Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

2004

(Submitted May 27, 2004)

Chapter 9

Proof of Theorem 3

Theorem 2 proves the generalized Foulkes' Conjecture for $c = 3$ without multiplicities. We can, however, strengthen this result to include multiplicities for those irreducibles corresponding to two row partitions. Namely,

Theorem 3. Let $n = 3b = cd$, with c, $d \geq 3$ and let $\lambda = [\lambda_1, \lambda_2]$ be a two row partition of n. Then every irreducible character χ^λ occurring in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ occurs in $1_{\mathcal{S}_d \wr \mathcal{S}_c}^{\mathcal{S}_n}$ with multiplicity at least as large.

The proof of this theorem involves constructing the appropriate number of tableaux, primarily out of the basis elements given in Section 4.3. The tableaux constructed are maximal so linear independence is assured from Lemma 3.4.12 provided the weights are distinct. Once we have the basis tableaux for $c = 4$, 5, and 6, the procedure generalizes to provide the appropriate number of tableaux for any c.

Take $\lambda = [r + s, s]$. The multiplicity m_λ of $\mathcal{S}^\lambda$ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ was determined by Thrall in [20], which we listed in Theorem 11.

Since the multiplicity depends on the relative sizes of r and s we will handle these cases separately. Moreover, if $s = 6k+j$ and $r = 6h+i$, we will often simply construct $k + 1$ or $h + 1$ tableaux when possible to avoid detailed case analysis.

9.1 Case: $s \leq r$

Let $\lambda = [r + s, s]$ be a partition of n with $s \leq r$, where $n = 3b = cd$. We wish to construct m_λ linearly independent tableaux, where m_λ is the multiplicity of χ^λ in

$1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ as described in Theorem 11. First we will construct these tableaux for $c = 4$, 5, and 6; then we will use these constructions in proving Theorem 3 for a general c. We will refer to the tableaux constructed in this way as basis tableaux. These constructions will make use of the $c = 3$ basis tableaux constructed in Section 4.3.

9.1.1 Basis Tableaux for $c = 4$, $s \le r$

Given $\lambda = [r + s, s]$ a partition of n, we have $2s + r = 4d = 3b$. From this equation and $s \le r$, we have $s \le d + \lfloor \frac{d}{3} \rfloor$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 4. These will be our $c = 4$ basis tableaux.

When $s \le r - d$, we can use the basis tableaux constructed in Section 4.3. Consider the partition $\lambda' = [r' + s, s]$ where $r' = r - d$. Since $s \le r'$, we have $m_\lambda = m_{\lambda'}$. In Section 4.3 we constructed $m_{\lambda'}$ linearly independent $\mathcal{B}_p$, where $\mathcal{B}_p$ are the basis tableaux for $c = 3$. Take $\mathcal{B}_p \vee \frac{\mathrm{d}}{4}$ as the basis tableaux for $c = 4$. This works for $s \le r - d$, so $r \ge s + d$. Hence $4d = 2s + r \ge 3s + d$ implies $s \le d$.

When $d < s \le d + \lfloor \frac{d}{3} \rfloor$, write $s = d + f$, with $1 \le f \le \lfloor \frac{d}{3} \rfloor$. Consider the tableau

$$T(\textsc{a}, \textsc{b}, \textsc{c}, \textsc{d}) = \begin{array}{cccc} \textsc{a} & \textsc{b} & \textsc{c} & \textsc{d} \\ \hline 4 & 3 & 4 & 3 \\ 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} \textsc{a} + \textsc{b} \le d \\ \textsc{a}, \textsc{b}, \textsc{c}, \textsc{d} > 0 \\ \textsc{a} > \textsc{d} \\ \textsc{b} > \textsc{c} \\ \text{or } \textsc{a} = \textsc{d}, \textsc{b} = \textsc{c} \end{array}$$

$$\mathsf{w}(T) = (\textsc{a} + \textsc{b}, \textsc{c} + \textsc{d}, 0, 0)$$

If $\textsc{a} > \textsc{d}$ and $\textsc{b} > \textsc{c}$, no other weight assignments are possible for T, hence this tableau is non-zero and maximal. If $\textsc{a} = \textsc{d}$ and $\textsc{b} = \textsc{c}$, we may also have the tableau obtained by exchanging the rows. However, this has sign $(-1)^{\textsc{a}+\textsc{b}+\textsc{c}+\textsc{d}} = 1$ and thus the tableau is still non-zero.

Let $\mathcal{C}_p = T(d - 2p,\ p + 1,\ p,\ f - 1)$. Then $\lambda_2(T) = \textsc{a} + \textsc{b} + \textsc{c} + \textsc{d} = d - 2p + p + 1 + p + f - 1 = d + f = s$. Hence these tableaux have the desired shape. Consider $\mathcal{C}_p$ for $p = 1, 2, \ldots, \lfloor \frac{d-f}{2} \rfloor$. To insure the $\mathcal{C}_p$ are non-zero and maximal we need to check that the constraints on T are satisfied. For $f \neq 1$ all the parameters are greater than zero. Obviously, $\text{B} = p+1 > \text{C} = p$. For $\text{A} > \text{D}$, we need $d-2p > f-1$. This is true provided $p < \frac{d-f+1}{2}$. Since $p \leq \lfloor \frac{d-f}{2} \rfloor$ this inequality holds.

The $\mathcal{C}_p$ are linearly independent by Lemma 3.4.12 if their max weights are distinct. We have $\mathsf{w}(\mathcal{C}_p) = (d-p+1, f+p-1, 0, 0)$. If $\mathsf{w}(\mathcal{C}_p) = \mathsf{w}(\mathcal{C}_{p'})$ for $p > p'$, then we must have $d-p+1 = f+p'-1$, that is $d-f+2 = p+p'$. But $p+p' \leq \frac{d-f}{2} + \frac{d-f}{2} - 1 = d-f-1$. Hence this cannot occur. Thus the $\mathcal{C}_p$ are linearly independent.

Since $s = d+f$, we have $m_\lambda \leq \lfloor \frac{d+f}{6} \rfloor + 1$, so it suffices to construct $\lfloor \frac{d+f}{6} \rfloor + 1$ tableaux. The $\mathcal{C}_p$ provide $\lfloor \frac{d-f}{2} \rfloor$ tableaux. To show $\lfloor \frac{d-f}{2} \rfloor \geq \lfloor \frac{d+f}{6} \rfloor + 1$, it suffices to show $\frac{d-f}{2} - \frac{1}{2} > \frac{d+f}{6}$, or equivalently, $d - \frac{3}{2} \geq 2f$. This holds for $d \geq 5$ as $f \leq \lfloor \frac{d}{3} \rfloor$. When $d = 3$ or $d = 4$ then $f = \lfloor \frac{d}{3} \rfloor = 1$, which is handled below. Hence for $s > d+1$ the $\mathcal{C}_p$ provided at least m_λ linearly independent tableaux.

For $s = d+1$, that is $f = 1$, take $\mathcal{C}_p = T(d-p-2, 2, 1, p)$, for $1 \leq p < \lfloor \frac{d-2}{2} \rfloor$. Then the conditions on T are satisfied and such p exist for $d \geq 6$. We have $\mathsf{w}(\mathcal{C}_p) = (d-p, p+1, 0, 0)$ so the max weights are distinct. These $\mathcal{C}_p$ provide at least $\lfloor \frac{d-2}{2} \rfloor - 1$ linearly independent tableaux and we need $\lfloor \frac{d+1}{6} \rfloor + 1$ tableaux. Now $\lfloor \frac{d-2}{2} \rfloor - 1 \geq \lfloor \frac{d+1}{6} \rfloor + 1$ provided $d \geq 8$. When $d = 7$, then $s = 8$ and two tableaux are needed. Use $T(4,2,1,1)$ and $T(3,2,1,2)$. When $d = 6$, then $s = 7$ and only one tableau is needed. In this case, use $\mathcal{C}_p$ described above for $p = 1$. When $d = 5$ then $s = 6$ so two tableaux are needed. Use $T(2,2,1,1)$ and $T(2,1,1,2)$. When $d = 4$ then $s = 5$ so one tableaux suffices. However, for $d = 4$, $s = 5$ there are no tableaux of maximal form. We will use the tableau U_8 constructed in Section 7.2. This tableau is non-zero but not maximal. Here we use maximal form only to show linear independence. Since only one tableau is needed for $s = 5$, this tableau works. (The $d = 4$, $s = 5$ case is actually not needed for the $c = 4$ basis tableaux since $n = 3b = 4d$ implies $3|d$, but we construct the basis tableaux for all $d \geq 3$ in order to simplify the construction process in Section 9.1.4. However we will not use U_8 in that construction.) When $d = 3$, then $s = 4$ and only one tableau is need, hence $T(1,1,1,1)$ suffices.

9.1.2 Basis Tableaux for $c = 5$, $s \le r$

Given $\lambda = [r+s, s]$ a partition of n, we have $2s + r = 5d = 3b$. From this equation and $s \le r$, we have $s \le \lfloor \frac{5d}{3} \rfloor$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 5. These will be our $c = 5$ basis tableaux.

When $s \le r - d$, we can use the $c = 4$ basis tableaux constructed in Section 9.1.1. Consider the partition $\lambda' = [r'+s, s]$ where $r' = r - d$. Since $s \le r'$, we have $m_\lambda = m_{\lambda'}$. In Section 9.1.1 we constructed $m_{\lambda'}$ linearly independent $\mathcal{C}_p$, where $\mathcal{C}_p$ are the basis tableaux for $c = 4$. Take $\mathcal{C}_p \vee \dfrac{\mathrm{d}}{5}$ as the basis tableaux for $c = 5$. This works for $s \le r - d$, so $r \ge s + d$. Hence $5d = 2s + r \ge 3s + d$ implies $s \le \lfloor \frac{4d}{3} \rfloor$.

For $\lfloor \frac{4d}{3} \rfloor < s \le \lfloor \frac{5d}{3} \rfloor$ consider

$$T_1(\text{A}, \text{B}) = \begin{array}{ccccccc} & & \text{A} & \lfloor \frac{d}{2} \rfloor & \lceil \frac{d}{2} \rceil & \text{B} & \\ \hline 4 & 4 & 5 & 4 & 3 & 4 & 5 \\ 2 & 2 & 2 & 1 & 1 & 3 & 3 \end{array}$$

$$1 \le \text{A} \le d - 2$$
$$1 \le \text{B} \le \lfloor \tfrac{d}{2} \rfloor - 3$$
$$\text{A} > \text{B}$$
$$s \ge d + 6$$
$$d \ge 8$$

$$\mathsf{w}(T_1) = (d, \text{A} + 2, \text{B} + 1, 0, 0)$$

T_1 is defined for $s \ge d+6$, $d \ge 8$. (When $d < 8$ then $s < d+6$ since $s \le \lfloor \frac{5d}{3} \rfloor$.) Since $\text{A} > \text{B}$, there are no other weight assignments possible and the tableau T_1 is maximal. Let $\mathcal{E}_p = T_1(s-d-3-p, p)$ for $1 \le p \le \lfloor \frac{s-d}{2} \rfloor - 2$. For $\mathcal{E}_p$ to be non-zero and maximal we need the conditions on T_1 to be satisfied. We have $\text{A} > \text{B}$ since $p \le \lfloor \frac{s-d}{2} \rfloor - 2$, while $p \ge 1$ implies $\text{A} = s - d - 3 - p < d - 2$. Since $d \ge 8$, then $\text{B} \le \lfloor \frac{d}{2} \rfloor - 3$. Note $\text{A} > \text{B}$ implies these max weights are distinct. Hence this construction provides $\lfloor \frac{s-d}{2} \rfloor - 2$ distinct, linearly independent tableaux when $s \ge d + 6$, $d \ge 8$.

We will need an additional tableau, so take:

$$T_2(\text{A}, \text{B}) = \begin{array}{cccc} & \text{A} & & \text{B} \\ \hline 5 & 3 & 5 & 4 \\ 1 & 1 & 2 & 2 \end{array}$$

$$1 \le \text{A}, \text{B} \le d - 1$$
$$s \ge 4$$

$$\mathsf{w}(T_2) = (\text{A} + 1, \text{B} + 1, 0, 0)$$

Clearly T_2 is non-zero and maximal. Given s, take $\mathcal{E}'_q = T_2(d-q, s-2-d+q)$ for $q = 1, 2, \ldots, d - \lfloor \frac{s-2}{2} \rfloor$. To insure $\mathcal{E}'_q$ is non-zero and maximal we need the conditions of T_2 are satisfied. Since $\lfloor \frac{4d}{3} \rfloor < s$ and $d \geq 3$, then $s \geq 4$. We have $1 \leq \mathrm{A} \leq d-1$ because $q < d$ since $s \leq \lfloor \frac{5d}{3} \rfloor$. Similarly the bounds on s show $1 \leq \mathrm{B} \leq d-1$. The $\mathcal{E}'_q$ are linearly independent because $\mathrm{A} < \mathrm{B}$ for $q < d - \lfloor \frac{s-2}{2} \rfloor$. This provides $d - \lfloor \frac{s}{2} \rfloor + 1$ tableaux.

We need $m_\lambda = \lfloor \frac{s}{6} \rfloor + 1$ linearly independent tableaux. By Lemma 3.4.12 the tableaux $\mathcal{E}_p$ and $\mathcal{E}'_q$ are linearly independent since they have different max weights. First consider $s \geq d + 6$ which implies $d \geq 8$. Then we have both $\mathcal{E}_p$ and $\mathcal{E}'_q$, for a total of $\lfloor \frac{s-d}{2} \rfloor - 2 + d - \lfloor \frac{s}{2} \rfloor + 1$ tableaux. This is greater than or equal to $\lfloor \frac{s}{6} \rfloor + 1$, since $s \leq \lfloor \frac{5d}{3} \rfloor$ and $d \geq 8$.

For $s < d + 6$ we only have $\mathcal{E}'_q$, which provides $d - \lfloor \frac{s}{2} \rfloor + 1$ tableaux. This is greater than or equal to $\lfloor \frac{s}{6} \rfloor + 1$ provided $d \geq 5$, since $s \leq d + 5$. When $d = 4$, then $s \leq \lfloor \frac{5d}{3} \rfloor = 6$ so the two $\mathcal{E}'_q$ suffice. When $d = 3$ then $s \leq 5$ and hence one tableau, $\mathcal{E}'_1$ is sufficient. Hence we have constructed at least m_λ tableaux as desired.

9.1.3 Basis Tableaux for $c = 6$, $s \leq r$

Given $\lambda = [r + s, s]$ a partition of n, we have $2s + r = 6d = 3b$. From this equation and $s \leq r$, we have $s \leq 2d$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 6. These will be our $c = 6$ basis tableaux.

When $s \leq r - d$, we can use the $c = 5$ basis tableaux constructed in Section 9.1.2. Consider the partition $\lambda' = [r' + s, s]$ where $r' = r - d$. Since $s \leq r'$, we have $m_\lambda = m_{\lambda'}$. In Section 9.1.2 we constructed $m_{\lambda'}$ linearly independent $\mathcal{E}_p$, where $\mathcal{E}_p$ are the basis tableaux for $c = 5$. Take $\mathcal{E}_p \vee \dfrac{\mathrm{d}}{6}$ as the basis tableaux for $c = 6$. This works for $s \leq r - d$, so $r \geq s + d$. Hence $6d = 2s + r \geq 3s + d$ implies $s \leq d + \lfloor \frac{2d}{3} \rfloor$.

For $d + \lfloor \frac{2d}{3} \rfloor < s \leq 2d$ we want to construct $m_\lambda \leq \lfloor \frac{s}{6} \rfloor + 1$ linearly independent tableaux. We will do this primarily by joining two $c = 3$ basis tableaux. In addition, we will use the following tableaux:

$$\mathcal{G}^1 = \begin{array}{l} \quad d-2\;f \\ \hline 5 \quad 3 \quad 4\;6 \\ 1 \quad 1 \quad 2\;2 \end{array} \qquad\qquad 1 \le f < d$$

$$\omega(\mathcal{G}^1) = (d-1, f+1, 0, 0, 0, 0)$$

$$\mathcal{G}^2 = \begin{array}{l} \quad d-2\;f-2 \quad 2 \\ \hline 4 \quad 5 \qquad 4 \quad 6\;6 \\ 1 \quad 1 \qquad 2 \quad 2\;3 \end{array} \qquad\qquad \begin{array}{l} 4 \le f < d \\ d \ge 5 \end{array}$$

$$\omega(\mathcal{G}^2) = (d-1, f-1, 2, 0, 0, 0)$$

$$\mathcal{G}^3 = \begin{array}{l} 2\;d-4\;d-3 \qquad 2 \\ \hline 6 \quad 4 \qquad 5 \quad 6\;4\;6 \\ 1 \quad 1 \qquad 2 \quad 2\;3\;3 \end{array} \qquad\qquad d > 6$$

$$\omega(\mathcal{G}^3) = (d-2, d-2, 3, 0, 0, 0)$$

$$\mathcal{G}^4 = \begin{array}{l} \quad d-2\;2\;d-5 \quad 2 \\ \hline 6 \quad 4 \quad 6 \quad 5 \quad 5\;6 \\ 1 \quad 1 \quad 2 \quad 2 \quad 3\;3 \end{array} \qquad\qquad d > 7$$

$$\omega(\mathcal{G}^4) = (d-1, d-3, 3, 0, 0, 0)$$

These tableaux are all maximal. Except for $\mathcal{G}^2$, no other weight assignments are possible, hence these tableaux are non-zero. For $\mathcal{G}^2$, the weight assignment $(d-1, f-1, 0, 0, 0, 2)$ is also valid. Since this has sign $(-1)^2$, $\mathcal{G}^2$ is also non-zero.

Consider $\lfloor \frac{5d}{3} \rfloor < s < 2d$. (The $s = 2d$ case will be handled separately.) Write $s = f + d$, then $\lfloor \frac{2d}{3} \rfloor < f < d$. This means $f > 2$ always, and $f < 6$ only for $d \ge 7$. Moreover $f \ge 4$ for $d \ge 5$ so the conditions on $\mathcal{G}^2$ are satisfied.

We will use the $c = 3$ basis tableaux for much of our construction. Let $\mathcal{D}_{p'}$ represent the basis elements for $c = 3$ of shape $[2d, d]$ as described in Section 4.3. Let $\mathcal{F}_{\overline{p}}$ represent the basis elements for $c = 3$ of shape $[3d-f, f]$ as described in Section 4.3. We will use the $\mathcal{D}_{p'}$, $\mathcal{F}_{\overline{p}}$, and the $\mathcal{G}^i$ to construct the m_λ linearly independent tableaux for $c = 6$. This construction will depend on d and $f \mod 6$. Write $d = 6k' + i'$ and $f = 6\overline{k} + \overline{i}$, with $0 \leq i', \overline{i} \leq 5$. For $i' \neq 1$ there exist $m_{[2d,d]} = k' + 1$ linearly independent $\mathcal{D}_{p'}$, with $p' = 0, 1, \ldots, k'$, and k' tableaux for $i' = 1$. Similarly, there exists $m_{[3d-f,f]} = \overline{k} + 1$ linearly independent $\mathcal{F}_{\overline{p}}$ with $\overline{p} = 0, 1, \ldots, \overline{k}$ for $\overline{i} \neq 1$ and $\overline{k}$ when $\overline{i} = 1$. Now $s = d + f = 6(k' + \overline{k}) + i' + \overline{i} = 6k + i$, so $m_\lambda \leq k + 1$. Since $k \leq k' + \overline{k} + 1$, it suffices to construct $k' + \overline{k} + 2$ linearly independent tableaux. We will first consider $d > 6$, $f \geq 6$. Since k', $\overline{k} > 0$ consider:

$$\begin{aligned}
&\mathcal{G}_{\overline{p}} = \mathcal{D}_{k'} \vee \mathcal{F}_{\overline{p}} && \overline{p} = 0, 1, \ldots, \overline{k} \\
&\mathcal{G}_{p'} = \mathcal{D}_{p'} \vee \mathcal{F}_{\overline{k}} && p' = 0, 1, \ldots, k' - 1 \\
&\mathcal{G}_0 = \mathcal{D}_0 \vee \mathcal{F}_0 &&
\end{aligned}$$

There are $\overline{k} + 1 + k' + 1 = \overline{k} + k' + 2$ tableaux listed here. By Lemma 3.4.12 if their max weights are distinct, these tableaux are linearly independent. Since these tableaux are in maximal form we can simply compare their weights. We have:

$$\begin{aligned}
&\omega(\mathcal{G}_{\overline{p}}) = (d, 0, 0, 2\overline{p} + \overline{i} + 4\overline{k}, 2(\overline{k} - \overline{p}), 0) && \overline{p} = 0, 1, \ldots, \overline{k} \\
&\omega(\mathcal{G}_{p'}) = (2p' + i' + 4k', 2(k' - p'), 0, f, 0, 0) && p' = 0, 1, \ldots, k' - 1 \\
&\omega(\mathcal{G}_0) = (4k' + i', 2k', 0, 4\overline{k} + \overline{i}, 2\overline{k}, 0) &&
\end{aligned}$$

Now $d > f$ and $p' \leq k' - 1$, so we have $\omega(\mathcal{G}_{\overline{p}}) \neq \omega(\mathcal{G}_{p'})$ since $\omega(\mathcal{G}_{p'})$ does not contain a weight of d. Both of these weights are distinct from $\omega(\mathcal{G}_0)$, since they each contain at least three 0's while $\omega(\mathcal{G}_0)$ contains only two 0's. The weights within each of these collections of tableaux are distinct because each collection $\{\mathcal{D}_{p'}\}$ and $\{\mathcal{F}_{\overline{p}}\}$ are linearly independent by Section 4.3.

When all these tableaux $\mathcal{G}$ exist we have a set of basis tableaux for $c = 6$. However, depending on the conditions on d and f, we may only have k' or $\overline{k}$ basis tableaux to work with. In those situtations we will need to use the appropriate $\mathcal{G}^i$ to complete our set of tableaux. Recall that we are taking $d > 6$ and that $f \geq 6$.

When $d \not\equiv 1 \pmod 6$ and $f \not\equiv 1 \pmod 6$, all the tableaux $\mathcal{G}$ exist. Hence we have the $k' + \overline{k} + 2$ linearly independent tableaux required.

When $d \not\equiv 1 \pmod 6$, $f \equiv 1 \pmod 6$, there exist only $\overline{k}$ linearly independent $\mathcal{F}_{\overline{p}}$ with $\overline{p} = 1, 2, \ldots, \overline{k}$. Hence we have all the tableaux listed above except for those with $\overline{p} = 0$ and $\mathcal{G}_0$. In place of those basis elements, use $\mathcal{G}_1$ and $\mathcal{G}_2$. This provided $\overline{k}+k'+2$ tableaux. They are linearly independent by Lemma 3.4.12 provided the weights of $\mathcal{G}^1$ and $\mathcal{G}^2$ are distinct from the weights of $\mathcal{G}_p$. We have $\omega(\mathcal{G}^1) = (d-1, f+1, 0, 0, 0, 0)$ and $\omega(\mathcal{G}^2) = (d-1, f-1, 2, 0, 0, 0)$. Clearly these weights are distinct from eachother. Now $\omega(\mathcal{G}^1) \neq \omega(\mathcal{G}_{p'})$ since the number of 0's differ. If $\omega(\mathcal{G}^1) = \omega(\mathcal{G}_{\overline{p}})$ we must have $f + 1 = d$ and $\overline{p} = \overline{k}$. Similarly $\omega(\mathcal{G}_{p'})$ will be distinct from $\omega(\mathcal{G}^2)$ unless $d - 1 = f$, $p' = k' - 1$. We have $\omega(\mathcal{G}^2)$ distinct from $\omega(\mathcal{G}_{\overline{p}})$ since d does not occur in its weight. Hence for $f \neq d - 1$ these $\overline{k} + k' + 2$ tableaux are linearly independent.

When $f = d-1$, use $\mathcal{G}^3$ and $\mathcal{G}^4$ in place of $\mathcal{G}^1$ and $\mathcal{G}^2$. Since $\omega(\mathcal{G}^3)$ contains neither d nor f it is clearly distinct from the weights of $\mathcal{G}_p$. Similarly, $\omega(\mathcal{G}^4)$ is distinct from $\omega(\mathcal{G}_{\overline{p}})$ since it does not contain d. While $\omega(\mathcal{G}^4)$ does contain f, $\mathcal{G}_{p'}$ cannot contain the weight 3 since $d > 7$. Thus we have sufficient linearly independent tableaux.

Now consider $d \equiv 1 \pmod 6$ and $f \not\equiv 1 \pmod 6$. We have the $\mathcal{G}_{\overline{p}}$ and $\mathcal{G}_{p'}$ listed earlier, for $p' \neq 0$, along with $\mathcal{G}^1$ and $\mathcal{G}^2$. The discussion in the $f \equiv 1 \pmod 6$ case above shows these are linearly independent provided $f \neq d - 1$. Similarly when $f = d - 1$ we can replace $\mathcal{G}^1$ and $\mathcal{G}^2$ with $\mathcal{G}^3$ and $\mathcal{G}^4$. If $d = 7$ then $\mathcal{G}^4$ does not exist. However, then $s = 13$ on only two tableaux, $\mathcal{D}_1 \vee \mathcal{F}_1$ and $\mathcal{G}^3$, are needed.

When $d \equiv 1 \pmod 6$ and $f \equiv 1 \pmod 6$, we can write $d = 6k' + 1$, $f = 6\overline{k} + 1$. Then $s = 6(k' + \overline{k}) + 2$ so $k' + \overline{k} + 1$ tableaux suffice. Use the $\mathcal{G}_{\overline{p}}$ and $\mathcal{G}_{p'}$ listed earlier, for $\overline{p}$, $p' \neq 0$, along with $\mathcal{G}^1$ and $\mathcal{G}^2$. This provides the requisite number of tableaux. They are linearly independent by the previous discussion since $f \neq d - 1$, as $d \equiv f \pmod 6$.

Now consider $3 \leq d \leq 6$. Then $f > \lfloor \frac{2d}{3} \rfloor$ implies $f > 2$. Since $s = d + f$, we must have $s \leq 11$. Hence two linearly independent tableaux will suffice, namely, $\mathcal{D}_0 \vee \mathcal{F}_0$ and $\mathcal{G}^1$. As in previous discussions, these weights are distinct provided $f \neq d - 1$. When $f = d - 1$, $s = 2d - 1$. We need two tableaux only when $d = 5$ or $d = 6$. Thus

$\mathcal{D}_0 \vee \mathcal{F}_0$ suffices for $d = 3$ and 4. When $d = 5$ or 6 use $\mathcal{D}_0 \vee \mathcal{F}_0$ and $\mathcal{T}$, where

$$\mathcal{T} = \begin{array}{c} 2\,4\,2\,2\,1 \\ \hline 5\,6\,4\,5\,6 \\ 1\,1\,3\,2\,2 \end{array} \qquad d = 6, \qquad \omega = (6, 3, 2, 0, 0, 0)$$

$$\mathcal{T} = \begin{array}{c} \;\;4\,2\,2 \\ \hline 5\,6\,5\,4 \\ 1\,1\,2\,3 \end{array} \qquad d = 5 \qquad \omega = (5, 2, 2, 0, 0, 0)$$

These tableaux are clearly maximal and non-zero. Since $f = d - 1$ is not a weight of $\mathcal{T}$, we have that $\mathcal{T}$ and $\mathcal{D}_0 \vee \mathcal{F}_0$ are linearly independent.

When $f < 6$ we have $d \leq 7$ because $f > \lfloor \frac{2d}{3} \rfloor$. Since all $d \leq 6$ cases were done above, only $d = 7$ remains. In this case we have only $f = 5$. Thus $s = 12$ and three tableaux are required. We can use $\mathcal{D}_1 \vee \mathcal{F}_0$, $\mathcal{G}^1$, and $\mathcal{G}^2$. These tableaux are linearly independent by previous discussions.

Now consider $s = 2d$. Write $d = 6k' + i$, so $s = 6(2k') + 2i'$. Hence $2k' + 2$ linearly independent tableaux will suffice. Let

$$\mathcal{A}_1 = \begin{array}{cccc} d-2 & & d-2 & 2 \\ \hline 4 & 5 & 5 & 4\,6 \\ 1 & 1 & 2 & 2\,3 \end{array}$$

$$\omega(\mathcal{A}_1) = (d - 1, d - 1, 2, 0, 0, 0)$$

This tableau is maximal. Although there are many valid weight assignments, all such assignments have positive sign, hence $\mathcal{A}_1$ is non-zero.

When $d \not\equiv 1 \pmod 6$ we have $k' + 1$ linearly independent tableaux $\mathcal{D}_{p'}$. Hence we can use:

$$\begin{aligned} &\mathcal{D}_p \vee \mathcal{D}_p && p = 0,\ 1, \ldots, k' \\ &\mathcal{D}_{k'} \vee \mathcal{D}_{p'} && p' = 0,\ 1, \ldots, k' - 1 \\ &\mathcal{A}_1 \end{aligned}$$

with weights:

$$\omega(\mathcal{D}_p \vee \mathcal{D}_p) = (2p + i' + 4k', 2(k' - p), 0, 2p + i' + 4k', 2(k' - p), 0)$$
$$\omega(\mathcal{D}_{k'} \vee \mathcal{D}_{p'}) = (d, 0, 0, 2p + i' + 4k', 2(k' - p), 0)$$
$$\omega(\mathcal{A}_1) = (d - 1, d - 1, 2, 0, 0, 0)$$

This provides $2k' + 2$ tableaux, provided $k' > 0$. Their weights are clearly distinct so they are linearly independent by Lemma 3.4.12. When $d \equiv 1 \pmod 6$, we have $s = 6(2k') + 2$ so only $2k' + 1$ tableaux are needed. All the tableaux listed above work except for $p = 0$ and $p' = 0$, providing $2k'$ linearly independent tableaux. In addition use:

$$\mathcal{A}_2 = \begin{array}{cccccc} 2 & d-3 & d-4 & 2 & 2 & \\ \hline 5 & 4 & 5 & 6 & 6 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad d \geq 6$$

$$\omega(\mathcal{A}_2) = (d - 1, d - 2, 3, 0, 0, 0)$$

This tableau is maximal and non-zero. Since $d \equiv 1 \pmod 6$ we have $d > 6$. Thus $\mathcal{A}_2$ provides the additional tableau and its weight is distinct from the other tableaux, insuring linear independence.

When $k' = 0$ we have $d < 6$. Then $s < 10$ so two tableaux, $\mathcal{A}_1$ and $\mathcal{D}_0 \vee \mathcal{D}_0$, suffice.

9.1.4 Basis Tableaux for $c > 6$, $s \leq r$

Let $\lambda = [r + s, s]$ be a partition of n, with $s \leq r$, where $2s + r = cd = n$. We want to construct m_λ linearly independent basis tableaux for an arbitrary c. In Sections 4.3, 9.1.1, 9.1.2, and 9.1.3 we constructed basis tableaux for $c \leq 6$, which we will make use of in this construction. In addition we will use the following tableaux:

$$\mathcal{A}_1 = \begin{array}{ccccc} \multicolumn{2}{c}{d-2} & \multicolumn{2}{c}{d-2} & 2 \\ \hline 4 & 5 & 5 & 4 & 6 \\ 1 & 1 & 2 & 2 & 3 \end{array}$$

$$\omega(\mathcal{A}_1) = (d-1, d-1, 2, 0, 0, 0)$$

$$\mathcal{A}_2 = \begin{array}{cccccc} 2 & d-3 & d-4 & 2 & 2 & \\ \hline 5 & 4 & 5 & 6 & 6 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad d \geq 6$$

$$\omega(\mathcal{A}_2) = (d-1, d-2, 3, 0, 0, 0)$$

$$\mathcal{A}_3 = \begin{array}{cccccc} 2 & d-4 & d-4 & 2 & 2 & 2 \\ \hline 5 & 5 & 6 & 6 & 4 & 4 \\ 1 & 2 & 1 & 3 & 3 & 2 \end{array} \qquad d \geq 6$$

$$\omega(\mathcal{A}_3) = (d-2, d-2, 4, 0, 0, 0)$$

$$\mathcal{A}_4 = \begin{array}{cccccc} 2 & d-4 & d-5 & 2 & 3 & 2 \\ \hline 5 & 4 & 5 & 6 & 6 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad d \geq 5$$

$$\omega(\mathcal{A}_4) = (d-2, d-3, 5, 0, 0, 0)$$

$$\mathcal{A}_5 = \begin{array}{ccc} 4 & 2 & 2 \\ \hline 4 & 5 & 6 \\ 1 & 2 & 3 \end{array} \qquad d = 4$$

$$\omega(\mathcal{A}_5) = (4, 2, 2, 0, 0, 0)$$

These tableaux are all maximal. Although some of these tableaux have additional valid weight assignments, all such assignments have positive sign. Hence these tableaux are non-zero.

To construct $\lambda = [r+s, s]$-tableaux with $s \leq r$, write $s = md + f$, with $0 \leq f < d$ and $r = md + pd + g$ with $0 \leq g \leq d$. Then $3md + pd + 2f + g = cd$, so $2f + g = xd$ for some x. This means $c = 3m + p + x$. If $p + x \geq 3$, a λ-tableau may be written $\mathcal{D} \vee \mathcal{B} \vee (c - 3m - 3)V(d)$, where $\mathcal{D}$ is a tableau of shape $[2md, md]$ filled with $3m$ elements, $\mathcal{B}$ is a $[3d - f, f]$ tableau filled with 3 elements, and $V(d)$ is the one row tableau. We will first consider this case and handle the $p + x < 3$ case later.

We have $s = md + f$, so writing $d = 6k' + i'$, $f = 6\bar{k} + \bar{i}$, with $0 \leq i', \bar{i} \leq 5$ gives $s = 6(mk' + \bar{k}) + mi' + \bar{i}$. Since $\lfloor \frac{mi'+\bar{i}}{6} \rfloor \leq m$, it suffices to construct $\lfloor \frac{s}{6} \rfloor + 1 \leq mk' + \bar{k} + m + 1$ linearly independent tableaux. If $m < 2$ we may simply use the basis tableaux constructed for $c = 6$ along with $V(d)'s$, so assume $m \geq 2$.

Let $\mathcal{D}_p$ be the $c = 3$ basis tableaux of shape $[2d, d]$ described in Section 4.3. There are $m_{[2d,d]} = k' + 1$ such tableaux when $i' \neq 1$ and k' for $i' = 1$. Let $\mathcal{B}_q$ be the $c = 3$ basis tableaux of shape $[3d - f, f]$ constructed in Section 4.3. This tableau has $s \leq r$ since $f < d$. There are $m_{[3d-f,f]} = k' + 1$ such tableaux for $\bar{i} \neq 1$ and k' tableaux when $\bar{i} = 1$. Take $f > 1$ and $d \geq 6$. The $d < 6$ and $f \leq 1$ cases will be handled separately. Consider the following tableaux forms (with the appropriate number of $V(d)$'s as necessary):

I.	$\ell\mathcal{D}_p \vee (m - \ell)\mathcal{D}_{k'} \vee \mathcal{B}_q$	$\ell = 1, 2, \ldots, m$ $p = 0, 1, \ldots, k' - 1$ $q = 0, 1, \ldots, \bar{k}$
II.	$m\mathcal{D}_{k'} \vee \mathcal{B}_q$	$q = 0, 1, \ldots, \bar{k}$
III.	$\ell\mathcal{A}_j \vee (m - 2\ell)\mathcal{D}_{k'} \vee \mathcal{B}_q$	$\ell = 1, 2, \ldots, \lfloor \frac{m}{2} \rfloor$ $q = 0, 1, \ldots, \bar{k}$ $j = 1, 2, 3, 4$ $d \geq 6$ if $j \neq 1$

Note that those tableaux with $p = 0$ or $q = 0$ exist only when $i' \neq 0$ or $\bar{i} \neq 0$ respectively. We will not make use of these tableaux unless necessary. However, even when $\overline{k} = 0$, at least one $\mathcal{B}_q$ exists, so if we regard $\overline{k}$ as the number of $\mathcal{B}_q$'s, we may assume $\overline{k} \geq 1$. When $d \not\equiv 1 \pmod 6$ we need $mk' + \overline{k} + m + 1$ tableaux. The list above (taking $q \neq 0$) provides at least $mk'\overline{k} + \overline{k} + 4\lfloor \frac{m}{2} \rfloor \overline{k}$ tableaux, which suffices since $4\lfloor \frac{m}{2} \rfloor \geq m + 1$. When $d \equiv 1 \pmod 6$, we need only $mk' + \lfloor \frac{m}{6} \rfloor + 2$ tableaux. If $k' \geq 2$ the list above provides at least $m(k' - 1)\overline{k} + \overline{k} + 4\lfloor \frac{m}{2} \rfloor$ tableaux. This is sufficient for $m \neq 3$. When $m = 3$ we need $3k' + \overline{k} + 2$ tableaux when $\bar{i} = 3$, 4, or 5 and $3k' + \overline{k} + 1$ tableaux for $\bar{i} = 0$, 1, and 2. When $\bar{i} \leq 2$ the tableaux listed suffice. For $\bar{i} \geq 3$ we need an additional tableau so use the tableau of the Form II with $q = 0$. When $k' = 1$, the tableaux of the Form I don't exist. Hence we have only $\overline{k} + 4\lfloor \frac{m}{2} \rfloor$ tableaux when $q \neq 0$ and we need $m + \overline{k} + \lfloor \frac{m}{6} \rfloor + 2$. For $m \neq 3$ this is sufficient. However, one additional tableau is needed for $m = 3$, when $\bar{i} = 3$, 4, or 5. In this case we may use $q = 0$ in Form II for the remaining tableau.

To show linear independence of these tableaux it suffices, by Lemma 3.4.12, to show that the max weights are distinct. For max weights we have:

$I.$	$(4k' + 2p + i', 2(k - p), 0)^{\ell} \vee (d, 0, 0)^{(m-\ell)} \vee (4\overline{k} + 2q + \bar{i}, 2(\overline{k} - q), 0)$
$II.$	$(d, 0, 0)^{m} \vee (4\overline{k} + 2q + \bar{i}, 2(\overline{k} - q), 0)$
$III - 1.$	$(d - 1, d - 1, 2, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{(m-2\ell)} \vee (4\overline{k} + 2q + \bar{i}, 2(\overline{k} - q), 0)$
$III - 2.$	$(d - 1, d - 2, 3, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{(m-2\ell)} \vee (4\overline{k} + 2q + \bar{i}, 2(\overline{k} - q), 0)$
$III - 3.$	$(d - 2, d - 2, 4, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{(m-2\ell)} \vee (4\overline{k} + 2q + \bar{i}, 2(\overline{k} - q), 0)$
$III - 4.$	$(d - 2, d - 3, 5, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{(m-2\ell)} \vee (4\overline{k} + 2q + \bar{i}, 2(\overline{k} - q), 0)$

Consider those tableaux of Form I. If $\omega(I(\ell, p, q)) = \omega(I(\ell', p', q'))$, then counting the number of d's shows $\ell = \ell'$, while counting the number of $4k' + p + i'$'s indicates $\ell = 1$. If $p = p'$ then $q = q'$, which is not possible for distinct tableaux. Hence by Lemma 3.4.13, the weights are distinct because $f < d$. Those of Form II are distinct due to the distinct weights of $\mathcal{B}_q$. Similarly those of the Form $III - j$ are distinct by the number of d's and distinctness of $\omega(\mathcal{B}_q)$. If $\omega(III - i(\ell, q)) = \omega(III - j(\ell', q'))$, then by counting the number of d's we have $\ell = \ell'$. Then by counting the number of 0's, $d-1$'s, and $d-2$'s, we find the weights must be distinct for $d > 6$. (If $d = 6$ the Forms $III - 2$ and $III - 4$ have the same weights.) To see the different forms have distinct weights, first count the number of d's. Obviously $\omega(II) \neq \omega(I)$ or $\omega(III)$. If $\omega(I(\ell)) = \omega(III(\ell'))$ then $\ell = 2\ell'$. However, counting the number of 0's shows the weights are distinct.

When $d = 6$ then $s \leq 6m + 5$, so $m + 1$ tableaux suffice. Since there are two $\mathcal{D}$'s and one $\mathcal{B}$, Forms I and II provide the requisite number of linearly independent tableaux.

Now consider the case where $d < 6$. Here $s = mi' + \bar{i}$ so $m + 1$ tableaux suffices. In this case the $\mathcal{A}_j$, $j = 2, 3$ do not exist. However for $d = 5$ we have $\mathcal{A}_4$ and for $d = 4$ we have $\mathcal{A}_5$. Moreover, we have exactly one $\mathcal{D}$ and one $\mathcal{B}$. Hence the appropriate Forms II, $III - 1$, and $III - 4$ or $III - 5$ provied $2\lfloor \frac{m}{2} \rfloor + 1$ tableaux. This suffices for even m. When m odd we need one additional tableau. For $m \geq 5$ use $\mathcal{A}_1 \vee \mathcal{A}_i \vee (m-4)\mathcal{D}_0 \vee \mathcal{B}_0$ where $i = 4$ or 5 as appropriate. If $m = 3$, then $s \leq 19$ so the three tableaux listed will suffice except when $d = 5$ and $f = 3$. In this case also use the non-zero maximal tableau:

$$\mathcal{T} = \begin{array}{cccccc} 4 & & 4 & & 4 & 4 \\ \hline 5 & 9 & 6 & 9 & 7 & 8 \\ 1 & 1 & 2 & 2 & 3 & 4 \end{array}$$

$$\omega(\mathcal{T}) = (5, 5, 4, 4, 0, 0, , 0, 0, 0)$$

When $d = 3$, we have $s = 3m + 2$ since $1 < f < d$. Hence $\lfloor \frac{3m+2}{6} \rfloor + 1$ tableaux are needed. We have $m\mathcal{D}_0 \vee \mathcal{B}_0$ and $\ell\mathcal{A}_1 \vee (m - 2\ell)\mathcal{D}_0 \vee \mathcal{B}_0$, which provide $\lfloor \frac{m}{2} \rfloor + 1$ tableaux. Since $m \geq 2$ we have $\lfloor \frac{m}{2} \rfloor \geq \lfloor \frac{3m+2}{6} \rfloor$, so these tableaux suffice.

For linear independence, we need only check distinctness of max weights by Lemma 3.4.12. First consider $d = 4$ or 5. Since $\omega(\mathcal{A}_5) = \omega(\mathcal{A}_3)$ our discussion on linear independence for $d \geq 6$ still holds. The only additional tableau is $\mathcal{A}_1 \vee \mathcal{A}_i \vee (m-4)\mathcal{D}_0 \vee \mathcal{B}_0$ which clearly has a distinct weight. Similarly, by counting the number of 5's and 4's, the weight of $\mathcal{T}$ is also distinct. Hence these tableaux are linearly independent. When $d = 3$ the tableaux listed are a subset of the tableaux for $d \geq 6$ and hence are linearly independent by the prior discussion. This covers all cases with $p + x \geq 3$ provided $f > 1$. The $f = 0$ and $f = 1$ cases will be handled after the $p + x < 3$ case.

Now assume that $p + x < 3$. Recall that if $s = md + f$, with $0 \leq f < d$ and $r = md + pd + g$, then $3md + pd + 2f + g = cd$ and $2f + g = xd$ for some x. This means $c = 3m + p + x$, so if $p + x < 3$ then λ-tableau with $s \leq r$ may be written as $\mathcal{D} \vee \mathcal{F} \vee (c - 3m - 3)V(d)$. Here $\mathcal{D}$ is a tableau of shape $[2(m-1)d, (m-1)d]$ filled with $3(m-1)$ elements and $\mathcal{F}$ is a $[(3 + x - 1)d - f, f + d]$ tableau filled with $3 + x$ elements. When $f > 0$, we have $x > 0$, which means $\mathcal{F}$ is a $c = 4$ or $c = 5$ tableau. We will first consider the case where $m \geq 3$, $d \geq 6$ and $f > 1$.

If $d = 6k' + i'$ and $f = 6\overline{k} + \overline{i}$, with $0 \leq i', \overline{i} \leq 5$, then $s = md + f$, so we still need $\lfloor \frac{s}{6} \rfloor + 1 \leq mk' + \overline{k} + m + 1$ tableaux. The number of tableaux $\mathcal{D}$ is the same as before. Let $\mathcal{F}_q$ be the $c = 4$ or 5 basis tableaux of shape $[cd - f - d, f + f]$. There are at least $k' + \overline{k}$ such tableaux. Moreover since $d \geq 6$ and $f > 1$ there is always at least 2 such tableaux.

Consider the tableaux of the following forms:

$$I. \qquad \ell\mathcal{D}_p \vee (m - \ell - 1)\mathcal{D}_{k'} \vee \mathcal{F}_q \qquad \begin{aligned} \ell &= 1,\, 2,\, \ldots,\, m-1 \\ p &= 0,\, 1,\, \ldots,\, k'-1 \\ q &= 1,\, 2,\, \ldots,\, k' + \overline{k} \end{aligned}$$

$$II. \qquad (m-1)\mathcal{D}_{k'} \vee \mathcal{F}_q \qquad q = 1,\, 2,\, \ldots,\, k' + \overline{k}$$

$$III. \qquad \ell\mathcal{A}_j \vee (m-2\ell-1)\mathcal{D}_{k'} \vee \mathcal{F}_q \qquad \begin{array}{l} \ell = 1, 2, \ldots, \lfloor\frac{m-1}{2}\rfloor \\ q = 1, 2, \ldots, k'+\overline{k} \\ j = 1, 2, 3, 4 \\ d \geq 6 \text{ if } j \neq 1 \end{array}$$

As before, the tableaux with $p = 0$ exist only for $d \not\equiv 1 \pmod 6$. When $d \not\equiv 1 \pmod 6$, the construction above provides $(m-1)k'(k'+\overline{k}) + \overline{k} + k' + 4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k})$. Now $k'+\overline{k}$ denotes the number of $\mathcal{F}$'s, which we can assume is at least 2. Then since $4\lfloor\frac{m-1}{2}\rfloor \geq m+1$ for $m \neq 4$, this construction provides sufficient tableaux. If $m = 4$ then $4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k}) \geq 8$, so this construction is sufficient.

When $d \equiv 1 \pmod 6$, we need only $mk' + \overline{k} + \lfloor\frac{m}{6}\rfloor + 2$ tableaux. For $k' \geq 2$, we have $(m-1)(k'-1)(k'+\overline{k}) + \overline{k} + k' + 4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k})$ tableaux by the construction above, which is sufficient. If $k' = 1$, the tableaux of Form I do not exist, so we have only $k' + \overline{k} + 4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k})$ tableaux. However, since we know that there are always at least two $\mathcal{F}_q$'s, we have $k'+\overline{k} \geq 2$. Thus the tableaux listed are sufficient.

When $d < 6$, the tableaux $\mathcal{A}_j$ with $j = 2$ and 3 do not exist. However we do have $\mathcal{A}_4$ for $d = 5$ and $\mathcal{A}_5$ for $d = 4$. Also, since $d < 6$, we may no longer assume that there are at least two $\mathcal{F}_q$'s (unless $d = 5$ and $f = 3$ or 4). However there is always at least one. In addition, there is only one $\mathcal{D}$. Under these constraints we have tableaux of the Forms II, $III-1$, and $III-4$ or $III-5$ when $d = 5$ or 4 respectively. For $m \geq 5$ also use $\mathcal{A}_1 \vee \mathcal{A}_j \vee (m-5)\mathcal{D}_0 \vee \mathcal{F}_0$. This provides $2\lfloor\frac{m-1}{2}\rfloor + 2$ tableaux for $m \geq 5$. For $d = 4$ we need at most $\lfloor\frac{4m+3}{2}\rfloor + 1$, hence this this suffices. For $d = 5$ and $f = 2$ we need at most $\lfloor\frac{5m+2}{6}\rfloor + 1$ which we have. Our construction suffices except for $m = 3$ and 4. When $m = 3$, then $s \leq 17$ and three tableaux, II, $III-1$, and $III-4$ or $III-5$ suffice. For $m = 4$ we need one additional non-zero maximal tableau, so use

$$\mathcal{T} = \begin{array}{ccccccccc} d-1 & 1 & d-1 & 1 & d-1 & 1 & d-1 & 1 & 2 \\ \hline 6 & 7 & 8 & 7 & 9 & 12 & 10 & 12 & 11 \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \end{array}$$

$$\omega(\mathcal{T}) = (d, d, d, d, 2, 0, 0, 0)$$

When $d = 5$, $f \geq 3$ there are two $\mathcal{F}_q$'s hence the Forms II, $III - 1$ and $III - 4$ give $4\lfloor \frac{m-1}{2} \rfloor + 1$ tableaux. Since at most $\lfloor \frac{5m+4}{6} \rfloor + 1$ tableaux are needed, this suffices.

When $d = 3$ we need $\lfloor \frac{3m+2}{6} \rfloor + 1$ tableaux. From Forms II and $III - 1$, we have $\lfloor \frac{m-1}{2} \rfloor + 1$ tableaux. This is sufficient except for m odd. When m even we need one additional tableau, so use $\mathcal{T}$ listed above.

Now to consider linear independence. By Lemma 3.4.12 it suffices to show that the max weights are distinct. First consider when the $\mathcal{F}$ are $c = 4$ basis tableaux. The max weights are:

$$\begin{array}{ll}
I. & (4k' + 2p + i', 2(k' - p), 0)^{\ell} \vee (d, 0, 0,)^{m-\ell-1} \vee (d - q + 1, f + q - 1, 0, 0) \\
II. & (d, 0, 0)^{m-1} \vee (d - q + 1, f + q - 1, 0, 0) \\
III - 1. & (d - 1, d - 1, 2, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d - q + 1, f + q - 1, 0, 0) \\
III - 2. & (d - 1, d - 2, 3, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d - q + 1, f + q - 1, 0, 0) \\
III - 3. & (d - 2, d - 2, 4, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d - q + 1, f + q - 1, 0, 0) \\
III - 4. & (d - 2, d - 3, 5, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d - q + 1, f + q - 1, 0, 0)
\end{array}$$

Consider those tableaux of Form I. If $\omega(I(\ell, p, q)) = \omega(I(\ell', p', q'))$, then counting the number of 0's shows $\ell = \ell'$. By counting the number of different weights we find $\ell = 1$. Now if $p = p'$ then we must have $q = q'$, which is not possible for distinct tableaux. Hence by Lemma 3.4.13, these weights are distinct. The weights of Form II are distinct by the construction of the $\mathcal{F}$'s in Section 9.1.1.

The weights of Form $III - j$ are distinct by counting the number of 0's and by the distinctness of the $\mathcal{F}$'s. If $\omega(III - j) = \omega(III' - i)$ counting the number of 0's and different numbers shows $\ell = \ell' = 1$. So if the weights are not distinct, then $\omega(A_j \vee \mathcal{F}_q) = \omega(A_i \vee \mathcal{F}_{q'})$, which implies $\mathcal{A}_i$ and $\mathcal{A}_j$ must have one non-zero element in common. Then by Lemma 3.4.13, these weights are distinct if $f > 2$. When $f = 2$ the tableaux have the same length and the lemma does not apply. If $f = 2$ then $\omega(III - 3(\ell = 1, q = 4)) = \omega(III - 4(\ell = 1, q = 3))$, though all other weights are distinct.

We have $\omega(I) \neq \omega(II)$, by counting the number of 0's. Forms II and III are also distinct by counting the number of 0's. If $\omega(I) = \omega(III')$, counting the number of 0's shows that $\ell = \ell'$. Then counting the number of d's shows that $\ell = 1$, $q' = 1$, and $q \neq 1$. If the weights are equal, then $(4k' + 2p + i', 2(k' - p), d - q + 1, f + q - 1) = (\omega(A_j), f)$. Since $q \leq \lfloor \frac{d-f}{2} \rfloor$, these weights are distinct unless $f = 2$. When $f = 2$ then $\omega(I(\ell = 1, p = k' - 1, q = j) = \omega(III - j(\ell' = 1, q' = 1))$, $j \neq 1$. Hence for $f > 2$ all the tableaux listed are linearly independent.

When $f = 2$ some of the tableaux in our list have the same max weights, and hence may be linearly dependent. If we eliminate these tableaux with duplicate max weights from our list we have $(m-1)k'(k'+\overline{k})+k'+\overline{k}+4\lfloor \frac{m-1}{2} \rfloor(k'+\overline{k})-4$ linearly independent tableaux when $d \not\equiv 1 \pmod 6$. We need at most $\lfloor \frac{m(6k'+i')+2}{6} \rfloor + 1 \leq mk' + m$ tableaux, which we have since we may still take $k' + \overline{k} \geq 2$. If $d \equiv 1 \pmod 6$ then we have $(m-1)(k'-1)(k'+\overline{k}) + k' + \overline{k} + 4\lfloor \frac{m-1}{2} \rfloor(k'+\overline{k}) - 4$ linearly independent tableaux for $k' > 1$. This is sufficient since only $mk' + \lfloor \frac{m+2}{6} \rfloor + 1$ tableaux are needed. When $k' = 1$ those tableaux of Form I don't exist, so we have $k' + \overline{k} + 4\lfloor \frac{m-1}{2} \rfloor(k' + \overline{k}) - 1$ linearly independent tableaux, which is sufficient. Thus the $d \geq 6$, $f > 1$, $m \geq 3$ case is finished for $c = 4$.

When $d < 6$ the same tableaux are used with the substitution of $\mathcal{A}_5$. However, $\mathcal{A}_5$ has the same weight as $\mathcal{A}_3$ so the above argument applies. (Note that by the conditions of $\mathcal{F}_q$, $q \leq \lfloor \frac{d-f}{2} \rfloor \leq 1$, so the max weight duplication does not occur.) In addition we also use the tableaux $\mathcal{A}_1 \vee \mathcal{A}_j \vee (m-5)\mathcal{D}_0 \vee \mathcal{F}_0$, for $j = 4$ or 5. However, counting the number of 0's and d's shows that this tableau is distinct from our previous collection; otherwise $\omega(\mathcal{A}_1 \vee \mathcal{A}_j \vee \mathcal{F}_0) = \omega(\mathcal{A}_i \vee \mathcal{A}_i \vee \mathcal{F}_q)$ which is impossible.

In addition, when $m = 4$, $d = 5$ or $d = 3$, m even, we also have the tableau $\mathcal{T}$. Counting the number of 0's and d's shows this tableau has a distinct max weight as well. Hence when $\mathcal{F}$ is a $c = 4$ tableau, we have linear independence for $f > 1$ and $m \geq 3$.

Now consider the case where $\mathfrak{F}$ is a $c = 5$ basis tableau. The $c = 5$ basis tableaux were constructed in Section 9.1.2. These tableaux have two different types of max weights corresponding to the T_1 and T_2 tableaux constructions used. Those of form T_1 have weights $(d, f-q-1, q+1, 0, 0)$ where $q = 1, 2, \ldots, \lfloor \frac{f}{2} \rfloor - 2$ (when $f \geq 6$). Those of the form T_2 have weights $(d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$ with $\overline{q} = 1,\ 2,\ d - \lfloor \frac{d+f-2}{2} \rfloor$. We will refer to the basis tableaux using the T_1 tableaux as q-forms and those using T_2 as $\overline{q}$-forms. Then the max weights for the general tableaux have the following forms:

$I(q)$. $(4k'+2p+i', 2(k'-p), 0)^{\ell} \vee (d, 0, 0,)^{m-\ell-1} \vee (d, f-q-1, q+1, 0, 0)$

$I(\overline{q})$. $(4k'+2p+i', 2(k'-p), 0)^{\ell} \vee (d, 0, 0,)^{m-\ell-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$II(q)$. $(d, 0, 0)^{m-1} \vee (d, f-q-1, q+1, 0, 0)$

$II(\overline{q})$. $(d, 0, 0)^{m-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$III-1(q)$. $(d-1, d-1, 2, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d, f-q-1, q+1, 0, 0)$

$III-1(\overline{q})$. $(d-1, d-1, 2, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$III-2(q)$. $(d-1, d-2, 3, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d, f-q-1, q+1, 0, 0)$

$III-2(\overline{q})$. $(d-1, d-2, 3, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$III-3(q)$. $(d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d, f-q-1, q+1, 0, 0)$

$III-3(\overline{q})$. $(d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

$III-4(q)$. $(d-2, d-3, 5, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d, f-q-1, q+1, 0, 0)$

$III-4(\overline{q})$. $(d-2, d-3, 5, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell-1} \vee (d-\overline{q}+1, f-1+\overline{q}, 0, 0, 0)$

These tableaux are linearly independent by Lemma 3.4.12 provided their max weights are distinct. To show these weights are distinct we will often count the number of 0's and d's in each weight. For convenience we will list these values below:

Consider those tableaux of Form $I(q)$. If $\omega(I(\ell, p, q)) = I(\ell', p', q')$, then Table 9.1.4 shows that $\ell = \ell'$. Then counting the number of different numbers shows $\ell = 1$. If $p = p'$ then we must have $q = q'$ and vice versa, but this

Form	# 0's	# d's
$I(q)$.	$2m-\ell$	$m-\ell$
$I(\overline{q})$.	$2m-\ell+$	$m-\ell-1$ $\quad \overline{q}\neq 1$
		$m-\ell$ $\quad \overline{q}=1$
$II(q)$.	$2m$	m
$II(\overline{q})$.	$2m+1$	$m-1$ $\quad \overline{q}\neq 1$
		m $\quad \overline{q}=1$
$III-j(q)$.	$2m-\ell$	$m-2\ell$
$III-j(\overline{q})$.	$2m-\ell+1$	$m-2\ell-1$ $\quad \overline{q}\neq 1$
		$m-2\ell$ $\quad \overline{q}=1$

Table 9.1: Weights of 0 or d.

can't happen since the tableaux are different. Then since $f < d$, the argument of Lemma 3.4.13 shows the weights are distinct. The same reason holds for Form $I(\overline{q})$. Table 9.1.4 also shows $\omega(I(q)) \neq \omega(I(\overline{q}))$. The tableaux of Form II are distinct since the max weights of the $\mathcal{F}$ are distinct for $c = 5$ by Section 9.1.2. Moreover, $\omega(II(q)) \neq \omega(II(\overline{q}))$ by Table 9.1.4. The distinctness of max weights for Form $III-j(q)$ or $III-j(\overline{q})$ follows from Table 9.1.4 and distinctness of $c = 5$ basis tableaux max weights. Also from Table 9.1.4 we have $\omega(III-j(q)) \neq \omega(III-j(\overline{q}))$. Now suppose $\omega(III-j(q,\ell)) = \omega(III-i(q',\ell'))$. Table 9.1.4 shows that $\ell = \ell'$ and counting the number of different numbers shows $\ell = 1$. Since $i \neq j$, the weights of $\mathcal{A}_i$ and $\mathcal{A}_j$ must have exactly one non-zero weight in common. Thus the argument of Lemma 3.4.13 applies and hence the weights are distinct. Similarly $\omega(III-j(\overline{q},\ell)) \neq \omega(III-i(\overline{q}',\ell'))$ for $f > 2$ by Lemma 3.4.13. (When $f = 2$, $\ell = 1$ the conditions of Lemma 3.4.13 are not met.) The values of Table 9.1.4 are sufficient to show $\omega(III-j(q,\ell)) \neq \omega(III-i(\overline{q}',\ell'))$. This shows the max weights within each tableau form are distinct for $f > 2$. Showing that $\omega(I) \neq \omega(II) \neq \omega(III)$ follows directly from Table 9.1.4.

When $f = 2$ our discussion on linear independence holds except for a few tableaux of the Form $III(\overline{q})$. Specifically, $\omega(III-2(\ell=1,\overline{q}=3)) = \omega(III-3(\ell=1,\overline{q}=2))$, $\omega(III-2(\ell=1,\overline{q}=4)) = \omega(III-4(\ell=1,\overline{q}=2))$, and $\omega(III-3(\ell=1,\overline{q}=4)) = \omega(III-4(\ell=1,\overline{q}=3))$. Thus we have three fewer linearly independent tableaux than we originally calculated. If we eliminate these tableaux with duplicate max weights from our list we have $(m-1)k'(k'+\overline{k})+k'+\overline{k}+4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k})-3$ linearly independent tableaux when $d \not\equiv 1 \pmod 6$. We need at most $\lfloor\frac{m(6k'+i')+2}{6}\rfloor+1 \le mk'+m$ tableaux, which we have since we may still take $k'+\overline{k} \ge 2$. If $d \equiv 1 \pmod 6$ then we have $(m-1)(k'-1)(k'+\overline{k})+k'+\overline{k}+4\lfloor\frac{m-1}{2}\rfloor(k'+\overline{k})-3$ linearly independent tableaux for $k'>1$. This is sufficient since only $mk'+\lfloor\frac{m+2}{6}\rfloor+1$ tableaux are needed. When $k'=1$ those tableaux of the Form I don't exist, but we still have $2+8\lfloor\frac{m-1}{2}\rfloor-3$ linearly independent tableaux, which is sufficient. Hence we have enough tableaux when $f=2$. Thus the $d \ge 6$, $f>1$, $m \ge 3$ case is finished for $c=5$.

When $d<6$ only the $\overline{q}$-form tableaux exist for $c=5$. These tableaux have the same max weights as the $c=4$ tableaux $\mathcal{F}$ (with the exception of an extra zero.) Thus by the same argument as in that case, these tableaux are linearly independent. Hence when $\mathcal{F}$ is a $c=5$ tableau, we have linear independence for $f>1$ and $m \ge 3$.

These constructions assumed $m \ge 3$. If $m=1$, The $c=4$ or $c=5$ basis tableaux (joined with sufficient $V(d)$'s) suffice. However, when $m=2$, we need tableaux with $c=7$ or 8 elements. This case must be dealt with separately. (We will still assume $f>1$.) Tableaux of Form I listed previously ($\mathcal{D}_p \vee \mathcal{F}_q$) still work for this case. These tableaux are linearly independent by our previous discussion. As before, let $d=6k'+i'$, $f=6\overline{k}+\overline{i}$, with $0 \le i'$, $\overline{i} \le 5$. Since $s=2d+f$, we need at most $2k'+\overline{k}+3$ tableaux. The tableaux $\mathcal{F}_q$ have $\lambda_2=d+f$ so there are at least $k'+\overline{k}$ such tableaux. So for $d \not\equiv 1 \pmod 6$, we have $(k'+1)(k'+\overline{k})$ tableaux. This provides sufficient tableaux unless $k'=1$, $\overline{k}=1$, or $\overline{k}=0$, $k'<2$. When $k'=\overline{k}=1$, then $d+f \ge 14$ so at least three $\mathcal{F}_q$'s exists. Hence the Form I construction is sufficient. In the remaining cases, computing exactly how many $\mathcal{F}_q$ exist and precisely how many tableaux are needed shows the Form I tableaux are sufficient except for: $d=11$, $f=2$, $d=3$, $f=2$, $d=4$, $f=3$, and $d=5$, $f=2,3,4$. (For instance, when $d=6$ and $f=5$ there are two $\mathcal{F}$'s and two $\mathcal{D}$'s, so Form I provides 4 tableaux. Since $s=17$, only three tableaux are needed.)

When $d = 11$ and $f = 2$, there are two $\mathcal{D}$'s and $\mathcal{F}$'s for a total of 4 Form I tableaux. Since $s = 24$, five tableaux are required. In addition to the Form I tableaux, use

$$\mathcal{T} = \begin{array}{ccc} 10 & 10 & 4 \\ \hline 4 & 5 & 6 \\ 1 & 2 & 3 \end{array} \qquad \omega(\mathcal{T}) = (10, 10, 4, 0, 0, 0)$$

which is non-zero, maximal, and linearly independent.

For $d = 3,\ f = 2$, we have $c = 8$. Since $s = 8$ two tableaux are needed. In addition to $\mathcal{D}_0 \vee \mathcal{F}_0$, use the non-zero maximal tableau:

$$\mathcal{T} = \begin{array}{cccc} 2 & 2 & 2 & 2 \\ \hline 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{array}$$

$$\omega(\mathcal{T}) = (2, 2, 2, 2, 0, 0, 0, 0)$$

When $d = 4$ and $f = 3$, then we must have $c = 9$ as $s \leq r$. Since $s = 11$, we need two tableaux. Use $\mathcal{D}_0 \vee \mathcal{F}_0$ and

$$\mathcal{T} = \begin{array}{ccccccc} 2 & & 2 & & 2 & & 2 \\ \hline 5 & 9 & 6 & 9 & 7 & 8 & 8 \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 \end{array} \qquad \omega(T) = (3, 3, 3, 2, 0, 0, 0, 0)$$

Counting the number of 4's shows these tableaux are linearly independent.

When $d = 5$ we have $c = 8$ and need three tableaux (except for $f = 3$ when two tableaux suffice). In addition to $\mathcal{D}_0 \vee \mathcal{F}_0$ use:

$$\mathcal{T} = \begin{array}{cccc} 4 & 4 & 4 & f-2 \\ \hline 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{array} \qquad\qquad f \neq 3$$

$$\omega(\mathcal{T}) = (4, 4, 4, f-2, 0, 0, 0, 0)$$

$$\mathcal{T}' = \begin{array}{cccccc} 4 & & & f & 2 & 2 \\ \hline 5 & 6 & 6 & 7 & 6 & 8 \\ 1 & 1 & 2 & 2 & 3 & 4 \end{array}$$

$$\omega(\mathcal{T}') = (5, f+1, 2, 2)$$

These tableaux are non-zero. Linear independence follows by counting the number of 5's and 2's. Thus we have sufficient tableaux for $d \not\equiv 1 \pmod 6$.

When $d \equiv 1 \pmod 6$ we need $2k' + \overline{k} + 2$ tableaux. The Form I construction $\mathcal{D}_p \vee \mathcal{F}_q$ discussed earlier provides at least $k'(k' + \overline{k} + 1)$ linearly independent tableaux for $f \not\equiv 0 \pmod 6$. (When $f \not\equiv 0 \pmod 6$ there are $k' + \overline{k} + 1$ tableaux $\mathcal{F}_q$.) This is sufficient for $k' \geq 2$. When $f \equiv 0 \pmod 6$ Form I provides $k'(k' + \overline{k})$ tableaux, but only $2k' + \overline{k} + 2$ tableaux are needed. This is sufficient for $k' \geq 2$ since $f \neq 0$.

Since $d \geq 3$, only $d = 7$ remains. We need at most 4 tableaux, as $s \leq 20$. Consider the tableaux $\mathcal{B}_p \vee \mathcal{C}_q$ where the $\mathcal{B}_p$ are $c = 3$ basis tableaux with $\lambda_2 = d - 1$ and the $\mathcal{C}_q$ are $c = 4$ or $c = 5$ basis tableaux with $\lambda_2 = d + f + 1$. There are two tableaux each for $\mathcal{B}_p$ and $\mathcal{C}_q$ so this construction is sufficient. The max weights for $\mathcal{B}_p$ are $(6, 0, 0)$ and $(4, 2, 0)$. The max weights for $\mathcal{C}_q$ are $(7, f+1, 0, 0)$ and $(6, f+2, 0, 0)$. Hence the weights of our construction are distinct unless $f = 5$. However when $f = 5$ only three tableaux are needed and the Form I construction $\mathcal{D} \vee \mathcal{F}_q$ provides three in this case.

When $f = 1$ we can proceed as in the $p + x < 3$ case. We have $s = md + 1$ and $r = md + pd + g$. Then $3md + pd + 2f + g = cd$, so $2 + g = xd$ for some x. Hence $x = 1$. This means that for both the $p + x \geq 3$ and $p + x < 3$ cases we can use the $c = 4$ basis tableaux, proceeding as in the $p + x < 3$ case when $f > 1$. As before, write $d = 6k' + i'$ with $0 \leq i' \leq 5$. We have at least k' linearly independent $\lambda = [3d - 1, d + 1]$ tableaux $\mathcal{F}_q$, which we use for the following forms:

$I.$	$\ell\mathcal{D}_p \vee (m - \ell - 1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1, 2, \ldots, m-1$ $p = 0, 1, \ldots, k'-1$ $q = 1, 2, \ldots, k'$
$II.$	$(m-1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$q = 1, 2, \ldots, k'$

$$III. \qquad \ell\mathcal{A}_j \vee (m-2\ell-1)\mathcal{D}_{k'} \vee \mathcal{F}_q \qquad \begin{array}{l} \ell = 1,\, 2,\, \ldots,\, \lfloor \frac{m-1}{2} \rfloor \\ q = 1,\, 2,\, \ldots,\, k' \\ j = 1,\, 2,\, 3,\, 4 \\ d \geq 6 \text{ if } j \neq 1 \end{array}$$

These tableaux have the following max weights:

$$\begin{array}{ll}
I. & (4k'+2p+i', 2(k'-p), 0)^{\ell} \vee (d,0,0,)^{m-\ell-1} \vee (d-q, q+1, 0, 0) \\
II. & (d,0,0)^{m-1} \vee (d-q, q+1, 0, 0) \\
III-1. & (d-1, d-1, 2, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell-1} \vee (d-q, q+1, 0, 0) \\
III-2. & (d-1, d-2, 3, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell-1} \vee (d-q, q+1, 0, 0) \\
III-3. & (d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell-1} \vee (d-q, q+1, 0, 0) \\
III-4. & (d-2, d-3, 5, 0, 0, 0)^{\ell} \vee (d,0,0)^{m-2\ell-1} \vee (d-q, q+1, 0, 0)
\end{array}$$

Counting the number of zeros and d's in these weights gives:

Form	#0's	#d's
$I.$	$2m-\ell$	$m-\ell-1$
$II.$	$2m$	$m-1$
$III-j.$	$2m-\ell$	$m-2\ell-1$

Then by arguments similar to previous cases and Lemma 3.4.13, these tableaux have distinct max weights except for $III-1(\ell=1, q=2)$ and $III-2(\ell=1, q=1)$ whose weights are the same. Thus by Lemma 3.4.12, these tableaux (omitting $III-1(\ell=1, q=2)$) are linearly independent.

When $d \geq 6$ and $m \geq 3$, we need $mk' + \lfloor \frac{5m+1}{6} \rfloor + 1$ tableaux. When $k' \geq 2$, the forms listed above provided $(m-1)k'k' + k' + 4\lfloor \frac{m-1}{2} \rfloor k' - 1$, which is sufficient. When $k'=1$ and $d \not\equiv 1 \pmod 6$, we still have these tableaux. If $d=7$, then there is exactly one $\mathcal{D}$ so Form I does not exist. However, there are two distinct $\mathcal{F}_q$'s. Thus Forms II and III provide $8\lfloor \frac{m-1}{2} \rfloor$ tableaux. Since only $m + \lfloor \frac{m+1}{6} \rfloor + 1$ tableaux are needed, this suffices.

If $m = 2$ then only Form I and II tableaux exist. Since $s = 2d + 1$, then $2k' + 2$ tableaux suffice. Forms I and II provided at least $(k' - 1)k' + k'$, tableaux. When $k' \geq 2$ this is enough. For $d > 7$, four tableaux are needed. Since there are at least two $\mathcal{D}$'s and two $\mathcal{F}$'s, this construction suffices. When $d = 6$, there are two $\mathcal{D}$'s, but only one $\mathcal{F}$. However, $s = 13$ so only two tableaux are needed, which we have. If $d = 7$, three tableaux are needed. Since there is only one $\mathcal{D}$ and two $\mathcal{F}$'s, an additional tableau is required. Use:

$$\mathcal{H} = \begin{array}{cccc} 6 & 6 & & 2 \\ \hline 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 3 \end{array} \qquad \omega(\mathcal{H}) = (6, 6, 3, 0, 0, 0, 0)$$

This tableau is clearly non-zero and maximal and its max weight does not contain d, so it is linearly independent from our previously constructed tableaux.

When $d = 5$, there are two distinct $\mathcal{F}_q$'s, which have weights of the form listed above. However, there is only one $\mathcal{D}$. Thus Forms II, $III - 1$, and $III - 4$ provide at least $1 + 4\lfloor \frac{m-1}{2} \rfloor$ tableaux. Since only $\lfloor \frac{5m+1}{6} \rfloor + 1$ tableaux are needed, this suffices for $m \geq 3$. When $m = 2$ then $s = 11$, so the two tableaux of Form II are sufficient.

When $d = 3$ there is only one $\mathcal{F}$ and one $\mathcal{D}$. Moreover, only $\mathcal{A}_1$ exists. However, all the tableaux are linearly independent. Thus we have $\lfloor \frac{m-1}{2} \rfloor + 1$ tableaux and we need $\lfloor \frac{3m+1}{6} \rfloor + 1$ tableaux. This is sufficient for m odd. When m even then $s = 3m + 1 \equiv 1 \pmod 6$ so only $\lfloor \frac{3m+1}{6} \rfloor$ tableaux are required, which we have when $m \geq 3$. If $m = 2$ then $s = 7$, which means the single tableau of Form II is sufficient.

When $d = 4$ we need to proceed differently since the $c = 4$ basis tableaux $\mathcal{F}$ is not in maximal form. First consider $m \geq 4$. We will replace $\mathcal{F}$ in the previous discussion with the following $c = 7$ tableaux, $\mathcal{H}_q$.

$$\mathcal{H}_0 = \begin{array}{cccc} 4 & 2 & & 2 \\ \hline 4 & 5 & 6 & 7 \\ 1 & 2 & 2 & 3 \end{array} \qquad\qquad \omega(\mathcal{H}_0) = (4, 3, 2, 0, 0, 0, 0)$$

$$\mathcal{H}_1 = \begin{array}{cccccc} 2 & & 2 & & 2 & \\ \hline 4 & 7 & 5 & 4 & 6 & 5 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad\qquad \omega(\mathcal{H}_0) = (3, 3, 3, 0, 0, 0, 0)$$

The the tableaux we will use are:

$$II. \qquad (m-2)\mathcal{D}_0 \vee \mathcal{H}_q \qquad q = 0,\ 1$$

$$III. \qquad \ell\mathcal{A}_j \vee (m-2\ell-2)\mathcal{D}_0 \vee \mathcal{H}_q \qquad \begin{array}{l} \ell = 1,\ 2,\ \ldots,\ \lfloor \frac{m-2}{2} \rfloor \\ q = 0,\ 1 \\ j = 1,\ 5 \end{array}$$

Counting the number of 0's and 4's in the max weights of these tableaux show they are linearly independent by Lemma 3.4.12. This can be seen from Table 9.1.4 below:

Form	# 0's	# 4's	q
$II.$	$2m$	$m-1$	0
	$2m$	$m-2$	1
$III-j.$	$2m-\ell$	$m-2\ell-1$	0
	$2m-\ell$	$m-2\ell-2$	1

Table 9.2: Weights of 0 and 4.

Hence for $m \geq 4$ this construction provides $4\lfloor \frac{m-2}{2} \rfloor + 2$ linearly independent tableaux. Since only $\lfloor \frac{4m+1}{6} \rfloor + 1$ tableaux are required, this suffices.

When $m = 3$, then $s = 13$, so only two tableaux are needed. The two tableaux of Form II suffices. Hence we have constructed sufficient linearly independent tableaux when $f = 1$ and $m \geq 3$.

For $m = 2$, two tableaux are needed, in which case use $\mathcal{H}_0$ and $\mathcal{H}_1$. Hence all $f = 1$ cases are accounted for, since $m = 1$ may be handled by $c \leq 6$ basis tableaux and $V(d)$'s.

When $f = 0$ the construction is similar to earlier cases, particularly the $f = 1$ case. However, instead of the $\mathcal{F}$ tableaux, we use only the $c = 3$ basis tableaux $\mathcal{D}$ of shape $[2d, d]$. Again, we may take $m \geq 2$, since $m = 1$ may be handled by $c \leq 6$ basis tableaux and $V(d)$'s. Writing $d = 6k' + i'$, $0 \leq i' \leq 5$, our general tableaux are:

$I.$	$\ell\mathcal{D}_p \vee (m-\ell)\mathcal{D}_{k'}$	$p = 0 \ldots k'-1$, $\ell = 1 \ldots m$
$II.$	$m\mathcal{D}_{k'}$	
$III.$	$\ell\mathcal{A}_j \vee (m-2\ell)\mathcal{D}_{k'}$	$\ell = 1 \ldots \lfloor \frac{m}{2} \rfloor$, $j = 1 \ldots 4$, $d \geq 6$

These tableaux have max weights:

$I.$	$(4k'+2p+i', 2(k'-p), 0)^{\ell} \vee (d, 0, 0,)^{m-\ell}$
$II.$	$(d, 0, 0)^m$
$III-1.$	$(d-1, d-1, 2, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell}$
$III-2.$	$(d-1, d-2, 3, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell}$
$III-3.$	$(d-2, d-2, 4, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell}$
$III-4.$	$(d-2, d-3, 5, 0, 0, 0)^{\ell} \vee (d, 0, 0)^{m-2\ell}$

Counting the number of zeros and d's in these weights gives:

Form	#0's	#d's
$I.$	$2m-\ell$	$m-\ell$
$II.$	$2m$	m
$III-j.$	$2m-\ell$	$m-2\ell$

Then by arguments similar to previous cases and Lemma 3.4.13, these tableaux have distinct max weights and are linearly independent by Lemma 3.4.12.

Consider $d \geq 6$. If $d \not\equiv 1 \pmod 6$, we need $mk' + m$ tableaux. The above construction provides $mk' + 1 + 4\lfloor \frac{m}{2} \rfloor$ tableaux, which is sufficient. When $d \equiv 1 \pmod 6$ we need $mk' + \lfloor \frac{m}{6} \rfloor + 1$ tableaux. In this case, $\mathcal{D}_0$ does not exists, so $p > 0$. For $k' \neq 1$, we have $m(k'-1) + 1 + 4\lfloor \frac{m}{2} \rfloor$ which is sufficient. For $k' = 1$, we have $4\lfloor \frac{m}{2} \rfloor + 1$ tableaux since Form I is no longer valid. However, this suffices.

When $d < 6$ not all of the $\mathcal{A}_j$ exist and there is exactly one $\mathcal{D}$. For $d = 4$ and $d = 5$ we have Forms II, $III - 1$, and $III - 5$ or $III - 4$ respectively. This provides $2\lfloor \frac{m}{2} \rfloor + 1$ tableaux. Since at most $\lfloor \frac{5m}{6} \rfloor + 1$ tableaux are needed, this is sufficient. When $d = 3$ we have Forms II and $III - 1$, which yield $\lfloor \frac{m}{2} \rfloor + 1$ tableaux. Since $\lfloor \frac{3m}{6} \rfloor + 1$ tableaux are required, we have enough. This construction work for $m \geq 2$, hence all $f = 0$ cases are accounted for.

9.2 Case: $r < s$

This proof of this case follows similarly to the $s \leq r$ case. Let $\lambda = [r + s, s]$ be a partition of n with $r < s$, where $n = 3b = cd$. We wish to construct m_λ linearly independent tableaux, where m_λ is the multiplicity of χ^λ in $1_{\mathcal{S}_b \wr \mathcal{S}_3}^{\mathcal{S}_n}$ as described in Theorem 11. First we will construct these tableaux for $c = 4$, 5, and 6; then we will use these constructions in proving Theorem 3 for a general c. We will refer to the tableaux constructed in this way as basis tableaux. These constructions will make use of the $c = 3$ basis tableaux constructed in Section 4.3 as well.

9.2.1 Basis Tableaux for $c = 4$, $r < s$

Given $\lambda = [r + s, s]$, a partition of n, we have $2s + r = 4d = 3b$. From this equation and $r < s$, we have $d + \lfloor \frac{d}{3} \rfloor < s \leq 2d$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 4. These will be our $c = 4$ basis tableaux.

When $d+\lfloor\frac{d}{3}\rfloor \leq s \leq 2d$, write $s=d+f$, with $\lfloor\frac{d}{3}\rfloor \leq f \leq d$. Consider the tableau T from Section 9.1.1. This tableau is non-zero and maximal.

$$T(\text{A},\text{B},\text{C},\text{D}) = \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \\ \hline 4 & 3 & 4 & 3 \\ 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} \text{A}+\text{B} \leq d \\ \text{A},\ \text{B},\ \text{C},\ \text{D} > 0 \\ \text{A} > \text{D} \\ \text{B} > \text{C} \\ \text{or A} = \text{D},\ \text{B} = \text{C} \end{array}$$

$$\mathsf{w}(T) = (\text{A}+\text{B}, \text{C}+\text{D}, 0, 0)$$

Let $\mathcal{C}_p = T(\lceil\frac{d}{2}\rceil - p, \lfloor\frac{d}{2}\rfloor - p, \lfloor\frac{f}{2}\rfloor + p, \lceil\frac{f}{2}\rceil + p)$ for $0 \leq p \leq \lfloor\frac{d-f}{4}\rfloor$. The constraints on T are satisfied for all p provided $f \geq 2$ when $p=0$. Hence $\mathcal{C}_p$ is non-zero for these parameters. Let $\mathcal{C}'_p = T(\lceil\frac{d}{2}\rceil - p+1, \lfloor\frac{d}{2}\rfloor - p, \lfloor\frac{f}{2}\rfloor + p, \lceil\frac{f}{2}\rceil + p-1)$, with $0 \leq p \leq \lfloor\frac{d-f}{4}\rfloor - 1$. Here the constraints on T are satisfied providedand $d-f \geq 4$ and $f \geq 4$ when $p=0$.

Hence $\mathcal{C}'_p$ is non-zero for these parameters. We have $\omega(\mathcal{C}_p) = (d-2p, f+2p, 0, 0)$ and $\omega(\mathcal{C}'_p) = (d-2p+1, f+2p-1, 0, 0)$. Thus the weights are distinct and Lemma 3.4.12 the tableaux are linearly independent.

By Theorem 11, $m_\lambda = \lfloor\frac{r}{6}\rfloor + 1$. Hence it suffices to construct $\lfloor\frac{r}{6}\rfloor + 1 \leq \lfloor\frac{2d-2f}{6}\rfloor + 1 \leq \frac{d-f}{3} + 1$ linearly independent tableaux since $r = 4d-2s$ and $s=d+f$. If $4 \leq f \leq d-4$ then $\mathcal{C}_p$ and $\mathcal{C}'_p$ together provide $2\lfloor\frac{d-f}{4}\rfloor + 1$ linearly independent tableaux. Since $2\lfloor\frac{d-f}{4}\rfloor \geq \lfloor\frac{d-f}{3}\rfloor$, this is sufficient. When $f > d-4$ we have $r \in \{0,2,4,6\}$ with s even for $r=0,4$ and odd for $r=2,6$. Therefor, at most one tableau is needed when $f > d-4$, which is $\mathcal{C}_0$. Now consider $f < 4$. Since $f \geq \lfloor\frac{d}{3}\rfloor$, when $d \geq 3$, only $f=2$ and $f=3$ remain. For $d=3$, then $f=1$ and we have $r=s$ which was done in Section 9.1.1. If $f=2$, then $d \leq 6$. However, by Theorem 11, $m_\lambda = 0$ for $d=3$. When $d=4$ or 5, $m_\lambda = 1$ and hence $\mathcal{C}_0$ suffices. For $d=6$, the tableaux $\mathcal{C}_0$ and $\mathcal{C}_1$ suffice. If $f=3$, then $d \leq 9$. For $d=9$ we have $m_\lambda = 3$, for $d=7$, we have $m_\lambda = 2$, while $m_\lambda \leq 1$ for the remaining $d \leq 8$. Now $\mathcal{C}_0$ suffices for those cases with $d \neq 7$ or 9. When $d=7$ we need an additional tableau, however, $\mathcal{C}_1$ exists. For $d=9$, then

$r = s$ and so this case was done in Section 9.1.1. Hence we have sufficient $c = 4$ basis tableaux for all partitions with $r < s$.

9.2.2 Basis Tableaux for $c = 5$, $r < s$

Given $\lambda = [r+s, s]$ a partition of n, we have $2s+r = 5d = 3b$. From this equation and $r < s$, we have $0 \le r \le \lfloor \frac{5d}{3} \rfloor$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 5. These will be our $c = 5$ basis tableaux.

First consider $0 \le r \le d$. If d is even, then $s - d \equiv s \pmod 2$. Moreover, $s - d \ge r$. Hence if $\lambda' = [r + s - d, s - d]$, then there are $m_{\lambda'}$ linearly independent $c = 3$ basis tableaux and $m_{\lambda'} = m_\lambda$ since $\lambda_1 = \lambda_2 = \lambda'_1 - \lambda'_2$. Therefore we can use $U_1(d) \vee T$, were T are the $c = 3$ basis tableaux of shape λ', as the $c = 5$ basis tableaux.

If d is odd, then $s \equiv d - 1 \pmod 2$ and $s - d + 1 \ge r$. Let T be the $c = 3$ basis tableaux of shape $\lambda' = [r + s - d - 1, s - d + 1]$. Consider the tableaux $U_1(d-1) \vee T$. There are $m_{\lambda'}$ such tableaux. Now $m_\lambda = m_{\lambda'}$ when $r \not\equiv 0, 3 \pmod 6$ since then $\lambda_1 - \lambda_2 - 2 = \lambda'_1 - \lambda'_2$. Thus for $r \not\equiv 0, 3 \pmod 6$ we have constructed sufficient tableaux. However, $2s + r = 5d$ and d odd implies r is odd, hence only $r \equiv 3 \pmod 6$ remains. In that case, $m_{\lambda'} = m_\lambda - 1$, so only one additional tableau is needed. For $d, r > 9$ use $U_1(d-3) \vee \mathcal{B}_0$ where $\mathcal{B}_0$ is the $c = 3$ basis tableau of shape $\lambda'' = [r + s - d - 6, s - d + 3]$. To show that $U_1(d-3) \vee \mathcal{B}_0$ is linearly independent from the $U_1(d-1) \vee T$ it suffices, by Lemma 3.4.12, to show that their max weights are distinct. Since $d - 1$ is a weight of $U_1(d-1) \vee T$ for all T, we need only show that $d - 1$ is not a weight of $U_1(d-3) \vee \mathcal{B}_0$. If we write $\lambda'' = [r' + s', s']$ and $s' = 6k + j$, $0 \le j \le 5$, then $\omega(U_1(d-3) \vee \mathcal{B}_0) = (d - 3, 4k + j, 2k, 0, 0)$. If $d - 1$ is in this weight then that means $d - 1 = 4k + j$. Since $2s' + r' = 3d$, we then must have $j = 0$ and $r' = 3$. As $r' = r - 6$, this implies $r = 9$. Thus for $r > 9, d > 3$ these tableaux are linearly independent.

If $d = 3$, then only $r = 3$ is needed because $r \le d$. Only one tableau is required, which is $U_2(2, 1)$. When $r = 9$ we need two tableaux basis tableaux. The $U_1(d-3) \vee \mathcal{B}_0$

we constructed above has weight $(d-1, d-3, \frac{d-1}{2})$ In addition, use

$$T = \begin{array}{cccccccc} 1 & \text{A} & d-\text{A}-1 & \text{B}+1 & d-\text{B}-2 & \text{A} & \text{B} \\ \hline 4 & 3 & 5 & 3 & 4 & 5 & 4 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 \end{array} \quad \text{A} = \lfloor \tfrac{d-7}{4} \rfloor, \text{B} = \lceil \tfrac{d-7}{4} \rceil, d \geq 11, d \text{ odd}$$

which has weight $(d, d-1, \frac{d-7}{2}, 0, 0)$. For $d < 11$, only $d = 9$ is needed since $9 = r \leq d$ and d is odd. In that case, $s = 2d$ so use $U_1(d-3) \vee \mathcal{B}_0$ and

$$T = \begin{array}{cccc} & d-1 & & d-1 \\ \hline 5 & 3 & 5 & 4 \\ 1 & 1 & 2 & 2 \end{array}$$

Thus we have sufficient tableaux for $r \leq d$.

Now consider $d < r \leq \lfloor \frac{5d}{3} \rfloor$. To construct the $c = 5$ basis tableaux for these r we will use two different types of tableaux. These are the same tableaux that were used in Section 9.1.2 and hence are non-zero and maximal.

$$T_1(\text{A}, \text{B}) = \begin{array}{ccccccc} & & \text{A} & \lfloor \frac{d}{2} \rfloor & \lceil \frac{d}{2} \rceil & \text{B} & \\ \hline 4 & 4 & 5 & 4 & 3 & 4 & 5 \\ 2 & 2 & 2 & 1 & 1 & 3 & 3 \end{array}$$

$1 \leq \text{A} \leq d-2$

$1 \leq \text{B} \leq \lfloor \frac{d}{2} \rfloor - 3$

$\text{A} > \text{B}$

$s \geq d+6$

$d \geq 8$

$$\mathsf{w}(T_1) = (d, \text{A}+2, \text{B}+1, 0, 0)$$

$$T_2(\text{A}, \text{B}) = \begin{array}{cccc} & \text{A} & & \text{B} \\ \hline 5 & 3 & 5 & 4 \\ 1 & 1 & 2 & 2 \end{array}$$

$1 \leq \text{A}, \text{B} \leq d-1$

$s \geq 4$

$$\mathsf{w}(T_2) = (\text{A}+1, \text{B}+1, 0, 0)$$

Since $d < r \leq \lfloor \frac{5d}{3} \rfloor$, then $2d > s \geq \lceil \frac{5d}{3} \rceil$. Consider $\mathcal{E}_p = T_1(s-d-3-p, p)$ for $1 \leq p \leq \lfloor \frac{s-d}{2} \rfloor - 2$. For $s < 2d-1$, the parameters on T_1 are satisfied provided $d \geq 8$. (When $s = 2d-1$, we take $p \leq \lfloor \frac{s-d}{2} \rfloor - 3$.) This provides $\lfloor \frac{s-d}{2} \rfloor - 2$

linearly independent tableaux. We will also use $\mathcal{E}'_q = T_2(d-q, s-2-d+q)$ for $1 \le q \le d - \lfloor \frac{s-2}{2} \rfloor$. The parameters on T_2 are satisfied provided $s \ge d+2$ (which holds for $d > 3$). Together, $\mathcal{E}_p$ and $\mathcal{E}'_q$ provide $\lfloor \frac{s-d}{2} \rfloor - 2 + d - \lfloor \frac{s-2}{2} \rfloor$ linearly independent tableaux when $d \ge 8$. We need $\lfloor \frac{r}{6} \rfloor + 1 = \lfloor \frac{5d-2s}{6} \rfloor + 1$ tableaux. When $s \ge d+6$, we have $\lfloor \frac{5d-2s}{6} \rfloor + 1 \le \lfloor \frac{s-d}{2} \rfloor - 2 + d - \lfloor \frac{s-2}{2} \rfloor$ so these tableaux are enough. Since $s \ge \lceil \frac{5d}{3} \rceil$ and $d \ge 8$, we have $s \ge d+6$. Thus for $d \ge 8$, $s < 2d-1$ these tableaux suffice

When $s = 2d-1$, $d \ge 8$ we have $\lfloor \frac{s-d}{2} \rfloor - 3 + d - \lfloor \frac{s-2}{2} \rfloor = \lfloor \frac{d-1}{2} \rfloor - 2$. This is greater than or equal to $\lfloor \frac{r}{6} \rfloor + 1 = \lfloor \frac{d+2}{6} \rfloor + 1$ provided $d > 8$. When $d = 8$, $s = 2d-1$ we have $r = 10$and only one tableau is needed, which $\mathcal{E}'_q$ provides. When $d = 8$ we also have $r = 12$. In that case two tableaux are needed, which $\mathcal{E}'_q$ provides.

Now consider $d < 8$ with $s \le 2d-1$. For $d \le 5$, or $d = 6$ and $r = 8$ only one tableau is needed. Here $\mathcal{E}'_q$ provides this tableau except if $d = 3$. When $d = 3$, then $r = s$ which was done in Section 9.1.2. When $d = 7$ or $d = 6$ and $r = 10$ two tableaux are needed. In these cases, $\mathcal{E}'_q$ suffices. Hence for $d < 8$ all tableaux are provided.

9.2.3 Basis Tableaux for $c = 6$, $r < s$

Given $\lambda = [r+s, s]$ a partition of n, we have $2s + r = 6d = 3b$. From this equation and $r < s$, we have $0 \le r < 2d$. For each λ we will construct m_λ linearly independent λ-tableaux filled with the numbers 1 to 6. These will be our $c = 6$ basis tableaux.

First consider $0 \le r \le d$. If d is even, then $s - d \equiv s \pmod 2$. Moreover, $s - d \ge r$. Hence if $\lambda' = [r+s-d, s-d]$, then there are at least $m_{\lambda'}$ linearly independent $c = 4$ basis tableaux and $m_{\lambda'} = m_\lambda$ since $\lambda_1 = \lambda_2 = \lambda'_1 - \lambda'_2$. Therefore we can use $U_1(d) \vee \mathcal{C}_p$, were $\mathcal{C}_p$ are the $c = 4$ basis tableaux of shape λ', as the $c = 6$ basis tableaux.

If d is odd, then $s \equiv d-1 \pmod 2$ and $s - d + 1 \ge r$. Let $\mathcal{C}_p$ be the $c = 4$ basis tableaux of shape $\lambda' = [r+s-d-1, s-d+1]$. Consider the tableaux $U_1(d-1) \vee \mathcal{C}_p$. There are $m_{\lambda'}$ such tableaux. Now $m_\lambda = m_{\lambda'}$ when $r \not\equiv 0, 3 \pmod 6$ since then $\lambda_1 - \lambda_2 - 2 = \lambda'_1 - \lambda'_2$. Thus for $r \not\equiv 0, 3 \pmod 6$ we have constructed sufficient tableaux. However, $2s + r = 6d$ implies r is even, hence only $r \equiv 0 \pmod 6$ remains.

In that case, $m_{\lambda'} = m_\lambda - 1$, so only one additional tableau is needed. Since $0 \le r \le d$, we have $\lfloor \frac{5d}{2} \rfloor \le s \le 3d$. For $r \ne 0$ we can write $s = 2d + f$ with $\lfloor \frac{d}{2} \rfloor \le f \le d - 3$. Then

$$T = \begin{array}{cccccc} d-2 & & d-1 & & f-1 & \\ \hline 4 & 6\;3 & 5 & 3 & 6 & 4 \\ 1 & 1\;1 & 2 & 2 & 3 & 3 \end{array} \quad \omega(T) = (d, d, f, 0, 0, 0)$$

provides the additional tableau needed. Since T does not have a weight of $d-1$ it must be linearly independent from $U_1(d-1) \vee \mathcal{C}_p$ by Lemma 3.4.12. When $r = 0$, then $s = 3d$. However, since d is odd, then so is s and hence no tableaux are required.

Now consider $d < r < 2d$ for arbitrary d. We have $2d < s \le \lfloor \frac{5d}{2} \rfloor$. Write $s = 2d + f$ for $1 \le f \le \lfloor \frac{d}{2} \rfloor$. Consider

$$\mathcal{H}_p = \begin{array}{cccccc} d-p-1 & p+1 & d-2p & p & p & f \\ \hline 4 & 6 & 5 & 4 & 5 & 6 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \qquad 0 < p \le \lfloor \tfrac{d-f}{2} \rfloor$$

$$\omega(\mathrm{H}_p) = (d, d-p, p+f)$$

Now $\mathcal{H}_p$ is non-zero and maximal. Moreover, for $0 < p \le \lfloor \frac{d-f}{2} \rfloor$, $\mathcal{H}_p$ are linearly independent by Lemma 3.4.12. Hence we have $\lfloor \frac{d-f}{2} \rfloor$ tableaux. We need $\lfloor \frac{r}{6} \rfloor + 1 = \lfloor \frac{2d-2f}{6} \rfloor + 1$. Hence it suffices to show that $\lfloor \frac{d-f}{2} \rfloor > \lfloor \frac{d-f}{3} \rfloor$. This holds except for $d = 6, f = 3$; $d = 5, f = 2$; and $d = 4, f = 1$. For $d = 4$ and 6, only one tableau is needed, so $\mathcal{H}_1$ suffices. When $d = 5, f = 2$, two tableaux are required. Use $\mathcal{H}_1$ and $U_1(d-1) \vee U_1(d-1) \vee U_1(d-1)$. Thus all the necessary $c = 6$ basis tableaux have been constructed.

9.2.4 Basis Tableaux for $c > 6$, $r < s$

Let $\lambda = [r + s, s]$ be a partition of n, with $r < s$, where $2s + r = cd = n$. We want to construct $m_\lambda \le \lfloor \frac{r}{6} \rfloor + 1$ linearly independent basis tableaux for an arbitrary c.

First write $s = s' + 2dq$ such that $s' - 2d < r \le s'$. If $c - 4q \ge 3$ then a λ-tableau $[r+s, s]$ may be written as $T = qU_2(d-1, 1) \vee T'$, where T' is a $\lambda' = [r+s', s']$ tableau filled with $c - 4q$ elements. Since $r \ge s'$ in λ', we have $m_\lambda = m_{\lambda'}$. Hence it suffices to construct m_λ tableaux T'. If $r = s'$, the tableaux T' were constructed in Section 9.1.4. Hence we only need to consider partitions $\lambda = [r+s, s]$ with $s - 2d < r < s$ and the case where $c - 4q < 3$.

If $c - 4q < 3$ we may write $T = (q-1)U_2(d-1,1) \vee T^*$ where T^* is a $\lambda' = [r + s' + d, s' + d]$ tableau filled with at most 6 elements. Since at least $m_\lambda = m_{\lambda'}$ tableaux T^* were constructed for $c \leq 6$ in previous sections, no additional construction is needed for this case.

To construct $\lambda = [r+s, s]$ tableaux with $s - 2d < r < s$, write $r = md + f$, with $0 \leq f < d$ and $s = md + pd + g$ with $0 \leq g < d$. Then $3md + 2pd + f + 2g = cd$, so $f + 2g = xd$ for some x. This means $c = 3m + 2p + x$. If $2p + x \geq 3$, a λ-tableau may be written $\mathcal{D} \vee \mathcal{F}$, where $\mathcal{D}$ is a tableau of shape $[2md, md]$ filled with $3m$ elements and $\mathcal{F}$ is a $[pd + g + f, pd + g]$ tableau filled with $2p + x$ elements. Given the constraints on r, we have $2p + x \leq 6$. Hence we can use the basis tableaux constructed in Sections 4.3, 9.2.1, 9.2.2, 9.2.3 as the $\mathcal{F}$ tableaux. We will first consider this case where $2p + x \geq 3$ and handle the $2p + x < 3$ case later.

We have $r = md + f$, so writing $d = 6k' + i'$, $f = 6\bar{k} + \bar{i}$, with $0 \leq i', \bar{i} \leq 5$ gives $r = 6(mk' + \bar{k}) + mi' + \bar{i}$. Since $\lfloor \frac{mi' + \bar{i}}{6} \rfloor \leq m$, it suffices to construct $\lfloor \frac{s}{6} \rfloor + 1 \leq mk' + \bar{k} + m + 1$ linearly independent tableaux. If $m \leq 1$ we have $c \leq 9$. For $c \leq 6$ we have constructed the necessary basis tableaux in previous section. $c = 7, 8$ and 9 will be handled later. Hence assume $m \geq 2$

Let $\mathcal{D}_p$ be the $c = 3$ basis tableaux of shape $[2d, d]$ described in Section 4.3. There are $m_{[2d,d]} = k' + 1$ such tableaux when $i' \neq 1$ and k' for $i' = 1$. Let $\mathcal{F}_q$ be the $c = 2p + x$ basis tableaux of shape $[pd + g + f, pd + g]$ constructed in Sections 4.3, 9.2.1, 9.2.2, 9.2.3. At least one such tableaux will always exist, provided $f \neq 0, 1, 2, 4$. There are at least $m_{[pd+g+f, pd+g]} \leq k'$ such tableaux, given the constraints on f. Take $f \neq 0, 1, 2, 4$ and $d \geq 6$. The $d < 6$ and f cases will be handled separately. Consider the following tableaux forms (where the $\mathcal{A}_i$ were defined in Section 9.1.4).

By Lemma 3.4.12, these tableaux are linearly independent provided their max weights are distinct. The max weights of these tableaux are:

$I.$	$\ell\mathcal{D}_p \vee (m-\ell)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1,\ 2, \ldots,\ m$ $p = 0,\ 1, \ldots,\ k'-1$ $q = 1,\ 2, \ldots,\ \overline{k}$
$II.$	$m\mathcal{D}_{k'} \vee \mathcal{F}_q$	$q = 1,\ 2, \ldots,\ \overline{k}$
$III.$	$\ell\mathcal{A}_j \vee (m-2\ell)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1,\ 2, \ldots,\ \lfloor \frac{m}{2} \rfloor$ $q = 1,\ 2, \ldots,\ \overline{k}$ $j = 1,\ 2,\ 3,\ 4$ $d \geq 6$ if $j \neq 1$

Table 9.3: Tableaux Forms.

$I.$	$(4k'+2p+i', 2(k'-p), 0)^{\ell} \vee (d,0,0)^{(m-\ell)} \vee \omega(\mathcal{F}_q)$
$II.$	$(d,0,0)^m \vee \omega(\mathcal{F}_q)$
$III-1.$	$(d-1,d-1,2,0,0,0)^{\ell} \vee (d,0,0)^{(m-2\ell)} \vee \omega(\mathcal{F}_q)$
$III-2.$	$(d-1,d-2,3,0,0,0)^{\ell} \vee (d,0,0)^{(m-2\ell)} \vee \omega(\mathcal{F}_q)$
$III-3.$	$(d-2,d-2,4,0,0,0)^{\ell} \vee (d,0,0)^{(m-2\ell)} \vee \omega(\mathcal{F}_q)$
$III-4.$	$(d-2,d-3,5,0,0,0)^{\ell} \vee (d,0,0)^{(m-2\ell)} \vee \omega(\mathcal{F}_q)$

The weights of $\mathcal{F}_q$ depend on which basis tableaux we are using for $\mathcal{F}$. We categorize these weights by the number of elements in $\mathcal{F}$.

c	$\omega(\mathcal{F}_q)$	Range	Conditions
3	$(4h+2q+j, 2(h-q), 0)$	$q = 1, \dots k'$	$s = 6h + j$
4	$(d-q, \frac{d-f}{2}+q, 0, 0)$	$q = 0, \dots, 2\lfloor \frac{d+f}{8} \rfloor$	
5	$(d, 4h+2q+j, 2(h-q), 0, 0)$	$q = 1, \dots, k'$	$s - d = 6h + j, r \leq d, d$ even
	$(d-1, 4h+2q+j, 2(h-q), 0, 0)$ $(d-3, 4h+j, 2h, 0, 0)$	$q = 1, \dots, k'$	$s - d + 1 = 6h + j, d$ odd $r \leq d.r > 9 if r \equiv 3 \pmod 6$
	$(d-1, d-3, \frac{d-1}{2}, 0, 0)$ $(d, d-1, \frac{d-7}{2}, 0, 0)$		$r = 9, d \geq 11, d$ odd
	$(d, d, 0, 0, 0)$ $(d-1, d-3, \frac{d-1}{2}, 0, 0)$		$r = 9, d = 9$
	$(d, d - \frac{f}{2} - q - 1, q+1, 0, 0)$ $(d - q' + 1, d - \frac{f}{2} - 1 + q', 0, 0, 0)$	$q = 1, \dots, \lfloor \frac{2d-f}{4} \rfloor - 2$ $q' = 1, \dots, \lfloor \frac{f}{4} \rfloor + 1$	$r > d$
6	$(d, d-q, \frac{d-f}{2}+q, 0, 0, 0)$	$q = 0, \dots, 2\lfloor \frac{d+f}{8} \rfloor$	$r \leq d, d$ even
	$(d-1, d-q, \frac{d-f}{2}+q+1, 0, 0, 0)$ $(d, d, \frac{d-f}{2}, 0, 0, 0)$	$q = 0, \dots, 2\lfloor \frac{d+f}{8} \rfloor$	$r \leq d, d$ odd $r \equiv 0 \pmod 6$
	$(d-1, d-q, \frac{d-f}{2}+q+1, 0, 0, 0)$	$q = 0, \dots, 2\lfloor \frac{d+f}{8} \rfloor$	$r \leq d, d$ odd
	$(d, d-q, q+\frac{d-f}{2}, 0, 0, 0)$	$q = 1, \dots, \lfloor \frac{d+f}{4} \rfloor$	$r > d$

The weights of Forms $I, II, III-j$ are distinct for each $\mathcal{F}_q$ listed provided $d > 7$, except for the following cases. When $\mathcal{F}$ is a $c = 5$ tableau, we have $\omega(I(\ell = 1, p = k'-1, \mathcal{F}_1)) = \omega(III - 4(\ell = 1, \mathcal{F}_2))$ when $d = 11, f = 9$. When $\mathcal{F}$ is $c = 4$ or $c = 6$, we have some duplicate tableau weights if $f = d-4$ or $f = d-2$. These weights are $\omega(I(\ell = 1, p = k'-1, q = j-1)) = \omega(III - j(\ell = 1, q = 0))$ for $j = 2, 3$, and 4. Also, when $f = d-4$, we have $\omega(III-2(\ell = 1, q = 2)) = \omega(III-3(\ell = 1, q = 1))$, $\omega(III-2(\ell = 1, q = 3)) = \omega(III-4(\ell = 1, q = 1))$, and $\omega(III-3(\ell = 1, q = 3)) = \omega(III-4(\ell = 1, q = 2))$. When $f = d-2$ we have $\omega(III-1(\ell = 1, q = 2)) = \omega(III-2(\ell = 1, q = 1))$ as well. In these cases we have (at most 6) fewer linearly independent tableaux available than listed. These will be called the constrained cases.

The linear independence of the remaining tableaux can be seen by counting the number of d's, 0's in each tableau, determining the number of distinct elements in each tableau, and applying Lemma 3.4.13 where appropriate. When $d = 6$, the Forms $III - 2$ and $III - 4$ have the same weight. When $d = 7$ the Forms $III - 3$ and $III - 4$ have the same weight. Hence for $d = 6$ and 7, Form $III - 4$ will not be used.

We wish to have $\lfloor \frac{r}{6} \rfloor + 1$ linearly independent tableaux. Recall that $r = md + f$, $d = 6k' + i'$ and $f = 6\overline{k} + \overline{i}$, so it suffices to construct $mk' + \overline{k} + m + 1$ linearly independent tableaux. We have $f \neq 0, 1, 2, 4$, $d \geq 6$ and $m \geq 2$. Note that there is always at least one $\mathcal{F}$ since $f \neq 0, 1, 2, 4$, so we will take $\overline{k} \geq 1$.

When $d > 6$, $d \not\equiv 1 \pmod 6$, all the tableaux listed in Table 9.2.4 exist and are linearly independent. This provides $mk'\overline{k} + \overline{k} + 4\lfloor \frac{m}{2} \rfloor \overline{k}$, which is at least $mk' + \overline{k} + m + 1$. Hence sufficient linearly independent tableaux exist. In the constrained cases, using the full set of $\mathcal{F}$ listed (as opposed to only the first k') will provide sufficient tableaux.

When $d = 6$, the tableaux of Forms $III - 2$ and $III - 4$ are not linearly independent. Hence we have $m\overline{k}+\overline{k}+3\lfloor \frac{m}{2} \rfloor \overline{k}$ linearly independent tableaux. Then $r = 6m+f$, so $m+\overline{k}+1$ tableaux are sufficient. Since we have $\overline{k} = 1$, we have listed sufficient tableaux.

When $d \equiv 1 \pmod 6$ the tableau D_0 does not exist. Hence Table 9.2.4 provides $m(k' - 1)\overline{k}+\overline{k}+4\lfloor \frac{m}{2} \rfloor \overline{k}$ linearly independent tableaux, provided $k' > 1$. In this case $r = 6k'm+m+f$, so $mk' + \overline{k} + \lfloor \frac{m}{6} \rfloor + 2$ tableaux suffices. Thus we have enough tableaux unless $m = 3, \overline{k} = 1$. In that case, specifically checking the number of tableaux needed and the number of $\mathcal{F}_q$ that exist, shows this construction is sufficient. When $k' = 1$, there are no tableaux of Form I and we do not use Form $III - 4$. Hence we have $1 + 3\lfloor \frac{m}{2} \rfloor$ tableaux, which is sufficient except in the following cases, where an additional tableau is needed If $m = 7, f = 5, c = 26$ use $6U_1(6) \vee U_1(4)$, if $m = 3, f = 3, c = 12$ use $5U_1(6)$, if $m = 3, f = 5, c = 14$ use $6U_1(6)$, if $m = 3, f = 6, c = 13$ use $4U_1(6) \vee U_4(3, 2)$, and if $m = 3, f = 6, c = 15$ use $6U_1(6)$. In the constrained cases, using the full set of $\mathcal{F}$ listed (as opposed to only the first k') will provide sufficient tableaux.

Hence for $d \geq 6$, $f \neq 0, 1, 2, 4$, $2p + x \geq 3$, we have constructed the requisite number of linearly independent tableaux. We will consider the $f = 0, 1, 2, 4$ case after doing the $2p + x < 3$ case.

When $2p + x < 3$, the procedure described above require $\mathcal{F}_q$ to have fewer than three elements. In that case we use $m - 1$ in place of m in our construction and take $\mathcal{F}_q$ to be basis tableaux filled with $2p+x+3$ elements with shape $[pd+2d+g+f, pd+d+g]$. There are at least $k' + \overline{k}$ such tableaux, though $\overline{k}$ may equal 0. Take $d \geq 6$, $2p + x < 3$, $m \geq 3$ Hence from the tableaux of Table 9.2.4 we get Table 9.2.4.

$I.$	$\ell\mathcal{D}_p \vee (m-\ell-1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1, 2, \ldots, m-1$ $p = 0, 1, \ldots, k'-1$ $q = 1, 2, \ldots, k' + \overline{k}$
$II.$	$(m-1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$q = 1, 2, \ldots, k' + \overline{k}$
$III.$	$\ell\mathcal{A}_j \vee (m-2\ell-1)\mathcal{D}_{k'} \vee \mathcal{F}_q$	$\ell = 1, 2, \ldots, \lfloor \frac{m-1}{2} \rfloor$ $q = 1, 2, \ldots, k' + \overline{k}$ $j = 1, 2, 3, 4$ $d \geq 6$ if $j \neq 1$

Table 9.4: Tableaux Forms

As before, these tableaux are linearly independent, except in the constrained cases where the same weight equalities occur. We do not use Form $III-4$ when $d = 6$ or 7. For the tableaux of Table 9.2.4, we require $m \geq 3$ for tableaux of Form III to exist. Since we still have $r = md + f$, we want $mk' + \overline{k} + m + 1$ linearly independent tableaux, when $d \not\equiv 1 \pmod 6$. When $d > 6$, $d \not\equiv 1 \pmod 6$, this construction provides $(m-1)k'(\overline{k} + k') + \overline{k} + k' + 4\lfloor \frac{m-1}{2} \rfloor(\overline{k} + k')$ linearly independent tableaux. This is larger than $mk' + \overline{k} + m + 1$, except when $m = 4, k' = 1, \overline{k} = 0$. However, computing precisely how many $\mathcal{F}_q$ exist in this case and the minimum number of tableaux required, shows these tableaux suffice. In the constrained cases, using the full set of $\mathcal{F}$ listed (as opposed to only the first $k' + \overline{k}$) will provide sufficient tableau.

When $d = 6$, Forms $III-2$ and $III-4$ are the same so we have only $(m-1)+1+3\lfloor \frac{m-1}{2} \rfloor$ linearly independent tableaux. Since $r = 6m + f$, $f < d$, we need $m + 1$ tableaux, which we have. In the $f = d - 4$ and $f = d - 2$ cases using the full set of $\mathcal{F}$ listed will provide sufficient tableau.

If $d \equiv 1 \pmod 6$, the tableau D_0 does not exist. Hence Table 9.2.4 provides $(m-1)(k'-1)(\overline{k}+k') + \overline{k} + k' + 4\lfloor \frac{m-1}{2} \rfloor(\overline{k} + k')$ linearly independent tableaux when $k' > 1$. In this case $r = 6m + m + f$, so $mk' + \overline{k} + \lfloor \frac{m}{6} \rfloor + 2$ tableaux suffice, which we have. In the constrained cases, using the full set of $\mathcal{F}$ listed will provide sufficient tableau. When $k' = 1$, there are no tableaux of Form I or $III-4$. Hence we have $\overline{k}+1+3\lfloor\frac{m-1}{2}\rfloor(\overline{k}+1)$. This is at least as large as $mk'+\overline{k}+\lfloor\frac{m}{6}\rfloor+2$ unless $m\leq 8$. However, computing precisely how many $\mathcal{F}_q$ exist in this case and the minimum number of tableaux required, shows these tableaux suffice in most cases. When $m=6, f=3, c=19$ we need an additional tableau; use $6U_1(6)\vee U_1(2)$. When $m=4, f=3$ we also need an additional tableau; use $U_4(3,2)\vee 7U_1(6)\vee U_1(4)$. In the constrained cases, using the full set of $\mathcal{F}$ listed will provide sufficient tableau.

Thus for $d\geq 6$, $2p+x<3$, $m>2$, all necessary tableaux have been constructed.

When $2p+x<3$ and $m=2$, we find that $c=7$ or $c=8$. In these cases, consider the tableaux of Forms I and II in Table 9.2.4. We have $r=2d+f$, so $2k'+\overline{k}+3$ tableaux are sufficient. For $d\not\equiv 1 \pmod 6$, we have at least $(k'+1)(k'+\overline{k})$ linearly independent tableaux of Forms I and II. When this isn't enough, checking precisely how many tableaux are needed and how many tableaux of Form I exist, shows that this construction is sufficient except for the following cases.

When $c=8, d=8, f=2$ three tableaux are needed; use $2U_1(8)\vee U_4(6,1)$, $U_1(8)\vee U_3(2,3)\vee U_4(6,1)$, and $U_1(8)\vee U_3(2,4)\vee U_4(4,1)$. When $c=8, d=11, f=2$ five tableaux are needed. They are $4U_1(8)$, $U_2(10,1)\vee U_1(10)$, $U_2(10,1)\vee\vee U_1(8)\vee U_1(2)$, $U_2(10,1)\vee U_1(6)\vee U_1(4)$, and $U_2(10,1)\vee U_2(2,3)$. When $c=7$, if $d=8$ and $f=2$, three tableaux are needed, but Form I provides only two. In addition to those tableaux, use $P_3(0,3,2)\vee U_1(8)\vee U_1(2)$. If $c=7$, $d=9$ and $f=1$, three tableaux are needed, but Form I provides only two. In addition to those tableaux, use $U_1(8)\vee U_1(8)\vee U_1(6)$. If $d=10$ and $f=0$, three tableaux are needed, but Form I provides only two. In addition to those tableaux, use $P_3(0,4,2)\vee U_1(10)\vee U_1(4)$.

When $d\equiv 1 \pmod 6$, it is sufficient to construct $2k'+\overline{k}+2$ tableaux. We have at least $(k')(k'+\overline{k})$ linearly independent tableaux of Forms I and II. When this is less than $2k'+\overline{k}+2$, checking precisely how many tableaux are needed and how many tableaux of Forms I and II exist, shows that this construction is sufficient except for the following cases. When $c=8, d=7, f=4$ we need three tableaux; use $U_2(6,1)\vee U_4(4,1)$, $2U_1(6)\vee U_4(6,1)$, and $U_1(6)\vee U_3(2,3)\vee U_4(4,1)$. When $d=7, f=2$ three tableaux are needed; use $U_2(6,1)\vee U_1(6)$, $U_2(6,1)\vee U_1(4)\vee U_1(2)$, and $3U_1(6)\vee U_1(2)$. When $d=7$, and $f=0$, two tableaux are needed, but Form II provides only one. In addition to that tableau, use $P_3(0,3,2)\vee 2U_1(6)$.

For $c = 7$ if $d = 13$ and $f = 1$, five tableaux are needed, but Forms I and II provide only four. In addition to those tableaux, use $U_1(12) \vee U_1(12) \vee U_1(8)$. If $d = 7$ and $f = 1$, three tableaux are needed, but Form I provides only two. In addition to those tableaux, use $P_3(0, 3, 2) \vee U_1(6) \vee U_1(2)$. This completes the cases of $c = 7$ and $c = 8$ for $d \geq 6$.

Now consider $c = 7$ and $c = 8$ when $d < 6$. When $c = 7$, $d = 5$ we need two tableaux of shapes $[23, 12]$ and $[22, 13]$, and one tableau of shape $[21, 14]$. Use $3U_1(4)$ and $U_2(4, 1) \vee U_1(2)$, for shape $[23, 12]$; $U_2(4, 1) \vee U_4(2, 1)$ and $U_4(3, 2) \vee 2U_1(4)$ for shape $[22, 13]$; and $U_2(4, 1) \vee U_1(4)$ for shape $[21, 14]$. When $c = 7$an $d = 4$ we need two tableaux of shape $[18, 10]$, $2U_1(4) \vee U_1(2)$ and $U_2(2, 1) \vee U_1(4)$, and one tableau of shape $[17, 11]$, $U_4(2, 1) \vee 2U_1(4)$. When $c = 7$ and $d = 3$, we need only one tableau of shape $[13, 8]$, $U_2(2, 1) \vee U_1(2)$.

When $c = 8$ and $d = 5$, three tableaux are needed for shape $[26, 14]$, one tableau for shape $[25, 15]$, and two tableaux for $[24, 16]$. Use $3U_1(4) \vee U_1(2), U_2(4, 1) \vee U_1(4)$, and $U_2(2, 1) \vee 2U_1(4)$ for shape $[26, 14]$. For shape $[25, 15]$ use $U_2(4, 1) \vee U_4(3, 2)$, while for shape $[24, 16]$ use $4U_1(4)$ and $U_2(4, 1) \vee U_1(4) \vee U_1(2)$. When $c = 8$ and $d = 4$ we need one tableau for shapes $[21, 11]$ and $[19, 13]$ and two tableaux for shape $[20, 12]$. In the first case, use $U_4(2, 1) \vee U_2(3, 1)$ and $U_1(4) \vee U_4(2, 1) \vee P_4(0, 2, 2)$, respectively. For $[20, 12]$ use $3U_1(4)$ and $2U_2(2, 1)$. When $c = 8$ and $d = 3$ we need only one tableau of shape $[15, 9]$, which is $U_2(2, 1) \vee U_4(2, 1)$. This completes the $m = 2, 2p + x < 3$ case.

Now consider $d \geq 6$, $2p + x \geq 3$, $f = 0, 1, 2, 4$. In this case we cannot guarantee that the tableaux $\mathcal{F}_q$ of shape $[2dp + g + f, 2dp + g]$ exists. However, since $d \geq 6$, tableaux $\mathcal{F}$ of shape $[pd + d + g + f, pd + d + g]$ will exist. Moreover, since $s < r + 2d$, this shape is fillable with $c \leq 6$ elements. Hence we can simply use the tableaux constructed in the $2p + x < 3$ case with this $\mathcal{F}$. Since we did not apply any restrictions of f in that case, those computations hold.

Now take $d < 6$. We have $r = dm + f$, $f < d$, $s - 2d < r < s$. Since d is small, only a few s are possible for each r. We will consider each case according to the value of d.

If $d = 3$, we have $r = 3m + f$, $s = 3m + 3 + f$, and $f < 3$. First consider $f = 0$ or 2. If $m = 2$ we have the shapes $[15, 9]$ and $[19, 11]$. In both cases, only one tableau is needed. Use

$U_2(2,1) \vee U_4(2,1)$ and $U_2(2,1) \vee U_4(2,1) \vee U_1(2)$. Now take $m \geq 3$. Let $\mathcal{F}$ be the tableau of shape $[9+2f, 6+f]$. Since $f = 0$ or 2, one such $\mathcal{F}$ always exists. Then consider the tableaux of Form II and $III-1$ in Table 9.2.4. They have weight $(2,2,2,0,0,0)^\ell \vee (3,0,0)^{m-2\ell-1} \vee \omega(\mathcal{F})$ for $\ell = 0, 1, \ldots, \lfloor \frac{m-1}{2} \rfloor$ and hence are linearly independent by Lemma 3.4.12. This construction provides $\lfloor \frac{m-1}{2} \rfloor + 1$ tableaux. Since we need at most $\lfloor \frac{3m+2}{6} \rfloor + 1$ tableaux for m odd and $\lfloor \frac{3m+2}{6} \rfloor$ for m even, this suffices. When $f = 1$, let $\mathcal{F} = U_2(2,1) \vee U_1(2) \vee U_1(2)$. If $m = 2$ or 3, one tableau will suffice. In those cases use $\mathcal{F}$ and $\mathcal{F} \vee U_4(2,1)$. When $m \geq 4$ we will use the tableaux $\ell A_1 \vee (m - 2\ell - 2)U_4(2,1) \vee \mathcal{F}$ for $\ell = 0, 1, \ldots, \lfloor \frac{m-2}{2} \rfloor$. There are $\lfloor \frac{m}{2} \rfloor$ such tableaux and they are linearly independent. Since only $\lfloor \frac{m}{2} \rfloor$ are needed, this suffices.

When $d = 4$ we have $r = 4m + f$, $s = 4m + h + f$, and $f < 3$. Since $s - 8 < r < s$, the only possibilities are $f = 0, h = 2, 4, 6$ and $f = 2, h = 1, 3, 5, 7$. Let $\mathcal{F}$ be a tableau of shape $[8 + 2f + h, 4 + f + h]$. Since $\mathcal{F}$ needs at most 8 elements it has already been constructed. If $m = 2$, we need one tableau when $f = 2$, so $\mathcal{F}$ suffices. When $f = 0$, two tableaux are needed. Use $U_1(4) \vee U_1(4) \vee U_1(2)$ and $U_2(2,1) \vee U_1(4)$; $U_1(4) \vee U_1(4) \vee U_1(4)$ and $U_2(2,1) \vee U_1(4) \vee U_1(2)$; or $U_1(4) \vee U_1(4) \vee U_1(4) \vee U_1(2)$ and $U_2(2,1) \vee U_1(4) \vee U_1(4)$; depending on h. For $m \geq 3$, use the tableaux of Forms II and $III-1$ of Table 9.2.4 along with those of form $III-1$, with $\mathcal{A}_5$ (given in Section 9.1.4) in place of $\mathcal{A}_1$. This provides $1 + 2\lfloor \frac{m-1}{2} \rfloor$ linearly independent tableaux, which is sufficient.

When $d = 5$ we have $r = 5m + f$, $s = 5m + h + f$, with $f < 5$, $0 < h < 10$, and $h \equiv f \pmod 5$. Let $\mathcal{F}$ be a tableau of shape $[10 + h + 2f, 5 + h + f]$. When $m \geq 3$, consider the tableaux of Forms II, $III-1$, $III-4$ of Table 9.2.4. This provides $1 + 2\lfloor \frac{m-1}{2} \rfloor$ linearly independent tableaux. Computing precisely how many tableaux are needed for each f and h we find that this is sufficient except in the following cases. When $f = 1, m = 4$ we need one additional tableau. For $h = 1$ use $4U_1(4) \vee U_1(2)$ and for $h = 6$ use $3U_4(3,2) \vee 3 \vee U_1(4)$. When $f = 2$ and $h = 2$ we need an additional tableau for $m = 8, 6$, and 4. Use $11U_1(4)$, $8U_1(4) \vee U_1(2)$, and $6U_1(4)$ respectively. When $f = 2$ and $h = 7$, this construction suffices. When $f = 3$ and $h = 3$, one additional tableau is needed for $m = 6$ and $m = 4$; use $9U_1(4)$ and $6U_1(4) \vee U_1(2)$, respectively. When $f = 3$ and $h = 8$ we can take $\mathcal{F}$ to be a tableau of shape $[14, 11]$ and use the tableaux of Forms II, $III-1$, and $III-4$ of Table 9.2.4. This provides $1+2\lfloor \frac{m}{2} \rfloor$ tableaux, which suffices except for $m = 3$. In that case three tableaux are needed; use $U_2(4,1) \vee 3U_1(2)$, $U_2(4,1) \vee 2U_4(2,1)$, and $U_2(4,1) \vee U_1(4) \vee U_1(2)$. When $f = 4$ and $h = 9$ this construction suffices except when $m = 4$. In that case we need an additional tableau, so use $3U_2(4,1) \vee U_4(3,2)$. However, when $h = 9$ the tableau $\mathcal{F}$ has 9 elements. Since we've only constructed the basis tableaux for $c \leq 8$, use $\mathcal{F} = U_2(4,1) \vee 2U_1(4)$. When $f = 4$ and $h = 4$ we can take $\mathcal{F}$ to be a tableau of shape $[12,8]$ and use the tableaux of Forms II, $III - 1$, and $III - 4$ of Table 9.2.4. This provides $1 + 2\lfloor \frac{m}{2} \rfloor$ tableaux, which suffices.

Now consider $m = 2$ for $d = 5$. The tableaux will have $c \leq 9$ elements unless $f = 2, h = 7$, $f = 3, h = 8$, or $f = 4$. The $c \leq 9$ will be constructed later. If $f = 2, h = 7$ we need two tableaux, $U_2(4,1) \vee U_4(3,2) \vee U_1(4)$ and $U_2(4,1) \vee U_4(2,1) \vee P_4(0,2,2)$. If $f = 3, h = 8$, two tableaux are required. Let $\mathcal{F}$ be a basis tableaux of shape $[14,11]$; use $\mathcal{A}_1 \vee \mathcal{F}$ and $\mathcal{A}_4 \vee \mathcal{F}$. If $f = 4, h = 9$, two tableaux are needed, $2U_2(4,1) \vee U_4(2,1)$ and $U_2(4,1) \vee U_4(2,1) \vee P_4(0,2,2) \vee U_1(4)$. If $f = 4, h = 4$ we need three tableaux. Let $\mathcal{F}$ be the basis tableau of shape $[12,8]$. Then $\mathcal{A}_4 \vee \mathcal{F}$, $\mathcal{A}_1 \vee \mathcal{F}$ and $2U_4(3,2) \vee \mathcal{F}$ provide the requisite tableaux. Thus all necessary tableaux for $d = 5$ have been constructed.

Now consider when $m < 2$ for arbitrary d. If $r < d$ then since $s - 2d < r < s$ we have $c \leq 6$, which has been done. If $r = d + f$ then we must have $c = 7, 8$ or 9. First consider when $c = 7$ with $r = d + f$. We get $s = 3d - \frac{f}{2}$. For d even use $U_1(d) \vee \mathcal{F}$ where $\mathcal{F}$ are the $c = 5$ tableaux with $r = d + f$. When d is odd use $U_1(d-1) \vee \mathcal{F}$ where $\mathcal{F}$ are the $c = 5$ tableaux with $r = d + f - 2$. This is sufficient unless $d \equiv 1 \pmod 6, f \equiv 2 \pmod 6$, $d \equiv 3 \pmod 6, f \equiv 0 \pmod 6$, or $d \equiv 5 \pmod 6, f \equiv 4 \pmod 6$, in which case we need one additional tableau. For that, use $U_2(d-1,1) \vee \mathcal{B}$ where $\mathcal{B}$ is a $c = 3$ basis tableau with $s \leq r$. This construction holds for $d > 3$. When $d = 3$, only the shape $[12,9]$ is needed. One tableau, $U_2(2,1) \vee U_4(2,1)$, suffices. Thus the $c = 7$ case is complete.

Now take $c = 8$. Since $s - 2d < r$, we have $f > \lfloor \frac{d}{3} \rfloor$ and $s = 3d + \frac{d-f}{2}$. For d even we can use $U_1(d) \vee \mathcal{F}$ where $\mathcal{F}$ are the $c = 6$ basis tableaux of shape $[3d - \frac{d+f}{2}, 2d + \frac{d-f}{2}]$. When d is odd, use $U_1(d-1) \vee \mathcal{F}$ where $\mathcal{F}$ is the $c = 6$ tableaux of shape $[3d - \frac{d+f}{2} - 1, 2d + \frac{d-f}{2} + 1]$. This construction suffices unless $d + f \equiv 0 \pmod 6$. (Since $\frac{d}{3} < f$, $d \equiv f \pmod 2$ and $d \geq 6$, such an $\mathcal{F}$ always exists.) If $d + f \equiv 0 \pmod 6$ use the tableau of Forms I and II of Table 9.2.4, with $\mathcal{F}$ the $c = 5$ tableaux of shape $[2d + \frac{d-f}{2} + f, 2d + \frac{d-f}{2}]$. This suffices unless $d = 13, f = 11$, $d \equiv 1 \pmod 6, f = 5$, or $d \equiv 5 \pmod 6, f = 7$. In the first case, five tableaux are needed; use $2U_2(9,1), U_2(9,1) \vee U_1(12) \vee U_1(8), 3U_1(12) \vee U_1(4), 2U_1(12) \vee U_1(10) \vee U_1(6), U_2(12,1) \vee U_1(12) \vee U_1(2)$. In the $f = 5$ case use the tableaux

$\mathcal{B}_p \vee \mathcal{F}$ where $\mathcal{B}_p$ are the $c = 3$ basis tableaux of shape $[2d+1, d-1]$ and $\mathcal{F}$ is the $c = 5$ tableau with $r = 3$. This suffices for $\frac{d-f}{2}$ even. When $\frac{d-f}{2}$ odd, an additional tableau, $U_2(d-1, 1) \vee U_1(d-1) \vee U_1(\frac{d-f}{2}+1)$, is needed. In the $f = 7$ case use the tableaux $\mathcal{B}_p \vee \mathcal{F}$ where $\mathcal{B}_p$ are the $c = 3$ basis tableaux of shape $[2d-1, d+1]$ and $\mathcal{F}$ are the $c = 5$ tableaux with $r = 9$.

Now consider when $c = 9$ with $r = d + f$. We get $s = 4d - \frac{f}{2}$. Since $s - 2d < r < s$ we have $\frac{2d}{3} < f < \frac{d}{2}$ and hence this case does not occur. This completes the $r < s$ case. Hence we have proven Theorem 3.

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