is counted in $\mathcal{P}(U)$. Similarly for the $\mathcal{P}(V)$ cases. The $\epsilon(\tau_{|_U}) = -1$ cases are counted in $\mathcal{N}(U)$. Thus $\mathcal{P}(T) \leq \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$.

Now any weight assignments $\omega(k_1, \ldots, k_a|\tau_{|_U}U) = \omega(1, \ldots, a|U)$ in $\mathcal{P}(U)$ and $\omega(k_{m+1}, \ldots, a|\tau_{|_V}V) = \omega(m+1, \ldots, a|V)$ in $\mathcal{P}(V)$ must also correspond to the valid weight assignment $\omega(k_1, \ldots, k_a|\tau T) = \omega(1, \ldots, a|T)$ with $\tau = \tau_{|_U} \times \tau_{|_V}$. Moreover $\epsilon(\tau) = \epsilon(\tau_{|_U})\dot{\epsilon}(\tau_{|_V}) = 1$. So this weight assignment is in $\mathcal{P}(T)$. Similarly for weight assignments in $\mathcal{N}(U)$ and $\mathcal{N}(V)$. Hence $\mathcal{P}(T) \geq \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$. Thus we have $\mathcal{P}(T) = \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$ as desired. □

Theorem 8 allows us to construct a non-zero tableau from smaller non-zero tableaux. The main difficulty in applying this theorem is showing that the weight-sets are disjoint. To deal with this, we develop an idea of maximality of weights which is sufficient for weight-set disjointness.

3.4 Maximal Weights

Since the action of $\mathcal{S}_a$ on a tableau T does not change the resulting $\mathbf{q}_T$, we generalize the definition of tableau weight to account for this. We put an order on this generic weight, thus defining maximal weights.

Definition 3.4.1. Given a tableau T, the *generic form* of $\omega_i(T)$ is $\mathsf{w}_i(T) = \omega_i(\pi T) = (x_1, \ldots, x_a)$ for any $\pi \in \mathcal{S}_a$ such that $x_j \geq x_{j+1}$ for all j. In essence, $\mathsf{w}_i(T)$ is the weights of $\omega_i(T)$ listed in decreasing order. This definition works for any row of T.

We define the generic form of a weight on the entire tableau (assuming T has at most three rows) by, $\mathsf{w}(T) = \omega_{2,3}(\pi T) = \left(\begin{smallmatrix} x_1 & \cdots & x_a \\ y_1 & \cdots & y_a \end{smallmatrix}\right)$ for any $\pi \in \mathcal{S}_a$ such that $y_j \geq y_{j+1}$ for all j and if $y_j = y_{j+1}$ then $x_j \geq x_{j+1}$. We consider only the weight vectors of the second and third rows and list the vectors so that the row three weights are decreasing. If two vectors have the same weight for row three, we list the vector with the larger weight in row two first.

For instance, if $T = \begin{matrix} 2\,2\,1 \\ 1\,1\,2 \\ 3\,3\,3 \end{matrix}$ then $\omega_{2,3}(T) = \left(\begin{smallmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right)$. Hence $\mathsf{w}(T) = \left(\begin{smallmatrix} 0 & 2 & 1 \\ 3 & 0 & 0 \end{smallmatrix}\right)$.

Definition 3.4.2. We put an order on generic weights by $\mathsf{w}_i(T_1) > \mathsf{w}_i(T_2)$ if when $\mathsf{w}_i(T_1) = (x_1, \ldots x_a)$, $\mathsf{w}_i(T_2) = (v_1, \ldots v_a)$, there exists $k \geq 1$ such that $x_j = v_j$ for $j < k$ and $x_k > v_k$. We say $\mathsf{w}(T_1) > \mathsf{w}(T_2)$ if

1. $\mathsf{w}_3(T_1) > \mathsf{w}_3(T_2)$ or
2. $\mathsf{w}_3(T_1) = \mathsf{w}_3(T_2)$ and $\mathsf{w}_2(T_1) > \mathsf{w}_2(T_2)$ or
3. $\mathsf{w}_3(T_1) = \mathsf{w}_3(T_2)$, $\mathsf{w}_2(T_1) = \mathsf{w}_2(T_2)$, and if we have $\mathsf{w}(T_1) = \left(\begin{smallmatrix} y_1 & \cdots & y_a \\ x_1 & \cdots & x_a \end{smallmatrix}\right)$ and $\mathsf{w}(T_2) = \left(\begin{smallmatrix} z_1 & \cdots & z_a \\ x_1 & \cdots & x_a \end{smallmatrix}\right)$ then there exists $k \geq 1$ such that $y_j = z_j$ for $j < k$ and $y_k > z_k$.

We also apply this ordering to sets of weight vectors, by associating to each set the generic weight vector formed by concatenating the given weights in order. So to the set $A = \{\binom{0}{2}, \binom{1}{1}\}$ we associate the weight $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 1 \end{smallmatrix}\right)$.

Definition 3.4.3. We define the *maximum generic weight* of row i of T to be the maximum with respect to $>$ of $\{\mathsf{w}_i(\tau T) | \tau \in C_T\}$, where w is the generic weight defined above. Similarly the *maximum generic weight* of T is the maximum with respect to $>$ of $\{\mathsf{w}(\tau T) | \tau \in C_T\}$. Note that the maximum generic weight of T is based only on the weights of rows two and three. As such we ignore the weight of the first row.

Example 3.4.4. Let $T = \begin{matrix} 1\,2 \\ 2\,1 \end{matrix}$. Then the generic weights $\mathsf{w}_2(\tau T)$ are $(1,1)$ and $(2,0)$, with $(2,0)$ (the generic weight of $\tau T = \begin{matrix} 1\,1 \\ 2\,2 \end{matrix}$ or $\begin{matrix} 2\,2 \\ 1\,1 \end{matrix}$) as maximum. Here we've suppressed writing $\mathsf{w}_3(T)$ since T has only two rows.

If $T = \begin{matrix} 1\,1\,4\,4\,4 \\ 2\,2\,3\,3\,3 \\ 3\,3 \end{matrix}$ then the maximum generic weight of T (of the second and third rows) is $\left(\begin{smallmatrix} 3 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{smallmatrix}\right)$ which is $\omega_{2,3}(T)$ in generic form, $\mathsf{w}(T)$.

Definition 3.4.5. Let T be a tableau having three or fewer rows. Let w^{m} be the maximum generic weight of T. We say w^{m} is the *max weight* for T if w^{m} occurs in $\mathbf{q}_T$. That is $\mathbf{q}_T \neq 0$ by weight-set counting on w^{m}.

Unlike the maximum weight, the max weight of a tableau may not exist since the weight may not occur in $\mathbf{q}_T$. For instance, consider $T = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{matrix}$. We know $\mathbf{q}_T = 0$ by Lemma 3.2.6, yet T has $\left(\begin{smallmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \end{smallmatrix}\right)$ as its maximum weight.

The max weight for T is always the maximum weight for row three of T, but it need not be the maximum weight for row two of T. Consider $T = \begin{matrix} 5 & 2 & 3 & 3 \\ 1 & 4 & 4 & 1 \\ 3 & 3 & & \end{matrix}$. The maximum generic weight of T is $\left(\begin{smallmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \end{smallmatrix}\right)$, but the maximum generic weight of row two of T is $(4, 0, 0, 0)$ which is not the generic form of $(0, 2, 2, 0)$.

Definition 3.4.6. If w^{m} is the max weight for T, we say T is in *maximal form* provided $\omega_{2,3}(\pi T) = w^{\mathsf{m}}$ for some $\pi \in \mathcal{S}_a$. This only requires that some permutation of the weight vectors of $\omega_{2,3}(T)$ be equal to the max weight of T.

While the max weight may not exist for all tableaux, it is easy to show weight-set disjointness for those tableaux which are in maximal form. In order to prove this, we use the following lemmas regarding our ordering.

Lemma 3.4.7. Given two weights, W_1 and W_2, of the same length, let $C_k = \{\binom{x}{y} | \binom{x}{y} \in W_k\}$, $k = 1, 2$ be the multisets of weight vectors in each of these weights. Let $A = C_1 \setminus (C_1 \cap C_2)$, the weight vectors of W_1 not in W_2. Similarly, let $B = C_2 \setminus (C_1 \cap C_2)$. If $W_1 \geq W_2$, then $A \geq B$.

Proof. Without loss of generality, assume W_i is written in maximal form (i.e., is equal to its generic weight). If $W_1 = W_2$ then $A = B = \varnothing$, so the result holds trivially. So suppose $W_1 > W_2$. Then either W_1 differs from W_2 at some place in the third row, or the third rows are equal and they differ at some place in the second row (at least vectorwise).

Define $A^y = \{\binom{x_i}{y_i} | \binom{x_i}{y_i} \in A,\, y_i = y\}$ and $B^v = \{\binom{u_i}{v_i} | \binom{u_i}{v_i} \in B,\, v_i = v\}$. Defining C_1^y and C_2^v similarly, we have $A^y = C_1^y \setminus (C_1^y \cap C_2^y)$ and $B^v = C_2^v \setminus (C_1^v \cap C_2^v)$.

Let $W_1 = \left(\begin{smallmatrix} x_1 & \cdots & x_m \\ y_1 & \cdots & y_m \end{smallmatrix}\right)$ and $W_2 = \left(\begin{smallmatrix} u_1 & \cdots & u_m \\ v_1 & \cdots & v_m \end{smallmatrix}\right)$. If W_1 differs from W_2 in the third row, then there exists j such that $y_j > v_j$ and $y_i = v_i$ for all $i < j$. Hence $|C_1^{y_i}| = |C_2^{v_i}| = |C_2^{y_i}|$ for $y_i > y_j$, so $|A^{y_i}| = |B^{v_i}| = |B^{y_i}|$. Then to show $A > B$ it suffices to show

$|A^{y_j}| > |B^{y_j}|$. But since $y_j > v_j$, we have $|C_1^{y_j}| > |C_2^{y_j}|$. Thus $\mathsf{w}_3(A) > \mathsf{w}_3(B)$ and the result follows. Note that when the third rows are equal, this argument shows $\mathsf{w}_3(A) = \mathsf{w}_3(B)$.

If W_1 and W_2 are equal in the third row but the generic weights of their second rows differ, we can apply the same argument as above, where A^x, B^x, C_i^x are the appropriate sets of weight vectors with the second row weight equal to x. This shows $\mathsf{w}_2(A) > \mathsf{w}_2(B)$. Since we've already have $\mathsf{w}_3(A) = \mathsf{w}_3(B)$, the result follows.

If W_1 and W_2 have the same generic weights in the second and third rows, then by above we know the generic weights of rows two and three of A and B are the same. By definition, $A \cap B = \varnothing$, so the second row of the first vectors in A and B are different. Now W_1 and W_2 agree in the third row, so the first vectors where they differ in the second row must be the first vector in A and B respectively. Since $W_1 > W_2$ we have the large vector occurring in W_1 and hence in A. Thus $A > B$. $\square$

The following lemma is an obvious property of the ordering, but is included due to the non-standard ordering used.

Lemma 3.4.8. Given sets A and B of weight vectors, if $A \geq B$ and $B \geq A$ then $A = B$.

Proof. View A and B as generic weight vectors W_A and W_B. Let $W_A = \left(\begin{smallmatrix} x_1 & \cdots & x_m \\ y_1 & \cdots & y_m \end{smallmatrix}\right)$ and $W_B = \left(\begin{smallmatrix} u_1 & \cdots & u_m \\ v_1 & \cdots & v_m \end{smallmatrix}\right)$. Since $A \geq B$ we have $(y_1, \ldots, y_m) \geq (v_1, \ldots, v_m)$. So either the rows are equal or there exists j such that $y_j > v_j$ and $y_i = v_i$ for all $i < j$. As $B \geq A$ we would similarly get $v_j \geq y_j$, which is a contradiction. Hence $\mathsf{w}_3(A) = \mathsf{w}_3(B)$. A similar argument shows $\mathsf{w}_2(A) = \mathsf{w}_2(B)$.

If $A \neq B$, let j be the first place where they differ. Then $A \geq B$ implies $\binom{x_j}{y_j} > \binom{u_j}{v_j}$. Since $y_j = v_j$, this means $x_j > u_j$. But the same argument on $B \geq A$ implies $u_j > x_j$, which is a contradiction. Thus $A = B$. $\square$

Now using these lemmas we can show that the max weights of tableaux are necessarily disjoint.

Lemma 3.4.9. If U and V are tableaux in maximal form, with U containing b copies of the elements 1 to m and V containing b copies of the elements $m+1$ to a (after renumbering as necessary), then $\omega(U)$ and $\omega(V)$ are disjoint.

Proof. Let U and V be in maximal form. Let $\omega_{2,3}(U \vee V) = \left(\begin{smallmatrix} x_1 & \cdots & x_m & x_{m+1} & \cdots & x_a \\ y_1 & \cdots & y_m & y_{m+1} & \cdots & y_a \end{smallmatrix}\right)$. To show that U and V are disjoint, we need to show that any valid weight assignment of $U \vee V$ restricts to a valid weight assignment of U and V. Let $\left(\begin{smallmatrix} x_{k_1} & \cdots & x_{k_m} & x_{k_{m+1}} & \cdots & x_{k_a} \\ y_{k_1} & \cdots & y_{k_m} & y_{k_{m+1}} & \cdots & y_{k_a} \end{smallmatrix}\right)$ be a valid weight assignment of $U \vee V$. That means there exists τ such that $\omega_{2,3}(\tau[U \vee V]) = \left(\begin{smallmatrix} x_{k_1} & \cdots & x_{k_m} & x_{k_{m+1}} & \cdots & x_{k_a} \\ y_{k_1} & \cdots & y_{k_m} & y_{k_{m+1}} & \cdots & y_{k_a} \end{smallmatrix}\right)$. We want to show it restricts to a valid weight assignment of U, that is $\omega_{2,3}(\tau_{|_U} U) = \left(\begin{smallmatrix} x_{k_1} & \cdots & x_{k_m} \\ y_{k_1} & \cdots & y_{k_m} \end{smallmatrix}\right) = \omega_{2,3}(\pi U)$ for some $\pi \in \mathcal{S}_m$. This is equivalent to showing $\{\binom{x_i}{y_i} | i = 1 \ldots m\} = \{\binom{x_{k_i}}{y_{k_i}} | i = 1 \ldots m\}$.

Let $C = \{\binom{x_i}{y_i} | i = 1 \ldots m\}$ be set of the weight vectors of $\omega_{2,3}(U)$ and $D = \{\binom{x_{k_i}}{y_{k_i}} | i = 1 \ldots m\}$ the set of weight vectors of $\omega_{2,3}(\tau_{|_U} U)$. Define $A = C \setminus (C \cap D)$ and $B = D \setminus (C \cap D)$. So A consists of those weight vectors of U assigned to V which are distinct from the weight vectors of V assigned to U under this weight assignment. That is, the vectors in A occur in $\omega_{2,3}(U)$ but not in $\omega_{2,3}(\tau_{|_U} U)$. The set B is the weight vectors are those vectors coming from $\omega_{2,3}(\tau_{|U} U)$ which are not in $\omega_{2,3}(U)$. To prove disjointness, we need to show that $C = D$, which is equivalent to showing $A = B = \varnothing$.

Now U is in maximal form, so $\omega_{2,3}(U) \geq \omega_{2,3}(\tau' U)$ for all τ'. Hence $\omega_{2,3}(U) \geq \omega_{2,3}(\tau_{|_U} U)$. So by Lemma 3.4.7, we have $A \geq B$. But we can also view A as the weight vectors of U in $\omega_{2,3}(\tau_{|_V} V)$ which are not in $\omega_{2,3}(V)$. Similarly B is the set of vectors from $\omega_{2,3}(V)$ which are not in $\omega_{2,3}(\tau_{|_V} V)$. Since V is also in maximal form, $\omega_{2,3}(V) \geq \omega_{2,3}(\tau_{|_V} V)$. Hence by Lemma 3.4.7, we have $B \geq A$. Thus Lemma 3.4.8 shows $A = B$. However, $A \cap B = \varnothing$ by definition, so $A = B = \varnothing$. Thus the weights are disjoint. $\square$

Example 3.4.10. Suppose U is a tableau in maximal form such that $\omega_{2,3}(1, 2, 3, 4|U) = \left(\begin{smallmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{smallmatrix}\right)$ and V is a tableau in maximal form with $\omega_{2,3}(5, 6, 7, 8, 9, 10, 11|V) = \left(\begin{smallmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$. Suppose these weights were not disjoint. That means we must be able to assign some weight $\binom{x}{y}$ of U to V and some weight

$\binom{x'}{y'}$ of V to U.

First consider the vector $\binom{1}{2}$ of U (i.e., the vector with the largest weight in row three). Since V is in maximal form, we know that there can be at most one copies of any element in row three of τV for any τ. Since $2 > 1$, this vector cannot be assigned to V. Similarly, once we know that $\binom{1}{2}$ remains a weight of U, the largest row three weight we can assign to U is 0. Hence $\binom{1}{1}$ remains with V.

Now consider $\binom{3}{0}$ of U. Having V in maximal form means that when $\binom{1}{1}$ are assigned to V, a vector $\binom{*}{0}$ assigned to V must have $* \leq 2$. Thus $\binom{3}{0}$ is assigned to U. Therefore the only vectors of U and V that can be assigned to each other are the $\binom{0}{0}$ vectors. However, since a weight assignment is based only on the vector and not its label, this is the same as a weight assignment arising from U and V. Hence the weights are disjoint.

Lemma 3.4.9 shows that if U and V are in maximal form, $\mathbf{q}_{U \vee V} \neq 0$ by Theorem 8. We will apply Theorem 8 repeatedly when constructing tableaux. As such, we want $U \vee V$ to be in maximal form whenever U and V are.

Lemma 3.4.11. If T_1 and T_2 are maximal tableaux filled with different elements, then $T_1 \vee T_2$ is maximal.

Proof. The tableaux have no elements in common so the weight splits over the join. Hence $\omega(\tau[T_1 \vee T_2]) = \omega(\tau_1 T_1 \vee \tau_2 T_2) = \omega(\tau_1 T_1) \vee \omega(\tau_2 T_2)$. Since each tableau weight was maximal, so too is their join. □

Hence maximality is preserved under the join operation. Through this join operation, we will construct collections of tableaux. To show these tableaux are linearly independent (over $\mathbb{C}$), we can simply compare max weights.

Lemma 3.4.12. Let $\{\mathcal{B}_p\}$ be a set of tableaux in maximal form. If the max weights of these tableaux are distinct, then $\{\mathbf{q}_{\mathcal{B}_p}\}$ is linearly independent.

Proof. Assume $\{\mathbf{q}_{\mathcal{B}_p}\}$ is not linearly independent. Let $\mathcal{B}_k$ be the tableau with the largest weight such that $\mathbf{q}_{\mathcal{B}_k}$ is not linearly independent from the rest of $\{\mathbf{q}_{\mathcal{B}_p}\}$. Write

$\mathbf{q}_{\mathcal{B}_k} = \sum a_p \mathbf{q}_{\mathcal{B}_p}$. Then $\omega(\mathbf{q}_{\mathcal{B}_k}) = \sum a_p \omega(\mathbf{q}_{\mathcal{B}_p})$. Since $\mathcal{B}_k$ is in maximal form, $\omega(\mathcal{B}_k)$ occurs with non-zero coefficient in $\omega(\mathbf{q}_{\mathcal{B}_k})$. Hence $\omega(\mathcal{B}_k)$ must occur with non-zero coefficient in $\omega(\mathbf{q}_{\mathcal{B}_p})$ for some p. However, $\mathcal{B}_k$ was chosen such that $\omega(\mathcal{B}_k) \geq \omega(\mathcal{B}_p)$. By hypothesis these weights are distinct, so the inequality is strict. But the $\mathcal{B}_p$ are in maximal form, so $\omega(\mathcal{B}_p)$ is the largest weight occurring in $\mathbf{q}_{\mathcal{B}_p}$. Hence $\omega(\mathcal{B}_k)$ does not occurring in $\sum a_p \omega(\mathbf{q}_{\mathcal{B}_p})$, contradicting the linear dependence. □

We will use Lemma 3.4.12 heavily in Chapter 9 to prove Theorem 3. To make use of this lemma, we need to have distinctness of max weights. When the tableaux are formed via the join operation, we can sometimes simplify the proof of max weight distinctness via the following lemma:

Lemma 3.4.13. Suppose we have the two row tableaux T_1, T_2, T_3, and T_4, were the non-zero max weights are as follows: $\omega(T_1) = (A, B)$, $\omega(T_2) = (C, D)$, $\omega(T_3) = (a, b)$, and $\omega(T_4) = (c, d)$. Assume $\omega(T_1) \neq \omega(T_3)$, $\omega(T_2) \neq \omega(T_4)$, but $\lambda_2(T_1) = \lambda_2(T_3)$ and $\lambda_2(T_2) = \lambda_2(T_4)$. If $\lambda_2(T_1) \neq \lambda_2(T_2)$, then $\omega(T_1 \vee T_2) \neq \omega(T_3 \vee T_4)$.

Proof. Since $\lambda_2(T_1) \neq \lambda_2(T_2)$ and $\omega(T_1) \neq \omega(T_3)$, assume $A+B > C+D$ and $A > a$. If $\omega(T_1 \vee T_2) = \omega(T_3 \vee T_4)$, either $A = c$ or $A = d$ since $a \geq b$ by maximality of $\omega(T_3)$. Consider $A = c$, then $B > d$ and $B \neq c$ because $C + D = c + d$. Then $A > a$ and $A + B = a + b$, implies $B < b \leq a$. Hence there is no weight equal to B, so this cannot occur. Similarly if $A = d$ then $c < B$ and hence there is no weight equal to B. Thus the weights are distinct. □

Although Lemma 3.4.13 applies directly to the join of only two tableaux, it may often be applied in a broader context. Namely, if many of the tableaux being joined are the same, the question of distinct weights reduces to looking only at the weights of those tableaux which differ. This approach will be used and discussed in Chapter 9.

Chapter 4

The Tableaux of $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$

In this chapter we completely classify and discuss those tableaux occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$. To begin with, we determine exactly which tableaux associated with this space have $\mathbf{q}_T \neq 0$. This is done in Section 4.1. In Section 4.2, we use this information and the results from Thrall's paper, [20], to determine precisely which partitions occur. Finally in Section 4.3, we construct a complete tableau basis for each the tableaux space $S^{\lambda,3} \cap M^{\lambda,3}$ associated to the irreducibles of $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$

4.1 Classification of $\mathbf{q}_T \neq 0$, for T filled with 1, 2, 3

Let T be a λ-tableau filled with b copies of the numbers 1, 2, and 3. By Lemma 3.1.5, T has at most three rows. By Remark 3.0.9 entry permutations, column permutations and column exchanges do not change whether $\mathbf{q}_T$ is non-zero. Hence we may take T to be in the following the form:

$$T = \begin{array}{ccccccc} \scriptstyle K & \scriptstyle L & \scriptstyle M & \scriptstyle N & \scriptstyle O & \scriptstyle P & \scriptstyle Q \\ \hline 1 & 1 & 1 & 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 3 & & & \\ 3 & & & & & & \end{array} \qquad \begin{cases} \mathrm{K} + \mathrm{L} + \mathrm{M} + \mathrm{O} = b \\ \mathrm{K} + \mathrm{L} + \mathrm{N} + \mathrm{P} = b \\ \mathrm{K} + \mathrm{M} + \mathrm{N} + \mathrm{Q} = b \\ \mathrm{L} \geq \mathrm{M} \geq \mathrm{N} \geq 0 \end{cases}$$

However, since we know there are exactly b copies of every number in T, we may omit the tail and simply write T as $T = \begin{array}{cccc} \scriptstyle \mathrm{K} & \scriptstyle \mathrm{L} & \scriptstyle \mathrm{M} & \scriptstyle \mathrm{N} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$, retaining the condition $\mathrm{L} \geq \mathrm{M} \geq \mathrm{N} \geq 0$ and assuming the tail.

Theorem 9. With T as described above,

$$\mathbf{q}_T = 0 \iff \begin{cases} \mathrm{K}+\mathrm{L} \text{ odd} & \mathrm{L} > \mathrm{M} = \mathrm{N} \geq 0 \\ \mathrm{K}+\mathrm{N} \text{ odd} & \mathrm{L} = \mathrm{M} \geq \mathrm{N} \geq 0 \\ \mathrm{K}+\mathrm{M} \text{ even} & \mathrm{L} = \mathrm{M}+2,\ \mathrm{N} = \mathrm{M}\text{-}1 \geq 0 \\ \mathrm{K}+\mathrm{L} \text{ even} & \mathrm{L} = \mathrm{M}+1,\ \mathrm{M}=\mathrm{N} \geq 0 \end{cases}$$

Moreover, when $\mathbf{q}_T \neq 0$, it is non-zero by weight-set counting.

Proof. $\Longleftarrow$ Using Remark 3.0.9, to show $\mathbf{q}_T = 0$ it suffices to exhibit $\pi \in \mathcal{S}_3$, $\tau \in C_T$, $\epsilon(\tau) = -1$, such that $\pi\tau T = T$ up to an exchange of columns.

For $\mathrm{L} > \mathrm{M} = \mathrm{N}$, $\mathrm{L}+\mathrm{K}$ odd, take $\tau = \overset{\mathrm{K}}{(12)}_T \times \overset{\mathrm{L}}{(12)}_T \times \overset{\mathrm{M}}{()}_T \times \overset{\mathrm{N}}{()}_T$ and $\pi = (12)$. So $\epsilon(\tau) = (-1)^{\mathrm{K}+\mathrm{L}} = -1$. Then $\tau T = \begin{array}{cccc} \scriptstyle \mathrm{K} & \scriptstyle \mathrm{L} & \scriptstyle \mathrm{M} & \scriptstyle \mathrm{N} \\ \hline 2 & 2 & 1 & 2 \\ 1 & 1 & 3 & 3 \\ 3 & & & \end{array}$ and $\pi\tau T = \begin{array}{cccc} \scriptstyle \mathrm{K} & \scriptstyle \mathrm{L} & \scriptstyle \mathrm{M} & \scriptstyle \mathrm{N} \\ \hline 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$. Since $\mathrm{M}=\mathrm{N}$, exchanging columns gives T.

For $\mathrm{L} = \mathrm{M} \geq \mathrm{N} \geq 0$, $\mathrm{K}+\mathrm{N}$ odd use $\pi = (23)$, $\tau = \overset{\mathrm{K}}{(23)}_T \times \overset{\mathrm{L}}{()}_T \times \overset{\mathrm{M}}{()}_T \times \overset{\mathrm{N}}{(12)}_T$ and interchange columns L and M.

Now consider when $\mathrm{L} = \mathrm{M}+1$, $\mathrm{M} = \mathrm{N}$ with $\mathrm{K}+\mathrm{L}$ even. Then $T = \begin{array}{ccccc} \scriptstyle \mathrm{K} & & \scriptstyle \mathrm{M} & \scriptstyle \mathrm{M} & \scriptstyle \mathrm{M} \\ \hline 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & & & & \end{array}$.

Write $T = T^* \vee T_1$ where $T^* = \begin{array}{cccc} \scriptstyle \mathrm{K} & \scriptstyle \mathrm{M} & \scriptstyle \mathrm{M} & \scriptstyle \mathrm{M} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$ and $T_1 = \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$. Let $T_2 = \begin{array}{cc} 2 & 3 \\ 1 & \end{array}$. Take $\sigma' \in R_{T_i}$ so $\sigma' T_2 = \begin{array}{cc} 3 & 2 \\ 1 & \end{array}$. Let $\pi' = (123) \in \mathcal{S}_3$ and $\tau^* = \overset{\mathrm{K}}{(132)} \times \overset{\mathrm{M}}{()} \times \overset{\mathrm{M}}{(12)} \times \overset{\mathrm{M}}{(12)} \in C_{T^*}$.

Then via reordering the columns by $\tilde{\sigma}$, we have $\tilde{\sigma}\tau^*\pi'\sigma'(T^* \vee T_2) = T^* \vee T_1$. Note that $\epsilon(\tau^*) = 1$ and $\tilde{\sigma}$, σ', π' commute with each other and all $\tau \in C_{T^*}$. Then

$$
\begin{aligned}
\mathbf{q}_T &= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_T} \sigma\pi\,\epsilon(\tau)\tau T \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_T} \sigma\pi\,\epsilon(\tau)\tau(T^*\vee T_1) \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3} \sigma\pi[(\sum_{\tau\in C_{T^*}} \epsilon(\tau)\tau T^*) \vee (\sum_{\tau'\in C_{T_1}} \epsilon(\tau')\tau' T_1)] \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3} \sigma\pi[(\sum_{\tau\in C_{T^*}} \epsilon(\tau)\tau T^*) \vee (T_1 - T_2)] \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau(T^*\vee T_1 - T^*\vee T_2) \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_1 - \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau\tilde{\sigma}\tau^*\pi'\sigma' T^*\vee T_2 - \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\,\epsilon(\tau^*)\tau T^*\vee T_2 - \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 - \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \sigma\pi\,\epsilon(\tau)\tau T^*\vee T_2 \\
&= 0
\end{aligned}
$$

Note that where appropriate we commute $\tilde{\sigma}$, σ', and π' to combine with σ and π and reparameterize. The $\epsilon(\tau^*)$ factor arises from the reparamterization of $\tau\tau^*$.

Now consider when L = M+2, N = M-1 with K + M even. So $T = \begin{array}{cccccc} \text{K} & \text{M-1} & & & \text{M-1} & \text{M-1} \\ \hline 1 & 1 & 1\,1\,1\,1 & & 1 & 2 \\ 2 & 2 & 2\,2\,2\,3 & & 3 & 3 \\ 3 & & & & & \end{array}$. Write $T = T^* \vee T_1$ where $T^* = \begin{array}{cccc} \text{K} & \text{M-1} & \text{M-1} & \text{M-1} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$ and $T_1 = \begin{array}{l} 1\,1\,1\,1\,2\,3\,3\,3 \\ 2\,2\,2\,3 \end{array}$.

We will use the idea of the previous case to show $\mathbf{q}_T = 0$. However, we need to sum over all $\epsilon(\tau)\tau T_1$ with τ in C_{T_1}. These tableaux have different symmetry relations with each other, so we will list all the $\epsilon(\tau)\tau T$ and their relations. Note that the tail is omitted for readability.

$$T_1 = \begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \end{matrix} \qquad T_2 = -\begin{matrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \end{matrix} \qquad T_3 = -\begin{matrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 3 \end{matrix} \qquad T_4 = -\begin{matrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 3 \end{matrix}$$

$$T_5 = -\begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 1 \end{matrix} \qquad T_6 = \begin{matrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{matrix} \qquad T_7 = \begin{matrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{matrix} \qquad T_8 = \begin{matrix} 2 & 1 & 1 & 3 \\ 1 & 2 & 2 & 1 \end{matrix}$$

$$T_9 = \begin{matrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 3 \end{matrix} \qquad T_{10} = \begin{matrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 2 & 1 \end{matrix} \qquad T_{11} = \begin{matrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 1 \end{matrix} \qquad T_{12} = -\begin{matrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 1 & 1 \end{matrix}$$

$$T_{13} = -\begin{matrix} 2 & 1 & 2 & 3 \\ 1 & 2 & 1 & 1 \end{matrix} \qquad T_{14} = -\begin{matrix} 2 & 2 & 1 & 3 \\ 1 & 1 & 2 & 1 \end{matrix} \qquad T_{15} = -\begin{matrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 3 \end{matrix} \qquad T_{16} = \begin{matrix} 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{matrix}$$

For appropriate $\sigma' \in R_{T_1}$ we have the following relations. All permutation listed are from $\mathcal{S}_3$.

$$T_1 = -(123)\sigma' T_{14} \quad T_2 = -(123)\sigma' T_9 \quad T_3 = -(123)\sigma' T_6 \quad T_4 = -(123)\sigma' T_7$$

$$T_5 = (12)\sigma' T_{13} \qquad T_8 = (12)\sigma' T_8 \qquad T_{10} = (12)\sigma' T_{10} \qquad T_{11} = (12)\sigma' T_{11}$$

$$T_{12} = (23)\sigma' T_{15} \qquad T_{16} = (23)\sigma' T_{16}$$

We also have some relations on T^*, namely that for $\pi^* \in \mathcal{S}_3$ a transposition, then $\pi^* T^* = \tau_{\pi^*} T^*$ for $\tau_{\pi^*} \in C_{T^*}$ with $\epsilon(\tau_{\pi^*}) = (-1)^{\mathrm{K}+\mathrm{M}-1}$. Also $(132)T^* = \tau_{(132)} T^*$ for $\tau_{(132)} \in C_{T^*}$ with $\epsilon(\tau_{(132)}) = (-1)^{2(\mathrm{M}-1)} = 1$. Then

$$\begin{aligned}
&\sum_{\sigma \in R_T} \sum_{\pi \in \mathcal{S}_3} \sum_{\tau \in C_{T^*}} \pi\sigma\, \epsilon(\tau)\tau(T^* \vee (\pi^*)^{-1}\sigma' T_i) \\
=& \sum_{\sigma \in R_T} \sum_{\pi \in \mathcal{S}_3} \sum_{\tau \in C_{T^*}} \pi\sigma\, \epsilon(\tau)\tau(\pi^* T^* \vee \sigma' T_i) \\
=& \sum_{\sigma \in R_T} \sum_{\pi \in \mathcal{S}_3} \sum_{\tau \in C_{T^*}} \pi\sigma\, \epsilon(\tau)\, \epsilon(\tau_{\pi^*})\tau(T^* \vee \sigma' T_i) \\
=& \sum_{\sigma \in R_T} \sum_{\pi \in \mathcal{S}_3} \sum_{\tau \in C_{T^*}} \pi\sigma\, \epsilon(\tau)\, \epsilon(\tau_{\pi^*})\tau(T^* \vee T_i)
\end{aligned}$$

Hence if $T_i = -(123)\sigma' T_j$ or $T_i = \pi^*\sigma' T_j$ for π^* a transposition, then $\sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \pi\sigma\,\epsilon(\tau)\tau[(T^*\vee T_i)+(T^*\vee T_j)] = 0$. Also, if $T_i = \pi^*\sigma' T_j$, then $\sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \pi\sigma\,\epsilon(\tau)\tau(T^*\vee T_i) = 0$. So using the cancellations above, we have

$$
\begin{aligned}
\mathbf{q}_T &= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_T} \pi\sigma\,\epsilon(\tau)\tau(T^*\vee T_1) \\
&= \sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3} \pi\sigma[(\sum_{\tau\in C_{T^*}} \epsilon(\tau)\tau T^*)\vee(\sum_{\tau'\in C_{T_1}} \epsilon(\tau')\tau T_1)] \\
&= \sum_{i=1}^{16}\sum_{\sigma\in R_T}\sum_{\pi\in\mathcal{S}_3}\sum_{\tau\in C_{T^*}} \pi\sigma\,\epsilon(\tau)\tau(T^*\vee T_i) \\
&= 0
\end{aligned}
$$

$\Longrightarrow$ To prove these are the only non-zero cases, we will use weight-set counting of Theorem 4 to show $\mathbf{q}_T \neq 0$ in the remaining cases. Given a weight, for every weight assignment $\pi\in\mathcal{S}_3$ we will count (with sign) the number of $\tau\in C_T$ such that $\pi\omega(T) = \omega(\tau T)$ and show the sum of these numbers is non-zero. In some cases it may be necessary to use $\omega(\tau' T)$ instead of $\omega(T)$ to show the weight-sum is non-zero. Since applying τ' affects only the sign of $\mathbf{q}_T$, this will not change our result.

Specifically we wish to show if:

$$
\left.\begin{array}{ll}
\text{K}+\text{L even} & \text{L} > \text{M} = \text{N} \geq 0 \\
\text{K}+\text{N even} & \text{L} = \text{M} \geq \text{N} \geq 0 \\
\text{L} > \text{M} > \text{N} &
\end{array}\right\} \text{ then } \mathbf{q}_T \neq 0
$$

unless L=M+1, M=N, K+L even, or L=M+1, M-1=N, K+M even.

Note that if $\text{K} = \text{L} = \text{M} = \text{N} = 0$ then T has only one row and $C_T = 1$. In this case $\mathbf{q}_{\sigma T} = \mathbf{q}_T$, so there is exactly one distinct T. Since $\text{K}+\text{N} = 0$, so the statement holds. Similarly, if $\text{L} = \text{M} = \text{N} = 0$, applying Lemma 3.2.6 gives K even. Hence we may assume $\text{L} > 0$.

We will use the first weight-set to illustrate the technique and notation of weight-set counting. This method of argument will be used extensively throughout the rest of the paper. We will list our weight-counting in a table of the following form:

$\omega_i = (\text{L}, \text{M}, \text{N})$	Tableau	#	ϵ	τ	bound
(x, y, z)	T'	j	$(-1)^\epsilon$	$\overset{\text{L}}{\tau_1} \times \overset{\text{M}}{\tau_2} \times \overset{\text{N}}{\tau_3}$	M=N

The column headings are: the weight being used, the form of the tableau, the number of τ corresponding to this weight, the sign of the τ, the form of τ, and any bounds required. The subsequent lines correspond to different weight assignments in line i. By (x, y, z), we mean $\omega_i(x, y, z) = (\text{L}, \text{M}, \text{N})$. When M=N, there are j distinct τ such that $\tau = \overset{\text{L}}{\tau_1} \times \overset{\text{M}}{\tau_2} \times \overset{\text{N}}{\tau_3}$. All the τ have sign $(-1)^\epsilon$ and τT is of the form T'.

The following is a standard formula that we will use in computing these weight sums. Its proof is a straightforward inductive application of Pascal's Identity. For notation purposes, we take $\binom{\mathsf{a}}{\mathsf{b}} = 0$ for $\mathsf{b} > \mathsf{a}$ or $\mathsf{b} < 0$. We also use the convention $\binom{0}{0} = 1$.

Lemma 4.1.1. $\binom{\mathsf{a+h}}{\mathsf{b}} - \binom{\mathsf{a}}{\mathsf{b}} = \sum_{i=1}^{\mathsf{h}} \binom{\mathsf{a+h}-i}{\mathsf{b}-1}$

Consider the cases where $\text{K} \geq 0$ and at least $\text{L} > 0$. We will apply weight-set counting to rows two and three. For the most part, the table should be self explanatory, though we will discuss the first weight-set table for clarity.

We'll start with $T = \begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & 2 \\ 3 & & & \end{array}$, $\quad \omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{K+L+N} & \text{M} \\ 0 & 0 & \text{K} \end{smallmatrix} \right)$

$\omega_{2,3}$	Tableau	#	ϵ	τ	bound
$(1,2,3)$	$T = \begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & 2 \\ 3 & & & \end{array}$	1	$(-1)^0$	$\overset{\text{K}}{()}_T \times \overset{\text{L}}{()}_T \times \overset{\text{M}}{()}_T \times \overset{\text{N}}{()}_T$	
$(1,3,2)$	$T = \begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 1 & 1 & 1 & 2 \\ 3 & 2 & 3 & 3 \\ 2 & & & \end{array}$	1	$(-1)^{\text{K}+\text{N}}$	$\overset{\text{K}}{(23)}_T \times \overset{\text{L}}{()}_T \times \overset{\text{M}}{()}_T \times \overset{\text{N}}{(12)}_T$	$\text{L} = \text{M}$
$(2,1,3)$	$T = \begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 2 & 2 & 31 & 2 \\ 1 & 1 & 13 & 3 \\ 3 & & & \end{array}$	$\binom{\text{M}}{\text{N}}$	$(-1)^{\text{K}+\text{L}}$	$\overset{\text{K}}{(12)}_T \times \overset{\text{L}}{(12)}_T \times \overset{\text{N}}{(12)}_T \times \overset{\text{N}}{(12)}_T$	
$(2,3,1)$	$T = \begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 2 & 2 & 1 & 2 \\ 3 & 1 & 3 & 3 \\ 1 & & & \end{array}$	1	$(-1)^{\text{L}+\text{N}}$	$\overset{\text{K}}{(123)}_T \times \overset{\text{L}}{(12)}_T \times \overset{\text{M}}{()}_T \times \overset{\text{N}}{(12)}_T$	$\text{L} = \text{M}$
$(3,1,2)$	$T = \begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 3 & 21 & 3 & 3 \\ 1 & 12 & 1 & 2 \\ 2 & & & \end{array}$	$\binom{\text{L}}{\text{M}-\text{N}}$	$(-1)^{\text{L}+\text{N}}$	$\overset{\text{K}}{(132)}_T \times \overset{\text{L}+\text{N}-\text{M}}{(12)}_T \times \overset{\text{M}}{(12)}_T \times \overset{\text{N}}{()}_T$	
$(3,2,1)$	$T = \begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 3 & 1 & 3 & 3 \\ 2 & 2 & 1 & 2 \\ 1 & & & \end{array}$	1	$(-1)^{\text{K}+\text{M}}$	$\overset{\text{K}}{(13)}_T \times \overset{\text{L}}{()}_T \times \overset{\text{M}}{(12)}_T \times \overset{\text{N}}{()}_T$	

To understand how these τ are obtained, first apply the permutations needed to have $\omega(x) = 0$, for the appropriate x. Additionally, apply necessary permutations so that row three of T has the correct weight. Once this is done, there will only be one column block whose permutations have not been specified. Apply the number of permutations needed to get the correct weight.

For the first line of this table, we see that T has the desired weight and any column permutations will change this. Thus there is exactly one τ and it is positive.

In line two, we have $\omega_3(2) = \text{K}$ and $\omega_{2,3}(1) = 0$. Hence we must apply $(23)_T$ to column block K. Columns $\frac{1}{2}$ and $\frac{1}{3}$ cannot move since $\omega_{2,3}(1) = 0$. This gives L 2's in row two, so we must have L=M and apply $(12)_T$ to block N. When L=M, this completely determines τ, and it has sign $(-1)^{\text{K}+\text{N}}$. No such τ exists for $\text{L} > \text{M}$.

Line three counts $\omega_{2,3}(2,1,3)$. Since there can be no 2's in either rows two or three, τ must contain $\overset{\text{K}}{(12)}_T \times \overset{\text{L}}{(12)}_T \times \overset{\text{N}}{(12)}_T$. As row three already contains K 3's, column block K needs no other permutation. Hence column $\substack{1\\3}$ is the only column where τ has not yet been determined. As it stands, we already have K+L 1's in row two, hence only N more are required. Thus we need to apply $\overset{\text{N}}{(13)}_T$ to $\substack{1\\3}$. There are $\binom{\text{M}}{\text{N}}$ ways to choose which N columns move within the M block. Hence we get $\binom{\text{M}}{\text{N}}$ distinct τ of the form described.

In line four we apply $(132)_T$ to block K in order to have $\omega_3(1) = \text{K}$ and $\omega_{2,3}(2) = 0$. Additionally, $\omega_{2,3}(2) = 0$ means we must apply $(12)_T$ to block L and $(12)_T$ to block N. This gives L 1's in row two, so we must have L=M and leave block M unchanged. In this case there is one such τ; it has sign $(-1)^{\text{L+N}}$.

Line five is similar to line three. From the constraints $\omega_3(1) = \text{K}$ and $\omega_{2,3}(3) = 0$ we have that τ contains $\overset{\text{K}}{(123)}_T \times \overset{\text{M}}{(12)}_T \times \overset{\text{N}}{()}_T$. In order to have the correct weight for row two, we need L+N-M more 2's. There are $\binom{\text{L}}{\text{M-N}}$ ways to choose $\overset{\text{L+N-M}}{(12)}_T$ from block L and all these τ have sign $(-1)^{\text{L+N}}$.

The last line has a similar argument to line one. We need only apply $\tau = \overset{\text{K}}{(13)}_T \times \overset{\text{M}}{(12)}_T$ to get the correct weight and this is the only possible τ.

From this table we get the following sums.

$$1 + (-1)^{\text{K+L}}\binom{\text{M}}{\text{M}-\text{N}} + (-1)^{\text{L+N}}\binom{\text{L}}{\text{M}-\text{N}} + (-1)^{\text{K+M}} \qquad \text{L} \neq \text{M} \qquad (4.1.1)$$

$$1 + (-1)^{\text{K+N}} + (-1)^{\text{K+L}}\binom{\text{L}}{\text{N}} + (-1)^{\text{L+N}} + (-1)^{\text{L+N}}\binom{\text{L}}{\text{N}} + (-1)^{\text{K+L}} \qquad \text{L} = \text{M} \qquad (4.1.2)$$

<u>Case I:</u> (L $>$ M $>$ N). Here, (4.1.1) equals zero only if $|\binom{\text{L}}{\text{M}-\text{N}} \pm \binom{\text{M}}{\text{M}-\text{N}}| = 0$ or 2. For it to equal 0, we must have $\text{M} = \text{N}$ or $\text{L} = \text{M}$.

To have $|\binom{\text{L}}{\text{M}-\text{N}} - \binom{\text{M}}{\text{M}-\text{N}}| = 2$, and (4.1.1) equal to zero, we must have $\text{K} + \text{M}$ even and $\text{K} + \text{N}$ odd. Applying Lemma 4.1.1, we get $\text{L} = \text{M} + 1$ or $\text{L} = \text{M} + 2$.

If $\text{L} = \text{M} + 1$, then (4.1.1) becomes $1 - \binom{\text{M}}{\text{M}-\text{N}} + \binom{\text{M}+1}{\text{M}-\text{N}} + 1$. Now $\binom{\text{M}+1}{\text{M}-\text{N}} - \binom{\text{M}}{\text{M}-\text{N}} = 2$ only if $\binom{\text{M}}{\text{M}-\text{N}-1} = 2$, that is $\text{M} = 2$ and $\text{N} = 0$. This contradicts $\text{K} + \text{M}$ even and $\text{K} + \text{N}$ odd. Hence (4.1.1) is non-zero for $\text{L} = \text{M} + 1$.

If $\text{L} = \text{M} + 2$ and $\text{K} + \text{M}$ even, (4.1.1) becomes $1 - \binom{\text{M}}{\text{M}-\text{N}} + (-1)^{\text{M+N}}\binom{\text{M}+2}{\text{M}-\text{N}} + 1$.

For this expression to be zero we must have $\text{N} = \text{M} - 1$, in which case we've already shown $\mathbf{q}_T = 0$. Thus (4.1.1) is non-zero unless $\text{L} = \text{M} + 2$ and $\text{N} = \text{M} - 1$.

Finally, $\binom{\text{L}}{\text{M}-\text{N}} + \binom{\text{M}}{\text{M}-\text{N}} = 2$ only when $\text{M} = \text{N}$.

Case II: $(\text{L} = \text{M} \geq \text{N}, \text{L} > 0)$. We need only show the expression (4.1.2) is non-zero for $\text{K}+\text{N}$ even. From this, (4.1.2) becomes $2+(-1)^{\text{K}+\text{L}}(1+\binom{\text{L}}{\text{N}})+(-1)^{\text{L}+\text{N}}(1+\binom{\text{L}}{\text{N}})$. The parity of K and N is the same, so we reduce to determining when $1+(-1)^{\text{K}+\text{L}}((1+\binom{\text{L}}{\text{N}}) = 0$. Since $\binom{\text{L}}{\text{N}} > 0$, this cannot occur.

Case III: $(\text{L} > \text{M} = \text{N})$. We want a non-zero weight sum for $\text{K}+\text{L}$ even. Under these conditions, expression (4.1.1) becomes $1 + 1 + (-1)^{\text{L}+\text{N}} + (-1)^{\text{K}+\text{N}}$. This is non-zero unless $\text{K} + \text{N}$ is odd.

It remains to show $\mathbf{q}_T \neq 0$ for $\text{L} > \text{N} = \text{M}$, $\text{K} + \text{L}$ even, $\text{K} + \text{M}$ odd.

For this, consider the following weight-set counting on $T = \begin{array}{cccc} \scriptstyle\text{K} & \scriptstyle\text{L} & \scriptstyle\text{M} & \scriptstyle\text{N} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$ with $\omega_{2,3} = \left(\begin{smallmatrix} \text{K+L+M-1} & 0 & \text{N+1} \\ 0 & 0 & \text{K} \end{smallmatrix} \right)$. Note that this weight-set is not the weight of T. We are counting which permutations τ will correspond to a weight assignment π, where $\omega_{2,3} = \pi\omega_{2,3}(\tau T)$.

$\omega_{2,3}$	Tableau	#	ϵ	τ	Bound
$(1,2,3)$	$T = \begin{array}{l} \text{K L M N} \\ \hline 2\ 2\ 31\ 2 \\ 1\ 1\ 13\ 3 \\ 3 \end{array}$	$\binom{M}{M-1}$	$(-1)^{K+L+M-1}$	$\overset{K}{(12)}_T \times \overset{L}{(12)}_T \times \overset{M-1}{(12)}_T \times \overset{N}{()}_T$	$M \geq 1$
$(1,3,2)$	$T = \begin{array}{l} \text{K L M N} \\ \hline 3\ 21\ 3\ 3 \\ 2\ 12\ 1\ 2 \\ 1 \end{array}$	$\binom{L}{L-1}$	$(-1)^{M+N+L-1}$	$\overset{K}{(132)}_T \times \overset{L-1}{(12)}_T \times \overset{M}{(12)}_T \times \overset{N}{(12)}_T$	$L \geq 1$
$(2,1,3)$	$T = \begin{array}{l} \text{K L M N} \\ \hline 1\ 1\ 1\ 32 \\ 2\ 2\ 3\ 23 \\ 3 \end{array}$	$\binom{N}{M-1}$	$(-1)^{M-1}$	$\overset{K}{()}_T \times \overset{L}{()}_T \times \overset{M}{()}_T \times \overset{M-1}{(12)}_T$	$M \geq 1$ $M-1 \leq N$
$(2,3,1)$	$T = \begin{array}{l} \text{K L M N} \\ \hline 3\ 21\ 3\ 2 \\ 2\ 12\ 1\ 3 \\ 1 \end{array}$	$\binom{L}{N+1-M}$	$(-1)^{K+1}$	$\overset{K}{(13)}_T \times \overset{N+1-M}{(12)}_T \times \overset{M}{(12)}_T \times \overset{N}{(12)}_T$	$M \leq N+1$ $N+1-M \leq L$
$(3,1,2)$	$T = \begin{array}{l} \text{K L M N} \\ \hline 1\ 1\ 1\ 32 \\ 3\ 2\ 3\ 23 \\ 2 \end{array}$	$\binom{N}{L-1}$	$(-1)^{K+N+L+1}$	$\overset{K}{(23)}_T \times \overset{L}{()}_T \times \overset{M}{()}_T \times \overset{N+1-L}{(12)}_T$	$L \geq 1$ $L-1 \leq N$
$(3,2,1)$	$T = \begin{array}{l} \text{K L M N} \\ \hline 2\ 2\ 31\ 2 \\ 3\ 1\ 13\ 3 \\ 1 \end{array}$	$\binom{M}{N+1-L}$	$(-1)^{N+1}$	$\overset{K}{(123)}_T \times \overset{L}{(12)}_T \times \overset{N+1-L}{(12)}_T \times \overset{N}{()}_T$	$L \leq N+1$ $N+1-L \leq M$

Since we previously dealt with the $L = N+1$ case, the last two lines of the table do not contribute. Hence the table gives the weight sum:

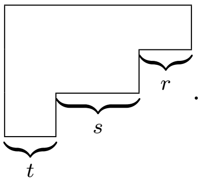
$$(-1)^{M-1}M + (-1)^{L-1}L + (-1)^{M-1}M + (-1)^{L+1}L \qquad L \neq N+1,\ M \neq 0 \tag{4.1.3}$$

Consider $L > M = N$, $K+L$ even, $K+N$ odd. For $L \neq N+1$, $M \neq 0$, (4.1.3) is $2((-1)^{M-1}M + (-1)^{L-1}L)$, which is non-zero as $L \neq M$. If $M = 0$, we must have K and L odd. This makes (4.1.3) $L + L$ which is not equal to zero since $L \neq 0$. For $L = N+1$ we've already shown $\mathbf{q}_T = 0$. Thus we determined all the non-zero tableaux of $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$. □

4.2 The Irreducibles Partitions of $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$

For Theorem 2, we need to know which irreducibles occur (i.e., have non-zero multiplicity) in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$. We call a shape (or partition) non-zero if the multiplicity of the corresponding irreducible in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$ is non-zero. By Lemma 2.2.7, the non-zero partitions are those partitions where $\dim(S^{\lambda,3} \cap M^{\lambda,3}) > 0$. Since $\mathbf{q}_T$ generates $S^{\lambda,a} \cap M^{\lambda,a}$ we need only determine from Theorem 9 which partitions have non-zero tableaux. These partitions and their multiplicities were completely determined by Thrall in [20]. We will first derive them from Theorem 9 and then confirm it with Thrall's result.

We will only consider shapes $[\lambda]$ which are partitions of $3b$. Consider $T =$



. This labeling will be useful in our later constructions, so we will derive the non-zero shapes in terms of it. When T is of the form $\begin{array}{cccc} \text{K} & \text{L} & \text{M} & \text{N} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array}$, we have $t = \text{K}$, $s = \text{L}+\text{M}+\text{N}$ and the tail $r = 3b - 2s - 3t$. Since 1, 2, and 3 all occur $\text{K} = t$ times in the part of T above t, we will sometimes replace b with $b' = b - \text{K}$ when considering the multiplicity of elements in columns r and s, since the subtableau formed by columns r, s and the tail will have the elements 1, 2, and 3 occurring b' times each. In partition notation we have $\lambda = [r + s + t, s + t, t]$.

4.2.1 Non-Zero Partitions from Theorem 9

Definition 4.2.1. For our purposes, we call a partition (or shape) λ of n *required* if there is a non-zero λ-tableau T filled with $\frac{n}{3}$ copies of the elements 1, 2, and 3. These are precisely the tableaux determined in Theorem 9. Specifically, a required shape is one for which we must construct an appropriately filled non-zero tableau in order to prove Theorems 1 and 2. These shapes are explicitly determined in Theorem 10.

To determine the non-zero $[\lambda] = [r + s + t, s + t, t]$, we analyze the required partitions that correspond to non-zero tableaux in Theorem 9. We find that:

Theorem 10. The only partitions $[r+s+t, s+t, t]$ of $n = 3b$ which do not occur in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ are those with s or $r = 1$ as well as those having $s+t$ odd and s or $r \in \{0, 2, 4\}$. Equivalently, a partition is non-zero if, for r, $s \neq 1$, when r or s is in $\{0, 2, 4\}$, then $s + t$ is even.

Proof. For a given partition $\lambda = [r + s + t, s + t, t]$, we need only find $(\text{L}, \text{M}, \text{N})$ with $\text{L} + \text{N} + \text{N} = s$, $\text{K} = t$, and the conditions of Theorem 9 satisfied to show $\mathbf{q}_T \neq 0$. This shows that $\mathcal{S}^\lambda$ must occur in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$. Since all values of $t = \text{K}$ can occur when $\text{L} > \text{M} > \text{N}$, (if $\text{L} \neq \text{M} + 2$), we consider tableaux of this form. Given T, the elements 1, 2 and 3 will occur $b' = b - t$ times in the remaining columns L, M, N, and the tail of T. For each t we need to determine which s for $0 \leq s \leq \frac{b'}{2}$ yield required non-zero partitions. To do so we will take the following parameterizations of $(\text{L}, \text{M}, \text{N})$ and determine the corresponding s.

$$
\begin{aligned}
(\text{L}, \text{M}, \text{N}) &= (i+2, i+1, i), \text{ we get} && s = 3i + 3 \text{ for} && 0 \leq i \leq \frac{b'-3}{2} \\
&= (i+3, i+2, i) && s = 3i + 5 && 0 \leq i \leq \frac{b'-5}{2} \\
&= (i+4, i+3, i) && s = 3i + 7 && 0 \leq i \leq \frac{b'-7}{2}
\end{aligned}
$$

For a given parameterization of $(\text{L}, \text{M}, \text{N})$ by i, we have $s = \text{L} + \text{M} + \text{N}$. Since there are at most b' 1's in the r and s sections, we must have $\text{L} + \text{M}] \leq b'$ which gives the upper bound on i.

These parameterizations of $(\text{L}, \text{M}, \text{N})$ are non-zero by Theorem 9 since $\text{L} > \text{M} > \text{N}$ and $\text{L} \neq \text{M} + 2$. Moreover, the parameterizations cover all equivalence classes of s (mod 3). Hence this tells us that all partitions with $s \geq 5$ or $s = 3$ are non-zero, leaving aside the upper bound on s for now. When $s = 4$, the possibilities for $(\text{L}, \text{M}, \text{N})$ are $(4, 0, 0)$, $(2, 2, 0)$, and $(2, 1, 1)$ which are non-zero only when $s+t$, (i.e., K) is even. For $(3, 1, 0)$ the tableau is always zero since $\text{L} = \text{M} + 2$, $\text{N} = \text{M} - 1$. For $s = 2$ the only possibilities are $(2, 0, 0)$ and $(1, 1, 0)$. The non-zero conditions of Theorem 9 require $\text{K} = t$, and hence $s + t$, to be even in both cases. Similarly for $s = 0$, we must have K even for T to be non-zero, as shown in Lemma 3.2.6. For $s = 1$, the only possibility is $(\text{L}, \text{M}, \text{N}) = (1, 0, 0)$, and such a tableau is always zero.

Determining the upper bounds on s corresponds to determining lower bounds on r. We will similarly give parameterizations of $(\text{L}, \text{M}, \text{N})$ which will cover all equivalence classes of $r \pmod 3$. Using the equation $3b' - 2s = r$ we get the corresponding equations and lower bounds.

$$\begin{aligned}
&\text{For } s = 3i+3, && i \leq \frac{b'-3}{2} && \text{then} && r \geq 3,\ r \equiv 0 \pmod 3 && (4.2.1)\\
&s = 3i+5, && i \leq \frac{b'-5}{2} && && r \geq 5,\ r \equiv 2 \pmod 3 && (4.2.2)\\
&s = 3i+7, && i \leq \frac{b'-7}{2} && && r \geq 7,\ r \equiv 1 \pmod 3 && (4.2.3)
\end{aligned}$$

This is the parameterization table for s rewritten in terms of r. So for $r \geq 5$, all partitions are non-zero, provided the conditions on s are met.

If $r = 0$, we must have $\text{L} = \text{M} = \text{N} = \frac{b'}{2}$. By Theorem 9, this shape is non-zero only if $\text{K}+\text{L}$ is even. Thus only the shapes with $s+t = \text{K}+\text{L}+\text{M}+\text{N}$ even are non-zero. For $r = 1$ we get the constraints $\text{L} + \text{M} = \text{L} + \text{N} = b'$ and $\text{M} + \text{N} = b' - 1$. This means $\text{M} = \text{N}$, and $\text{L}=\text{M}+1$. These tableaux are always zero.

When $r = 2$, we must have either $\text{L} = \text{M}$ or $\text{M} = \text{N}$. These shapes will be non-zero if $\text{K}+\text{N}$ or $\text{K}+\text{L}$ is even respectively. In either case, $s + t$ is even.

For $r = 4$, we must have either $\text{M} = \text{N}$, $\text{L} = \text{M}$, or $(\text{L}, \text{M}, \text{N}) = (\frac{b'+2}{2}, \frac{b'-2}{2}, \frac{b'-4}{2})$. In the first two cases, the non-zero conditions of Theorem 9 force $s + t$ to be even. In the last case, the tableau is always zero.

If $r = 3$ or 5 then $i = \frac{b'-3}{2}$ or $i = \frac{b'-5}{2}$ is an integer for b' odd so the parameterization listed works. For b' even, these partitions are not required since $3b' - 2s = r$, would make r even.

In addition, Theorem 9 shows the remaining partitions are zero.

□

4.2.2 Partition Multiplicities according to Thrall

In [20], Thrall determines the partitions occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$ with multiplicity, which he calls $f(\lambda)$. If $\lambda = [\lambda_1, \lambda_2, \lambda_3]$, he gives the following method to compute $f(\lambda)$:

> To the minimum of $1 + \lambda_1 - \lambda_2$ and $1 + \lambda_2 - \lambda_3$, add whichever of -2, 0, $+2$ will give a result divisible by 3. If this result is even, divide by 6 to get $f(\lambda)$. If the result is odd, add or subtract 3 according to λ_2 being even or odd and then divide by 6 to get $f(\lambda)$.

Letting $\lambda = [r+s+t, s+t, t]$ we have $\min(1+\lambda_1-\lambda_2, 1+\lambda_2-\lambda_3) = 1+\min(r, s)$. Then $f(\lambda) = \frac{1+\min(r,s)+x+y}{6}$ where $x \in \{-2, 0, 2\}$ such that $1 + \min(r, s) + x \equiv 0 \pmod 3$, $y \in \{-3, 0, 3\}$ such that $1 + \min(r, s) + x + y \equiv 0 \pmod 6$, and $y \geq 0$ if $s + t$ even and $y \leq 0$ if $s + t$ odd.

Theorem 11. Writing $s = 6k + j$, $r = 6h + i$ with $0 \leq i, j \leq 5$, then by [20], the multiplicity of $\mathcal{S}^\lambda$ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_{3b}}$ is $f(\lambda)$, where

$$f(\lambda) = \begin{cases} k & s \leq r \quad i = 0,\ 2,\ 4, \quad s+t \text{ odd} \\ k & s \leq r \quad i = 1 \\ k+1 & s \leq r \quad i = 0,\ 2,\ 4, \quad s+t \text{ even} \\ k+1 & s \leq r \quad i = 3,\ 5 \\ h & r < s \quad i = 0,\ 2,\ 4, \quad s+t \text{ odd} \\ h & r < s \quad i = 1 \\ h+1 & r < s \quad i = 0,\ 2,\ 4, \quad s+t \text{ even} \\ h+1 & r < s \quad i = 3,\ 5 \end{cases}$$

Hence $f(\lambda) \neq 0$ for $r, s \neq 1$, provided $s+t$ is even when r or s is in $\{0, 2, 4\}$. This agrees with our results in Section 4.1.

4.3 Construction of Basis Tableaux for $c = 3$

Recall that the space $S^{\lambda,c} \cap M^{\lambda,c}$ is spanned by $\{\mathbf{q}_T\}$ where the T are λ-tableaux filled with the numbers 1 to c. By Lemma 2.2.7, $\dim(S^{\lambda,c} \cap M^{\lambda,c})$ equals the multiplicity of $\mathcal{S}^\lambda$ in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$. Given a partition λ of $n = 3b$, we want a set of tableaux $\{\mathcal{B}_p\}$ such that $\{\mathbf{q}_{\mathcal{B}_p}\}$ is linearly independent and that $|\{\mathcal{B}_p\}|$ is the multiplicity of the irreducible corresponding to λ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$. We call these $\mathcal{B}_p$ the basis tableaux for $c = 3$. These tableaux will be used in Chapter 9 for the proof of Theorem 3. We will build these tableaux from the following components:

$$\mathcal{M}_1 = \begin{array}{l} 1\;2\;3 \end{array} \qquad \mathcal{M}_2 = \begin{array}{l} 2\;2\;3\;3\;3\;3 \\ 1\;1\;1\;1\;2\;2 \end{array}$$

$$\mathcal{M}_3 = \begin{array}{l} 3\;3 \\ 1\;1 \\ 2\;2 \end{array} \qquad \mathcal{M}_4 = \begin{array}{l} 3\;3\;2\;3 \\ 1\;1\;1\;2 \\ 2 \end{array}$$

$$\mathcal{N}_{\text{A}} = \begin{array}{l} \overline{\overset{\text{A-1}}{2}\;\;3\;2\;\;\overset{\text{A-1}}{3}} \\ \;1\;\;\;1 \end{array} \qquad 1 < \text{A} \leq \tfrac{n}{3} = b$$

In constructing these basis tableaux, we want tableaux filled with only the numbers 1, 2, and 3. We will use $\veebar$ to denote the joining of tableaux without renumbering them. For example, $\mathcal{M}_2 \veebar \mathcal{M}_1 = \begin{array}{l} 2\;2\;3\;3\;3\;3\;1\;2\;3 \\ 1\;1\;1\;1\;2\;2 \end{array}$.

Let $\lambda = [r + s + t, s + t, t]$. When t is even, write $s = 6k + j$ and $r = 6h + i$, with $0 \leq i, j \leq 5$. Let $g = min(k, h)$. Since λ is a partition of $n = 3b$ we have $3t + 2s + r = 3b$. Hence $2j + i \equiv 0 \pmod 3$. Let $\delta = \frac{i-j}{3}$. (So $\delta = 0$ for $i = j$, $\delta = 1$ for $i = j + 3$, and $\delta = -1$ for $i = j - 3$.) When t is odd, we proceed as above, except let $s - 3 = 6k' + j'$, $\delta = \frac{i-j'}{3}$, and $g = min(k', h)$.

For $p = 1, 2, \ldots, g$, we define the basis tableaux:

$$\mathcal{B}_p = \tfrac{t}{2}\mathcal{M}_3 \veebar \mathcal{N}_{6p+j} \veebar (k - p)\mathcal{M}_2 \veebar (2h - 2p + \delta)\mathcal{M}_1 \qquad (t \text{ even})$$

$$\mathcal{B}_p = \tfrac{t-1}{2}\mathcal{M}_3 \veebar \mathcal{M}_4 \veebar \mathcal{N}_{6p+j'} \veebar (k' - p)\mathcal{M}_2 \veebar (2h - 2p + \delta)\mathcal{M}_1 \qquad (t \text{ odd})$$

Additionally, for j, $j' \neq 1$ we have the tableau $\mathcal{B}_0$, which is $\mathcal{B}_p$ with $p = 0$ under certain conditions.

$$\mathcal{B}_0 = \tfrac{t}{2}\mathcal{M}_3 \veebar \mathcal{N}_j \veebar k\mathcal{M}_2 \veebar (2h+\delta)\mathcal{M}_1, \qquad j > 1 \qquad (t \text{ even})$$

$$\mathcal{B}_0 = \tfrac{t}{2}\mathcal{M}_3 \veebar k\mathcal{M}_2 \veebar (2h+\delta)\mathcal{M}_1 \qquad j = 0 \qquad (t \text{ even})$$

$$\mathcal{B}_0 = \tfrac{t-1}{2}\mathcal{M}_3 \veebar \mathcal{M}_4 \veebar \mathcal{N}_{j'} \veebar k'\mathcal{M}_2 \veebar (2h+\delta)\mathcal{M}_1, \qquad j' > 1 \qquad (t \text{ odd})$$

$$\mathcal{B}_0 = \tfrac{t-1}{2}\mathcal{M}_3 \vee \mathcal{M}_4 \veebar k'\mathcal{M}_2 \veebar (2h+\delta)\mathcal{M}_1 \qquad j' = 0 \qquad (t \text{ odd})$$

Note that if $\delta = -1$, then $\mathcal{B}_0$ exists only for $h \geq 1$ and $\mathcal{B}_g$ exists only for $g < h$.

To demonstrate that the $\{\mathcal{B}_p\}$ is a basis, we need to verify that they:

- Have the correct shape,
- Are non-zero and maximal,
- Are linearly independent,
- Span the space.

Shape: First consider the shape of these tableaux. We need to show these tableaux have shape $\lambda = [r+s+t, s+t, t]$. For t even:

$$\begin{aligned}\lambda_3(\mathcal{B}_p) &= \tfrac{t}{2}\lambda_3(\mathcal{M}_3) + \lambda_3(\mathcal{N}_{6p+j}) + (k-p)\lambda_3(\mathcal{M}_2) + (2h-2p+\delta)\lambda_3(\mathcal{M}_1) \\ &= \tfrac{t}{2} * 2 + (k-p) * 0 + (2h-2p+\delta) * 0 \\ &= t\end{aligned}$$

$$\begin{aligned}\lambda_2(\mathcal{B}_p) &= \tfrac{t}{2}\lambda_2(\mathcal{M}_3) + \lambda_2(\mathcal{N}_{6p+j}) + (k-p)\lambda_2(\mathcal{M}_2) + (2h-2p+\delta)\lambda_2(\mathcal{M}_1) \\ &= t + 6p + j + (k-p) * 6 + (2h-2p+\delta) * 0 \\ &= t + 6k + j \\ &= t + s\end{aligned}$$

$$\begin{aligned}\lambda_1(\mathcal{B}_p) &= \tfrac{t}{2}\lambda_1(\mathcal{M}_3) + \lambda_1(\mathcal{N}_{6p+j}) + (k-p)\lambda_1(\mathcal{M}_2) + (2h-2p+\delta)\lambda_1(\mathcal{M}_1)\\ &= t + (6p+j)*2 + (k-p)*6 + (2h-2p+\delta)*3\\ &= t + 6k + j + 6h + j + 3\delta\\ &= t + s + 6h + j + 3\delta\\ &= t + s + 6h + i\\ &= t + s + r\end{aligned}$$

When t is odd we have:

$$\begin{aligned}\lambda_3(\mathcal{B}_p) &= \tfrac{t-1}{2}\lambda_3(\mathcal{M}_3) + \lambda_3(\mathcal{M}_4) + \lambda_3(\mathcal{N}_{6p+j'}) + (k'-p)\lambda_3(\mathcal{M}_2) + (2h-2p+\delta)\lambda_3(\mathcal{M}_1)\\ &= \tfrac{t-1}{2}*2 + 1 + (k'-p)*0 + (2h-2p+\delta)*0\\ &= t\end{aligned}$$

$$\begin{aligned}\lambda_2(\mathcal{B}_p) &= \tfrac{t-1}{2}\lambda_2(\mathcal{M}_3) + \lambda_2(\mathcal{M}_4) + \lambda_2(\mathcal{N}_{6p+j'}) + (k'-p)\lambda_2(\mathcal{M}_2) + (2h-2p+\delta)\lambda_2(\mathcal{M}_1)\\ &= t - 1 + 4 + 6p + j + (k-p)*6 + (2h-2p+\delta)*0\\ &= t + 3 + 6k' + j'\\ &= t + s\end{aligned}$$

$$\begin{aligned}\lambda_1(\mathcal{B}_p) &= \tfrac{t}{2}\lambda_1(\mathcal{M}_3) + \lambda_1(\mathcal{M}_4) + \lambda_1(\mathcal{N}_{6p+j'}) + (k'-p)\lambda_1(\mathcal{M}_2) + (2h-2p+\delta)\lambda_1(\mathcal{M}_1)\\ &= t - 1 + 4 + (6p+j')*2 + (k-p)*6 + (2h-2p+\delta)*3\\ &= t + 3 + 6k' + j' + 6h + j' + 3\delta\\ &= t + s + 6h + j' + 3\delta\\ &= t + s + 6h + i\\ &= t + s + r\end{aligned}$$

A similar computation works for the shape of $\mathcal{B}_0$. Hence these tableaux have the correct shape. Moreover, within each component, the same number of 1's, 2's and 3's were used. Hence the $\mathcal{B}_p$ have the correct number of 1's 2's and 3's.

Maximality: When t is even, a generic basis element (with the tail suppressed) looks like:

$$\mathcal{B}_p = \begin{array}{cccc} \text{t} & \text{A} & \text{B} & \text{C} \\ \hline 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \qquad \begin{array}{l} \text{A} = 2(k-p)+6p+j-1 \\ \text{B} = 2(k-p)+1 \\ \text{C} = 2(k-p) \end{array}$$

Then $\omega_{2,3}(\mathcal{B}_p) = \left(\begin{smallmatrix} t+6p+j+4(k-p) & 2(k-p) & 0 \\ 0 & t & 0 \end{smallmatrix}\right)$. Since $\text{B} > \text{C}$ and $\text{A} > \text{B}$ (for $p \geq 1$), this weight is maximal. $\mathcal{B}_p$ is also non-zero, since the only other possible weight assignment is $\left(\begin{smallmatrix} t+6p+j+4(k-p) & 0 & 2(k-p) \\ 0 & 0 & t \end{smallmatrix}\right)$. This has sign $(-1)^{t+2(k-p)} = 1$ as t is even.

When t is even, the tableau $\mathcal{B}_0$ is the same as $\mathcal{B}_p$ with $p = 0$ when $j > 1$. In this case $\text{A} > \text{B}$ and the above argument holds. There is no $\mathcal{B}_0$ for $j = 1$. When $j = 0$, we have (suppressing the tail):

$$\mathcal{B}_0 = \begin{array}{cccc} \text{t} & \text{A} & \text{A} & \text{A} \\ \hline 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \qquad \text{A} = 2k$$

Then $\omega_{2,3}(\mathcal{B}_0) = \left(\begin{smallmatrix} t+4k & 2k & 0 \\ 0 & t & 0 \end{smallmatrix}\right)$. This weight is clearly maximal. Although there are many different weight assignments possible for $\mathcal{B}_0$, all weight assignments are positive since each column block is even. Hence $\mathcal{B}_0$ is non-zero.

When t is odd, a generic basis element (with the tail suppressed) looks like:

$$\mathcal{B}_p = \begin{array}{cccc} \text{t} & \text{A} & \text{B} & \text{C} \\ \hline 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \qquad \begin{array}{l} \text{A} = 2(k'-p)+6p+j' \\ \text{B} = 2(k'-p)+2 \\ \text{C} = 2(k'-p)+1 \end{array}$$

Then $\omega_{2,3}(\mathcal{B}_p) = \left(\begin{smallmatrix} t+6p+j'+4(k'-p)+2 & 2(k'-p)+1 & 0 \\ 0 & t & 0 \end{smallmatrix}\right)$. Since $\text{B} > \text{C}$ and $\text{A} \geq \text{B}$ (for $p \geq 1$), this weight is maximal. $\mathcal{B}_p$ is also non-zero since the only other possible weight assignment is $\left(\begin{smallmatrix} t+6p+j'+4(k'-p)+2 & 0 & 2(k'-p)+1 \\ 0 & 0 & t \end{smallmatrix}\right)$. This has sign $(-1)^{t+2(k'-p)+1} = 1$ as t is odd.

When t is odd, the tableau $\mathcal{B}_0$ is the same as $\mathcal{B}_p$ with $p = 0$ when $j' > 1$. In this case $\text{A} \geq \text{B}$ and the above argument holds. There is no $\mathcal{B}_0$ for $j' = 1$ (since $\mathbf{q}_{\mathcal{B}_0}$ is zero when $j' = 1$). When $j' = 0$, we have (suppressing the tail):

$$\mathcal{B}_0 = \begin{array}{cccc} \text{t} & \text{A} & \text{A} & \text{A} \\ \hline 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \qquad \text{A} = 2k' + 1$$

Then $\omega_{2,3}(\mathcal{B}_0) = \left(\begin{smallmatrix} t+4k'+2 & 2k'+1 & 0 \\ 0 & t & 0 \end{smallmatrix}\right)$. This weight is clearly maximal. Although there are many different weight assignments possible for $\mathcal{B}_0$, These weight assignments always move exactly two or four column blocks. Since the size of the column blocks is odd, all weight assignments are positive. Hence $\mathcal{B}_0$ is non-zero.

Linear Independence: To show the tableaux $\mathcal{B}_p$ are linearly independent, by Lemma 3.4.12 it suffices to show their max weights are distinct. First consider t even. Say $\mathsf{w}_{2,3}(\mathcal{B}_p) = \mathsf{w}_{2,3}(\mathcal{B}_{p'})$, with $p < p'$. Then we must have $t + 6p + j + 4(k - p) = 2(k - p')$, which forces $k = p = j = t = 0$. Then our partition is just $[n]$, so only one tableau is needed. If $\mathsf{w}_{2,3}(\mathcal{B}_p) = \mathsf{w}_{2,3}(\mathcal{B}_0)$ with $p > 0$, then we get $t + 4k = 2(k - p)$ which implies $t = k = p = 0$. Hence the tableaux are linearly independent. When t is odd, the max weights must be distinct, since an argument similar to the one above shows $t = 0$ which is not possible.

Span: Since the tableaux $\mathcal{B}_p$ are linearly independent they will span the space $S^{\lambda,3} \cap M^{\lambda,3}$ if $|\{\mathcal{B}_p\}| = m_\lambda$, where $m_\lambda = f(\lambda)$ as determined by [20]. (We listed $f(\lambda)$ explicitly in Theorem 11.)

First consider the case of t even. Given $\lambda = [r + s + t+, s + t, t]$ with t even, then m_λ depends on the relative sizes of r and s. For $s \leq r$, we need $k + 1$ tableaux for $j \neq 1$ and k tableaux for $j = 1$. When $s \leq r$, we have $g = k$. Thus we get k different $\mathcal{B}_p$'s and when $j \neq 1$ we have $\mathcal{B}_0$ as well. The restriction on the tableaux when $\delta = -1$ occurs only when $g = h$. However, then $h = k$ and $i = j - 3$ which contradicts $s \leq r$, so this case does not occur here. Hence we have a full set of basis tableaux.

If $r < s$, we have $g = h$. How many tableaux we have depends on δ. Note that $\mathcal{B}_h$ exists only for $\delta \neq -1$, and $\mathcal{B}_0$ requires $h > 0$ for $\delta = -1$. The number of tableaux needed according to Theorem 11 also depends on δ.

Since $s \equiv j \pmod 2$ and $3\delta = i - j$, when $\delta = 0$ we have $i \equiv s \pmod 2$. Then s is even for $i = 0$, 2, and 4, hence by [20] we need we need $h + 1$ tableaux for $i \neq 1$ and h tableaux for $i = 1$. Since $\delta \neq -1$, there are h distinct $\mathcal{B}_p$. We also have $\mathcal{B}_0$ when $i \neq 1$ since $i = j$. Hence we have a full set of basis tableaux.

If $\delta = 1$, all the tableaux described when $\delta = 0$ occur. Since $\delta = 1$, we have $i = j + 3$ which implies $i = 3, 4$ or 5. For $i = 4$ we need need h tableaux since s is odd, while for $i = 3, 5$ we need $h + 1$ tableaux. When $i = 4$ we have h different $\mathcal{B}_p$ (though no $\mathcal{B}_0$ since $j = 1$). When $i = 3, 5$ then $j = 0, 2$. Hence $\mathcal{B}_0$ exists and we obtain the complete set of basis tableaux.

For $\delta = -1$ we have $i = j - 3$, so $i = 0, 1, 2$. We need exactly h tableaux since either $i = 1$ or s is odd. However, we no longer have $\mathcal{B}_h$, so we get only $h-1$ tableaux from the $\mathcal{B}_p$. In addition, when $h \geq 1$ we have $\mathcal{B}_0$ since $j = i + 3 \geq 3$ and so $j > 1$. Hence we have h tableaux for $h \geq 1$. If $h = 0$ then no tableaux are needed since either $r = 1$ or $r = 0$ or $r = 2$ with s odd. Thus the correct number of tableaux is given.

Now consider the case when t is odd. When $s \leq r$ we need k tableaux for $j = 0$, 1, 2, or 4 and $k + 1$ tableaux for $j = 3$ or 5. If $j = 3$ or 5, then $k' = k = g$ so there are k tableaux $\mathcal{B}_p$, in addition to the tableau $\mathcal{B}_0$ (since $j' = j - 3 \neq 1$). If $j < 3$ then $g = k' = k - 1$, so there are $k - 1$ tableaux $\mathcal{B}_p$ in addition to the tableau $\mathcal{B}_0$. For $j = 4$, we have k tableaux $\mathcal{B}_p$, since $g = k' = k$. However since $j' = 1$, $\mathcal{B}_0$ does not exists. The restriction on the tableaux when $\delta = -1$ occurs only when $g = h$. However, then $h = k' = k$ and $i = j - 3$ which contradicts $s \leq r$, so this case does not occur here. Hence we have a full set of basis tableaux.

When $r < s$ we need h tableaux when $i = 1$. For $i = 0, 2$, and 4 we need h tableaux when s is even and $h + 1$ tableaux when s is odd. For $i = 3$ or 5 we need $h + 1$ tableaux. Note that $s \not\equiv j' \pmod 2$.

If $\delta = 0$ then $i = j'$. We have h tableaux $\mathcal{B}_p$, along with $\mathcal{B}_0$ for $j' \neq 1$. Hence we have h tableaux when $i = 1$, and $h + 1$ tableaux otherwise. Since s is odd when $i = 0$, 2, or 4, this is the correct number.

If $\delta = 1$ then $i = j' + 3$, so $i = 3$, 4 or 5. If $i = 3$ or 5, then $j' \neq 1$ and there are $h + 1$ tableaux as desired. If $i = 4$ then $j' = 1$, hence there are only h tableaux $\mathcal{B}_p$. However, only h tableaux are needed here since s is even.

If $\delta = -1$ then $i = j' - 3$, so $i = 0$, 1 or 2. Since s is even when $i \neq 1$, only h tableaux are required in this case. However, we no longer have $\mathcal{B}_h$, so there are $h - 1$ tableaux $\mathcal{B}_p$. In addition, when $h \geq 1$, we have $\mathcal{B}_0$, since $j' > 1$, so the correct number of tableaux are obtained. If $h = 0$ then no tableaux are needed since either $r = 1$ or $r = 0$ or $r = 2$ with $s+t$ odd. Thus the correct number of tableaux is given.

Chapter 5

Proof of Theorem 1

Recall, Theorem 1 says that every irreducible occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with equal or greater multiplicity, where $n = 2b = cd$ and $b, c, d \geq 2$.

In Section 3.2 we proved Theorem 7, which showed that the irreducibles occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_2}^{\mathcal{S}_n}$ were exactly those corresponding to partitions $\lambda = [n-s, s]$ for s even and they occur with multiplicity one. (Since $n = 2b$ is even, it suffices to consider only the even values of s.) By Remark 2.2.8, to prove Theorem 1, it suffices to construct a non-zero tableau filled with d copies of c elements for each partition $[n-s, s]$, where $0 \leq s \leq \frac{n}{2}$, s even and $n = cd$.

To do this we will construct some non-zero generic tableaux that when assembled via Theorem 8 will produce all the shapes and fillings needed. Since we are constructing generic tableaux for many partitions and fillings, we will not use a fixed c. However, we assume that every element listed in the body of the tableau occurs d times, filling out the tail as needed. We apply weight-set counting to prove a tableau is non-zero. The tableaux we need are:

<u>Tableau U_1</u>

$$U_1 = \begin{array}{ccc} \text{A} & \text{d-A} & \text{d-A} \\ \hline 1 & 1 & 2 \\ 2 & & \end{array} \sim \begin{array}{c} \text{A} \\ \hline 1 \\ 2 \end{array} \qquad \begin{array}{l} \text{A even} \\ \text{A} \leq d \end{array}$$

$\omega_2(U_1) = (0, \text{A})$

$$\lambda = [2d - \text{A}, \text{A}]$$

For this first tableau, we listed U_1 both with and without the tail. Normally we will suppress the tail when writing these tableaux. U_1 is non-zero by Lemma 3.2.6 since A is even. It is maximal since $(\text{A}, 0)$ is the largest possible weight-set for this shape.

Tableau U_2

$$U_2 = \begin{array}{cccc} \text{A} & \text{A} & \text{B} & \text{B} \\ \hline 1 & 3 & 1 & 3 \\ 2 & 4 & 4 & 2 \end{array} \qquad \begin{array}{l} \text{A} + \text{B} \leq d \\ \text{A},\ \text{B} > 0 \end{array}$$

$$\omega_2 = (0, \text{A+B}, 0, \text{A+B})$$

$$\lambda = [4d - 2(\text{A+B}), 2(\text{A+B})]$$

Examining the filling of U_2 and A,B > 0 we find the following constraints on any valid weight assignment: (Recall that U^* corresponds to a possible tableau τU_2.)

- If $\omega_2(1|U^*) = 0$ then $\omega_2(2 \text{ and } 4|U^*) > 0$.
- If $\omega_2(2|U^*) = 0$ then $\omega_2(1 \text{ and } 3\ |U^*) > 0$.
- We must have $\omega_2(1 \text{ or } 2|U^*) > 0$ and $\omega_2(3 \text{ or } 4|U^*) > 0$.

Since any valid weight assignment of $(0, \text{A} + \text{B}, 0, \text{A} + \text{B})$ has exactly two zeros, the restrictions above show that $(1, 2, 3, 4)$ and $(2, 1, 4, 3)$ are the only valid weight assignments. These weights-sets correspond to applying $\tau = \overset{\text{A}}{()}_T \times \overset{\text{A}}{()}_T \times \overset{\text{B}}{()}_T \times \overset{\text{B}}{()}_T$ and $\tau = \overset{\text{A}}{(12)}_T \times \overset{\text{A}}{(12)}_T \times \overset{\text{B}}{(12)}_T \times \overset{\text{B}}{(12)}_T$ respectively. As both of these τ have positive sign, $\mathbf{q}_{U_2} \neq 0$. This tableau is maximal since every element x must have $\omega_2(x) \leq \text{A+B}$.

Tableau U_3

$$U_3 = \frac{\begin{matrix} \text{A} & \text{B} & \text{B} \end{matrix}}{\begin{matrix} 1 & 1 & 2 \\ 2 & 3 & 3 \end{matrix}} \qquad \begin{matrix} \text{A even} \\ \text{A} + \text{B} \le d \\ d \text{ even} \\ \text{B} = \frac{d}{2} \end{matrix}$$

$$\omega_2 = (0, \text{A}, d)$$

$$\lambda = [2d - \text{A}, \text{A} + d]$$

To show U_3 is non-zero we will use weight-set counting on $\omega_2 = (0, \text{A}, d)$. There are two cases for which we need to determine weight assignments, $\text{A} + \text{B} < d$ and $\text{A} + \text{B} = d$.

When $\text{A} + \text{B} < d$, only the element 3 may be assigned a row two weight of d. So the distinct weight assignments are $(1, 2, 3)$ and $(2, 1, 3)$, which occur with $\tau = ()_T$ and $\tau = \overset{\text{A}}{(12)}_T \times \overset{\text{B}}{()}_t \times \overset{\text{B}}{()}_T$ respectively. Since A is even, both τ have positive sign. Hence U_3 is non-zero.

If $\text{A} + \text{B} = d$, then $\text{A} = \text{B} = \frac{d}{2}$ and $d \equiv 0 \pmod 4$. While every permutation corresponds to a distinct weight assignment, every weight assignment can only be obtained by having τ move complete column blocks. Since all of these blocks are even, τ is positive for every weight assignment and hence U_3 is non-zero. This tableau is maximal since $(d, \text{A}, 0)$ is the largest possible weight.

Tableau $V(d)$

$$V = \frac{\text{d}}{1}$$

$$\omega_1(V) = (d)$$

$$\lambda = [d]$$

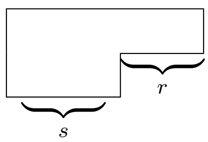
This is just a single row with d ones. Since there are no column permutations, this tableau is always non-zero. It is obviously maximal.

Having constructed these generic tableaux, we will use the notation $U_i(x)$ to denote the tableau U_i with the parameter $\mathrm{A} = x$ or $U_i(x, y)$ for $x = \mathrm{A}$ and $y = \mathrm{B}$ in U_i. We will use $\mathsf{f}U_i$ to denote the join of f copies of U_i. Note that these tableaux are all in maximal form.

For the proof of Theorem 1, the parity effects the construction process. To simplify notation, we define the *-function.

$$x^* = \begin{cases} x & x \text{ even} \\ x - 1 & x \text{ odd} \end{cases}$$

We analyze T by the parameters $r = n - 2s$ and s, where $\lambda = [r + s, s]$. For reference, we consider tableau of the following shape, with r and s even.

$T =$ 

Proof of Theorem 1. To prove Theorem 1 we need to construct a non-zero tableau of shape $\lambda = [n - s, s]$ for $s \leq \frac{n}{2}$, with s even and $n = cd$, $c, d \geq 2$. First we construct a general tableau that covers most s. Suppose $s \leq \frac{c^* d^*}{2}$. We know s is even, so write $s = \mathsf{f}d^* + e$, where $0 \leq e < d^*$, e even. Since s, d^*, and e are even, this is possible by the Euclidean algorithm.

Let $T = \mathsf{f}U_1(d^*) \vee U_1(e)$. Note that the bound on s guarantees that $2(\mathsf{f} + 1) \leq c$ when $e > 0$, and $2\mathsf{f} \leq c$ when $e = 0$. This insures that there are at most c distinct elements in T. If there are fewer than c elements in T add all the remaining elements to the tail of T by joining the appropriate number of $V(d)$'s. Suppressing the tail elements from the U_1's and $V(d)$'s, T looks like:

$$T = \begin{array}{ccccccc} d^* & d^* & \cdots & d^* & v & d & \cdots\ d \\ \hline 1 & 3 & \cdots & 2\mathsf{f}-1 & 2\mathsf{f}+1 & 2\mathsf{f}+3 & \cdots\ c \\ 2 & 4 & \cdots & 2\mathsf{f} & 2\mathsf{f}+2 & & \end{array}$$

Theorem 8 shows T is non-zero, provided the weight-sets are disjoint. Since the tableaux are in maximal form, the weights must be disjoint by Lemma 3.4.9. This covers the majority of the s. The remaining tableaux will be constructed according to the parity of c and d.

Case I: (c, d even) In this case $\frac{c^*d^*}{2} = \frac{cd}{2}$, so T constructed above covers all partitions.

Case II: (d even, c odd) By the above construction, we have all tableaux with s up to $\frac{(c-1)d}{2}$. Thus we only need those even partitions with $s = \frac{cd-k}{2}$ for $0 \le k \le d-2$, $k \equiv d \pmod 4$. Take $T = \frac{\mathsf{c}-3}{2}U_1(d) \vee U_3(\text{A})$ for $0 \le \text{A} \le \frac{d}{2}$ with A even. Then $s = \frac{c-3}{2}d + \text{A} + d = \frac{cd-d+2\text{A}}{2}$. Thus we have $k = d - 2\text{A}$, which ranges over the correct parameters. Since U_1 and U_3 are in maximal form, Lemma 3.4.9 implies disjointness and Theorem 8 shows T is non-zero.

Case III: (c even, d odd) Since $r = n - 2s = cd - 2s$ s we need $\lambda = [r+s, s]$ for $r \le cd$ with $r \equiv cd \pmod 4$. It suffices to construct a non-zero tableau for $r < 4d$. When $r \ge 4d$, let $r' = r - 4d\mathsf{z}$ with $r' < 4d$. Then if we construct a $\lambda' = [s + r', s]$ tableau T' filled with d copies of $c - 4\mathsf{z}$ elements, we get the needed tableau by $T = T' \vee 4\mathsf{z}V(d)$. Hence we will take $r < 4d$.

When $c \equiv 0 \pmod 4$ then $r \equiv 0 \pmod 4$. Take $T = \frac{c-4}{4}U_2(d-1,1) \vee U_2(d - \frac{r}{4} - 1, 1)$. This construction gives the shape $\frac{c-4}{4}[2d, 2d] + [2d + \frac{r}{2}, 2d - \frac{r}{2}] = [\frac{cd}{2} + \frac{r}{2}, \frac{cd}{2} - \frac{r}{2}]$ as desired. The parameters of these tableaux are positive unless $r = 4d-4$ since $r < 4d$, $r \equiv 0 \pmod 4$, and $d \ge 2$. If $r = 4d-4$ then $d - \frac{r}{4} - 1 = 0$, so use $U_1(2) \vee 2V(d)$ instead of of $U_2(d - \frac{r}{4} - 1, 1)$.

For $c \equiv 2 \pmod 4$ we will assume $r < 2d$. When $2d \le r < 4d$ let $r' = r - 2d$. Then can construct a $\lambda' = [r'+s, s]$ tableau T' with $c \equiv 0 \pmod 4$ and use $T = T' \vee 2V(d)$. Take $T = \frac{c-2}{4}U_2(d-1,1) \vee U_1(\frac{2d-r}{2})$ with $V(d)$'s as needed. Note that $cd \equiv r \pmod 4$ implies that $\frac{2d-r}{2}$ is even, while $r < 2d$ insures it is positive. So we get the shape $[\frac{cd}{2} + \frac{r}{2}, \frac{cd}{2} - \frac{r}{2}]$ as needed. Theorem 8 shows these T's are non-zero provided the weight-sets are disjoint, which follows from maximality.

Note that since $cd = n$, n even, then c or d is even. Thus we have constructed all cases. □

Although it is not directly apparent from this construction, c or d even is often a necessary requirement for any non-zero two row tableau with s even to exist. For instance, when $c = 3$ and $d = 7$, the shape $[11, 10]$ has s even, but all tableaux are zero by Theorem 9.

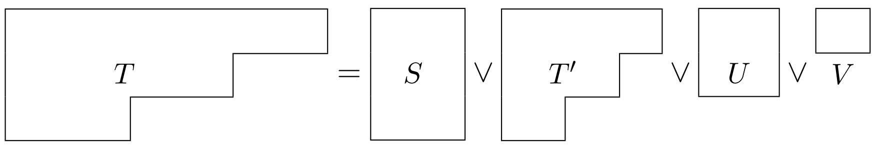
Chapter 6

Proof of Theorem 2

From Remark 2.2.8, to prove Theorem 2, it suffices to construct non-zero tableaux filled with d copies of c elements for all required partitions of $n = cd$, $c, d \geq 3$. These partitions were determined in Theorem 10.

Our approach is similar to the proof of Theorem 1 in Chapter 5. Using some generic non-zero tableaux (like U_i and V in Chapter 5) with c elements, we join them together by Theorem 8 to form a tableau of the appropriate shape and filling. However, unlike in Chapter 5, a large number of generic tableaux are needed. Since the cataloging of non-zero tableaux is quite tedious, we post-pone the construction until Chapter 7. Namely, our proof here will presuppose the construction of all tableaux of the required shapes for $c \leq 8$.

The general idea is to write a tableau T as follows:



for an appropriate T', where S, U, and V generic constructions based on the parity of d. This reduces the construction of T to a construction of T' where the shape parameters, (r, s, and t), of T' are small. Thus we only need to construct tableaux for a limited number of cases corresponding to small shapes. The tableaux S, U, and V are based on the following non-zero maximal tableaux: (Here U_1, U_2 and V occurred in Chapter 5.)

$$S_0 = P_1(d) = \begin{array}{c} d \\ \hline 1 \\ 2 \\ 3 \end{array} \quad d \text{ even}, \qquad S_1 = \begin{array}{cccccc} \text{A} & \text{A} & \text{B} & \text{B} & \text{B} & \text{B} \\ \hline 5 & 6 & 6 & 6 & 5 & 5 \\ 4 & 3 & 3 & 4 & 3 & 4 \\ 1 & 2 & 1 & 1 & 2 & 2 \end{array} \quad \begin{array}{l} \text{A} = \frac{d-\mathsf{x}}{3} + \mathsf{x} \\ \text{B} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3, \\ \mathsf{x} \in \{0,1,2\} \end{array}$$

$$U_1(d) = \begin{array}{c} d \\ \hline 1 \\ 2 \end{array} \quad d \text{ even}, \qquad U_2 = \begin{array}{cccc} \text{A} & \text{A} & \text{B} & \text{B} \\ \hline 1 & 3 & 1 & 3 \\ 2 & 4 & 4 & 2 \end{array} \quad \begin{array}{l} \text{A} = d - 1 \\ \text{B} = 1 \end{array}$$

$$V = \begin{array}{c} d \\ \hline 1 \end{array}$$

Let $\lambda = [r+s+t, s+t, t]$. We can write $T = S \vee T' \vee U \vee V$, for appropriate T' provided T' is maximail. Then T' will be filled with d copies of c' elements, for some $c' < c$, which will eventually allow us to reduce to $c \leq 8$. If T' is non-zero and maximal then by Lemma 3.4.9 and the Theorem 8 $\mathbf{q}_T \neq 0$ as desired. For simplicity, we will base our construction on the parity of d.

6.1 Case: d even

To see how to write T as $T = S \vee U \vee V \vee T'$ for an appropriate T' we first discuss the individual reductions allowing us to write $T = S \vee T'$, $T = U \vee T'$, or $T = V \vee T'$. Then successive applications of these reductions yield our desired decomposition. An analysis of these reductions also computes the resulting bounds on the shape of T'. An example application follows the reductions listed below. The reader may wish to refer to Example 6.1.1 while reading these reductions.

Reduction 1: Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take f to be the maximum integer such that $\mathsf{f}d \leq \lambda_3$ and $c - 3\mathsf{f} \geq 3$. Let $S = \mathsf{f}P_1(d)$ be the join of f copies of $P_1(d)$. Then by Theorem 8, we may write $T = S \vee T'$ for T' a $\lambda' = (\lambda_1 - d\mathsf{f}, \lambda_2 - d\mathsf{f}, \lambda_3 - d\mathsf{f})$-tableau filled with d copies of $c' = c - 3\mathsf{f}$ elements, provided the weight-sets of S and T' are disjoint. The choice of f means that in T', $t' = \lambda_3' = \lambda_3 - d\mathsf{f} < d$ or $c' = c - 3\mathsf{f} < 6$. Thus we need only consider tableaux with $t = \lambda_3 < d$ or $c < 6$. The $c < 6$ condition coresponds to the requirement $c - 3\mathsf{f} \geq 3$. We need this requirement so that there are at least three elements available with which to fill the remaining tableau, T'.

Reduction 2: Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take g to be the maximum integer such that $\mathsf{g}d \leq \lambda_2 - \lambda_3 = s$ and $c - 2\mathsf{g} \geq 3$. Let $U = \mathsf{g}U_1(d)$ be the join of g copies of $U_1(d)$. Then by Theorem 8, we may write $T = U \vee T'$ for T' a $\lambda' = (\lambda_1 - d\mathsf{g}, \lambda_2 - d\mathsf{g}, \lambda_3)$-tableau filled with d copies of $c' = c - 2\mathsf{g}$ elements, provided the weight-sets of U and T' are disjoint. The choice of g means that in T', $s' = \lambda_2' - \lambda_3' = \lambda_2 - d\mathsf{g} - \lambda'\_3 < d$ or $c' = c - 2\mathsf{g} < 5$. However, we will need the existence of a non-zero T' in the specified shape. As was shown in Theorem 9, this is not always the case for some s. Specifically, when $s < 5$ non-zero tableaux do not exist for certain shapes when $c = 3$. (Consider $\lambda = [6 + d, 2 + d, 1] = [9, 5, 1]$ with $d = 3$ and $c = 5$. Applying Reduction 2 yields $\lambda' = [5, 2, 1]$ with $c = 3$. All such tableaux are zero by Theorem 10 since $s = 1$.) To account for this, we modify the construction above to use $\mathsf{g} - 1$ copies of $U_1(d)$ when $\mathsf{g} > 0$ and $s' < 5$. In such a case, the modified T' now has $s' < d + 5$. Thus we need only consider arbitrary tableaux with $s < d + 5$ or $c < 5$.

Reduction 3: Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take h to be the maximum integer such that $\mathsf{h}d \leq \lambda_1 - \lambda_2 = r$ and $c - \mathsf{h} \geq 3$. Let $V = \mathsf{h}V_1(d)$ be the join of h copies of $V_1(d)$. Then by Theorem 8, we may write $T = V \vee T'$ for T' a $\lambda' = (\lambda_1 - d\mathsf{h}, \lambda_2, \lambda_3)$-tableau filled with d copies of $c' = c - \mathsf{h}$ elements, provided the weight-sets of V and T' are disjoint. The choice of h means that in T', $r' = \lambda_1' - \lambda_2' = \lambda_1 - d\mathsf{h} - \lambda_2 < d$ or $c' = c - \mathsf{h} < 4$. However, we will need the existence of non-zero T' in the specified shape. As was shown in Theorem 9, this is not always the case for some r. Specifically, when $r < 5$ non-zero tableaux do not exist for certain shapes when $c = 3$. To account for this, we modify the construction above to use $\mathsf{h} - 1$ copies of $U_1(d)$ when $\mathsf{h} > 0$ and $r' < 5$. In that case, the modified T' now has $r' < d+5$. Thus we need only consider arbitrary tableaux with $r < d+5$ or $c < 4$.

Conclusion: When d is even, we can apply these reductions successively. Take an arbitrary λ-tableau T filled with d copies of c elements and assume $c \geq 6$. We use $T^{(i)}$ to represent the appropriate T' obtained in these reductions. By Reduction 1, $T = S \vee T^{(1)}$, where $S = \mathsf{f}P_1(d)$ and $T^{(1)}$has $t = \lambda_3(T^{(1)}) < d$ and is filled with $c^{(1)} = c - 3\mathsf{f}$ elements.

Now, if $c^{(1)} \geq 6$ apply Reduction 2 to $T^{(1)}$. Since $c^{(1)} \geq 6$, then by Reduction 2, write $T^{(1)} = U \vee T^{(2)}$ where $U = \mathsf{g}U_1(d)$ and $T^{(2)}$ has $t < d$ (since $T^{(1)}$ does) and $s < d+5$. Here $T^{(2)}$ is filled with $c^{(2)} = c^{(1)} - 2\mathsf{g}$ elements.

Finally if $c^{(2)} \geq 6$ apply Reduction 3. This gives $T^{(2)} = V \vee T^{(3)}$, where $V = \mathsf{h}V_1(d)$ and $T^{(3)}$ has $t < d$, $s < d+5$, and $r < d+5$. Here $T^{(3)}$ is filled with $c^{(3)} = c^{(2)} - \mathsf{h}$ elements.

Hence $T = S \vee U \vee V \vee T^{(i)}$ where either $T^{(i)}$ is filled with fewer than 6 elements, or $T^{(i)}$ has $t < d$, $s < d+5$, and $r < d+5$. In the second case, $T^{(i)}$ must be filled with $3t + 2s + r = cd$ elements. This is less than or equal to $3(d-1) + 2(d+4) + (d+4) = 6d + 9 \leq 8d$ if $d > 4$. (If $d = 4$ we have $6d + 8 \leq 8d$ and it's not possible to have $6d + 9 = 9d$ when $d = 4$. For $d = 3$ additional reductions apply.) Hence we only need those tableaux with $c \leq 8$. Moreover, if r or $s < 5$ in $T^{(i)}$, then r or $s < 5$ in T, because the reductions do not reduce r or s to less than 5. Hence $T^{(i)} = T'$ has a shape occurring in Theorem 10 since all partitions of n with r and $s \geq 5$ are needed.

This reduction uses Theorem 8. Our usage only requires verification that the weight-sets are disjoint. However, the tableaux S, U, and V are in maximal form. Hence for appropriately chosen tableaux (i.e., ones in maximal form), an application of Lemma 3.4.9 can easily prove weight-set disjointness.

Example 6.1.1. To see how this reduction works, let us consider a specific shape, $\lambda = [9d-2, 5d, d+2]$ where $d \geq 6$, d even and $c = 15$. This shape has $t = d+2$, $s = 4d-2$, and $r = 4d-2$. First we apply Reduction 1, which joins $P_1(d)$ in order to have $t < d$.

$$[9d-2, 5d, d+2] = P_1(d) \vee [8d-2, 4d, 2]$$

Then we apply Reduction 2 to the shape $[8d-2, 4d, 2]$, which has $s = 4d-2$ to reduce to $s < d+5$ by joining three copies of $U_1(d)$.

$$[8d-2, 4d, 2] = 3U_1(d) \vee [5d-2, d, 2]$$

Applying Reduction 3 to shape $[5d-2, d, 2]$, which has $r = 4d-2$ we normally want to reduce r to be between 5 and $d+5$. Here we won't necessarily reduce r fully, so that the resulting tableau will be familiar. Instead we will reduce to $r = 2d-2$ (which may be reduced further depending on d) by joining two copies of $V(d)$.

$$[5d-2, d, 2] = 2V(d) \vee [3d-2, d, 2]$$

Hence, when we combine all these reductions, we get

$$[9d-2, 5d, d+2] = P_1(d) \vee 3U_1(d) \vee 2V(d) \vee [3d-2, d, 2]$$

A non-zero tableau of shape $[3d-2, d, 2]$ is Q^* of Example 3.2.7 with $\textsc{a} = 1$, $\textsc{b} = 1$, $\textsc{c} = d-2$. Therefore, writing

$$T = P_1(d) \vee Q^*(1, 1, d-2) \vee 3U_1(d) \vee 2V(d)$$

and omitting the extra tail of Q^* we have

$$T = \begin{array}{ccccccccccc} d & 1 & 1 & 1 & 1 & d\text{-}2 & d & d & d & d & d \\ \hline 1 & 5 & 5 & 5 & 6 & 4 & 8 & 10 & 12 & 14 & 15 \\ 2 & 4 & 7 & 7 & 4 & 7 & 9 & 11 & 13 & & \\ 3 & 6 & 6 & & & & & & & & \end{array}$$

As Q^* is in maximal form, $\mathbf{q}_T \neq 0$.

6.2 Case: d odd

When d is odd, we proceed exactly as in the even case, except the tableaux we use are slightly different. Namely, we use S_1 instead of P_1 and U_2 instead of U_1. These adjustments are necessary for Reductions 1 and 2 since $P_1(d)$ and $U_1(d)$ are zero for d odd. Reduction 3 remains unchanged however. For completeness, we rewrite these reductions in terms of d odd. However, these reductions alone are not enough to reduce to $c \leq 8$. So after these reductions, we apply a few more in order to reduce the size of tableaux we need to consider.

Reduction $1'$**:** Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take w to be the maximum integer such that $\mathsf{w} \cdot 2d \leq \lambda_3$ and $c - 6\mathsf{w} \geq 3$. Let $S = \mathsf{w}S_1(d)$ be the join of w copies of $S_1(d)$. Then by Theorem 8, we may write $T = S \vee T'$ for T' a $\lambda' = (\lambda_1 - 2d \cdot \mathsf{w}, \lambda_2 - 2d \cdot \mathsf{w}, \lambda_3 - 2d \cdot \mathsf{w})$-tableau filled with d copies of $c' = c - 6\mathsf{w}$ elements, provided the weight-sets of S and T' are disjoint. The choice of w means that in T', $t' = \lambda_3' = \lambda_3 - 2d \cdot \mathsf{w} < 2d$ or $c' = c - 6\mathsf{w} < 9$. Thus we need only consider tableaux with $t = \lambda_3 < 2d$ or $c < 9$.

Reduction $2'$**:** Let T be any λ-tableau with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take v to be the maximum integer such that $\mathsf{v} \cdot 2d \leq \lambda_2 - \lambda_3 = s$ and $c - 4\mathsf{v} \geq 3$. Let $U = \mathsf{v}U_2(d)$ be the join of v copies of $U_2(d)$. Then by Theorem 8, we may write $T = U \vee T'$ for T' a $\lambda' = (\lambda_1 - 2d \cdot \mathsf{v}, \lambda_2 - 2d \cdot \mathsf{v}, \lambda_3)$-tableau filled with d copies of $c' = c - 4\mathsf{v}$ elements, provided the weight-sets of U and T' are disjoint. The choice of v means that in T', $s' = \lambda_2' - \lambda_3' = \lambda_2 - d \cdot \mathsf{v} - \lambda_3' < 2d$ or $c' = c - 4\mathsf{v} < 8$. As in the even case, to account for the shapes $s < 5$, we modify this reduction to use $\mathsf{v} - 1$ copies of $U_1(d)$ when $\mathsf{v} > 0$ and $s' < 5$. Then the modified T' now has $s' < 2d + 5$. Thus we need only consider arbitrary tableaux with $s < 2d + 5$ or $c < 8$.

Summary: The same argument as in the even case works for the d odd cases, though the numbers are adjusted slightly. Take an arbitrary λ-tableau T with filled with d copies of c elements, but this time assume $c \geq 9$. Then by applications of

Reductions 1', 2' and 3, $T = S \vee U \vee V \vee T^{(i)}$ where either $T^{(i)}$ is filled with fewer than 9 elements, or $T^{(i)}$ has $t < 2d$, $s < 2d + 5$, and $r < d + 5$. In the second case, $T^{(i)}$ must be filled with $3t + 2s + r$ elements, which is less than or equal to $3(2d - 1) + 2(2d + 4) + (d + 4) = 11d + 9$ as $d \geq 3$. Moreover, if r or $s < 5$ in $T^{(i)}$, then r or $s < 5$ in T. Hence $T^{(i)}$ has a required shape of Theorem 10. However, we wish to have $T^{(i)}$ fillable with $c \leq 8$. To do this we have additional reduction techniques. However, these techniques are very sensitive to the parameters in $T^{(i)}$, so we will categorize them by such. The additional non-zero maximal tableaux we use are

$$U_1(d-1) = \begin{array}{c} d-1 \\ \hline 1 \\ 2 \end{array} \qquad d \text{ odd} \qquad \omega_2 = (0, d-1)$$

$$P_1(d-1) = \begin{array}{c} d-1 \\ \hline 1 \\ 2 \\ 3 \end{array} \qquad d \text{ odd} \qquad \omega_{2,3} = \left(\begin{smallmatrix} 0 & d-1 & 0 \\ 0 & 0 & d-1 \end{smallmatrix}\right)$$

$$P_4(d-2,1,1) = \begin{array}{cc} \multicolumn{2}{c}{d-2} \\ \hline 1 & 1\,1\,3 \\ 2 & 2\,3\,2 \\ 3 & \end{array} \qquad d \text{ odd} \qquad \omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 1 \\ 0 & 0 & d-2 \end{smallmatrix}\right)$$

Start with a tableau T where $t \leq 2d-1$, $s \leq d+4$, $s \neq 1$, $r \leq d+4$, $r \neq 1$, d odd and $s+t$ even if r or s in $\{0, 2, 4\}$. (These are the partitions required by Theorem 10 after the previous reductions have been applied.) First consider those tableaux with $r \geq 10$, which implies $d \geq 6$.

Case A: Assume $r \geq 10$, $s < d+4$, $t < d-1$. Then $3t + 2s + r \leq 3(d-2) + 2(d+3) + d + 4 = 6d + 4 \leq 8d$. Hence this case is covered by $c \leq 8$.

Case B: Assume $r \geq 10$, $s \geq d+4$, $t \geq d-1$. Write $T = P_1(d-1) \vee U_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r-5, s-(d+1), t-(d-1))$. Thus $5 \leq r' \leq d-1$, $5 \leq s' \leq d+5$, and $0 \leq t' \leq d$. Then

$3t + 2s + r \leq 3d + 2(d+5) + d - 1 = 6d + 9 \leq 8d$. Note that no exceptional r or s cases occur in T'. Hence this case is covered by $c \leq 8$.

Case C: Assume $r \geq 10$, $s < d+4$, $t \geq d-1$. Write $T = P_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r-3, s, t-(d-1))$. Thus $7 \leq r' \leq d+1$, $s' \leq d+3$, and $0 \leq t' \leq d$. Then $3t+2s+r \leq 3d+2(d+3)+d+1 = 6d+7 \leq 8d$. Note that no exceptional r or s cases occur in T'. Hence this case is covered by $c \leq 8$.

Case D: Assume $r \geq 10$, $s \geq d+4$, $t < d-1$. Write $T = U_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r-2, s-(d-1), t)$. Thus $8 \leq r' \leq d+2$, $s' \leq d+5$, and $0 \leq t' \leq d-1$. Then $3t+2s+r \leq 3(d-1)+2(d+5)+d+2 = 6d+9 \leq 8d$. Note that no exceptional r or s cases occur in T'. Hence this case is covered by $c \leq 8$.

For $r < 10$ the arguments depend more on the values of r, but the general idea is the same.

Case E: Assume $r < 10$, $s < d+4$, $t < d-1$. Then $3t+2s+r \leq 3(d-1) + 2(d+3) + 9 = 5d + 12 \leq 8d$ for $d \geq 4$. When $d = 3$, $c = 9$ is a possibility. Now, when $d = 3$, we have $r \leq d+4 = 7$, $s \leq d+3 = 6$, and $t \leq d-2 = 1$. So here $3t+2s+r \leq 3 + 2 \cdot 6 + 7 = 22 < 8d$. Hence $c \leq 8$ tableaux will suffice to cover this case. Note that the $s+t$ parity is preserved in the exceptional r and s cases.

Case F: Assume $r < 10$, $s \geq d+4$, $t < d-1$. If $r = 9, 8, 7, 5$ then write $T = U_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r-2, s-(d-1), t)$. Then $r' \in \{7, 6, 5, 3\}$, $s' \leq d+5$, and $0 \leq t' \leq d-1$. Then $3t' + 2s' + r' \leq 3(d-2) + 2(d+5) + 7 = 5d + 11 \leq 8d$ for $d > 3$. When $d = 3$, then $s' \leq 8$, $t' \leq 1$. We have $3t' + 2s' + r' \leq 3 + 16 + 7 = 26 < 9d$. Hence $c \leq 8$ tableaux will suffice to cover these case.

If $r = 2, 4$, or 6, then write $T = U_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r-2, s-(d-1), t)$. Thus $r' \in \{4, 2, 0\}$, $s' \leq d+5$, and $0 \leq t' \leq d-1$. Then $3t' + 2s' + r' \leq 3(d-2) + 2(d+5) + 4 = 5d + 8 \leq 8d$. Hence $c \leq 8$ tableaux will suffice to cover these cases if the tableau exists.

Note that $U_1(d-1)$ preserves the parity of $s + t$, so if $s + t$ are even (always the case when $r = 2$ or 4), we will have $s' + t'$ even which is necessary for $r' = 0$, 2, or 4. Hence this construction works except when $r = 6$ and $s + t$ odd. Then $3t + 2s + r \leq 3(d-2) + 2(2d+4) + 6 = 7d + 8 \leq 8d$ for $d \geq 8$ otherwise it is less than $9d$ unless $d = 3$. If $3t + 2s + 6 = 9d$, then t is odd so write $t = d - 2k$ with $k \geq 1$. Then we have $s = 3d + 3k - 3$. Since $s \leq 2d + 4$, we have $d + 3k \leq 7$, so $d = 3, k = 1$ is the only solution. This corresponds to $t = 1$, $s = 9$, $r = 6$. Since $s + t$ even, this case has already been done. For $d = 3$ we also need to consider $c = 10$. However, $3 + 20 + 6 < 10d$ so no such partition will occur.

For $r = 0$ we have $s+t$ even. Then $3t+2s+r \leq 3(d-2)+2(2d+4) = 7d+2 \leq 8d$, so this case is covered by $c \leq 8$ tableaux.

For $r = 3$ we $3t + 2s + r \leq 3(d-2) + 2(2d+4) + 3 = 7d + 5 \leq 8d$ for $d \geq 5$. When $d = 3$ we have $3 + 20 + 3 < 9d$, so such a partition does not occur. Hence $r = 3$ is covered by $c \leq 8$.

Case G: Assume $r < 10$, $s < d + 4$, $t \geq d - 1$. If $r = 9, 8, 6$ then write $T = P_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r - 3, s, t - (d-1))$. Thus $r' \in \{6, 5, 3\}$, $s' \leq d + 3$, and $0 \leq t' \leq d$. Then $3t' + 2s' + r' \leq 3d + 2(d+3) + 6 = 5d + 12 \leq 8d$ for $d \geq 5$. When $d = 3$, then $s' \leq 6$, $t' \leq 3$. Hence we have $3t' + 2s' + r' \leq 9 + 12 + 6 = 27 = 9d$, so the only solution is $t' = 3$, $s' = 6$, $r' = 6$. This we can further reduce by writing $T' = V(d) \vee T''$ where T'' is a $c = 8$ tableau of parameters $t = 3$, $s = 6$, and $r = 3$. Hence $c \leq 8$ tableaux will suffice.

If $r = 7, 5$, or 3 and $s + t$ even then write $T = P_1(d-1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r - 3, s, t - (d-1))$ and the parity of $s' + t'$ is preserved. Thus $r' \in \{0, 2, 4\}$, $s' \leq d + 3$, and $0 \leq t' \leq d$. Then $3t' + 2s' + r' \leq 3d + 2(d+3) + 4 = 5d + 10 \leq 8d$ for $d \geq 5$. For $d = 3$ we have

$3t' + 2s' + r' \leq 9 + 12 + 4 = 25 < 9d$, so this does not occur. Hence the $c \geq 8$ tableaux will suffice.

Consider $r = 0, 2, 4$ with $s + t$ even, or $r = 3, 5, 7$ with $s + t$ odd. If $s \geq 8$ write $T = P_4(d-2, 1, 1) \vee T'$. If (r, s, t) are the parameters of T, then T' has parameters $(r', s', t') = (r, s-3, t-(d-2))$ and the parity of $s'+t'$ is preserved. Thus , $5 \leq s' \leq d$, and $0 \leq t' \leq d+1$. Then $3t' + 2s' + r' \leq 3(d+1) + 2(d) + 7 = 5d + 10 \leq 8d$ for $d \geq 5$. For $d = 3$ we have $3t' + 2s' + r' \leq 12 + 6 + 7 = 25 < 9d$, so this does not occur. Hence the $c \geq 8$ tableaux will suffice.

For $r = 3, 5$, or 7, $s+t$ odd, and $s \leq 7$, we have $3t + 3s + r \leq 3(2d-1) + 2 \cdot 7 + 7 = 6d + 18 \leq 8d$ for $d \geq 9$. If $3t + 2s + r = 9d$, then since r is odd, we have $t = 2d - 2k$ for $k \geq 1$. This implies $s = d + 3k + \frac{d-r}{2}$. Since $s \leq 7$, the only possible solutions are $(r, s, t) = (7, 4, 4)$, $(5, 5, 4)$, $(3, 6, 4)$, $(7, 7, 2)$ when $d = 3$ and $(7, 4, 8)$ when $d = 5$. For those with $s + t$ even, the case has already been done. For $(r, s, t) = (7, 7, 2)$ write $T = U_1(2) \vee T'$ where T' has parameters $(5, 5, 2)$ and is a $c = 7$ tableau. Hence we still need $(r, s, t) = (5, 5, 4)$ for $d = 3$. But this is $U_1(2) \vee T'$, where T' is a $c = 7$ tableau of parameters $(3, 3, 4)$.

We also need to consider those partitions with $c = 10$. Then we have $3t+2s+r = 10d$ which implies $t = 2d - 2k - 1$ as t is odd. So $s = 2d + 3k + 1 - \frac{r-1}{2}$. The only solutions with $s \leq 7$ are $(r, s, t) = (3, 6, 5)$, $(5, 5, 5)$, $(7, 4, 5)$, and $(7, 7, 3)$ all with $d = 3$. The cases with $s + t$ even have been done already. For $(r, s, t) = (3, 6, 5)$, use $T = P_4(1, 1, 1) \vee T'$ where T' is a $c = 7$ tableau with $(r', s,', t') = (3, 3, 4)$. For $(7, 4, 5)$ we have $s = 4$ and $s + t$ odd, so this case partition is not needed.

When $d = 3$ we also may have $c = 11$ or $c = 12$. Proceeding as above, the only solutions are $(r, s, t) = (7, 7, 4)$ and $(7, 7, 5)$. The second case has $s+t$ even and hence is not needed. The first case can be reduced to $T = U_1(2) \vee T'$, where T' is a $c = 9$ case with parameters $(5, 5, 4)$. But this is $U_1(2) \vee T'$, where T' is a $c = 7$ tableau of parameters $(3, 3, 4)$.

If $r = 0, 2, 4$, $s + t$ even, and $s \leq 7$, we have $3t + 3s + r \leq 3(2d-1) + 2 \cdot 7 + 4 = 6d + 15 \leq 8d$ for $d \geq 9$. For $c = 9$, we have $t = 2d - 2k - 1$ as t is odd. Then $s = d + 3k + 1 + \frac{d+1-r}{2}$. Since $s \leq 7$ the only solutions are $(r, s, t) = (0, 6, 5)$, $(2, 5, 5)$, $(4, 4, 5)$, and $(4,7,3)$ with $d=3$ and $(4,7,9)$. Note that only those with $s+t$ even are needed. For these cases write $T = U_1(d-1) \vee T'$, where T' is a $c'=7$ tableau with parameters $(r-2, s-(d-1), t)$. Since this preserves the parity of $s+t$ and does not cause $r', s' = 1$, the tableau exists.

For $d=3$ we may also have $c=10$ or 11. Proceeding as above, the only solutions are $(r,s,t) = (4,7,4)$ which has $s+t$ odd, hence it is not need, and $(4,7,5)$ which is $U_1(2) \vee U_1(2) \vee T'$ where T' is a $c=7$ tableau with $(r,s,t) = (0,3,5)$.

Case H: Assume $r < 10$, $s \geq d+4$, $t \geq d-1$. If $r=8$ or $r = 9,7,5$ and $s+t$ even then write $T = P_1(d-1) \vee U_1(d-1) \vee T'$ where T' has parameters $(r', s', t') = (r-5, s-(d-1), t-(d-1))$. Thus $r' \leq 4$, $s' \leq d+5$, and $t' \leq d$. Then $3t' + 2s' + r' \leq 3d + 2(d+5) + 4 = 5d + 14 \leq 8d$ for $d > 3$. When $d=3$ we can have $3t' + 2s' + r' = 9d$ only for $(r', s', t') = (2,8,3)$ or $(4,7,3)$. But $s+t$ even means only $(4,7,3)$ is needed. This is $U_1(d-1) \vee T''$ where T'' is a $c=7$ tableau with $(r,s,t) = (2,5,3)$. Hence we've reduced to $c \leq 8$ cases.

For $r = 9,7,5$ with $s+t$ odd, write $T = P_4(d-2,1,1) \vee U_1(d-1) \vee T'$ where T' has parameters $(r', s', t') = (r-2, s-(d-1)-3, t-(d-2))$. Hence $r' \leq 7$, $2 \leq s' \leq d+2$, and $t' \leq d+1$. We will have $s' \notin \{0,1,2,4\}$ provided $s \neq d+4$ or $d+6$. Then $3t' + 2s' + r' \leq 3(d+1) + 2(d+2) + 7 = 5d + 14 \leq 8d$ for $d > 3$. When $s = d+4$ or $d+6$ write $T = P_4(d-2,1,1) \vee T'$ where T' has parameters $(r', s', t') = (r, s-3, t-(d-2))$. Then $3t' + 2s' + r \leq 3(d+1) + 2(d+3) + 9 = 5d + 18 \leq 8d$ for $d \geq 5$.

For $d=5$, $s' = d+3$ or $d+1$ there are no partitions with $t' \leq d+1$. Now consider $d=3$ with $s' \leq d+3$, $t' \leq d+1$. Since $r \leq d+4$ we have $r = 3, 5$, or 7. Given these parameters, solving $3t' + 2s' + r' = 3c$ for $c \geq 9$, the only solutions are $(r', s', t') = (3,6,4)$, $(5,5,4)$, and $(7,4,4)$. Since $s+t$ odd the only partition we need to construct is $(r', s', t') = (5,5,4)$ with $c=9$. But this is $U_1(2) \vee T'$ where T' is a $c=7$ tableau of parameters $(3,3,4)$.

For $r = 2,4,6$ with $s+t$ even, write $T = P_4(d-2,1,1) \vee U_1(d-1) \vee T'$ where T' has parameters $(r', s', t') = (r-2, s-(d-1)-3, t-(d-2))$. Thus $r' \leq 4$, $2 \leq s' \leq d+2$, and $t' \leq d+1$. Since $s+t$ even, we need only that $s' \neq 1$. Then

$3t' + 2s' + r' \leq 3(d + 1) + 2(d + 2) + 4 = 5d + 11 \leq 8d$ for $d > 3$. For $d = 3$, $d \cdot 4 + 2 \cdot 5 + 4 = 26 < 9d$ so that case is not needed.

For $r = 6$, $s + t$ odd write $T = P_1(d-1) \vee T'$, where T' has parameters $(r', s', t') = (3, s, t - (d - 1))$. Thus $3t' + 2s' + r' \leq 3d + 2(2d + 4) + 3 = 7d + 11 \leq 8d$ for $d \geq 11$. Solving $3t' + 2s' + r' = 9d$ given the parameters, the only solution is $(r', s', t') = (3, 9, 2)$. This is $P_4(1, 1, 1) \vee T''$ where T'' is a $c = 6$ tableau with parameters $(3, 6, 1)$. When $c = 10$, we also have $(3, 9, 3)$ as a solution, but this does not have $s + t$ odd.

For $r = 3$, write $T = P_4(d - 2, 1, 1) \vee T'$ where T' has parameters $(r', s', t') = (3, s - 3, t - (d - 2))$. Thus $3t' + 2s' + r' \leq 3(d + 1) + 2(2d + 1) + 3 = 7d + 8 \leq 8d$ for $d > 5$. (The $c = 8$ constructions will hold unless $s - 3 \in \{0, 1, 2, 4\}$, but since $s \geq d + 4$ this can only occur with $s = d + 1$, $d = 3$. Then our original tableau will have $t \leq 2d - 1$, $s = d + 1$, $r = 3$ which is satisfiable by a $c \leq 8$ tableau.) Solving $3t' + 2s' + r' = 9d$ given the parameters, the only solutions are $(r', s', t') = (3, 6, 4)$, $(3, 9, 2)$, and $(3, 12, 6)$ with $d = 5$. The $(3, 6, 4)$ case is $P_4(1, 1, 1) \vee T''$ where T'' is a $c = 6$ tableau with parameters $(3, 3, 3)$. The $(3, 9, 2)$ case is $P_4(1, 1, 1) \vee T''$ where T'' is a $c = 6$ tableau with parameters $(3, 6, 1)$. When $d = 5$, the $(3, 12, 6)$ case is $P_4(3, 1, 1) \vee T''$ where T'' is a $c = 6$ tableau with parameters $(3, 9, 3)$. There are no tableaux with $c > 9$.

For $r = 0$ we have $s+t$ even. Write $T = P_4(d-2, 1, 1) \vee T'$ where T' has parameters $(r', s', t') = (3, s-3, t-(d-2))$. So $3t' + 2s' + r' \leq 3(d+1) + 2(2d+1) = 7d+5 \leq 8d$ for $d > 3$. As P_4 preserves parity and $s + t$ even, the $c = 8$ constructions will work provided $s' \neq 1$. But $s \geq d + 4$ forces $s' \geq d + 1 \geq 4$. When $d = 3$ we have $3 \cdot 4 + 2 \cdot 7 = 26 < 9d$, hence all cases are covered.

Conclusion: When d is odd, we need all tableaux with $c \leq 8$. The basic reduction requires those tableaux to be disjoint from multiple copies of S_1, $U_2(d - 1, 1)$ and $V(d)$. The further reductions also require the tableau to be disjoint from $P_1(d-1)$, $U_1(d-1)$, $P_1(d-1) \vee U_1(d-1)$, $P_4(d-2, 1, 1)$, and $U_1(d-1) \vee P_4(d-2, 1, 1)$.

For d even we need all tableaux with $c \leq 6$, along with those tableaux having $t \leq d - 1$, $s \leq d + 4$, $r \leq d + 4$ when $c = 7$ or 8. These tableaux need to be disjoint from $P_1(d)$, $U_1(d)$, and $V(d)$. These tableaux will be listed in Chapter 7. In Chapter 8, we verify that all necessary tableaux have been produced.

Chapter 7

Tableaux Construction

The proof of Theorem 2 in Chapter 6 requires the construction of non-zero maximal tableaux with $c \leq 8$ for the shapes discussed in Theorem 10. In this chapter we construct all the necessary tableaux and show they are both non-zero and maximal. Some basic properties of maximality are listed in Section 7.1 and are used throughout our construction. We construct those tableaux with two rows in Section 7.2. In Sections 7.3, 7.4, 7.5, and 7.6 we construction the necessary tableaux with $c = 3$, 4, 5, and 6, respectively. Section 7.7 contains additional tableaux need when $c = 7$ or 8.

7.1 Maximality of Tableaux

Maximality, as discussed in Lemma 3.4.9, is an important property of the tableaux we construct to prove Theorem 2. Given certain conditions of a weight, it is easy to verify that a tableau is in maximal form. We discuss maximality here in general, in order to simplify the proof of maximal form for the specific tableaux we construct in the next sections.

Recall that a tableau T is in maximal form if $\mathbf{q}_T \neq 0$ by weight-set counting on $\omega(T)$ and $\omega(T)$ is the largest weight of all $\omega(\tau T)$ for $\tau \in C_T$. All tableaux that we consider in the next sections are shown to be non-zero by weight-set counting prior to addressing the maximality issue. As such, we only consider the weight condition here. To summarize the basic conditions of Definition 3.4.2, a tableau weight is maximal if it has the largest weight possible for row three, and given this, the largest weight possible for row two. When multiple weights satisfy this, the weight where rows two and three have elements in common (e.g., the element 4 appears in both rows) is considered larger.

In general, to determine the maximal form of a tableau, we start with a given filling of T and use column operations to produce tableaux τT with larger weights. The following procedure provides an overview of how to determine the maximality of a filling.

- First maximize the weight of row three. This usually involves having a weight of d for as many elements as possible. (This is discussed more extensively in Lemma 7.1.2.) There may be multiple different fillings having the same maximum weight for row three. All such fillings should be considered for the next step.

- Now maximize the weight of row two, given the filling(s) of row three determined previously. To do this, determine the largest possible row two weight of each element, provided those elements assigned to row three are not used. For a given element this is equivalent to the number of copies in the body of T minus the number of copies used in row three. Fill row two with the element having the largest weight, then repeat the procedure with the remaining elements and positions. (This is summarized for weights in Lemma 7.1.4.) There may be multiple such fillings having the largest weight.

- If there are multiple fillings after step two that have different generic weights, choose the filling in which an element in row three has the largest weight in row two. (The weight of their common elements is maximized.) There may not be a unique such filling, but all such maximal fillings will have the same generic weight. As such these fillings will differ on by an action of $\mathcal{S}_a$.

Example 7.1.1. Consider

$$T = Q_2 = \begin{array}{c|ccc} \textsc{z}+\mathsf{x} & \textsc{z} & \textsc{z} & \textsc{a} \\ \hline 3 & 3 & 2 & 3 \\ 4 & 2 & 4 & 4 \\ 1 & 1 & 1 & 2 \end{array} \qquad \omega_{2,3}(T) = \left(\begin{smallmatrix} 0 & \textsc{z} & 0 & d-\textsc{z}+\textsc{a}+\mathsf{x} \\ d & \textsc{a} & 0 & 0 \end{smallmatrix}\right)$$

where $\textsc{z} = \frac{d-\mathsf{x}}{3}$, $\textsc{a} < \textsc{z}$, and some conditions on $\textsc{z}, \textsc{a}$, and x to insure $\mathbf{q}_T$ is non-zero.

First we want to maximize row three. Since this occurs when the non-zero weights of row three are d and $\textsc{a}$, we check to see which elements can have weight d in row three. Here, the only option is the element 1. Since any of the remaining elements can have a weight of $\textsc{a}$, we leave that column unfilled for now. Thus the maximal (third row) form for T is $111{*}$.

To determine the maximal second row form, we consider how many of each element is available given the third row is partially determined. There are zero 1's available, $2\textsc{z}+\textsc{a}$ 2's available, and $2\textsc{z}+\textsc{a}+\mathsf{x}$ 3's and 4's available. Hence we can maximally fill the second row with either 3's or 4's. Since the tableau is symmetric in 3's and 4's, we will use 4's without loss of generality. Thus we get a maximal (second and third row) form of $\begin{smallmatrix} 4 & * & 4 & 4 \\ 1 & 1 & 1 & * \end{smallmatrix}$.

This form gives rise to two different generic weights depending on whether the $*$'s are the same. These generic weights are $\left(\begin{smallmatrix} 0 & \textsc{z} & d-\textsc{z}+\textsc{a} & 0 \\ d & \textsc{a} & 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 & d-\textsc{z}+\textsc{a} & \textsc{z} \\ d & \textsc{a} & 0 & 0 \end{smallmatrix}\right)$. The first one, corresponding to the same element for both $*$'s, is larger. Hence a maximal (second and third row) form is $\begin{smallmatrix} 4 & 2 & 4 & 4 \\ 1 & 1 & 1 & 2 \end{smallmatrix}$. Note that this filling is not unique. We could have used $\begin{smallmatrix} 4 & 3 & 4 & 4 \\ 1 & 1 & 1 & 3 \end{smallmatrix}$, $\begin{smallmatrix} 3 & 4 & 3 & 3 \\ 1 & 1 & 1 & 4 \end{smallmatrix}$, or $\begin{smallmatrix} 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \end{smallmatrix}$ instead. All would be in maximal form.

Some fillings can be seen as maximal strictly by examining the weight of the tableau.

Lemma 7.1.2. If a row has at most one weight not equal to 0 or d, then its row weight is maximal.

This follows directly from the ordering on weights and the limit of d copies of an element in a tableau. If we call the weight not equal to 0 or d the non-d weight for the row, we get the following result on tableau maximality.

Lemma 7.1.3. A tableau is maximal if rows two and three satisfy Lemma 7.1.2 and either their non-d weights come from the same element or the sum of the non-d weights is greater than d.

Proof. Given a tableau satisfying Lemma 7.1.2 for rows two and three, its maximal generic weight must be either $\left(\begin{smallmatrix} 0 & \mathrm{B} & d & 0 \\ d & \mathrm{A} & 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 0 & d & \mathrm{B} \\ d & \mathrm{A} & 0 & 0 \end{smallmatrix}\right)$ with the appropriate number of 0 and d weights. By our ordering, the first weight is strictly larger than the second weight. Hence when the non-d weights come from the same element the tableau is maximal. If $\mathrm{A} + \mathrm{B}$ is greater than d, the first weight is not possible and so the second one is maximal. □

This lemma is directly applicable to the tableau weights. A more generalized form of this, depending on the actual filling, is also useful. The following lemma reflects the technique used in Example 7.1.1.

Lemma 7.1.4. A row is maximal if all except one non-zero weight corresponds to the largest weights possible for any elements.

This lemma is a generalization of Lemma 7.1.2, where the maximum weight for each element is no longer d. Using this in Lemma 7.1.3 gives:

Lemma 7.1.5. A tableau is maximal if row three satisfies Lemma 7.1.2, row two satisfies Lemma 7.1.4, and the non-d weight of row three has the largest possible weight in row two of all such weights satisfying Lemma 7.1.4.

Example 7.1.1 represents an appropriate use of Lemma 7.1.5. The weight of row three is $(d, \mathrm{A}, 0, 0)$, clearly satisfying Lemma 7.1.2. The weight of row two is $(0, \mathrm{Z}, 0, d - \mathrm{Z} + \mathrm{A} + \mathsf{x})$. That satisfies Lemma 7.1.4, since we checked that $d - \mathrm{Z} + \mathrm{A} + \mathsf{x}$ was the largest weight possible. Finally, $\omega_3(2) = \mathrm{A}$ is the non-d weight of row three. Since $\omega_2(2) = \mathrm{Z}$ is larger than $\omega_2(2) = 0$, the weight is maximal by Lemma 7.1.5.

In general, for the application of Lemma 7.1.5 it is to check that Lemma 7.1.2 and Lemma 7.1.4 apply and then to show that the non-d weight cannot be assigned a larger row two weight without changing the generic weight of row two. In particular the non-d weight conditions of Lemma 7.1.5 are satisfied if all extra (non-third row) copies an element in row three are contained in row two. We will refer to these lemmas for maximality justification of the tableaux constructed in the next section.

7.2 Tableaux for Two Row Partitions

In this section we construct all the two row tableaux needed for the proof of Theorem 2. For many of these constructions, the parity of d is relevant. Recall the notation:

$$d^* = \begin{cases} d & \text{if } d \text{ is even} \\ d-1 & \text{if } d \text{ is odd} \end{cases}$$

Since we are constructing tableaux for many partitions, we will not use a fixed c. However, for every element that is listed in the tableau, we assume it occurs d times, filling out the tail as needed using only those numbers in the body of T.

Tableau U_1

$$U_1 = \begin{array}{ccc} \textsc{a} & d-\textsc{a} & d-\textsc{a} \\ \hline 1 & 1 & 2 \\ 2 & & \end{array} \sim \begin{array}{c} \textsc{a} \\ \hline 1 \\ 2 \end{array} \qquad \begin{array}{l} \textsc{a} \text{ even} \\ \textsc{a} \le d \end{array}$$

$$\omega_2(U_1) = (0, \textsc{a})$$

$$\lambda = [2d - \textsc{a}, \textsc{a}]$$

$$r = 2d - 2\textsc{a},\ s = \textsc{a},\ t = 0$$

We showed U_1 non-zero in the proof of Theorem 1.

Maximality: This tableau is maximal by Lemma 7.1.4.

Tableau U_2

$$U_2 = \begin{array}{cccc} \text{A} & \text{A} & \text{B} & \text{B} \\ \hline 1 & 3 & 1 & 3 \\ 2 & 4 & 4 & 2 \end{array} \qquad \begin{array}{l} \text{A} + \text{B} \le d \\ \text{A},\ \text{B} > 0 \end{array}$$

$$\omega_2 = (0, \text{A}+\text{B}, 0, \text{A}+\text{B})$$

$$\lambda = [4d - 2\text{A} - 2\text{B}, 2\text{A} + 2\text{B}]$$

$$r = 4d - 4\text{A} - 4\text{B},\ s = 2\text{A} + 2\text{B},\ t = 0$$

We showed U_2 non-zero in the proof of Theorem 1.

Maximality: This tableau is maximal by Lemma 7.1.4.

Tableau U_3

$$U_3 = \begin{array}{ccc} \text{A} & \text{B} & \text{B} \\ \hline 1 & 1 & 2 \\ 2 & 3 & 3 \end{array} \qquad \begin{array}{l} \text{A even} \\ \text{A} + \text{B} \le d \\ 0 < 2\text{B} \le d \\ \text{B} \ge \text{A} \end{array}$$

$$\omega_2 = (0, \text{A}, 2\text{B})$$

$$\lambda = [3d - \text{A} - 2\text{B}, \text{A} + 2\text{B}]$$

$$r = 3d - 2\text{A} - 4\text{B},\ s = \text{A} + 2\text{B},\ t = 0$$

If $\text{B} > \text{A}$, any valid weight assignment must have $\omega_2(3) = 2\text{B}$. Hence $(1,2,3)$ with sign 1 and $(2,1,3)$ with sign $(-1)^{\text{A}}$ are the only possible weight assignments. Thus U_3 is non-zero. If $\text{A} = \text{B}$, we may also have weight assignments: $(1,3,2)$ with sign $(-1)^{\text{B}}$; $(2,1,3)$ with sign $(-1)^{\text{A}+\text{B}}$; $(3,1,2)$ with sign $(-1)^{2\text{B}}$; and $(3,2,1)$ with sign $(-1)^{\text{A}+2\text{B}}$. Since $\text{A} = \text{B}$ and A is even, these are all positive. Hence U_3 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.4 since $2\text{B} \ge \text{A} + \text{B}$.

Tableau U_4

$$U_4 = \begin{array}{cc} \text{A} & \text{B} \\ \hline 2 & 3 \\ 1 & 1 \end{array} \qquad \text{A} \geq \text{B} > 0$$

$$\omega_2 = (\text{A} + \text{B}, 0, 0)$$

$$\lambda = [3d - \text{A} - \text{B}, \text{A} + \text{B}]$$

$$r = 3d - 2\text{A} - 2\text{B},\ s = \text{A} + \text{B},\ t = 0$$

Since only the element 1 can have $\omega_2 = \text{A} + \text{B}$, there is exactly one valid weight assignment, $(1, 2, 3)$. Thus U_4 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.4.

Tableau U_5

$$U_5 = \begin{array}{cccc} \text{A} & \text{B} - 2 & \text{A} - 1 & \text{B} \\ \hline 1 & 1 & 4 & 4 \\ 2 & 3 & 3 & 2 \end{array} \qquad \begin{array}{l} \text{A} \geq 2 \\ \text{B} \geq 3 \\ \text{A} + \text{B} \leq d \end{array}$$

$$\omega_2 = (0, \text{A} + \text{B}, \text{A} + \text{B} - 3, 0)$$

$$\lambda = [4d - 2\text{A} - 2\text{B} + 3, 2\text{A} + 2\text{B} - 3]$$

$$r = 4d - 4\text{A} - 4\text{B} + 6,\ s = 2\text{A} + 2\text{B} - 3,\ t = 0$$

Only the element 2 may have $\omega_2(2)$=A+B. Since $\text{A} \geq 2$ and $\text{B} \geq 3$, we must have $\omega_2(3) = \text{A} + \text{B} - 3$. Hence $(1, 2, 3, 4)$ is the only valid weight assignment, so U_5 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.4 since A+B is the largest weight possible.

Tableau U_6

$$U_6 = \begin{array}{ccccccccccc} \scriptstyle A & \scriptstyle B & \scriptstyle C & & & \scriptstyle D & & & & & \\ \hline 1 & 3 & 5 & 5 & 5 & 3 & 1 & 3 & 1 & 3 & 5 \\ 2 & 4 & 4 & 2 & 4 & 5 & 5 & 2 & & & \end{array} \qquad \begin{array}{l} \mathrm{A} = d-2 \\ \mathrm{B} = \frac{d+1}{2} \\ \mathrm{C} = \frac{d-3}{2} \\ \mathrm{D} = \frac{d-5}{2} \end{array} \qquad \begin{array}{l} d \equiv 1 \pmod 2 \\ d > 5 \end{array}$$

$$\omega_2 = (0, d, 0, d, \tfrac{d-3}{2})$$

$$\lambda = [3d - \tfrac{d-3}{2}, 2d + \tfrac{d+3}{2}]$$

$$r = 3,\ s = 2d + \tfrac{d-3}{2},\ t = 0,$$

Only the elements 2 and 4 may have $\omega_2 = d$. If $\omega_2(5) = 0$, then $\omega_2(1 \text{ and } 3) > 0$. Hence $(1, 2, 3, 4, 5)$ is the only valid weight assignment and so U_6 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.2.

Tableau U_7

$$U_7 = \begin{array}{ccccccccc} 2 & & 2 & 3 & & 2 & & & \\ \hline 4 & 3 & 5 & 5 & 3 & 4 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 & 2 & 3 & & & \end{array} \qquad d = 5$$

$$\omega_2 = (5, 4, 2, 0, 0)$$

$$\lambda = [14, 11]$$

$$r = 3,\ s = 11,\ t = 0,$$

The valid weight assignments are $(5, 4, 2, 0, 0, 0)$, $(5, 4, 0, 2, 0)$, $(2, 0, 4, 0, 5)$, $(0, 0, 2, 4, 5)$, and $(0, 0, 4, 2, 5)$. Since there are an odd number of weight assignments, this tableau is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.4.

Tableau U_8

$$U_8 = \begin{array}{ccccccccccc} 4 & 3 & 3 & 3 & 3 & 1 & 2 & 2 & 4 & 4 & 4 \\ 1 & 1 & 1 & 2 & 2 & & & & & & \end{array} \qquad d = 4$$

$\omega_2 = (3, 2, 0, 0)$

$\lambda = [11, 5]$

$r = 6,\ s = 5,\ t = 0$

Then only valid weight assignments are $(1, 2, 3, 4)$, with sign 1, $(1, 3, 2, 4)$, with sign $(-1)^2$, and $(3, 1, 2, 4)$ with sign $(-1)^3$. Hence the weight sum is 1 and U_8 is non-zero.

Maximality: This tableau is not maximal since $\omega_2 = (0, 0, 4, 1)$ is larger. However, in $\mathbf{q}_T$, the weight $(0, 0, 4, 1)$ always cancels. This tableau cannot be put in maximal form, hence we will need to prove directly that it is disjoint from the requisite tableaux. This will be done in Section 8.7.

7.3 Tableaux for $c = 3$

We know by Theorem 9 which tableaux are non-zero for $c = 3$. However, using Theorem 8 on tableaux requires the tableaux to be non-zero by weight-set counting on $\omega(T)$. We also want the tableaux to be maximal, in order to obtain the disjointness of Lemma 3.4.9. Here we will list the tableaux used, briefly showing they are non-zero and maximal. For all these tableaux, any valid weight assignment corresponds to a unique tableau, so we will not explicitly state how many tableaux correspond to each weight assignment.

Tableau P_1

$$P_1 = \begin{array}{c} \text{A} \\ \hline 1 \\ 2 \\ 3 \end{array} \qquad \begin{array}{l} 0 \leq \text{A} \leq d \\ \text{A even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A} & 0 \\ 0 & 0 & \text{A} \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\text{A}, \text{A}, \text{A}]$$

$$r = 3d - 3\text{A},\ s = 0,\ t = \text{A}$$

P_1 is non-zero by the Lemma 3.2.6.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau P_2

$$P_2 = \begin{array}{cc} \text{A} & \text{B} \\ \hline 1 & 1 \\ 2 & 2 \\ 3 & \end{array} \qquad \begin{array}{l} \text{A} + \text{B} \leq d \\ \text{A, B even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A+B} & 0 \\ 0 & 0 & \text{A} \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\text{A} - \text{B}, \text{A+B}, \text{A}]$$

$$r = 3d - 3\text{A} - 2\text{B},\ s = \text{B},\ t = \text{A}$$

Only the elements 1 and 2 may have $\omega_2 = \text{A}+\text{B}$. Hence the valid weight assignments are: $(1,2,3)$ with sign 1; $(3,2,1)$ with sign $(-1)^{\text{A}}$; $(2,1,3)$ with sign $(-1)^{\text{A}+\text{B}}$; and $(3,1,2)$ with sign $(-1)^{\text{B}}$. Since A and B are even, this weight sum is 4 and $\mathbf{q}_{P_2} \neq 0$.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau P_3

$$P_3 = \begin{array}{cccc} \text{A} & \text{B}+1 & \text{B} & \text{C} \\ \hline 2 & 2 & 3 & 2 \\ 1 & 1 & 1 & 3 \\ 3 & & & \end{array} \qquad \begin{array}{l} 0 \le \text{A} \le d \\ 2\text{B}+1 \le d-\text{A} \\ 0 \le \text{C} < \text{B} \\ \text{A}+\text{C} \text{ even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A}+2\text{B}+1 & \text{C} \\ 0 & 0 & \text{A} \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\text{A} - 2\text{B} - \text{C} - 1, \text{A} + 2\text{B} + \text{C} + 1, \text{A}]$$

$$r = 3d - 3\text{A} - 4\text{B} - 2\text{C} - 2,\ s = 2\text{B} + \text{C} + 1,\ t = \text{A}$$

Only the element 1 may have $\omega_2 = \text{A}+2\text{B}+1$. Hence the valid weight assignments are $(1,2,3)$ with sign 1 and $(1,3,2)$ with sign $(-1)^{\text{A}+\text{C}}$. Since $\text{A}+\text{C}$ is even, this weight sum is positive and hence P_3 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau P_4

$$P_4 = \begin{array}{cccc} \text{A} & \text{B} & \text{B} & \text{C} \\ \hline 2 & 2 & 3 & 2 \\ 1 & 1 & 1 & 3 \\ 3 & & & \end{array} \qquad \begin{array}{l} 0 \le \text{A} \le d \\ 2\text{B} \le d-\text{A} \\ 0 \le \text{C} \le \text{B} \\ \text{A}+\text{C} \text{ even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A}+2\text{B} & \text{C} \\ 0 & 0 & \text{A} \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\text{A} - 2\text{B} - \text{C}, \text{A} + 2\text{B} + \text{C}, \text{A}]$$

$$r = 3d - 3\text{A} - 4\text{B} - 2\text{C},\ s = 2\text{B} + \text{C},\ t = \text{A}$$

Unless $\text{C} = \text{B}$, the only valid weight assignments are $(1, 2, 3)$ with sign 1 and $(1, 3, 2)$ with sign $(-1)^{\text{A}+\text{C}} = 1$. If $\text{B} = \text{C}$ the tableau is symmetric in 1, 2, and 3. Hence we also get the weight assignments: $(2, 1, 3)$ with sign $(-1)^{\text{A}+\text{B}+\text{B}+\text{C}} = 1$; $(2, 3, 1)$ with sign $(-1)^{2\text{B}} = 1$; $(3, 2, 1)$ with sign $(-1)^{\text{A}+\text{B}} = 1$; and $(3, 1, 2)$ with sign $(-1)^{2\text{B}} = 1$. Thus the weight sum is always positive and P_4 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

7.4 Tableaux for $c = 4$

Tableau Q_1

$$Q_1 = \begin{array}{c|cccccc} \text{Z+X} & \text{Z} & \text{Z} & \text{A} & \text{B} & \text{C} & \text{D} \\ \hline 3 & 3 & 2 & 3 & 3 & 2 & 3 \\ 4 & 2 & 4 & 4 & 4 & 4 & 2 \\ 1 & 1 & 1 & 2 & & & \end{array}$$

$0 \leq \text{A} < \text{Z}$

$\text{B} + \text{X} \geq \text{D}$

$\text{C} \geq \text{D}$

$\text{B} + \text{C} \leq \text{Z} - \text{A}$

$\text{Z} + \text{A} + \text{D}$ even

$\text{C} + \text{D}$ even, if $\text{D} = \text{B} + \text{X}$

$\text{D} + \text{B} + \text{X}$ even, if $\text{D} = \text{C}$

$\text{Z} = \frac{d-\text{X}}{3}$

$d \equiv \text{X} \pmod 3$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{Z+D} & 0 & \text{2Z+X+A+B+C} \\ d & \text{A} & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [2d - 2\text{A} - \text{B} - \text{C} - \text{D}, d + \text{A} + \text{B} + \text{C} + \text{D}, d + \text{A}]$

$r = d - 3\text{A} - 2\text{B} - 2\text{C} - 2\text{D}$, $s = \text{B} + \text{C} + \text{D}$, $t = d + \text{A}$

Since $\text{A} < \text{Z}$, any valid weight assignment must have $\omega_3(1) = d$. We must have $\omega_2(4) = 2\text{Z} + \text{X} + \text{A} + \text{B} + \text{C}$ unless $\text{D} = \text{C}$ or $\text{D} = \text{B} + \text{X}$. If $\text{D} = \text{C}$, then we may also have $\omega_2(3) = 2\text{Z}+\text{X}+\text{A}+\text{B}+\text{C}$. If $\text{D} = \text{B}+\text{X}$, then we may also have $\omega_2(2) = 2\text{Z}+\text{X}+\text{A}+\text{B}+\text{C}$. Given these restrictions, we list the valid weight assignments in the table below, along with the sign corresponding to that assignment. Note that there is a unique tableau associated to each of these assignments.

Assignment	Sign	Condition
$(1, 2, 3, 4)$	1	
$(1, 3, 2, 4)$	$(-1)^{\text{Z+A+D}}$	
$(1, 2, 4, 3)$	$(-1)^{\text{Z+X+A+B+C+D}}$	$\text{D} = \text{C}$
$(1, 4, 2, 3)$	$(-1)^{\text{2Z+X+B+D}}$	$\text{D} = \text{C}$
$(1, 3, 4, 2)$	$(-1)^{\text{2Z+X+B+C}}$	$\text{D} = \text{B} + \text{X}$
$(1, 4, 3, 2)$	$(-1)^{\text{Z+A+C}}$	$\text{D} = \text{B} + \text{X}$

Given the parity conditions on the parameters, the sign of these terms reduces to 1 in all cases. Hence the weight sum is positive and Q_1 is non-zero.

Maximality: Because $\text{C} \geq \text{D}$ and $\text{B} + \text{X} \geq \text{D}$, the weight $2\text{Z} + \text{X} + \text{A} + \text{B} + \text{C}$ of 4 in row two is the largest possible weight given the Maximality of row three by

Lemma 7.1.2. Hence this tableau is maximal by Lemma 7.1.5.

Tableau Q_2

$$Q_2 = \begin{array}{c|ccc} \text{Z+x} & \text{Z} & \text{Z} & \text{A} \\ \hline 3 & 3 & 2 & 3 \\ 4 & 2 & 4 & 4 \\ 1 & 1 & 1 & 2 \end{array} \qquad \begin{array}{l} 0 < \text{A} \leq \text{Z} \\ \text{Z} + \text{A} \text{ even} \end{array} \qquad \begin{array}{l} \text{Z} = \frac{d-\text{x}}{3} \\ d \equiv \text{x} \pmod 3 \\ \text{x} \in \{0, 2, 4\} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{Z} & 0 & d-\text{Z}+\text{A} \\ d & \text{A} & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [2d - 2\text{A}, d + \text{A}, d + \text{A}]$$

$$r = d - 3\text{A},\ s = 0,\ t = d + \text{A}$$

For $\text{A} < \text{Z}$ any valid weight assignment must $\omega_3(1) = d$. The tableau is symmetric in 3 and 4, as well as 2 if $\text{x} = 0$. This gives the following signed weight table:

Assignment	Sign	Condition
$(1, 2, 3, 4)$	1	
$(1, 3, 2, 4)$	$(-1)^{\text{Z}+\text{A}}$	
$(1, 2, 4, 3)$	$(-1)^{3\text{Z}+\text{x}+\text{A}}$	
$(1, 4, 2, 3)$	$(-1)^{2\text{Z}+\text{x}}$	
$(1, 3, 4, 2)$	$(-1)^{2\text{Z}}$	$\text{x} = 0$
$(1, 4, 3, 2)$	$(-1)^{\text{Z}+\text{A}}$	$\text{x} = 0$

Since x is even, all the terms are positive and Q_2 is non-zero. If $\text{Z} = \text{A}$ the tableau is symmetric in 1, 3, and 4, as well as 2 if $\text{x} = 0$. We get all the weight assignments listed above, in addition to those obtained by interchanging rows or allowing $\omega_{2,3}(1) = \left(\begin{smallmatrix} \text{Z} \\ \text{Z} \end{smallmatrix}\right)$. Interchanging rows has sign $(-1)^{d+\text{Z}} = 1$ since x is even. The other possibility changes the sign by $(-1)^{\text{Z}+\text{A}} = 1$. Hence all the terms are positive and Q_2 is non-zero.

Maximality: This tableau is maximal as shown in Example 7.1.1.

Tableau Q_3

$$Q_3 = \begin{array}{lll} \text{A} & \text{B} & \text{B} \\ \hline 1 & 4 & 4 \\ 2 & 3 & 2 \\ 3 & & \end{array} \qquad \begin{array}{l} 0 < \text{A} \leq d - \text{B} \\ 2\text{B} \leq d \\ \text{A} \geq \text{B} \\ \text{A even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A}+\text{B} & \text{B} & 0 \\ 0 & 0 & \text{A} & 0 \end{smallmatrix}\right)$$

$$\lambda = [4d - 2\text{A} - 2\text{B}, \text{A} + 2\text{B}, \text{A}]$$

$$r = 4d - 3\text{A} - 4\text{B},\ s = 2\text{B},\ t = \text{A}$$

Since $\text{A} \geq \text{B}$ only the elements 2 and 3 may have $\omega_2 = \text{A} + \text{B}$. Hence the only valid weight assignments are $(1, 2, 3, 4)$ with sign 1 and $(1, 3, 2, 4)$ with sign $(-1)^{\text{A}} = 1$. Thus all the valid weight assignments are positive and therefore Q_3 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau Q_4

$$Q_4 = \begin{array}{llllll} \text{A} & \text{A} & \text{B} & \text{C} & \text{C} & \text{D} \\ \hline 3 & 4 & 1 & 3 & 1 & 3 \\ 2 & 2 & 2 & 2 & 4 & 4 \\ 1 & 1 & & & & \end{array} \qquad \begin{array}{l} 2\text{A} + \text{B} + \text{C} = d \\ \text{A} + \text{C} + \text{D} \leq d \\ \text{A}, \text{B}, \text{C}, \text{D} > 0 \\ \text{B} + \text{D even} \\ \text{A even, if } \text{A} + \text{C} + \text{D} = d \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & \text{C}+\text{D} \\ 2\text{A} & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\text{A} - \text{C} - \text{D}, d + \text{C} + \text{D}, 2\text{A}]$$

$$r = 2d - 2\text{A} - 2\text{C} - 2\text{D},\ s = \text{B} + 2\text{C} + \text{D},\ t = 2\text{A}$$

Any weight assignment must have either $\omega_3(1) = 2\text{A}$ or $\omega_3(2) = 2\text{A}$. Unless $\text{A} + \text{C} + \text{D} = d$, only the elements 1 and 2 may have $\omega_2 = d$. This gives the following weight table:

Assignment	Sign	Condition
$(1,2,3,4)$	1	
$(2,1,4,3)$	$(-1)^{\text{B}+\text{D}}$	
$(2,3,4,1)$	$(-1)^{\text{A}+\text{B}+\text{D}}$	$\text{A}+\text{C}+\text{D}=d$
$(1,4,3,2)$	$(-1)^{\text{A}}$	$\text{A}+\text{C}+\text{D}=d$

By our parity constraints, the weight sum is always positive. Hence Q_4 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau Q_5

$$Q_5 = \begin{array}{cccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} \\ \hline 4 & 3 & 3 & 4 & 3 & 1 \\ 2 & 2 & 2 & 2 & 4 & 4 \\ 1 & 1 & & & & \end{array}$$

$\text{A}+\text{B}+\text{C}+\text{D}=d$

$\text{A},\text{B},\text{E},\text{F}>0$

$\text{A}+\text{B}+\text{F}\leq d$

$\text{B}+\text{C}+\text{E}<d$

$\text{A}+\text{D}+\text{E}+\text{F}<d$

If $\text{A}+\text{B}+\text{F}=d$ then:

either $\text{C},\text{D}>0$

or $\text{D}=0$ and $d+\text{C}+\text{E}$ even

or $\text{C}=0$ and $d+\text{D}$ even

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & \text{E}+\text{F} \\ \text{A}+\text{B} & 0 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [3d-\text{A}-\text{B}-\text{E}-\text{F}, d+\text{E}+\text{F}, \text{A}+\text{B}]$

$r = 2d-\text{A}-\text{B}-2\text{E}-2\text{F}$, $s=\text{C}+\text{D}+\text{E}+\text{F}$, $t=\text{A}+\text{B}$

Unless $\text{A}+\text{B}+\text{F}=d$, we must have $\omega_2(2)=d$. Then since $\text{E},\text{F}>0$ we have $\omega_2(4)=\text{E}+\text{F}$. As $\text{A},\text{B}>0$, we have $\omega_3(1)=\text{A}+\text{B}$. Therefore no other valid weight assignments exist.

If $\text{A}+\text{B}+\text{F}=d$, then it is possible to have $\omega_2(1)=d$. However, unless C or D equals zero, there is no element with $\omega_2=\text{E}+\text{F}$. Hence no such weight assignment can exist. If $\text{D}=0$ we can have the weight assignment $(2,1,4,3)$ which has sign $(-1)^{d+\text{C}+\text{E}}=1$. If $\text{C}=0$ then we can have the weight assignment $(2,1,3,4)$ which has sign $(-1)^{d+\text{D}}=1$. Thus, in either case, the weight sum is positive and Q_5 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. As no other fillings may have this row two weight and common elements between rows two and three, the tableau is maximal.

Tableau Q_6

$$Q_6 = \begin{array}{ccccc} \text{A} & \text{B} & \text{C} & & \\ \hline 1 & 4 & 1 & 1 & 4 \\ 2 & 2 & 3 & 2 & 3 \\ 3 & & & & \end{array} \qquad \begin{array}{l} \text{A} > 0 \\ \text{A} + \text{B} \le d - 1 \\ \text{B} \ge \text{C} \\ \text{A even, if B} = \text{C} \\ \text{B, C not both } 0 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A}+\text{B}+1 & \text{C}+1 & 0 \\ 0 & 0 & \text{A} & 0 \end{smallmatrix}\right)$$

$$\lambda = [4d - 2\text{A} - \text{B} - \text{C} - 2, \text{A} + \text{B} + \text{C} + 2, \text{A}]$$

$$r = 4d - 3\text{A} - 2\text{B} - 2\text{C} - 4,\ s = \text{B} + \text{C} + 2,\ t = \text{A}$$

Unless $\text{B} = \text{C}$, we must have $\omega_2(2) = \text{A} + \text{B} + 1$. This force $\omega_{2,3}(3) = \left(\begin{smallmatrix} \text{C}+1 \\ \text{A} \end{smallmatrix}\right)$ and hence there are no other valid weight assignments.

When $\text{C}{=}\text{B}$ then the elements 1, 2, or 3 may have $\omega_2 = \text{A}{+}\text{B}{+}1$. If $\omega_2(2) = \text{A}{+}\text{B}{+}1$ then only the element 3 may have $\omega_{2,3} = \left(\begin{smallmatrix} \text{C}+1 \\ \text{A} \end{smallmatrix}\right)$. If $\omega_2(1) = \text{A} + \text{B} + 1$, then a valid weight assignment exists only for $\text{B} = 0$. In this case $\omega_2(3) = \text{C}{+}1$. If $\omega_2(3) = \text{A}{+}\text{B}{+}1$ then we may have $\omega_2(2) = \text{C} + 1$ or, if $\text{C} = 0$, $\omega_2(1) = \text{C} + 1$. Since these conditions are subject to $\text{B} = \text{C}$ and B and C are never simultaneously zero, the only valid weight assignments are $(1, 2, 3, 4)$ with sign 1 and $(1, 3, 2, 4)$ with sign $(-1)^{\text{A}} = 1$. Hence the weight sum is positive and Q_6 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau Q_7

$$Q_7 = \begin{array}{cccccc} \text{A} & \text{B} & & \text{C} & & 2 \\ \hline 3 & 4 & 4 & 4 & 1 & 4 \\ 2 & 2 & 3 & 3 & 3 & 2 \\ 1 & 1 & 1 & & & \end{array} \qquad \begin{array}{l} \text{A} + \text{B} = d - 2 \\ \text{A} + \text{C} \le d - 2 \\ \text{B} + \text{C} \le d - 3 \\ \text{A}, \text{B} > 0 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & \text{C}+2 & 0 \\ d-1 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [2d - \text{C} - 1, d + \text{C} + 2, d - 1]$$

$$r = d - 2\text{C} - 3,\ s = \text{C} + 3,\ t = d - 1$$

Any valid weight assignment must have $\omega_3(1) = d - 1$. Therefore $\omega_2(1) = 0$, so we must have $\omega_2(3) > 0$. Unless $\text{B} = \text{C}$ only the element 2 may have $\omega_2 = d$, so the only valid weight assignment is $(1, 2, 3, 4)$ with sign 1. When $\text{B} = \text{C}$, we may also have $(1, 3, 2, 4)$ with sign $(-1)^{\text{A}}$ and $(1, 3, 4, 2)$ with sign $(-1)^{\text{A}+\text{B}+2}$. Since this sum is odd, we must have Q_7 non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

7.5 Tableaux for $c = 5$

Tableau R_1

$$R_1 = \begin{array}{cccccc} \mathsf{x} & \text{Z} & \text{Z} & \text{Z} & \text{A} & \text{B} \\ \hline 5 & 2 & 4 & 5 & 5 & 5 \\ 3 & 3 & 3 & 4 & 4 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} 0 < \text{A} \leq \text{B} \leq \text{Z} \\ d + \text{A} + \text{Z} \text{ even, if } \text{B} = \text{Z} \\ d \text{ even, if } \text{A} = \text{B}, \mathsf{x} = 0 \end{array} \qquad \begin{array}{l} \text{Z} = \frac{d - \mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & 0 & 2\text{Z}+\mathsf{x}+\text{B} & \text{Z}+\text{A} & 0 \\ d & \text{A}+\text{B} & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [3d - 2\text{A} - 2\text{B}, d + \text{A} + \text{B}, d + \text{A} + \text{B}]$$

$$r = 2d - 3\text{A} - 3\text{B},\ s = 0,\ t = d + \text{A} + \text{B}$$

First consider the valid weight assignments with $\text{A} < \text{Z}$. If $\text{B} < \text{Z}$ then we must have $\omega_3(1) = d$; if $\text{B} = \text{Z}$ we may have $\omega_3(3) = d$ as well. If we have $\omega_3(1) = d$, then $\omega_3(2 \text{ or } 5) = \text{A}+\text{B}$ because $\text{A}, \text{B} > 0$. Since $\text{B} \geq \text{A}$ only the elements 1 and 3 may have $\omega_2 = 2\text{Z} + \mathsf{x} + \text{B}$ unless $\text{A} = \text{B}$ and $\mathsf{x} = 0$. In that case, the element 4 may also have this weight. If $\omega_2(3) = 2\text{Z} + \mathsf{x} + \text{B}$, then $\omega_2(5 \text{ or } 4) = \text{Z} + \text{A}$. If $\omega_2(4) = 2\text{Z} + \text{B}$, then $\omega_2(2 \text{ or } 3) = \text{Z} + \text{A}$. These constraints give the following table of weight assignments:

Assignment	Sign	Condition
$(1, 2, 3, 4, 5)$	1	
$(1, 5, 3, 4, 2)$	$(-1)^{\text{A}+\text{B}}$	
$(1, 2, 3, 5, 4)$	$(-1)^{\text{Z}+\text{A}}$	
$(1, 2, 4, 3, 5)$	$(-1)^{\text{Z}}$	$\text{A} = \text{B}, \mathsf{x} = 0$
$(1, 5, 4, 3, 2)$	$(-1)^{\text{Z}+\text{B}+\text{A}}$	$\text{A} = \text{B}, \mathsf{x} = 0$
$(1, 5, 4, 2, 3)$	$(-1)^{2\text{Z}+\text{A}}$	$\text{A} = \text{B}, \mathsf{x} = 0$
$(3, 4, 1, 2, 5)$	$(-1)^{d+\text{Z}+\text{A}}$	$\text{B} = \text{Z}$
$(3, 5, 1, 2, 4)$	$(-1)^{d}$	$\text{B} = \text{Z}$
$(3, 4, 1, 5, 2)$	$(-1)^{d}$	$\text{B} = \text{Z}$

For $\text{B} < \text{Z}$ and $\text{A} \neq \text{B} + \mathsf{x}$, we get a weight sum of $1 + (-1)^{\text{A}+\text{B}} + (-1)^{\text{Z}+\text{A}}$ which is non-zero. When $\text{A} = \text{B}$, $\mathsf{x} = 0$, we get a weight sum of $1 + 1 + (-1)^{\text{Z}+\text{A}} + (-1)^{\text{Z}} + (-1)^{\text{Z}} +$

$(-1)^{\mathrm{A}}$. This is non-zero by the parity constraints which imply Z even. Similarly, if $\mathrm{A} < \mathrm{B} = \mathrm{Z}$, then the weight sum is $1+(-1)^{\mathrm{A}+\mathrm{Z}}+(-1)^{\mathrm{Z}+\mathrm{A}}+(-1)^{d+\mathrm{Z}+\mathrm{A}}+(-1)^d+(-1)^d$, which is also non-zero by the parity constraints.

When $\mathrm{A} = \mathrm{B} = \mathrm{Z}$ the tableau has many symmetries. Since the weight of rows two and three are the same, interchanging rows yields a new weight assignment with a difference in sign of $(-1)^{d+\mathrm{A}+\mathrm{B}} = 1$. Hence we will not count those weight assignments which are inversions of rows two and three. First consider the possible element pairs (x, y) that can have $\omega_3(x, y) = (d, 2\mathrm{Z})$. These are $(1, 2)$, $(1, 5)$, $(3, 4)$, $(3, 5)$, $(5, 3)$, and $(5, 1)$, along with $(2, 1)$, $(2, 4)$, $(4, 3)$, $(4, 2)$ when $\mathsf{x} = 0$. Two of these pairs make up a weight assignment (not counting row inversion). If $\mathsf{x} \neq 0$ the possible pair assignments are: $(1, 2)(3, 4)$; $(1, 2)(3, 5)$; $(1, 2)(5, 3)$; $(1, 5)(3, 4)$; and $(3, 4)(5, 1)$. Since there is an odd number, this weight sum in non-zero. If $\mathsf{x} = 0$, the condition d even, implies Z is even. Hence all the column blocks are even. Since any valid weight assignment moves full column blocks, all weight assignments are positive and thus R_1 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau R_2

$$R_2 = \begin{array}{c|cccccc} \mathsf{x}+\mathrm{Z} & \mathrm{Z} & \mathrm{Z} & \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{C} \\ \hline 2 & 4 & 4 & 5 & 4 & 4 & 2 \\ 3 & 5 & 3 & 3 & 5 & 3 & 5 \\ 1 & 1 & 1 & 2 & 2 & & \end{array} \qquad \begin{array}{l} 0 \le \mathrm{A} \le \mathrm{Z} - \mathrm{C} \\ 0 \le \mathrm{B} \le \mathrm{Z} - \mathrm{C} \\ 0 < \mathrm{C} \\ \mathrm{B} \le \mathrm{A} \end{array} \qquad \begin{array}{l} \mathrm{Z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ d \text{ even, if } \mathrm{B} = \mathrm{A} + \mathsf{x} \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & 0 & 2\mathrm{Z}+\mathrm{A}+\mathrm{C}+\mathsf{x} & 0 & \mathrm{Z}+\mathrm{B}+\mathrm{C} \\ d & \mathrm{B}+\mathrm{A} & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [3d - 2\mathrm{A} - 2\mathrm{B} - 2\mathrm{C}, d + \mathrm{A} + \mathrm{B} + 2\mathrm{C}, d + \mathrm{A} + \mathrm{B}]$$

$$r = 2d - 3\mathrm{A} - 3\mathrm{B} - 4\mathrm{C},\ s = 2\mathrm{C},\ t = d + \mathrm{A} + \mathrm{B}$$

A valid weight assignment must have $\omega_3(1) = d$ since $\mathrm{C} > 0$. Then we must have $\omega_3(2 \text{ or } 5) = \mathrm{A} + \mathrm{B}$, or if $\mathrm{B} = 0$, then $\omega_2(3) = \mathrm{A}$ is also possible. There are not enough 2's to have $\omega_2(2) = 2\mathrm{Z} + \mathrm{A} + \mathrm{C} + \mathsf{x}$. If we have $\omega_2(3) = 2\mathrm{Z} + \mathrm{A} + \mathrm{C} + \mathsf{x}$, then

$\omega_2(5) = \mathrm{Z} + \mathrm{B} + \mathrm{C}$. If we have $\omega_2(4) = 2\mathrm{Z} + \mathrm{A} + \mathrm{C} + \mathsf{x}$, then $\omega_2(2) = \mathrm{Z} + \mathrm{B} + \mathrm{C}$, and $\mathrm{B} = \mathrm{A}$, $\mathsf{x} = 0$. There are not enough 5's to have $\omega_2(5) = 2\mathrm{Z} + \mathrm{A} + \mathrm{C} + \mathsf{x}$.

Hence there are two possible weight assignments: $(1, 2, 3, 4, 5)$ with sign 1 and $(1, 5, 4, 3, 2)$ with sign $(-1)^d$, which occurs only when $\mathrm{B} = \mathrm{A}$, $\mathsf{x} = 0$. By our parity constraint, this sum is positive. If $\mathrm{B} = 0$, we also have the weight assignment $(1, 3, 4, 5, 2)$ when $\mathrm{A} = \mathrm{B}$, $\mathsf{x} = 0$. The weight sum is odd and hence the tableau is non-zero.

Maximality: The tableau is maximal by Lemma 7.1.5. To see this, note that when row three has the maximum weight d, the largest possible weights for row two are: 0 for the element 1, $\mathrm{Z} + \mathsf{x} + \mathrm{A} + \mathrm{B} + \mathrm{C}$ for the element 2, $2\mathrm{Z} + \mathsf{x} + \mathrm{A} + \mathrm{C}$ for the element 3, $2\mathrm{Z} + \mathrm{B} + \mathrm{C}$ for the element 4, $\mathrm{Z} + \mathrm{A} + \mathrm{B} + \mathrm{C}$ for the element 5. Since $\mathrm{Z} - \mathrm{C} \geq \mathrm{A} \geq \mathrm{B}$, the weight $2\mathrm{Z} + \mathsf{x} + \mathrm{A} + \mathrm{C}$ is the largest. Given this, Lemma 7.1.4 shows that row two is maximal. Moreover rows two and three cannot have elements in common, given the columns remaining after the largest weight elements have been assigned. Hence the tableau is maximal.

<u>Tableau R_3</u>

$$R_3 = \begin{array}{c|cccccccc} \multicolumn{1}{c}{\mathrm{Z}+\mathsf{x}} & \mathrm{Z} & \mathrm{Z} & \mathrm{B} & \mathrm{A} & & & & \\ \hline 5 & 2 & 5 & 4 & 5 & 2 & 5 & 2 \\ 3 & 3 & 4 & 3 & 4 & 3 & 4 & 4 \\ 1 & 1 & 1 & 2 & 2 & & & \end{array} \qquad 0 \leq \mathrm{A} < \mathrm{B} \leq \mathrm{Z} - 1 \qquad \mathrm{Z} = \tfrac{d-\mathsf{x}}{3}, \quad d \equiv \mathsf{x} \pmod 3$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & 2\mathrm{Z}+\mathsf{x}+\mathrm{B}+1 & \mathrm{Z}+\mathrm{A}+2 & 0 \\ d & \mathrm{A}+\mathrm{B} & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\mathrm{A} - 2\mathrm{B} - 3, d + \mathrm{A} + \mathrm{B} + 3, d + \mathrm{A} + \mathrm{B}]$$

$$r = 2d - 3\mathrm{A} - 3\mathrm{B} - 6,\ s = 3,\ t = d + \mathrm{A} + \mathrm{B},$$

We must have $\omega_3(1) = d$. Since $\mathrm{A} < \mathrm{B}$, only the element 3 may have $\omega_2 = 2\mathrm{Z} + \mathsf{x} + \mathrm{B} + 1$. Then $\omega_2(4) = \mathrm{Z} + \mathrm{A} + 2$. Hence there are no other valid weight assignments. Thus R_3 is non-zero.

Maximality: Row two is maximal by Lemma 7.1.2 and row three is maximal by Lemma 7.1.4. Inspection shows that it is not possible to have the non-d weights assigned to the same element. Hence the tableau is maximal by Lemma 7.1.5.

Tableau R_4

$$R_4 = \begin{array}{cccccccccc} \mathsf{x} & \mathrm{Z}-1 & \mathrm{Z}{+}1 & \mathrm{Z} & \mathrm{Z}-\mathrm{A} & \mathrm{Z}-\mathrm{A} & \mathrm{A}-1 & \mathrm{B} & \mathrm{A}{+}1 & \mathrm{C} \\ \hline 2 & 2 & 5 & 5 & 4 & 5 & 5 & 2 & 2 & 5 \\ 3 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 3 & 4 \\ 1 & 1 & 1 & 1 & 2 & 2 & & & & \end{array} \qquad \begin{array}{l} 1 \leq \mathrm{A} \leq \mathrm{Z} \\ \mathrm{B} \leq \mathrm{A} \\ \mathrm{C} \leq \mathsf{x} \end{array} \qquad \begin{array}{l} d + \mathrm{A} + \mathrm{C} \text{ even, if } \mathrm{A} = \mathrm{B} \\ \mathrm{A} + \mathrm{B} \text{ even, if } \mathrm{C} = \mathsf{x} \\ \mathrm{Z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ \mathrm{Z} \geq 2 \end{array}$$

$\omega_{2,3} = \begin{pmatrix} 0 & 0 & d & 2\mathrm{Z}{+}\mathrm{B}{+}\mathrm{C} & 0 \\ d & 2\mathrm{Z}{-}2\mathrm{A} & 0 & 0 & 0 \end{pmatrix}$

$\lambda = [3d - 4\mathrm{Z} - \mathrm{B} - \mathrm{C} + 2\mathrm{A}, d + 2\mathrm{Z} + \mathrm{B} + \mathrm{C}, d + 2\mathrm{Z} - 2\mathrm{A}]$

$r = 2d - 6\mathrm{Z} + 2\mathrm{A} - 2\mathrm{B} - 2\mathrm{C}$, $s = 2\mathrm{A} + \mathrm{B} + \mathrm{C}$, $t = d + 2\mathrm{Z} - 2\mathrm{A}$,

By construction, only the element 1 may have $\omega_3(1) = d$. This means any valid weight assignment must have $\omega_3(2 \text{ or } 3) = 2\mathrm{Z} - 2\mathrm{A}$ (unless $\mathrm{A} = \mathrm{Z}$ in which case the weight assignments are equivalent to those using 2 or 3). Furthermore, the elements appearing in row two must either be both 3 and 4 or both 2 and 5. Now we apply weight-set counting.

Assignment	Sign	Condition
$(1, 2, 3, 4, 5)$	1	
$(1, 2, 4, 3, 5)$	$(-1)^{\mathrm{Z}-\mathrm{A}}$	$\mathrm{B} = \mathrm{A}$, $\mathrm{C} = \mathsf{x}$
$(1, 3, 2, 5, 4)$	$(-1)^{d+\mathrm{B}+\mathrm{C}}$	$\mathrm{B} = \mathrm{A}$
$(1, 3, 5, 2, 4)$	$(-1)^{\mathsf{x}+\mathrm{A}+\mathrm{B}+\mathrm{C}}$	$\mathrm{C} = \mathsf{x}$

By our parity constraints, all these terms are positive. Hence R_4 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. Inspection shows that it is not possible to have the non-d weights assigned to the same element. Hence the tableau is maximal.

Tableau R_5

$$R_5 = \begin{array}{cccccccc} \textsc{y}+\textsc{w}-\textsc{a} & \textsc{y}-\textsc{a} & \textsc{a} & \textsc{a} & \textsc{b} & \textsc{y} & \textsc{y} & \textsc{w} \\ \hline 4 & 5 & 1 & 4 & 5 & 5 & 4 & 5 \\ 3 & 2 & 3 & 2 & 1 & 3 & 2 & 2 \\ 1 & 1 & & & & & & \end{array} \qquad \begin{array}{ll} \textsc{a} \text{ even, if } \textsc{a}=\textsc{b} & \textsc{y} = \frac{d^*}{2} \\ 0 \leq \textsc{b} \leq \textsc{a} < \textsc{y} & 2\textsc{y}+\textsc{w} = d \end{array}$$

$\omega_{2,3} = \left(\begin{smallmatrix} \textsc{b} & d & d & 0 & 0 \\ d-2\textsc{a} & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [2d - \textsc{b} + 2\textsc{a}, 2d + \textsc{b}, d - 2\textsc{a}]$

$r = 2\textsc{a} - 2\textsc{b}$, $s = d + 2\textsc{a} + \textsc{b}$, $t = d - 2\textsc{a}$

Since $\textsc{y} - \textsc{a} > 0$, any weight assignment must have $\omega_3(1) = d - 2\textsc{a}$. If $\omega_2(3) = d$, then $\omega_2(2) = d$ and vice versa. Similarly for the elements 4 and 5, however we may only have $\omega_2(5) = d$ if $\textsc{a} = \textsc{b}$. So unless $\textsc{a} = \textsc{b}$ there is only one valid weight assignment. When $\textsc{a} = \textsc{b}$, we also have $(1, 4, 5, 3, 2)$ which has sign $(-1)^{2d+\textsc{a}}$. Since $\textsc{a}$ is even in this case, the sum is non-zero. Therefore R_5 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_6

$$R_6 = \begin{array}{cccccccccccccc} \textsf{x} & \textsc{z} & \textsc{z} & \textsc{z} & \textsc{z}-\textsc{a} & \textsc{z}-\textsc{a}-1 & \textsc{a} & \textsc{b} & \textsc{a}+1 & \textsc{c} & & & \\ \hline 2 & 2 & 5 & 5 & 4 & 5 & 5 & 2 & 2 & 5 & 2 & 5 & 4 \\ 3 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 3 & 4 & & & \\ 1 & 1 & 1 & 1 & 2 & 2 & & & & & & & \end{array} \qquad \begin{array}{ll} 1 \leq \textsc{a} \leq \textsc{z}-2 & \textsc{z} = \frac{d-\textsf{x}}{3} \\ 0 \leq \textsc{b} < \textsc{a} & d \equiv \textsf{x} \pmod 3 \\ \textsc{c} \leq \textsf{x} & \textsc{z} \geq 3 \end{array}$$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & d & 2\textsc{z}+\textsc{b}+\textsc{c} & 0 \\ d & 2\textsc{z}-2\textsc{a}-1 & 0 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [3d - 4\textsc{z} - \textsc{b} + 2\textsc{a} + 1, d + 2\textsc{z} + \textsc{b} + \textsf{x}, d + 2\textsc{z} - 2\textsc{a} - 1]$

$r = 2d - 6\textsc{z} + 2\textsc{a} - 2\textsc{b} - 2\textsc{c} + 1$, $s = 2\textsc{a} + \textsc{b} + \textsc{c} + 1$, $t = d + 2\textsc{z} - 2\textsc{a} - 1$

Only the element 1 can have ω_3=d and then any weight assignment must have $\omega_2(3) = d$. This forces $\omega_2(4) = 2\textsc{z}+\textsc{a}+1+\textsf{x}$. Hence $\omega_3(4)$=0 which implies $\omega_3(2) > 0$. Thus there are no other valid weight-assignments and R_6 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. Inspection shows that it is non possible to have the non-d weights assigned to the same element. Hence the tableau is maximal.

Tableau R_7

$$R_7 = \begin{array}{cccccccc} \textsc{y}-\textsc{a} & \textsc{y}-\textsc{a} & \textsc{a}{+}1 & \textsc{a} & \textsc{a}-1 & \textsc{y} & \textsc{y}{+}1 & \\ \hline 4 & 5 & 1 & 4 & 5 & 4 & 5 & 1\;4\;5 \\ 3 & 2 & 3 & 2 & 1 & 3 & 2 & \\ 1 & 1 & & & & & & \end{array} \qquad \begin{array}{l} 1 \leq \textsc{a} < \textsc{y} \\ \textsc{y} = \frac{d-1}{2} \\ d \text{ odd} \\ d \geq 5 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} \textsc{a}-1 & d\ d\ 0\ 0 \\ d-1-2\textsc{a} & 0\ 0\ 0\ 0 \end{pmatrix}$$

$$\lambda = [2d + \textsc{a} + 2, 2d + \textsc{a} - 1, d - 2\textsc{a} - 1]$$

$$r = 3,\ s = d + 3\textsc{a},\ t = d - 2\textsc{a} - 1$$

As $\textsc{y} - \textsc{a} > 0$, any weight assignment must have $\omega_3(1) = d - 1 - 2\textsc{a}$. Moreover, only the elements 2 and 3 can have $\omega_2 = d$. Hence there are no other valid weight assignments, so R_7 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_8

$$R_8 = \begin{array}{ccccccc} \textsc{y}-\textsc{a}-1 & \textsc{y}-\textsc{a} & \textsc{a}{+}1 & \textsc{a} & \textsc{a}-1 & \textsc{y} & \textsc{y} \\ \hline 4 & 5 & 1 & 4 & 5 & 4 & 5\;1\;4\;5 \\ 3 & 2 & 3 & 2 & 1 & 3 & 2 \\ 1 & 1 & & & & & \end{array} \qquad \begin{array}{l} 1 \leq \textsc{a} \leq \textsc{y} - 2 \\ \textsc{y} = \frac{d}{2} \\ d \text{ even} \\ d \geq 6 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} \textsc{a}-1 & d\ d\ 0\ 0 \\ d-1-2\textsc{a} & 0\ 0\ 0\ 0 \end{pmatrix}$$

$$\lambda = [2d + \textsc{a} + 2, 2d + \textsc{a} - 1, d - 2\textsc{a} - 1]$$

$$r = 3,\ s = d + 3\textsc{a},\ t = d - 2\textsc{a} - 1$$

Since $\textsc{a} < \textsc{y} - 1$, any weight assignment must have $\omega_3(1) = d - 1 - 2\textsc{a}$. Moreover, the only elements 2 and 3 can both have ω_2= d. Therefore there are no other valid weight assignments, so R_8 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_9

$$R_9 = \begin{array}{cccccccccc} \mathsf{x} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \\ \hline 3 & 4 & 4 & 3 & 2 & 4 & 3 & 4 & 4 & 4 \\ 2 & 2 & 5 & 5 & 5 & 2 & 2 & 2 & 3 & 5 \\ 1 & 1 & 1 & 1 & 3 & 3 \end{array}$$

$$0 \le \text{A} < \text{Z}$$
$$\text{B} + \text{C},\ \text{B} + \text{D} < \text{Z} - \text{A} \qquad \text{Z} = \tfrac{d-\mathsf{x}}{3}$$
$$\text{C} + \text{D} + \text{E} < \text{Z} - \text{A} + \mathsf{x} \qquad d \equiv \mathsf{x} \pmod 3$$
$$0 \le \text{E} \le \mathsf{x} \qquad \mathsf{x} \neq 0$$
$$0 \le \text{D} < \mathsf{x} + \text{B}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{Z+A+B+C+}\mathsf{x} & \text{D} & 0 & 3\text{Z+E} \\ d & 0 & \text{Z+A} & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\text{A} - \text{B} - \text{C} - \text{D} - \text{E} - 2\text{Z}, d + \text{Z} + \text{A} + \text{B} + \text{C} + \text{D} + \text{E}, d + \text{Z} + \text{A}]$$

$$r = 2d - 3\text{A} - 2\text{B} - 2\text{C} - 2\text{D} - 2\text{E} - 3\text{Z},\ s = \text{B} + \text{C} + \text{D} + \text{E},\ t = d + \text{Z} + \text{A}$$

Any valid weight assignment must have $\omega_3(1) = d$. Only the elements 1 and 5 may have $\omega_2 = 3\text{Z} + \text{E}$. If $\omega_2(5) = 3\text{Z} + \text{E}$, then only the element 2 may have $\omega_2 = \mathsf{x} + \text{Z} + \text{A} + \text{B} + \text{C}$ due to the condition $\text{D} < \text{B} + \mathsf{x}$.

Examining the tableau in light of these constraints shows that $(1, 2, 3, 4, 5)$ is the only valid weight assignment. Thus the tableau is non-zero.

Maximality: As the discussion above shows, the tableau weights are as large as possible. Hence by Lemma 7.1.5, this tableau is maximal.

<u>Tableau R_{10}</u>

$$R_{10} = \begin{array}{ccccccccccc} \mathsf{x} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z}-\mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathsf{x} \\ \hline 3 & 4 & 4 & 3 & 3 & 2 & 4 & 3 & 4 & 3 & 4 \\ 2 & 2 & 5 & 5 & 5 & 5 & 3 & 2 & 2 & 5 & 5 \\ 1 & 1 & 1 & 1 & 2 & & & & & & \end{array}$$

$$1 \le \mathrm{A} \le \mathrm{Z} \qquad \mathrm{B} \le \lfloor \tfrac{\mathrm{A}}{2} \rfloor \qquad \mathrm{C}, \mathrm{E} \le \lfloor \tfrac{\mathrm{Z}}{2} \rfloor \qquad \mathrm{D} \le \lceil \tfrac{\mathrm{Z}}{2} \rceil + \lfloor \tfrac{\mathrm{A}}{2} \rfloor \qquad \mathrm{F} \le \lceil \tfrac{\mathrm{A}}{2} \rceil \qquad \mathrm{C} \text{ even}$$

$$\mathrm{D} > \mathrm{C} \qquad \mathrm{Z} + \mathrm{B} > \mathrm{C} + \mathrm{D} \qquad \mathrm{Z} - \mathrm{A} + \mathrm{B} + \mathrm{F} > \mathrm{C} + \mathrm{E} \qquad \mathrm{Z} + \mathrm{F} > \mathrm{D} + \mathrm{E} \qquad \mathrm{Z} = \tfrac{d-\mathsf{x}}{3} \qquad d \equiv \mathsf{x} \pmod 3 \qquad \mathrm{Z} \ge 2$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \mathsf{x}+\mathrm{Z}+\mathrm{D}+\mathrm{E} & \mathrm{C} & 0 & 3\mathrm{Z}-\mathrm{A}+\mathrm{B}+\mathrm{F}+\mathsf{x} \\ d & \mathrm{Z}-\mathrm{A} & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\mathrm{Z} + 2\mathrm{A} - \mathrm{B} - \mathrm{C} - \mathrm{D} - \mathrm{E} - \mathrm{F} - \mathsf{x}, d + \mathrm{Z} - \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + \mathsf{x}, d + \mathrm{Z} - \mathrm{A}]$$

$$r = d + 3\mathrm{A} - 2\mathrm{B} - 2\mathrm{C} - 2\mathrm{D} - 2\mathrm{E} - 2\mathrm{F} - \mathsf{x},\ s = \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + \mathsf{x},\ t = d + \mathrm{Z} - \mathrm{A}$$

Any valid weight assignment must have $\omega_3(1) = d$. Given the bounding parameters, only the element 5 can have $\omega_2 = 2\mathrm{Z} + \mathrm{A} + \mathrm{B} + \mathrm{F} + \mathsf{x}$. When $\omega_2(5)$ is maximal, only the element 2 may have $\omega_2 = \mathsf{x} + \mathrm{Z} + \mathrm{D} + \mathrm{E}$.

From this we find that the valid weight assignments are $(1, 2, 3, 4, 5)$ with sign 1, and $(1, 2, 4, 3, 5)$ with sign $(-1)^{\mathrm{C}} = 1$.

Maximality: The discussion above shows that the row two weights are maximized. Hence by Lemma 7.1.5, this tableau is maximal.

<u>Tableau R_{11}</u>

$$R_{11} = \begin{array}{ccccccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & & & \\ \hline 3 & 5 & 3 & 5 & 1 & 5 & 3 & 1 & 5 \\ 4 & 2 & 4 & 2 & 2 & 1 & 4 & 2 & 4 \\ 1 & 1 & & & & & & & \end{array}$$

$$0 < \mathrm{A}, \mathrm{B} \qquad \mathrm{C} \le d - \mathrm{A} - 2 \qquad \mathrm{D} \le d - \mathrm{B} - \mathrm{E} - 1 \qquad 0 < \mathrm{F} \le \mathrm{E}$$

$$\mathrm{E} + \mathrm{F} \le d - \mathrm{A} - \mathrm{B} - 1 \qquad \mathrm{B} + \mathrm{D} > \mathrm{F} \qquad \mathrm{A} + \mathrm{C} \ge \mathrm{E} + \mathrm{F}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} \mathrm{F} & \mathrm{B}+\mathrm{D}+\mathrm{E}+1 & \mathrm{A}+\mathrm{C}+2 & 0 & 0 \\ \mathrm{A}+\mathrm{B} & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [5d - 2\mathrm{A} - 2\mathrm{B} - \mathrm{C} - \mathrm{D} - \mathrm{E} - \mathrm{F} - 3, \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + 3, \mathrm{A} + \mathrm{B}]$$

$$r = 5d - 3\mathrm{A} - 3\mathrm{B} - 2\mathrm{C} - 2\mathrm{D} - 2\mathrm{E} - 2\mathrm{F} - 6,\ s = \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + 3,\ t = \mathrm{A} + \mathrm{B}$$

Any valid weight assignment must have $\omega_3(1) = \mathrm{A} + \mathrm{B}$ and hence $\omega_2(1) = \mathrm{F}$. This implies $\omega_2(2) > 0$. But to have exactly three non-zero weights in row two, we then must have $\omega_2(4) > 0$. Hence no other weight assignments are possible and hence the tableau is non-zero.

Maximality: The possible row two weights, given the maximization of row three, are $\mathrm{E} + \mathrm{F} + 1$ for the element 1, $\mathrm{B} + \mathrm{D} + \mathrm{E} + 1$ for the element 2, $\mathrm{A} + \mathrm{C} + 1$ for the element 3, $\mathrm{A} + \mathrm{C} + 2$ for the element 4, and $\mathrm{B} + \mathrm{D} + \mathrm{F} + 1$ for the element 5. Given the conditions on these parameters we see that the weights for 2 and 4 are the largest. Hence by Lemma 7.1.5, this tableau is maximal.

<u>Tableau R_{12}</u>

$$R_{12} = \begin{array}{cccccccc} \mathrm{Y}-1+\mathrm{W} & \mathrm{Y} & & \mathrm{Y} & \mathrm{Y} & \mathrm{Y} & \mathrm{A} & \mathrm{W} \\ \hline 1 & 1 & & 5 & 4 & 1 & 5 & 5 \\ 3 & 3 & & 3 & 2 & 2 & 4 & 2 \\ 4 & & & & & & & \end{array} \qquad \begin{array}{ll} 0 < \mathrm{A} \le d - \mathrm{Y} - 1 & \mathrm{Y} = \frac{d^*}{2} \\ \mathrm{Y} \text{ even, if } \mathrm{Y} = \mathrm{A} & 2\mathrm{Y} + \mathrm{W} = d \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & d & \mathrm{A} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d - \mathrm{A} - 1, 2d + \mathrm{A}, 1]$$

$$r = d - 2\mathrm{A} - 1,\ s = 2d + \mathrm{A} - 1,\ t = 1$$

Any valid weight assignment must have two of the elements 1, 3, and 4 having non-zero weight in row two. Hence at least one of them must have $\omega_2 = d$. Moreover, we can not have a row two weight of d for both of these elements. If $\omega_2(1) = d$ we may not have $\omega_2(2) = d$, so the only option is $\omega_2(5) = d$ when $\mathrm{A} = \mathrm{Y}$. This corresponds to a weight assignment of $(3, 1, 5, 4, 2)$ with sign $(-1)^{2d+\mathrm{A}}$ which equals 1 due to the parity constraint. If $\omega_2(3) = d$ we may not have $\omega_2(5) = d$, so the only option is $\omega_2(2) = d$. This corresponds to a weight assignment of $(1, 2, 3, 4, 5)$ with sign 1. When $\omega_2(4) = d$, neither the element 2 nor the element 5 may have $\omega_2 = d$, so there is no weight assignment with this option. Hence the weight sum is always positive and thus R_{12} is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_{13}

$$R_{13} = \begin{array}{ccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \\ \hline 1 & 3 & 1 & 3 & 1 \\ 2 & 4 & 4 & 2 & 2 \\ 5 & 5 & & & \end{array} \qquad \begin{array}{ll} \text{A}+\text{D}+\text{E} \le d & \text{D} \ge \text{C} \\ \text{A}+\text{B} \le d & \text{A}+\text{E} \ge \text{B} \\ \text{B}+\text{D} \le d & \text{If C} = \text{D then A}+\text{B}+\text{C}+\text{D}+\text{E even} \\ \text{A, B, C, D, E} > 0 & \text{If B} = \text{A}+\text{E then A}+\text{B}+\text{C}+\text{D}+\text{E even} \end{array}$$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A+D+E} & 0 & \text{B+C} & 0 \\ 0 & 0 & 0 & 0 & \text{A+B} \end{smallmatrix}\right)$

$\lambda = [5d - 2\text{A} - 2\text{B} - \text{C} - \text{D} - \text{E}, \text{A} + \text{B} + \text{C} + \text{D} + \text{E}, \text{A} + \text{B}]$

$r = 5d - 3\text{A} - 3\text{B} - 2\text{C} - 2\text{D} - 2\text{E},$, $s = \text{C} + \text{D} + \text{E}$, $t = \text{A} + \text{B}$

Since $\text{A}, \text{B} > 0$ any valid weight assignment must have $\omega_3(5) = \text{A} + \text{B}$. We must have $\omega_2(1 \text{ or } 2) > 0$. If $\omega_2(2) > 0$, then $\omega_2(4) > 0$; if $\omega_2(1) > 0$, then $\omega_2(3) > 0$.

This shows the only valid weight assignments are $(1, 2, 3, 4, 5)$ and $(2, 1, 4, 3, 5)$. The second one only occurs when $\text{D} = \text{C}$ or $\text{B} = \text{A} + \text{E}$. If that happens, the sign is $(-1)^{\text{A+B+C+D+E}} = 1$. Hence R_{13} is non-zero.

Maximality: The possible maximal weights for row two are: $\text{A} + \text{C} + \text{E}$ for the element 1; $\text{A} + \text{D} + \text{E}$ for the element 2; $\text{B} + \text{D}$ for the element 3; $\text{B} + \text{C}$ for the element 4; and 0 for the element 5. By our parameter conditions, $\text{A} + \text{D} + \text{E}$ is the largest. Hence by Lemma 7.1.5 the tableau is maximal.

Tableau R_{14}

$$R_{14} = \begin{array}{cccccc} d-3 & 2 & d-4 & & & \\ \hline 1 & 4 & 1 & 5 & 1\,5\,1 \\ 2 & 2 & 2 & 3 & 3\,4\,4 \\ 3 & & & & \end{array} \qquad d \ge 5$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & d-3 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d, 2d-1, 1]$$

$$r = d+1,\ s = 2d-2,\ t = 1$$

For $d > 5$, any valid weight assignment must have $\omega_2(2) = d$. This forces $\omega_2(3) = d-3$ and $\omega_2(4) = 2$. Hence there are no other valid weight assignments. When $d = 5$, we may have $\omega_2(1) = d$ or $\omega_2(1) = d-3$ as well. Then the possible weight assignments are $(1,2,3,4,5)$, $(3,2,1,5,4,)$, and $(3,1,2,5,4)$. Since there is an odd number, this sum is non-zero and thus so is R_{14}.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau R_{15}

$$R_{15} = \begin{array}{c|ccccc} \multicolumn{6}{l}{\mathsf{z}+\mathsf{x}\ \mathsf{z}\ \mathsf{z}} \\ \hline 2 & 5 & 5 & 2 & 2 & 2 \\ 3 & 4 & 3 & 4 & 3 & 3 \\ 1 & 1 & 1 & & & \end{array} \qquad \begin{array}{l} \mathsf{z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ \mathsf{z} \geq 2 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & 2\mathsf{z}+\mathsf{x}+2 & \mathsf{z}+1 & 0 \\ d & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d-3, d+3, d]$$

$$r = 2d-6,\ s = 3,\ t = d$$

Since only the element 1 may have $\omega_3 = d$ and only the element 3 may have $\omega_2 = 2\mathsf{z}+\mathsf{x}+2$, there are no other valid weight assignments. Hence the tableau is non-zero.

Maximality: Given the row three weight of d, the row two weights are as large as possible. Thus by Lemma 7.1.5 this tableau is maximal.

Tableau R_{16}

$$R_{16} = \begin{array}{cc} d-2 & \textsc{a} \\ \hline 1 & 5\,5\,5\,3\,1 \\ 2 & 4\,4\,2\,4\,2 \\ 3 & 3 \end{array} \qquad \begin{array}{l} 1 \leq \textsc{a} \leq d-4 \\ d \geq 5 \end{array}$$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & \textsc{a}+2 & 0 \\ 0 & 0 & d-1 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [3d - \textsc{a} - 1, d + \textsc{a} + 2, d - 1]$

$r = 2d - 2\textsc{a} - 3,\ s = \textsc{a} + 3,\ t = d - 1$

Any valid weight assignment must have $\omega_3(3) = d - 1$ and $\omega_2(2) = d$. This forces $\omega_2(4) = \textsc{a}+2$. Hence there are no other valid weight assignments and R_{16} is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_{17}

$$R_{17} = \begin{array}{ccc} \mathsf{x}\ \textsc{z}\ \textsc{z}\ \textsc{z} & \textsc{z}-1 & \textsc{a}\ \textsc{z}\ \textsc{b} \\ \hline 2\,2\,5\,5\,2 & 5 & 2\,2\,5\,2\,5\,4 \\ 3\,4\,4\,3\,3 & 3 & 4\,3\,4 \\ 1\,1\,1\,1\,4 & & \end{array} \qquad \begin{array}{l} 0 \leq \textsc{a} \leq \textsc{z} - 2 \\ \textsc{b} \leq \mathsf{x} \end{array} \qquad \begin{array}{l} \textsc{z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ \textsc{z} \geq 2 \end{array}$$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & d & 2\textsc{z}+\textsc{a}+\textsc{b} & 0 \\ d & 0 & 0 & 1 & 0 \end{smallmatrix}\right)$

$\lambda = [3d - 2\textsc{z} - \textsc{a} - 1, d + 2\textsc{z} + \textsc{a} + \mathsf{x}, d + 1]$

$r = 2d - 4\textsc{z} - 2\textsc{a} - 2\textsc{b} - 1,\ s = 2\textsc{z} + \textsc{a} + \textsc{b} - 1,\ t = d + 1$

Only the element 1 can have $\omega_3{=}d$ and then any weight assignment must have $\omega_2(3) = d$. This forces $\omega_2(4) = 2\textsc{z}{+}\textsc{a} + 1 + \mathsf{x}$. Thus there are no other valid weight assignments and R_{17} is non-zero.

Maximality: The tableau is maximal by Lemma 7.1.3.

Tableau R_{18}

$$R_{18} = \begin{array}{ccccc} \mathsf{x} & \mathrm{z} & \mathrm{z} & \mathrm{z} & \mathrm{z} \\ \hline 5 & 2 & 4 & 4 & 2 \\ 3 & 3 & 3 & 5 & 3 \\ 1 & 1 & 1 & 1 & 5 \end{array} \qquad \mathsf{x} \text{ even}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & d & 0 & \mathrm{z} \\ d & 0 & 0 & 0 & \mathrm{z} \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\mathrm{z}, d + \mathrm{z}, d + \mathrm{z}]$$

$$r = 2d - 3\mathrm{z},\ s = 0,\ t = d + \mathrm{z}$$

Only the elements 1 and 3 may have a weight of d. Hence the only valid weight assignments are $(1, 2, 3, 4, 5)$ and $(3, 2, 1, 4, 5)$, both of which have sign 1 since x is even. Hence R_{18} is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_{19}

$$R_{19} = \begin{array}{cccccc} d-2 & & & & d-3 & \\ \hline 1 & 5 & 5 & 3 & 5 & 1 \\ 2 & 4 & 2 & 2 & 4 & 4 \\ 3 & 3 & & & & \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & d-1 \\ 0 & 0 & d-1 & 0 \end{smallmatrix}\right)$$

$$\lambda = [2d + 2, 2d - 1, d - 1]$$

$$r = 3,\ s = d,\ t = d - 1$$

Any valid weight assignment must have $\omega_3(3) = d$. Then $\omega_2(2) = d$. Hence there are no other weight assignments possible. Thus R_{19} is non-zero. **Maximality:** This tableau is maximal by Lemma 7.1.3.

7.6 Tableaux for $c = 6$

Tableau S_1

$$S_1 = \begin{array}{cccccc} \mathsf{x}+\textsc{z} & \mathsf{x}+\textsc{z} & \textsc{z} & \textsc{z} & \textsc{a} & \textsc{a} \\ \hline 5 & 6 & 5 & 6 & 5 & 6 \\ 4 & 3 & 3 & 4 & 3 & 4 \\ 1 & 2 & 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} 0 \le \textsc{a} \le \textsc{z} \\ \textsc{z}+\mathsf{x} \text{ even, if } \textsc{a} = 0 \end{array} \qquad \begin{array}{l} \textsc{z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & 0 & 2\textsc{z}+\textsc{a}+\mathsf{x} & 2\textsc{z}+\textsc{a}+\mathsf{x} & 0 & 0 \\ d & \textsc{z}+2\textsc{a}+\mathsf{x} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [4d - 2\textsc{z} - 2\mathsf{x} - 4\textsc{a}, d + \textsc{z} + 2\textsc{a} + \mathsf{x}, d + \textsc{z} + 2\textsc{a} + \mathsf{x}]$$

$$r = 2d - 2\mathsf{x} - 6\textsc{a},\ s = 0,\ t = d + 2\textsc{a}+\textsc{z} + \mathsf{x}$$

The construction of S_1 means that any valid weight assignment with $\textsc{a} > 0$ must have $\omega_i(j) > 0$ if and only if $\omega_i(k) > 0$ for the pairs $(j,k) = (1,2)$, $(3,4)$, or $(5,6)$. These constraints show that the only valid weight assignments are those that interchange complete rows in the body of S_1. Since the length of these rows $t = 4\textsc{z}+2\mathsf{x}+2\textsc{a}$ is even, all valid weight assignments are positive and hence S_1 is non-zero.

When $\textsc{a} = 0$ we must have $\omega_3(1) = d$ and $\omega_2(3,4) > 0$ or $\omega_2(5,6) > 0$. Hence the valid weight assignments are: $(1,2,3,4,5,6)$ with sign 1; $(1,6,3,4,5,2)$ with sign $(-1)^{\textsc{z}+\mathsf{x}}$; $(1,2,5,6,3,4)$ with sign $(-1)^{4\textsc{z}+2\mathsf{x}}$; and $(1,3,5,6,2,4)$ with sign $(-1)^{5\textsc{z}+3\mathsf{x}}$. Since $\textsc{z} + \mathsf{x}$ is even, these assignments are all positive. Hence the tableau is non-zero.

Maximality: By Lemma 7.1.5 this tableau is maximal.

Tableau S_2

$$S_2 = \begin{array}{c|cccccc} d-1 & \textsc{a} & & & & & \\ \hline 4 & 2 & 2 & 2 & 4 & 2 & 3 \\ 5 & 1 & 5 & 1 & 1 & 1 & 1 \\ 6 & 3 & 3 & 6 & 3 & & \end{array} \qquad \begin{array}{l} 0 \le \textsc{a} \le d-4 \\ \textsc{a} \equiv d \pmod 2 \\ d \ge 4 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} \textsc{a}+4 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & \textsc{a}+2 & 0 & 0 & d \end{pmatrix}$$

$$\lambda = [4d - 2\textsc{a} - 6, d + \textsc{a} + 4, d + \textsc{a} + 2]$$

$$r = 3d - 3\textsc{a} - 10,\ s = 2,\ t = d + \textsc{a} + 2$$

Any valid weight assignment must have $\omega_2(1) = \text{A} + 4$ since if $\omega_2(1) = 0$ then we must have $\omega_2(2,3) = (\text{A} + 4, d)$ which is not possible. If $\omega_2(1) = \text{A} + 4$, then $\omega_2(5) = d$. Thus the only valid weight assignments are $(1,2,3,4,5,6)$ with sign 1 and $(1,3,2,6,5,4)$ with sign $(-1)^{d+\text{A}+2}$. Since $\text{A} \equiv d \pmod 2$ this sum is positive. Hence S_2 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. Since it is not possible for the non-d element of row three to have a row two weight of $\text{A} + 2$, the tableau is maximal.

Tableau S_3

$$S_3 = \begin{array}{c|cccccccc} \multicolumn{9}{l}{d-4 \;\; \text{A} \;\; 2 \;\; 2} \\ \hline 5 & 1 & 1 & 3 & 5 & 5 & 5 & 1 & 1 \\ 2 & 4 & 2 & 4 & 4 & 2 & 4 & 2 & 4 \\ 6 & 3 & 6 & 6 & 3 & 3 & & & \end{array} \qquad \begin{array}{l} 0 \le \text{A} \le d-5 \\ d \ge 5 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & \text{A}+5 & 0 & 0 \\ 0 & 0 & \text{A}+2 & 0 & 0 & d \end{smallmatrix}\right)$$

$$\lambda = [4d - 2\text{A} - 7, d + \text{A} + 5, d + \text{A} + 2]$$

$$r = 3d - 3\text{A} - 12,\ s = 3,\ t = d + \text{A} + 2$$

Any valid weight assignment must have $\omega_3(6) = d$ and $\omega_3(3) = \text{A}+2$ (or $\omega_3(5) = 2$ if $\text{A} = 0$). Then only the element 2 may have $\omega_2 = d$ (or 4 if $\text{A} = d - 5$). Moreover if $\omega_2(2) = d$, then $\omega_2(4) = \text{A} + 5$. Hence the only valid weight assignments are $(1,2,3,4,5,6)$ with sign 1 and, when $\text{A} = 0$, $(1,2,5,4,3,6)$ with sign $(-1)^2$. Therefore the weight sum is positive and S_3 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. Since the non-d element of row two has weight $\text{A} + 5$ and there are only two copies of the element 3 available for row two, the tableau is maximal.

Tableau S_4

$$S_4 = \begin{array}{c|c|c} d-3 & \mathrm{A} & 2 \\ \hline 5 & 1\ 5\ 3\ 5\ 1\ 1\ 1\ 1\ 1 & \\ 2 & 4\ 2\ 4\ 4\ 2\ 2\ 3\ 4\ 4 & \\ 6 & 3\ 6\ 6\ 3\ 3 & \end{array} \qquad \begin{array}{l} 0 \le \mathrm{A} \le d-5 \\ d \ge 5 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 1 & \mathrm{A}+5 & 0 & 0 \\ 0 & 0 & \mathrm{A}+2 & 0 & 0 & d \end{smallmatrix}\right)$$

$$\lambda = [4d - 2\mathrm{A} - 8, d + \mathrm{A} + 6, d + \mathrm{A} + 2]$$

$$r = 3d - 3\mathrm{A} - 14,\ s = 4,\ t = d + \mathrm{A} + 2$$

Any valid weight assignment must have $\omega_3(6) = d$. Now $\omega_3(6) = d$ implies $\omega_3(3) = \mathrm{A}+2$ and $\omega_2(3) = 1$. Unless $\mathrm{A} = d-5$, we must have $\omega_2(2) = d$ and so $\omega_2(4) = \mathrm{A}+5$. When $\mathrm{A} = d-5$ we may have $\omega_2(1) = d$ or $\omega_2(4) = d$. However, if $\omega_2(1) = d$, then there is no element with $\omega_2 = \mathrm{A}+5$. If $\omega_2(4) = d$, then $\omega_2(2) = \mathrm{A}+5$. This shows the only weight assignment is $(1, 2, 3, 4, 5, 6)$. Hence S_4 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5 and the discussion above.

Tableau S_5

$$S_5 = \begin{array}{c|c|c|c|c|c|c|c|c} \mathsf{x}+\mathrm{z}' & \mathsf{x}+\mathrm{z}' & \mathrm{z}' & \mathrm{z}' & \mathrm{z}' & \mathrm{z}' & \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \hline 5 & 6 & 5 & 6 & 5 & 6 & 6 & 5 & 1 \\ 4 & 3 & 3 & 4 & 3 & 4 & 2 & 4 & 3 \\ 1 & 2 & 1 & 1 & 2 & 2 & & & \end{array}$$

$0 < \mathrm{A}, \mathrm{B}, \mathrm{C} \le d - e$

A even

C even if $\mathrm{A} = \mathrm{B}$

B even if $\mathrm{A} = \mathrm{C}$

$\mathrm{B}, \mathrm{C} \ge \mathrm{A}$

$\mathrm{z}' = \frac{e-\mathsf{x}}{3}$

$e \equiv \mathsf{x} \pmod 3$

$3 \le e < d$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \mathrm{A} & e+\mathrm{C} & e+\mathrm{B} & 0 & 0 \\ e & e & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [6d - 4e - \mathrm{A} - \mathrm{B} - \mathrm{C}, 2e + \mathrm{A} + \mathrm{B} + \mathrm{C}, 2e]$$

$$r = 6d - 6e - 2\mathrm{A} - 2\mathrm{B} - 2\mathrm{C},\ s = \mathrm{A} + \mathrm{B} + \mathrm{C},\ t = 2e$$

The construction of S_5 forces the following constraints on any valid weight assignment.

- $\omega_3(1) > 0 \iff \omega_3(2) > 0.$
- $\omega_3(3) > 0 \iff \omega_3(4) > 0.$
- $\omega_3(5) > 0 \iff \omega_3(6) > 0.$
- If $\omega_2(j) = \text{A}$, then $\omega_3(j) > 0.$
- If $\omega_2(1) = \text{A}$, then $\omega_2(5,6) > 0$ and $\text{A} = \text{C}$.
- If $\omega_2(2) = \text{A}$, then $\omega_2(3,4) > 0.$
- If $\omega_2(3) = \text{A}$, then $\omega_2(5,6) > 0$ and $\text{A} = \text{C}$.
- If $\omega_2(4) = \text{A}$, then $\omega_2(1,2) > 0$ and $\text{A} = \text{B}$.
- If $\omega_2(5) = \text{A}$, then $\omega_2(1,2) > 0$ and $\text{A} = \text{B}$.
- If $\omega_2(6) = \text{A}$, then $\omega_2(3,4) > 0.$

From this we can derive a signed weight table.

Assignment	Sign	Condition
$(1,2,3,4,5,6)$	1	
$(2,1,6,5,4,3)$	$(-1)^{2e+\text{A}+\text{B}+\text{C}}$	$\text{A} = \text{C}$
$(3,4,1,2,5,6)$	$(-1)^{2e+\text{C}}$	$\text{A} = \text{B}$
$(4,3,6,5,1,2)$	$(-1)^{4e+\text{A}+\text{B}}$	$\text{A} = \text{C}$
$(5,6,3,4,1,2)$	$(-1)^{2e+\text{A}}$	
$(6,5,1,2,3,4)$	$(-1)^{4e+\text{B}+\text{C}}$	$\text{A} = \text{B}$

Computing the weight sum we obtain $3+3(-1)^{\text{A}} = 6$ when $\text{A} = \text{B} = \text{C}$ as A is always even. For $\text{A} = \text{B} \neq \text{C}$, the sum is $2 + 2(-1)^{\text{C}}$. This is non-zero as C is even when $\text{A} = \text{B}$. For $\text{A} = \text{C} \neq \text{B}$, we have $2 + 2(-1)^{\text{B}}$. This is non-zero as B even when $\text{A} = \text{C}$. Finally if $\text{A} \neq \text{B}$, C the sum is $1 + (-1)^{\text{A}}$ which is non-zero. Hence S_5 is non-zero.

Maximality: By Lemma 7.1.4, row three is maximal. Since B, $\mathrm{C} \geq \mathrm{A}$, row two is maximal by Lemma 7.1.4. As the number of row three elements available for row two is bounded by A, this is the largest weight and hence the tableau is maximal.

<u>Tableau S_6</u>

$$S_6 = \begin{array}{cccccccccc} \mathrm{x}+\mathrm{z}' & \mathrm{x}+\mathrm{z}' & \mathrm{z}' & \mathrm{z}' & \mathrm{z}' & \mathrm{z}'-1 & \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \hline 3 & 4 & 4 & 3 & 3 & 4 & 2 & 3 & 4 \\ 5 & 6 & 5 & 6 & 5 & 6 & 6 & 5 & 1 \\ 1 & 2 & 1 & 1 & 2 & 2 & & & \end{array}$$

$0 < \mathrm{A},\ \mathrm{B},\ \mathrm{C} \leq d - e$	$\mathrm{z}' = \frac{e-\mathrm{x}}{3}$
$\mathrm{A},\ \mathrm{B} \geq \mathrm{C}$	$e \equiv \mathrm{x} \pmod 3$
$\mathrm{B} \geq \mathrm{A} - 1$	$3 \leq e \leq d$

$\omega_{2,3} = \begin{pmatrix} \mathrm{C} & 0 & e+\mathrm{B} & e+\mathrm{A}-1 & 0 & 0 \\ e & e-1 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\lambda = [6d - 4e - \mathrm{A} - \mathrm{B} - \mathrm{C} + 2, 2e + \mathrm{A} + \mathrm{B} + \mathrm{C} - 1, 2e - 1]$

$r = 6d - 6e - 2\mathrm{A} - 2\mathrm{B} - 2\mathrm{C} + 3,\ s = \mathrm{A} + \mathrm{B} + \mathrm{C},\ t = 2e - 1$

For any valid weight assignment we can have $\omega_3(j) = e$ only for $j \in \{1, 3, 5\}$. Moreover, $\omega_3(j) = e$ if and only if $\omega_3(j+1) = e - 1$. If $\omega_3(j) = e$ then $\omega_2(j) = \mathrm{C}$. If $\omega_2(1) = \mathrm{C}$, then $\omega_2(5, 6) > 0$. If $\omega_2(3) = \mathrm{C}$, then $\omega_2(1, 2) > 0$ and $\mathrm{B} = \mathrm{C}$. If $\omega_2(5) = \mathrm{C}$, then $\omega_2(1, 2) > 0$ and $\mathrm{B} = \mathrm{C}$.

This means $(1, 2, 3, 4, 5, 6)$ is the only valid weight assignment when $\mathrm{B} \neq \mathrm{C}$. If $\mathrm{B} = \mathrm{C}$, then we additionally have weight assignments $(3, 4, 1, 2, 5, 6)$ and $(5, 6, 1, 2, 3, 4)$. In either case the weight sum is odd and hence non-zero. Thus S_6 is non-zero.

Maximality: By Lemma 7.1.4 row three is maximal. Since A, $\mathrm{B} \geq \mathrm{C}$, row two is maximal by Lemma 7.1.4. As the number of row three elements available for row two is bounded by C, this is the largest weight and the tableau is maximal.

<u>Tableau S_7</u>

$$S_7 = \begin{array}{cccccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & & & & \mathrm{E} \\ \hline 1 & 4 & 3 & 1 & 1 & 1 & 3 & 5 \\ 2 & 2 & 2 & 4 & 2 & 2 & 4 & 6 \\ 3 & 3 & & & & & & \end{array}$$

$\mathrm{A} + \mathrm{B} + \mathrm{C} = d - 2$	$\mathrm{A}, \mathrm{B}, \mathrm{D} > 0$
$\mathrm{B} + \mathrm{D} \leq d - 2$	$0 \leq \mathrm{E} \leq d$
$\mathrm{A} + \mathrm{D} < d - 2$	E even

$\omega_{2,3} = \begin{pmatrix} 0 & d & 0 & \mathrm{D}+1 & 0 & \mathrm{E} \\ 0 & 0 & \mathrm{A}+\mathrm{B} & 0 & 0 & 0 \end{pmatrix}$

$$\lambda = [5d - \mathrm{A} - \mathrm{B} - \mathrm{D} - \mathrm{E} - 1, d + \mathrm{D} + \mathrm{E} + 1, \mathrm{A} + \mathrm{B}]$$

$$r = 4d - \mathrm{A} - \mathrm{B} - 2\mathrm{D} - 2\mathrm{E} - 2,\ s = \mathrm{C} + \mathrm{D} + \mathrm{E} + 3,\ t = \mathrm{A} + \mathrm{B}$$

Any valid weight assignment must have $\omega_2(5 \text{ or } 6) = \mathrm{E}$ and $\omega_3(3) = \mathrm{A} + \mathrm{B}$. Since $\mathrm{A} + \mathrm{D} < d - 2$, we must have $\omega_2(2) = d$. Then, as $\mathrm{D} > 0$ we must have $\omega_2(4) = \mathrm{D} + 1$. Hence the only valid weight assignments are $(1, 2, 3, 4, 5, 6)$ with sign 1 and $(1, 2, 3, 4, 6, 5)$ with sign $(-1)^{\mathrm{E}}$. Since is E is even, this weight sum is positive. Hence S_7 is non-zero.

Maximality: Inspection shows that rows two and three are maximal. Since $\mathrm{D} > 0$ we cannot have any 3's in a maximal row two. Thus S_7 is maximal by Lemma 7.1.5.

<u>Tableau S_8</u>

$$S_8 = \begin{array}{ccccccccccc} & & d-4 & & d-2 & d-4 & & 2 & & & \\ \hline 6 & 6 & 6 & 3 & 2 & 1 & 2 & 3 & 6 & 6 & 2 \\ 3 & 4 & 3 & 4 & 4 & 5 & 5 & 5 & 1 & 5 & 1 \\ 1 & 1 & & & & & & & & & \end{array} \qquad d \geq 5$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 2 & 0 & d-3 & d & d & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d - 1, 3d - 1, 2]$$

$$r = 0,\ s = 3d - 3,\ t = 2$$

Examining the tableau shows that we can only have $\omega_2(j, k) = (d, d)$ for $(j, k) = (2, 3)$, $(2, 6)$ or $(4, 5)$. Also, we must have $\omega_{2,3}(1 \text{ or } 6) > 0$, so $\omega_2(2, 6) = (d, d)$ and $\omega_2(1) = 2$. This is possible only when $d = 6$. Hence the valid weight assignments are: $(1, 2, 3, 4, 5, 6)$ with sign 1; $(6, 4, 1, 2, 3, 5)$ with sign $(-1)^{2d+4}$; $(1, 4, 3, 2, 6, 5)$ with sign $(-1)^{3d-1}$ when $d = 6$; and $(6, 2, 1, 4, 5, 3)$ with sign $(-1)^3$ when $d = 5$. In all cases, the weight sum is positive and hence S_8 is non-zero.

Maximality: This tableau is not maximal since $\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & d-1 & d & d & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$. However, this weight is zero in $\mathbf{q}_T$. This tableau cannot be put in maximal form, hence we will need to prove directly that it is disjoint from the requisite tableaux. This will be done in Section 8.7.

Tableau S_9

$$S_9 = \begin{array}{l} \scriptstyle \text{A B C D E F G} \\ \hline 1\ 5\ 1\ 5\ 3\ 6\ 6\ 3\ 6 \\ 2\ 4\ 2\ 2\ 5\ 4\ 5\ 4\ 2 \\ 3\ 3 \end{array} \qquad \begin{array}{l} \text{A} + \text{C} + \text{D} = d - 1 \\ \text{B} + \text{F} = d - 1 \\ \text{F} + \text{G} \leq d - 1 \\ \text{A} + \text{B} + \text{E} \leq d - 1 \\ \text{B} + \text{D} + \text{E} + \text{G} \leq d - 1 \\ \text{A, B} > 0 \\ \text{E, G} > 0 \text{ or E} = 0 \text{ and G even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & d & \text{E+G} & 0 \\ 0 & 0 & \text{A+B} & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [4d - \text{A} - \text{B} - \text{E} - \text{G}, 2d + \text{E} + \text{G}, \text{A} + \text{B}]$$

$$r = 2d - \text{A} - \text{B} - 2\text{E} - 2\text{G},\ s = \text{C} + \text{D} + \text{E} + \text{F} + \text{G} + 2,\ t = \text{A} + \text{B}$$

Any valid weight assignment must have $\omega_2(3) = \text{A} + \text{B}$ because $\text{A}, \text{B} > 0$. Only the elements 2 and 4 may simultaneously have $\omega_2 = d$. Then $\omega_2(5) = \text{E} + \text{G}$ unless E or G is 0. If $\text{E} = 0$ we may also have $\omega_2(6) = \text{G}$. Hence the weight assignments are are $(1, 2, 3, 4, 5, 6)$ with sign 1 and, if $\text{E} = 0$, $(1, 2, 3, 4, 6, 5)$ with sign $(-1)^{\text{G}} = 1$. Thus S_9 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5 since $\text{G} > 0$.

Tableau S_{10}

$$S_{10} = \begin{array}{l} 2 \qquad \scriptstyle \text{A} \\ \hline 1\ 5\ 5\ 1\ 6\ 5 \\ 2\ 4\ 2\ 2\ 4\ 3 \\ 3\ 3 \end{array} \qquad \begin{array}{l} 2 \leq \text{A} \leq 3 \\ d = 4 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 4 & 1 & \text{A+1} & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [15 - \text{A}, \text{A} + 6, 3]$$

$$r = 9 - 2\text{A},\ s = \text{A} + 3,\ t = 3$$

Any valid weight assignment must have $\omega_3(3) = 3$. Other than 3, the only elements that can have $\omega_2 = d$ is 2, or 4 if $\text{A} = 3$. Thus there are no other valid weight assignments possible and the tableau is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau S_{11}

$$S_{11} = \begin{array}{cccccc} d-1 & & d-3 & d-1 & & \\ \hline 1 & 5 & 6 & 1 & 6 & 5 \\ 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & & & & & \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & d-2 & d & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d+1, 3d-2, 1]$$

$$r = 3,\ s = 3d-3,\ t = 1$$

Any valid weight assignment must have two of the elements 1, 2, and 3 with $\omega_2 > 0$. Since there are not enough 1's in the body for this to happen, we must have $\omega_2(2) = d$ and $\omega_2(3) = d-2$. This force $\omega_2(4) = d$. Hence there are no other valid weight assignments. Thus the tableau is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau S_{12}

$$S_{12} = \begin{array}{ccccccc} \text{A} & \text{B} & & & & & \text{C} \\ \hline 3 & 3 & 1 & 1 & 1 & 3 & 5 \\ 2 & 2 & 4 & 2 & 2 & 4 & 6 \\ 4 & & & & & & \end{array} \qquad \begin{array}{l} 0 \le \text{A} \le d-3 \\ 0 \le \text{B} \le d-2 \\ \text{A} \ge \text{B} \\ 0 \le \text{C} \le d \\ \text{C even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A}+3 & 0 & \text{B}+1 & 0 & \text{C} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [6d - \text{A} - \text{B} - \text{C} - 5, \text{A} + \text{B} + \text{C} + 4, 1]$$

$$r = 6d - 2\text{A} - 2\text{B} - 2\text{C} - 9,\ s = \text{A} + \text{B} + \text{C} + 3,\ t = 1$$

Any valid weight assignment must have $\omega_2(5 \text{ or } 6) = \text{C}$. We also must have

$\omega_2(2) = \text{A} + 3$ since $\text{A} \geq \text{B}$. Then $\omega_2(4) = \text{B} + 1$ unless $\text{B} = 0$, in which case $\omega_2(3) = 1$ is possible. Hence the weight assignments are $(1,2,3,4,5,6)$ with sign 1 and $(1,2,3,4,6,5)$ with sign $(-1)^{\text{C}}$. When $\text{B} = 0$ we also have $(1,2,4,3,5,6)$ with sign $(-1)^2$, and $(1,2,4,3,6,5)$ with sign $(-1)^{\text{C}+2}$. Since C is even, this sum is positive. Hence S_{12} is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

7.7 Tableaux for $c = 7$ and $c = 8$

For $c = 7$ and 8, nearly all the required tableaux can be obtained by joining those tableaux already constructed. This is demonstrated in Chapter 8. However, we do need to construct one additional tableau, which is listed below.

Tableau W_1

$$S_1 = \begin{array}{cccccccccccc} \mathsf{x} & \mathsf{x} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{A} \\ \hline 7 & 8 & 6 & 7 & 8 & 8 & 3 & 7 & 8 & 7 & 3 \\ 4 & 5 & 4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 \\ 1 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \end{array} \qquad \begin{array}{l} \text{A} \leq \mathsf{x} \\ \text{Z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ \text{A} = 0, 2 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & 0 & 0 & d & d & 2\text{Z}+\text{A} & 0 & 0 & \\ d & d & 2\text{Z} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [4d - 4\text{Z} - \text{A}, 2d + 2\text{Z} + \text{A}, 2d + 2\text{Z}]$$

$$r = 2d - 6\text{Z} - 2\text{A},\ s = \text{A},\ t = 2d + 2\text{Z}$$

Only the triples $(1,2,3)$ and $(4,5,6)$ may have $\omega_i = (d, d, 2\text{Z})$ or larger. Hence the only valid weight assignments are $(1,2,3,4,5,6,7,8)$ with sign 1, and $(4,5,6,1,2,3,7,8)$ with sign $(-1)^{2d+2\text{Z}+\text{A}}$. Since A is even, the weight sum in positive. Hence W_1 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Chapter 8

Tableau Sufficiency

Our proof of Theorem 2 in Chapter 6 presupposed we had constructed all tableaux with $c \leq 8$ for the shapes of Theorem 10. In Chapter 7 we constructed many non-zero tableaux. In this chapter we will demonstrate that all the necessary tableaux have been constructed. Specifically we need all shapes in Theorem 10, that is all partitions $[r+s+t, s+t, t]$ of n, with $r, s \neq 1$, such that if r or s is in $\{0, 2, 4\}$ then $s+t$ is even. Recall that those required shapes with r or s less than 5 are called exceptional cases.

8.1 Sufficiency when $c = 3$

The tableaux we will use for $c = 3$ are the P_i described in Section 7.3. These tableaux are all maximal and non-zero by weight-set counting. We will show that every necessary partition of $n = 3d$ has a corresponding P_i.

First consider the exceptional r cases. These are tableaux having $s+t$ even with $r \in \{0, 2, 4\}$, or $r = 3$ with no constraints on s and t. Since our shape is a partition of $3d$, we have the condition $3t + 2s + r = 3d$. Hence for a given r, we need only to verify that all the appropriate s in the range $0 \leq s \leq \frac{3d-r}{2}$, with $s \neq 1$ are obtained. This condition shows $s \equiv r \pmod 3$.

Table 8.1 below lists the tableaux we are using for these cases. The column 'Parameters' indicates the restrictions on the tableaux arising from their construction in Section 7.3. The column 's, t values' indicates their values in terms of the tableau parameters, while the 's covered' lists those cases covered by the given tableau. The restrictions given on the s covered reflect the listed conditions on the parameters.

	Tableau	Parameters	s, t values	s covered
$r = 0$	$P_4(d - 2\mathrm{B}, \mathrm{B}, \mathrm{B})$	$0 \leq \mathrm{B} \leq \frac{d}{2}$ $d - \mathrm{B}$ even	$s = 3\mathrm{B}$ $t = d - 2\mathrm{B}$	$0 \leq s \leq 3\lfloor \frac{d}{2} \rfloor$ $s + t$ even
$r = 2$	$P_4(d - 2\mathrm{B}, \mathrm{B}, \mathrm{B} - 1)$	$1 \leq \mathrm{B} \leq \frac{d}{2}$ $d - \mathrm{B}$ odd	$s = 3\mathrm{B} - 1$ $t = d - 2\mathrm{B}$	$2 \leq s \leq 3\lfloor \frac{d}{2} \rfloor - 1$ $s + t$ even
$r = 3$	$P_4(d - 2\mathrm{B} - 1, \mathrm{B}, \mathrm{B}, \mathrm{B})$	$0 \leq \mathrm{B} \leq \frac{d-1}{2}$ $d - \mathrm{B}$ odd	$s = 3\mathrm{B}$ $t = d - 2\mathrm{B} - 1$	$0 \leq s \leq 3\lfloor \frac{d-1}{2} \rfloor$ $s + t$ even
	$P_3(d - 2\mathrm{B} - 1, \mathrm{B}, \mathrm{B} - 1)$	$1 \leq \mathrm{B} \leq \frac{d-1}{2}$ $d - \mathrm{B}$ even	$s = 3\mathrm{B}$ $t = d - 2\mathrm{B} - 1$	$3 \leq s \leq 3\lfloor \frac{d-1}{2} \rfloor$ $s + t$ odd
$r = 4$	$P_4(d - 2\mathrm{B}, \mathrm{B}, \mathrm{B} - 2)$	$2 \leq \mathrm{B} \leq \frac{d}{2}$ $d - \mathrm{B}$ even	$s = 3\mathrm{B} - 2$ $t = d - 2\mathrm{B}$	$4 \leq s \leq 3\lfloor \frac{d}{2} \rfloor - 2$ $s + t$ even

Table 8.1: Exceptional r cases for $c = 3$.

To see why Table 8.1 reaches the necessary upper bounds on s, we need to consider the parity of d. For d and r even, the maximum s needed is $\frac{3d}{2} - \frac{r}{2}$, which is obtained in the table. When d is odd, we need $s \leq \lfloor \frac{3d-r}{2} \rfloor = \frac{3d-r-1}{2}$. However, $s = \frac{3d-r-1}{2} \not\equiv r \pmod 3$, thus the largest s we need is $s = \frac{3d-r-3}{2} = 3\lfloor \frac{d}{2} \rfloor - \frac{r}{2}$. When $r = 3$, we need $s \leq \frac{3d-3}{2}$, which equals $3\lfloor \frac{d-1}{2} \rfloor$ when d is odd. But for d even, the largest $s \equiv r \pmod 3$ is $s = \frac{3d-9}{2} = 3\lfloor \frac{d-1}{2} \rfloor$. Hence the s bounds in Table 8.1 are correct.

For the lower bounds, Table 8.1 shows that all the necessary s are covered, except possibly some $s < 5$. Since $s \equiv r \pmod 3$, all s are covered in the $r = 2$ case. In the $r = 3$ case, $s = 0$ is only necessary when the $s + t$ is even. Similarly, no additional tableaux are needed in the $r = 4$ case because $s = 1$ is not a shape of Theorem 10.

Now consider the exceptional s cases. These are tableaux having t even with $s \in \{0, 2, 4\}$, or $s = 3$ with no constraints on t. We still have the condition $3t + 2s + r = 3d$, so for a given s we need only verify that all the appropriate t in the range $0 \leq t \leq d - \lceil \frac{2s}{3} \rceil$, (with t even if $s \neq 3$) are obtained.

Table 8.2 belows lists the tableaux we are using for these cases. The columns 'Parameters' and 'Conditions' indicate the restrictions on the tableaux arising from their construction in Section 7.3. The column 't values' indicates t's value in terms of the tableau parameters, while the 't covered' lists those cases covered by the given tableau. The restrictions given on the t covered reflect the conditions listed on the parameters.

	Tableau	Parameters	Conditions	t value	t covered
$s = 0$	$P_2(\text{A}, 0)$	$0 \leq \text{A} \leq d$	A even	$t = \text{A}$	$0 \leq t \leq d$ t even
$s = 2$	$P_2(\text{A}, 2)$	$0 \leq \text{A} \leq d - 2$	A even	$t = \text{A}$	$0 \leq t \leq d - 2$ t even
$s = 3$	$P_3(\text{A}, 1, 0)$	$0 \leq \text{A} \leq d - 3$	A even	$t = \text{A}$	$0 \leq t \leq d - 3$ t even
	$P_4(\text{A}, 1, 1)$	$0 \leq \text{A} \leq d - 2$	A odd	$t = \text{A}$	$0 \leq t \leq d - 2$ t odd
$s = 4$	$P_2(\text{A}, 4)$	$0 \leq \text{A} \leq d - 4$	A even	$t = \text{A}$	$0 \leq t \leq d - 4$ t even

Table 8.2: Exceptional s cases for $c = 3$.

Table 8.2 shows that all the necessary t are covered, except possibly when $s = 3$ or 4. When $s = 3$, the $t = d - 2$, t even case does not appear. In this case, $r = 0$ and $s + t$ is odd, so by Theorem 10 this case is not needed. For $s = 4$ and $t = d - 3$, we have $r = 1$ which is not a required shape. Hence all the exceptional cases have been covered.

Finally, consider the general cases remaining. These are tableaux having $r, s \geq 5$ and no additional constraints. We still have the condition $3t + 2s + r = 3d$, so for a fixed t we need only verify that all the appropriate s in the range $5 \leq s \leq \frac{3d-3t-5}{2}$, are obtained. (This accounts for the bounds both on r and on s.)

Table 8.3 belows lists the tableaux we are using for this case. The columns 'Parameters' and 'Conditions' indicate the restrictions on the tableaux arising from their construction in Section 7.3. The column 's values' indicates its value in terms of the tableau parameters, while the 's covered' lists those cases covered by the given tableau.

To see how Table 8.3 covers all the necessary shapes, first consider P_3. As C varies

Tableau	Parameters	Conditions	s value	s covered
$P_3(t,\textsc{b},\textsc{c})$	$2 \le \textsc{b} \le \frac{d-t-1}{2}$ $0 \le \textsc{c} < \textsc{b}$	$t+\textsc{c}$ even	$s = 2\textsc{b}+\textsc{c}+1$	$5 \le s \le 3\lfloor\frac{d-t-1}{2}\rfloor$ $s+t$ odd
$P_4(t,\textsc{b},\textsc{c})$	$1 \le \textsc{b} \le \frac{d-t}{2}$ $0 \le \textsc{c} \le \textsc{b}$	$t+\textsc{c}$ even	$s = 2\textsc{b}+\textsc{c}$	$2 \le s \le 3\lfloor\frac{d-t}{2}\rfloor$ $s+t$ even

Table 8.3: General $c = 3$ cases.

between 0 and $\textsc{b}-1$, we get $2\textsc{b}+1 \le s \le 3\textsc{b}$. When $\textsc{b}$ increases to $\textsc{b}+1$, we go from $s = 3\textsc{b}$ to $s = 2\textsc{b}+3 = 2(\textsc{b}+1)+1$. There is no gap between these provided $\textsc{b} \ge 2$. Since $\textsc{b} = 2$ yields a minimum $s = 5$, we don't need any smaller cases. Hence P_3 will cover the cases, provided $s+t$ is odd (equivalently $t+\textsc{c}$ is even). Given a case where $s = 2\textsc{b}+\textsc{c}+1$ but $t+\textsc{c}$ is odd, use $P_4(t,\textsc{b},\textsc{c}+1)$. Then $t+\textsc{c}+1$ is even, and $\textsc{c} < \textsc{b}$ implies $\textsc{c}+1 \le \textsc{b}$, so the conditions of P_4 are satisfied. (As similar analysis on P_4 shows all the s in the range do occur.) To see that the upper bound of $s \le \lfloor\frac{3d-3t-5}{2}\rfloor$ is met, first consider P_4. Since $3\lfloor\frac{d-t}{2}\rfloor \ge \lfloor\frac{3d-3t}{2}\rfloor - 1 = \lfloor\frac{3d-3t-2}{2}\rfloor$, the upper bound is obtained. For P_3, $3\lfloor\frac{d-t-1}{2}\rfloor \ge \frac{3d-3t-3}{2}$ when $d-t$ is odd. If $d-t$ is even, we have $\lfloor\frac{3d-3t-5}{2}\rfloor = \frac{3d-3t-6}{2}$, which is $3\lfloor\frac{d-t-1}{2}\rfloor$ as desired. Thus all the required shapes are listed.

8.2 Sufficiency when $c = 4$

When $c = 4$, the tableaux we will use are the Q_i listed in Section 7.4. We will show that every partition of $n = 4d$ described in Theorem 10 has a corresponding Q_i. Note that shapes with $r \ge d+5$ can be obtained by $P_i \vee V(d)$ for the appropriate P_i filled with $c = 3$ elements. As such, we will not include these shapes in the following compilation. Throughout, we will use the convention $d = 3\textsc{z}+\mathsf{x}$, where $d \equiv \mathsf{x} \pmod 3$. Unless otherwise specified, take $\mathsf{x} \in \{0,1,2\}$. We also use the notation d^* from previous chapters ($d^* = 2\lfloor\frac{d}{2}\rfloor$).

First consider the exceptional r cases. These are tableaux having $s+t$ even with $r \in \{0,2,4\}$ or $r = 3$ with no constraints on s and t. Since our shape is a partition of

$4d$ we have the condition $3t + 2s + r = 4d$. Hence for a given r, we need only verify that all the appropriate t in the range $0 \leq t \leq \lfloor \frac{4d-r}{3} \rfloor$, (with $s+t$ even for $r \neq 3$) are obtained.

For the exceptional r cases with $s + t$ even, we only need those t with $t \equiv r \pmod 4$. Given a shape (r, s, t) the next shape needed is $(r, s-6, t+4)$. Consider Table 8.4. For $r = 0$ we have all $t \leq \frac{4d}{3}$ when $d \equiv 0 \pmod 3$. When $d \equiv 1 \pmod 3$, the shape $t = d + \frac{d-1}{3}$ is not possible and the table provides all $t \leq d + \frac{d-1}{2} - 1$. Similarly, when $d \equiv 2 \pmod 3$, the shape $t = d + \frac{d-2}{3}$ corresponds to $s = 1$, while $t = d + \frac{d-2}{3} - 1$ is not possible. This covers the $r = 0$ cases. Note when $d = 5$, the only shapes needed are $t = d - 1$ and $t = 0$. Moreover, $t = d - 1$ and $t = d - 2$ are not possible for $d \leq 4$.

For $r = 2$, Table 8.4 provides tableaux for $0 < t \leq d + \mathsf{z} - 2 - \mathsf{x}$ or $t = d + \mathsf{z}$ if $d \equiv 2 \pmod 3$. Since $t \equiv 2 \pmod 4$, $t < 2$ is not needed. When $d \equiv 2 \pmod 3$, $t = d + \mathsf{z} - 1$ and $t = d + \mathsf{z} - 3$ are not congruent to $2 \pmod 4$. Similarly, $t = d + \mathsf{z} - \mathsf{x}$ and $t = d + \mathsf{z} - \mathsf{x} - 1$ are not congruent to $2 \pmod 4$ for $\mathsf{x} = 0$ and 1.

When $r = 3$ we no longer have the conditions $t \equiv r \pmod 4$; instead t must be odd. Table 8.5 accounts for all tableaux with $0 < t \leq d + \frac{d}{3} - 3$ for $d \equiv 0 \pmod 3$. For $d \equiv 1 \pmod 3$ Table 8.5 accounts for all $t \leq d + \frac{d-1}{3} - 4$ and for $t \leq d + \frac{d-2}{3} - 5$ when $d \equiv 2 \pmod 3$. Tableaux with larger t correspond to shapes having exceptional s cases ($s \neq 3$). Since t is odd, these shapes are not needed according to Theorem 10. When $t \leq d - 1$ and d small, the shapes are either not required by Theorem 10 or are not possible.

For $r = 4$, we need all $t \equiv 0 \pmod 4$, where $t \leq d + \mathsf{z} - 4$ if $\mathsf{x} = 0$, $t \leq d + \mathsf{z} - 1$ if $\mathsf{x} = 1$, and $t \leq d + \mathsf{z} - 2$ if $\mathsf{x} = 2$. Table 8.6 provides all these tableaux. Again, the bounds on d are necessary to produce a valid shape. When $d = 5$ and $t = 4$, the shape has $s = 2$ and can be found in Table 8.7. Hence these tables cover all the exceptional r cases.

Now consider the exceptional s cases. These are tableaux having t even with $s \in \{0, 2, 4\}$, or $s = 3$ with no constraints on t. When $r \geq d$, $r \neq d + 1$, we may reduced to $P_i \vee V(d)$ for some P_i filled with $c = 3$ elements. (If $s = 3$ we may only

	Tableau	Parameters	s, t values	t covered
$r = 0$	$Q_2(\mathsf{z})$	$\mathsf{x} = 0$	$s = 0$ $t = d + \mathsf{z}$	$t = d + \mathsf{z}$ $d \equiv 0 \pmod 3$
	$Q_1(\mathsf{z} - k, \frac{k-\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2})$	$3k + \mathsf{x} \equiv 0 \pmod 4$ $\mathsf{x} \le k \le \mathsf{z}$, $0 < k$	$s = k + \frac{k+\mathsf{x}}{2}$ $t = d + \mathsf{z} - k$	$d \le t \le d + \mathsf{z} - 1$, $\mathsf{x} \ne 2$ $d \le t \le d + \mathsf{z} - 2$, $\mathsf{x} = 2$ $d \ne 5$ $t \equiv 0 \pmod 4$
	$Q_7(\frac{d-1}{2}, \frac{d-3}{2}, \frac{d-3}{2})$	$d \ge 5$, d odd	$s = \frac{d+3}{2}$ $t = d - 1$	$t = d - 1$ $t \equiv 0 \pmod 2$
	$Q_4(\frac{d^*-2k}{2}, k, k + W, \frac{d^*}{2})$	$0 < 2k < d^*$ $2k \equiv d^* \pmod 4$ $W = d - d^*$	$s = \frac{d^*}{2} + 3k + 2W$ $t = d^* - 2k$	$4 \le t \le d - 2$ $t \equiv 0 \pmod 4$, $d \ge 6$
	$U_2(d - 1, 1)$		$s = 2d$ $t = 0$	$t = 0$
$r = 2$	$Q_2(\mathsf{z})$	$\mathsf{x} = 2$	$s = 0$ $t = d + \mathsf{z}$	$t = d + \mathsf{z}$ $d \equiv 2 \pmod 3$
	$Q_1(\mathsf{z} - k, \frac{k-\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2} - 1)$	$3k + \mathsf{x} \equiv 2 \pmod 4$ $\mathsf{x} \le k \le \mathsf{z}$, $0 < k$	$s = k + \frac{k+\mathsf{x}}{2} - 1$ $t = d + \mathsf{z} - k$	$d \le t \le d + \mathsf{z} - \mathsf{x} - 2$ $t \equiv 2 \pmod 4$
	$Q_5(\frac{d-1}{2}, \frac{d-1}{2}, 1, 0, \frac{d-3}{2}, 1)$	$d \ge 5$ $d \equiv 3 \pmod 4$	$s = \frac{d+1}{2}$ $t = d - 1$	$t = d - 1$ $t \equiv 2 \pmod 4$
	$Q_4(\frac{d^*-2k}{2}, k, k + W, \frac{d^*}{2} - 1)$	$0 < 2k < d^*$ $d^* - 2k \equiv 2 \pmod 4$ $d \ge 4$ $W = d - d^*$	$s = \frac{d^*}{2} + 3k + 2W - 1$ $t = d^* - 2k$	$2 \le t \le d - 2$ $d \ge 4$ $t \equiv 2 \pmod 4$
	$Q_3(2, 1)$	$d = 3$	$s = 2$ $t = 2$	$t = 2$ $d = 3$

Table 8.4: Exceptional $r = 0$ and $r = 2$ cases for $c = 4$.

Tableau	**Parameters**	s, t **values**	t **covered**
$Q_1(\mathrm{z}-4,1,2,2)$	$\mathsf{x}=1$ $d \geq 13$	$s=5$ $t=d+\mathrm{z}-4$	$t=d+\mathrm{z}-4$ $d \equiv 1 \pmod 3$, $d \geq 13$
$Q_1(\mathrm{z}-5,1,4,3)$	$\mathsf{x}=4$ $d \geq 19$	$s=8$ $t=d+\mathrm{z}-5$	$t=d+\frac{d-1}{3}-6$ $d \equiv 1 \pmod 3$, $d \geq 19$
$Q_1(\mathrm{z}-2k-1, \frac{2k-\mathsf{x}}{2}, \frac{2k+\mathsf{x}}{2}, \frac{2k+\mathsf{x}}{2})$	$k+\frac{\mathsf{x}}{2} \equiv 1 \pmod 2$ $\mathsf{x} \in \{0,2,4\}$ $d \neq 4, 5, 8$ $\mathsf{x} \leq 2k \leq \mathrm{z}-1$	$s=3k+\frac{\mathsf{x}}{2}$ $t=d+\mathrm{z}-2k-1$	$d \leq t \leq d+\mathrm{z}-3$, $\mathsf{x}=0$, $d \leq t \leq d+\mathrm{z}-5$, $\mathsf{x}=2$ $d \leq t \leq d+\mathrm{z}-7$, $\mathsf{x}=4$ t odd, $d \neq 4, 5, 8$
$Q_5(\frac{d}{2}, \frac{d}{2}, 1, 0, \frac{d-4}{2}, 1)$	d even $d \geq 6$	$s=\frac{d}{2}$ $t=d-1$	$t=d-1$ t odd $d \geq 6$
$Q_5(\frac{d-1}{2}, \frac{d-3}{2}, 2, 0, \frac{d-3}{2}, 1)$	d odd $d \geq 5$	$s=\frac{d+3}{2}$ $t=d-2$	$t=d-2$ t odd $d \geq 5$
$Q_5(k, k+1, d-2k-2, 1, k-1, d-2k-1)$	$2 \leq k \leq \frac{d-3}{2}$ $d \geq 8$	$s=2d-3k-3$ $t=2k+1$	$5 \leq t \leq d-3$ t odd $d \geq 8$
$Q_5(2,1,d-3,0,1,d-4)$	$d \geq 5$	$s=2d-6$ $t=3$	$t=3$ $d \geq 5$
$Q_6(1,d-2,d-3)$		$s=2d-3$ $t=1$	$t=1$

Table 8.5: Exceptional $r=3$ cases for $c=4$.

	Tableau	Parameters	s, t values	t covered
$r = 4$	$Q_2(\mathrm{z})$	$\mathsf{x} = 4$ $d \neq 4$	$s = 0$ $t = d + \frac{d-1}{3} - 1$	$t = d + \frac{d-1}{3} - 1$ $d \equiv 1 \pmod 3$ $d \neq 4$
	$Q_1(\mathrm{z} - k, \frac{k-\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2} - 2)$	$3k + \mathsf{x} \equiv 0 \pmod 4$ $\mathsf{x} \leq k \leq \mathrm{z}$, $k \geq 2$ $d \geq 6$	$s = k + \frac{k+\mathsf{x}}{2} - 2$ $t = d + \mathrm{z} - k$	$d \leq t \leq d + \mathrm{z} - 2$, $\mathsf{x} = 2$ $d \leq t \leq d + \mathrm{z} - 4 - \mathsf{x}$, $\mathsf{x} \neq 2$ $t \equiv 0 \pmod 4$ $d \geq 6$
	$Q_5(\frac{d-1}{2}, \frac{d-1}{2}, 0, 1, \frac{d-5}{2}, 1)$	$d \equiv 1 \pmod 4$ $d \geq 7$	$s = \frac{d-1}{2}$ $t = d - 1$	$t = d - 1$ $t \equiv 0 \pmod 4$ $d \geq 7$
	$Q_4(\frac{d^*-2k}{2}, k, k + W, \frac{d^*}{2} - 2)$	$0 < 2k \leq d^*$ $2k \equiv d^* \pmod 4$ $W = d - d^*$ $d \geq 6$	$s = \frac{d^*}{2} + 3k + 2W - 2$ $t = d^* - 2k$	$4 \leq t \leq d - 2$ $t \equiv 0 \pmod 4$ $d \geq 6$
	$U_2(d - 2, 1)$		$s = 2d - 2$ $t = 0$	$t = 0$

Table 8.6: Exceptional $r = 4$ cases for $c = 4$.

reduce when $r \geq d+5$.) Since we have already listed those table with exceptional r cases, we will take $r \geq 5$. Hence for a given s, we need only verify that all the appropriate t in the range $\lceil \frac{3d-2s-1}{3} \rceil \leq t \leq \lfloor \frac{4d-2s-5}{3} \rfloor$ (with t even for $s \neq 3$) are obtained.

For the exceptional s cases, consider Table 8.7. When $s = 0$, all t are covered except $t = d + \frac{d-1}{3}$ when $\mathsf{x} = 4$. However, this case is unnecessary as $r = 1$. If $t \leq d$, the shape is reducible to a $c = 3$ case.

For $s = 2$, all $t \geq d-1$ except $t = d + \frac{d-\mathsf{x}}{3} - 1$ (with $\mathsf{x} \in \{0, 2, 4\}$) are given in Table 8.7. When that occurs, either the shape is invalid or $r < 5$. If $t \leq d-2$, the shape is reducible to a $c = 3$ case. When $d \leq 7$ and $t \geq d$ the shapes are covered by the exceptional r cases or are reducible to $c = 3$ cases.

For $s = 3$ we want $t \leq d + \frac{d-\mathsf{x}}{3} - 2$. However, when $t = d + \frac{d-\mathsf{x}}{3} - 2$ ($\mathsf{x} \in \{0, 2, 4\}$), then $r < 5$. Hence these cases have already been covered. When $t < d-2$ the shape is reducible to a $c = 3$ case. For $d \leq 6$, the cases are covered by $c = 3$ or exceptional r cases. The bounds on d for $t \geq d$ are needed to produce a valid shape.

For $s = 4$, Table 8.7 provides all tableaux with $t \leq d + \frac{d-\mathsf{x}}{3} - 3$ with $\mathsf{x} \in \{1, 3, 5\}$. Since any t larger than this has $r < 5$, this covers all shapes not already listed. Note that for $d \leq 12$, all necessary shapes have $t < d$; those shapes with $t = d-1$ are not needed for $d \leq 6$. Hence all the exceptional s cases are accounted for in Table 8.7.

The general cases of $r, s \geq 5$ are classified in Table 8.8. When $r \geq d+5$ we can reduce to a $c = 3$ tableau. Fix $t = d + \textsc{z} - k$. Since r and s are greater than 5, we need all t with $5 \leq k \leq \textsc{z}$. For a t of this form, we need all shapes with $5 \leq s \leq k + \lfloor \frac{\mathsf{x}+k-5}{2} \rfloor$. This range is covered in Table 8.8. To see why all such s are obtained, note that for any fixed $\textsc{c}$, we always get $2\textsc{c} \leq s \leq 3\textsc{c}-1$. Since $\textsc{c} \geq 2$, there are no gaps as we increment $\textsc{c}$. The parameters between the cases are comparable, so writing $s = 2\textsc{c} + \textsc{d} + 1$ and using the case corresponding to the parity of $s + k$ will yield the appropriate tableau. Since $\mathsf{x} \leq 2$, we find $s \leq k + \lfloor \frac{\mathsf{x}+k-5}{2} \rfloor$ implies $s \leq k + \lfloor \frac{k-1}{2} \rfloor - 1$ for $\mathsf{x} = 0$ or 2, and $s \leq k + \lfloor \frac{k}{2} \rfloor - 2$ for $\mathsf{x} = 1$. Comparing the bounds shows all s are obtained. This takes care of all shapes with $t \geq d$.

When $0 < t < d$, we require all shapes with $5 \leq s \leq 2d - t - \lfloor \frac{t+5}{2} \rfloor$. The tableaux

	Tableau	Parameters	t value	t covered
$s=0$	$Q_2(\textsc{a})$	$0 < \textsc{a} \leq \textsc{z}$ $\textsc{z}+\textsc{a}$ even $\mathsf{x} \in \{0,2,4\}$	$t = d + \textsc{a}$	$d < t \leq d + \textsc{z}$ $\mathsf{x} \in \{0,2,4\}$ t even, $d \neq 4$
$s=2$	$Q_1(\textsc{a},1,1,0)$	$0 \leq \textsc{a} \leq \textsc{z}-2$ $\mathsf{x} \in \{0,2,4\}$ $\textsc{z}+\textsc{a}$ even	$t = d + \textsc{a}$	$d \leq t \leq d + \textsc{z} - 2$ $\mathsf{x} \in \{0,2,4\}$ t even, $d \geq 8$
	$Q_3(d-1,1)$	d odd	$t = d-1$	$t = d-1$ t even
$s=3$	$Q_1(\textsc{a},2,1,0)$	$0 \leq \textsc{a} \leq \textsc{z}-3$ $\mathsf{x} \in \{0,2,4\}$ $\textsc{z}+\textsc{a}$ even	$t = d + \textsc{a}$	$d \leq t \leq d + \textsc{z} - 3$ $\mathsf{x} \in \{0,2,4\}$ t even, $d \geq 11$
	$Q_1(\textsc{a},1,1,1)$	$0 \leq \textsc{a} \leq \textsc{z}-2$ $\textsc{z}+\textsc{a}$ odd $\mathsf{x} \in \{0,2,4\}$	$t = d + \textsc{a}$	$d \leq t \leq d + \textsc{z} - 2$ $\mathsf{x} \in \{0,2,4\}$ t odd, $d \geq 8$
	$Q_5(\frac{d-1}{2}, \frac{d-1}{2}, 0, 1, 1, 1)$	d odd $d \geq 7$	$t = d-1$	$t = d-1$, $d \geq 7$, d odd
	$Q_5(\frac{d}{2}-1, \frac{d}{2}, 1, 0, 1, 1)$	d even $d \geq 6$	$t = d-1$	$t = d-1$, $d \geq 6$, d even
	$Q_6(d-2,1,0)$		$t = d-2$	$t = d-2$
$s=4$	$Q_1(\textsc{a},1,2,1)$	$0 \leq \textsc{a} \leq \textsc{z}-3$ $\textsc{z}+\textsc{a}$ odd $\mathsf{x} \in \{1,3,5\}$	$t = d + \textsc{a}$	$d \leq t \leq d + \textsc{z} - 3$ $\mathsf{x} \in \{1,3,5\}$ t even, $d \geq 12$
	$Q_5(\frac{d-1}{2}, \frac{d-1}{2}, 1, 0, 2, 1)$	d odd $d \geq 7$	$t = d-1$	$t = d-1$ d odd $d \geq 7$
	$Q_3(\textsc{a},2)$	$2 \leq \textsc{a} \leq d-2$ $\textsc{a}$ even, $d \geq 4$	$t = \textsc{a}$	$2 \leq t \leq d-2$ t even, $d \geq 4$

Table 8.7: Exceptional s cases for $c = 4$.

Tableau		Parameters	s value	s covered
$d \le t \le d + \textsc{z} - 5$ $t = d + \textsc{z} - k$	$Q_1(\textsc{z} - k, \textsc{c} + 1, \textsc{c}, \textsc{d})$	$2 \le \textsc{c} \le \lfloor \frac{k-1}{2} \rfloor$ $5 \le k \le \textsc{z}$ $0 \le \textsc{d} < \textsc{c}$ $k \equiv \textsc{d} \pmod 2$	$s = 2\textsc{c} + \textsc{d} + 1$	$5 \le s \le 3\lfloor \frac{k-1}{2} \rfloor$ $s + k$ odd
	$Q_1(\textsc{z} - k, \textsc{c}, \textsc{c}, \textsc{d} + 1)$	$2 \le \textsc{c} \le \lfloor \frac{k}{2} \rfloor$ $5 \le k \le \textsc{z}$ $0 \le \textsc{d} \le \textsc{c} - 2$ $\textsc{d} \le \textsc{c} + 1$ if $\textsc{x} = 0, 2$ $k \not\equiv \textsc{d} \pmod 2$	$s = 2\textsc{c} + \textsc{d} + 1$	$4 \le s \le 3\lfloor \frac{k}{2} \rfloor - 1$ $s \le 3\lfloor \frac{k}{2} \rfloor$ if $\textsc{x} = 0, 2$ $s + k$ even
$t = d - 1$	$Q_5(\frac{d-1}{2}, \frac{d-1}{2}, 0, 1, \textsc{e}, 1)$	$1 \le \textsc{e} < \frac{d-1}{2} - 2$ $d \ge 7$ d odd	$s = \textsc{e} + 2$	$4 \le s \le \frac{d-1}{2}$ $d \ge 7$ d odd
	$Q_7(\frac{d}{2} - 1, \frac{d}{2} - 1, \textsc{c})$	$1 \le \textsc{c} \le \frac{d}{2} - 2$ d even	$s = \textsc{c} + 3$	$3 \le s \le \frac{d}{2} + 1$ d even
$t = d - 2$	$Q_5(\lfloor \frac{d-2}{2} \rfloor, \lceil \frac{d-2}{2} \rceil, 1, 1, \textsc{e}, 1)$	$1 \le \textsc{e} \le \lceil \frac{d-2}{2} \rceil - 1$ $d \ge 5$	$s = \textsc{e} + 3$	$4 \le s \le \lceil \frac{d}{2} \rceil + 1$ $d \ge 5$
$3 \le t \le d - 3$ $t = 2k + 1$	$Q_5(k, k + 1, d - 2k - 2, 1, \textsc{e}, \textsc{f})$	$1 \le k \le \lfloor \frac{d-3}{2} \rfloor$ $1 \le \textsc{e} \le k$ $1 \le \textsc{f} \le d - 2k - 2$ $d \ge 5$	$s = d - 2k - 1 + \textsc{e} + \textsc{f}$	$d - 2k + 1 \le s \le 2d - 3k - 3$ t odd $d \ge 5$
$2 \le t \le d - 3$ $t = 2k$	$Q_5(k, k, d - 2k - 1, 1, \textsc{e}, \textsc{f})$	$1 \le k \le \lfloor \frac{d-2}{2} \rfloor$ $1 \le \textsc{e} \le k$ $1 \le \textsc{f} \le d - 2k - 2$	$s = d - 2k + \textsc{e} + \textsc{f}$	$d - 2k + 2 \le s \le 2d - 3k - 2$ t even
$1 \le t \le d - 3$	$Q_6(t, \textsc{b}, \textsc{c})$	$\textsc{b} \le d - t - 1$ $\textsc{b} > \textsc{c} \ge 0$ $d \ge 4$	$s = \textsc{b} + \textsc{c} + 2$	$5 \le s \le 2d - 2t - 1$ $d \ge 4$
$t = 0$	$U_2(d - k - 1, 1)$	$0 \le k \le d - 2$	$s = 2(d - k)$	$4 \le s \le 2d$ s even
	$U_5(2, k - 2)$	$5 \le k \le d$	$s = 2k - 3$	$7 \le s \le 2d - 3$ $d \ge 5$, s odd
	U_8	$d = 4$	$s = 5$	$s = 5$, $d = 4$

Table 8.8: General $c = 4$ cases.

listed in Table 8.8 satisfy this. Note that for $t \leq d-3$ the bounds on s for Q_5 and Q_6 overlap, thus guaranteeing all s are covered. For the small d not listed, the shapes are either not possible, not needed, or result in exceptional cases done previously. When $t = 0$ we require all $d + \frac{d-5}{2} \leq s \leq 2d-3$. As Table 8.8 shows, this is satisfied. Hence all necessary cases for $c = 4$ have been covered.

8.3 Sufficiency for $c = 5$

When $c = 5$, the tableaux we will use are the R_i listed in Section 7.5. We will show that every partition of $n = 5d$ described in Theorem 10 has a corresponding R_i. Note that shapes with $r \geq d+5$ can be obtained by $Q_i \vee V(d)$ for the appropriate Q_i filled with $c = 4$ elements. As such, we will not include these shapes in the following compilation. Throughout, we will use the convention $d = 3\mathrm{z} + \mathsf{x}$, where $d \equiv \mathsf{x} \pmod 3$. Unless otherwise specified, take $\mathsf{x} \in \{0, 1, 2\}$.

First consider the exceptional r cases. These are tableaux having $s + t$ even with $r \in \{0, 2, 4\}$ or $r = 3$ with no constraints on s and t. Since our shape is a partition of $5d$ we have the condition $3t + 2s + r = 5d$. Hence for a given r, we need only verify that all the appropriate t in the range $0 \leq t \leq \lfloor \frac{5d-r}{3} \rfloor$, (with $s + t$ even if $r \neq 3$) are obtained.

For the exceptional r cases with $s + t$ even, we only need those t with $t \equiv d + r \pmod 4$. Hence given a shape (r, s, t) the next shape needed is $(r, s-6, t+4)$. Consider Table 8.9. For $r = 0$, this table provides all the t, except $t = \frac{5d-\mathsf{x}}{3}$, $\mathsf{x} = 1, 2$ and $t = d + 2\mathrm{z} - 1$. These are not a possible shapes. When $d < 6$ we only need those shapes with $t \leq d$.

For $r = 2$, Table 8.9 yields all $t \leq d + 2\mathrm{z} - 2$ with $\mathsf{x} \in \{1, 2, 3\}$. Since $t \equiv d + 2 \pmod 4$. When $t = d + 2\mathrm{z}$, for $\mathsf{x} = 0, 2$ we do not get a shape required by Theorem 10; for $\mathsf{x} = 1$ the tableau is listed in the table. Also, $t = d + 2\mathrm{z} - 1$ is not a shape and $t = d + 2\mathrm{z} - 2$ with d odd is not needed (the d even case is listed). If $d \leq 6$, then all necessary shapes have $t \leq d - 2$. Hence we have accounted for all necessary tableaux with $r = 2$.

Tableau		Conditions	t value	t covered
$r=0$	$R_1(\textsc{z},\textsc{z})$	$\mathsf{x}=0$ d even	$t=\frac{5d}{3}$	$t=\frac{5d}{3}$ $d\equiv 0 \pmod 3$ d even
	$R_4(\textsc{a},\textsc{a},\mathsf{x})$	$1\le \textsc{a}\le \textsc{z}$ $\textsc{z}+\textsc{a}$ even $d\ge 6$	$t=d+2\textsc{z}-2\textsc{a}$	$d\le t\le d+2\textsc{z}-2$ $t\equiv d \pmod 4$ $d\ge 6$
	$R_5(\textsc{a},\textsc{a})$	$0\le \textsc{a}\le \frac{d^*}{2}-1$ $\textsc{a}$ even	$t=d-2\textsc{a}$	$2\le t\le d$ $t\equiv d \pmod 4$
	$R_{12}(\frac{d-1}{2})$	$d\equiv 1 \pmod 4$	$t=1$	$t=1$ $d\equiv 1 \pmod 4$
	$U_1(d)\vee U_3(\frac{d}{2},\frac{d}{2})$	$d\equiv 0 \pmod 4$	$t=0$	$t=0$ $d\equiv 0 \pmod 4$
$r=2$	$R_1(\textsc{z},\textsc{z})$	$\mathsf{x}=1$ d even	$t=\frac{5d-2}{3}$	$t=\frac{5d-2}{3}$ $d\equiv 1 \pmod 3$ d even
	$R_2(\textsc{z}-1,\textsc{z}-1,1)$	$\mathsf{x}=0$ d even	$t=d+\frac{2d}{3}-2$	$t=d+\frac{2d}{3}-2$ $t\equiv d+2 \pmod 4$ d even $d\equiv 0 \pmod 3$
	$R_4(\textsc{a},\textsc{a},\mathsf{x}-1)$	$\mathsf{x}\in\{1,2,3\}$ $1\le \textsc{a}\le \textsc{z}$ $d\ge 7$ $\textsc{z}+\textsc{a}$ odd	$t=d+2\textsc{z}-2\textsc{a}$	$d\le t\le d+2\textsc{z}-2$ $t\equiv d+2 \pmod 4$ $d\ge 7$
	$R_5(\textsc{a},\textsc{a}-1)$	$1\le \textsc{a}\le \frac{d^*}{2}-1$	$t=d-2\textsc{a}$	$3\le t\le d-2$ $t=2$, d even
	$P_4(1,\frac{d-1}{2},\frac{d-1}{2})$	$d\equiv 3 \pmod 4$	$t=1$	$t=1$, $d\equiv 3 \pmod 4$
	$U_1(d)\vee U_3(\frac{d}{2}-1,\frac{d}{2})$	$d\equiv 2 \pmod 4$	$t=0$	$t=0$, $d\equiv 2 \pmod 4$

Table 8.9: Exceptional $r=0$ and $r=2$ cases for $c=5$.

For $r=3$ consider Table 8.10. In this case, we need $t\not\equiv d \pmod 2$. All cases with $t\le d+2\textsc{z}-3$ are covered in the table. In addition we need $t=d+2\textsc{z}-1$ for $d\equiv 0, 2 \pmod 3$ with d even, which are listed as well. When $d<9$ we have $t\le d+1$ or $t=d+2\textsc{z}-1$, and so those cases are covered. Finally, when $d=4$, $t=d+1$ has $s=1$, so that shape is no needed. When $t=0$ and $d=3$ then s is even, so this case is reducible to $c = 4$ case with $r = 0$. Hence all necessary tableaux with $r = 3$ are provided by Table 8.10.

For $r = 4$ we need those tableaux with $t \equiv d \pmod 4$, $t \leq d+2\text{z}-4$. In addition, we need $t = d+2\text{z}$ when $d \equiv 2 \pmod 3$ and $t = d+2\text{z}-2$ when $d \equiv 1 \pmod 3$. These are all found in Table 8.10. For $d = 5$ the needed tableaux are listed individually. For $d = 4$, $t = d + 2\text{z} - 2$ and $t = 0$ are the only shapes required, while $d = 3$ does not need any shapes. Thus all the exceptional r cases are contained in Table 8.9 and Table 8.10.

Now consider the exceptional s cases. These are tableaux having t even with $s \in \{0, 2, 4\}$, or $s = 3$ with no constraints on t. When $r \geq d$, $r \neq d + 1$, we may reduced to $Q_i \vee V(d)$ for some Q_i filled with $c = 4$ elements. (If $s = 3$ we may only reduce when $r \geq d + 5$.) Since we have already listed the exceptional r cases, we will also take $r \geq 5$. Hence for a given s, we need only verify that all the appropriate t in the range $\lceil \frac{4d-2s-1}{3} \rceil \leq t \leq \lfloor \frac{5d-2s-5}{3} \rfloor$ (with t even if $s \neq 3$) are obtained.

When $s = 0$, we need all even $t \leq d + 2\text{z}$. Table 8.11 provides all tableaux with $t \geq d+2$, t even. For $t \leq d+1$ we can use the reduction to a $c = 4$ case. When $s = 2$, we need all tableaux with $d + 1 \leq t \leq d + 2\text{z} - 2$ and t even, which Table 8.11 lists. Those with $t \leq d$ are reducible to a $c = 4$ case, while the $d \leq 5$ cases correspond to exceptional r cases or are similarly reducible.

When $s = 3$ we need those shapes with $t \leq d + 2\text{z} - 3$ and $d + 5 \geq r \geq 5$. For other r or $r = d + 3$ we can reduce to a $c = 4$ case. Table 8.11 provides tableaux for all $t \geq d$. When $t \leq d - 1$, we can reduce to a $c = 4$ case when $d \geq 8$ or $d = 6$. The remaining tableaux with $r \geq 5$ and $d \leq 7$ which are not reducible to a $c = 4$ case are listed as well.

For $s = 4$ we need all even $t \leq d + 2\text{z} - 5$, along with $t = d + 2\text{z} - 4$ when $d \not\equiv 0 \pmod 3$. Table 8.11 provides all those tableaux with $t \geq d + 1$ and $d \geq 9$. When $t \leq d$ we may reduce directly to a $c = 4$ case when $d \geq 10$. Those remaining tableaux with $d \leq 9$ have $r \geq d + 2$, $r = d$ or $r < 5$ and hence are either reducible or listed previously. Thus Table 8.11 suffices for the exceptional s cases.

The general tableaux with $r, s \geq 5$ are classified in Table 8.12 and Table 8.13.

Since $r, s \geq 5$, we only need those tableaux with $0 \leq t \leq d + 2\text{Z} - 5$. For a fixed t, we need all tableaux with $\frac{4d-3t-4}{2} \leq s \leq \frac{5d-3t-5}{2}$ because for $r \geq d+5$ we can reduce to a $c = 4$ tableau. First consider $t = d + \text{Z} + \text{A}$ of Table 8.12. We need all $s \leq \text{Z} - \text{A} + \text{x} - 2 + \lfloor \frac{\text{Z}-\text{A}-1}{2} \rfloor$. Letting the parameters of R_9 vary in order over their bounds yields all s up to $\text{B} + \text{Z} - \text{A} + 2\text{x} - 3$. Since B ranges to $\lfloor \frac{\text{Z}-\text{A}-\text{x}+1}{2} \rfloor$, we get all $s \leq \text{Z} - \text{A} + \text{x} - 3 + \lfloor \frac{\text{Z}-\text{A}+\text{x}-1}{2} \rfloor$. This covers all the necessary s for $\text{x} > 1$. When $\text{x} = 1$ we also get $s = \text{Z} - \text{A} - 1 + \lfloor \frac{\text{Z}-\text{A}}{2} \rfloor$ as needed. This tableau requires $d \geq 16$, for $d < 15$ we find there are no shapes with $t \geq d + \text{Z}$, $5 \leq r \leq d + 4$ and $s \geq 5$. When $d = 15$, R_6 provides $t = d + \text{Z}$, the largest t required.

When $d \leq t < d + \text{Z}$, Table 8.12 provides tableau R_{10}. Fix $t = d + \text{Z} - \text{A}$, then we need all $s \leq \text{A} + \text{Z} + \text{x} + \lfloor \frac{\text{A}+\text{Z}-5}{2} \rfloor$. To see why this tableau suffices, begin by taking $\text{C} = 0$. First let B and F vary over their ranges. Taking them maximal, vary D up to its maximum and then take E up to $\lfloor \frac{\text{Z}}{2} \rfloor - 1$. This satisfies all the required inequalities. Then take $R_{10}(\text{A}, \lfloor \frac{\text{A}}{2} \rfloor, \text{C}, \lceil \frac{\text{Z}}{2} \rceil + \lfloor \frac{\text{A}}{2} \rfloor, \lfloor \frac{\text{Z}}{2} \rfloor - 1, \lceil \frac{\text{A}}{2} \rceil)$ with $0 \leq \text{C} \leq \lfloor \frac{\text{A}}{2} \rfloor - 1$, C even. Similarly, use $R_{10}(\text{A}, \lfloor \frac{\text{A}}{2} \rfloor, \text{C}, \lceil \frac{\text{Z}}{2} \rceil + \lfloor \frac{\text{A}}{2} \rfloor - 1, \lfloor \frac{\text{Z}}{2} \rfloor - 1, \lceil \frac{\text{A}}{2} \rceil)$ with $0 \leq \text{C} \leq \lfloor \frac{\text{A}}{2} \rfloor$, C even. This gives all $5 \leq s \leq \text{A} + \text{Z} + \lfloor \frac{\text{A}}{2} \rfloor + \text{x} + \lfloor \frac{\text{Z}}{2} \rfloor - 2$. Since $\lfloor \frac{\text{A}+\text{Z}-5}{2} \rfloor \leq \lfloor \frac{\text{A}}{2} \rfloor + \lfloor \frac{\text{Z}}{2} \rfloor - 2$, all the necessary s are obtained. This tableau required $d \geq 6$. When $d < 6$, we find there are no shapes with $t \geq d$ and r, s within the needed bounds. Table 8.12 also contains $t = d - 1$, which requires all $s \leq d - 1$, as listed.

When $t \leq d - 2$, consider Table 8.13. For $t \leq d - 2$, we need all $s \leq 2d - t + \lfloor \frac{d-t-1}{2} \rfloor - 2$. Tableau R_{11} provides this by taking the parameters through their ranges in order, using $\text{B} = 1$ and $\text{B} = 2$ (with $\text{E} \geq \text{F}$). (We need $\text{B} = 2$ in order to obtain $s = 5$, otherwise $\text{B} = 1$ suffices.) The only snag is when $t = 2$. Then we cannot have $\text{C} = 1$, hence $s = 5$ is not obtainable in this case. However, $t = 2, s = 5$ is needed only when $d = 5$ and hence is listed separately. Also, $d = 3$ needs only $t = 0$, accounting for $d \leq 4$.

For $t = 1$, we need all $2d - 2 \leq s \leq 2d + \frac{d}{2} - 6$. (The $s = 2d - 3$ case is a $r = d + 3$ reduction to $c = 4$.) All these shapes are obtained in Table 8.13. For small d, those tableaux not listed have $s < 5$.

For $t = 0$ we need all $2d - 2 \leq s \leq 2d + \lfloor \frac{d-1}{2} \rfloor - 2$. Table 8.13 provides all the necessary tableaux. When d is even, we get all $2d \leq s \leq 2d + \frac{d}{2}$, with s even and $2d - 1 \leq s \leq 2d + \frac{d}{2} - 3$ with s odd. While $s = 2d - 2$ is not listed, this shape has $r = d + 4$ and $s + t$ even, so we may reduce to a $c = 4$ case. When d is odd, we get all $2d - 2 \leq s \leq 2d + \frac{d-1}{2} - 2$ with s even and $2d - 1 \leq s \leq 2d + \frac{d-1}{2} - 2$ with s odd. Hence all the required s are listed. Therefore Tables 8.12 and 8.13 provide the required tableaux for the non-exceptional cases with $c = 5$.

8.4 Sufficiency for $c = 6$

When $c = 6$, the tableaux we will use are the S_i listed in Section 7.6. We will show that every partition of $n = 6d$ described in Theorem 10 has a corresponding S_i. Note that shapes with $r \geq d + 5$ can be obtained by $R_i \vee V(d)$ for the appropriate R_i filled with $c = 5$ elements. As such, we will not include these shapes in the following compilation. Throughout, we will use the convention $d = 3\mathsf{z} + \mathsf{x}$, where $d \equiv \mathsf{x} \pmod 3$. Unless otherwise specified, take $\mathsf{x} \in \{0, 1, 2\}$.

First consider the exceptional r cases. These are tableaux having $s + t$ even with $r \in \{0, 2, 4\}$ or $r = 3$ with no constraints on s and t. Since our shape is a partition of $6d$, we have the condition $3t + 2s + r = 6d$. Hence for a given r, we need only verify that all the appropriate t in the range $0 \leq t \leq \lfloor \frac{6d-r}{3} \rfloor$, (with $s + t$ even for $r \neq 3$) are obtained.

For the exceptional r cases with $s + t$ even, we only need those t with $t + r \equiv 2d \pmod 4$. Hence given a shape (r, s, t) the next shape needed is $(r, s - 6, t + 4)$. Moreover, for r even, both s and t must be even. Consider Table 8.14. For $r = 0$, this table provides all the all the required partitions. When $t = 4$ we must have d even, while we need d odd for $t = 2$. For $d = 3$ and 4, only $t = 2d$, $2d - 4$, and $t = 0$ are required. Hence all the $r = 0$ cases are provided. When $r = 2$, Table 8.14 gives all the necessary tableaux since only those with $t \leq 2d - 2$ and $t \equiv 2d + 2 \pmod 4$ are required.

For $r = 3$ consider Table 8.15. In this case we need all odd $t \leq 2d - 3$ because $t = 2d - 1$ has $s = 0$ and thus is not required by Theorem 10. All such tableaux are listed except $d = 3$ with $t \geq 3$. When $t = 3$ we reduce to a $c = 5$ case with $s = 3$ and $r = 0$. When $t = 5$ and $d = 3$ we have $s = 0$ and hence the shape is not needed. For $r = 4$ Table 8.15 provides all necessary tableaux. In this cases we have s and t even with $t \leq 2d - 4$, $t \equiv 2d \pmod 4$. When $d = 3$ and $t = 6$ we have $s = 1$, so that shape is not required. Hence all the exceptional r cases are listed.

Now consider the exceptional s cases. These are tableaux having t even with $s \in \{0, 2, 4\}$, or $s = 3$ with no constraints on t. When $r \geq d$, $r \neq d + 1$, we may reduced to $R_i \vee V(d)$ for some R_i filled with $c = 5$ elements. (If $s = 3$ we may only reduce when $r \geq d + 5$.) Since we have already listed the exceptional r cases, we will also take $r \geq 5$. Hence for a given s, we need only verify that all the appropriate t in the range $\lceil \frac{5d-2s-1}{3} \rceil \leq t \leq \lfloor \frac{6d-2s-5}{3} \rfloor$ (with t even for $s \neq 3$) are obtained.

Consider Table 8.16. When $s = 0$ this provides all tableaux since those tableaux with $t < d^*$ have $r \geq d + 2$. For $s = 2$ the table provides all tableaux with $t \geq d + 2$. When $t < d + 2$ we have $r \geq d + 2$ provided $d \geq 5$. When $d = 3$ or $d = 4$ either $r < 5$ or we can reduce to a $c = 5$ case as well.

Similarly when $s = 3$, those tableaux with $t < d + 2$ are reducible to a $c = 5$ case provided $d \geq 7$. When $d = 6$, $t = 7$ we may reduce to a $c = 5$ case with $r = 3$; for smaller t we have $r \geq d + 5$. When $d = 5$ and $t = 5$, we may reduce to a $c = 5$ case with $r = 4$ as $s + t$ even; for smaller t, $r \geq d + 5$. When $d = 4$ and $t = 5$ we have $r < 5$. When $d = 3$ and $t = 2$ we may reduce to a $c = 5$ case with $r = 3$; for smaller t we have $r \geq d + 5$.

When $s = 4$ we need all even $t \leq 2d - 3$. Table 8.16 lists all tableaux with $t \geq d + 2$ and $d \geq 5$. When $t \leq d + 1$, t even, we may reduce to a $c = 5$ case provided $d \geq 6$. When $d = 5$, $t = 6$ we have $r = 4$ and we may reduce using $P_1(4)$ to a $c = 3$ case. For $d = 3$ or 4 there are no needed shapes with $5 \leq r \leq d + 1$. Hence Table 8.16 lists all the exceptional s cases.

The general cases of $r, s \geq 5$ are classified in Table 8.17 and Table 8.19. Since $r, s \geq 5$, we only need those tableaux with $0 \leq t \leq 2d - 5$. For a fixed t, we need all tableaux with $\frac{5d-3t-4}{2} \leq s \leq \frac{6d-3t-5}{2}$ since if $r \geq d + 5$ we can reduce to a $c = 5$ tableau. When t is odd, write $t = 2e - 1$ then we need all $s \leq 3d - 3e - 1$. Consider

S_6 of Table 8.17. First we let B, A and then C vary over their parameters This yields all $s \leq 3d - 3e$ with $t \geq 5$. For $t = 3$, we use S_7. First let D vary to its bound of $d-4$. Then take $S_7(d-4, \text{E})$ and $S_7(d-5, \text{E})$ with E varying over the even numbers. This yields all the required s. For $t = 1$ we use S_{12}. First vary B over its bounds to $d-2$. Then use $S_{12}(d-3, d-2, \text{C})$ and $S_{12}(d-3, d-3, \text{C})$ with C even to obtain the needed s. Note the bounds on d in these tableaux are necessary for coherence. Those tableaux with $d \leq 5$ and shapes with r, $s \geq 5$ which are not reducible to a $c = 5$ case are listed in Table 8.18. Hence all odd t are covered.

When t is even consider Table 8.19. Write $t = 2e$, so we want all $2d + \frac{d}{2} - 3e - 2 \leq s \leq 3d - 3e - 3$. Consider S_5. First let B and C vary. Then take $S_5(\text{A}, d-e, d-e)$ and $S_5(\text{A}, d-e, d-e-1)$ as A varies over all even numbers up to $d-e-1$ and $d-e-2$ respectively. This yields all $8 \leq s \leq 3d - 3e - 3$ as needed when $t \geq 6$. For $5 \leq s \leq 7$ the tableaux are listed individually. For $t = 2$ or 4, we use S_7 which covers all necessary s (as shown above for $t = 3$). When $t = 0$ we need $2d + \frac{d}{2} - 2 \leq s \leq 3d - 3$ which is provided in the table. Note the bounds on d in these tableaux are necessary for coherence. Those tableau with $d \leq 5$ and shapes with r, $s \geq 5$ which are not reducible to a $c = 5$ case are listed in Table 8.18. Hence all even t are covered by Table 8.19. Therefor all the required tableaux with $c = 6$ have been listed.

	Tableau	Conditions	t value	t covered
$r = 3$	$R_1(\textsc{z}-1,\textsc{z})$	$\mathsf{x} = 0$ $d \geq 6$, d odd	$t = d + 2\textsc{z} - 1$	$t = 2d + 2\textsc{z} - 1$ $d \geq 6$, $d \equiv 3 \pmod 4$
	$R_9(\textsc{z}-1, 1, 0, 0, 1)$	$\mathsf{x} = 2$	$t = d + 2\textsc{z} - 1$	$t = d + 2\textsc{z} - 1$ $d \equiv 2 \pmod 3$
	$R_6(\textsc{a}, \textsc{a}-1, \mathsf{x})$	$1 \leq \textsc{a} \leq \textsc{z} - 2$ $d \geq 9$	$t = d + 2\textsc{z} - 2\textsc{a} - 1$	$d + 1 \leq t \leq d + 2\textsc{z} - 3$ $d \geq 9$
	$R_{17}(\textsc{z}-2, \mathsf{x})$	$d \geq 6$	$t = d + 1$	$t = d + 1$, $d \geq 6$
	R_{18}	$d = 3$ or 5	$t = d + 1$	$t = d + 1$, $d = 3, 5$
	R_{19}		$t = d - 1$	$t = d - 1$
	$R_7(\textsc{a})$	$1 \leq \textsc{a} \leq \frac{d-1}{2} - 1$ d odd, $d \geq 5$	$t = d - 2\textsc{a} - 1$	$2 \leq t \leq d - 3$ $t \not\equiv d \pmod 2$ d odd, $d \geq 5$
	$R_8(\textsc{a})$	$1 \leq \textsc{a} \leq \frac{d}{2} - 2$ d even, $d \geq 6$	$t = d - 2\textsc{a} - 1$	$3 \leq t \leq d - 3$ $t \not\equiv d \pmod 2$ d even, $d \geq 6$
	$R_{12}(\frac{d}{2} - 2)$	d even, $d \neq 4$	$t = 1$	$t = 1$, d even, $d \neq 4$
	$P_4(1,1,1) \vee U_1(4)$	$d = 4$	$t = 1$	$t = 1$, $d = 4$
	U_6 U_7	$d > 5, d$ odd $d = 5$	$t = 0$ $t = 0$	$t = 0$, $d > 5, d$ odd $t = 0$, $d = 5$
$r = 4$	$R_1(\textsc{z}, \textsc{z})$	$\mathsf{x} = 2$ $\textsc{z}$ even	$t = d + \frac{2d-4}{3}$	$t = d + \frac{2d-4}{3}$ t even $d \equiv 2 \pmod 3$
	$R_2(\textsc{z}-1, \textsc{z}-1, 1)$	$\mathsf{x} = 1$	$t = d + \frac{2d-2}{3} - 2$	$t = d + \frac{2d-2}{3} - 2$ $d \equiv 1 \pmod 3$,
	$R_4(\textsc{a}, \textsc{a}-2, \mathsf{x})$	$2 \leq \textsc{a} \leq \textsc{z}$ $d \geq 6$	$t = d + 2\textsc{z} - 2\textsc{a}$	$d \leq t \leq d + 2\textsc{z} - 4$ $d \geq 6$
	$R_5(\textsc{a}, \textsc{a}-2)$	$2 \leq \textsc{a} \leq \frac{d^*}{2} - 1$	$t = d - 2\textsc{a}$	$2 \leq t \leq d - 4$, $d \geq 6$
	$R_{13}(3,2,1,1,1)$	$d = 5$	$t = 5$	$t = 5$, $d = 5$
	$R_{12}(\frac{d-5}{2})$	$d \geq 7$ d odd	$t = 1$	$t = 1$ $d \geq 7$ d odd
	$P_4(1,2,1) \vee U_1(4)$	$d = 5$	$t = 1$	$t = 1$, $d = 5$
	$U_1(d) \vee U_3(\frac{d}{2} - 2, \frac{d}{2})$	$d \equiv 0 \pmod 4$	$t = 0$	$t = 0$ $d \equiv 0 \pmod 4$

Table 8.10: Exceptional $r = 3$ and $r = 4$ cases for $c = 5$.

	Tableau	Conditions	t value	t covered
$s=0$	$R_1(\text{A},\text{B})$	$0<\text{A}\le\text{B}\le\text{Z}$ $\text{B}\ge\text{A}$ $d+\text{A}+\text{B}$ even	$t=d+\text{B}+\text{A}$	$d+2\le t\le d+2\text{Z}$ t even
$s=2$	$R_2(\text{A},\text{B},1)$	$0\le\text{B}\le\text{A}\le\text{Z}-1$ d even, if $\text{A}=\text{B}$	$t=d+\text{A}+\text{B}$	$d+1\le t\le d+2\text{Z}-3$ t even $d\ge 6$
$s=3$	$R_3(\text{A},\text{B})$	$0\le\text{A}<\text{B}\le\text{Z}-1$ $d\ge 6$	$t=d+\text{A}+\text{B}$	$d+1\le t\le d+2\text{Z}-3$ $d\ge 6$
	R_{15}	$d\ge 6$	$t=d$	$t=d,\ d\ge 6$
	$Q_5(3,3,0,1,1,1)$	$d=7$	$t=6$	$t=6,\ d=7$
	$Q_5(1,3,0,1,1,1)$	$d=5$	$t=4$	$t=4,\ d=5$
	$R_{13}(1,2,1,1,1)$	$d=4$	$t=3$	$t=3,\ d=4$
	$Q_6(2,1,0)$	$d=4$	$t=2$	$t=2,\ d=4$
$s=4$	$R_2(\text{Z}-2,\text{Z}-2,2)$	$\mathsf{x}=1,\ 2$ $d\ge 7$	$t=d+2\text{Z}-4$	$t=d+2\text{Z}-4$ $d\not\equiv 0 \pmod 3$ $d\ge 7$
	$R_2(\text{A},\text{B},2)$	$0\le\text{B}<\text{A}\le\text{Z}-2$ $d\ge 9$	$t=d+\text{A}+\text{B}$	$d+1\le t\le d+2\text{Z}-5$ $d\ge 9$

Table 8.11: Exceptional s cases for $c=5$.

	Tableau	Parameters	s value	s covered
$t = d + \mathrm{Z} + \mathrm{A}$ $d + \mathrm{Z} \le t \le d + 2\mathrm{Z} - 5$	$R_9(\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D},\mathrm{E})$	$0 \le \mathrm{A} \le \mathrm{Z} - 5$ $1 \le \mathrm{B} \le \lfloor \frac{\mathrm{Z}-\mathrm{A}-\mathsf{x}}{2} \rfloor$ $0 \le \mathrm{C} \le \mathrm{Z} - \mathrm{A} - \mathrm{B} - 1$ $0 \le \mathrm{D} \le \mathrm{B} + \mathsf{x} - 2$ or $\mathrm{D} = \mathrm{B} + \mathsf{x} - 1$, $\mathrm{E} \le 1$ $0 \le \mathrm{E} \le \mathsf{x}$ $\mathsf{x} \ge 1$, $d \ge 16$	$s = \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E}$	$5 \le s \le \frac{3\mathrm{Z}-3\mathrm{A}+3\mathsf{x}-6}{2}$ $s \le \frac{3\mathrm{Z}-3\mathrm{A}-1}{2}$ if $\mathsf{x} = 1$ $d \ge 16$
$t = d + \mathrm{Z}$, $d = 15$	$R_6(2,0,0)$	$d = 15$	$t = d + \mathrm{Z}$	$t = d + \mathrm{Z}$, $d = 15$
$t = d + \mathrm{Z} - \mathrm{A}$ $d \le t \le d + \mathrm{Z} - 1$	$R_{10}(\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D},\mathrm{E},\mathrm{F})$	$1 \le \mathrm{A} \le \mathrm{Z}$ $0 \le \mathrm{B} \le \lfloor \frac{\mathrm{A}}{2} \rfloor$ $0 \le \mathrm{C} \le \lfloor \frac{\mathrm{Z}}{2} \rfloor$ $1 \le \mathrm{D} \le \lceil \frac{\mathrm{Z}}{2} \rceil + \lfloor \frac{\mathrm{A}}{2} \rfloor$ $0 \le \mathrm{E} \le \lfloor \frac{\mathrm{Z}}{2} \rfloor$ $0 \le \mathrm{F} \le \lceil \frac{\mathrm{A}}{2} \rceil$ C even $d \ge 6$	$s = \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + \mathsf{x}$ $\mathrm{C} < \mathrm{D}$ $\mathrm{C} + \mathrm{D} < \mathrm{Z} + \mathrm{B}$ c $\mathrm{C} + \mathrm{E} < \mathrm{Z} - \mathrm{A} + \mathrm{B} + \mathrm{F}$ $\mathrm{D} + \mathrm{E} < \mathrm{Z} + \mathrm{F}$	$5 \le s$ $s \le \mathsf{x} + \lfloor \frac{3\mathrm{A}}{2} \rfloor + \lfloor \frac{3\mathrm{Z}}{2} \rfloor - 1$ $d \ge 6$
$t = d - 1$	$R_{16}(\mathrm{A})$	$1 \le \mathrm{A} \le d - 4$ $d \ge 5$	$s = \mathrm{A} + 3$	$4 \le s \le d - 1$ $d \ge 5$
$t = \mathrm{A} + \mathrm{B}$ $2 \le t \le d - 2$	$R_{11}(\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D},\mathrm{E},\mathrm{F})$	$1 \le \mathrm{A} \le d - \mathrm{B}$ $\mathrm{B} = 1$, or 2 $0 \le \mathrm{C} \le d - \mathrm{A} - 2$ $0 \le \mathrm{D} \le d - \mathrm{E} - \mathrm{B} - 1$ $1 \le \mathrm{E} \le \lceil \frac{d-\mathrm{A}-\mathrm{B}-1}{2} \rceil$ $1 \le \mathrm{F} \le \lfloor \frac{d-\mathrm{A}-\mathrm{B}-1}{2} \rfloor$ $\mathrm{F} \le \mathrm{E}$ $\mathrm{B} + \mathrm{D} > \mathrm{F}$ $\mathrm{A} + \mathrm{C} > \mathrm{E} + \mathrm{F}$ $d \ge 4$ $\mathrm{C} \ge 1$ if $t = 2$	$s = \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + 3$	$5 \le s$ $s \le 2d - t + \lfloor \frac{d-t}{2} \rfloor + 2$ $d \ge 4$ $s \ne 5$ if $t = 2$, $d = 5$

Table 8.12: General $c = 5$ cases for $t > 2$.

	Tableau	Parameters	s value	s covered
$t=2$	$P_3(2,1,0) \vee U_1(2)$	$d=5$	$s=5$	$s=5$, $d=5$
$t=1$	$R_{12}(\text{A})$	$1 \le \text{A} \le \lfloor \frac{d}{2} \rfloor - 1$	$s = 2d + \text{A} - 1$	$2d \le s$ $s \le 2d + \lfloor \frac{d}{2} \rfloor - 1$
	$P_4(1, \frac{d}{2} - 1, 1) \vee U_1(d)$	d even	$s = 2d - 1$	$s = 2d - 1$ d even
	$P_4(1, \frac{d-1}{2}, 1) \vee U_1(d-1)$	d odd	$s = 2d - 1$	$s = 2d - 1$ odd
	R_{14}	$d \ge 5$	$s = 2d - 2$	$s = 2d - 2$ $d \ge 5$
$t=0$	$U_1(d^*) \vee U_3(\text{A}, \lfloor \frac{d}{2} \rfloor)$	$0 \le \text{A} \le \lfloor \frac{\text{A}}{2} \rfloor$ A even	$s = 2d^* + \text{A}$	$2d^* \le s$ $s \le 2d^* + \lfloor \frac{d}{2} \rfloor$ s even
	$U_1(d^*) \vee P_3(0, \lfloor \frac{d-1}{2} \rfloor, \text{C})$	$0 \le \text{C} \le \lfloor \frac{d-1}{2} \rfloor - 1$ C even	$s = d^* + 2\lfloor \frac{d-1}{2} \rfloor + \text{C} + 1$	$d^* + 2\lfloor \frac{d-1}{2} \rfloor + 1 \le s$ $s \le d^* + 3\lfloor \frac{d-1}{2} \rfloor$ s odd

Table 8.13: General $c = 5$ cases for $t = 0$, 1, and 2.

	Tableau	Conditions	t value	t covered
$r = 0$	$S_1(\mathrm{z})$		$t = 2d$	$t = 2d$
	$S_5(d-e, d-e, d-e)$	$3 \leq e \leq d-2$ $d \equiv e \pmod 2$	$t = 2e$	$6 \leq t \leq 2d-4$ $t \equiv 2d \pmod 4$ $d \geq 5$
	$Q_4(2, 1, d-5, 3) \vee U_1(d)$	$d \geq 6$, d even	$t = 4$	$t = 4$ $d \geq 6$, d even
	$P_1(4) \vee U_3(2, 2)$	$d = 4$	$t = 4$	$t = 4$, $d = 4$
	S_8	$d \geq 5$, d odd	$t = 2$	$t = 2$ $d \geq 5$, d odd
	$U_1(d) \vee U_1(d) \vee U_1(d)$	d even	$t = 0$	$t = 0$, d even
$r = 2$	$S_5(d-e-1, d-e, d-e)$	$e \not\equiv d \pmod 2$ $3 \leq e \leq d-1$ $d \geq 4$	$t = 2e$	$6 \leq t \leq 2d-2$ $t \equiv 2 + 2d \pmod 4$ $d \geq 4$
	$Q_4(2, \frac{d-7}{2}, \frac{d-1}{2}, \frac{d-5}{2}) \vee U_1(d-1)$	$d \geq 9$, d odd	$t = 4$	$t = 4$ $d \geq 9$, d odd
	$P_4(3, 3, 3) \vee P_4(1, 2, 1)$	$d = 7$	$t = 4$	$t = 4$, $d = 7$
	$P_4(3, 1, 1) \vee P_4(1, 2, 1)$	$d = 5$	$t = 4$	$t = 4$, $d = 5$
	$Q_2(1) \vee U_1(2)$	$d = 3$	$t = 4$	$t = 4$, $d = 3$
	$Q_4(1, \frac{d-2}{2}, \frac{d-2}{2}, \frac{d-2}{2}) \vee U_1(d)$	d even	$t = 2$	$t = 2$, d even
	$U_2(d-1, 1) \vee U_1(d-1)$	d odd	$t = 0$	$t = 0$, d odd

Table 8.14: Exceptional $r = 0$ and $r = 2$ cases for $c = 6$.

	Tableau	Conditions	t value	t covered
$r=3$	$S_6(d-e,d-e,d-e)$	$3 \le e \le d-1$ $d \ge 4$	$t=2e-1$	$5 \le t \le 2d-3$ $d \ge 4$
	$S_9(2,1,d-4,1,d-4,d-2,1)$	$d \ge 5$	$t=3$	$t=3$, $d \ge 5$
	$S_{10}(3)$	$d=4$	$t=3$	$t=3$, $d=4$
	S_{11}		$t=1$	$t=1$
$r=4$	$S_4(d-6)$	$d \ge 6$	$t=2d-4$	$t=2d-4$ $d \ge 6$
	$S_5(d-e-2,d-e,d-e)$	$3 \le e \le d-4$ $e \equiv d \pmod 2$ $d \ge 7$	$t=2e$	$6 \le t \le 2d-8$ $t \equiv 2d \pmod 4$ $d \ge 7$
	$Q_2(1) \vee U_1(4)$	$d=5$	$t=6$	$t=6$, $d=5$
	$Q_4(2, \frac{d-4}{2}, \frac{d-4}{2}, \frac{d-4}{2}) \vee U_1(d)$	$d \ge 6$, d even	$t=4$	$t=4$ $d \ge 6$, d even
	$P_1(4) \vee U_4(2,2)$	$d=4$	$t=4$	$t=4$, $d=4$
	$Q_4(1, \frac{d-3}{2}, \frac{d-1}{2}, \frac{d-3}{2}) \vee U_1(d-1)$	$d \ge 5$, d odd	$t=2$	$t=2$ $d \ge 5$, d odd
	$Q_3(2,1,1) \vee U_1(2)$	$d=3$	$t=2$	$t=2$, $d=3$
	$U_1(d) \vee U_1(d) \vee U_1(d-2)$	d even	$t=0$	$t=0$ d even

Table 8.15: Exceptional $r=3$ and $r=4$ cases for $c=6$.

	Tableau	Conditions	t value	t covered
$s=0$	$P_1(d^*) \vee P_1(\text{A})$	$0 \le \text{A} \le d$ A even	$t=d^*+\text{A}$	$d^* \le t \le 2d^*$ t even
$s=2$	$S_2(\text{A})$	$0 \le \text{A} \le d-4$ $\text{A} \equiv d \pmod 2$ $d \ge 4$	$t=d+\text{A}+2$	$d+2 \le t \le 2d-2$ t even $d \ge 4$
$s=3$	$S_3(\text{A})$	$0 \le \text{A} \le d-5$ $d \ge 5$	$t=d+\text{A}+2$	$d+2 \le t \le 2d-3$ $d \ge 5$
	$P_1(4) \vee P_3(2,1,0)$	$d=5$	$t=6$	$t=6$, $d=5$
	$P_1(4) \vee U_4(2,1)$	$d=4$	$t=4$	$t=4$, $d=4$
$s=4$	$S_4(\text{A})$	$0 \le \text{A} \le d-5$ $d \ge 5$	$t=d+\text{A}+2$	$d+2 \le t \le 2d-3$ $d \ge 5$

Table 8.16: Exceptional s cases for $c=6$.

	Tableau	Parameters	s value	s covered
$t = 2e-1$ $5 \leq t \leq 2d-5$ t odd	$S_6(\text{A}, \text{B}, \text{C})$	$1 \leq \text{A}, \text{B}, \text{C} \leq d-e$ $\text{A}, \text{B} \geq \text{C}$ $\text{B} \geq \text{A}-1$ $3 \leq e \leq d-2$ $d \geq 5$	$s = \text{A}+\text{B}+\text{C}$	$5 \leq s \leq 3d-3e$ $d \geq 5$
$t = 3$	$S_7(1, 2, d-5, \text{D}, \text{E})$	$0 \leq \text{E} \leq d$ E even $1 \leq \text{D} \leq d-4$ $d \geq 5$	$s = d-2+\text{D}+\text{E}$	$d-1 \leq s \leq 3d-7$ $d \geq 5$
$t = 1$	$S_{12}(d-3, \text{B}, \text{C})$	$0 \leq \text{C} \leq d$ C even $0 \leq \text{B} \leq d-2$	$s = d+\text{B}+\text{C}$	$d \leq s \leq 3d-3$

Table 8.17: General $c = 6$ cases for odd t.

d	Tableau	Shape
$d = 5$	$P_3(2,1,0) \vee P_3(2,1,0)$	$t = 4,\ s = 6$
$d = 4$	$S_{10}(2)$	$t = 3,\ s = 5$
	$P_4(2,1,0) \vee U_4(2,1)$	$t = 2,\ s = 5$
	$U_3(2,2) \vee U_4(2,1)$	$t = 0,\ s = 9$

Table 8.18: General $c = 6$ cases for $d \leq 5$.

	Tableau	Parameters	s value	s covered
$t = 2e$ $6 \le t \le 2d-6$ t even	$S_5(\text{A}, \text{B}, \text{C})$	$1 \le \text{A}, \text{B}, \text{C} \le d-e$ $\text{B}, \text{C} > \text{A}$ A even, $d \ge 6$ $3 \le e \le d-3$	$s = \text{A} + \text{B} + \text{C}$	$8 \le s \le 3d - 3e - 3$ $d \ge 6$
	$P_3(e,1,0) \vee P_4(e,2,0)$	$0 \le e \le d-4$ e even	$s = 7$	$s = 7$, $d \ge 5$ e even
	$P_3(e+1,1,0) \vee P_4(e-1,2,0)$	$e \le d-4$ e odd	$s = 7$	$s = 7$, $d \ge 5$ e odd
	$S_5(2,2,2)$	$3 \le e \le d-3$	$s = 6$	$s = 6$, $d \ge 6$
	$P_3(e,1,0) \vee P_4(e,1,0)$	$e \le d-3$ e even	$s = 5$	$s = 5$, $d \ge 5$ e even
	$P_4(e+1,1,0) \vee P_4(e-1,1,0)$	$e \le d-3$ e odd	$s = 5$	$s = 5$, $d \ge 5$ e odd
$t = 4$	$S_7(2,2,d-6,\text{D},\text{E})$	$0 \le \text{E} \le d$ E even, $d \ge 6$ $1 \le \text{D} \le d-5$	$s = d - 3 + \text{D} + \text{E}$	$d - 2 \le s \le 3d - 9$ $d \ge 6$
$t = 2$	$S_7(1,1,d-4,\text{D},\text{E})$	$0 \le \text{E} \le d$ E even, $d \ge 5$ $1 \le \text{D} \le d-4$	$s = d - 1 + \text{D} + \text{E}$	$d \le s \le 3d - 6$ $d \ge 5$
$t = 0$	$U_1(d^*) \vee U_1(d^*) \vee U_1(\text{A})$	$0 \le \text{A} \le d$ A even	$s = 2d^* + \text{A}$	$2d^* \le s \le 3d^*$ s even
	$U_5(\frac{d}{2}, \frac{d}{2}) \vee U_1(\text{A})$	$d \ge 6$, d even $0 \le \text{A} \le d$ A even	$s = 2d - 3 + \text{A}$	$2d - 3 \le s \le 3d - 3$ s odd $d \ge 6$, d even
	$U_5(\frac{d-1}{2}, \frac{d+3}{2}) \vee U_1(\text{A})$	d odd, A even $0 \le \text{A} \le d$	$s = 2d - 3 + \text{A}$	$2d - 3 \le s \le 3d - 4$ s odd, d odd

Table 8.19: General $c = 6$ cases for even t.

8.5 Sufficiency for $c > 6$, d even

In Chapter 6 we algorithmically demonstrate how to reduce an arbitrary tableau to one of those tableaux filled with fewer elements. In the case where d is even, we reduced all tableaux to joins of tableaux with $c \leq 6$ to or those tableaux with $t < d$, $s < d+5$, $r < d+5$. Previously we showed all tableaux with $c \leq 6$ where constructed in Chapter 7. Now we will address the remaining cases with d even.

Since any tableau must satisfy $cd = 3t + 2s + r$, applying the bounds on r, s, and t we find $cd \leq 3(d-1) + 2(d+4) + d + 4 = 6d + 9$. Hence for $d > 9$ all such tableaux will have $c \leq 6$. For $d = 8$ and $d = 6$ it may be possible to have $c = 7$, while for $d = 4$, both $c = 7$ and $c = 8$ are possible.

When d=8 and c=7, only the shape with $(r,s,t) = (11,12,7)$ satisfy the constraints. However, we can reduce this shape by $V(d)$ to a $c = 6$ case with $r = 3$.

When $d = 6$ and $c = 7$, only the shapes with $(r,s,t) = (10,10,4)$, $(9,9,5)$ and $(7,10,5)$ satisfy the constraints. For $(10,10,4)$ we may reduce by $V(d)$ to a $c = 6$ case with $(r,s,t) = (4,10,4)$ since $s+t$ is even. For $(9,9,5)$ we may reduce by $V(d)$ to a $c = 6$ case with $(r,s,t) = (4,9,5)$ since $s+t$ is even. For $(7,10,5)$ we may reduce by $U_1(4)$ to a $c = 5$ case with $(r,s,t) = (3,6,5)$.

When $d = 4$ and $c = 7$, only the shapes with $(r,s,t) = (7,6,3)$, $(5,7,3)$, $(3,8,3)$, $(6,8,2)$, and $(8,7,2)$ satisfy the constraints. For $(7,6,3)$ we may reduce by $V(d)$ to a $c = 6$ case of $(3,6,3)$. For $(3,8,3)$ use $R_{19} \vee U_1(4)$. For $(5,7,3)$ we may reduce by $U_1(4)$ to the $c = 5$ cases $(5,3,3)$. For $(6,8,2)$ we may reduce by $V(d)$ to a $c = 6$ case of $(2,8,2)$ since $s+t$ is even. For $(8,7,2)$ we may reduce by $U_1(4)$ to the $c = 5$ case $(8,3,2)$.

When $d = 4$ and $c = 8$, only the shape with $(r,s,t) = (7,8,3)$ satisfies the constraints. It can be obtained by $S_{10}(3) \vee U_1(2)$. Thus when d is even, all shapes are reducible to tableaux filled with less than or equal to six elements and all the cases with $c \leq 6$ were obtained previously.

8.6 Sufficiency for $c > 6$, d odd

When d is odd, the reduction techniques of Chapter 6 work to reduce a tableau to $c \leq 8$. As we've already constructed those tableaux with $c \leq 6$, we will focus on those with $c = 7$ or 8.

First, assume $t \geq d-1$. So long as $r \notin \{0,2,3,4,5,7\}$ we may use $P_1(d-1)$ to reduce to a $c-3$ case. If $t \geq d-1$ and $r \in \{0,2,3,4,5,7\}$ we may use $P_4(d-2,1,1)$ to reduce to a $c-3$ case unless $s \in \{0,2,3,4,5,7\}$. For $s = 3,5,7$ we may still reduce by $P_4(d-2,1,1)$ when t is odd. For $r = 0,2,4$, $s = 3,5,7$ and t even, the shape is not needed by Theorem 10. For $r = 3,5,7$ and $s = 3,5,7$ with t even, no shapes are possible for $c = 8$. When $c = 7$ we have $d \geq 5$ so we may use $P_3(d-3,1,0)$ to reduce to a $c = 4$ case with $s+t$ even. For $s = 0,2,4$, we only need those shapes with t even. For $r = 3,5,7$ we may reduce by $P_1(d-1)$ to a $c-3$ case. This leaves those cases with $s = 0,2,4$, $r = 0,2,4$ and t even.

When $c = 7$ there are no shapes having $s = 0,2,4$, $r = 0,2,4$ and t even as d is odd. For $c = 8$, these shapes are obtainable, depending on $d \pmod 3$. We list the appropriate tableaux in Table 8.20. This completes all cases with $t \geq d-1$

If $t < d-1$ and $s < d+4$, but $r = d+3$ or $r \geq d+5$ we may use $V(d)$ to reduce to a $c-1$ case. When $c = 8$, there are no valid shapes with $r \leq d+4$ and $t < d-1$, $s < d+4$. For $c = 7$ this is also true provided $d \geq 5$. When $d = 3$ we need $(r,s,t) = (6,6,1)$ which is reducible by $V(d)$ to a $c = 6$ case with $r = 3$.

If $t < d-1$ and $s \geq d+4$ we may use $U_1(d-1)$ to reduce to a $c-2$ case, provided $r \notin \{0,2,3,4,6\}$. Consider those cases with $r \in \{0,2,3,4,6\}$ If $s \geq 2d+5$ or $s = 2d+3$ we may use $U_2(d,d)$ to reduce to a $c-4$ case. For $s \leq 2d-1$ there are no shapes with $t \leq d-2$ and $r \leq 6$. If $s = 2d$, $2d+2$ or $2d+4$, then we may still reduce via $U_2(d,d)$, provided t is even (which always occurs if $r = 0,2,4$). Thus we need only consider those tableaux with $s = 2d+1$, or $s = 2d$, $2d+2$, $2d+4$ with $r = 3$ or 6.

For $c = 7$ and $s \geq 2d$, $s \neq 2d+1$ only $r = 6$, $s = 2d$, $t = d-2$ is possible. This can be obtained by $P_4(d-2,1,1) \vee U_5(2,d-2)$ provided $d \geq 5$. When $d = 3$ we may

(r,s)	d	Tableau
$(0,0)$	$d \equiv 0 \pmod 3$	$Q_2(\mathrm{z}) \vee Q_2(\mathrm{z})$
$(0,2)$	$d \equiv 2 \pmod 3$	$W_1(2)$
$(0,4)$	$d \equiv 1 \pmod 3$	$Q_1(\mathrm{z}-1,0,1,1) \vee Q_1(\mathrm{z}-1,0,1,1)$
$(2,0)$	$d \equiv 1 \pmod 3$	$W_1(0)$
$(2,2)$	$d \equiv 0 \pmod 3$, $d > 3$	$Q_2(\mathrm{z}) \vee Q_1(\mathrm{z}-2,1,1,0)$
	$d = 3$	$S_1(1) \vee U_1(2)$
$(2,4)$	$d \equiv 2 \pmod 3$, $d > 5$	$Q_2(\mathrm{z}) \vee Q_1(\mathrm{z}-2,0,2,2)$
	$d = 5$	$S_1(1) \vee U_1(4)$
$(4,0)$	$d \equiv 2 \pmod 3$	$Q_2(\mathrm{z}) \vee Q_2(\mathrm{z})$
$(4,2)$	$d \equiv 1 \pmod 3$	$Q_2(\mathrm{z};\mathsf{x}=4) \vee Q_1(\mathrm{z}-1,0,1,1;\mathsf{x}=1)$
$(4,4)$	$d \equiv 0 \pmod 3$, $d > 9$	$Q_2(\mathrm{z}) \vee Q_1(\mathrm{z}-4,2,2,0)$
	$d = 9$	$Q_2(\mathrm{z}) \vee Q_5(4,4,1,0,2,1)$
	$d = 3$	$Q_1(1) \vee U_1(2) \vee U_1(2)$

Table 8.20: Exceptional r and s cases for $c = 8$.

reduced by $V(d)$ to a $c = 6$ case with $r = 3$. For $c = 8$ and $s \geq 2d$, $s \neq 2d+1$ only $s = 2d+4$, $r = 3$, $d = 5$ can occur. In that case, use $Q_5(2,1,2,0,1,1) \vee U_2(5,5)$.

This leaves those tableaux with $t < d-1$, $s = 2d+1$ and $r \in \{0,2,3,4,6\}$. When $r = 0,2,4$, we must have t odd, so in the $c = 8$ case there are no possible shapes. For $c = 7$, we get a valid shape only for $r = 4$, in which case we have $t = d-2$. For this use $U_1(d-1)$ to reduce to a $c = 5$ case with $r = 2$, $s = d+1$ which will still have $s+t$ even. For $r = 3$, there are no shapes satisfying $t < d-1$ and $s = 2d+1$ for either $c = 7$ or 8; similarly for $r = 6$. Thus all required shapes may be reduced to those filled with $c \leq 6$ elements.

8.7 Tableaux Disjointness

Our proof of Theorem 2 requires the tableaux we constructed to be disjoint. Since Lemma 3.4.9 showed that maximal tableaux are always disjoint, we need only be concerned with those tableaux which could not be put in maximal form, namely, U_8 and S_8.

Recall that $U_8 = \begin{array}{ccccccccccc} 4 & 3 & 3 & 3 & 3 & 1 & 2 & 2 & 4 & 4 & 4 \\ 1 & 1 & 1 & 2 & 2 \end{array}$ has $d = 4$ and $\omega_2 = (3, 2, 0, 0)$. When $d = 4$ the tableaux used in the reduction techniques are $P_1(4)$, $U_1(4)$, and $V(d)$. These are the only tableaux that would be joined with U_8, so it suffices to show U_8 is disjoint from these tableaux. Note that the weights of these tableaux consist only of 4's and 0's.

If U_8 were not disjoint from these tableaux, then there is some weight assignment of U_8, not equivalent to $\omega_2 = (3, 2, 0, 0)$ which uses a weight from at least one of these tableaux. However, since the only additional weights we may use are 4's and 0's, there is no way to have a weight assignment of length 5 using 0's, 2's, 3's, and 4's without being equivalent to $(3, 2, 0, 0)$. Hence the weights are disjoint from U_8.

Recall that $S_8 = \begin{array}{ccccccccccc} & & d-4 & & d-2 & d-4 & & 2 & & & \\ \hline 6 & 6 & 6 & 3 & 2 & 1 & 2 & 3 & 6 & 6 & 2 \\ 3 & 4 & 3 & 4 & 4 & 5 & 5 & 5 & 1 & 5 & 1 \\ 1 & 1 \end{array}$ with $\omega_{2,3} = \left(\begin{smallmatrix} 2 & 0 & d-3 & d & d & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ and $d \geq 5$. When d is even, the tableaux used in the reduction techniques are $P_1(d)$, $U_1(d)$, and $V(d)$. When d is odd, the tableaux used in the reduction techniques are $S_1(d)$, $U_2(d)$, whose weights are 0's and d's, $P_1(d-1)$, $U_1(d-1)$ whose weights are 0's and $d-1$'s, and $P_4(d-2, 1, 1)$ with $\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 1 \\ 0 & 0 & d-2 \end{smallmatrix}\right)$. Hence it suffices to show S_8 disjoint from these tableaux.

Now any weight assigned to S_8 must have $\lambda_3 = 2$. Since $d \geq 5$, there are no weights other of the listed tableaux than the weight $\binom{2}{2}$ of S_8 for which this is possible. This means if S_8 were not disjoint from these tableaux there would be a weight assignment of S_8 of the form $\left(\begin{smallmatrix} 2 & * & * & * & * & * \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ which is not equivalent to $\left(\begin{smallmatrix} 2 & 0 & d-3 & d & d & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$. The only weights we may use for the $*$'s are the weights of the listed tableaux, namely, 0's d's, $d-1$'s and $d-3$. Since the $*$'s sum to $3d-3$, they would have to be three $d-1$'s and two

0's. That is, the weights of $U_1(d-1)$ or $P_1(d-1)$. Then the weights of S_8 would need to be assigned to other tableaux, so a weight of d would need to be assigned to either $U_1(d-1)$ or $P_1(d-1)$, which is not possible as these tableaux are maximal. Hence there is no other weight assignment for S_8 and so the tableau is disjoint as required. Thus all the tableaux constructed in the proof of Theorem 2 are disjoint as desired.

Chapter 9

Proof of Theorem 3

Theorem 2 proves the generalized Foulkes' Conjecture for $c = 3$ without multiplicities. We can, however, strengthen this result to include multiplicities for those irreducibles corresponding to two row partitions. Namely,

Theorem 3. Let $n = 3b = cd$, with c, $d \geq 3$ and let $\lambda = [\lambda_1, \lambda_2]$ be a two row partition of n. Then every irreducible character χ^λ occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with multiplicity at least as large.

The proof of this theorem involves constructing the appropriate number of tableaux, primarily out of the basis elements given in Section 4.3. The tableaux constructed are maximal so linear independence is assured from Lemma 3.4.12 provided the weights are distinct. Once we have the basis tableaux for $c = 4$, 5, and 6, the procedure generalizes to provide the appropriate number of tableaux for any c.

Take $\lambda = [r + s, s]$. The multiplicity m_λ of $\mathcal{S}^\lambda$ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ was determined by Thrall in [20], which we listed in Theorem 11.

Since the multiplicity depends on the relative sizes of r and s we will handle these cases separately. Moreover, if $s = 6k + j$ and $r = 6h + i$, we will often simply construct $k + 1$ or $h + 1$ tableaux when possible to avoid detailed case analysis.

9.1 Case: $s \leq r$

Let $\lambda = [r + s, s]$ be a partition of n with $s \leq r$, where $n = 3b = cd$. We wish to construct m_λ linearly independent tableaux, where m_λ is the multiplicity of χ^λ in

$1^{\mathcal{S}_n}_{\mathcal{S}_b \wr \mathcal{S}_3}$ as described in Theorem 11. First we will construct these tableaux for $c = 4$, 5, and 6; then we will use these constructions in proving Theorem 3 for a general c. We will refer to the tableaux constructed in this way as basis tableaux. These constructions will make use of the $c = 3$ basis tableaux constructed in Section 4.3.

9.1.1 Basis Tableaux for $c = 4$, $s \leq r$

Given $\lambda = [r + s, s]$ a partition of n, we have $2s + r = 4d = 3b$. From this equation and $s \leq r$, we have $s \leq d + \lfloor \frac{d}{3} \rfloor$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 4. These will be our $c = 4$ basis tableaux.

When $s \leq r - d$, we can use the basis tableaux constructed in Section 4.3. Consider the partition $\lambda' = [r' + s, s]$ where $r' = r - d$. Since $s \leq r'$, we have $m_\lambda = m_{\lambda'}$. In Section 4.3 we constructed $m_{\lambda'}$ linearly independent $\mathcal{B}_p$, where $\mathcal{B}_p$ are the basis tableaux for $c = 3$. Take $\mathcal{B}_p \vee \frac{\mathrm{d}}{4}$ as the basis tableaux for $c = 4$. This works for $s \leq r - d$, so $r \geq s + d$. Hence $4d = 2s + r \geq 3s + d$ implies $s \leq d$.

When $d < s \leq d + \lfloor \frac{d}{3} \rfloor$, write $s = d + f$, with $1 \leq f \leq \lfloor \frac{d}{3} \rfloor$. Consider the tableau

$$T(\text{A}, \text{B}, \text{C}, \text{D}) = \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \\ \hline 4 & 3 & 4 & 3 \\ 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} \text{A} + \text{B} \leq d \\ \text{A, B, C, D} > 0 \\ \text{A} > \text{D} \\ \text{B} > \text{C} \\ \text{or A} = \text{D}, \text{B} = \text{C} \end{array}$$

$$\mathsf{w}(T) = (\text{A} + \text{B}, \text{C} + \text{D}, 0, 0)$$

If $\text{A} > \text{D}$ and $\text{B} > \text{C}$, no other weight assignments are possible for T, hence this tableau is non-zero and maximal. If $\text{A} = \text{D}$ and $\text{B} = \text{C}$, we may also have the tableau obtained by exchanging the rows. However, this has sign $(-1)^{\text{A}+\text{B}+\text{C}+\text{D}} = 1$ and thus the tableau is still non-zero.

Let $\mathcal{C}_p = T(d - 2p,\ p + 1,\ p,\ f - 1)$. Then $\lambda_2(T) = \text{A} + \text{B} + \text{C} + \text{D} = d - 2p + p +$

$1+p+f-1=d+f=s$. Hence these tableaux have the desired shape. Consider $\mathcal{C}_p$ for $p=1, 2, \ldots, \lfloor\frac{d-f}{2}\rfloor$. To insure the $\mathcal{C}_p$ are non-zero and maximal we need to check that the constraints on T are satisfied. For $f \neq 1$ all the parameters are greater than zero. Obviously, $\text{B}=p+1>\text{C}=p$. For $\text{A}>\text{D}$, we need $d-2p>f-1$. This is true provided $p<\frac{d-f+1}{2}$. Since $p \leq \lfloor\frac{d-f}{2}\rfloor$ this inequality holds.

The $\mathcal{C}_p$ are linearly independent by Lemma 3.4.12 if their max weights are distinct. We have $\mathsf{w}(\mathcal{C}_p)=(d-p+1, f+p-1, 0, 0)$. If $\mathsf{w}(\mathcal{C}_p)=\mathsf{w}(\mathcal{C}_{p'})$ for $p>p'$, then we must have $d-p+1=f+p'-1$, that is $d-f+2=p+p'$. But $p+p' \leq \frac{d-f}{2}+\frac{d-f}{2}-1=d-f-1$. Hence this cannot occur. Thus the $\mathcal{C}_p$ are linearly independent.

Since $s=d+f$, we have $m_\lambda \leq \lfloor\frac{d+f}{6}\rfloor+1$, so it suffices to construct $\lfloor\frac{d+f}{6}\rfloor+1$ tableaux. The $\mathcal{C}_p$ provide $\lfloor\frac{d-f}{2}\rfloor$ tableaux. To show $\lfloor\frac{d-f}{2}\rfloor \geq \lfloor\frac{d+f}{6}\rfloor+1$, it suffices to show $\frac{d-f}{2}-\frac{1}{2}>\frac{d+f}{6}$, or equivalently, $d-\frac{3}{2} \geq 2f$. This holds for $d \geq 5$ as $f \leq \lfloor\frac{d}{3}\rfloor$. When $d=3$ or $d=4$ then $f=\lfloor\frac{d}{3}\rfloor=1$, which is handled below. Hence for $s>d+1$ the $\mathcal{C}_p$ provided at least m_λ linearly independent tableaux.

For $s=d+1$, that is $f=1$, take $\mathcal{C}_p=T(d-p-2, 2, 1, p)$, for $1 \leq p < \lfloor\frac{d-2}{2}\rfloor$. Then the conditions on T are satisfied and such p exist for $d \geq 6$. We have $\mathsf{w}(\mathcal{C}_p)=(d-p, p+1, 0, 0)$ so the max weights are distinct. These $\mathcal{C}_p$ provide at least $\lfloor\frac{d-2}{2}\rfloor-1$ linearly independent tableaux and we need $\lfloor\frac{d+1}{6}\rfloor+1$ tableaux. Now $\lfloor\frac{d-2}{2}\rfloor-1 \geq \lfloor\frac{d+1}{6}\rfloor+1$ provided $d \geq 8$. When $d=7$, then $s=8$ and two tableaux are needed. Use $T(4,2,1,1)$ and $T(3,2,1,2)$. When $d=6$, then $s=7$ and only one tableau is needed. In this case, use $\mathcal{C}_p$ described above for $p=1$. When $d=5$ then $s=6$ so two tableaux are needed. Use $T(2,2,1,1)$ and $T(2,1,1,2)$. When $d=4$ then $s=5$ so one tableaux suffices. However, for $d=4$, $s=5$ there are no tableaux of maximal form. We will use the tableau U_8 constructed in Section 7.2. This tableau is non-zero but not maximal. Here we use maximal form only to show linear independence. Since only one tableau is needed for $s=5$, this tableau works. (The $d=4$, $s=5$ case is actually not needed for the $c=4$ basis tableaux since $n=3b=4d$ implies $3|d$, but we construct the basis tableaux for all $d \geq 3$ in order to simplify the construction process in Section 9.1.4. However we will not use U_8 in that construction.) When $d=3$, then $s=4$ and only one tableau is need, hence

$T(1,1,1,1)$ suffices.

9.1.2 Basis Tableaux for $c = 5$, $s \leq r$

Given $\lambda = [r+s, s]$ a partition of n, we have $2s + r = 5d = 3b$. From this equation and $s \leq r$, we have $s \leq \lfloor \frac{5d}{3} \rfloor$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 5. These will be our $c = 5$ basis tableaux.

When $s \leq r - d$, we can use the $c = 4$ basis tableaux constructed in Section 9.1.1. Consider the partition $\lambda' = [r'+s, s]$ where $r' = r-d$. Since $s \leq r'$, we have $m_\lambda = m_{\lambda'}$. In Section 9.1.1 we constructed $m_{\lambda'}$ linearly independent $\mathcal{C}_p$, where $\mathcal{C}_p$ are the basis tableaux for $c = 4$. Take $\mathcal{C}_p \vee \frac{\mathrm{d}}{5}$ as the basis tableaux for $c = 5$. This works for $s \leq r - d$, so $r \geq s + d$. Hence $5d = 2s + r \geq 3s + d$ implies $s \leq \lfloor \frac{4d}{3} \rfloor$.

For $\lfloor \frac{4d}{3} \rfloor < s \leq \lfloor \frac{5d}{3} \rfloor$ consider

$$T_1(\text{A},\text{B}) = \begin{array}{ccccccc} \multicolumn{3}{c}{\text{A}} & \lfloor\frac{d}{2}\rfloor & \lceil\frac{d}{2}\rceil & \multicolumn{2}{c}{\text{B}} \\ \hline 4\,4 & 5 & 4 & 3 & 4 & 5 \\ 2\,2 & 2 & 1 & 1 & 3 & 3 \end{array}$$

$$1 \leq \text{A} \leq d-2$$

$$1 \leq \text{B} \leq \lfloor \tfrac{d}{2} \rfloor - 3$$

$$\text{A} > \text{B}$$

$$s \geq d + 6$$

$$d \geq 8$$

$$\mathsf{w}(T_1) = (d, \text{A}+2, \text{B}+1, 0, 0)$$

T_1 is defined for $s \geq d+6$, $d \geq 8$. (When $d < 8$ then $s < d+6$ since $s \leq \lfloor \frac{5d}{3} \rfloor$.) Since $\text{A} > \text{B}$, there are no other weight assignments possible and the tableau T_1 is maximal. Let $\mathcal{E}_p = T_1(s-d-3-p, p)$ for $1 \leq p \leq \lfloor \frac{s-d}{2} \rfloor - 2$. For $\mathcal{E}_p$ to be non-zero and maximal we need the conditions on T_1 to be satisfied. We have $\text{A} > \text{B}$ since $p \leq \lfloor \frac{s-d}{2} \rfloor - 2$, while $p \geq 1$ implies $\text{A} = s - d - 3 - p < d - 2$. Since $d \geq 8$, then $\text{B} \leq \lfloor \frac{d}{2} \rfloor - 3$. Note $\text{A} > \text{B}$ implies these max weights are distinct. Hence this construction provides $\lfloor \frac{s-d}{2} \rfloor - 2$ distinct, linearly independent tableaux when $s \geq d+6$, $d \geq 8$.

We will need an additional tableau, so take:

$$T_2(\text{A},\text{B}) = \begin{array}{cccc} & \text{A} & \text{B} & \\ \hline 5 & 3 & 5 & 4 \\ 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} 1 \le \text{A}, \text{B} \le d-1 \\ s \ge 4 \end{array}$$

$$\mathsf{w}(T_2) = (\text{A}+1, \text{B}+1, 0, 0)$$

Clearly T_2 is non-zero and maximal. Given s, take $\mathcal{E}'_q = T_2(d-q, s-2-d+q)$ for $q = 1, 2, \ldots, d - \lfloor \frac{s-2}{2} \rfloor$. To insure $\mathcal{E}'_q$ is non-zero and maximal we need the conditions of T_2 are satisfied. Since $\lfloor \frac{4d}{3} \rfloor < s$ and $d \ge 3$, then $s \ge 4$. We have $1 \le \text{A} \le d-1$ because $q < d$ since $s \le \lfloor \frac{5d}{3} \rfloor$. Similarly the bounds on s show $1 \le \text{B} \le d-1$. The $\mathcal{E}'_q$ are linearly independent because $\text{A} < \text{B}$ for $q < d - \lfloor \frac{s-2}{2} \rfloor$. This provides $d - \lfloor \frac{s}{2} \rfloor + 1$ tableaux.

We need $m_\lambda = \lfloor \frac{s}{6} \rfloor + 1$ linearly independent tableaux. By Lemma 3.4.12 the tableaux $\mathcal{E}_p$ and $\mathcal{E}'_q$ are linearly independent since they have different max weights. First consider $s \ge d+6$ which implies $d \ge 8$. Then we have both $\mathcal{E}_p$ and $\mathcal{E}'_q$, for a total of $\lfloor \frac{s-d}{2} \rfloor - 2 + d - \lfloor \frac{s}{2} \rfloor + 1$ tableaux. This is greater than or equal to $\lfloor \frac{s}{6} \rfloor + 1$, since $s \le \lfloor \frac{5d}{3} \rfloor$ and $d \ge 8$.

For $s < d+6$ we only have $\mathcal{E}'_q$, which provides $d - \lfloor \frac{s}{2} \rfloor + 1$ tableaux. This is greater than or equal to $\lfloor \frac{s}{6} \rfloor + 1$ provided $d \ge 5$, since $s \le d+5$. When $d = 4$, then $s \le \lfloor \frac{5d}{3} \rfloor = 6$ so the two $\mathcal{E}'_q$ suffice. When $d = 3$ then $s \le 5$ and hence one tableau, $\mathcal{E}'_1$ is sufficient. Hence we have constructed at least m_λ tableaux as desired.

9.1.3 Basis Tableaux for $c = 6$, $s \le r$

Given $\lambda = [r+s, s]$ a partition of n, we have $2s + r = 6d = 3b$. From this equation and $s \le r$, we have $s \le 2d$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 6. These will be our $c = 6$ basis tableaux.

When $s \le r - d$, we can use the $c = 5$ basis tableaux constructed in Section 9.1.2. Consider the partition $\lambda' = [r'+s, s]$ where $r' = r - d$. Since $s \le r'$, we have $m_\lambda = m_{\lambda'}$. In Section 9.1.2 we constructed $m_{\lambda'}$ linearly independent $\mathcal{E}_p$, where $\mathcal{E}_p$ are the basis tableaux for $c = 5$. Take $\mathcal{E}_p \vee \begin{array}{c} \text{d} \\ \hline 6 \end{array}$ as the basis tableaux for $c = 6$. This works for $s \leq r - d$, so $r \geq s + d$. Hence $6d = 2s + r \geq 3s + d$ implies $s \leq d + \lfloor \frac{2d}{3} \rfloor$.

For $d + \lfloor \frac{2d}{3} \rfloor < s \leq 2d$ we want to construct $m_\lambda \leq \lfloor \frac{s}{6} \rfloor + 1$ linearly independent tableaux. We will do this primarily by joining two $c = 3$ basis tableaux. In addition, we will use the following tableaux:

$$\mathcal{G}^1 = \begin{array}{cccc} \multicolumn{4}{c}{d-2\ f} \\ \hline 5 & 3 & 4 & 6 \\ 1 & 1 & 2 & 2 \end{array} \qquad 1 \leq f < d$$

$$\omega(\mathcal{G}^1) = (d-1, f+1, 0, 0, 0, 0)$$

$$\mathcal{G}^2 = \begin{array}{ccccc} \multicolumn{5}{c}{d-2\ f-2\ \ 2} \\ \hline 4 & 5 & 4 & 6 & 6 \\ 1 & 1 & 2 & 2 & 3 \end{array} \qquad 4 \leq f < d, \quad d \geq 5$$

$$\omega(\mathcal{G}^2) = (d-1, f-1, 2, 0, 0, 0)$$

$$\mathcal{G}^3 = \begin{array}{cccccc} \multicolumn{6}{c}{2\ d-4\ d-3\ \ \ 2} \\ \hline 6 & 4 & 5 & 6 & 4 & 6 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad d > 6$$

$$\omega(\mathcal{G}^3) = (d-2, d-2, 3, 0, 0, 0)$$

$$\mathcal{G}^4 = \begin{array}{cccccc} \multicolumn{6}{c}{d-2\ 2\ d-5\ \ 2} \\ \hline 6 & 4 & 6 & 5 & 5 & 6 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad d > 7$$

$$\omega(\mathcal{G}^4) = (d-1, d-3, 3, 0, 0, 0)$$

These tableaux are all maximal. Except for $\mathcal{G}^2$, no other weight assignments are possible, hence these tableaux are non-zero. For $\mathcal{G}^2$, the weight assignment $(d-1, f-1, 0, 0, 0, 2)$ is also valid. Since this has sign $(-1)^2$, $\mathcal{G}^2$ is also non-zero.

Consider $\lfloor\frac{5d}{3}\rfloor < s < 2d$. (The $s = 2d$ case will be handled separately.) Write $s = f + d$, then $\lfloor\frac{2d}{3}\rfloor < f < d$. This means $f > 2$ always, and $f < 6$ only for $d \geq 7$. Moreover $f \geq 4$ for $d \geq 5$ so the conditions on $\mathcal{G}^2$ are satisfied.

We will use the $c = 3$ basis tableaux for much of our construction. Let $\mathcal{D}_{p'}$ represent the basis elements for $c = 3$ of shape $[2d, d]$ as described in Section 4.3. Let $\mathcal{F}_{\overline{p}}$ represent the basis elements for $c = 3$ of shape $[3d-f, f]$ as described in Section 4.3. We will use the $\mathcal{D}_{p'}$, $\mathcal{F}_{\overline{p}}$, and the $\mathcal{G}^i$ to construct the m_λ linearly independent tableaux for $c = 6$. This construction will depend on d and $f \mod 6$. Write $d = 6k' + i'$ and $f = 6\overline{k} + \overline{i}$, with $0 \leq i', \overline{i} \leq 5$. For $i' \neq 1$ there exist $m_{[2d,d]} = k' + 1$ linearly independent $\mathcal{D}_{p'}$, with $p' = 0, 1, \ldots, k'$, and k' tableaux for $i' = 1$. Similarly, there exists $m_{[3d-f,f]} = \overline{k} + 1$ linearly independent $\mathcal{F}_{\overline{p}}$ with $\overline{p} = 0, 1, \ldots, \overline{k}$ for $\overline{i} \neq 1$ and $\overline{k}$ when $\overline{i} = 1$. Now $s = d + f = 6(k' + \overline{k}) + i' + \overline{i} = 6k + i$, so $m_\lambda \leq k + 1$. Since $k \leq k' + \overline{k} + 1$, it suffices to construct $k' + \overline{k} + 2$ linearly independent tableaux. We will first consider $d > 6$, $f \geq 6$. Since $k', \overline{k} > 0$ consider:

$$\begin{aligned}
&\mathcal{G}_{\overline{p}} = \mathcal{D}_{k'} \vee \mathcal{F}_{\overline{p}} && \overline{p} = 0, 1, \ldots, \overline{k} \\
&\mathcal{G}_{p'} = \mathcal{D}_{p'} \vee \mathcal{F}_{\overline{k}} && p' = 0, 1, \ldots, k' - 1 \\
&\mathcal{G}_0 = \mathcal{D}_0 \vee \mathcal{F}_0 &&
\end{aligned}$$

There are $\overline{k} + 1 + k' + 1 = \overline{k} + k' + 2$ tableaux listed here. By Lemma 3.4.12 if their max weights are distinct, these tableaux are linearly independent. Since these tableaux are in maximal form we can simply compare their weights. We have:

$$\begin{aligned}
&\omega(\mathcal{G}_{\overline{p}}) = (d, 0, 0, 2\overline{p} + \overline{i} + 4\overline{k}, 2(\overline{k} - \overline{p}), 0) && \overline{p} = 0, 1, \ldots, \overline{k} \\
&\omega(\mathcal{G}_{p'}) = (2p' + i' + 4k', 2(k' - p'), 0, f, 0, 0) && p' = 0, 1, \ldots, k' - 1 \\
&\omega(\mathcal{G}_0) = (4k' + i', 2k', 0, 4\overline{k} + \overline{i}, 2\overline{k}, 0) &&
\end{aligned}$$

Now $d > f$ and $p' \leq k' - 1$, so we have $\omega(\mathcal{G}_{\overline{p}}) \neq \omega(\mathcal{G}_{p'})$ since $\omega(\mathcal{G}_{p'})$ does not contain a weight of d. Both of these weights are distinct from $\omega(\mathcal{G}_0)$, since they each contain at least three 0's while $\omega(\mathcal{G}_0)$ contains only two 0's. The weights within each of these collections of tableaux are distinct because each collection $\{\mathcal{D}_{p'}\}$ and $\{\mathcal{F}_{\overline{p}}\}$ are linearly independent by Section 4.3.

When all these tableaux $\mathcal{G}$ exist we have a set of basis tableaux for $c = 6$. However, depending on the conditions on d and f, we may only have k' or $\overline{k}$ basis tableaux to work with. In those situtations we will need to use the appropriate $\mathcal{G}^i$ to complete our set of tableaux. Recall that we are taking $d > 6$ and that $f \geq 6$.

When $d \not\equiv 1 \pmod 6$ and $f \not\equiv 1 \pmod 6$, all the tableaux $\mathcal{G}$ exist. Hence we have the $k' + \overline{k} + 2$ linearly independent tableaux required.

When $d \not\equiv 1 \pmod 6$, $f \equiv 1 \pmod 6$, there exist only $\overline{k}$ linearly independent $\mathcal{F}_{\overline{p}}$ with $\overline{p} = 1, 2, \ldots, \overline{k}$. Hence we have all the tableaux listed above except for those with $\overline{p} = 0$ and $\mathcal{G}_0$. In place of those basis elements, use $\mathcal{G}_1$ and $\mathcal{G}_2$. This provided $\overline{k}+k'+2$ tableaux. They are linearly independent by Lemma 3.4.12 provided the weights of $\mathcal{G}^1$ and $\mathcal{G}^2$ are distinct from the weights of $\mathcal{G}_p$. We have $\omega(\mathcal{G}^1) = (d-1, f+1, 0, 0, 0, 0)$ and $\omega(\mathcal{G}^2) = (d-1, f-1, 2, 0, 0, 0)$. Clearly these weights are distinct from eachother. Now $\omega(\mathcal{G}^1) \neq \omega(\mathcal{G}_{p'})$ since the number of 0's differ. If $\omega(\mathcal{G}^1) = \omega(\mathcal{G}_{\overline{p}})$ we must have $f + 1 = d$ and $\overline{p} = \overline{k}$. Similarly $\omega(\mathcal{G}_{p'})$ will be distinct from $\omega(\mathcal{G}^2)$ unless $d - 1 = f$, $p' = k' - 1$. We have $\omega(\mathcal{G}^2)$ distinct from $\omega(\mathcal{G}_{\overline{p}})$ since d does not occur in its weight. Hence for $f \neq d - 1$ these $\overline{k} + k' + 2$ tableaux are linearly independent.

When $f = d-1$, use $\mathcal{G}^3$ and $\mathcal{G}^4$ in place of $\mathcal{G}^1$ and $\mathcal{G}^2$. Since $\omega(\mathcal{G}^3)$ contains neither d nor f it is clearly distinct from the weights of $\mathcal{G}_p$. Similarly, $\omega(\mathcal{G}^4)$ is distinct from $\omega(\mathcal{G}_{\overline{p}})$ since it does not contain d. While $\omega(\mathcal{G}^4)$ does contain f, $\mathcal{G}_{p'}$ cannot contain the weight 3 since $d > 7$. Thus we have sufficient linearly independent tableaux.

Now consider $d \equiv 1 \pmod 6$ and $f \not\equiv 1 \pmod 6$. We have the $\mathcal{G}_{\overline{p}}$ and $\mathcal{G}_{p'}$ listed earlier, for $p' \neq 0$, along with $\mathcal{G}^1$ and $\mathcal{G}^2$. The discussion in the $f \equiv 1 \pmod 6$ case above shows these are linearly independent provided $f \neq d - 1$. Similarly when $f = d - 1$ we can replace $\mathcal{G}^1$ and $\mathcal{G}^2$ with $\mathcal{G}^3$ and $\mathcal{G}^4$. If $d = 7$ then $\mathcal{G}^4$ does not exist. However, then $s = 13$ on only two tableaux, $\mathcal{D}_1 \vee \mathcal{F}_1$ and $\mathcal{G}^3$, are needed.

When $d \equiv 1 \pmod 6$ and $f \equiv 1 \pmod 6$, we can write $d = 6k' + 1$, $f = 6\overline{k} + 1$. Then $s = 6(k' + \overline{k}) + 2$ so $k' + \overline{k} + 1$ tableaux suffice. Use the $\mathcal{G}_{\overline{p}}$ and $\mathcal{G}_{p'}$ listed earlier, for $\overline{p}$, $p' \neq 0$, along with $\mathcal{G}^1$ and $\mathcal{G}^2$. This provides the requisite number of tableaux.

They are linearly independent by the previous discussion since $f \neq d-1$, as $d \equiv f \pmod{6}$.

Now consider $3 \leq d \leq 6$. Then $f > \lfloor \frac{2d}{3} \rfloor$ implies $f > 2$. Since $s = d + f$, we must have $s \leq 11$. Hence two linearly independent tableaux will suffice, namely, $\mathcal{D}_0 \vee \mathcal{F}_0$ and $\mathcal{G}^1$. As in previous discussions, these weights are distinct provided $f \neq d-1$. When $f = d-1$, $s = 2d-1$. We need two tableaux only when $d = 5$ or $d = 6$. Thus $\mathcal{D}_0 \vee \mathcal{F}_0$ suffices for $d = 3$ and 4. When $d = 5$ or 6 use $\mathcal{D}_0 \vee \mathcal{F}_0$ and $\mathcal{T}$, where

$$\mathcal{T} = \begin{array}{c} 2\,4\,2\,2\,1 \\ \hline 5\,6\,4\,5\,6 \\ 1\,1\,3\,2\,2 \end{array} \qquad d = 6, \qquad \omega = (6,3,2,0,0,0)$$

$$\mathcal{T} = \begin{array}{c} \;\;4\,2\,2 \\ \hline 5\,6\,5\,4 \\ 1\,1\,2\,3 \end{array} \qquad d = 5 \qquad \omega = (5,2,2,0,0,0)$$

These tableaux are clearly maximal and non-zero. Since $f = d-1$ is not a weight of $\mathcal{T}$, we have that $\mathcal{T}$ and $\mathcal{D}_0 \vee \mathcal{F}_0$ are linearly independent.

When $f < 6$ we have $d \leq 7$ because $f > \lfloor \frac{2d}{3} \rfloor$. Since all $d \leq 6$ cases were done above, only $d = 7$ remains. In this case we have only $f = 5$. Thus $s = 12$ and three tableaux are required. We can use $\mathcal{D}_1 \vee \mathcal{F}_0$, $\mathcal{G}^1$, and $\mathcal{G}^2$. These tableaux are linearly independent by previous discussions.

Now consider $s = 2d$. Write $d = 6k' + i$, so $s = 6(2k') + 2i'$. Hence $2k' + 2$ linearly independent tableaux will suffice. Let

$$\mathcal{A}_1 = \begin{array}{cccc} d-2 & d-2 & 2 & \\ \hline 4 & 5 & 5 & 4\,6 \\ 1 & 1 & 2 & 2\,3 \end{array}$$

$$\omega(\mathcal{A}_1) = (d-1, d-1, 2, 0, 0, 0)$$

This tableau is maximal. Although there are many valid weight assignments, all such assignments have positive sign, hence $\mathcal{A}_1$ is non-zero.

When $d \not\equiv 1 \pmod 6$ we have $k'+1$ linearly independent tableaux $\mathcal{D}_{p'}$. Hence we can use:

$$\begin{aligned}
&\mathcal{D}_p \vee \mathcal{D}_p \qquad p = 0,\, 1, \ldots, k' \\
&\mathcal{D}_{k'} \vee \mathcal{D}_{p'} \qquad p' = 0,\, 1, \ldots, k'-1 \\
&\mathcal{A}_1
\end{aligned}$$

with weights:

$$\begin{aligned}
&\omega(\mathcal{D}_p \vee \mathcal{D}_p) = (2p + i' + 4k', 2(k'-p), 0, 2p + i' + 4k', 2(k'-p), 0) \\
&\omega(\mathcal{D}_{k'} \vee \mathcal{D}_{p'}) = (d, 0, 0, 2p + i' + 4k', 2(k'-p), 0) \\
&\omega(\mathcal{A}_1) = (d-1, d-1, 2, 0, 0, 0)
\end{aligned}$$

This provides $2k'+2$ tableaux, provided $k' > 0$. Their weights are clearly distinct so they are linearly independent by Lemma 3.4.12. When $d \equiv 1 \pmod 6$, we have $s = 6(2k')+2$ so only $2k'+1$ tableaux are needed. All the tableaux listed above work except for $p = 0$ and $p' = 0$, providing $2k'$ linearly independent tableaux. In addition use:

$$\mathcal{A}_2 = \begin{array}{cccccc} 2 & d-3 & d-4 & 2 & 2 \\ \hline 5 & 4 & 5 & 6 & 6 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad\qquad d \geq 6$$

$$\omega(\mathcal{A}_2) = (d-1, d-2, 3, 0, 0, 0)$$

This tableau is maximal and non-zero. Since $d \equiv 1 \pmod 6$ we have $d > 6$. Thus $\mathcal{A}_2$ provides the additional tableau and its weight is distinct from the other tableaux, insuring linear independence.

When $k' = 0$ we have $d < 6$. Then $s < 10$ so two tableaux, $\mathcal{A}_1$ and $\mathcal{D}_0 \vee \mathcal{D}_0$, suffice.

9.1.4 Basis Tableaux for $c > 6$, $s \leq r$

Let $\lambda = [r+s, s]$ be a partition of n, with $s \leq r$, where $2s + r = cd = n$. We want to construct m_λ linearly independent basis tableaux for an arbitrary c. In Sections 4.3, 9.1.1, 9.1.2, and 9.1.3 we constructed basis tableaux for $c \leq 6$, which we will make use of in this construction. In addition we will use the following tableaux:

$$\mathcal{A}_1 = \begin{array}{cccc} d-2 & & d-2 & 2 \\ \hline 4 & 5 & 5 & 4\ 6 \\ 1 & 1 & 2 & 2\ 3 \end{array}$$

$$\omega(\mathcal{A}_1) = (d-1, d-1, 2, 0, 0, 0)$$

$$\mathcal{A}_2 = \begin{array}{cccccc} 2 & d-3 & d-4 & 2 & 2 & \\ \hline 5 & 4 & 5 & 6 & 6 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad d \geq 6$$

$$\omega(\mathcal{A}_2) = (d-1, d-2, 3, 0, 0, 0)$$

$$\mathcal{A}_3 = \begin{array}{cccccc} 2 & d-4 & d-4 & 2 & 2 & 2 \\ \hline 5 & 5 & 6 & 6 & 4 & 4 \\ 1 & 2 & 1 & 3 & 3 & 2 \end{array} \qquad d \geq 6$$

$$\omega(\mathcal{A}_3) = (d-2, d-2, 4, 0, 0, 0)$$

$$\mathcal{A}_4 = \begin{array}{cccccc} 2 & d-4 & d-5 & 2 & 3 & 2 \\ \hline 5 & 4 & 5 & 6 & 6 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad d \geq 5$$

$$\omega(\mathcal{A}_4) = (d-2, d-3, 5, 0, 0, 0)$$

$$\mathcal{A}_5 = \begin{array}{ccc} 4 & 2 & 2 \\ \hline 4 & 5 & 6 \\ 1 & 2 & 3 \end{array} \qquad d = 4$$

$\omega(\mathcal{A}_5) = (4, 2, 2, 0, 0, 0)$

These tableaux are all maximal. Although some of these tableaux have additional valid weight assignments, all such assignments have positive sign. Hence these tableaux are non-zero.

To construct $\lambda = [r+s, s]$-tableaux with $s \leq r$, write $s = md + f$, with $0 \leq f < d$ and $r = md + pd + g$ with $0 \leq g \leq d$. Then $3md + pd + 2f + g = cd$, so $2f + g = xd$ for some x. This means $c = 3m + p + x$. If $p + x \geq 3$, a λ-tableau may be written $\mathcal{D} \vee \mathcal{B} \vee (c - 3m - 3)V(d)$, where $\mathcal{D}$ is a tableau of shape $[2md, md]$ filled with $3m$ elements, $\mathcal{B}$ is a $[3d - f, f]$ tableau filled with 3 elements, and $V(d)$ is the one row tableau. We will first consider this case and handle the $p + x < 3$ case later.

We have $s = md + f$, so writing $d = 6k' + i'$, $f = 6\overline{k} + \overline{i}$, with $0 \leq i', \overline{i} \leq 5$ gives $s = 6(mk' + \overline{k}) + mi' + \overline{i}$. Since $\lfloor \frac{mi' + \overline{i}}{6} \rfloor \leq m$, it suffices to construct $\lfloor \frac{s}{6} \rfloor + 1 \leq mk' + \overline{k} + m + 1$ linearly independent tableaux. If $m < 2$ we may simply use the basis tableaux constructed for $c = 6$ along with $V(d)'s$, so assume $m \geq 2$.

Let $\mathcal{D}_p$ be the $c = 3$ basis tableaux of shape $[2d, d]$ described in Section 4.3. There are $m_{[2d,d]} = k' + 1$ such tableaux when $i' \neq 1$ and k' for $i' = 1$. Let $\mathcal{B}_q$ be the $c = 3$ basis tableaux of shape $[3d - f, f]$ constructed in Section 4.3. This tableau has $s \leq r$ since $f < d$. There are $m_{[3d-f,f]} = k' + 1$ such tableaux for $\overline{i} \neq 1$ and k' tableaux when $\overline{i} = 1$. Take $f > 1$ and $d \geq 6$. The $d < 6$ and $f \leq 1$ cases will be handled separately. Consider the following tableaux forms (with the appropriate number of $V(d)$'s as necessary):

$$I. \qquad \ell\mathcal{D}_p \vee (m - \ell)\mathcal{D}_{k'} \vee \mathcal{B}_q \qquad \begin{array}{l} \ell = 1, 2, \ldots, m \\ p = 0, 1, \ldots, k' - 1 \\ q = 0, 1, \ldots, \overline{k} \end{array}$$

$II.$	$m\mathcal{D}_{k'} \vee \mathcal{B}_q$	$q = 0, 1, \ldots, \overline{k}$
$III.$	$\ell\mathcal{A}_j \vee (m-2\ell)\mathcal{D}_{k'} \vee \mathcal{B}_q$	$\ell = 1, 2, \ldots, \lfloor \frac{m}{2} \rfloor$ $q = 0, 1, \ldots, \overline{k}$ $j = 1, 2, 3, 4$ $d \geq 6$ if $j \neq 1$

Note that those tableaux with $p = 0$ or $q = 0$ exist only when $i' \neq 0$ or $\overline{i} \neq 0$ respectively. We will not make use of these tableaux unless necessary. However, even when $\overline{k} = 0$, at least one $\mathcal{B}_q$ exists, so if we regard $\overline{k}$ as the number of $\mathcal{B}_q$'s, we may assume $\overline{k} \geq 1$. When $d \not\equiv 1 \pmod 6$ we need $mk' + \overline{k} + m + 1$ tableaux. The list above (taking $q \neq 0$) provides at least $mk'\overline{k} + \overline{k} + 4\lfloor \frac{m}{2} \rfloor \overline{k}$ tableaux, which suffices since $4\lfloor \frac{m}{2} \rfloor \geq m + 1$. When $d \equiv 1 \pmod 6$, we need only $mk' + \lfloor \frac{m}{6} \rfloor + 2$ tableaux. If $k' \geq 2$ the list above provides at least $m(k'-1)\overline{k} + \overline{k} + 4\lfloor \frac{m}{2} \rfloor$ tableaux. This is sufficient for $m \neq 3$. When $m = 3$ we need $3k' + \overline{k} + 2$ tableaux when $\overline{i} = 3$, 4, or 5 and $3k' + \overline{k} + 1$ tableaux for $\overline{i} = 0$, 1, and 2. When $\overline{i} \leq 2$ the tableaux listed suffice. For $\overline{i} \geq 3$ we need an additional tableau so use the tableau of the Form II with $q = 0$. When $k' = 1$, the tableaux of the Form I don't exist. Hence we have only $\overline{k} + 4\lfloor \frac{m}{2} \rfloor$ tableaux when $q \neq 0$ and we need $m + \overline{k} + \lfloor \frac{m}{6} \rfloor + 2$. For $m \neq 3$ this is sufficient. However, one additional tableau is needed for $m = 3$, when $\overline{i} = 3$, 4, or 5. In this case we may use $q = 0$ in Form II for the remaining tableau.

To show linear independence of these tableaux it suffices, by Lemma 3.4.12, to show that the max weights are distinct. For max weights we have:

$I.$ $\quad (4k' + 2p + i', 2(k-p), 0)^{\ell} \vee (d, 0, 0)^{(m-\ell)} \vee (4\overline{k} + 2q + \overline{i}, 2(\overline{k} - q), 0)$

$II.$ $\quad (d, 0, 0)^m \vee (4\overline{k} + 2q + \overline{i}, 2(\overline{k} - q), 0)$

$III-1.\quad (d-1,d-1,2,0,0,0)^{\ell}\vee(d,0,0)^{(m-2\ell)}\vee(4\bar{k}+2q+\bar{i},2(\bar{k}-q),0)$

$III-2.\quad (d-1,d-2,3,0,0,0)^{\ell}\vee(d,0,0)^{(m-2\ell)}\vee(4\bar{k}+2q+\bar{i},2(\bar{k}-q),0)$

$III-3.\quad (d-2,d-2,4,0,0,0)^{\ell}\vee(d,0,0)^{(m-2\ell)}\vee(4\bar{k}+2q+\bar{i},2(\bar{k}-q),0)$

$III-4.\quad (d-2,d-3,5,0,0,0)^{\ell}\vee(d,0,0)^{(m-2\ell)}\vee(4\bar{k}+2q+\bar{i},2(\bar{k}-q),0)$

Consider those tableaux of Form I. If $\omega(I(\ell,p,q))=\omega(I(\ell',p',q'))$, then counting the number of d's shows $\ell=\ell'$, while counting the number of $4k'+p+i'$'s indicates $\ell=1$. If $p=p'$ then $q=q'$, which is not possible for distinct tableaux. Hence by Lemma 3.4.13, the weights are distinct because $f<d$. Those of Form II are distinct due to the distinct weights of $\mathcal{B}_q$. Similarly those of the Form $III-j$ are distinct by the number of d's and distinctness of $\omega(\mathcal{B}_q)$. If $\omega(III-i(\ell,q))=\omega(III-j(\ell',q'))$, then by counting the number of d's we have $\ell=\ell'$. Then by counting the number of 0's, $d-1$'s, and $d-2$'s, we find the weights must be distinct for $d>6$. (If $d=6$ the Forms $III-2$ and $III-4$ have the same weights.) To see the different forms have distinct weights, first count the number of d's. Obviously $\omega(II)\neq\omega(I)$ or $\omega(III)$. If $\omega(I(\ell))=\omega(III(\ell'))$ then $\ell=2\ell'$. However, counting the number of 0's shows the weights are distinct.

When $d=6$ then $s\leq 6m+5$, so $m+1$ tableaux suffice. Since there are two $\mathcal{D}$'s and one $\mathcal{B}$, Forms I and II provide the requisite number of linearly independent tableaux.

Now consider the case where $d<6$. Here $s=mi'+\bar{i}$ so $m+1$ tableaux suffices. In this case the $\mathcal{A}_j$, $j=2,3$ do not exist. However for $d=5$ we have $\mathcal{A}_4$ and for $d=4$ we have $\mathcal{A}_5$. Moreover, we have exactly one $\mathcal{D}$ and one $\mathcal{B}$. Hence the appropriate Forms II, $III-1$, and $III-4$ or $III-5$ provied $2\lfloor\frac{m}{2}\rfloor+1$ tableaux. This suffices for even m. When m odd we need one additional tableau. For $m\geq 5$ use $\mathcal{A}_1\vee\mathcal{A}_i\vee(m-4)\mathcal{D}_0\vee\mathcal{B}_0$ where $i=4$ or 5 as appropriate. If $m=3$, then $s\leq 19$ so the three tableaux listed will suffice except when $d=5$ and $f=3$. In this case also use the non-zero maximal tableau:

$$\mathcal{T} = \begin{array}{c} 4\ \ 4\ \ 4\,4 \\ \hline 5\,9\,6\,9\,7\,8 \\ 1\,1\,2\,2\,3\,4 \end{array}$$

$$\omega(\mathcal{T}) = (5, 5, 4, 4, 0, 0, , 0, 0, 0)$$

When $d = 3$, we have $s = 3m + 2$ since $1 < f < d$. Hence $\lfloor \frac{3m+2}{6} \rfloor + 1$ tableaux are needed. We have $m\mathcal{D}_0 \vee \mathcal{B}_0$ and $\ell\mathcal{A}_1 \vee (m - 2\ell)\mathcal{D}_0 \vee \mathcal{B}_0$, which provide $\lfloor \frac{m}{2} \rfloor + 1$ tableaux. Since $m \geq 2$ we have $\lfloor \frac{m}{2} \rfloor \geq \lfloor \frac{3m+2}{6} \rfloor$, so these tableaux suffice.

For linear independence, we need only check distinctness of max weights by Lemma 3.4.12. First consider $d = 4$ or 5. Since $\omega(\mathcal{A}_5) = \omega(\mathcal{A}_3)$ our discussion on linear independence for $d \geq 6$ still holds. The only additional tableau is $\mathcal{A}_1 \vee \mathcal{A}_i \vee (m - 4)\mathcal{D}_0 \vee \mathcal{B}_0$ which clearly has a distinct weight. Similarly, by counting the number of 5's and 4's, the weight of $\mathcal{T}$ is also distinct. Hence these tableaux are linearly independent. When $d = 3$ the tableaux listed are a subset of the tableaux for $d \geq 6$ and hence are linearly independent by the prior discussion. This covers all cases with $p + x \geq 3$ provided $f > 1$. The $f = 0$ and $f = 1$ cases will be handled after the $p + x < 3$ case.

Now assume that $p + x < 3$. Recall that if $s = md + f$, with $0 \leq f < d$ and $r = md + pd + g$, then $3md + pd + 2f + g = cd$ and $2f + g = xd$ for some x. This means $c = 3m + p + x$, so if $p + x < 3$ then λ-tableau with $s \leq r$ may be written as $\mathcal{D} \vee \mathcal{F} \vee (c - 3m - 3)V(d)$. Here $\mathcal{D}$ is a tableau of shape $[2(m - 1)d, (m - 1)d]$ filled with $3(m - 1)$ elements and $\mathcal{F}$ is a $[(3 + x - 1)d - f, f + d]$ tableau filled with $3 + x$ elements. When $f > 0$, we have $x > 0$, which means $\mathcal{F}$ is a $c = 4$ or $c = 5$ tableau. We will first consider the case where $m \geq 3$, $d \geq 6$ and $f > 1$.

If $d = 6k' + i'$ and $f = 6\overline{k} + \overline{i}$, with $0 \leq i', \overline{i} \leq 5$, then $s = md + f$, so we still need $\lfloor \frac{s}{6} \rfloor + 1 \leq mk' + \overline{k} + m + 1$ tableaux. The number of tableaux $\mathcal{D}$ is the same as before. Let $\mathcal{F}_q$ be the $c = 4$ or 5 basis tableaux of shape $[cd - f - d, f + f]$. There are at least $k' + \overline{k}$ such tableaux. Moreover since $d \geq 6$ and $f > 1$ there is always at least 2 such tableaux.

Consider the tableaux of the following forms:

$I.$	$\ell\mathcal{D}_p \vee (m-\ell-1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1, 2, \ldots, m-1$ $p = 0, 1, \ldots, k'-1$ $q = 1, 2, \ldots, k'+\overline{k}$
$II.$	$(m-1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$q = 1, 2, \ldots, k'+\overline{k}$
$III.$	$\ell\mathcal{A}_j \vee (m-2\ell-1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1, 2, \ldots, \lfloor\frac{m-1}{2}\rfloor$ $q = 1, 2, \ldots, k'+\overline{k}$ $j = 1, 2, 3, 4$ $d \geq 6$ if $j \neq 1$

As before, the tableaux with $p = 0$ exist only for $d \not\equiv 1 \pmod 6$. When $d \not\equiv 1 \pmod 6$, the construction above provides $(m-1)k'(k'+\overline{k}) + \overline{k} + k' + 4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k})$. Now $k'+\overline{k}$ denotes the number of $\mathcal{F}$'s, which we can assume is at least 2. Then since $4\lfloor\frac{m-1}{2}\rfloor \geq m+1$ for $m \neq 4$, this construction provides sufficient tableaux. If $m = 4$ then $4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k}) \geq 8$, so this construction is sufficient.

When $d \equiv 1 \pmod 6$, we need only $mk' + \overline{k} + \lfloor\frac{m}{6}\rfloor + 2$ tableaux. For $k' \geq 2$, we have $(m-1)(k'-1)(k'+\overline{k}) + \overline{k} + k' + 4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k})$ tableaux by the construction above, which is sufficient. If $k' = 1$, the tableaux of Form I do not exist, so we have only $k'+\overline{k}+4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k})$ tableaux. However, since we know that there are always at least two $\mathcal{F}_q$'s, we have $k'+\overline{k} \geq 2$. Thus the tableaux listed are sufficient.

When $d < 6$, the tableaux $\mathcal{A}_j$ with $j = 2$ and 3 do not exist. However we do have $\mathcal{A}_4$ for $d = 5$ and $\mathcal{A}_5$ for $d = 4$. Also, since $d < 6$, we may no longer assume that there are at least two $\mathcal{F}_q$'s (unless $d = 5$ and $f = 3$ or 4). However there is always at least one. In addition, there is only one $\mathcal{D}$. Under these constraints we have tableaux of the Forms II, $III-1$, and $III-4$ or $III-5$ when $d=5$ or 4 respectively. For $m \geq 5$ also use $\mathcal{A}_1 \vee \mathcal{A}_j \vee (m-5)\mathcal{D}_0 \vee \mathcal{F}_0$. This provides $2\lfloor \frac{m-1}{2} \rfloor + 2$ tableaux for $m \geq 5$. For $d=4$ we need at most $\lfloor \frac{4m+3}{2} \rfloor + 1$, hence this this suffices. For $d=5$ and $f=2$ we need at most $\lfloor \frac{5m+2}{6} \rfloor + 1$ which we have. Our construction suffices except for $m=3$ and 4. When $m=3$, then $s \leq 17$ and three tableaux, II, $III-1$, and $III-4$ or $III-5$ suffice. For $m=4$ we need one additional non-zero maximal tableau, so use

$$\mathcal{T} = \begin{array}{ccccccccc} d-1 & 1 & d-1 & 1 & d-1 & 1 & d-1 & 1 & 2 \\ \hline 6 & 7 & 8 & 7 & 9 & 12 & 10 & 12 & 11 \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \end{array}$$

$$\omega(\mathcal{T}) = (d,d,d,d,2,0,0,0)$$

When $d=5$, $f \geq 3$ there are two $\mathcal{F}_q$'s hence the Forms II, $III-1$ and $III-4$ give $4\lfloor \frac{m-1}{2} \rfloor + 1$ tableaux. Since at most $\lfloor \frac{5m+4}{6} \rfloor + 1$ tableaux are needed, this suffices.

When $d=3$ we need $\lfloor \frac{3m+2}{6} \rfloor + 1$ tableaux. From Forms II and $III-1$, we have $\lfloor \frac{m-1}{2} \rfloor + 1$ tableaux. This is sufficient except for m odd. When m even we need one additional tableau, so use $\mathcal{T}$ listed above.

Now to consider linear independence. By Lemma 3.4.12 it suffices to show that the max weights are distinct. First consider when the $\mathcal{F}$ are $c=4$ basis tableaux. The max weights are:

$I.$ $\quad (4k'+2p+i', 2(k'-p), 0)^{\ell} \vee (d,0,0,)^{m-\ell-1} \vee (d-q+1, f+q-1, 0, 0)$

$II.$ $\quad (d,0,0)^{m-1} \vee (d-q+1, f+q-1, 0, 0)$

$III-1.$ $\quad (d-1, d-1, 2, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell-1} \vee (d-q+1, f+q-1, 0, 0)$

$III-2.$ $\quad (d-1, d-2, 3, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell-1} \vee (d-q+1, f+q-1, 0, 0)$

$III-3.$ $\quad (d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell-1} \vee (d-q+1, f+q-1, 0, 0)$

$III-4.$ $\quad (d-2, d-3, 5, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell-1} \vee (d-q+1, f+q-1, 0, 0)$

Consider those tableaux of Form I. If $\omega(I(\ell,p,q)) = \omega(I(\ell',p',q')$, then counting the number of 0's shows $\ell = \ell'$. By counting the number of different weights we find $\ell = 1$. Now if $p = p'$ then we must have $q = q'$, which is not possible for distinct tableaux. Hence by Lemma 3.4.13, these weights are distinct. The weights of Form II are distinct by the construction of the $\mathcal{F}$'s in Section 9.1.1.

The weights of Form $III - j$ are distinct by counting the number of 0's and by the distinctness of the $\mathcal{F}$'s. If $\omega(III - j) = \omega(III' - i)$ counting the number of 0's and different numbers shows $\ell = \ell' = 1$. So if the weights are not distinct, then $\omega(A_j \vee \mathcal{F}_q) = \omega(A_i \vee \mathcal{F}_{q'})$, which implies $\mathcal{A}_i$ and $\mathcal{A}_j$ must have one non-zero element in common. Then by Lemma 3.4.13, these weights are distinct if $f > 2$. When $f = 2$ the tableaux have the same length and the lemma does not apply. If $f = 2$ then $\omega(III - 3(\ell = 1, q = 4)) = \omega(III - 4(\ell = 1, q = 3))$, though all other weights are distinct.

We have $\omega(I) \neq \omega(II)$, by counting the number of 0's. Forms II and III are also distinct by counting the number of 0's. If $\omega(I) = \omega(III')$, counting the number of 0's shows that $\ell = \ell'$. Then counting the number of d's shows that $\ell = 1$, $q' = 1$, and $q \neq 1$. If the weights are equal, then $(4k' + 2p + i', 2(k' - p), d - q + 1, f + q - 1) = (\omega(A_j), f)$. Since $q \leq \lfloor \frac{d-f}{2} \rfloor$, these weights are distinct unless $f = 2$. When $f = 2$ then $\omega(I(\ell = 1, p = k' - 1, q = j) = \omega(III - j(\ell' = 1, q' = 1))$, $j \neq 1$. Hence for $f > 2$ all the tableaux listed are linearly independent.

When $f = 2$ some of the tableaux in our list have the same max weights, and hence may be linearly dependent. If we eliminate these tableaux with duplicate max weights from our list we have $(m-1)k'(k'+\overline{k})+k'+\overline{k}+4\lfloor \frac{m-1}{2} \rfloor (k'+\overline{k})-4$ linearly independent tableaux when $d \not\equiv 1 \pmod 6$. We need at most $\lfloor \frac{m(6k'+i')+2}{6} \rfloor + 1 \leq mk' + m$ tableaux, which we have since we may still take $k' + \overline{k} \geq 2$. If $d \equiv 1 \pmod 6$ then we have $(m-1)(k'-1)(k'+\overline{k}) + k' + \overline{k} + 4\lfloor \frac{m-1}{2} \rfloor (k'+\overline{k}) - 4$ linearly independent tableaux for $k' > 1$. This is sufficient since only $mk' + \lfloor \frac{m+2}{6} \rfloor + 1$ tableaux are needed. When $k' = 1$ those tableaux of Form I don't exist, so we have $k' + \overline{k} + 4\lfloor \frac{m-1}{2} \rfloor (k'+\overline{k}) - 1$ linearly independent tableaux, which is sufficient. Thus the $d \geq 6$, $f > 1$, $m \geq 3$ case is finished for $c = 4$.

When $d < 6$ the same tableaux are used with the substitution of $\mathcal{A}_5$. However, $\mathcal{A}_5$ has the same weight as $\mathcal{A}_3$ so the above argument applies. (Note that by the conditions of $\mathcal{F}_q$, $q \leq \lfloor \frac{d-f}{2} \rfloor \leq 1$, so the max weight duplication does not occur.) In addition we also use the tableaux $\mathcal{A}_1 \vee \mathcal{A}_j \vee (m-5)\mathcal{D}_0 \vee \mathcal{F}_0$, for $j = 4$ or 5. However, counting the number of 0's and d's shows that this tableau is distinct from our previous collection; otherwise $\omega(\mathcal{A}_1 \vee \mathcal{A}_j \vee \mathcal{F}_0) = \omega(\mathcal{A}_i \vee \mathcal{A}_i \vee \mathcal{F}_q)$ which is impossible.

In addition, when $m = 4$, $d = 5$ or $d = 3$, m even, we also have the tableau $\mathcal{T}$. Counting the number of 0's and d's shows this tableau has a distinct max weight as well. Hence when $\mathcal{F}$ is a $c = 4$ tableau, we have linear independence for $f > 1$ and $m \geq 3$.

Now consider the case where $\mathcal{F}$ is a $c = 5$ basis tableau. The $c = 5$ basis tableaux were constructed in Section 9.1.2. These tableaux have two different types of max weights corresponding to the T_1 and T_2 tableaux constructions used. Those of form T_1 have weights $(d, f-q-1, q+1, 0, 0)$ where $q = 1, 2, \ldots, \lfloor \frac{f}{2} \rfloor - 2$ (when $f \geq 6$). Those of the form T_2 have weights $(d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$ with $\overline{q} = 1,\ 2,\ d - \lfloor \frac{d+f-2}{2} \rfloor$. We will refer to the basis tableaux using the T_1 tableaux as q-forms and those using T_2 as $\overline{q}$-forms. Then the max weights for the general tableaux have the following forms:

$I(q)$. $\quad (4k'+2p+i', 2(k'-p), 0)^{\ell} \vee (d, 0, 0,)^{m-\ell-1} \vee (d, f-q-1, q+1, 0, 0)$

$I(\overline{q})$. $\quad (4k'+2p+i', 2(k'-p), 0)^{\ell} \vee (d, 0, 0,)^{m-\ell-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$II(q)$. $\quad (d, 0, 0)^{m-1} \vee (d, f-q-1, q+1, 0, 0)$

$II(\overline{q})$. $\quad (d, 0, 0)^{m-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$III-1(q)$. $\quad (d-1, d-1, 2, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d, f-q-1, q+1, 0, 0)$

$III-1(\overline{q})$. $\quad (d-1, d-1, 2, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$III-2(q)$. $\quad (d-1, d-2, 3, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d, f-q-1, q+1, 0, 0)$

$III-2(\overline{q})$. $\quad (d-1, d-2, 3, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$III-3(q)$. $\quad (d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d, f-q-1, q+1, 0, 0)$

$III-3(\overline{q})$. $\quad (d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$III-4(q).\quad (d-2,d-3,5,0,0,0)^{\ell}\vee(d,0,0)^{m-2\ell-1}\vee(d,f-q-1,q+1,0,0)$

$III-4(\overline{q}).\quad (d-2,d-3,5,0,0,0)^{\ell}\vee(d,0,0)^{m-2\ell-1}\vee(d-\overline{q}+1,f-1+\overline{q},0,0,0)$

These tableaux are linearly independent by Lemma 3.4.12 provided their max weights are distinct. To show these weights are distinct we will often count the number of 0's and d's in each weight. For convenience we will list these values below:

Form	# 0's	# d's
$I(q)$.	$2m-\ell$	$m-\ell$
$I(\overline{q})$.	$2m-\ell+$	$m-\ell-1$ $\quad \overline{q}\neq 1$ $m-\ell$ $\quad \overline{q}=1$
$II(q)$.	$2m$	m
$II(\overline{q})$.	$2m+1$	$m-1$ $\quad \overline{q}\neq 1$ m $\quad \overline{q}=1$
$III-j(q)$.	$2m-\ell$	$m-2\ell$
$III-j(\overline{q})$.	$2m-\ell+1$	$m-2\ell-1$ $\quad \overline{q}\neq 1$ $m-2\ell$ $\quad \overline{q}=1$

Table 9.1: Weights of 0 or d.

Consider those tableaux of Form $I(q)$. If $\omega(I(\ell,p,q))=I(\ell',p',q')$, then Table 9.1.4 shows that $\ell=\ell'$. Then counting the number of different numbers shows $\ell=1$. If $p=p'$ then we must have $q=q'$ and vice versa, but this can't happen since the tableaux are different. Then since $f<d$, the argument of Lemma 3.4.13 shows the weights are distinct. The same reason holds for Form $I(\overline{q})$. Table 9.1.4 also shows $\omega(I(q))\neq\omega(I(\overline{q}))$. The tableaux of Form II are distinct since the max weights of the $\mathfrak{F}$ are distinct for $c=5$ by Section 9.1.2. Moreover, $\omega(II(q))\neq\omega(II(\overline{q}))$ by Table 9.1.4. The distinctness of max weights for Form $III-j(q)$ or $III-j(\overline{q})$ follows from Table 9.1.4 and distinctness of $c=5$ basis tableaux max weights. Also from Table 9.1.4 we have $\omega(III-j(q))\neq\omega(III-j(\overline{q}))$. Now suppose $\omega(III-j(q,\ell))=\omega(III-i(q',\ell'))$. Table 9.1.4 shows that $\ell=\ell'$

and counting the number of different numbers shows $\ell = 1$. Since $i \neq j$, the weights of $\mathcal{A}_i$ and $\mathcal{A}_j$ must have exactly one non-zero weight in common. Thus the argument of Lemma 3.4.13 applies and hence the weights are distinct. Similarly $\omega(III - j(\overline{q}, \ell)) \neq \omega(III - i(\overline{q}', \ell'))$ for $f > 2$ by Lemma 3.4.13. (When $f = 2$, $\ell = 1$ the conditions of Lemma 3.4.13 are not met.) The values of Table 9.1.4 are sufficient to show $\omega(III - j(q, \ell)) \neq \omega(III - i(\overline{q}', \ell'))$. This shows the max weights within each tableau form are distinct for $f > 2$. Showing that $\omega(I) \neq \omega(II) \neq \omega(III)$ follows directly from Table 9.1.4.

When $f = 2$ our discussion on linear independence holds except for a few tableaux of the Form $III(\overline{q})$. Specifically, $\omega(III - 2(\ell = 1, \overline{q} = 3)) = \omega(III - 3(\ell = 1, \overline{q} = 2))$, $\omega(III - 2(\ell = 1, \overline{q} = 4)) = \omega(III - 4(\ell = 1, \overline{q} = 2))$, and $\omega(III - 3(\ell = 1, \overline{q} = 4)) = \omega(III - 4(\ell = 1, \overline{q} = 3))$. Thus we have three fewer linearly independent tableaux than we originally calculated. If we eliminate these tableaux with duplicate max weights from our list we have $(m-1)k'(k'+\overline{k})+k'+\overline{k}+4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k})-3$ linearly independent tableaux when $d \not\equiv 1 \pmod 6$. We need at most $\lfloor\frac{m(6k'+i')+2}{6}\rfloor+1 \leq mk'+m$ tableaux, which we have since we may still take $k' + \overline{k} \geq 2$. If $d \equiv 1 \pmod 6$ then we have $(m - 1)(k' - 1)(k' + \overline{k}) + k' + \overline{k} + 4\lfloor\frac{m-1}{2}\rfloor(k' + \overline{k}) - 3$ linearly independent tableaux for $k' > 1$. This is sufficient since only $mk' + \lfloor\frac{m+2}{6}\rfloor + 1$ tableaux are needed. When $k' = 1$ those tableaux of the Form I don't exist, but we still have $2 + 8\lfloor\frac{m-1}{2}\rfloor - 3$ linearly independent tableaux, which is sufficient. Hence we have enough tableaux when $f = 2$. Thus the $d \geq 6$, $f > 1$, $m \geq 3$ case is finished for $c = 5$.

When $d < 6$ only the $\overline{q}$-form tableaux exist for $c = 5$. These tableaux have the same max weights as the $c = 4$ tableaux $\mathcal{F}$ (with the exception of an extra zero.) Thus by the same argument as in that case, these tableaux are linearly independent. Hence when $\mathcal{F}$ is a $c = 5$ tableau, we have linear independence for $f > 1$ and $m \geq 3$.

These constructions assumed $m \geq 3$. If $m = 1$, The $c = 4$ or $c = 5$ basis tableaux (joined with sufficient $V(d)$'s) suffice. However, when $m = 2$, we need tableaux with $c = 7$ or 8 elements. This case must be dealt with separately. (We will still assume

$f > 1$.) Tableaux of Form I listed previously ($\mathcal{D}_p \vee \mathcal{F}_q$) still work for this case. These tableaux are linearly independent by our previous discussion. As before, let $d = 6k' + i'$, $f = 6\overline{k} + \overline{i}$, with $0 \leq i', \overline{i} \leq 5$. Since $s = 2d + f$, we need at most $2k' + \overline{k} + 3$ tableaux. The tableaux $\mathcal{F}_q$ have $\lambda_2 = d + f$ so there are at least $k' + \overline{k}$ such tableaux. So for $d \not\equiv 1 \pmod 6$, we have $(k'+1)(k'+\overline{k})$ tableaux. This provides sufficient tableaux unless $k' = 1$, $\overline{k} = 1$, or $\overline{k} = 0$, $k' < 2$. When $k' = \overline{k} = 1$, then $d + f \geq 14$ so at least three $\mathcal{F}_q$'s exists. Hence the Form I construction is sufficient. In the remaining cases, computing exactly how many $\mathcal{F}_q$ exist and precisely how many tableaux are needed shows the Form I tableaux are sufficient except for: $d = 11$, $f = 2$, $d = 3$, $f = 2$, $d = 4$, $f = 3$, and $d = 5$, $f = 2, 3, 4$. (For instance, when $d = 6$ and $f = 5$ there are two $\mathcal{F}$'s and two $\mathcal{D}$'s, so Form I provides 4 tableaux. Since $s = 17$, only three tableaux are needed.)

When $d = 11$ and $f = 2$, there are two $\mathcal{D}$'s and $\mathcal{F}$'s for a total of 4 Form I tableaux. Since $s = 24$, five tableaux are required. In addition to the Form I tableaux, use

$$\mathcal{T} = \begin{array}{ccc} 10 & 10 & 4 \\ \hline 4 & 5 & 6 \\ 1 & 2 & 3 \end{array} \qquad \omega(\mathcal{T}) = (10, 10, 4, 0, 0, 0)$$

which is non-zero, maximal, and linearly independent.

For $d = 3$, $f = 2$, we have $c = 8$. Since $s = 8$ two tableaux are needed. In addition to $\mathcal{D}_0 \vee \mathcal{F}_0$, use the non-zero maximal tableau:

$$\mathcal{T} = \begin{array}{cccc} 2 & 2 & 2 & 2 \\ \hline 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{array}$$

$$\omega(\mathcal{T}) = (2, 2, 2, 2, 0, 0, 0, 0)$$

When $d = 4$ and $f = 3$, then we must have $c = 9$ as $s \leq r$. Since $s = 11$, we need two tableaux. Use $\mathcal{D}_0 \vee \mathcal{F}_0$ and

$$\mathcal{T} = \begin{array}{c} 2\ \ 2\ \ 2\ \ 2 \\ \hline 5\,9\,6\,9\,7\,8\,8 \\ 1\,1\,2\,2\,3\,3\,4 \end{array} \quad \omega(T) = (3,3,3,2,0,0,0,0)$$

Counting the number of 4's shows these tableaux are linearly independent.

When $d = 5$ we have $c = 8$ and need three tableaux (except for $f = 3$ when two tableaux suffice). In addition to $\mathcal{D}_0 \vee \mathcal{F}_0$ use:

$$\mathcal{T} = \begin{array}{c} 4\,4\,4\ f-2 \\ \hline 5\,6\,7\quad 8 \\ 1\,2\,3\quad 4 \end{array} \qquad f \neq 3$$

$$\omega(\mathcal{T}) = (4,4,4,f-2,0,0,0,0)$$

$$\mathcal{T}' = \begin{array}{c} 4\qquad f\ 2\ 2 \\ \hline 5\,6\,6\,7\,6\,8 \\ 1\,1\,2\,2\,3\,4 \end{array}$$

$$\omega(\mathcal{T}') = (5, f+1, 2, 2)$$

These tableaux are non-zero. Linear independence follows by counting the number of 5's and 2's. Thus we have sufficient tableaux for $d \not\equiv 1 \pmod 6$.

When $d \equiv 1 \pmod 6$ we need $2k' + \overline{k} + 2$ tableaux. The Form I construction $\mathcal{D}_p \vee \mathcal{F}_q$ discussed earlier provides at least $k'(k' + \overline{k} + 1)$ linearly independent tableaux for $f \not\equiv 0 \pmod 6$. (When $f \not\equiv 0 \pmod 6$ there are $k' + \overline{k} + 1$ tableaux $\mathcal{F}_q$.) This is sufficient for $k' \geq 2$. When $f \equiv 0 \pmod 6$ Form I provides $k'(k' + \overline{k})$ tableaux, but only $2k' + \overline{k} + 2$ tableaux are needed. This is sufficient for $k' \geq 2$ since $f \neq 0$.

Since $d \geq 3$, only $d = 7$ remains. We need at most 4 tableaux, as $s \leq 20$. Consider the tableaux $\mathcal{B}_p \vee \mathcal{C}_q$ where the $\mathcal{B}_p$ are $c = 3$ basis tableaux with $\lambda_2 = d - 1$ and the $\mathcal{C}_q$ are $c = 4$ or $c = 5$ basis tableaux with $\lambda_2 = d + f + 1$. There are two tableaux each for $\mathcal{B}_p$ and $\mathcal{C}_q$ so this construction is sufficient. The max weights for $\mathcal{B}_p$ are $(6,0,0)$ and $(4,2,0)$. The max weights for $\mathcal{C}_q$ are $(7, f+1, 0, 0)$ and $(6, f+2, 0, 0)$. Hence the weights of our construction are distinct unless $f = 5$. However when $f = 5$ only three tableaux are needed and the Form I construction $\mathcal{D} \vee \mathcal{F}_q$ provides three in this case.

When $f = 1$ we can proceed as in the $p + x < 3$ case. We have $s = md + 1$ and $r = md + pd + g$. Then $3md + pd + 2f + g = cd$, so $2 + g = xd$ for some x. Hence $x = 1$. This means that for both the $p + x \geq 3$ and $p + x < 3$ cases we can use the $c = 4$ basis tableaux, proceeding as in the $p + x < 3$ case when $f > 1$. As before, write $d = 6k' + i'$ with $0 \leq i' \leq 5$. We have at least k' linearly independent $\lambda = [3d - 1, d + 1]$ tableaux $\mathcal{F}_q$, which we use for the following forms:

$I.$	$\ell\mathcal{D}_p \vee (m - \ell - 1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1, 2, \ldots, m - 1$ $p = 0, 1, \ldots, k' - 1$ $q = 1, 2, \ldots, k'$
$II.$	$(m - 1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$q = 1, 2, \ldots, k'$
$III.$	$\ell\mathcal{A}_j \vee (m - 2\ell - 1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1, 2, \ldots, \lfloor \frac{m-1}{2} \rfloor$ $q = 1, 2, \ldots, k'$ $j = 1, 2, 3, 4$ $d \geq 6$ if $j \neq 1$

These tableaux have the following max weights:

$I.$	$(4k' + 2p + i', 2(k' - p), 0)^{\ell} \vee (d, 0, 0,)^{m-\ell-1} \vee (d - q, q + 1, 0, 0)$
$II.$	$(d, 0, 0)^{m-1} \vee (d - q, q + 1, 0, 0)$
$III - 1.$	$(d - 1, d - 1, 2, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d - q, q + 1, 0, 0)$
$III - 2.$	$(d - 1, d - 2, 3, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d - q, q + 1, 0, 0)$

$III-3.$ $(d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d-q, q+1, 0, 0)$

$III-4.$ $(d-2, d-3, 5, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d-q, q+1, 0, 0)$

Counting the number of zeros and d's in these weights gives:

Form	#0's	#d's
$I.$	$2m-\ell$	$m-\ell-1$
$II.$	$2m$	$m-1$
$III-j.$	$2m-\ell$	$m-2\ell-1$

Then by arguments similar to previous cases and Lemma 3.4.13, these tableaux have distinct max weights except for $III-1(\ell=1, q=2)$ and $III-2(\ell=1, q=1)$ whose weights are the same. Thus by Lemma 3.4.12, these tableaux (omitting $III-1(\ell=1, q=2)$) are linearly independent.

When $d \geq 6$ and $m \geq 3$, we need $mk' + \lfloor \frac{5m+1}{6} \rfloor + 1$ tableaux. When $k' \geq 2$, the forms listed above provided $(m-1)k'k' + k' + 4\lfloor \frac{m-1}{2} \rfloor k' - 1$, which is sufficient. When $k' = 1$ and $d \not\equiv 1 \pmod 6$, we still have these tableaux. If $d = 7$, then there is exactly one $\mathcal{D}$ so Form I does not exist. However, there are two distinct $\mathcal{F}_q$'s. Thus Forms II and III provide $8\lfloor \frac{m-1}{2} \rfloor$ tableaux. Since only $m + \lfloor \frac{m+1}{6} \rfloor + 1$ tableaux are needed, this suffices.

If $m = 2$ then only Form I and II tableaux exist. Since $s = 2d+1$, then $2k'+2$ tableaux suffice. Forms I and II provided at least $(k'-1)k' + k'$, tableaux. When $k' \geq 2$ this is enough. For $d > 7$, four tableaux are needed. Since there are at least two $\mathcal{D}$'s and two $\mathcal{F}$'s, this construction suffices. When $d = 6$, there are two $\mathcal{D}$'s, but only one $\mathcal{F}$. However, $s = 13$ so only two tableaux are needed, which we have. If $d = 7$, three tableaux are needed. Since there is only one $\mathcal{D}$ and two $\mathcal{F}$'s, an additional tableau is required. Use:

$$\mathcal{H} = \begin{array}{c} 6\,6\quad 2 \\ \hline 4\,5\,6\,7 \\ 1\,2\,3\,3 \end{array} \quad \omega(\mathcal{H}) = (6,6,3,0,0,0,0)$$

This tableau is clearly non-zero and maximal and its max weight does not contain d, so it is linearly independent from our previously constructed tableaux.

When $d = 5$, there are two distinct $\mathcal{F}_q$'s, which have weights of the form listed above. However, there is only one $\mathcal{D}$. Thus Forms II, $III - 1$, and $III - 4$ provide at least $1 + 4\lfloor\frac{m-1}{2}\rfloor$ tableaux. Since only $\lfloor\frac{5m+1}{6}\rfloor + 1$ tableaux are needed, this suffices for $m \geq 3$. When $m = 2$ then $s = 11$, so the two tableaux of Form II are sufficient.

When $d = 3$ there is only one $\mathcal{F}$ and one $\mathcal{D}$. Moreover, only $\mathcal{A}_1$ exists. However, all the tableaux are linearly independent. Thus we have $\lfloor\frac{m-1}{2}\rfloor + 1$ tableaux and we need $\lfloor\frac{3m+1}{6}\rfloor + 1$ tableaux. This is sufficient for m odd. When m even then $s = 3m + 1 \equiv 1 \pmod 6$ so only $\lfloor\frac{3m+1}{6}\rfloor$ tableaux are required, which we have when $m \geq 3$. If $m = 2$ then $s = 7$, which means the single tableau of Form II is sufficient.

When $d = 4$ we need to proceed differently since the $c = 4$ basis tableaux $\mathcal{F}$ is not in maximal form. First consider $m \geq 4$. We will replace $\mathcal{F}$ in the previous discussion with the following $c = 7$ tableaux, $\mathcal{H}_q$.

$$\mathcal{H}_0 = \begin{array}{c} 4\,2\quad 2 \\ \hline 4\,5\,6\,7 \\ 1\,2\,2\,3 \end{array} \qquad \omega(\mathcal{H}_0) = (4,3,2,0,0,0,0)$$

$$\mathcal{H}_1 = \begin{array}{c} 2\quad 2\quad 2 \\ \hline 4\,7\,5\,4\,6\,5 \\ 1\,1\,2\,2\,3\,3 \end{array} \qquad \omega(\mathcal{H}_0) = (3,3,3,0,0,0,0)$$

The the tableaux we will use are:

$II.$ $\qquad (m-2)\mathcal{D}_0 \vee \mathcal{H}_q \qquad q = 0,\, 1$

$$III. \qquad \ell\mathcal{A}_j \vee (m-2\ell-2)\mathcal{D}_0 \vee \mathcal{H}_q \qquad \begin{array}{l} \ell = 1,\, 2,\, \ldots,\, \lfloor \frac{m-2}{2} \rfloor \\ q = 0,\, 1 \\ j = 1,\, 5 \end{array}$$

Counting the number of 0's and 4's in the max weights of these tableaux show they are linearly independent by Lemma 3.4.12. This can be seen from Table 9.1.4 below:

Form	# 0's	# 4's	q
$II.$	$2m$	$m-1$	0
	$2m$	$m-2$	1
$III-j.$	$2m-\ell$	$m-2\ell-1$	0
	$2m-\ell$	$m-2\ell-2$	1

Table 9.2: Weights of 0 and 4.

Hence for $m \geq 4$ this construction provides $4\lfloor \frac{m-2}{2} \rfloor + 2$ linearly independent tableaux. Since only $\lfloor \frac{4m+1}{6} \rfloor + 1$ tableaux are required, this suffices.

When $m = 3$, then $s = 13$, so only two tableaux are needed. The two tableaux of Form II suffices. Hence we have constructed sufficient linearly independent tableaux when $f = 1$ and $m \geq 3$.

For $m = 2$, two tableaux are needed, in which case use $\mathcal{H}_0$ and $\mathcal{H}_1$. Hence all $f = 1$ cases are accounted for, since $m = 1$ may be handled by $c \leq 6$ basis tableaux and $V(d)$'s.

When $f = 0$ the construction is similar to earlier cases, particularly the $f = 1$ case. However, instead of the $\mathcal{F}$ tableaux, we use only the $c = 3$ basis tableaux $\mathcal{D}$ of shape $[2d, d]$. Again, we may take $m \geq 2$, since $m = 1$ may be handled by $c \leq 6$ basis tableaux and $V(d)$'s. Writing $d = 6k' + i'$, $0 \leq i' \leq 5$, our general tableaux are:

$I.$	$\ell\mathcal{D}_p \vee (m-\ell)\mathcal{D}_{k'}$	$p = 0\ldots k'-1,\ \ell = 1\ldots m$
$II.$	$m\mathcal{D}_{k'}$	
$III.$	$\ell\mathcal{A}_j \vee (m-2\ell)\mathcal{D}_{k'}$	$\ell = 1\ldots \lfloor \frac{m}{2} \rfloor,\ j = 1\ldots 4,\ d \geq 6$

These tableaux have max weights:

$I.$	$(4k'+2p+i', 2(k'-p), 0)^{\ell} \vee (d,0,0,)^{m-\ell}$
$II.$	$(d,0,0)^m$
$III-1.$	$(d-1, d-1, 2, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell}$
$III-2.$	$(d-1, d-2, 3, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell}$
$III-3.$	$(d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell}$
$III-4.$	$(d-2, d-3, 5, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell}$

Counting the number of zeros and d's in these weights gives:

Form	#0's	#d's
$I.$	$2m-\ell$	$m-\ell$
$II.$	$2m$	m
$III-j.$	$2m-\ell$	$m-2\ell$

Then by arguments similar to previous cases and Lemma 3.4.13, these tableaux have distinct max weights and are linearly independent by Lemma 3.4.12.

Consider $d \geq 6$. If $d \not\equiv 1 \pmod 6$, we need $mk'+m$ tableaux. The above construction provides $mk'+1+4\lfloor \frac{m}{2} \rfloor$ tableaux, which is sufficient. When $d \equiv 1 \pmod 6$ we need $mk'+\lfloor \frac{m}{6} \rfloor+1$ tableaux. In this case, $\mathcal{D}_0$ does not exists, so $p > 0$. For $k' \neq 1$, we have $m(k'-1)+1+4\lfloor \frac{m}{2} \rfloor$ which is sufficient. For $k' = 1$, we have

$4\lfloor \frac{m}{2} \rfloor + 1$ tableaux since Form I is no longer valid. However, this suffices.

When $d < 6$ not all of the $\mathcal{A}_j$ exist and there is exactly one $\mathcal{D}$. For $d = 4$ and $d = 5$ we have Forms II, $III - 1$, and $III - 5$ or $III - 4$ respectively. This provides $2\lfloor \frac{m}{2} \rfloor + 1$ tableaux. Since at most $\lfloor \frac{5m}{6} \rfloor + 1$ tableaux are needed, this is sufficient. When $d = 3$ we have Forms II and $III - 1$, which yield $\lfloor \frac{m}{2} \rfloor + 1$ tableaux. Since $\lfloor \frac{3m}{6} \rfloor + 1$ tableaux are required, we have enough. This construction work for $m \geq 2$, hence all $f = 0$ cases are accounted for.

9.2 Case: $r < s$

This proof of this case follows similarly to the $s \leq r$ case. Let $\lambda = [r + s, s]$ be a partition of n with $r < s$, where $n = 3b = cd$. We wish to construct m_λ linearly independent tableaux, where m_λ is the multiplicity of χ^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ as described in Theorem 11. First we will construct these tableaux for $c = 4$, 5, and 6; then we will use these constructions in proving Theorem 3 for a general c. We will refer to the tableaux constructed in this way as basis tableaux. These constructions will make use of the $c = 3$ basis tableaux constructed in Section 4.3 as well.

9.2.1 Basis Tableaux for $c = 4$, $r < s$

Given $\lambda = [r + s, s]$, a partition of n, we have $2s + r = 4d = 3b$. From this equation and $r < s$, we have $d + \lfloor \frac{d}{3} \rfloor < s \leq 2d$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 4. These will be our $c = 4$ basis tableaux.

When $d + \lfloor \frac{d}{3} \rfloor \leq s \leq 2d$, write $s = d + f$, with $\lfloor \frac{d}{3} \rfloor \leq f \leq d$. Consider the tableau T from Section 9.1.1. This tableau is non-zero and maximal.

$$T(\text{A},\text{B},\text{C},\text{D}) = \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \\ \hline 4 & 3 & 4 & 3 \\ 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} \text{A}+\text{B} \le d \\ \text{A},\ \text{B},\ \text{C},\ \text{D} > 0 \\ \text{A} > \text{D} \\ \text{B} > \text{C} \\ \text{or } \text{A}=\text{D},\ \text{B}=\text{C} \end{array}$$

$$\mathsf{w}(T) = (\text{A}+\text{B}, \text{C}+\text{D}, 0, 0)$$

Let $\mathcal{C}_p = T(\lceil \frac{d}{2} \rceil - p, \lfloor \frac{d}{2} \rfloor - p, \lfloor \frac{f}{2} \rfloor + p, \lceil \frac{f}{2} \rceil + p)$ for $0 \le p \le \lfloor \frac{d-f}{4} \rfloor$. The constraints on T are satisfied for all p provided $f \ge 2$ when $p = 0$. Hence $\mathcal{C}_p$ is non-zero for these parameters. Let $\mathcal{C}'_p = T(\lceil \frac{d}{2} \rceil - p + 1, \lfloor \frac{d}{2} \rfloor - p, \lfloor \frac{f}{2} \rfloor + p, \lceil \frac{f}{2} \rceil + p - 1)$, with $0 \le p \le \lfloor \frac{d-f}{4} \rfloor - 1$. Here the constraints on T are satisfied providedand $d - f \ge 4$ and $f \ge 4$ when $p = 0$.

Hence $\mathcal{C}'_p$ is non-zero for these parameters. We have $\omega(\mathcal{C}_p) = (d - 2p, f + 2p, 0, 0)$ and $\omega(\mathcal{C}'_p) = (d - 2p + 1, f + 2p - 1, 0, 0)$. Thus the weights are distinct and Lemma 3.4.12 the tableaux are linearly independent.

By Theorem 11, $m_\lambda = \lfloor \frac{r}{6} \rfloor + 1$. Hence it suffices to construct $\lfloor \frac{r}{6} \rfloor + 1 \le \lfloor \frac{2d-2f}{6} \rfloor + 1 \le \frac{d-f}{3} + 1$ linearly independent tableaux since $r = 4d - 2s$ and $s = d + f$. If $4 \le f \le d - 4$ then $\mathcal{C}_p$ and $\mathcal{C}'_p$ together provide $2\lfloor \frac{d-f}{4} \rfloor + 1$ linearly independent tableaux. Since $2\lfloor \frac{d-f}{4} \rfloor \ge \lfloor \frac{d-f}{3} \rfloor$, this is sufficient. When $f > d - 4$ we have $r \in \{0, 2, 4, 6\}$ with s even for $r = 0, 4$ and odd for $r = 2, 6$. Therefor, at most one tableau is needed when $f > d - 4$, which is $\mathcal{C}_0$. Now consider $f < 4$. Since $f \ge \lfloor \frac{d}{3} \rfloor$, when $d \ge 3$, only $f = 2$ and $f = 3$ remain. For $d = 3$, then $f = 1$ and we have $r = s$ which was done in Section 9.1.1. If $f = 2$, then $d \le 6$. However, by Theorem 11, $m_\lambda = 0$ for $d = 3$. When $d = 4$ or 5, $m_\lambda = 1$ and hence $\mathcal{C}_0$ suffices. For $d = 6$, the tableaux $\mathcal{C}_0$ and $\mathcal{C}_1$ suffice. If $f = 3$, then $d \le 9$. For $d = 9$ we have $m_\lambda = 3$, for $d = 7$, we have $m_\lambda = 2$, while $m_\lambda \le 1$ for the remaining $d \le 8$. Now $\mathcal{C}_0$ suffices for those cases with $d \ne 7$ or 9. When $d = 7$ we need an additional tableau, however, $\mathcal{C}_1$ exists. For $d = 9$, then $r = s$ and so this case was done in Section 9.1.1. Hence we have sufficient $c = 4$ basis tableaux for all partitions with $r < s$.

9.2.2 Basis Tableaux for $c = 5$, $r < s$

Given $\lambda = [r+s, s]$ a partition of n, we have $2s+r = 5d = 3b$. From this equation and $r < s$, we have $0 \le r \le \lfloor \frac{5d}{3} \rfloor$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 5. These will be our $c = 5$ basis tableaux.

First consider $0 \le r \le d$. If d is even, then $s - d \equiv s \pmod 2$. Moreover, $s - d \ge r$. Hence if $\lambda' = [r + s - d, s - d]$, then there are $m_{\lambda'}$ linearly independent $c = 3$ basis tableaux and $m_{\lambda'} = m_\lambda$ since $\lambda_1 = \lambda_2 = \lambda'_1 - \lambda'_2$. Therefore we can use $U_1(d) \vee T$, were T are the $c = 3$ basis tableaux of shape λ', as the $c = 5$ basis tableaux.

If d is odd, then $s \equiv d - 1 \pmod 2$ and $s - d + 1 \ge r$. Let T be the $c = 3$ basis tableaux of shape $\lambda' = [r + s - d - 1, s - d + 1]$. Consider the tableaux $U_1(d-1) \vee T$. There are $m_{\lambda'}$ such tableaux. Now $m_\lambda = m_{\lambda'}$ when $r \not\equiv 0, 3 \pmod 6$ since then $\lambda_1 - \lambda_2 - 2 = \lambda'_1 - \lambda'_2$. Thus for $r \not\equiv 0, 3 \pmod 6$ we have constructed sufficient tableaux. However, $2s + r = 5d$ and d odd implies r is odd, hence only $r \equiv 3 \pmod 6$ remains. In that case, $m_{\lambda'} = m_\lambda - 1$, so only one additional tableau is needed. For $d, r > 9$ use $U_1(d-3) \vee \mathcal{B}_0$ where $\mathcal{B}_0$ is the $c = 3$ basis tableau of shape $\lambda'' = [r + s - d - 6, s - d + 3]$. To show that $U_1(d-3) \vee \mathcal{B}_0$ is linearly independent from the $U_1(d-1) \vee T$ it suffices, by Lemma 3.4.12, to show that their max weights are distinct. Since $d-1$ is a weight of $U_1(d-1) \vee T$ for all T, we need only show that $d - 1$ is not a weight of $U_1(d-3) \vee \mathcal{B}_0$. If we write $\lambda'' = [r' + s', s']$ and $s' = 6k + j$, $0 \le j \le 5$, then $\omega(U_1(d-3) \vee \mathcal{B}_0) = (d - 3, 4k + j, 2k, 0, 0)$. If $d - 1$ is in this weight then that means $d - 1 = 4k + j$. Since $2s' + r' = 3d$, we then must have $j = 0$ and $r' = 3$. As $r' = r - 6$, this implies $r = 9$. Thus for $r > 9, d > 3$ these tableaux are linearly independent.

If $d = 3$, then only $r = 3$ is needed because $r \le d$. Only one tableau is required, which is $U_2(2, 1)$. When $r = 9$ we need two tableaux basis tableaux. The $U_1(d-3) \vee \mathcal{B}_0$

we constructed above has weight $(d-1, d-3, \frac{d-1}{2})$ In addition, use

$$T = \begin{array}{cccccc} 1\ \text{A} & d-\text{A}-1 & \text{B}+1 & d-\text{B}-2 & \text{A}\ \text{B} \\ \hline 4\ 3 & 5 & 3 & 4 & 5\ 4 \\ 1\ 1 & 1 & 2 & 2 & 3\ 3 \end{array} \quad \text{A} = \lfloor \tfrac{d-7}{4} \rfloor, \text{B} = \lceil \tfrac{d-7}{4} \rceil, d \geq 11, d \text{ odd}$$

which has weight $(d, d-1, \frac{d-7}{2}, 0, 0)$. For $d < 11$, only $d = 9$ is needed since $9 = r \leq d$ and d is odd. In that case, $s = 2d$ so use $U_1(d-3) \vee \mathcal{B}_0$ and

$$T = \begin{array}{cccc} \multicolumn{2}{c}{d-1} & \multicolumn{2}{c}{d-1} \\ \hline 5 & 3 & 5 & 4 \\ 1 & 1 & 2 & 2 \end{array}$$

Thus we have sufficient tableaux for $r \leq d$.

Now consider $d < r \leq \lfloor \frac{5d}{3} \rfloor$. To construct the $c = 5$ basis tableaux for these r we will use two different types of tableaux. These are the same tableaux that were used in Section 9.1.2 and hence are non-zero and maximal.

$$T_1(\text{A}, \text{B}) = \begin{array}{cccc} \text{A} & \lfloor \frac{d}{2} \rfloor & \lceil \frac{d}{2} \rceil & \text{B} \\ \hline 4\ 4\ 5 & 4 & 3 & 4\ 5 \\ 2\ 2\ 2 & 1 & 1 & 3\ 3 \end{array}$$

$1 \leq \text{A} \leq d-2$

$1 \leq \text{B} \leq \lfloor \frac{d}{2} \rfloor - 3$

$\text{A} > \text{B}$

$s \geq d+6$

$d \geq 8$

$\mathsf{w}(T_1) = (d, \text{A}+2, \text{B}+1, 0, 0)$

$$T_2(\text{A}, \text{B}) = \begin{array}{cc} \text{A} & \text{B} \\ \hline 5\ 3 & 5\ 4 \\ 1\ 1 & 2\ 2 \end{array}$$

$1 \leq \text{A}, \text{B} \leq d-1$

$s \geq 4$

$\mathsf{w}(T_2) = (\text{A}+1, \text{B}+1, 0, 0)$

Since $d < r \leq \lfloor \frac{5d}{3} \rfloor$, then $2d > s \geq \lceil \frac{5d}{3} \rceil$. Consider $\mathcal{E}_p = T_1(s-d-3-p,p)$ for $1 \leq p \leq \lfloor \frac{s-d}{2} \rfloor - 2$. For $s < 2d-1$, the parameters on T_1 are satisfied provided $d \geq 8$. (When $s = 2d-1$, we take $p \leq \lfloor \frac{s-d}{2} \rfloor - 3$.) This provides $\lfloor \frac{s-d}{2} \rfloor - 2$ linearly independent tableaux. We will also use $\mathcal{E}'_q = T_2(d-q, s-2-d+q)$ for $1 \leq q \leq d - \lfloor \frac{s-2}{2} \rfloor$. The parameters on T_2 are satisfied provided $s \geq d+2$ (which holds for $d > 3$). Together, $\mathcal{E}_p$ and $\mathcal{E}'_q$ provide $\lfloor \frac{s-d}{2} \rfloor - 2 + d - \lfloor \frac{s-2}{2} \rfloor$ linearly independent tableaux when $d \geq 8$. We need $\lfloor \frac{r}{6} \rfloor + 1 = \lfloor \frac{5d-2s}{6} \rfloor + 1$ tableaux. When $s \geq d+6$, we have $\lfloor \frac{5d-2s}{6} \rfloor + 1 \leq \lfloor \frac{s-d}{2} \rfloor - 2 + d - \lfloor \frac{s-2}{2} \rfloor$ so these tableaux are enough. Since $s \geq \lceil \frac{5d}{3} \rceil$ and $d \geq 8$, we have $s \geq d+6$. Thus for $d \geq 8$, $s < 2d-1$ these tableaux suffice

When $s = 2d-1$, $d \geq 8$ we have $\lfloor \frac{s-d}{2} \rfloor - 3 + d - \lfloor \frac{s-2}{2} \rfloor = \lfloor \frac{d-1}{2} \rfloor - 2$. This is greater than or equal to $\lfloor \frac{r}{6} \rfloor + 1 = \lfloor \frac{d+2}{6} \rfloor + 1$ provided $d > 8$. When $d = 8$, $s = 2d-1$ we have $r = 10$and only one tableau is needed, which $\mathcal{E}'_q$ provides. When $d = 8$ we also have $r = 12$. In that case two tableaux are needed, which $\mathcal{E}'_q$ provides.

Now consider $d < 8$ with $s \leq 2d-1$. For $d \leq 5$, or $d = 6$ and $r = 8$ only one tableau is needed. Here $\mathcal{E}'_q$ provides this tableau except if $d = 3$. When $d = 3$, then $r = s$ which was done in Section 9.1.2. When $d = 7$ or $d = 6$ and $r = 10$ two tableaux are needed. In these cases, $\mathcal{E}'_q$ suffices. Hence for $d < 8$ all tableaux are provided.

9.2.3 Basis Tableaux for $c = 6$, $r < s$

Given $\lambda = [r+s, s]$ a partition of n, we have $2s + r = 6d = 3b$. From this equation and $r < s$, we have $0 \leq r < 2d$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 6. These will be our $c = 6$ basis tableaux.

First consider $0 \leq r \leq d$. If d is even, then $s - d \equiv s \pmod 2$. Moreover, $s - d \geq r$. Hence if $\lambda' = [r+s-d, s-d]$, then there are at least $m_{\lambda'}$ linearly independent $c = 4$ basis tableaux and $m_{\lambda'} = m_\lambda$ since $\lambda_1 = \lambda_2 = \lambda'_1 - \lambda'_2$. Therefore we can use $U_1(d) \vee \mathcal{C}_p$, were $\mathcal{C}_p$ are the $c = 4$ basis tableaux of shape λ', as the $c = 6$ basis tableaux.

If d is odd, then $s \equiv d-1 \pmod 2$ and $s - d + 1 \geq r$. Let $\mathcal{C}_p$ be the $c = 4$ basis tableaux of shape $\lambda' = [r+s-d-1, s-d+1]$. Consider the tableaux $U_1(d-1) \vee \mathcal{C}_p$.

There are $m_{\lambda'}$ such tableaux. Now $m_\lambda = m_{\lambda'}$ when $r \not\equiv 0, 3 \pmod 6$ since then $\lambda_1 - \lambda_2 - 2 = \lambda_1' - \lambda_2'$. Thus for $r \not\equiv 0, 3 \pmod 6$ we have constructed sufficient tableaux. However, $2s + r = 6d$ implies r is even, hence only $r \equiv 0 \pmod 6$ remains. In that case, $m_{\lambda'} = m_\lambda - 1$, so only one additional tableau is needed. Since $0 \le r \le d$, we have $\lfloor \frac{5d}{2} \rfloor \le s \le 3d$. For $r \neq 0$ we can write $s = 2d + f$ with $\lfloor \frac{d}{2} \rfloor \le f \le d - 3$. Then

$$T = \begin{array}{ccccccc} \multicolumn{2}{c}{d-2} & & d-1 & & f-1 & \\ \hline 4 & 6\,3 & 5 & 3 & 6 & 4 \\ 1 & 1\,1 & 2 & 2 & 3 & 3 \end{array} \qquad \omega(T) = (d, d, f, 0, 0, 0)$$

provides the additional tableau needed. Since T does not have a weight of $d - 1$ it must be linearly independent from $U_1(d-1) \vee \mathcal{C}_p$ by Lemma 3.4.12. When $r = 0$, then $s = 3d$. However, since d is odd, then so is s and hence no tableaux are required.

Now consider $d < r < 2d$ for arbitrary d. We have $2d < s \le \lfloor \frac{5d}{2} \rfloor$. Write $s = 2d + f$ for $1 \le f \le \lfloor \frac{d}{2} \rfloor$. Consider

$$\mathcal{H}_p = \begin{array}{cccccc} d-p-1 & p+1 & d-2p & p & p & f \\ \hline 4 & 6 & 5 & 4 & 5 & 6 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad 0 < p \le \lfloor \tfrac{d-f}{2} \rfloor$$

$$\omega(\mathrm{H}_p) = (d, d-p, p+f)$$

Now $\mathcal{H}_p$ is non-zero and maximal. Moreover, for $0 < p \le \lfloor \frac{d-f}{2} \rfloor$, $\mathcal{H}_p$ are linearly independent by Lemma 3.4.12. Hence we have $\lfloor \frac{d-f}{2} \rfloor$ tableaux. We need $\lfloor \frac{r}{6} \rfloor + 1 = \lfloor \frac{2d-2f}{6} \rfloor + 1$. Hence it suffices to show that $\lfloor \frac{d-f}{2} \rfloor > \lfloor \frac{d-f}{3} \rfloor$. This holds except for $d = 6, f = 3$; $d = 5, f = 2$; and $d = 4, f = 1$. For $d = 4$ and 6, only one tableau is needed, so $\mathcal{H}_1$ suffices. When $d = 5, f = 2$, two tableaux are required. Use $\mathcal{H}_1$ and $U_1(d-1) \vee U_1(d-1) \vee U_1(d-1)$. Thus all the necessary $c = 6$ basis tableaux have been constructed.

9.2.4 Basis Tableaux for $c > 6$, $r < s$

Let $\lambda = [r + s, s]$ be a partition of n, with $r < s$, where $2s + r = cd = n$. We want to construct $m_\lambda \le \lfloor \frac{r}{6} \rfloor + 1$ linearly independent basis tableaux for an arbitrary c.

First write $s = s' + 2dq$ such that $s' - 2d < r \leq s'$. If $c - 4q \geq 3$ then a λ-tableau $[r+s, s]$ may be written as $T = qU_2(d-1,1) \vee T'$, where T' is a $\lambda' = [r+s', s']$ tableau filled with $c-4q$ elements. Since $r \geq s'$ in λ', we have $m_\lambda = m_{\lambda'}$. Hence it suffices to construct m_λ tableaux T'. If $r = s'$, the tableaux T' were constructed in Section 9.1.4. Hence we only need to consider partitions $\lambda = [r+s, s]$ with $s - 2d < r < s$ and the case where $c - 4q < 3$.

If $c - 4q < 3$ we may write $T = (q-1)U_2(d-1,1) \vee T^*$ where T^* is a $\lambda' = [r + s' + d, s' + d]$ tableau filled with at most 6 elements. Since at least $m_\lambda = m_{\lambda'}$ tableaux T^* were constructed for $c \leq 6$ in previous sections, no additional construction is needed for this case.

To construct $\lambda = [r+s, s]$ tableaux with $s - 2d < r < s$, write $r = md + f$, with $0 \leq f < d$ and $s = md + pd + g$ with $0 \leq g < d$. Then $3md + 2pd + f + 2g = cd$, so $f + 2g = xd$ for some x. This means $c = 3m + 2p + x$. If $2p + x \geq 3$, a λ-tableau may be written $\mathcal{D} \vee \mathcal{F}$, where $\mathcal{D}$ is a tableau of shape $[2md, md]$ filled with $3m$ elements and $\mathcal{F}$ is a $[pd+g+f, pd+g]$ tableau filled with $2p+x$ elements. Given the constraints on r, we have $2p + x \leq 6$. Hence we can use the basis tableaux constructed in Sections 4.3, 9.2.1, 9.2.2, 9.2.3 as the $\mathcal{F}$ tableaux. We will first consider this case where $2p + x \geq 3$ and handle the $2p + x < 3$ case later.

We have $r = md + f$, so writing $d = 6k' + i'$, $f = 6\bar{k} + \bar{i}$, with $0 \leq i', \bar{i} \leq 5$ gives $r = 6(mk' + \bar{k}) + mi' + \bar{i}$. Since $\lfloor \frac{mi' + \bar{i}}{6} \rfloor \leq m$, it suffices to construct $\lfloor \frac{s}{6} \rfloor + 1 \leq mk' + \bar{k} + m + 1$ linearly independent tableaux. If $m \leq 1$ we have $c \leq 9$. For $c \leq 6$ we have constructed the necessary basis tableaux in previous section. $c = 7, 8$ and 9 will be handled later. Hence assume $m \geq 2$

Let $\mathcal{D}_p$ be the $c = 3$ basis tableaux of shape $[2d, d]$ described in Section 4.3. There are $m_{[2d,d]} = k' + 1$ such tableaux when $i' \neq 1$ and k' for $i' = 1$. Let $\mathcal{F}_q$ be the $c = 2p + x$ basis tableaux of shape $[pd + g + f, pd + g]$ constructed in Sections 4.3, 9.2.1, 9.2.2, 9.2.3. At least one such tableaux will always exist, provided $f \neq 0, 1, 2, 4$. There are at least $m_{[pd+g+f,pd+g]} \leq k'$ such tableaux, given the constraints on f. Take $f \neq 0, 1, 2, 4$ and $d \geq 6$. The $d < 6$ and f cases will be handled separately. Consider the following tableaux forms (where the $\mathcal{A}_i$ were defined in Section 9.1.4).

$I.$	$\ell\mathcal{D}_p \vee (m-\ell)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1,\ 2, \dots,\ m$ $p = 0,\ 1, \dots,\ k'-1$ $q = 1,\ 2, \dots,\ \overline{k}$
$II.$	$m\mathcal{D}_{k'} \vee \mathcal{F}_q$	$q = 1,\ 2, \dots,\ \overline{k}$
$III.$	$\ell\mathcal{A}_j \vee (m-2\ell)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1,\ 2, \dots,\ \lfloor \frac{m}{2} \rfloor$ $q = 1,\ 2, \dots,\ \overline{k}$ $j = 1,\ 2,\ 3,\ 4$ $d \geq 6$ if $j \neq 1$

Table 9.3: Tableaux Forms.

By Lemma 3.4.12, these tableaux are linearly independent provided their max weights are distinct. The max weights of these tableaux are:

$I.$	$(4k' + 2p + i', 2(k'-p), 0)^{\ell} \vee (d, 0, 0)^{(m-\ell)} \vee \omega(\mathcal{F}_q)$
$II.$	$(d, 0, 0)^m \vee \omega(\mathcal{F}_q)$
$III-1.$	$(d-1, d-1, 2, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{(m-2\ell)} \vee \omega(\mathcal{F}_q)$
$III-2.$	$(d-1, d-2, 3, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{(m-2\ell)} \vee \omega(\mathcal{F}_q)$
$III-3.$	$(d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{(m-2\ell)} \vee \omega(\mathcal{F}_q)$
$III-4.$	$(d-2, d-3, 5, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{(m-2\ell)} \vee \omega(\mathcal{F}_q)$

The weights of $\mathcal{F}_q$ depend on which basis tableaux we are using for $\mathcal{F}$. We categorize these weights by the number of elements in $\mathcal{F}$.

c	$\omega(\mathcal{F}_q)$	Range	Conditions
3	$(4h+2q+j, 2(h-q), 0)$	$q = 1, \ldots k'$	$s = 6h + j$
4	$(d-q, \frac{d-f}{2}+q, 0, 0)$	$q = 0, \ldots, 2\lfloor \frac{d+f}{8} \rfloor$	
5	$(d, 4h+2q+j, 2(h-q), 0, 0)$	$q = 1, \ldots, k'$	$s - d = 6h + j, r \leq d, d$ even
	$(d-1, 4h+2q+j, 2(h-q), 0, 0)$	$q = 1, \ldots, k'$	$s - d + 1 = 6h + j, d$ odd
	$(d-3, 4h+j, 2h, 0, 0)$		$r \leq d. r > 9 if r \equiv 3 \pmod 6$
	$(d-1, d-3, \frac{d-1}{2}, 0, 0)$		$r = 9, d \geq 11, d$ odd
	$(d, d-1, \frac{d-7}{2}, 0, 0)$		
	$(d, d, 0, 0, 0)$		$r = 9, d = 9$
	$(d-1, d-3, \frac{d-1}{2}, 0, 0)$		
	$(d, d-\frac{f}{2}-q-1, q+1, 0, 0)$	$q = 1, \ldots, \lfloor \frac{2d-f}{4} \rfloor - 2$	$r > d$
	$(d-q'+1, d-\frac{f}{2}-1+q', 0, 0, 0)$	$q' = 1, \ldots, \lfloor \frac{f}{4} \rfloor + 1$	
6	$(d, d-q, \frac{d-f}{2}+q, 0, 0, 0)$	$q = 0, \ldots, 2\lfloor \frac{d+f}{8} \rfloor$	$r \leq d, d$ even
	$(d-1, d-q, \frac{d-f}{2}+q+1, 0, 0, 0)$	$q = 0, \ldots, 2\lfloor \frac{d+f}{8} \rfloor$	$r \leq d, d$ odd
	$(d, d, \frac{d-f}{2}, 0, 0, 0)$		$r \equiv 0 \pmod 6$
	$(d-1, d-q, \frac{d-f}{2}+q+1, 0, 0, 0)$	$q = 0, \ldots, 2\lfloor \frac{d+f}{8} \rfloor$	$r \leq d, d$ odd
	$(d, d-q, q+\frac{d-f}{2}, 0, 0, 0)$	$q = 1, \ldots, \lfloor \frac{d+f}{4} \rfloor$	$r > d$

The weights of Forms $I, II, III - j$ are distinct for each $\mathcal{F}_q$ listed provided $d > 7$, except for the following cases. When $\mathcal{F}$ is a $c = 5$ tableau, we have $\omega(I(\ell = 1, p = k' - 1, \mathcal{F}_1)) = \omega(III - 4(\ell = 1, \mathcal{F}_2))$ when $d = 11, f = 9$. When $\mathcal{F}$ is $c = 4$ or $c = 6$, we have some duplicate tableau weights if $f = d - 4$ or $f = d - 2$. These weights are $\omega(I(\ell = 1, p = k' - 1, q = j - 1)) = \omega(III - j(\ell = 1, q = 0))$ for $j = 2, 3$, and 4. Also, when $f = d - 4$, we have $\omega(III - 2(\ell = 1, q = 2)) = \omega(III - 3(\ell = 1, q = 1))$, $\omega(III - 2(\ell = 1, q = 3)) = \omega(III - 4(\ell = 1, q = 1))$, and $\omega(III - 3(\ell = 1, q = 3)) = \omega(III - 4(\ell = 1, q = 2))$. When $f = d - 2$ we have $\omega(III - 1(\ell = 1, q = 2)) = \omega(III - 2(\ell = 1, q = 1))$ as well. In these cases we have (at most 6) fewer linearly independent tableaux available than listed.

These will be called the constrained cases.

The linear independence of the remaining tableaux can be seen by counting the number of d's, 0's in each tableau, determining the number of distinct elements in each tableau, and applying Lemma 3.4.13 where appropriate. When $d = 6$, the Forms $III - 2$ and $III - 4$ have the same weight. When $d = 7$ the Forms $III - 3$ and $III - 4$ have the same weight. Hence for $d = 6$ and 7, Form $III - 4$ will not be used.

We wish to have $\lfloor \frac{r}{6} \rfloor + 1$ linearly independent tableaux. Recall that $r = md + f$, $d = 6k' + i'$ and $f = 6\overline{k} + \overline{i}$, so it suffices to construct $mk' + \overline{k} + m + 1$ linearly independent tableaux. We have $f \neq 0, 1, 2, 4$, $d \geq 6$ and $m \geq 2$. Note that there is always at least one $\mathcal{F}$ since $f \neq 0, 1, 2, 4$, so we will take $\overline{k} \geq 1$.

When $d > 6$, $d \not\equiv 1 \pmod 6$, all the tableaux listed in Table 9.2.4 exist and are linearly independent. This provides $mk'\overline{k} + \overline{k} + 4\lfloor \frac{m}{2} \rfloor \overline{k}$, which is at least $mk' + \overline{k} + m + 1$. Hence sufficient linearly independent tableaux exist. In the constrained cases, using the full set of $\mathcal{F}$ listed (as opposed to only the first k') will provide sufficient tableaux.

When $d = 6$, the tableaux of Forms $III - 2$ and $III - 4$ are not linearly independent. Hence we have $m\overline{k}+\overline{k}+3\lfloor \frac{m}{2} \rfloor \overline{k}$ linearly independent tableaux. Then $r = 6m+f$, so $m+\overline{k}+1$ tableaux are sufficient. Since we have $\overline{k} = 1$, we have listed sufficient tableaux.

When $d \equiv 1 \pmod 6$ the tableau D_0 does not exist. Hence Table 9.2.4 provides $m(k' - 1)\overline{k}+\overline{k}+4\lfloor \frac{m}{2} \rfloor \overline{k}$ linearly independent tableaux, provided $k' > 1$. In this case $r = 6k'm+m+f$, so $mk' + \overline{k} + \lfloor \frac{m}{6} \rfloor + 2$ tableaux suffices. Thus we have enough tableaux unless $m = 3, \overline{k} = 1$. In that case, specifically checking the number of tableaux needed and the number of $\mathcal{F}_q$ that exist, shows this construction is sufficient. When $k' = 1$, there are no tableaux of Form I and we do not use Form $III - 4$. Hence we have $1 + 3\lfloor \frac{m}{2} \rfloor$ tableaux, which is sufficient except in the following cases, where an additional tableau is needed If $m = 7, f = 5, c = 26$ use $6U_1(6) \vee U_1(4)$, if $m = 3, f = 3, c = 12$ use $5U_1(6)$, if $m = 3, f = 5, c = 14$ use $6U_1(6)$, if $m = 3, f = 6, c = 13$ use $4U_1(6) \vee U_4(3, 2)$, and if $m = 3, f = 6, c = 15$ use $6U_1(6)$. In the constrained cases, using the full set of $\mathcal{F}$ listed (as opposed to only the first k') will provide sufficient tableaux.

Hence for $d \geq 6$, $f \neq 0, 1, 2, 4$, $2p + x \geq 3$, we have constructed the requisite number of linearly independent tableaux. We will consider the $f = 0, 1, 2, 4$ case after doing the $2p + x < 3$ case.

When $2p + x < 3$, the procedure described above require $\mathcal{F}_q$ to have fewer than three elements. In that case we use $m - 1$ in place of m in our construction and take $\mathcal{F}_q$ to be basis tableaux filled with $2p+x+3$ elements with shape $[pd+2d+g+f, pd+d+g]$. There are at least $k' + \overline{k}$ such tableaux, though $\overline{k}$ may equal 0. Take $d \geq 6$, $2p + x < 3$, $m \geq 3$ Hence from the tableaux of Table 9.2.4 we get Table 9.2.4.

$I.$	$\ell\mathcal{D}_p \vee (m - \ell - 1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1, 2, \ldots, m - 1$ $p = 0, 1, \ldots, k' - 1$ $q = 1, 2, \ldots, k' + \overline{k}$
$II.$	$(m - 1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$q = 1, 2, \ldots, k' + \overline{k}$
$III.$	$\ell\mathcal{A}_j \vee (m - 2\ell - 1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1, 2, \ldots, \lfloor\frac{m-1}{2}\rfloor$ $q = 1, 2, \ldots, k' + \overline{k}$ $j = 1, 2, 3, 4$ $d \geq 6$ if $j \neq 1$

Table 9.4: Tableaux Forms

As before, these tableaux are linearly independent, except in the constrained cases where the same weight equalities occur. We do not use Form $III - 4$ when $d = 6$ or 7. For the tableaux of Table 9.2.4, we require $m \geq 3$ for tableaux of Form III to exist. Since we still have $r = md + f$, we want $mk' + \overline{k} + m + 1$ linearly independent tableaux, when $d \not\equiv 1 \pmod 6$. When $d > 6$, $d \not\equiv 1 \pmod 6$, this construction provides $(m - 1)k'(\overline{k} + k') + \overline{k} + k' + 4\lfloor\frac{m-1}{2}\rfloor(\overline{k} + k')$ linearly independent tableaux. This is larger than $mk' + \overline{k} + m + 1$, except when $m = 4, k' = 1, \overline{k} = 0$. However, computing precisely how many $\mathcal{F}_q$ exist in this case and the minimum number of tableaux required, shows these tableaux suffice. In the constrained cases, using the full set of $\mathcal{F}$ listed (as opposed to only the first $k' + \overline{k}$) will provide sufficient tableau.

When $d = 6$, Forms $III-2$ and $III-4$ are the same so we have only $(m-1)+1+3\lfloor\frac{m-1}{2}\rfloor$ linearly independent tableaux. Since $r = 6m + f$, $f < d$, we need $m + 1$ tableaux, which we have. In the $f = d - 4$ and $f = d - 2$ cases using the full set of $\mathcal{F}$ listed will provide sufficient tableau.

If $d \equiv 1 \pmod 6$, the tableau D_0 does not exist. Hence Table 9.2.4 provides $(m-1)(k'-1)(\overline{k}+k')+\overline{k}+k'+4\lfloor\frac{m-1}{2}\rfloor(\overline{k}+k')$ linearly independent tableaux when $k' > 1$. In this case $r = 6m + m + f$, so $mk' + \overline{k} + \lfloor\frac{m}{6}\rfloor + 2$ tableaux suffice, which we have. In the constrained cases, using the full set of $\mathcal{F}$ listed will provide sufficient tableau. When $k' = 1$, there are no tableaux of Form I or $III-4$. Hence we have $\overline{k}+1+3\lfloor\frac{m-1}{2}\rfloor(\overline{k}+1)$. This is at least as large as $mk' + \overline{k} + \lfloor\frac{m}{6}\rfloor + 2$ unless $m \leq 8$. However, computing precisely how many $\mathcal{F}_q$ exist in this case and the minimum number of tableaux required, shows these tableaux suffice in most cases. When $m = 6, f = 3, c = 19$ we need an additional tableau; use $6U_1(6) \vee U_1(2)$. When $m = 4, f = 3$ we also need an additional tableau; use $U_4(3,2) \vee 7U_1(6) \vee U_1(4)$. In the constrained cases, using the full set of $\mathcal{F}$ listed will provide sufficient tableau.

Thus for $d \geq 6$, $2p + x < 3$, $m > 2$, all necessary tableaux have been constructed.

When $2p + x < 3$ and $m = 2$, we find that $c = 7$ or $c = 8$. In these cases, consider the tableaux of Forms I and II in Table 9.2.4. We have $r = 2d + f$, so $2k' + \overline{k} + 3$ tableaux are sufficient. For $d \not\equiv 1 \pmod 6$, we have at least $(k'+1)(k'+\overline{k})$ linearly independent tableaux of Forms I and II. When this isn't enough, checking precisely how many tableaux are needed and how many tableaux of Form I exist, shows that this construction is sufficient except for the following cases.

When $c = 8, d = 8, f = 2$ three tableaux are needed; use $2U_1(8) \vee U_4(6,1)$, $U_1(8) \vee U_3(2,3) \vee U_4(6,1)$, and $U_1(8) \vee U_3(2,4) \vee U_4(4,1)$. When $c = 8, d = 11, f = 2$ five tableaux are needed. They are $4U_1(8)$, $U_2(10,1) \vee U_1(10)$, $U_2(10,1) \vee\vee U_1(8) \vee U_1(2)$, $U_2(10,1) \vee U_1(6) \vee U_1(4)$, and $U_2(10,1) \vee U_2(2,3)$. When $c = 7$, if $d = 8$ and $f = 2$, three tableaux are needed, but Form I provides only two. In addition to those tableaux, use $P_3(0,3,2) \vee U_1(8) \vee U_1(2)$. If $c = 7$, $d = 9$ and $f = 1$, three tableaux are needed, but Form I provides only two. In addition to those tableaux, use $U_1(8) \vee U_1(8) \vee U_1(6)$. If $d = 10$ and $f = 0$, three tableaux are needed, but Form I provides only two. In addition to those tableaux, use $P_3(0,4,2) \vee U_1(10) \vee U_1(4)$.

When $d \equiv 1 \pmod 6$, it is sufficient to construct $2k' + \overline{k} + 2$ tableaux. We have at least $(k')(k'+\overline{k})$ linearly independent tableaux of Forms I and II. When this is less than $2k' + \overline{k} + 2$, checking precisely how many tableaux are needed and how many tableaux of Forms I and II exist, shows that this construction is sufficient except for the following cases. When $c = 8, d = 7, f = 4$ we need three tableaux; use $U_2(6,1) \vee U_4(4,1)$, $2U_1(6) \vee$

$U_4(6,1)$, and $U_1(6) \vee U_3(2,3) \vee U_4(4,1)$. When $d = 7, f = 2$ three tableaux are needed; use $U_2(6,1) \vee U_1(6)$, $U_2(6,1) \vee U_1(4) \vee U_1(2)$, and $3U_1(6) \vee U_1(2)$. When $d = 7$, and $f = 0$, two tableaux are needed, but Form II provides only one. In addition to that tableau, use $P_3(0,3,2) \vee 2U_1(6)$.

For $c = 7$ if $d = 13$ and $f = 1$, five tableaux are needed, but Forms I and II provide only four. In addition to those tableaux, use $U_1(12) \vee U_1(12) \vee U_1(8)$. If $d = 7$ and $f = 1$, three tableaux are needed, but Form I provides only two. In addition to those tableaux, use $P_3(0,3,2) \vee U_1(6) \vee U_1(2)$. This completes the cases of $c = 7$ and $c = 8$ for $d \geq 6$.

Now consider $c = 7$ and $c = 8$ when $d < 6$. When $c = 7$, $d = 5$ we need two tableaux of shapes $[23,12]$ and $[22,13]$, and one tableau of shape $[21,14]$. Use $3U_1(4)$ and $U_2(4,1) \vee U_1(2)$, for shape $[23,12]$; $U_2(4,1) \vee U_4(2,1)$ and $U_4(3,2) \vee 2U_1(4)$ for shape $[22,13]$; and $U_2(4,1) \vee U_1(4)$ for shape $[21,14]$. When $c = 7$an $d = 4$ we need two tableaux of shape $[18,10]$, $2U_1(4) \vee U_1(2)$ and $U_2(2,1) \vee U_1(4)$, and one tableau of shape $[17,11]$, $U_4(2,1) \vee 2U_1(4)$. When $c = 7$ and $d = 3$, we need only one tableau of shape $[13,8]$, $U_2(2,1) \vee U_1(2)$.

When $c = 8$ and $d = 5$, three tableaux are needed for shape $[26,14]$, one tableau for shape $[25,15]$, and two tableaux for $[24,16]$. Use $3U_1(4) \vee U_1(2), U_2(4,1) \vee U_1(4)$, and $U_2(2,1) \vee 2U_1(4)$ for shape $[26,14]$. For shape $[25,15]$ use $U_2(4,1) \vee U_4(3,2)$, while for shape $[24,16]$ use $4U_1(4)$ and $U_2(4,1) \vee U_1(4) \vee U_1(2)$. When $c = 8$ and $d = 4$ we need one tableau for shapes $[21,11]$ and $[19,13]$ and two tableaux for shape $[20,12]$. In the first case, use $U_4(2,1) \vee U_2(3,1)$ and $U_1(4) \vee U_4(2,1) \vee P_4(0,2,2)$, respectively. For $[20,12]$ use $3U_1(4)$ and $2U_2(2,1)$. When $c = 8$ and $d = 3$ we need only one tableau of shape $[15,9]$, which is $U_2(2,1) \vee U_4(2,1)$. This completes the $m = 2, 2p + x < 3$ case.

Now consider $d \geq 6$, $2p + x \geq 3$, $f = 0,1,2,4$. In this case we cannot guarantee that the tableaux $\mathcal{F}_q$ of shape $[2dp + g + f, 2dp + g]$ exists. However, since $d \geq 6$, tableaux $\mathcal{F}$ of shape $[pd + d + g + f, pd + d + g]$ will exist. Moreover, since $s < r + 2d$, this shape is fillable with $c \leq 6$ elements. Hence we can simply use the tableaux constructed in the $2p + x < 3$ case with this $\mathcal{F}$. Since we did not apply any restrictions of f in that case, those computations hold.

Now take $d < 6$. We have $r = dm + f$, $f < d$, $s - 2d < r < s$. Since d is small, only a few s are possible for each r. We will consider each case according to the value of d.

If $d = 3$, we have $r = 3m + f$, $s = 3m + 3 + f$, and $f < 3$. First consider $f = 0$ or 2. If $m = 2$ we have the shapes $[15, 9]$ and $[19, 11]$. In both cases, only one tableau is needed. Use $U_2(2,1) \vee U_4(2,1)$ and $U_2(2,1) \vee U_4(2,1) \vee U_1(2)$. Now take $m \geq 3$. Let $\mathcal{F}$ be the tableau of shape $[9+2f, 6+f]$. Since $f = 0$ or 2, one such $\mathcal{F}$ always exists. Then consider the tableaux of Form II and $III - 1$ in Table 9.2.4. They have weight $(2,2,2,0,0,0)^{\ell} \vee (3,0,0)^{m-2\ell-1} \vee \omega(\mathcal{F})$ for $\ell = 0, 1, \ldots, \lfloor \frac{m-1}{2} \rfloor$ and hence are linearly independent by Lemma 3.4.12. This construction provides $\lfloor \frac{m-1}{2} \rfloor + 1$ tableaux. Since we need at most $\lfloor \frac{3m+2}{6} \rfloor + 1$ tableaux for m odd and $\lfloor \frac{3m+2}{6} \rfloor$ for m even, this suffices. When $f = 1$, let $\mathcal{F} = U_2(2,1) \vee U_1(2) \vee U_1(2)$. If $m = 2$ or 3, one tableau will suffice. In those cases use $\mathcal{F}$ and $\mathcal{F} \vee U_4(2,1)$. When $m \geq 4$ we will use the tableaux $\ell A_1 \vee (m - 2\ell - 2)U_4(2,1) \vee \mathcal{F}$ for $\ell = 0, 1, \ldots, \lfloor \frac{m-2}{2} \rfloor$. There are $\lfloor \frac{m}{2} \rfloor$ such tableaux and they are linearly independent. Since only $\lfloor \frac{m}{2} \rfloor$ are needed, this suffices.

When $d = 4$ we have $r = 4m + f$, $s = 4m + h + f$, and $f < 3$. Since $s - 8 < r < s$, the only possibilities are $f = 0, h = 2, 4, 6$ and $f = 2, h = 1, 3, 5, 7$. Let $\mathcal{F}$ be a tableau of shape $[8 + 2f + h, 4 + f + h]$. Since $\mathcal{F}$ needs at most 8 elements it has already been constructed. If $m = 2$, we need one tableau when $f = 2$, so $\mathcal{F}$ suffices. When $f = 0$, two tableaux are needed. Use $U_1(4) \vee U_1(4) \vee U_1(2)$ and $U_2(2,1) \vee U_1(4)$; $U_1(4) \vee U_1(4) \vee U_1(4)$ and $U_2(2,1) \vee U_1(4) \vee U_1(2)$; or $U_1(4) \vee U_1(4) \vee U_1(4) \vee U_1(2)$ and $U_2(2,1) \vee U_1(4) \vee U_1(4)$; depending on h. For $m \geq 3$, use the tableaux of Forms II and $III - 1$ of Table 9.2.4 along with those of form $III - 1$, with $\mathcal{A}_5$ (given in Section 9.1.4) in place of $\mathcal{A}_1$. This provides $1 + 2\lfloor \frac{m-1}{2} \rfloor$ linearly independent tableaux, which is sufficient.

When $d = 5$ we have $r = 5m + f$, $s = 5m + h + f$, with $f < 5$, $0 < h < 10$, and $h \equiv f \pmod 5$. Let $\mathcal{F}$ be a tableau of shape $[10 + h + 2f, 5 + h + f]$. When $m \geq 3$, consider the tableaux of Forms II, $III - 1$, $III - 4$ of Table 9.2.4. This provides $1 + 2\lfloor \frac{m-1}{2} \rfloor$ linearly independent tableaux. Computing precisely how many tableaux are needed for each f and h we find that this is sufficient except in the following cases. When $f = 1, m = 4$ we need one additional tableau. For $h = 1$ use $4U_1(4) \vee U_1(2)$ and for $h = 6$ use $3U_4(3,2) \vee 3 \vee U_1(4)$. When $f = 2$ and $h = 2$ we need an additional tableau for $m = 8, 6$, and 4. Use $11U_1(4)$, $8U_1(4) \vee U_1(2)$, and $6U_1(4)$ respectively. When $f = 2$ and $h = 7$, this construction suffices. When $f = 3$ and $h = 3$, one additional tableau is needed for $m = 6$ and $m = 4$; use $9U_1(4)$ and $6U_1(4) \vee U_1(2)$, respectively. When $f = 3$ and $h = 8$ we can take $\mathcal{F}$ to be a tableau of shape $[14, 11]$ and use the tableaux of Forms II, $III - 1$, and $III - 4$ of Table 9.2.4. This provides $1+2\lfloor \frac{m}{2} \rfloor$ tableaux, which suffices except for $m = 3$. In that case three tableaux are needed; use $U_2(4, 1) \vee 3U_1(2)$, $U_2(4, 1) \vee 2U_4(2, 1)$, and $U_2(4, 1) \vee U_1(4) \vee U_1(2)$. When $f = 4$ and $h = 9$ this construction suffices except when $m = 4$. In that case we need an additional tableau, so use $3U_2(4, 1) \vee U_4(3, 2)$. However, when $h = 9$ the tableau $\mathcal{F}$ has 9 elements. Since we've only constructed the basis tableaux for $c \leq 8$, use $\mathcal{F} = U_2(4, 1) \vee 2U_1(4)$. When $f = 4$ and $h = 4$ we can take $\mathcal{F}$ to be a tableau of shape $[12, 8]$ and use the tableaux of Forms II, $III - 1$, and $III - 4$ of Table 9.2.4. This provides $1 + 2\lfloor \frac{m}{2} \rfloor$ tableaux, which suffices.

Now consider $m = 2$ for $d = 5$. The tableaux will have $c \leq 9$ elements unless $f = 2, h = 7$, $f = 3, h = 8$, or $f = 4$. The $c \leq 9$ will be constructed later. If $f = 2, h = 7$ we need two tableaux, $U_2(4, 1) \vee U_4(3, 2) \vee U_1(4)$ and $U_2(4, 1) \vee U_4(2, 1) \vee P_4(0, 2, 2)$. If $f = 3, h = 8$, two tableaux are required. Let $\mathcal{F}$ be a basis tableaux of shape $[14, 11]$; use $\mathcal{A}_1 \vee \mathcal{F}$ and $\mathcal{A}_4 \vee \mathcal{F}$. If $f = 4, h = 9$, two tableaux are needed, $2U_2(4, 1) \vee U_4(2, 1)$ and $U_2(4, 1) \vee U_4(2, 1) \vee P_4(0, 2, 2) \vee U_1(4)$. If $f = 4, h = 4$ we need three tableaux. Let $\mathcal{F}$ be the basis tableau of shape $[12, 8]$. Then $\mathcal{A}_4 \vee \mathcal{F}$, $\mathcal{A}_1 \vee \mathcal{F}$ and $2U_4(3, 2) \vee \mathcal{F}$ provide the requisite tableaux. Thus all necessary tableaux for $d = 5$ have been constructed.

Now consider when $m < 2$ for arbitrary d. If $r < d$ then since $s - 2d < r < s$ we have $c \leq 6$, which has been done. If $r = d + f$ then we must have $c = 7, 8$ or 9. First consider when $c = 7$ with $r = d + f$. We get $s = 3d - \frac{f}{2}$. For d even use $U_1(d) \vee \mathcal{F}$ where $\mathcal{F}$ are the $c = 5$ tableaux with $r = d + f$. When d is odd use $U_1(d - 1) \vee \mathcal{F}$ where $\mathcal{F}$ are the $c = 5$ tableaux with $r = d + f - 2$. This is sufficient unless $d \equiv 1 \pmod 6, f \equiv 2 \pmod 6$, $d \equiv 3 \pmod 6, f \equiv 0 \pmod 6$, or $d \equiv 5 \pmod 6, f \equiv 4 \pmod 6$, in which case we need one additional tableau. For that, use $U_2(d - 1, 1) \vee \mathcal{B}$ where $\mathcal{B}$ is a $c = 3$ basis tableau with $s \leq r$. This construction holds for $d > 3$. When $d = 3$, only the shape $[12, 9]$ is needed. One tableau, $U_2(2, 1) \vee U_4(2, 1)$, suffices. Thus the $c = 7$ case is complete.

Now take $c = 8$. Since $s - 2d < r$, we have $f > \lfloor \frac{d}{3} \rfloor$ and $s = 3d + \frac{d-f}{2}$. For d even we can use $U_1(d) \vee \mathcal{F}$ where $\mathcal{F}$ are the $c = 6$ basis tableaux of shape $[3d - \frac{d+f}{2}, 2d + \frac{d-f}{2}]$. When d is odd, use $U_1(d - 1) \vee \mathcal{F}$ where $\mathcal{F}$ is the $c = 6$ tableaux of shape $[3d - \frac{d+f}{2} - 1, 2d + \frac{d-f}{2} + 1]$. This construction suffices unless $d + f \equiv 0 \pmod 6$. (Since $\frac{d}{3} < f$, $d \equiv f \pmod 2$ and $d \geq 6$, such an $\mathcal{F}$ always exists.) If $d + f \equiv 0 \pmod 6$ use the tableau of Forms I and

II of Table 9.2.4, with $\mathcal{F}$ the $c = 5$ tableaux of shape $[2d + \frac{d-f}{2} + f, 2d + \frac{d-f}{2}]$. This suffices unless $d = 13, f = 11$, $d \equiv 1 \pmod 6), f = 5$, or $d \equiv 5 \pmod 6), f = 7$. In the first case, five tableaux are needed; use $2U_2(9,1), U_2(9,1) \vee U_1(12) \vee U_1(8), 3U_1(12) \vee U_1(4), 2U_1(12) \vee U_1(10) \vee U_1(6), U_2(12,1) \vee U_1(12) \vee U_1(2)$. In the $f = 5$ case use the tableaux $\mathcal{B}_p \vee \mathcal{F}$ where $\mathcal{B}_p$ are the $c = 3$ basis tableaux of shape $[2d+1, d-1]$ and $\mathcal{F}$ is the $c = 5$ tableau with $r = 3$. This suffices for $\frac{d-f}{2}$ even. When $\frac{d-f}{2}$ odd, an additional tableau, $U_2(d-1,1) \vee U_1(d-1) \vee U_1(\frac{d-f}{2}+1)$, is needed. In the $f = 7$ case use the tableaux $\mathcal{B}_p \vee \mathcal{F}$ where $\mathcal{B}_p$ are the $c = 3$ basis tableaux of shape $[2d-1, d+1]$ and $\mathcal{F}$ are the $c = 5$ tableaux with $r = 9$.

Now consider when $c = 9$ with $r = d + f$. We get $s = 4d - \frac{f}{2}$. Since $s - 2d < r < s$ we have $\frac{2d}{3} < f < \frac{d}{2}$ and hence this case does not occur. This completes the $r < s$ case. Hence we have proven Theorem 3.

Chapter 10

Two Row Partitions and the Gaussian Polynomial

Let $n = ab$ with $a, b \in \mathbb{N}$ and take $\ell \in \mathbb{N}$ such that $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$. Let $\mathcal{P}_a^b(\ell)$ be the numbers of partitions of n having at most a parts each of size less that or equal to b, that is partitions of n fitting inside a $b \times a$ rectangle. Then $\mathcal{P}_a^b(\ell)$ is the co-efficient of q^ℓ in the Gaussian polynomial $\left[{a+b \atop b} \right]_q$, as in [1]. The Gaussian polynomial, $\left[{a+b \atop b} \right]_q$, is also called the Gaussian co-efficient or the generalized q-binomial coefficient.

Lemma 10.0.1. Take $n = ab$ and $\lambda = [n-\ell, \ell]$. Let $K = \mathcal{S}_b \wr \mathcal{S}_a$. The multiplicity of χ^λ in $1_K^{\mathcal{S}_n}$ equals $\mathcal{P}_a^b(\ell) - \mathcal{P}_a^b(\ell - 1)$.

Proof. Let $H = \mathcal{S}_{n-\ell} \times \mathcal{S}_\ell$ and $H' = \mathcal{S}_{n-\ell+1} \times \mathcal{S}_{\ell-1}$. Then $\chi^{[n-\ell,\ell]} = 1_H^{\mathcal{S}_n} - 1_{H'}^{\mathcal{S}_n}$ by the determinantal formula [14]. So $\langle 1_K^{\mathcal{S}_n}, \chi^{[n-\ell,\ell]} \rangle_{\mathcal{S}_n} = \langle 1_K^{\mathcal{S}_n}, 1_H^{\mathcal{S}_n} \rangle_{\mathcal{S}_n} - \langle 1_K^{\mathcal{S}_n}, 1_{H'}^{\mathcal{S}_n} \rangle_{\mathcal{S}_n}$. Hence it suffices to show $\langle 1_K^{\mathcal{S}_n}, 1_H^{\mathcal{S}_n} \rangle_{\mathcal{S}_n} = \mathcal{P}_a^b(\ell)$.

Now $\langle 1_K^{\mathcal{S}_n}, 1_H^{\mathcal{S}_n} \rangle_{\mathcal{S}_n}$ is the number of orbits of K acting on the cosets of H in $\mathcal{S}_n$ [11]. View the numbers 1 to n in blocks of size b, that is

$$|1, 2, \ldots, b|b+1, \ldots 2b| \cdots |(a-1)b+1, \ldots ab|$$

The copies of H in $\mathcal{S}_n$ correspond to the different ways $\mathcal{S}_\ell$ sits in $\mathcal{S}_n$, that is subsets of $\{1, \ldots n\}$ of size ℓ. Given such a subset L (corresponding to a copy of H) it will be broken into a parts by intersection with the blocks above. Let μ_i be the size of the part of L in the ith block. Since K acts by $\mathcal{S}_b$ on each of the blocks, L is equivalent (under K) to a subset L' where the first μ_i numbers $\{i \cdot b + 1, \ldots i \cdot b + \mu_i\}$ are chosen from block i (starting with the 0^{th} block). Since K also has the wreath product action by $\mathcal{S}_a$ acting on the blocks, L'

is equivalent to the subset L^*, where the blocks are reordered so the $\mu_i \geq \mu_{i+1}$. Hence L^* corresponds to a partition of the number ℓ into a parts of size at most b and every such partition corresponds to a copy of H in $\mathcal{S}_n$.

So every such partition is contained in some orbit of K on $\mathcal{S}_n/H$, and every orbit contains some such partition. Hence it suffices to shows that no two partitions are in the same orbit. Say $\mu = [\mu_0, \dots \mu_{a-1}]$ and $\nu = [\nu_0, \dots \nu_{a-1}]$ are partitions of ℓ where we allow $0 \leq \mu_i, \nu_i \leq b$. If μ and ν are in the same orbit, then there exists $g \in K$ such that $g \cdot \{i \cdot b + j | 0 \leq i \leq a-1, 1 \leq j \leq \mu_i\} = \{i \cdot b + j | 0 \leq i \leq a-1, 1 \leq j \leq \nu_i\}$. So $g(ib + j) = k_{i,j}b + c_{i,j}$. Since g moves complete blocks, we must have $k_{i,j} = k_{i,j'}$ for all $1 \leq j, j' \leq \mu_i$. As the action is injective, we must then have $c_{i,j} \neq c_{i,j'}$ for $j \neq j'$. Hence looking at the image, we have $\mu_i = |\{c_{i,j}\}| \leq \nu_{k_i}$.

Take $\mu \trianglerighteq \nu$, $\mu \neq \nu$. There exists i such that $\mu_i > \nu_i$ and $\mu_{i'} = \nu_{i'}$ for all $i' < i$. Then there are i such $\nu_{i'}$ with $\nu_{i'} \geq \mu_i$. But if $g \cdot \mu = \nu$ then $\mu_i = |\{c_{i,j}\}| \leq \nu_{k_i}$ implies there are at least $i+1$ such $\nu_{i'}$, which is a contradiction. Hence no orbit contains two such partitions, which finishes the proof. □

The ideas behind this proof are due to J. Saxl stemming from discussions of his paper [13].

Since the multiplicity of irreducibles in induced characters is non-negative [16], this lemma implies the well-known unimodality of the Gaussian coefficients [1]. Now $\mathcal{P}^b_a(\ell) = \mathcal{P}^a_b(\ell)$, since $\left[{a+b \atop b}\right]_q = \left[{a+b \atop a}\right]_q$ by taking conjugate partitions. Hence this lemma shows that Foulkes' Conjecture always holds for two row partitions, which is discussed in [14].

We can also interpret our results on the generalized Foulkes' Conjecture in terms of the Gaussian coefficient. From Theorem 1 we have:

Theorem 12. If $n = 2b = cd$, with $c, d \geq 2$, then for $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$,

$$\mathcal{P}^d_c(\ell) - \mathcal{P}^d_c(\ell - 1) \geq \mathcal{P}^b_2(\ell) - \mathcal{P}^b_2(\ell - 1)$$

Similarly, Theorem 3 gives:

Theorem 13. If $n = 3b = cd$, with $c, d \geq 3$, then for $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$,

$$\mathcal{P}^d_c(\ell) - \mathcal{P}^d_c(\ell - 1) \geq \mathcal{P}^b_3(\ell) - \mathcal{P}^b_3(\ell - 1)$$

Hence our results give insight into the relationship between the rates of growth of different Gaussian coefficients.

Chapter 11

The Alternating Character

Since Foulkes' Conjecture is based on the trivial character, it is natural to ask whether the ideas hold for the alternating character. The first question is, what we mean by the alternating character in terms of induced modules.

Consider, the 'alternating' character of the form $((-1)_{\mathcal{S}_{ab}} \downarrow_{\mathcal{S}_a \wr \mathcal{S}_b}) \uparrow^{\mathcal{S}_{ab}}$, that is the usual alternating character of $\mathcal{S}_{ab}$ restricted to the subgroup $\mathcal{S}_a \wr \mathcal{S}_b$, which is induced back up to $\mathcal{S}_{ab}$. A brief computer check of Foulkes' Conjecture using this character shows it holds for some small values of a and b. In fact, Foulkes' Conjecture is equivalent to the following conjecture using the alternating character.

Conjecture 3 (Foulkes' Conjecture for Alternating Characters). If $a \leq b$ then every irreducible character occurring in $((-1)_{\mathcal{S}_{ab}} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_a}) \uparrow^{\mathcal{S}_{ab}}$ occurs in $((-1)_{\mathcal{S}_{ab}} \downarrow_{\mathcal{S}_a \wr \mathcal{S}_b}) \uparrow^{\mathcal{S}_{ab}}$ with multiplicity greater than or equal to its multiplicity in $((-1)_{\mathcal{S}_{ab}} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_a}) \uparrow^{\mathcal{S}_{ab}}$.

Naturally, this conjecture also generalizes to:

Conjecture 4 (Generalized Foulkes' Conjecture for Alternating Characters). Given $n = ab$, $a \leq b$, if c, d are such that $cd = n$, and $c, d \geq a$, then every irreducible character occurring in $((-1)_{\mathcal{S}_n} \downarrow_{\mathcal{S}_b \wr \mathcal{S}_a}) \uparrow^{\mathcal{S}_n}$ occurs in $((-1)_{\mathcal{S}_n} \downarrow_{\mathcal{S}_d \wr \mathcal{S}_c}) \uparrow^{\mathcal{S}_n}$ with multiplicity at least as large.

Showing the equivalences of Conjecture 1 or Conjecture 2, (Foulkes' Conjecture for trivial characters), and Conjectures 3 or 4 (Foulkes' Conjecture for alternating characters) is straightforward. We will assume Conjecture 1 (or Conjecture 2) holds and prove the alternating character version. The same argument shows the reverse equivalence.

Proof. First recall that if S is a subgroup of finite index in G, F an S-module and E a G-module over a field, then there is an isomorphism $\mathrm{Ind}_S^G(\mathrm{Res}_S(E)\otimes F)\simeq E\otimes \mathrm{Ind}_S^G(F)$. (See Chapter XVIII §7 of [16].)

Note that here we have used Ind for induction and Res for restriction of modules. Also, let $G=\mathcal{S}_n$, $S=\mathcal{S}_a\wr\mathcal{S}_b$ and $T=\mathcal{S}_b\wr\mathcal{S}_a$ or $\mathcal{S}_d\wr\mathcal{S}_c$ as appropriate. Let E be the G-module corresponding to the character (-1) on G. Since we are working over $\mathbb{C}$, we will use $\mathbb{C}$ to denote the trivial module over any group.

The characters we're comparing are $\chi_S=\mathrm{Ind}_S^G(\mathrm{Res}_S(E))=(-1)_{\mathcal{S}_{ab}}\downarrow_{\mathcal{S}_b\wr\mathcal{S}_a}\uparrow^{\mathcal{S}_{ab}}$ and $\chi_T=\mathrm{Ind}_T^G(\mathrm{Res}_T(E))$. Then $\chi_S\simeq\mathrm{Ind}_S^G(\mathrm{Res}_S(E)\otimes\mathbb{C})\simeq E\otimes\mathrm{Ind}_S^G(\mathbb{C})$ by the isomorphism mentioned above. Similarly for χ_T. Switching notation back to characters, we get $\chi_S\simeq(-1)_G 1_S^G$ and $\chi_T\simeq(-1)_G 1_T^G$.

Since we've assumed Foulkes' Conjecture on trivial characters, we have $1_S^G=1_T^G+\psi$ for some character ψ. Then $\chi_S=(-1)_G(1_T^G+\psi)=(-1)_T^G 1_T^G+(-1)_G\psi$. Hence $\chi_S\leq\chi_T$ as desired.

□

Since we've proven Theorem 1 the argument above shows:

Theorem 14. If $2\leq b$ then every irreducible character occurring in $((-1)_{\mathcal{S}_{2b}}\downarrow_{\mathcal{S}_b\wr\mathcal{S}_2})\uparrow^{\mathcal{S}_{2b}}$ occurs in $((-1)_{\mathcal{S}_{2b}}\downarrow_{\mathcal{S}_2\wr\mathcal{S}_b})\uparrow^{\mathcal{S}_{2b}}$ with multiplicity greater than or equal to its multiplicity in $((-1)_{\mathcal{S}_{2b}}\downarrow_{\mathcal{S}_b\wr\mathcal{S}_2})\uparrow^{\mathcal{S}_{2b}}$.

Similarly, Theorem 2 gives:

Theorem 15. Given $n=3b$, $3\leq b$, if c, d are such that $cd=n$, and c, $d\geq 3$, then every irreducible character occurring in $((-1)_{\mathcal{S}_n}\downarrow_{\mathcal{S}_b\wr\mathcal{S}_3})\uparrow^{\mathcal{S}_n}$ occurs in $((-1)_{\mathcal{S}_n}\downarrow_{\mathcal{S}_d\wr\mathcal{S}_c})\uparrow^{\mathcal{S}_n}$.

While Theorem 3 shows:

Theorem 16. Let $n=3b=cd$, with c, $d\geq 3$ and let $\lambda=[\lambda_1,\lambda_2]$ be a two row partition of n. Then every irreducible character χ^λ occurring in $((-1)_{\mathcal{S}_n}\downarrow_{\mathcal{S}_b\wr\mathcal{S}_3})\uparrow^{\mathcal{S}_n}$ occurs in $((-1)_{\mathcal{S}_n}\downarrow_{\mathcal{S}_d\wr\mathcal{S}_c})\uparrow^{\mathcal{S}_n}$ with multiplicity at least as large.

Given the success of replacing the trivial character in Foulkes' Conjecture with this 'alternating' character, it is natural to investigate if other definitions of an alternating character yield similar results. One suggestion was to try $(-1)_{\mathcal{S}_a\wr\mathcal{A}_b}^{\mathcal{S}_{ab}}-(-1)_{\mathcal{S}_a\wr\mathcal{S}_b}^{\mathcal{S}_{ab}}$ for an induced alternating character in place of $1_{\mathcal{S}_a \wr \mathcal{S}_b}^{\mathcal{S}_{ab}}$ in Foulkes' Conjecture. Alas, a simple computer check via GAP [9] shows Foulkes' Conjecture for this character fails when $a = 3$ and $b = 4$. Other variations on this character, such as $(-1)_{\mathcal{A}_a \wr \mathcal{S}_b}^{\mathcal{S}_{ab}} - (-1)_{\mathcal{S}_a \wr \mathcal{S}_b}^{\mathcal{S}_{ab}}$ also fail at those values.

Chapter 12

Discussion of General Results

Theorems 1, 2, and 3 extend the current research on Foulkes' Conjecture. Although, the proof used combinatorial techniques on Young tableaux, the results correspondingly apply to areas such as Shur functions, Rational Homotopy Theory, and other means of interpreting Foulkes' Conjecture. In addition we may interpret the Foulkes' Conjecture using the alternating character via these theorems, which we discussed in Chapter 11.

While the construction of the tableaux themselves are cumbersome, the development of the theory illustrates new approaches to Young tableaux. These concepts could be carried forth in contexts involving tableaux other than its usage here. Although the main theorems are specific to the cases $a = 2$ and $a = 3$, some general results arise from this study.

The main theoretical techniques of this paper are that of weight-set counting in Theorem 4, the application of Theorem 8, and the use of maximality to show linear independence.

The theory and technique of weight-set counting developed in this paper can be implemented in general, as can the concept of maximal form. While we only used tableaux with three or fewer rows, the theoretical foundations of weight-set counting have been laid for tableaux of an arbitrary number of rows. Although the computations are impractical for a random tableau, the counting works smoothly for tableaux with suitable symmetries, particularly those tableaux in maximal form.

Moreover, the technique of weight-set counting is not dependent on a filling of content $[b^a]$. However, for non-uniform contents, one must watch carefully the action of $\mathcal{S}_a$; the weight-set counting may need to count all rows and the definition of maximality will need adjustment.

Similarly, the usage of Theorem 8 in constructing larger tableaux will also work for other contents and row quantities. The use of the Lemma 3.4.9, to show weight-set disjointness by maximality, however, has only been defined for three row partitions. It should be possible to generalize it for other partitions.

Tableau maximality is a very useful concept for showing weight-set disjointness and applying Theorem 8. It is vital in proving linear independence of tableaux. Linear independence through tableau maximality should allow more progress on issues such as multiplicity.

The methods of proving Theorem 2 could also apply to proving Conjecture 2 with other a's, not including multiplicities. Unfortunately, the computations are likely to be somewhat cumbersome, especially the establishments of non-zero shapes as done by Theorem 9. However, should those parameters be established through other techniques, the tableaux constructed for Theorem 2 should provide nearly all the needed shapes with three or fewer rows, thus reducing the work substantially. Moreover, the reduction procedures will also apply.

Specifically, given any $n = ab = cd$, if the shapes having multiplicity zero in $\mathcal{S}_b \wr \mathcal{S}_a$ are bounded, then to prove the generalized Foulkes' Conjecture for arbitrary c, $d \geq a$, we should only need to prove it for a limited number of c's. For instance, suppose a shape had multiplicity zero only if $\lambda_i - \lambda_{i+1} \leq f_i$. (For $n = 3b$, we had $f_1 = f_2 = 4$.) Assume d is even (the odd case, though more cumbersome should follows analogously). Then given a tableau, we can 'peel off' a column block of size d with the appropriate row length, for instance, $U_1(d)$ and $P_1(d)$ are the two and three row versions. We can repeat this process so long as $\lambda_i - \lambda_{i+1} > d + f_i$ (and there are at least as many elements as there are rows). Hence in the end, we need only construct a tableau with $\lambda_i - \lambda_{i+1} \leq d + f_i$. This tableau will need at most $\frac{1}{d} \sum_i i(d + f_i)$ elements, hence c will be bounded by this number. This should imply, given the f_i, if the generalized Foulkes' Conjecture is true for c up to some bound, it is true for all c. (Presumably, if these tableaux exists, we can find versions with maximal/disjoint weight sets as needed.) The existence of the f_i seems probable since Theorem 2 implies f_1, $f_2 \leq 4$ if $3|n$. It may be the case, as in Theorem 1 that the parity of λ_i strongly effects the multiplicity. However, since the 'peeling off' does not change the parity, this process should still go through.

In addition to this procedure, the investigations of Theorems 1 and 2 yield some general results. Take the character $1_{\mathcal{S}_a \wr \mathcal{S}_b}^{\mathcal{S}_{ab}}$ and consider the irreducible $\mathbb{S}^\lambda$ corresponding to $\lambda = [\lambda_1, \lambda_2]$. Then $\mathbb{S}^\lambda$ always has non-zero multiplicity whenever λ_1 and λ_2 are even.

Moreover, this multiplicity is zero whenever $\lambda_2 = 1$ regardless of the choice of a and b. These theorems also have implications regarding the generalized Gaussian polynomial, as discussed in Chapter 10.

Finally, the techniques within the proof of Theorem 3 should extend beyond two row tableaux. Specifically, Theorem 2 can probably be strengthened to include multiplicities for all partitions. Such a result should follow the ideas of Theorem 3, though sufficient linearly independent three row tableaux for $c = 4$, 5, and 6 must first be established. However, weight-set maximality should be sufficient to demonstrate linear independence. In all, these results provide a strong foundation for those wishing to study the representation theory of wreath products of symmetric groups via tableaux.

Appendix A

Association between Tableaux Spaces and Irreducibles

Let $n = ab$ and $H = \mathcal{S}_b \wr \mathcal{S}_a$. Recall that $\mathcal{W}^{\lambda,a} = \{T|T$ a λ-tableau filled with 1 to a, each b times$\}$. We will explicitly show why the multiplicity of χ_λ in $1_H^{\mathcal{S}_n}$ equals the dimension of $\mathbb{C}\{\mathbf{q}_T|T \in \mathcal{W}^{\lambda,a}\}$.

View H as a subgroup of $\mathcal{S}_n$, where H acts on

$$1, 2, \dots b|b+1, \dots, 2b| \dots |(a-1)b+1, \dots, ab$$

by $\mathcal{S}_b$ on each block and by $\mathcal{S}_a$ permuting the blocks. The elements of H are the form $(\pi_1, \dots \pi_a, \sigma)$ with $\pi_i \in \mathcal{S}_b$, $\sigma \in \mathcal{S}_a$. Now $\mathcal{S}_a \times \mathcal{S}_n$ acts on $\mathcal{W}^{\lambda,a}$ with $\mathcal{S}_a$ acting on the numbers 1 to a and $\mathcal{S}_n$ acting on the positions (corresponding to labelling across the rows).

Let $K = \{(\sigma^{-1}, (\pi_1, \dots \pi_a, \sigma))|\pi_i \in \mathcal{S}_b, \sigma \in \mathcal{S}_a\}$. So $K \leq \mathcal{S}_a \times H \leq \mathcal{S}_a \times \mathcal{S}_n$. Let T be the λ-tableau filled across the rows with b 1's, then b 2's, etc. Then $\mathcal{S}_a \times \mathcal{S}_n$ acting on T gives $\mathcal{W}^{\lambda,a}$ and K fixes T. Specifically, $Stab_{\mathcal{S}_a \times \mathcal{S}_n}(T) = K$. Hence as $\mathcal{S}_a \times \mathcal{S}_n$ modules, $\mathcal{W}^{\lambda,a} \simeq 1_{Stab(T)}^{\mathcal{S}_a \times \mathcal{S}_n} = 1_K^{\mathcal{S}_a \times \mathcal{S}_n}$.

Proposition A.0.2. $\mathcal{W}^{\lambda,a} \simeq \sum_{\mu \vdash a} \varphi_{\mathcal{S}_a}(\mu) \otimes (\varphi_H(\mu))^{\mathcal{S}_n}$ where $\varphi_{\mathcal{S}_a}(\mu)$ is the irreducible of $\mathcal{S}_a$ indexed by μ and $\varphi_H(\mu)$ is the irreducible of $H/(\mathcal{S}_b \times \dots \times \mathcal{S}_b) \simeq \mathcal{S}_a$ indexed by μ.

Proof. Since $1 \times (\mathcal{S}_b \times \dots \times \mathcal{S}_b) \leq K$, it is in the kernel of 1_K. As $\mathcal{S}_b \times \dots \times \mathcal{S}_b \trianglelefteq H$, it is in the kernel of $1_K^{\mathcal{S}_a \times H}$. So we can view $1_K^{\mathcal{S}_a \times H}$ as an $\mathcal{S}_a \times H/(\mathcal{S}_b \times \dots \times \mathcal{S}_b) \simeq \mathcal{S}_a \times \mathcal{S}_a$ module. Let $D = \{(\sigma^{-1}, \sigma)|\sigma \in \mathcal{S}_a\}$ be the image of K in $\mathcal{S}_a \times H/(\mathcal{S}_b \times \dots \times \mathcal{S}_b)$. Hence $1_K^{\mathcal{S}_a \times H} \simeq 1_D^{\mathcal{S}_a \times \mathcal{S}_a}$ as $\mathcal{S}_a \times H/(\mathcal{S}_b \times \dots \times \mathcal{S}_b)$ modules. Thus we can write $1_D^{\mathcal{S}_a \times \mathcal{S}_a} = \sum a_{\mu,\nu} \phi_\mu \otimes \phi_\nu$ for $\mu, \nu \vdash a$ and some $a_{\mu,\nu}$, where ϕ is the corresponding irreducible of $\mathcal{S}_a$.

By Frobenius reciprocity $a_{\mu,\nu} = (\phi_\mu \otimes \phi_\nu, 1_D^{\mathcal{S}_a\times\mathcal{S}_a}) = (\phi_\mu \otimes \phi_\nu|_D, 1_D)_D$. Now $\phi_\mu \otimes \phi_\nu|_D = \phi_\mu\overline{\phi_\nu}$. So $(\phi_\mu \otimes \phi_\nu|_D, 1_D)_D = \frac{1}{|D|}\sum_{\sigma\in\mathcal{S}_a}\phi_\mu(\sigma)\overline{\phi_\nu(\sigma)} = \frac{|\mathcal{S}_a|}{|D|}\cdot(\phi_\mu,\phi_\nu)_{\mathcal{S}_a}$. Using row orthogonality and $|\mathcal{S}_a| = |D|$, we have $a_{\mu,\nu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ otherwise} \end{cases}$.

So $1_D^{\mathcal{S}_a\times\mathcal{S}_a} = \sum_{\mu\vdash a}\phi_\mu\otimes\phi_\mu$. If we lift back to the original module $1_K^{\mathcal{S}_a\times H}$ we have $1_K^{\mathcal{S}_a\times H} = \sum_\mu \phi_{\mathcal{S}_a}(\mu)\otimes\phi_H(\mu)$ where $\phi_{\mathcal{S}_a}(\mu)$ is the irreducible of $\mathcal{S}_a$ indexed by μ and $\phi_H(\mu)$ is the irreducible of $H/(\mathcal{S}_b\times\cdots\times\mathcal{S}_b)\simeq\mathcal{S}_a$ indexed by μ. Since $1_K^{\mathcal{S}_a\times\mathcal{S}_n} = (1_K^{\mathcal{S}_a\times H})^{\mathcal{S}_a\times\mathcal{S}_n}$ we get

$$1_K^{\mathcal{S}_a\times\mathcal{S}_n} = (\sum_\mu \phi_{\mathcal{S}_a}(\mu)\otimes\phi_H(\mu))^{\mathcal{S}_a\times\mathcal{S}_n} = \sum_\mu \phi_{\mathcal{S}_a}(\mu)\otimes(\phi_H(\mu))^{\mathcal{S}_n}.$$

□

By this proposition we have $\mathcal{W}^{\lambda,a}\simeq\sum_{\mu\vdash a}\varphi_{\mathcal{S}_a}(\mu)\otimes(\varphi_H(\mu))^{\mathcal{S}_n}$ as $\mathcal{S}_a\times\mathcal{S}_n$ modules. Consider the submodule on which $\mathcal{S}_a$ is trivial, that is, $\mu = (a)$. This corresponds to $1_{\mathcal{S}_a}\otimes(1_H)^{\mathcal{S}_n}$. If $1_H^{\mathcal{S}_n} = \sum_{\nu\vdash n} m_\nu\chi_\nu$, this module corresponds to $\sum_{\nu\vdash n} 1_{\mathcal{S}_a}\otimes m_\nu\chi_\nu$. Now $e_\lambda = \sum_{\sigma\in R_T}\sum_{\tau\in C_T}\epsilon(\tau)\sigma\tau$ is an idempotent of $\mathcal{S}_n$ on λ-tableau T. So the action of e_λ on $\sum 1\otimes m_\nu\chi_\nu$ is the same as the action of $q_\lambda = \sum_{\pi\in\mathcal{S}_a}\pi e_\lambda$ on $\mathcal{W}^{\lambda,a}$. Then $q_\lambda\cdot\mathcal{W}^{\lambda,a}\simeq m_\lambda(e_\lambda\cdot\mathcal{S}^\lambda)$ as $\mathcal{S}_n$ modules, as $e_\lambda\cdot\mathcal{S}^\nu = 0$ for $\lambda\neq\nu$. Now $\mathcal{S}^\lambda$ is is a cyclic $\mathcal{S}_n$-module generated by $e_\lambda(T)$. (Correspondingly, the semi-standard tableaux which span $\mathcal{S}^\lambda$ are equivalent under the action of $\mathcal{S}_n$.) Therefore $\dim(e_\lambda\mathcal{S}^\lambda) = 1$ and $\dim(q_\lambda\mathcal{W}^\lambda) = m_\lambda$. Hence $\{\mathbf{q}_T | T\in\mathcal{W}^{\lambda,a}\}$, spans a module of dimension m_λ, the multiplicity of χ_λ in $1_{\mathcal{S}_b\wr\mathcal{S}_a}^{\mathcal{S}_n}$. This proof is due to Wales, [22].

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