Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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Chapter 8

Tableau Sufficiency

Our proof of Theorem 2 in Chapter 6 presupposed we had constructed all tableaux with $c \leq 8$ for the shapes of Theorem 10. In Chapter 7 we constructed many non-zero tableaux. In this chapter we will demonstrate that all the necessary tableaux have been constructed. Specifically we need all shapes in Theorem 10, that is all partitions $[r+s+t, s+t, t]$ of n, with $r, s \neq 1$, such that if r or s is in $\{0, 2, 4\}$ then $s+t$ is even. Recall that those required shapes with r or s less than 5 are called exceptional cases.

8.1 Sufficiency when $c = 3$

The tableaux we will use for $c = 3$ are the P_i described in Section 7.3. These tableaux are all maximal and non-zero by weight-set counting. We will show that every necessary partition of $n = 3d$ has a corresponding P_i.

First consider the exceptional r cases. These are tableaux having $s+t$ even with $r \in \{0, 2, 4\}$, or $r = 3$ with no constraints on s and t. Since our shape is a partition of $3d$, we have the condition $3t + 2s + r = 3d$. Hence for a given r, we need only to verify that all the appropriate s in the range $0 \leq s \leq \frac{3d-r}{2}$, with $s \neq 1$ are obtained. This condition shows $s \equiv r \pmod 3$.

Table 8.1 below lists the tableaux we are using for these cases. The column 'Parameters' indicates the restrictions on the tableaux arising from their construction in Section 7.3. The column 's, t values' indicates their values in terms of the tableau parameters, while the 's covered' lists those cases covered by the given tableau. The restrictions given on the s covered reflect the listed conditions on the parameters.

Tableau		Parameters	s, t values	s covered
$r=0$	$P_4(d-2\textsc{b}, \textsc{b}, \textsc{b})$	$0 \leq \textsc{b} \leq \frac{d}{2}$	$s = 3\textsc{b}$	$0 \leq s \leq 3\lfloor \frac{d}{2} \rfloor$
		$d-\textsc{b}$ even	$t = d-2\textsc{b}$	$s+t$ even
$r=2$	$P_4(d-2\textsc{b}, \textsc{b}, \textsc{b}-1)$	$1 \leq \textsc{b} \leq \frac{d}{2}$	$s = 3\textsc{b}-1$	$2 \leq s \leq 3\lfloor \frac{d}{2} \rfloor - 1$
		$d-\textsc{b}$ odd	$t = d-2\textsc{b}$	$s+t$ even
$r=3$	$P_4(d-2\textsc{b}-1, \textsc{b}, \textsc{b}, \textsc{b})$	$0 \leq \textsc{b} \leq \frac{d-1}{2}$	$s = 3\textsc{b}$	$0 \leq s \leq 3\lfloor \frac{d-1}{2} \rfloor$
		$d-\textsc{b}$ odd	$t = d-2\textsc{b}-1$	$s+t$ even
	$P_3(d-2\textsc{b}-1, \textsc{b}, \textsc{b}-1)$	$1 \leq \textsc{b} \leq \frac{d-1}{2}$	$s = 3\textsc{b}$	$3 \leq s \leq 3\lfloor \frac{d-1}{2} \rfloor$
		$d-\textsc{b}$ even	$t = d-2\textsc{b}-1$	$s+t$ odd
$r=4$	$P_4(d-2\textsc{b}, \textsc{b}, \textsc{b}-2)$	$2 \leq \textsc{b} \leq \frac{d}{2}$	$s = 3\textsc{b}-2$	$4 \leq s \leq 3\lfloor \frac{d}{2} \rfloor - 2$
		$d-\textsc{b}$ even	$t = d-2\textsc{b}$	$s+t$ even

Table 8.1: Exceptional r cases for $c = 3$.

To see why Table 8.1 reaches the necessary upper bounds on s, we need to consider the parity of d. For d and r even, the maximum s needed is $\frac{3d}{2} - \frac{r}{2}$, which is obtained in the table. When d is odd, we need $s \leq \lfloor \frac{3d-r}{2} \rfloor = \frac{3d-r-1}{2}$. However, $s = \frac{3d-r-1}{2} \not\equiv r \pmod 3$, thus the largest s we need is $s = \frac{3d-r-3}{2} = 3\lfloor \frac{d}{2} \rfloor - \frac{r}{2}$. When $r = 3$, we need $s \leq \frac{3d-3}{2}$, which equals $3\lfloor \frac{d-1}{2} \rfloor$ when d is odd. But for d even, the largest $s \equiv r \pmod 3$ is $s = \frac{3d-9}{2} = 3\lfloor \frac{d-1}{2} \rfloor$. Hence the s bounds in Table 8.1 are correct.

For the lower bounds, Table 8.1 shows that all the necessary s are covered, except possibly some $s < 5$. Since $s \equiv r \pmod 3$, all s are covered in the $r = 2$ case. In the $r = 3$ case, $s = 0$ is only necessary when the $s + t$ is even. Similarly, no additional tableaux are needed in the $r = 4$ case because $s = 1$ is not a shape of Theorem 10.

Now consider the exceptional s cases. These are tableaux having t even with $s \in \{0, 2, 4\}$, or $s = 3$ with no constraints on t. We still have the condition $3t+2s+r = 3d$, so for a given s we need only verify that all the appropriate t in the range $0 \leq t \leq d - \lceil \frac{2s}{3} \rceil$, (with t even if $s \neq 3$) are obtained.

Table 8.2 belows lists the tableaux we are using for these cases. The columns 'Parameters' and 'Conditions' indicate the restrictions on the tableaux arising from their construction in Section 7.3. The column 't values' indicates t's value in terms of the tableau parameters, while the 't covered' lists those cases covered by the given tableau. The restrictions given on the t covered reflect the conditions listed on the parameters.

	Tableau	**Parameters**	**Conditions**	t **value**	t **covered**
$s = 0$	$P_2(\text{A}, 0)$	$0 \le \text{A} \le d$	A even	$t = \text{A}$	$0 \le t \le d$ t even
$s = 2$	$P_2(\text{A}, 2)$	$0 \le \text{A} \le d - 2$	A even	$t = \text{A}$	$0 \le t \le d - 2$ t even
$s = 3$	$P_3(\text{A}, 1, 0)$	$0 \le \text{A} \le d - 3$	A even	$t = \text{A}$	$0 \le t \le d - 3$ t even
	$P_4(\text{A}, 1, 1)$	$0 \le \text{A} \le d - 2$	A odd	$t = \text{A}$	$0 \le t \le d - 2$ t odd
$s = 4$	$P_2(\text{A}, 4)$	$0 \le \text{A} \le d - 4$	A even	$t = \text{A}$	$0 \le t \le d - 4$ t even

Table 8.2: Exceptional s cases for $c = 3$.

Table 8.2 shows that all the necessary t are covered, except possibly when $s = 3$ or 4. When $s = 3$, the $t = d - 2$, t even case does not appear. In this case, $r = 0$ and $s + t$ is odd, so by Theorem 10 this case is not needed. For $s = 4$ and $t = d - 3$, we have $r = 1$ which is not a required shape. Hence all the exceptional cases have been covered.

Finally, consider the general cases remaining. These are tableaux having $r, s \ge 5$ and no additional constraints. We still have the condition $3t + 2s + r = 3d$, so for a fixed t we need only verify that all the appropriate s in the range $5 \le s \le \frac{3d-3t-5}{2}$, are obtained. (This accounts for the bounds both on r and on s.)

Table 8.3 belows lists the tableaux we are using for this case. The columns 'Parameters' and 'Conditions' indicate the restrictions on the tableaux arising from their construction in Section 7.3. The column 's values' indicates its value in terms of the tableau parameters, while the 's covered' lists those cases covered by the given tableau.

To see how Table 8.3 covers all the necessary shapes, first consider P_3. As C varies

Tableau	Parameters	Conditions	s value	s covered
$P_3(t, \textsc{b}, \textsc{c})$	$2 \le \textsc{b} \le \frac{d-t-1}{2}$ $0 \le \textsc{c} < \textsc{b}$	$t + \textsc{c}$ even	$s = 2\textsc{b} + \textsc{c} + 1$	$5 \le s \le 3\lfloor \frac{d-t-1}{2} \rfloor$ $s + t$ odd
$P_4(t, \textsc{b}, \textsc{c})$	$1 \le \textsc{b} \le \frac{d-t}{2}$ $0 \le \textsc{c} \le \textsc{b}$	$t + \textsc{c}$ even	$s = 2\textsc{b} + \textsc{c}$	$2 \le s \le 3\lfloor \frac{d-t}{2} \rfloor$ $s + t$ even

Table 8.3: General $c = 3$ cases.

between 0 and $\textsc{b} - 1$, we get $2\textsc{b} + 1 \le s \le 3\textsc{b}$. When $\textsc{b}$ increases to $\textsc{b} + 1$, we go from $s = 3\textsc{b}$ to $s = 2\textsc{b} + 3 = 2(\textsc{b} + 1) + 1$. There is no gap between these provided $\textsc{b} \ge 2$. Since $\textsc{b} = 2$ yields a minimum $s = 5$, we don't need any smaller cases. Hence P_3 will cover the cases, provided $s + t$ is odd (equivalently $t + \textsc{c}$ is even). Given a case where $s = 2\textsc{b} + \textsc{c} + 1$ but $t + \textsc{c}$ is odd, use $P_4(t, \textsc{b}, \textsc{c} + 1)$. Then $t + \textsc{c} + 1$ is even, and $\textsc{c} < \textsc{b}$ implies $\textsc{c} + 1 \le \textsc{b}$, so the conditions of P_4 are satisfied. (As similar analysis on P_4 shows all the s in the range do occur.) To see that the upper bound of $s \le \lfloor \frac{3d-3t-5}{2} \rfloor$ is met, first consider P_4. Since $3\lfloor \frac{d-t}{2} \rfloor \ge \lfloor \frac{3d-3t}{2} \rfloor - 1 = \lfloor \frac{3d-3t-2}{2} \rfloor$, the upper bound is obtained. For P_3, $3\lfloor \frac{d-t-1}{2} \rfloor \ge \frac{3d-3t-3}{2}$ when $d - t$ is odd. If $d - t$ is even, we have $\lfloor \frac{3d-3t-5}{2} \rfloor = \frac{3d-3t-6}{2}$, which is $3\lfloor \frac{d-t-1}{2} \rfloor$ as desired. Thus all the required shapes are listed.

8.2 Sufficiency when $c = 4$

When $c = 4$, the tableaux we will use are the Q_i listed in Section 7.4. We will show that every partition of $n = 4d$ described in Theorem 10 has a corresponding Q_i. Note that shapes with $r \ge d + 5$ can be obtained by $P_i \vee V(d)$ for the appropriate P_i filled with $c = 3$ elements. As such, we will not include these shapes in the following compilation. Throughout, we will use the convention $d = 3\textsc{z} + \mathsf{x}$, where $d \equiv \mathsf{x} \pmod 3$. Unless otherwise specified, take $\mathsf{x} \in \{0, 1, 2\}$. We also use the notation d^* from previous chapters ($d^* = 2\lfloor \frac{d}{2} \rfloor$).

First consider the exceptional r cases. These are tableaux having $s + t$ even with $r \in \{0, 2, 4\}$ or $r = 3$ with no constraints on s and t. Since our shape is a partition of

$4d$ we have the condition $3t + 2s + r = 4d$. Hence for a given r, we need only verify that all the appropriate t in the range $0 \le t \le \lfloor \frac{4d-r}{3} \rfloor$, (with $s+t$ even for $r \ne 3$) are obtained.

For the exceptional r cases with $s + t$ even, we only need those t with $t \equiv r \pmod 4$. Given a shape (r, s, t) the next shape needed is $(r, s-6, t+4)$. Consider Table 8.4. For $r = 0$ we have all $t \le \frac{4d}{3}$ when $d \equiv 0 \pmod 3$. When $d \equiv 1 \pmod 3$, the shape $t = d + \frac{d-1}{3}$ is not possible and the table provides all $t \le d + \frac{d-1}{2} - 1$. Similarly, when $d \equiv 2 \pmod 3$, the shape $t = d + \frac{d-2}{3}$ corresponds to $s = 1$, while $t = d + \frac{d-2}{3} - 1$ is not possible. This covers the $r = 0$ cases. Note when $d = 5$, the only shapes needed are $t = d - 1$ and $t = 0$. Moreover, $t = d - 1$ and $t = d - 2$ are not possible for $d \le 4$.

For $r = 2$, Table 8.4 provides tableaux for $0 < t \le d + \mathsf{z} - 2 - \mathsf{x}$ or $t = d + \mathsf{z}$ if $d \equiv 2 \pmod 3$. Since $t \equiv 2 \pmod 4$, $t < 2$ is not needed. When $d \equiv 2 \pmod 3$, $t = d+\mathsf{z}-1$ and $t = d+\mathsf{z}-3$ are not congruent to $2 \pmod 4$. Similarly, $t = d+\mathsf{z}-\mathsf{x}$ and $t = d + \mathsf{z} - \mathsf{x} - 1$ are not congruent to $2 \pmod 4$ for $\mathsf{x} = 0$ and 1.

When $r = 3$ we no longer have the conditions $t \equiv r \pmod 4$; instead t must be odd. Table 8.5 accounts for all tableaux with $0 < t \le d + \frac{d}{3} - 3$ for $d \equiv 0 \pmod 3$. For $d \equiv 1 \pmod 3$ Table 8.5 accounts for all $t \le d + \frac{d-1}{3} - 4$ and for $t \le d + \frac{d-2}{3} - 5$ when $d \equiv 2 \pmod 3$. Tableaux with larger t correspond to shapes having exceptional s cases ($s \ne 3$). Since t is odd, these shapes are not needed according to Theorem 10. When $t \le d-1$ and d small, the shapes are either not required by Theorem 10 or are not possible.

For $r = 4$, we need all $t \equiv 0 \pmod 4$, where $t \le d + \mathsf{z} - 4$ if $\mathsf{x} = 0$, $t \le d + \mathsf{z} - 1$ if $\mathsf{x} = 1$, and $t \le d + \mathsf{z} - 2$ if $\mathsf{x} = 2$. Table 8.6 provides all these tableaux. Again, the bounds on d are necessary to produce a valid shape. When $d = 5$ and $t = 4$, the shape has $s = 2$ and can be found in Table 8.7. Hence these tables cover all the exceptional r cases.

Now consider the exceptional s cases. These are tableaux having t even with $s \in \{0, 2, 4\}$, or $s = 3$ with no constraints on t. When $r \ge d$, $r \ne d+1$, we may reduced to $P_i \vee V(d)$ for some P_i filled with $c = 3$ elements. (If $s = 3$ we may only

	Tableau	Parameters	s, t values	t covered
$r = 0$	$Q_2(\mathsf{z})$	$\mathsf{x} = 0$	$s = 0$ $t = d + \mathsf{z}$	$t = d + \mathsf{z}$ $d \equiv 0 \pmod 3$
	$Q_1(\mathsf{z} - k, \frac{k-\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2})$	$3k + \mathsf{x} \equiv 0 \pmod 4$ $\mathsf{x} \le k \le \mathsf{z}$, $0 < k$	$s = k + \frac{k+\mathsf{x}}{2}$ $t = d + \mathsf{z} - k$	$d \le t \le d + \mathsf{z} - 1$, $\mathsf{x} \ne 2$ $d \le t \le d + \mathsf{z} - 2$, $\mathsf{x} = 2$ $d \ne 5$ $t \equiv 0 \pmod 4$
	$Q_7(\frac{d-1}{2}, \frac{d-3}{2}, \frac{d-3}{2})$	$d \ge 5$, d odd	$s = \frac{d+3}{2}$ $t = d - 1$	$t = d - 1$ $t \equiv 0 \pmod 2$
	$Q_4(\frac{d^*-2k}{2}, k, k + W, \frac{d^*}{2})$	$0 < 2k < d^*$ $2k \equiv d^* \pmod 4$ $W = d - d^*$	$s = \frac{d^*}{2} + 3k + 2W$ $t = d^* - 2k$	$4 \le t \le d - 2$ $t \equiv 0 \pmod 4$, $d \ge 6$
	$U_2(d - 1, 1)$		$s = 2d$ $t = 0$	$t = 0$
$r = 2$	$Q_2(\mathsf{z})$	$\mathsf{x} = 2$	$s = 0$ $t = d + \mathsf{z}$	$t = d + \mathsf{z}$ $d \equiv 2 \pmod 3$
	$Q_1(\mathsf{z} - k, \frac{k-\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2} - 1)$	$3k + \mathsf{x} \equiv 2 \pmod 4$ $\mathsf{x} \le k \le \mathsf{z}$, $0 < k$	$s = k + \frac{k+\mathsf{x}}{2} - 1$ $t = d + \mathsf{z} - k$	$d \le t \le d + \mathsf{z} - \mathsf{x} - 2$ $t \equiv 2 \pmod 4$
	$Q_5(\frac{d-1}{2}, \frac{d-1}{2}, 1, 0, \frac{d-3}{2}, 1)$	$d \ge 5$ $d \equiv 3 \pmod 4$	$s = \frac{d+1}{2}$ $t = d - 1$	$t = d - 1$ $t \equiv 2 \pmod 4$
	$Q_4(\frac{d^*-2k}{2}, k, k + W, \frac{d^*}{2} - 1)$	$0 < 2k < d^*$ $d^* - 2k \equiv 2 \pmod 4$ $d \ge 4$ $W = d - d^*$	$s = \frac{d^*}{2} + 3k + 2W - 1$ $t = d^* - 2k$	$2 \le t \le d - 2$ $d \ge 4$ $t \equiv 2 \pmod 4$
	$Q_3(2, 1)$	$d = 3$	$s = 2$ $t = 2$	$t = 2$ $d = 3$

Table 8.4: Exceptional $r = 0$ and $r = 2$ cases for $c = 4$.

Tableau	**Parameters**	s, t **values**	t **covered**
$Q_1(\mathrm{z}-4, 1, 2, 2)$	$\mathsf{x} = 1$ $d \geq 13$	$s = 5$ $t = d + \mathrm{z} - 4$	$t = d + \mathrm{z} - 4$ $d \equiv 1 \pmod 3$, $d \geq 13$
$Q_1(\mathrm{z}-5, 1, 4, 3)$	$\mathsf{x} = 4$ $d \geq 19$	$s = 8$ $t = d + \mathrm{z} - 5$	$t = d + \frac{d-1}{3} - 6$ $d \equiv 1 \pmod 3$, $d \geq 19$
$Q_1(\mathrm{z} - 2k - 1, \frac{2k-\mathsf{x}}{2}, \frac{2k+\mathsf{x}}{2}, \frac{2k+\mathsf{x}}{2})$	$k + \frac{\mathsf{x}}{2} \equiv 1 \pmod 2$ $\mathsf{x} \in \{0, 2, 4\}$ $d \neq 4, 5, 8$ $\mathsf{x} \leq 2k \leq \mathrm{z} - 1$	$s = 3k + \frac{\mathsf{x}}{2}$ $t = d + \mathrm{z} - 2k - 1$	$d \leq t \leq d + \mathrm{z} - 3$, $\mathsf{x} = 0$, $d \leq t \leq d + \mathrm{z} - 5$, $\mathsf{x} = 2$ $d \leq t \leq d + \mathrm{z} - 7$, $\mathsf{x} = 4$ t odd, $d \neq 4, 5, 8$
$Q_5(\frac{d}{2}, \frac{d}{2}, 1, 0, \frac{d-4}{2}, 1)$	d even $d \geq 6$	$s = \frac{d}{2}$ $t = d - 1$	$t = d - 1$ t odd $d \geq 6$
$Q_5(\frac{d-1}{2}, \frac{d-3}{2}, 2, 0, \frac{d-3}{2}, 1)$	d odd $d \geq 5$	$s = \frac{d+3}{2}$ $t = d - 2$	$t = d - 2$ t odd $d \geq 5$
$Q_5(k, k+1, d-2k-2, 1, k-1, d-2k-1)$	$2 \leq k \leq \frac{d-3}{2}$ $d \geq 8$	$s = 2d - 3k - 3$ $t = 2k + 1$	$5 \leq t \leq d - 3$ t odd $d \geq 8$
$Q_5(2, 1, d-3, 0, 1, d-4)$	$d \geq 5$	$s = 2d - 6$ $t = 3$	$t = 3$ $d \geq 5$
$Q_6(1, d-2, d-3)$		$s = 2d - 3$ $t = 1$	$t = 1$

Table 8.5: Exceptional $r = 3$ cases for $c = 4$.

	Tableau	Parameters	s, t values	t covered
$r = 4$	$Q_2(\mathrm{z})$	$\mathsf{x} = 4$ $d \neq 4$	$s = 0$ $t = d + \frac{d-1}{3} - 1$	$t = d + \frac{d-1}{3} - 1$ $d \equiv 1 \pmod 3$ $d \neq 4$
	$Q_1(\mathrm{z} - k, \frac{k-\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2}, \frac{k+\mathsf{x}}{2} - 2)$	$3k + \mathsf{x} \equiv 0 \pmod 4$ $\mathsf{x} \leq k \leq \mathrm{z}$, $k \geq 2$ $d \geq 6$	$s = k + \frac{k+\mathsf{x}}{2} - 2$ $t = d + \mathrm{z} - k$	$d \leq t \leq d + \mathrm{z} - 2$, $\mathsf{x} = 2$ $d \leq t \leq d + \mathrm{z} - 4 - \mathsf{x}$, $\mathsf{x} \neq 2$ $t \equiv 0 \pmod 4$ $d \geq 6$
	$Q_5(\frac{d-1}{2}, \frac{d-1}{2}, 0, 1, \frac{d-5}{2}, 1)$	$d \equiv 1 \pmod 4$ $d \geq 7$	$s = \frac{d-1}{2}$ $t = d - 1$	$t = d - 1$ $t \equiv 0 \pmod 4$ $d \geq 7$
	$Q_4(\frac{d^*-2k}{2}, k, k + W, \frac{d^*}{2} - 2)$	$0 < 2k \leq d^*$ $2k \equiv d^* \pmod 4$ $W = d - d^*$ $d \geq 6$	$s = \frac{d^*}{2} + 3k + 2W - 2$ $t = d^* - 2k$	$4 \leq t \leq d - 2$ $t \equiv 0 \pmod 4$ $d \geq 6$
	$U_2(d - 2, 1)$		$s = 2d - 2$ $t = 0$	$t = 0$

Table 8.6: Exceptional $r = 4$ cases for $c = 4$.

reduce when $r \geq d+5$.) Since we have already listed those table with exceptional r cases, we will take $r \geq 5$. Hence for a given s, we need only verify that all the appropriate t in the range $\lceil \frac{3d-2s-1}{3} \rceil \leq t \leq \lfloor \frac{4d-2s-5}{3} \rfloor$ (with t even for $s \neq 3$) are obtained.

For the exceptional s cases, consider Table 8.7. When $s = 0$, all t are covered except $t = d + \frac{d-1}{3}$ when $\mathsf{x} = 4$. However, this case is unnecessary as $r = 1$. If $t \leq d$, the shape is reducible to a $c = 3$ case.

For $s = 2$, all $t \geq d-1$ except $t = d + \frac{d-\mathsf{x}}{3} - 1$ (with $\mathsf{x} \in \{0, 2, 4\}$) are given in Table 8.7. When that occurs, either the shape is invalid or $r < 5$. If $t \leq d-2$, the shape is reducible to a $c = 3$ case. When $d \leq 7$ and $t \geq d$ the shapes are covered by the exceptional r cases or are reducible to $c = 3$ cases.

For $s = 3$ we want $t \leq d + \frac{d-\mathsf{x}}{3} - 2$. However, when $t = d + \frac{d-\mathsf{x}}{3} - 2$ ($\mathsf{x} \in \{0, 2, 4\}$), then $r < 5$. Hence these cases have already been covered. When $t < d-2$ the shape is reducible to a $c = 3$ case. For $d \leq 6$, the cases are covered by $c = 3$ or exceptional r cases. The bounds on d for $t \geq d$ are needed to produce a valid shape.

For $s = 4$, Table 8.7 provides all tableaux with $t \leq d + \frac{d-\mathsf{x}}{3} - 3$ with $\mathsf{x} \in \{1, 3, 5\}$. Since any t larger than this has $r < 5$, this covers all shapes not already listed. Note that for $d \leq 12$, all necessary shapes have $t < d$; those shapes with $t = d-1$ are not needed for $d \leq 6$. Hence all the exceptional s cases are accounted for in Table 8.7.

The general cases of $r, s \geq 5$ are classified in Table 8.8. When $r \geq d+5$ we can reduce to a $c = 3$ tableau. Fix $t = d + \textsc{z} - k$. Since r and s are greater than 5, we need all t with $5 \leq k \leq \textsc{z}$. For a t of this form, we need all shapes with $5 \leq s \leq k + \lfloor \frac{\mathsf{x}+k-5}{2} \rfloor$. This range is covered in Table 8.8. To see why all such s are obtained, note that for any fixed $\textsc{c}$, we always get $2\textsc{c} \leq s \leq 3\textsc{c} - 1$. Since $\textsc{c} \geq 2$, there are no gaps as we increment $\textsc{c}$. The parameters between the cases are comparable, so writing $s = 2\textsc{c} + \textsc{d} + 1$ and using the case corresponding to the parity of $s + k$ will yield the appropriate tableau. Since $\mathsf{x} \leq 2$, we find $s \leq k + \lfloor \frac{\mathsf{x}+k-5}{2} \rfloor$ implies $s \leq k + \lfloor \frac{k-1}{2} \rfloor - 1$ for $\mathsf{x} = 0$ or 2, and $s \leq k + \lfloor \frac{k}{2} \rfloor - 2$ for $\mathsf{x} = 1$. Comparing the bounds shows all s are obtained. This takes care of all shapes with $t \geq d$.

When $0 < t < d$, we require all shapes with $5 \leq s \leq 2d - t - \lfloor \frac{t+5}{2} \rfloor$. The tableaux

	Tableau	Parameters	t value	t covered
$s=0$	$Q_2(\textsc{a})$	$0<\textsc{a}\le\textsc{z}$ $\textsc{z}+\textsc{a}$ even $\mathsf{x}\in\{0,2,4\}$	$t=d+\textsc{a}$	$d<t\le d+\textsc{z}$ $\mathsf{x}\in\{0,2,4\}$ t even, $d\ne 4$
$s=2$	$Q_1(\textsc{a},1,1,0)$	$0\le\textsc{a}\le\textsc{z}-2$ $\mathsf{x}\in\{0,2,4\}$ $\textsc{z}+\textsc{a}$ even	$t=d+\textsc{a}$	$d\le t\le d+\textsc{z}-2$ $\mathsf{x}\in\{0,2,4\}$ t even, $d\ge 8$
	$Q_3(d-1,1)$	d odd	$t=d-1$	$t=d-1$ t even
$s=3$	$Q_1(\textsc{a},2,1,0)$	$0\le\textsc{a}\le\textsc{z}-3$ $\mathsf{x}\in\{0,2,4\}$ $\textsc{z}+\textsc{a}$ even	$t=d+\textsc{a}$	$d\le t\le d+\textsc{z}-3$ $\mathsf{x}\in\{0,2,4\}$ t even, $d\ge 11$
	$Q_1(\textsc{a},1,1,1)$	$0\le\textsc{a}\le\textsc{z}-2$ $\textsc{z}+\textsc{a}$ odd $\mathsf{x}\in\{0,2,4\}$	$t=d+\textsc{a}$	$d\le t\le d+\textsc{z}-2$ $\mathsf{x}\in\{0,2,4\}$ t odd, $d\ge 8$
	$Q_5(\frac{d-1}{2},\frac{d-1}{2},0,1,1,1)$	d odd $d\ge 7$	$t=d-1$	$t=d-1$, $d\ge 7$, d odd
	$Q_5(\frac{d}{2}-1,\frac{d}{2},1,0,1,1)$	d even $d\ge 6$	$t=d-1$	$t=d-1$, $d\ge 6$, d even
	$Q_6(d-2,1,0)$		$t=d-2$	$t=d-2$
$s=4$	$Q_1(\textsc{a},1,2,1)$	$0\le\textsc{a}\le\textsc{z}-3$ $\textsc{z}+\textsc{a}$ odd $\mathsf{x}\in\{1,3,5\}$	$t=d+\textsc{a}$	$d\le t\le d+\textsc{z}-3$ $\mathsf{x}\in\{1,3,5\}$ t even, $d\ge 12$
	$Q_5(\frac{d-1}{2},\frac{d-1}{2},1,0,2,1)$	d odd $d\ge 7$	$t=d-1$	$t=d-1$ d odd $d\ge 7$
	$Q_3(\textsc{a},2)$	$2\le\textsc{a}\le d-2$ $\textsc{a}$ even, $d\ge 4$	$t=\textsc{a}$	$2\le t\le d-2$ t even, $d\ge 4$

Table 8.7: Exceptional s cases for $c=4$.

Tableau		Parameters	s value	s covered
$d \le t \le d + \textsc{z} - 5$ $t = d + \textsc{z} - k$	$Q_1(\textsc{z} - k, \textsc{c} + 1, \textsc{c}, \textsc{d})$	$2 \le \textsc{c} \le \lfloor \frac{k-1}{2} \rfloor$ $5 \le k \le \textsc{z}$ $0 \le \textsc{d} < \textsc{c}$ $k \equiv \textsc{d} \pmod 2$	$s = 2\textsc{c} + \textsc{d} + 1$	$5 \le s \le 3\lfloor \frac{k-1}{2} \rfloor$ $s + k$ odd
	$Q_1(\textsc{z} - k, \textsc{c}, \textsc{c}, \textsc{d} + 1)$	$2 \le \textsc{c} \le \lfloor \frac{k}{2} \rfloor$ $5 \le k \le \textsc{z}$ $0 \le \textsc{d} \le \textsc{c} - 2$ $\textsc{d} \le \textsc{c} + 1$ if $\mathsf{x} = 0, 2$ $k \not\equiv \textsc{d} \pmod 2$	$s = 2\textsc{c} + \textsc{d} + 1$	$4 \le s \le 3\lfloor \frac{k}{2} \rfloor - 1$ $s \le 3\lfloor \frac{k}{2} \rfloor$ if $\mathsf{x} = 0, 2$ $s + k$ even
$t = d - 1$	$Q_5(\frac{d-1}{2}, \frac{d-1}{2}, 0, 1, \textsc{e}, 1)$	$1 \le \textsc{e} < \frac{d-1}{2} - 2$ $d \ge 7$ d odd	$s = \textsc{e} + 2$	$4 \le s \le \frac{d-1}{2}$ $d \ge 7$ d odd
	$Q_7(\frac{d}{2} - 1, \frac{d}{2} - 1, \textsc{c})$	$1 \le \textsc{c} \le \frac{d}{2} - 2$ d even	$s = \textsc{c} + 3$	$3 \le s \le \frac{d}{2} + 1$ d even
$t = d - 2$	$Q_5(\lfloor \frac{d-2}{2} \rfloor, \lceil \frac{d-2}{2} \rceil, 1, 1, \textsc{e}, 1)$	$1 \le \textsc{e} \le \lceil \frac{d-2}{2} \rceil - 1$ $d \ge 5$	$s = \textsc{e} + 3$	$4 \le s \le \lceil \frac{d}{2} \rceil + 1$ $d \ge 5$
$3 \le t \le d - 3$ $t = 2k + 1$	$Q_5(k, k + 1, d - 2k - 2, 1, \textsc{e}, \textsc{f})$	$1 \le k \le \lfloor \frac{d-3}{2} \rfloor$ $1 \le \textsc{e} \le k$ $1 \le \textsc{f} \le d - 2k - 2$ $d \ge 5$	$s = d - 2k - 1 + \textsc{e} + \textsc{f}$	$d - 2k + 1 \le s \le 2d - 3k - 3$ t odd $d \ge 5$
$2 \le t \le d - 3$ $t = 2k$	$Q_5(k, k, d - 2k - 1, 1, \textsc{e}, \textsc{f})$	$1 \le k \le \lfloor \frac{d-2}{2} \rfloor$ $1 \le \textsc{e} \le k$ $1 \le \textsc{f} \le d - 2k - 2$	$s = d - 2k + \textsc{e} + \textsc{f}$	$d - 2k + 2 \le s \le 2d - 3k - 2$ t even
$1 \le t \le d - 3$	$Q_6(t, \textsc{b}, \textsc{c})$	$\textsc{b} \le d - t - 1$ $\textsc{b} > \textsc{c} \ge 0$ $d \ge 4$	$s = \textsc{b} + \textsc{c} + 2$	$5 \le s \le 2d - 2t - 1$ $d \ge 4$
$t = 0$	$U_2(d - k - 1, 1)$	$0 \le k \le d - 2$	$s = 2(d - k)$	$4 \le s \le 2d$ s even
	$U_5(2, k - 2)$	$5 \le k \le d$	$s = 2k - 3$	$7 \le s \le 2d - 3$ $d \ge 5$, s odd
	U_8	$d = 4$	$s = 5$	$s = 5$, $d = 4$

Table 8.8: General $c = 4$ cases.

listed in Table 8.8 satisfy this. Note that for $t \leq d-3$ the bounds on s for Q_5 and Q_6 overlap, thus guaranteeing all s are covered. For the small d not listed, the shapes are either not possible, not needed, or result in exceptional cases done previously. When $t = 0$ we require all $d + \frac{d-5}{2} \leq s \leq 2d-3$. As Table 8.8 shows, this is satisfied. Hence all necessary cases for $c = 4$ have been covered.

8.3 Sufficiency for $c = 5$

When $c = 5$, the tableaux we will use are the R_i listed in Section 7.5. We will show that every partition of $n = 5d$ described in Theorem 10 has a corresponding R_i. Note that shapes with $r \geq d+5$ can be obtained by $Q_i \vee V(d)$ for the appropriate Q_i filled with $c = 4$ elements. As such, we will not include these shapes in the following compilation. Throughout, we will use the convention $d = 3\mathrm{z} + \mathsf{x}$, where $d \equiv \mathsf{x} \pmod 3$. Unless otherwise specified, take $\mathsf{x} \in \{0, 1, 2\}$.

First consider the exceptional r cases. These are tableaux having $s+t$ even with $r \in \{0, 2, 4\}$ or $r = 3$ with no constraints on s and t. Since our shape is a partition of $5d$ we have the condition $3t + 2s + r = 5d$. Hence for a given r, we need only verify that all the appropriate t in the range $0 \leq t \leq \lfloor \frac{5d-r}{3} \rfloor$, (with $s+t$ even if $r \neq 3$) are obtained.

For the exceptional r cases with $s+t$ even, we only need those t with $t \equiv d + r \pmod 4$. Hence given a shape (r, s, t) the next shape needed is $(r, s-6, t+4)$. Consider Table 8.9. For $r = 0$, this table provides all the t, except $t = \frac{5d-\mathsf{x}}{3}$, $\mathsf{x} = 1, 2$ and $t = d + 2\mathrm{z} - 1$. These are not a possible shapes. When $d < 6$ we only need those shapes with $t \leq d$.

For $r = 2$, Table 8.9 yields all $t \leq d + 2\mathrm{z} - 2$ with $\mathsf{x} \in \{1, 2, 3\}$. Since $t \equiv d+2 \pmod 4$. When $t = d + 2\mathrm{z}$, for $\mathsf{x} = 0$, 2 we do not get a shape required by Theorem 10; for $\mathsf{x} = 1$ the tableau is listed in the table. Also, $t = d + 2\mathrm{z} - 1$ is not a shape and $t = d + 2\mathrm{z} - 2$ with d odd is not needed (the d even case is listed). If $d \leq 6$, then all necessary shapes have $t \leq d-2$. Hence we have accounted for all necessary tableaux with $r = 2$.

	Tableau	Conditions	t value	t covered
$r = 0$	$R_1(\text{Z}, \text{Z})$	$\mathsf{x} = 0$ d even	$t = \frac{5d}{3}$	$t = \frac{5d}{3}$ $d \equiv 0 \pmod 3$ d even
	$R_4(\text{A}, \text{A}, \mathsf{x})$	$1 \leq \text{A} \leq \text{Z}$ $\text{Z} + \text{A}$ even $d \geq 6$	$t = d + 2\text{Z} - 2\text{A}$	$d \leq t \leq d + 2\text{Z} - 2$ $t \equiv d \pmod 4$ $d \geq 6$
	$R_5(\text{A}, \text{A})$	$0 \leq \text{A} \leq \frac{d^*}{2} - 1$ A even	$t = d - 2\text{A}$	$2 \leq t \leq d$ $t \equiv d \pmod 4$
	$R_{12}(\frac{d-1}{2})$	$d \equiv 1 \pmod 4$	$t = 1$	$t = 1$ $d \equiv 1 \pmod 4$
	$U_1(d) \vee U_3(\frac{d}{2}, \frac{d}{2})$	$d \equiv 0 \pmod 4$	$t = 0$	$t = 0$ $d \equiv 0 \pmod 4$
$r = 2$	$R_1(\text{Z}, \text{Z})$	$\mathsf{x} = 1$ d even	$t = \frac{5d-2}{3}$	$t = \frac{5d-2}{3}$ $d \equiv 1 \pmod 3$ d even
	$R_2(\text{Z} - 1, \text{Z} - 1, 1)$	$\mathsf{x} = 0$ d even	$t = d + \frac{2d}{3} - 2$	$t = d + \frac{2d}{3} - 2$ $t \equiv d + 2 \pmod 4$ d even $d \equiv 0 \pmod 3$
	$R_4(\text{A}, \text{A}, \mathsf{x} - 1)$	$\mathsf{x} \in \{1, 2, 3\}$ $1 \leq \text{A} \leq \text{Z}$ $d \geq 7$ Z+A odd	$t = d + 2\text{Z} - 2\text{A}$	$d \leq t \leq d + 2\text{Z} - 2$ $t \equiv d + 2 \pmod 4$ $d \geq 7$
	$R_5(\text{A}, \text{A} - 1)$	$1 \leq \text{A} \leq \frac{d^*}{2} - 1$	$t = d - 2\text{A}$	$3 \leq t \leq d - 2$ $t = 2$, d even
	$P_4(1, \frac{d-1}{2}, \frac{d-1}{2})$	$d \equiv 3 \pmod 4$	$t = 1$	$t = 1$, $d \equiv 3 \pmod 4$
	$U_1(d) \vee U_3(\frac{d}{2} - 1, \frac{d}{2})$	$d \equiv 2 \pmod 4$	$t = 0$	$t = 0$, $d \equiv 2 \pmod 4$

Table 8.9: Exceptional $r = 0$ and $r = 2$ cases for $c = 5$.

For $r = 3$ consider Table 8.10. In this case, we need $t \not\equiv d \pmod 2$. All cases with $t \leq d + 2\text{Z} - 3$ are covered in the table. In addition we need $t = d + 2\text{Z} - 1$ for $d \equiv 0, 2 \pmod 3$ with d even, which are listed as well. When $d < 9$ we have $t \leq d+1$ or $t = d + 2\text{Z} - 1$, and so those cases are covered. Finally, when $d = 4$, $t = d + 1$ has $s = 1$, so that shape is no needed. When $t = 0$ and $d = 3$ then s is even, so this case is reducible to $c = 4$ case with $r = 0$. Hence all necessary tableaux with $r = 3$ are provided by Table 8.10.

For $r = 4$ we need those tableaux with $t \equiv d \pmod 4$, $t \leq d + 2\text{z} - 4$. In addition, we need $t = d + 2\text{z}$ when $d \equiv 2 \pmod 3$ and $t = d + 2\text{z} - 2$ when $d \equiv 1 \pmod 3$. These are all found in Table 8.10. For $d = 5$ the needed tableaux are listed individually. For $d = 4$, $t = d + 2\text{z} - 2$ and $t = 0$ are the only shapes required, while $d = 3$ does not need any shapes. Thus all the exceptional r cases are contained in Table 8.9 and Table 8.10.

Now consider the exceptional s cases. These are tableaux having t even with $s \in \{0, 2, 4\}$, or $s = 3$ with no constraints on t. When $r \geq d$, $r \neq d + 1$, we may reduced to $Q_i \vee V(d)$ for some Q_i filled with $c = 4$ elements. (If $s = 3$ we may only reduce when $r \geq d + 5$.) Since we have already listed the exceptional r cases, we will also take $r \geq 5$. Hence for a given s, we need only verify that all the appropriate t in the range $\lceil \frac{4d-2s-1}{3} \rceil \leq t \leq \lfloor \frac{5d-2s-5}{3} \rfloor$ (with t even if $s \neq 3$) are obtained.

When $s = 0$, we need all even $t \leq d + 2\text{z}$. Table 8.11 provides all tableaux with $t \geq d + 2$, t even. For $t \leq d + 1$ we can use the reduction to a $c = 4$ case. When $s = 2$, we need all tableaux with $d + 1 \leq t \leq d + 2\text{z} - 2$ and t even, which Table 8.11 lists. Those with $t \leq d$ are reducible to a $c = 4$ case, while the $d \leq 5$ cases correspond to exceptional r cases or are similarly reducible.

When $s = 3$ we need those shapes with $t \leq d + 2\text{z} - 3$ and $d + 5 \geq r \geq 5$. For other r or $r = d + 3$ we can reduce to a $c = 4$ case. Table 8.11 provides tableaux for all $t \geq d$. When $t \leq d - 1$, we can reduce to a $c = 4$ case when $d \geq 8$ or $d = 6$. The remaining tableaux with $r \geq 5$ and $d \leq 7$ which are not reducible to a $c = 4$ case are listed as well.

For $s = 4$ we need all even $t \leq d + 2\text{z} - 5$, along with $t = d + 2\text{z} - 4$ when $d \not\equiv 0 \pmod 3$. Table 8.11 provides all those tableaux with $t \geq d + 1$ and $d \geq 9$. When $t \leq d$ we may reduce directly to a $c = 4$ case when $d \geq 10$. Those remaining tableaux with $d \leq 9$ have $r \geq d + 2$, $r = d$ or $r < 5$ and hence are either reducible or listed previously. Thus Table 8.11 suffices for the exceptional s cases.

The general tableaux with $r, s \geq 5$ are classified in Table 8.12 and Table 8.13.

Since $r, s \geq 5$, we only need those tableaux with $0 \leq t \leq d + 2\text{Z} - 5$. For a fixed t, we need all tableaux with $\frac{4d-3t-4}{2} \leq s \leq \frac{5d-3t-5}{2}$ because for $r \geq d+5$ we can reduce to a $c = 4$ tableau. First consider $t = d + \text{Z} + \text{A}$ of Table 8.12. We need all $s \leq \text{Z} - \text{A} + \mathsf{x} - 2 + \lfloor \frac{\text{Z}-\text{A}-1}{2} \rfloor$. Letting the parameters of R_9 vary in order over their bounds yields all s up to $\text{B} + \text{Z} - \text{A} + 2\mathsf{x} - 3$. Since B ranges to $\lfloor \frac{\text{Z}-\text{A}-\mathsf{x}+1}{2} \rfloor$, we get all $s \leq \text{Z} - \text{A} + \mathsf{x} - 3 + \lfloor \frac{\text{Z}-\text{A}+\mathsf{x}-1}{2} \rfloor$. This covers all the necessary s for $\mathsf{x} > 1$. When $\mathsf{x} = 1$ we also get $s = \text{Z} - \text{A} - 1 + \lfloor \frac{\text{Z}-\text{A}}{2} \rfloor$ as needed. This tableau requires $d \geq 16$, for $d < 15$ we find there are no shapes with $t \geq d + \text{Z}$, $5 \leq r \leq d + 4$ and $s \geq 5$. When $d = 15$, R_6 provides $t = d + \text{Z}$, the largest t required.

When $d \leq t < d + \text{Z}$, Table 8.12 provides tableau R_{10}. Fix $t = d + \text{Z} - \text{A}$, then we need all $s \leq \text{A} + \text{Z} + \mathsf{x} + \lfloor \frac{\text{A}+\text{Z}-5}{2} \rfloor$. To see why this tableau suffices, begin by taking $\text{C} = 0$. First let B and F vary over their ranges. Taking them maximal, vary D up to its maximum and then take E up to $\lfloor \frac{\text{Z}}{2} \rfloor - 1$. This satisfies all the required inequalities. Then take $R_{10}(\text{A}, \lfloor \frac{\text{A}}{2} \rfloor, \text{C}, \lceil \frac{\text{Z}}{2} \rceil + \lfloor \frac{\text{A}}{2} \rfloor, \lfloor \frac{\text{Z}}{2} \rfloor - 1, \lceil \frac{\text{A}}{2} \rceil)$ with $0 \leq \text{C} \leq \lfloor \frac{\text{A}}{2} \rfloor - 1$, C even. Similarly, use $R_{10}(\text{A}, \lfloor \frac{\text{A}}{2} \rfloor, \text{C}, \lceil \frac{\text{Z}}{2} \rceil + \lfloor \frac{\text{A}}{2} \rfloor - 1, \lfloor \frac{\text{Z}}{2} \rfloor - 1, \lceil \frac{\text{A}}{2} \rceil)$ with $0 \leq \text{C} \leq \lfloor \frac{\text{A}}{2} \rfloor$, C even. This gives all $5 \leq s \leq \text{A} + \text{Z} + \lfloor \frac{\text{A}}{2} \rfloor + \mathsf{x} + \lfloor \frac{\text{Z}}{2} \rfloor - 2$. Since $\lfloor \frac{\text{A}+\text{Z}-5}{2} \rfloor \leq \lfloor \frac{\text{A}}{2} \rfloor + \lfloor \frac{\text{Z}}{2} \rfloor - 2$, all the necessary s are obtained. This tableau required $d \geq 6$. When $d < 6$, we find there are no shapes with $t \geq d$ and r, s within the needed bounds. Table 8.12 also contains $t = d - 1$, which requires all $s \leq d - 1$, as listed.

When $t \leq d - 2$, consider Table 8.13. For $t \leq d - 2$, we need all $s \leq 2d - t + \lfloor \frac{d-t-1}{2} \rfloor - 2$. Tableau R_{11} provides this by taking the parameters through their ranges in order, using $\text{B} = 1$ and $\text{B} = 2$ (with $\text{E} \geq \text{F}$). (We need $\text{B} = 2$ in order to obtain $s = 5$, otherwise $\text{B} = 1$ suffices.) The only snag is when $t = 2$. Then we cannot have $\text{C} = 1$, hence $s = 5$ is not obtainable in this case. However, $t = 2, s = 5$ is needed only when $d = 5$ and hence is listed separately. Also, $d = 3$ needs only $t = 0$, accounting for $d \leq 4$.

For $t = 1$, we need all $2d - 2 \leq s \leq 2d + \frac{d}{2} - 6$. (The $s = 2d - 3$ case is a $r = d + 3$ reduction to $c = 4$.) All these shapes are obtained in Table 8.13. For small d, those tableaux not listed have $s < 5$.

For $t = 0$ we need all $2d - 2 \leq s \leq 2d + \lfloor \frac{d-1}{2} \rfloor - 2$. Table 8.13 provides all the necessary tableaux. When d is even, we get all $2d \leq s \leq 2d + \frac{d}{2}$, with s even and $2d - 1 \leq s \leq 2d + \frac{d}{2} - 3$ with s odd. While $s = 2d - 2$ is not listed, this shape has $r = d + 4$ and $s + t$ even, so we may reduce to a $c = 4$ case. When d is odd, we get all $2d - 2 \leq s \leq 2d + \frac{d-1}{2} - 2$ with s even and $2d - 1 \leq s \leq 2d + \frac{d-1}{2} - 2$ with s odd. Hence all the required s are listed. Therefore Tables 8.12 and 8.13 provide the required tableaux for the non-exceptional cases with $c = 5$.

8.4 Sufficiency for $c = 6$

When $c = 6$, the tableaux we will use are the S_i listed in Section 7.6. We will show that every partition of $n = 6d$ described in Theorem 10 has a corresponding S_i. Note that shapes with $r \geq d + 5$ can be obtained by $R_i \vee V(d)$ for the appropriate R_i filled with $c = 5$ elements. As such, we will not include these shapes in the following compilation. Throughout, we will use the convention $d = 3\mathsf{z} + \mathsf{x}$, where $d \equiv \mathsf{x} \pmod 3$. Unless otherwise specified, take $\mathsf{x} \in \{0, 1, 2\}$.

First consider the exceptional r cases. These are tableaux having $s + t$ even with $r \in \{0, 2, 4\}$ or $r = 3$ with no constraints on s and t. Since our shape is a partition of $6d$, we have the condition $3t + 2s + r = 6d$. Hence for a given r, we need only verify that all the appropriate t in the range $0 \leq t \leq \lfloor \frac{6d-r}{3} \rfloor$, (with $s + t$ even for $r \neq 3$) are obtained.

For the exceptional r cases with $s + t$ even, we only need those t with $t + r \equiv 2d \pmod 4$. Hence given a shape (r, s, t) the next shape needed is $(r, s - 6, t + 4)$. Moreover, for r even, both s and t must be even. Consider Table 8.14. For $r = 0$, this table provides all the all the required partitions. When $t = 4$ we must have d even, while we need d odd for $t = 2$. For $d = 3$ and 4, only $t = 2d$, $2d - 4$, and $t = 0$ are required. Hence all the $r = 0$ cases are provided. When $r = 2$, Table 8.14 gives all the necessary tableaux since only those with $t \leq 2d - 2$ and $t \equiv 2d + 2 \pmod 4$ are required.

For $r = 3$ consider Table 8.15. In this case we need all odd $t \leq 2d - 3$ because $t = 2d - 1$ has $s = 0$ and thus is not required by Theorem 10. All such tableaux are listed except $d = 3$ with $t \geq 3$. When $t = 3$ we reduce to a $c = 5$ case with $s = 3$ and $r = 0$. When $t = 5$ and $d = 3$ we have $s = 0$ and hence the shape is not needed. For $r = 4$ Table 8.15 provides all necessary tableaux. In this cases we have s and t even with $t \leq 2d - 4$, $t \equiv 2d \pmod 4$. When $d = 3$ and $t = 6$ we have $s = 1$, so that shape is not required. Hence all the exceptional r cases are listed.

Now consider the exceptional s cases. These are tableaux having t even with $s \in \{0, 2, 4\}$, or $s = 3$ with no constraints on t. When $r \geq d$, $r \neq d + 1$, we may reduced to $R_i \vee V(d)$ for some R_i filled with $c = 5$ elements. (If $s = 3$ we may only reduce when $r \geq d + 5$.) Since we have already listed the exceptional r cases, we will also take $r \geq 5$. Hence for a given s, we need only verify that all the appropriate t in the range $\lceil \frac{5d-2s-1}{3} \rceil \leq t \leq \lfloor \frac{6d-2s-5}{3} \rfloor$ (with t even for $s \neq 3$) are obtained.

Consider Table 8.16. When $s = 0$ this provides all tableaux since those tableaux with $t < d^*$ have $r \geq d + 2$. For $s = 2$ the table provides all tableaux with $t \geq d + 2$. When $t < d + 2$ we have $r \geq d + 2$ provided $d \geq 5$. When $d = 3$ or $d = 4$ either $r < 5$ or we can reduce to a $c = 5$ case as well.

Similarly when $s = 3$, those tableaux with $t < d + 2$ are reducible to a $c = 5$ case provided $d \geq 7$. When $d = 6$, $t = 7$ we may reduce to a $c = 5$ case with $r = 3$; for smaller t we have $r \geq d + 5$. When $d = 5$ and $t = 5$, we may reduce to a $c = 5$ case with $r = 4$ as $s + t$ even; for smaller t, $r \geq d + 5$. When $d = 4$ and $t = 5$ we have $r < 5$. When $d = 3$ and $t = 2$ we may reduce to a $c = 5$ case with $r = 3$; for smaller t we have $r \geq d + 5$.

When $s = 4$ we need all even $t \leq 2d - 3$. Table 8.16 lists all tableaux with $t \geq d + 2$ and $d \geq 5$. When $t \leq d + 1$, t even, we may reduce to a $c = 5$ case provided $d \geq 6$. When $d = 5$, $t = 6$ we have $r = 4$ and we may reduce using $P_1(4)$ to a $c = 3$ case. For $d = 3$ or 4 there are no needed shapes with $5 \leq r \leq d + 1$. Hence Table 8.16 lists all the exceptional s cases.

The general cases of $r, s \geq 5$ are classified in Table 8.17 and Table 8.19. Since $r, s \geq 5$, we only need those tableaux with $0 \leq t \leq 2d - 5$. For a fixed t, we need all tableaux with $\frac{5d-3t-4}{2} \leq s \leq \frac{6d-3t-5}{2}$ since if $r \geq d + 5$ we can reduce to a $c = 5$ tableau. When t is odd, write $t = 2e - 1$ then we need all $s \leq 3d - 3e - 1$. Consider

S_6 of Table 8.17. First we let B, A and then C vary over their parameters This yields all $s \leq 3d - 3e$ with $t \geq 5$. For $t = 3$, we use S_7. First let D vary to its bound of $d - 4$. Then take $S_7(d-4, \text{E})$ and $S_7(d-5, \text{E})$ with E varying over the even numbers. This yields all the required s. For $t = 1$ we use S_{12}. First vary B over its bounds to $d - 2$. Then use $S_{12}(d-3, d-2, \text{C})$ and $S_{12}(d-3, d-3, \text{C})$ with C even to obtain the needed s. Note the bounds on d in these tableaux are necessary for coherence. Those tableaux with $d \leq 5$ and shapes with r, $s \geq 5$ which are not reducible to a $c = 5$ case are listed in Table 8.18. Hence all odd t are covered.

When t is even consider Table 8.19. Write $t = 2e$, so we want all $2d + \frac{d}{2} - 3e - 2 \leq s \leq 3d - 3e - 3$. Consider S_5. First let B and C vary. Then take $S_5(\text{A}, d-e, d-e)$ and $S_5(\text{A}, d-e, d-e-1)$ as A varies over all even numbers up to $d - e - 1$ and $d - e - 2$ respectively. This yields all $8 \leq s \leq 3d - 3e - 3$ as needed when $t \geq 6$. For $5 \leq s \leq 7$ the tableaux are listed individually. For $t = 2$ or 4, we use S_7 which covers all necessary s (as shown above for $t = 3$). When $t = 0$ we need $2d + \frac{d}{2} - 2 \leq s \leq 3d - 3$ which is provided in the table. Note the bounds on d in these tableaux are necessary for coherence. Those tableau with $d \leq 5$ and shapes with r, $s \geq 5$ which are not reducible to a $c = 5$ case are listed in Table 8.18. Hence all even t are covered by Table 8.19. Therefor all the required tableaux with $c = 6$ have been listed.

	Tableau	Conditions	t value	t covered
$r=3$	$R_1(\text{z}-1,\text{z})$	$\text{x}=0$ $d \geq 6$, d odd	$t=d+2\text{z}-1$	$t=2d+2\text{z}-1$ $d \geq 6$, $d \equiv 3 \pmod 4$
	$R_9(\text{z}-1,1,0,0,1)$	$\text{x}=2$	$t=d+2\text{z}-1$	$t=d+2\text{z}-1$ $d \equiv 2 \pmod 3$
	$R_6(\text{A},\text{A}-1,\text{x})$	$1 \leq \text{A} \leq \text{z}-2$ $d \geq 9$	$t=d+2\text{z}-2\text{A}-1$	$d+1 \leq t \leq d+2\text{z}-3$ $d \geq 9$
	$R_{17}(\text{z}-2,\text{x})$	$d \geq 6$	$t=d+1$	$t=d+1$, $d \geq 6$
	R_{18}	$d=3$ or 5	$t=d+1$	$t=d+1$, $d=3$, 5
	R_{19}		$t=d-1$	$t=d-1$
	$R_7(\text{A})$	$1 \leq \text{A} \leq \frac{d-1}{2}-1$ d odd, $d \geq 5$	$t=d-2\text{A}-1$	$2 \leq t \leq d-3$ $t \not\equiv d \pmod 2$ d odd, $d \geq 5$
	$R_8(\text{A})$	$1 \leq \text{A} \leq \frac{d}{2}-2$ d even, $d \geq 6$	$t=d-2\text{A}-1$	$3 \leq t \leq d-3$ $t \not\equiv d \pmod 2$ d even, $d \geq 6$
	$R_{12}(\frac{d}{2}-2)$	d even, $d \neq 4$	$t=1$	$t=1$, d even, $d \neq 4$
	$P_4(1,1,1) \vee U_1(4)$	$d=4$	$t=1$	$t=1$, $d=4$
	U_6 U_7	$d>5, d$ odd $d=5$	$t=0$ $t=0$	$t=0$, $d>5, d$ odd $t=0$, $d=5$
$r=4$	$R_1(\text{z},\text{z})$	$\text{x}=2$ z even	$t=d+\frac{2d-4}{3}$	$t=d+\frac{2d-4}{3}$ t even $d \equiv 2 \pmod 3$
	$R_2(\text{z}-1,\text{z}-1,1)$	$\text{x}=1$	$t=d+\frac{2d-2}{3}-2$	$t=d+\frac{2d-2}{3}-2$ $d \equiv 1 \pmod 3$,
	$R_4(\text{A},\text{A}-2,\text{x})$	$2 \leq \text{A} \leq \text{z}$ $d \geq 6$	$t=d+2\text{z}-2\text{A}$	$d \leq t \leq d+2\text{z}-4$ $d \geq 6$
	$R_5(\text{A},\text{A}-2)$	$2 \leq \text{A} \leq \frac{d^*}{2}-1$	$t=d-2\text{A}$	$2 \leq t \leq d-4$, $d \geq 6$
	$R_{13}(3,2,1,1,1)$	$d=5$	$t=5$	$t=5$, $d=5$
	$R_{12}(\frac{d-5}{2})$	$d \geq 7$ d odd	$t=1$	$t=1$ $d \geq 7$ d odd
	$P_4(1,2,1) \vee U_1(4)$	$d=5$	$t=1$	$t=1$, $d=5$
	$U_1(d) \vee U_3(\frac{d}{2}-2,\frac{d}{2})$	$d \equiv 0 \pmod 4$	$t=0$	$t=0$ $d \equiv 0 \pmod 4$

Table 8.10: Exceptional $r=3$ and $r=4$ cases for $c=5$.

	Tableau	Conditions	t value	t covered
$s=0$	$R_1(\text{A},\text{B})$	$0<\text{A}\le\text{B}\le\text{Z}$ $\text{B}\ge\text{A}$ $d+\text{A}+\text{B}$ even	$t=d+\text{B}+\text{A}$	$d+2\le t\le d+2\text{Z}$ t even
$s=2$	$R_2(\text{A},\text{B},1)$	$0\le\text{B}\le\text{A}\le\text{Z}-1$ d even, if $\text{A}=\text{B}$	$t=d+\text{A}+\text{B}$	$d+1\le t\le d+2\text{Z}-3$ t even $d\ge 6$
$s=3$	$R_3(\text{A},\text{B})$	$0\le\text{A}<\text{B}\le\text{Z}-1$ $d\ge 6$	$t=d+\text{A}+\text{B}$	$d+1\le t\le d+2\text{Z}-3$ $d\ge 6$
	R_{15}	$d\ge 6$	$t=d$	$t=d,\ d\ge 6$
	$Q_5(3,3,0,1,1,1)$	$d=7$	$t=6$	$t=6,\ d=7$
	$Q_5(1,3,0,1,1,1)$	$d=5$	$t=4$	$t=4,\ d=5$
	$R_{13}(1,2,1,1,1)$	$d=4$	$t=3$	$t=3,\ d=4$
	$Q_6(2,1,0)$	$d=4$	$t=2$	$t=2,\ d=4$
$s=4$	$R_2(\text{Z}-2,\text{Z}-2,2)$	$\mathsf{x}=1,\ 2$ $d\ge 7$	$t=d+2\text{Z}-4$	$t=d+2\text{Z}-4$ $d\not\equiv 0 \pmod 3$ $d\ge 7$
	$R_2(\text{A},\text{B},2)$	$0\le\text{B}<\text{A}\le\text{Z}-2$ $d\ge 9$	$t=d+\text{A}+\text{B}$	$d+1\le t\le d+2\text{Z}-5$ $d\ge 9$

Table 8.11: Exceptional s cases for $c=5$.

	Tableau	Parameters	s value	s covered
$t = d + \text{Z} + \text{A}$ $d + \text{Z} \le t \le d + 2\text{Z} - 5$	$R_9(\text{A}, \text{B}, \text{C}, \text{D}, \text{E})$	$0 \le \text{A} \le \text{Z} - 5$ $1 \le \text{B} \le \lfloor \frac{\text{Z}-\text{A}-\text{x}}{2} \rfloor$ $0 \le \text{C} \le \text{Z} - \text{A} - \text{B} - 1$ $0 \le \text{D} \le \text{B} + \text{x} - 2$ or $\text{D} = \text{B} + \text{x} - 1$, $\text{E} \le 1$ $0 \le \text{E} \le \text{x}$ $\text{x} \ge 1$, $d \ge 16$	$s = \text{B} + \text{C} + \text{D} + \text{E}$	$5 \le s \le \frac{3\text{Z}-3\text{A}+3\text{x}-6}{2}$ $s \le \frac{3\text{Z}-3\text{A}-1}{2}$ if $\text{x} = 1$ $d \ge 16$
$t = d + \text{Z}$, $d = 15$	$R_6(2, 0, 0)$	$d = 15$	$t = d + \text{Z}$	$t = d + \text{Z}$, $d = 15$
$t = d + \text{Z} - \text{A}$ $d \le t \le d + \text{Z} - 1$	$R_{10}(\text{A}, \text{B}, \text{C}, \text{D}, \text{E}, \text{F})$	$1 \le \text{A} \le \text{Z}$ $0 \le \text{B} \le \lfloor \frac{\text{A}}{2} \rfloor$ $0 \le \text{C} \le \lfloor \frac{\text{Z}}{2} \rfloor$ $1 \le \text{D} \le \lceil \frac{\text{Z}}{2} \rceil + \lfloor \frac{\text{A}}{2} \rfloor$ $0 \le \text{E} \le \lfloor \frac{\text{Z}}{2} \rfloor$ $0 \le \text{F} \le \lceil \frac{\text{A}}{2} \rceil$ C even $d \ge 6$	$s = \text{B} + \text{C} + \text{D} + \text{E} + \text{F} + \text{x}$ $\text{C} < \text{D}$ $\text{C} + \text{D} < \text{Z} + \text{B}$ c $\text{C} + \text{E} < \text{Z} - \text{A} + \text{B} + \text{F}$ $\text{D} + \text{E} < \text{Z} + \text{F}$	$5 \le s$ $s \le \text{x} + \lfloor \frac{3\text{A}}{2} \rfloor + \lfloor \frac{3\text{Z}}{2} \rfloor - 1$ $d \ge 6$
$t = d - 1$	$R_{16}(\text{A})$	$1 \le \text{A} \le d - 4$ $d \ge 5$	$s = \text{A} + 3$	$4 \le s \le d - 1$ $d \ge 5$
$t = \text{A} + \text{B}$ $2 \le t \le d - 2$	$R_{11}(\text{A}, \text{B}, \text{C}, \text{D}, \text{E}, \text{F})$	$1 \le \text{A} \le d - \text{B}$ $\text{B} = 1$, or 2 $0 \le \text{C} \le d - \text{A} - 2$ $0 \le \text{D} \le d - \text{E} - \text{B} - 1$ $1 \le \text{E} \le \lceil \frac{d-\text{A}-\text{B}-1}{2} \rceil$ $1 \le \text{F} \le \lfloor \frac{d-\text{A}-\text{B}-1}{2} \rfloor$ $\text{F} \le \text{E}$ $\text{B} + \text{D} > \text{F}$ $\text{A} + \text{C} > \text{E} + \text{F}$ $d \ge 4$ $\text{C} \ge 1$ if $t = 2$	$s = \text{C} + \text{D} + \text{E} + \text{F} + 3$	$5 \le s$ $s \le 2d - t + \lfloor \frac{d-t}{2} \rfloor + 2$ $d \ge 4$ $s \ne 5$ if $t = 2$, $d = 5$

Table 8.12: General $c = 5$ cases for $t > 2$.

	Tableau	Parameters	s value	s covered
$t=2$	$P_3(2,1,0) \vee U_1(2)$	$d=5$	$s=5$	$s=5$, $d=5$
$t=1$	$R_{12}(\text{A})$	$1 \leq \text{A} \leq \lfloor \frac{d}{2} \rfloor - 1$	$s = 2d + \text{A} - 1$	$2d \leq s$ $s \leq 2d + \lfloor \frac{d}{2} \rfloor - 1$
	$P_4(1, \frac{d}{2} - 1, 1) \vee U_1(d)$	d even	$s = 2d - 1$	$s = 2d - 1$ d even
	$P_4(1, \frac{d-1}{2}, 1) \vee U_1(d-1)$	d odd	$s = 2d - 1$	$s = 2d - 1$ odd
	R_{14}	$d \geq 5$	$s = 2d - 2$	$s = 2d - 2$ $d \geq 5$
$t=0$	$U_1(d^*) \vee U_3(\text{A}, \lfloor \frac{d}{2} \rfloor)$	$0 \leq \text{A} \leq \lfloor \frac{\text{A}}{2} \rfloor$ A even	$s = 2d^* + \text{A}$	$2d^* \leq s$ $s \leq 2d^* + \lfloor \frac{d}{2} \rfloor$ s even
	$U_1(d^*) \vee P_3(0, \lfloor \frac{d-1}{2} \rfloor, \text{C})$	$0 \leq \text{C} \leq \lfloor \frac{d-1}{2} \rfloor - 1$ C even	$s = d^* + 2\lfloor \frac{d-1}{2} \rfloor + \text{C} + 1$	$d^* + 2\lfloor \frac{d-1}{2} \rfloor + 1 \leq s$ $s \leq d^* + 3\lfloor \frac{d-1}{2} \rfloor$ s odd

Table 8.13: General $c = 5$ cases for $t = 0$, 1, and 2.

	Tableau	Conditions	t value	t covered
$r = 0$	$S_1(\mathrm{z})$		$t = 2d$	$t = 2d$
	$S_5(d-e, d-e, d-e)$	$3 \leq e \leq d-2$ $d \equiv e \pmod 2$	$t = 2e$	$6 \leq t \leq 2d-4$ $t \equiv 2d \pmod 4$ $d \geq 5$
	$Q_4(2, 1, d-5, 3) \vee U_1(d)$	$d \geq 6$, d even	$t = 4$	$t = 4$ $d \geq 6$, d even
	$P_1(4) \vee U_3(2, 2)$	$d = 4$	$t = 4$	$t = 4$, $d = 4$
	S_8	$d \geq 5$, d odd	$t = 2$	$t = 2$ $d \geq 5$, d odd
	$U_1(d) \vee U_1(d) \vee U_1(d)$	d even	$t = 0$	$t = 0$, d even
$r = 2$	$S_5(d-e-1, d-e, d-e)$	$e \not\equiv d \pmod 2$ $3 \leq e \leq d-1$ $d \geq 4$	$t = 2e$	$6 \leq t \leq 2d-2$ $t \equiv 2 + 2d \pmod 4$ $d \geq 4$
	$Q_4(2, \frac{d-7}{2}, \frac{d-1}{2}, \frac{d-5}{2}) \vee U_1(d-1)$	$d \geq 9$, d odd	$t = 4$	$t = 4$ $d \geq 9$, d odd
	$P_4(3, 3, 3) \vee P_4(1, 2, 1)$	$d = 7$	$t = 4$	$t = 4$, $d = 7$
	$P_4(3, 1, 1) \vee P_4(1, 2, 1)$	$d = 5$	$t = 4$	$t = 4$, $d = 5$
	$Q_2(1) \vee U_1(2)$	$d = 3$	$t = 4$	$t = 4$, $d = 3$
	$Q_4(1, \frac{d-2}{2}, \frac{d-2}{2}, \frac{d-2}{2}) \vee U_1(d)$	d even	$t = 2$	$t = 2$, d even
	$U_2(d-1, 1) \vee U_1(d-1)$	d odd	$t = 0$	$t = 0$, d odd

Table 8.14: Exceptional $r = 0$ and $r = 2$ cases for $c = 6$.

	Tableau	Conditions	t value	t covered
$r = 3$	$S_6(d-e, d-e, d-e)$	$3 \le e \le d-1$ $d \ge 4$	$t = 2e-1$	$5 \le t \le 2d-3$ $d \ge 4$
	$S_9(2, 1, d-4, 1, d-4, d-2, 1)$	$d \ge 5$	$t = 3$	$t = 3$, $d \ge 5$
	$S_{10}(3)$	$d = 4$	$t = 3$	$t = 3$, $d = 4$
	S_{11}		$t = 1$	$t = 1$
$r = 4$	$S_4(d-6)$	$d \ge 6$	$t = 2d-4$	$t = 2d-4$ $d \ge 6$
	$S_5(d-e-2, d-e, d-e)$	$3 \le e \le d-4$ $e \equiv d \pmod 2$ $d \ge 7$	$t = 2e$	$6 \le t \le 2d-8$ $t \equiv 2d \pmod 4$ $d \ge 7$
	$Q_2(1) \vee U_1(4)$	$d = 5$	$t = 6$	$t = 6$, $d = 5$
	$Q_4(2, \frac{d-4}{2}, \frac{d-4}{2}, \frac{d-4}{2}) \vee U_1(d)$	$d \ge 6$, d even	$t = 4$	$t = 4$ $d \ge 6$, d even
	$P_1(4) \vee U_4(2, 2)$	$d = 4$	$t = 4$	$t = 4$, $d = 4$
	$Q_4(1, \frac{d-3}{2}, \frac{d-1}{2}, \frac{d-3}{2}) \vee U_1(d-1)$	$d \ge 5$, d odd	$t = 2$	$t = 2$ $d \ge 5$, d odd
	$Q_3(2, 1, 1) \vee U_1(2)$	$d = 3$	$t = 2$	$t = 2$, $d = 3$
	$U_1(d) \vee U_1(d) \vee U_1(d-2)$	d even	$t = 0$	$t = 0$ d even

Table 8.15: Exceptional $r = 3$ and $r = 4$ cases for $c = 6$.

	Tableau	Conditions	t value	t covered
$s = 0$	$P_1(d^*) \vee P_1(\text{A})$	$0 \le \text{A} \le d$ A even	$t = d^* + \text{A}$	$d^* \le t \le 2d^*$ t even
$s = 2$	$S_2(\text{A})$	$0 \le \text{A} \le d-4$ $\text{A} \equiv d \pmod 2$ $d \ge 4$	$t = d + \text{A} + 2$	$d+2 \le t \le 2d-2$ t even $d \ge 4$
$s = 3$	$S_3(\text{A})$	$0 \le \text{A} \le d-5$ $d \ge 5$	$t = d + \text{A} + 2$	$d+2 \le t \le 2d-3$ $d \ge 5$
	$P_1(4) \vee P_3(2, 1, 0)$	$d = 5$	$t = 6$	$t = 6$, $d = 5$
	$P_1(4) \vee U_4(2, 1)$	$d = 4$	$t = 4$	$t = 4$, $d = 4$
$s = 4$	$S_4(\text{A})$	$0 \le \text{A} \le d-5$ $d \ge 5$	$t = d + \text{A} + 2$	$d+2 \le t \le 2d-3$ $d \ge 5$

Table 8.16: Exceptional s cases for $c = 6$.

	Tableau	**Parameters**	**s value**	**s covered**
$t = 2e - 1$ $5 \leq t \leq 2d - 5$ t odd	$S_6(\text{A}, \text{B}, \text{C})$	$1 \leq \text{A}, \text{B}, \text{C} \leq d - e$ $\text{A}, \text{B} \geq \text{C}$ $\text{B} \geq \text{A} - 1$ $3 \leq e \leq d - 2$ $d \geq 5$	$s = \text{A} + \text{B} + \text{C}$	$5 \leq s \leq 3d - 3e$ $d \geq 5$
$t = 3$	$S_7(1, 2, d - 5, \text{D}, \text{E})$	$0 \leq \text{E} \leq d$ E even $1 \leq \text{D} \leq d - 4$ $d \geq 5$	$s = d - 2 + \text{D} + \text{E}$	$d - 1 \leq s \leq 3d - 7$ $d \geq 5$
$t = 1$	$S_{12}(d - 3, \text{B}, \text{C})$	$0 \leq \text{C} \leq d$ C even $0 \leq \text{B} \leq d - 2$	$s = d + \text{B} + \text{C}$	$d \leq s \leq 3d - 3$

Table 8.17: General $c = 6$ cases for odd t.

d	Tableau	Shape
$d = 5$	$P_3(2, 1, 0) \vee P_3(2, 1, 0)$	$t = 4,\ s = 6$
$d = 4$	$S_{10}(2)$	$t = 3,\ s = 5$
	$P_4(2, 1, 0) \vee U_4(2, 1)$	$t = 2,\ s = 5$
	$U_3(2, 2) \vee U_4(2, 1)$	$t = 0,\ s = 9$

Table 8.18: General $c = 6$ cases for $d \leq 5$.

Tableau		Parameters	s value	s covered
$t = 2e$ $6 \le t \le 2d-6$ t even	$S_5(\text{A}, \text{B}, \text{C})$	$1 \le \text{A}, \text{B}, \text{C} \le d-e$ $\text{B}, \text{C} > \text{A}$ A even, $d \ge 6$ $3 \le e \le d-3$	$s = \text{A}+\text{B}+\text{C}$	$8 \le s \le 3d-3e-3$ $d \ge 6$
	$P_3(e,1,0) \vee P_4(e,2,0)$	$0 \le e \le d-4$ e even	$s = 7$	$s = 7$, $d \ge 5$ e even
	$P_3(e+1,1,0) \vee P_4(e-1,2,0)$	$e \le d-4$ e odd	$s = 7$	$s = 7$, $d \ge 5$ e odd
	$S_5(2,2,2)$	$3 \le e \le d-3$	$s = 6$	$s = 6$, $d \ge 6$
	$P_3(e,1,0) \vee P_4(e,1,0)$	$e \le d-3$ e even	$s = 5$	$s = 5$, $d \ge 5$ e even
	$P_4(e+1,1,0) \vee P_4(e-1,1,0)$	$e \le d-3$ e odd	$s = 5$	$s = 5$, $d \ge 5$ e odd
$t = 4$	$S_7(2,2,d-6,\text{D},\text{E})$	$0 \le \text{E} \le d$ E even, $d \ge 6$ $1 \le \text{D} \le d-5$	$s = d-3+\text{D}+\text{E}$	$d-2 \le s \le 3d-9$ $d \ge 6$
$t = 2$	$S_7(1,1,d-4,\text{D},\text{E})$	$0 \le \text{E} \le d$ E even, $d \ge 5$ $1 \le \text{D} \le d-4$	$s = d-1+\text{D}+\text{E}$	$d \le s \le 3d-6$ $d \ge 5$
$t = 0$	$U_1(d^*) \vee U_1(d^*) \vee U_1(\text{A})$	$0 \le \text{A} \le d$ A even	$s = 2d^*+\text{A}$	$2d^* \le s \le 3d^*$ s even
	$U_5(\frac{d}{2}, \frac{d}{2}) \vee U_1(\text{A})$	$d \ge 6$, d even $0 \le \text{A} \le d$ A even	$s = 2d-3+\text{A}$	$2d-3 \le s \le 3d-3$ s odd $d \ge 6$, d even
	$U_5(\frac{d-1}{2}, \frac{d+3}{2}) \vee U_1(\text{A})$	d odd, A even $0 \le \text{A} \le d$	$s = 2d-3+\text{A}$	$2d-3 \le s \le 3d-4$ s odd, d odd

Table 8.19: General $c = 6$ cases for even t.

8.5 Sufficiency for $c > 6$, d even

In Chapter 6 we algorithmically demonstrate how to reduce an arbitrary tableau to one of those tableaux filled with fewer elements. In the case where d is even, we reduced all tableaux to joins of tableaux with $c \leq 6$ to or those tableaux with $t < d$, $s < d+5$, $r < d+5$. Previously we showed all tableaux with $c \leq 6$ where constructed in Chapter 7. Now we will address the remaining cases with d even.

Since any tableau must satisfy $cd = 3t + 2s + r$, applying the bounds on r, s, and t we find $cd \leq 3(d-1) + 2(d+4) + d + 4 = 6d + 9$. Hence for $d > 9$ all such tableaux will have $c \leq 6$. For $d = 8$ and $d = 6$ it may be possible to have $c = 7$, while for $d = 4$, both $c = 7$ and $c = 8$ are possible.

When d=8 and c=7, only the shape with $(r, s, t) = (11, 12, 7)$ satisfy the constraints. However, we can reduce this shape by $V(d)$ to a $c = 6$ case with $r = 3$.

When $d = 6$ and $c = 7$, only the shapes with $(r, s, t) = (10, 10, 4)$, $(9, 9, 5)$ and $(7, 10, 5)$ satisfy the constraints. For $(10, 10, 4)$ we may reduce by $V(d)$ to a $c = 6$ case with $(r, s, t) = (4, 10, 4)$ since $s + t$ is even. For $(9, 9, 5)$ we may reduce by $V(d)$ to a $c = 6$ case with $(r, s, t) = (4, 9, 5)$ since $s+t$ is even. For $(7, 10, 5)$ we may reduce by $U_1(4)$ to a $c = 5$ case with $(r, s, t) = (3, 6, 5)$.

When $d = 4$ and $c = 7$, only the shapes with $(r, s, t) = (7, 6, 3)$, $(5, 7, 3)$, $(3, 8, 3)$, $(6, 8, 2)$, and $(8, 7, 2)$ satisfy the constraints. For $(7, 6, 3)$ we may reduce by $V(d)$ to a $c = 6$ case of $(3, 6, 3)$. For $(3, 8, 3)$ use $R_{19} \vee U_1(4)$. For $(5, 7, 3)$ we may reduce by $U_1(4)$ to the $c = 5$ cases $(5, 3, 3)$. For $(6, 8, 2)$ we may reduce by $V(d)$ to a $c = 6$ case of $(2, 8, 2)$ since $s + t$ is even. For $(8, 7, 2)$ we may reduce by $U_1(4)$ to the $c = 5$ case $(8, 3, 2)$.

When $d = 4$ and $c = 8$, only the shape with $(r, s, t) = (7, 8, 3)$ satisfies the constraints. It can be obtained by $S_{10}(3) \vee U_1(2)$. Thus when d is even, all shapes are reducible to tableaux filled with less than or equal to six elements and all the cases with $c \leq 6$ were obtained previously.

8.6 Sufficiency for $c > 6$, d odd

When d is odd, the reduction techniques of Chapter 6 work to reduce a tableau to $c \leq 8$. As we've already constructed those tableaux with $c \leq 6$, we will focus on those with $c = 7$ or 8.

First, assume $t \geq d-1$. So long as $r \notin \{0,2,3,4,5,7\}$ we may use $P_1(d-1)$ to reduce to a $c-3$ case. If $t \geq d-1$ and $r \in \{0,2,3,4,5,7\}$ we may use $P_4(d-2,1,1)$ to reduce to a $c-3$ case unless $s \in \{0,2,3,4,5,7\}$. For $s = 3,5,7$ we may still reduce by $P_4(d-2,1,1)$ when t is odd. For $r = 0,2,4$, $s = 3,5,7$ and t even, the shape is not needed by Theorem 10. For $r = 3,5,7$ and $s = 3,5,7$ with t even, no shapes are possible for $c = 8$. When $c = 7$ we have $d \geq 5$ so we may use $P_3(d-3,1,0)$ to reduce to a $c = 4$ case with $s+t$ even. For $s = 0,2,4$, we only need those shapes with t even. For $r = 3,5,7$ we may reduce by $P_1(d-1)$ to a $c-3$ case. This leaves those cases with $s = 0,2,4$, $r = 0,2,4$ and t even.

When $c = 7$ there are no shapes having $s = 0,2,4$, $r = 0,2,4$ and t even as d is odd. For $c = 8$, these shapes are obtainable, depending on $d \pmod 3$. We list the appropriate tableaux in Table 8.20. This completes all cases with $t \geq d-1$

If $t < d-1$ and $s < d+4$, but $r = d+3$ or $r \geq d+5$ we may use $V(d)$ to reduce to a $c-1$ case. When $c = 8$, there are no valid shapes with $r \leq d+4$ and $t < d-1$, $s < d+4$. For $c = 7$ this is also true provided $d \geq 5$. When $d = 3$ we need $(r,s,t) = (6,6,1)$ which is reducible by $V(d)$ to a $c = 6$ case with $r = 3$.

If $t < d-1$ and $s \geq d+4$ we may use $U_1(d-1)$ to reduce to a $c-2$ case, provided $r \notin \{0,2,3,4,6\}$. Consider those cases with $r \in \{0,2,3,4,6\}$ If $s \geq 2d+5$ or $s = 2d+3$ we may use $U_2(d,d)$ to reduce to a $c-4$ case. For $s \leq 2d-1$ there are no shapes with $t \leq d-2$ and $r \leq 6$. If $s = 2d$, $2d+2$ or $2d+4$, then we may still reduce via $U_2(d,d)$, provided t is even (which always occurs if $r = 0,2,4$). Thus we need only consider those tableaux with $s = 2d+1$, or $s = 2d$, $2d+2$, $2d+4$ with $r = 3$ or 6.

For $c = 7$ and $s \geq 2d$, $s \neq 2d+1$ only $r = 6$, $s = 2d$, $t = d-2$ is possible. This can be obtained by $P_4(d-2,1,1) \vee U_5(2,d-2)$ provided $d \geq 5$. When $d = 3$ we may

(r,s)	d	Tableau
$(0,0)$	$d \equiv 0 \pmod 3$	$Q_2(\mathrm{z}) \vee Q_2(\mathrm{z})$
$(0,2)$	$d \equiv 2 \pmod 3$	$W_1(2)$
$(0,4)$	$d \equiv 1 \pmod 3$	$Q_1(\mathrm{z}-1,0,1,1) \vee Q_1(\mathrm{z}-1,0,1,1)$
$(2,0)$	$d \equiv 1 \pmod 3$	$W_1(0)$
$(2,2)$	$d \equiv 0 \pmod 3$, $d > 3$	$Q_2(\mathrm{z}) \vee Q_1(\mathrm{z}-2,1,1,0)$
	$d = 3$	$S_1(1) \vee U_1(2)$
$(2,4)$	$d \equiv 2 \pmod 3$, $d > 5$	$Q_2(\mathrm{z}) \vee Q_1(\mathrm{z}-2,0,2,2)$
	$d = 5$	$S_1(1) \vee U_1(4)$
$(4,0)$	$d \equiv 2 \pmod 3$	$Q_2(\mathrm{z}) \vee Q_2(\mathrm{z})$
$(4,2)$	$d \equiv 1 \pmod 3$	$Q_2(\mathrm{z}; \mathsf{x}=4) \vee Q_1(\mathrm{z}-1,0,1,1; \mathsf{x}=1)$
$(4,4)$	$d \equiv 0 \pmod 3$, $d > 9$	$Q_2(\mathrm{z}) \vee Q_1(\mathrm{z}-4,2,2,0)$
	$d = 9$	$Q_2(\mathrm{z}) \vee Q_5(4,4,1,0,2,1)$
	$d = 3$	$Q_1(1) \vee U_1(2) \vee U_1(2)$

Table 8.20: Exceptional r and s cases for $c = 8$.

reduced by $V(d)$ to a $c = 6$ case with $r = 3$. For $c = 8$ and $s \geq 2d$, $s \neq 2d+1$ only $s = 2d+4$, $r = 3$, $d = 5$ can occur. In that case, use $Q_5(2,1,2,0,1,1) \vee U_2(5,5)$.

This leaves those tableaux with $t < d-1$, $s = 2d+1$ and $r \in \{0,2,3,4,6\}$. When $r = 0,2,4$, we must have t odd, so in the $c = 8$ case there are no possible shapes. For $c = 7$, we get a valid shape only for $r = 4$, in which case we have $t = d-2$. For this use $U_1(d-1)$ to reduce to a $c = 5$ case with $r = 2$, $s = d+1$ which will still have $s+t$ even. For $r = 3$, there are no shapes satisfying $t < d-1$ and $s = 2d+1$ for either $c = 7$ or 8; similarly for $r = 6$. Thus all required shapes may be reduced to those filled with $c \leq 6$ elements.

8.7 Tableaux Disjointness

Our proof of Theorem 2 requires the tableaux we constructed to be disjoint. Since Lemma 3.4.9 showed that maximal tableaux are always disjoint, we need only be concerned with those tableaux which could not be put in maximal form, namely, U_8 and S_8.

Recall that $U_8 = \begin{array}{c} 4\,3\,3\,3\,3\,1\,2\,2\,4\,4\,4 \\ 1\,1\,1\,2\,2 \end{array}$ has $d = 4$ and $\omega_2 = (3, 2, 0, 0)$. When $d = 4$ the tableaux used in the reduction techniques are $P_1(4)$, $U_1(4)$, and $V(d)$. These are the only tableaux that would be joined with U_8, so it suffices to show U_8 is disjoint from these tableaux. Note that the weights of these tableaux consist only of 4's and 0's.

If U_8 were not disjoint from these tableaux, then there is some weight assignment of U_8, not equivalent to $\omega_2 = (3, 2, 0, 0)$ which uses a weight from at least one of these tableaux. However, since the only additional weights we may use are 4's and 0's, there is no way to have a weight assignment of length 5 using 0's, 2's, 3's, and 4's without being equivalent to $(3, 2, 0, 0)$. Hence the weights are disjoint from U_8.

Recall that $S_8 = \begin{array}{ccccccccccc} & & d-4 & & d-2 & d-4 & & 2 & & & \\ \hline 6 & 6 & 6 & 3 & 2 & 1 & 2 & 3 & 6 & 6 & 2 \\ 3 & 4 & 3 & 4 & 4 & 5 & 5 & 5 & 1 & 5 & 1 \\ 1 & 1 & & & & & & & & & \end{array}$ with $\omega_{2,3} = \left(\begin{smallmatrix} 2 & 0 & d-3 & d & d & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ and $d \geq 5$. When d is even, the tableaux used in the reduction techniques are $P_1(d)$, $U_1(d)$, and $V(d)$. When d is odd, the tableaux used in the reduction techniques are $S_1(d)$, $U_2(d)$, whose weights are 0's and d's, $P_1(d-1)$, $U_1(d-1)$ whose weights are 0's and $d-1$'s, and $P_4(d-2, 1, 1)$ with $\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 1 \\ 0 & 0 & d-2 \end{smallmatrix}\right)$. Hence it suffices to show S_8 disjoint from these tableaux.

Now any weight assigned to S_8 must have $\lambda_3 = 2$. Since $d \geq 5$, there are no weights other of the listed tableaux than the weight $\binom{2}{2}$ of S_8 for which this is possible. This means if S_8 were not disjoint from these tableaux there would be a weight assignment of S_8 of the form $\left(\begin{smallmatrix} 2 & * & * & * & * & * \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ which is not equivalent to $\left(\begin{smallmatrix} 2 & 0 & d-3 & d & d & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$. The only weights we may use for the $*$'s are the weights of the listed tableaux, namely, 0's d's, $d-1$'s and $d-3$. Since the $*$'s sum to $3d-3$, they would have to be three $d-1$'s and two

0's. That is, the weights of $U_1(d-1)$ or $P_1(d-1)$. Then the weights of S_8 would need to be assigned to other tableaux, so a weight of d would need to be assigned to either $U_1(d-1)$ or $P_1(d-1)$, which is not possible as these tableaux are maximal. Hence there is no other weight assignment for S_8 and so the tableau is disjoint as required. Thus all the tableaux constructed in the proof of Theorem 2 are disjoint as desired.

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