Generalized Foulkes' Conjecture and Tableaux Construction

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Chapter 7

Tableaux Construction

The proof of Theorem 2 in Chapter 6 requires the construction of non-zero maximal tableaux with $c \leq 8$ for the shapes discussed in Theorem 10. In this chapter we construct all the necessary tableaux and show they are both non-zero and maximal. Some basic properties of maximality are listed in Section 7.1 and are used throughout our construction. We construct those tableaux with two rows in Section 7.2. In Sections 7.3, 7.4, 7.5, and 7.6 we construction the necessary tableaux with $c = 3$, 4, 5, and 6, respectively. Section 7.7 contains additional tableaux need when $c = 7$ or 8.

7.1 Maximality of Tableaux

Maximality, as discussed in Lemma 3.4.9, is an important property of the tableaux we construct to prove Theorem 2. Given certain conditions of a weight, it is easy to verify that a tableau is in maximal form. We discuss maximality here in general, in order to simplify the proof of maximal form for the specific tableaux we construct in the next sections.

Recall that a tableau T is in maximal form if $\mathbf{q}_T \neq 0$ by weight-set counting on $\omega(T)$ and $\omega(T)$ is the largest weight of all $\omega(\tau T)$ for $\tau \in C_T$. All tableaux that we consider in the next sections are shown to be non-zero by weight-set counting prior to addressing the maximality issue. As such, we only consider the weight condition here. To summarize the basic conditions of Definition 3.4.2, a tableau weight is maximal if it has the largest weight possible for row three, and given this, the largest weight possible for row two. When multiple weights satisfy this, the weight where rows two and three have elements in common (e.g., the element 4 appears in both rows) is considered larger.

In general, to determine the maximal form of a tableau, we start with a given filling of T and use column operations to produce tableaux τT with larger weights. The following procedure provides an overview of how to determine the maximality of a filling.

- First maximize the weight of row three. This usually involves having a weight of d for as many elements as possible. (This is discussed more extensively in Lemma 7.1.2.) There may be multiple different fillings having the same maximum weight for row three. All such fillings should be considered for the next step.

- Now maximize the weight of row two, given the filling(s) of row three determined previously. To do this, determine the largest possible row two weight of each element, provided those elements assigned to row three are not used. For a given element this is equivalent to the number of copies in the body of T minus the number of copies used in row three. Fill row two with the element having the largest weight, then repeat the procedure with the remaining elements and positions. (This is summarized for weights in Lemma 7.1.4.) There may be multiple such fillings having the largest weight.

- If there are multiple fillings after step two that have different generic weights, choose the filling in which an element in row three has the largest weight in row two. (The weight of their common elements is maximized.) There may not be a unique such filling, but all such maximal fillings will have the same generic weight. As such these fillings will differ on by an action of $\mathcal{S}_a$.

Example 7.1.1. Consider

$$T = Q_2 = \begin{array}{c|ccc} \textsc{z}+\mathsf{x} & \textsc{z} & \textsc{z} & \textsc{a} \\ \hline 3 & 3 & 2 & 3 \\ 4 & 2 & 4 & 4 \\ 1 & 1 & 1 & 2 \end{array} \qquad \omega_{2,3}(T) = \left(\begin{smallmatrix} 0 & \textsc{z} & 0 & d-\textsc{z}+\textsc{a}+\mathsf{x} \\ d & \textsc{a} & 0 & 0 \end{smallmatrix}\right)$$

where $\textsc{z} = \frac{d-\mathsf{x}}{3}$, $\textsc{a} < \textsc{z}$, and some conditions on $\textsc{z}, \textsc{a}$, and x to insure $\mathbf{q}_T$ is non-zero.

First we want to maximize row three. Since this occurs when the non-zero weights of row three are d and $\textsc{a}$, we check to see which elements can have weight d in row three. Here, the only option is the element 1. Since any of the remaining elements can have a weight of $\textsc{a}$, we leave that column unfilled for now. Thus the maximal (third row) form for T is $111*$.

To determine the maximal second row form, we consider how many of each element is available given the third row is partially determined. There are zero 1's available, $2\textsc{z}+\textsc{a}$ 2's available, and $2\textsc{z}+\textsc{a}+\mathsf{x}$ 3's and 4's available. Hence we can maximally fill the second row with either 3's or 4's. Since the tableau is symmetric in 3's and 4's, we will use 4's without loss of generality. Thus we get a maximal (second and third row) form of $\begin{smallmatrix} 4 & * & 4 & 4 \\ 1 & 1 & 1 & * \end{smallmatrix}$.

This form gives rise to two different generic weights depending on whether the $*$'s are the same. These generic weights are $\left(\begin{smallmatrix} 0 & \textsc{z} & d-\textsc{z}+\textsc{a} & 0 \\ d & \textsc{a} & 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 & d-\textsc{z}+\textsc{a} & \textsc{z} \\ d & \textsc{a} & 0 & 0 \end{smallmatrix}\right)$. The first one, corresponding to the same element for both $*$'s, is larger. Hence a maximal (second and third row) form is $\begin{smallmatrix} 4 & 2 & 4 & 4 \\ 1 & 1 & 1 & 2 \end{smallmatrix}$. Note that this filling is not unique. We could have used $\begin{smallmatrix} 4 & 3 & 4 & 4 \\ 1 & 1 & 1 & 3 \end{smallmatrix}$, $\begin{smallmatrix} 3 & 4 & 3 & 3 \\ 1 & 1 & 1 & 4 \end{smallmatrix}$, or $\begin{smallmatrix} 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \end{smallmatrix}$ instead. All would be in maximal form.

Some fillings can be seen as maximal strictly by examining the weight of the tableau.

Lemma 7.1.2. If a row has at most one weight not equal to 0 or d, then its row weight is maximal.

This follows directly from the ordering on weights and the limit of d copies of an element in a tableau. If we call the weight not equal to 0 or d the non-d weight for the row, we get the following result on tableau maximality.

Lemma 7.1.3. A tableau is maximal if rows two and three satisfy Lemma 7.1.2 and either their non-d weights come from the same element or the sum of the non-d weights is greater than d.

Proof. Given a tableau satisfying Lemma 7.1.2 for rows two and three, its maximal generic weight must be either $\left(\begin{smallmatrix} 0 & \text{B} & d & 0 \\ d & \text{A} & 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 0 & d & \text{B} \\ d & \text{A} & 0 & 0 \end{smallmatrix}\right)$ with the appropriate number of 0 and d weights. By our ordering, the first weight is strictly larger than the second weight. Hence when the non-d weights come from the same element the tableau is maximal. If $\text{A} + \text{B}$ is greater than d, the first weight is not possible and so the second one is maximal. □

This lemma is directly applicable to the tableau weights. A more generalized form of this, depending on the actual filling, is also useful. The following lemma reflects the technique used in Example 7.1.1.

Lemma 7.1.4. A row is maximal if all except one non-zero weight corresponds to the largest weights possible for any elements.

This lemma is a generalization of Lemma 7.1.2, where the maximum weight for each element is no longer d. Using this in Lemma 7.1.3 gives:

Lemma 7.1.5. A tableau is maximal if row three satisfies Lemma 7.1.2, row two satisfies Lemma 7.1.4, and the non-d weight of row three has the largest possible weight in row two of all such weights satisfying Lemma 7.1.4.

Example 7.1.1 represents an appropriate use of Lemma 7.1.5. The weight of row three is $(d, \text{A}, 0, 0)$, clearly satisfying Lemma 7.1.2. The weight of row two is $(0, \text{Z}, 0, d - \text{Z} + \text{A} + \mathsf{x})$. That satisfies Lemma 7.1.4, since we checked that $d - \text{Z} + \text{A} + \mathsf{x}$ was the largest weight possible. Finally, $\omega_3(2) = \text{A}$ is the non-d weight of row three. Since $\omega_2(2) = \text{Z}$ is larger than $\omega_2(2) = 0$, the weight is maximal by Lemma 7.1.5.

In general, for the application of Lemma 7.1.5 it is to check that Lemma 7.1.2 and Lemma 7.1.4 apply and then to show that the non-d weight cannot be assigned a larger row two weight without changing the generic weight of row two. In particular the non-d weight conditions of Lemma 7.1.5 are satisfied if all extra (non-third row) copies an element in row three are contained in row two. We will refer to these lemmas for maximality justification of the tableaux constructed in the next section.

7.2 Tableaux for Two Row Partitions

In this section we construct all the two row tableaux needed for the proof of Theorem 2. For many of these constructions, the parity of d is relevant. Recall the notation:

$$d^* = \begin{cases} d & \text{if } d \text{ is even} \\ d-1 & \text{if } d \text{ is odd} \end{cases}$$

Since we are constructing tableaux for many partitions, we will not use a fixed c. However, for every element that is listed in the tableau, we assume it occurs d times, filling out the tail as needed using only those numbers in the body of T.

<u>Tableau U_1</u>

$$U_1 = \begin{array}{ccc} \overset{\textsc{a}}{\overline{1}} & \overset{d-\textsc{a}}{\overline{1}} & \overset{d-\textsc{a}}{\overline{2}} \\ 2 & & \end{array} \sim \begin{array}{c} \overset{\textsc{a}}{\overline{1}} \\ 2 \end{array} \qquad \begin{array}{l} \textsc{a} \text{ even} \\ \textsc{a} \leq d \end{array}$$

$$\omega_2(U_1) = (0, \textsc{a})$$

$$\lambda = [2d - \textsc{a}, \textsc{a}]$$

$$r = 2d - 2\textsc{a},\ s = \textsc{a},\ t = 0$$

We showed U_1 non-zero in the proof of Theorem 1.

Maximality: This tableau is maximal by Lemma 7.1.4.

Tableau U_2

$$U_2 = \begin{array}{cccc} \text{A} & \text{A} & \text{B} & \text{B} \\ \hline 1 & 3 & 1 & 3 \\ 2 & 4 & 4 & 2 \end{array} \qquad \begin{array}{l} \text{A} + \text{B} \leq d \\ \text{A},\ \text{B} > 0 \end{array}$$

$$\omega_2 = (0, \text{A} + \text{B}, 0, \text{A} + \text{B})$$

$$\lambda = [4d - 2\text{A} - 2\text{B}, 2\text{A} + 2\text{B}]$$

$$r = 4d - 4\text{A} - 4\text{B},\ s = 2\text{A} + 2\text{B},\ t = 0$$

We showed U_2 non-zero in the proof of Theorem 1.

Maximality: This tableau is maximal by Lemma 7.1.4.

Tableau U_3

$$U_3 = \begin{array}{ccc} \text{A} & \text{B} & \text{B} \\ \hline 1 & 1 & 2 \\ 2 & 3 & 3 \end{array} \qquad \begin{array}{l} \text{A even} \\ \text{A} + \text{B} \leq d \\ 0 < 2\text{B} \leq d \\ \text{B} \geq \text{A} \end{array}$$

$$\omega_2 = (0, \text{A}, 2\text{B})$$

$$\lambda = [3d - \text{A} - 2\text{B}, \text{A} + 2\text{B}]$$

$$r = 3d - 2\text{A} - 4\text{B},\ s = \text{A} + 2\text{B},\ t = 0$$

If $\text{B} > \text{A}$, any valid weight assignment must have $\omega_2(3) = 2\text{B}$. Hence $(1, 2, 3)$ with sign 1 and $(2, 1, 3)$ with sign $(-1)^{\text{A}}$ are the only possible weight assignments. Thus U_3 is non-zero. If $\text{A} = \text{B}$, we may also have weight assignments: $(1, 3, 2)$ with sign $(-1)^{\text{B}}$; $(2, 1, 3)$ with sign $(-1)^{\text{A}+\text{B}}$; $(3, 1, 2)$ with sign $(-1)^{2\text{B}}$; and $(3, 2, 1)$ with sign $(-1)^{\text{A}+2\text{B}}$. Since $\text{A} = \text{B}$ and A is even, these are all positive. Hence U_3 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.4 since $2\text{B} \geq \text{A} + \text{B}$.

Tableau U_4

$$U_4 = \begin{array}{cc} \text{A} & \text{B} \\ \hline 2 & 3 \\ 1 & 1 \end{array} \qquad \text{A} \geq \text{B} > 0$$

$$\omega_2 = (\text{A} + \text{B}, 0, 0)$$

$$\lambda = [3d - \text{A} - \text{B}, \text{A} + \text{B}]$$

$$r = 3d - 2\text{A} - 2\text{B},\ s = \text{A} + \text{B},\ t = 0$$

Since only the element 1 can have $\omega_2 = \text{A} + \text{B}$, there is exactly one valid weight assignment, $(1, 2, 3)$. Thus U_4 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.4.

Tableau U_5

$$U_5 = \begin{array}{cccc} \text{A} & \text{B} - 2 & \text{A} - 1 & \text{B} \\ \hline 1 & 1 & 4 & 4 \\ 2 & 3 & 3 & 2 \end{array} \qquad \begin{array}{l} \text{A} \geq 2 \\ \text{B} \geq 3 \\ \text{A} + \text{B} \leq d \end{array}$$

$$\omega_2 = (0, \text{A} + \text{B}, \text{A} + \text{B} - 3, 0)$$

$$\lambda = [4d - 2\text{A} - 2\text{B} + 3, 2\text{A} + 2\text{B} - 3]$$

$$r = 4d - 4\text{A} - 4\text{B} + 6,\ s = 2\text{A} + 2\text{B} - 3,\ t = 0$$

Only the element 2 may have $\omega_2(2)$=A+B. Since $\text{A} \geq 2$ and $\text{B} \geq 3$, we must have $\omega_2(3) = \text{A} + \text{B} - 3$. Hence $(1, 2, 3, 4)$ is the only valid weight assignment, so U_5 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.4 since A+B is the largest weight possible.

Tableau U_6

$$U_6 = \begin{array}{l} \scriptstyle A\ B\ C\qquad D \\ \hline 1\ 3\ 5\ 5\ 5\ 3\ 1\ 3\ 1\ 3\ 5 \\ 2\ 4\ 4\ 2\ 4\ 5\ 5\ 2 \end{array} \qquad \begin{array}{l} \mathrm{A} = d-2 \\ \mathrm{B} = \frac{d+1}{2} \\ \mathrm{C} = \frac{d-3}{2} \\ \mathrm{D} = \frac{d-5}{2} \end{array} \qquad \begin{array}{l} d \equiv 1 \pmod 2 \\ d > 5 \end{array}$$

$\omega_2 = (0, d, 0, d, \frac{d-3}{2})$

$\lambda = [3d - \frac{d-3}{2}, 2d + \frac{d+3}{2}]$

$r = 3,\ s = 2d + \frac{d-3}{2},\ t = 0,$

Only the elements 2 and 4 may have $\omega_2 = d$. If $\omega_2(5) = 0$, then $\omega_2(1 \text{ and } 3) > 0$. Hence $(1, 2, 3, 4, 5)$ is the only valid weight assignment and so U_6 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.2.

Tableau U_7

$$U_7 = \begin{array}{l} 2\quad 2\ 3\quad 2 \\ \hline 4\ 3\ 5\ 5\ 3\ 4\ 2\ 3\ 4 \\ 1\ 1\ 1\ 2\ 2\ 3 \end{array} \qquad d = 5$$

$\omega_2 = (5, 4, 2, 0, 0)$

$\lambda = [14, 11]$

$r = 3,\ s = 11,\ t = 0,$

The valid weight assignments are $(5, 4, 2, 0, 0, 0)$, $(5, 4, 0, 2, 0)$, $(2, 0, 4, 0, 5)$, $(0, 0, 2, 4, 5)$, and $(0, 0, 4, 2, 5)$. Since there are an odd number of weight assignments, this tableau is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.4.

Tableau U_8

$$U_8 = \begin{array}{l} 4\ 3\ 3\ 3\ 3\ 1\ 2\ 2\ 4\ 4\ 4 \\ 1\ 1\ 1\ 2\ 2 \end{array} \qquad d = 4$$

$\omega_2 = (3, 2, 0, 0)$

$\lambda = [11, 5]$

$r = 6,\ s = 5,\ t = 0$

Then only valid weight assignments are $(1, 2, 3, 4)$, with sign 1, $(1, 3, 2, 4)$, with sign $(-1)^2$, and $(3, 1, 2, 4)$ with sign $(-1)^3$. Hence the weight sum is 1 and U_8 is non-zero.

Maximality: This tableau is not maximal since $\omega_2 = (0, 0, 4, 1)$ is larger. However, in $\mathbf{q}_T$, the weight $(0, 0, 4, 1)$ always cancels. This tableau cannot be put in maximal form, hence we will need to prove directly that it is disjoint from the requisite tableaux. This will be done in Section 8.7.

7.3 Tableaux for $c = 3$

We know by Theorem 9 which tableaux are non-zero for $c = 3$. However, using Theorem 8 on tableaux requires the tableaux to be non-zero by weight-set counting on $\omega(T)$. We also want the tableaux to be maximal, in order to obtain the disjointness of Lemma 3.4.9. Here we will list the tableaux used, briefly showing they are non-zero and maximal. For all these tableaux, any valid weight assignment corresponds to a unique tableau, so we will not explicitly state how many tableaux correspond to each weight assignment.

Tableau P_1

$$P_1 = \begin{array}{c} \text{A} \\ \hline 1 \\ 2 \\ 3 \end{array} \qquad \begin{array}{l} 0 \leq \text{A} \leq d \\ \text{A even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A} & 0 \\ 0 & 0 & \text{A} \end{smallmatrix} \right)$$

$$\lambda = [3d - 2\text{A}, \text{A}, \text{A}]$$

$$r = 3d - 3\text{A},\ s = 0,\ t = \text{A}$$

P_1 is non-zero by the Lemma 3.2.6.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau P_2

$$P_2 = \begin{array}{cc} \text{A} & \text{B} \\ \hline 1 & 1 \\ 2 & 2 \\ 3 & \end{array} \qquad \begin{array}{l} \text{A} + \text{B} \leq d \\ \text{A, B even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A+B} & 0 \\ 0 & 0 & \text{A} \end{smallmatrix} \right)$$

$$\lambda = [3d - 2\text{A} - \text{B}, \text{A+B}, \text{A}]$$

$$r = 3d - 3\text{A} - 2\text{B},\ s = \text{B},\ t = \text{A}$$

Only the elements 1 and 2 may have $\omega_2 = \text{A}+\text{B}$. Hence the valid weight assignments are: $(1,2,3)$ with sign 1; $(3,2,1)$ with sign $(-1)^{\text{A}}$; $(2,1,3)$ with sign $(-1)^{\text{A}+\text{B}}$; and $(3,1,2)$ with sign $(-1)^{\text{B}}$. Since A and B are even, this weight sum is 4 and $\mathbf{q}_{P_2} \neq 0$.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau P_3

$$P_3 = \begin{array}{cccc} \text{A} & \text{B}+1 & \text{B} & \text{C} \\ \hline 2 & 2 & 3 & 2 \\ 1 & 1 & 1 & 3 \\ 3 & & & \end{array} \qquad \begin{array}{l} 0 \leq \text{A} \leq d \\ 2\text{B}+1 \leq d - \text{A} \\ 0 \leq \text{C} < \text{B} \\ \text{A}+\text{C} \text{ even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A}+2\text{B}+1 & \text{C} \\ 0 & 0 & \text{A} \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\text{A} - 2\text{B} - \text{C} - 1, \text{A} + 2\text{B} + \text{C} + 1, \text{A}]$$

$$r = 3d - 3\text{A} - 4\text{B} - 2\text{C} - 2,\ s = 2\text{B} + \text{C} + 1,\ t = \text{A}$$

Only the element 1 may have $\omega_2 = \text{A}+2\text{B}+1$. Hence the valid weight assignments are $(1,2,3)$ with sign 1 and $(1,3,2)$ with sign $(-1)^{\text{A}+\text{C}}$. Since $\text{A}+\text{C}$ is even, this weight sum is positive and hence P_3 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau P_4

$$P_4 = \begin{array}{cccc} \text{A} & \text{B} & \text{B} & \text{C} \\ \hline 2 & 2 & 3 & 2 \\ 1 & 1 & 1 & 3 \\ 3 & & & \end{array} \qquad \begin{array}{l} 0 \leq \text{A} \leq d \\ 2\text{B} \leq d - \text{A} \\ 0 \leq \text{C} \leq \text{B} \\ \text{A}+\text{C} \text{ even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A}+2\text{B} & \text{C} \\ 0 & 0 & \text{A} \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\text{A} - 2\text{B} - \text{C}, \text{A} + 2\text{B} + \text{C}, \text{A}]$$

$$r = 3d - 3\text{A} - 4\text{B} - 2\text{C},\ s = 2\text{B} + \text{C},\ t = \text{A}$$

Unless $\text{C} = \text{B}$, the only valid weight assignments are $(1,2,3)$ with sign 1 and $(1,3,2)$ with sign $(-1)^{\text{A}+\text{C}} = 1$. If $\text{B} = \text{C}$ the tableau is symmetric in 1, 2, and 3. Hence we also get the weight assignments: $(2,1,3)$ with sign $(-1)^{\text{A}+\text{B}+\text{B}+\text{C}} = 1$; $(2,3,1)$ with sign $(-1)^{2\text{B}} = 1$; $(3,2,1)$ with sign $(-1)^{\text{A}+\text{B}} = 1$; and $(3,1,2)$ with sign $(-1)^{2\text{B}} = 1$. Thus the weight sum is always positive and P_4 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

7.4 Tableaux for $c = 4$

Tableau Q_1

$$Q_1 = \begin{array}{c|cccccc} \text{Z+X} & \text{Z} & \text{Z} & \text{A} & \text{B} & \text{C} & \text{D} \\ \hline 3 & 3 & 2 & 3 & 3 & 2 & 3 \\ 4 & 2 & 4 & 4 & 4 & 4 & 2 \\ 1 & 1 & 1 & 2 & & & \end{array}$$

$0 \le \text{A} < \text{Z}$

$\text{B} + \text{X} \ge \text{D}$

$\text{C} \ge \text{D}$

$\text{B} + \text{C} \le \text{Z} - \text{A}$

$\text{Z} + \text{A} + \text{D}$ even

$\text{C} + \text{D}$ even, if $\text{D} = \text{B} + \text{X}$

$\text{D} + \text{B} + \text{X}$ even, if $\text{D} = \text{C}$

$\text{Z} = \frac{d-\text{X}}{3}$

$d \equiv \text{X} \pmod 3$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{Z+D} & 0 & 2\text{Z+X+A+B+C} \\ d & \text{A} & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [2d - 2\text{A} - \text{B} - \text{C} - \text{D}, d + \text{A} + \text{B} + \text{C} + \text{D}, d + \text{A}]$

$r = d - 3\text{A} - 2\text{B} - 2\text{C} - 2\text{D}$, $s = \text{B} + \text{C} + \text{D}$, $t = d + \text{A}$

Since $\text{A} < \text{Z}$, any valid weight assignment must have $\omega_3(1) = d$. We must have $\omega_2(4) = 2\text{Z}+\text{X}+\text{A}+\text{B}+\text{C}$ unless $\text{D} = \text{C}$ or $\text{D} = \text{B}+\text{X}$. If $\text{D} = \text{C}$, then we may also have $\omega_2(3) = 2\text{Z}+\text{X}+\text{A}+\text{B}+\text{C}$. If $\text{D} = \text{B}+\text{X}$, then we may also have $\omega_2(2) = 2\text{Z}+\text{X}+\text{A}+\text{B}+\text{C}$. Given these restrictions, we list the valid weight assignments in the table below, along with the sign corresponding to that assignment. Note that there is a unique tableau associated to each of these assignments.

Assignment	Sign	Condition
$(1,2,3,4)$	1	
$(1,3,2,4)$	$(-1)^{\text{Z+A+D}}$	
$(1,2,4,3)$	$(-1)^{\text{Z+X+A+B+C+D}}$	$\text{D} = \text{C}$
$(1,4,2,3)$	$(-1)^{2\text{Z+X+B+D}}$	$\text{D} = \text{C}$
$(1,3,4,2)$	$(-1)^{2\text{Z+X+B+C}}$	$\text{D} = \text{B} + \text{X}$
$(1,4,3,2)$	$(-1)^{\text{Z+A+C}}$	$\text{D} = \text{B} + \text{X}$

Given the parity conditions on the parameters, the sign of these terms reduces to 1 in all cases. Hence the weight sum is positive and Q_1 is non-zero.

Maximality: Because $\text{C} \ge \text{D}$ and $\text{B} + \text{X} \ge \text{D}$, the weight $2\text{Z} + \text{X} + \text{A} + \text{B} + \text{C}$ of 4 in row two is the largest possible weight given the Maximality of row three by

Lemma 7.1.2. Hence this tableau is maximal by Lemma 7.1.5.

Tableau Q_2

$$Q_2 = \begin{array}{cccc} \mathrm{Z}+\mathsf{x} & \mathrm{Z} & \mathrm{Z} & \mathrm{A} \\ \hline 3 & 3 & 2 & 3 \\ 4 & 2 & 4 & 4 \\ 1 & 1 & 1 & 2 \end{array} \qquad \begin{array}{l} 0 < \mathrm{A} \leq \mathrm{Z} \\ \mathrm{Z} + \mathrm{A} \text{ even} \end{array} \qquad \begin{array}{l} \mathrm{Z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ \mathsf{x} \in \{0, 2, 4\} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \mathrm{Z} & 0 & d-\mathrm{Z}+\mathrm{A} \\ d & \mathrm{A} & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [2d - 2\mathrm{A}, d + \mathrm{A}, d + \mathrm{A}]$$

$$r = d - 3\mathrm{A},\ s = 0,\ t = d + \mathrm{A}$$

For $\mathrm{A} < \mathrm{Z}$ any valid weight assignment must $\omega_3(1) = d$. The tableau is symmetric in 3 and 4, as well as 2 if $\mathsf{x} = 0$. This gives the following signed weight table:

Assignment	Sign	Condition
$(1, 2, 3, 4)$	1	
$(1, 3, 2, 4)$	$(-1)^{\mathrm{Z}+\mathrm{A}}$	
$(1, 2, 4, 3)$	$(-1)^{3\mathrm{Z}+\mathsf{x}+\mathrm{A}}$	
$(1, 4, 2, 3)$	$(-1)^{2\mathrm{Z}+\mathsf{x}}$	
$(1, 3, 4, 2)$	$(-1)^{2\mathrm{Z}}$	$\mathsf{x} = 0$
$(1, 4, 3, 2)$	$(-1)^{\mathrm{Z}+\mathrm{A}}$	$\mathsf{x} = 0$

Since x is even, all the terms are positive and Q_2 is non-zero. If $\mathrm{Z} = \mathrm{A}$ the tableau is symmetric in 1, 3, and 4, as well as 2 if $\mathsf{x} = 0$. We get all the weight assignments listed above, in addition to those obtained by interchanging rows or allowing $\omega_{2,3}(1) = \left(\begin{smallmatrix} \mathrm{Z} \\ \mathrm{Z} \end{smallmatrix}\right)$. Interchanging rows has sign $(-1)^{d+\mathrm{Z}} = 1$ since x is even. The other possibility changes the sign by $(-1)^{\mathrm{Z}+\mathrm{A}} = 1$. Hence all the terms are positive and Q_2 is non-zero.

Maximality: This tableau is maximal as shown in Example 7.1.1.

Tableau Q_3

$$Q_3 = \begin{array}{ccc} \text{A} & \text{B} & \text{B} \\ \hline 1 & 4 & 4 \\ 2 & 3 & 2 \\ 3 & & \end{array} \qquad \begin{array}{l} 0 < \text{A} \leq d - \text{B} \\ 2\text{B} \leq d \\ \text{A} \geq \text{B} \\ \text{A even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A}+\text{B} & \text{B} & 0 \\ 0 & 0 & \text{A} & 0 \end{smallmatrix}\right)$$

$$\lambda = [4d - 2\text{A} - 2\text{B}, \text{A} + 2\text{B}, \text{A}]$$

$$r = 4d - 3\text{A} - 4\text{B},\ s = 2\text{B},\ t = \text{A}$$

Since $\text{A} \geq \text{B}$ only the elements 2 and 3 may have $\omega_2 = \text{A} + \text{B}$. Hence the only valid weight assignments are $(1, 2, 3, 4)$ with sign 1 and $(1, 3, 2, 4)$ with sign $(-1)^{\text{A}} = 1$. Thus all the valid weight assignments are positive and therefore Q_3 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau Q_4

$$Q_4 = \begin{array}{cccccc} \text{A} & \text{A} & \text{B} & \text{C} & \text{C} & \text{D} \\ \hline 3 & 4 & 1 & 3 & 1 & 3 \\ 2 & 2 & 2 & 2 & 4 & 4 \\ 1 & 1 & & & & \end{array} \qquad \begin{array}{l} 2\text{A} + \text{B} + \text{C} = d \\ \text{A} + \text{C} + \text{D} \leq d \\ \text{A}, \text{B}, \text{C}, \text{D} > 0 \\ \text{B} + \text{D even} \\ \text{A even, if } \text{A} + \text{C} + \text{D} = d \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & \text{C}+\text{D} \\ 2\text{A} & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\text{A} - \text{C} - \text{D}, d + \text{C} + \text{D}, 2\text{A}]$$

$$r = 2d - 2\text{A} - 2\text{C} - 2\text{D},\ s = \text{B} + 2\text{C} + \text{D},\ t = 2\text{A}$$

Any weight assignment must have either $\omega_3(1) = 2\text{A}$ or $\omega_3(2) = 2\text{A}$. Unless $\text{A} + \text{C} + \text{D} = d$, only the elements 1 and 2 may have $\omega_2 = d$. This gives the following weight table:

Assignment	Sign	Condition
$(1,2,3,4)$	1	
$(2,1,4,3)$	$(-1)^{\text{B}+\text{D}}$	
$(2,3,4,1)$	$(-1)^{\text{A}+\text{B}+\text{D}}$	$\text{A}+\text{C}+\text{D}=d$
$(1,4,3,2)$	$(-1)^{\text{A}}$	$\text{A}+\text{C}+\text{D}=d$

By our parity constraints, the weight sum is always positive. Hence Q_4 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau Q_5

$$Q_5 = \begin{array}{cccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} \\ \hline 4 & 3 & 3 & 4 & 3 & 1 \\ 2 & 2 & 2 & 2 & 4 & 4 \\ 1 & 1 & & & & \end{array}$$

$\text{A}+\text{B}+\text{C}+\text{D}=d$

$\text{A},\text{B},\text{E},\text{F}>0$

$\text{A}+\text{B}+\text{F}\leq d$

$\text{B}+\text{C}+\text{E}<d$

$\text{A}+\text{D}+\text{E}+\text{F}<d$

If $\text{A}+\text{B}+\text{F}=d$ then:

either $\text{C},\text{D}>0$

or $\text{D}=0$ and $d+\text{C}+\text{E}$ even

or $\text{C}=0$ and $d+\text{D}$ even

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & \text{E}+\text{F} \\ \text{A}+\text{B} & 0 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [3d-\text{A}-\text{B}-\text{E}-\text{F}, d+\text{E}+\text{F}, \text{A}+\text{B}]$

$r = 2d-\text{A}-\text{B}-2\text{E}-2\text{F}$, $s=\text{C}+\text{D}+\text{E}+\text{F}$, $t=\text{A}+\text{B}$

Unless $\text{A}+\text{B}+\text{F}=d$, we must have $\omega_2(2)=d$. Then since E, $\text{F}>0$ we have $\omega_2(4)=\text{E}+\text{F}$. As $\text{A},\text{B}>0$, we have $\omega_3(1)=\text{A}+\text{B}$. Therefore no other valid weight assignments exist.

If $\text{A}+\text{B}+\text{F}=d$, then it is possible to have $\omega_2(1)=d$. However, unless C or D equals zero, there is no element with $\omega_2=\text{E}+\text{F}$. Hence no such weight assignment can exist. If $\text{D}=0$ we can have the weight assignment $(2,1,4,3)$ which has sign $(-1)^{d+\text{C}+\text{E}}=1$. If $\text{C}=0$ then we can have the weight assignment $(2,1,3,4)$ which has sign $(-1)^{d+\text{D}}=1$. Thus, in either case, the weight sum is positive and Q_5 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. As no other fillings may have this row two weight and common elements between rows two and three, the tableau is maximal.

Tableau Q_6

$$Q_6 = \begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & & \\ \hline 1 & 4 & 1 & 1 & 4 \\ 2 & 2 & 3 & 2 & 3 \\ 3 & & & & \end{array} \qquad \begin{array}{l} \mathrm{A} > 0 \\ \mathrm{A} + \mathrm{B} \leq d - 1 \\ \mathrm{B} \geq \mathrm{C} \\ \mathrm{A} \text{ even, if } \mathrm{B} = \mathrm{C} \\ \mathrm{B}, \mathrm{C} \text{ not both } 0 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \mathrm{A}+\mathrm{B}+1 & \mathrm{C}+1 & 0 \\ 0 & 0 & \mathrm{A} & 0 \end{smallmatrix}\right)$$

$$\lambda = [4d - 2\mathrm{A} - \mathrm{B} - \mathrm{C} - 2, \mathrm{A} + \mathrm{B} + \mathrm{C} + 2, \mathrm{A}]$$

$$r = 4d - 3\mathrm{A} - 2\mathrm{B} - 2\mathrm{C} - 4,\ s = \mathrm{B} + \mathrm{C} + 2,\ t = \mathrm{A}$$

Unless $\mathrm{B} = \mathrm{C}$, we must have $\omega_2(2) = \mathrm{A} + \mathrm{B} + 1$. This force $\omega_{2,3}(3) = \left(\begin{smallmatrix} \mathrm{C}+1 \\ \mathrm{A} \end{smallmatrix}\right)$ and hence there are no other valid weight assignments.

When $\mathrm{C}{=}\mathrm{B}$ then the elements 1, 2, or 3 may have $\omega_2 = \mathrm{A}{+}\mathrm{B}{+}1$. If $\omega_2(2) = \mathrm{A}{+}\mathrm{B}{+}1$ then only the element 3 may have $\omega_{2,3} = \left(\begin{smallmatrix} \mathrm{C}+1 \\ \mathrm{A} \end{smallmatrix}\right)$. If $\omega_2(1) = \mathrm{A} + \mathrm{B} + 1$, then a valid weight assignment exists only for $\mathrm{B} = 0$. In this case $\omega_2(3) = \mathrm{C}{+}1$. If $\omega_2(3) = \mathrm{A}{+}\mathrm{B}{+}1$ then we may have $\omega_2(2) = \mathrm{C} + 1$ or, if $\mathrm{C} = 0$, $\omega_2(1) = \mathrm{C} + 1$. Since these conditions are subject to $\mathrm{B} = \mathrm{C}$ and B and C are never simultaneously zero, the only valid weight assignments are $(1, 2, 3, 4)$ with sign 1 and $(1, 3, 2, 4)$ with sign $(-1)^{\mathrm{A}} = 1$. Hence the weight sum is positive and Q_6 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau Q_7

$$Q_7 = \begin{array}{cccccc} \mathrm{A} & \mathrm{B} & & \mathrm{C} & & 2 \\ \hline 3 & 4 & 4 & 4 & 1 & 4 \\ 2 & 2 & 3 & 3 & 3 & 2 \\ 1 & 1 & 1 & & & \end{array} \qquad \begin{array}{l} \mathrm{A} + \mathrm{B} = d - 2 \\ \mathrm{A} + \mathrm{C} \leq d - 2 \\ \mathrm{B} + \mathrm{C} \leq d - 3 \\ \mathrm{A}, \mathrm{B} > 0 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & \mathrm{C}+2 & 0 \\ d-1 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [2d - \mathrm{C} - 1, d + \mathrm{C} + 2, d - 1]$$

$$r = d - 2\text{C} - 3,\ s = \text{C} + 3,\ t = d - 1$$

Any valid weight assignment must have $\omega_3(1) = d - 1$. Therefore $\omega_2(1) = 0$, so we must have $\omega_2(3) > 0$. Unless $\text{B} = \text{C}$ only the element 2 may have $\omega_2 = d$, so the only valid weight assignment is $(1, 2, 3, 4)$ with sign 1. When $\text{B} = \text{C}$, we may also have $(1, 3, 2, 4)$ with sign $(-1)^{\text{A}}$ and $(1, 3, 4, 2)$ with sign $(-1)^{\text{A}+\text{B}+2}$. Since this sum is odd, we must have Q_7 non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

7.5 Tableaux for $c = 5$

Tableau R_1

$$R_1 = \begin{array}{cccccc} \mathsf{x} & \textsc{z} & \textsc{z} & \textsc{z} & \textsc{a} & \textsc{b} \\ \hline 5 & 2 & 4 & 5 & 5 & 5 \\ 3 & 3 & 3 & 4 & 4 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} 0 < \textsc{a} \le \textsc{b} \le \textsc{z} \\ d + \textsc{a} + \textsc{z} \text{ even, if } \textsc{b} = \textsc{z} \\ d \text{ even, if } \textsc{a} = \textsc{b},\ \mathsf{x} = 0 \end{array} \qquad \begin{array}{l} \textsc{z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & 0 & 2\textsc{z}+\mathsf{x}+\textsc{b} & \textsc{z}+\textsc{a} & 0 \\ d & \textsc{a}+\textsc{b} & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [3d - 2\textsc{a} - 2\textsc{b}, d + \textsc{a} + \textsc{b}, d + \textsc{a} + \textsc{b}]$$

$$r = 2d - 3\textsc{a} - 3\textsc{b},\ s = 0,\ t = d + \textsc{a} + \textsc{b}$$

First consider the valid weight assignments with $\textsc{a} < \textsc{z}$. If $\textsc{b} < \textsc{z}$ then we must have $\omega_3(1) = d$; if $\textsc{b} = \textsc{z}$ we may have $\omega_3(3) = d$ as well. If we have $\omega_3(1) = d$, then $\omega_3(2 \text{ or } 5) = \textsc{a}+\textsc{b}$ because $\textsc{a}, \textsc{b} > 0$. Since $\textsc{b} \ge \textsc{a}$ only the elements 1 and 3 may have $\omega_2 = 2\textsc{z} + \mathsf{x} + \textsc{b}$ unless $\textsc{a} = \textsc{b}$ and $\mathsf{x} = 0$. In that case, the element 4 may also have this weight. If $\omega_2(3) = 2\textsc{z} + \mathsf{x} + \textsc{b}$, then $\omega_2(5 \text{ or } 4) = \textsc{z} + \textsc{a}$. If $\omega_2(4) = 2\textsc{z} + \textsc{b}$, then $\omega_2(2 \text{ or } 3) = \textsc{z} + \textsc{a}$. These constraints give the following table of weight assignments:

Assignment	Sign	Condition
$(1,2,3,4,5)$	1	
$(1,5,3,4,2)$	$(-1)^{\textsc{a}+\textsc{b}}$	
$(1,2,3,5,4)$	$(-1)^{\textsc{z}+\textsc{a}}$	
$(1,2,4,3,5)$	$(-1)^{\textsc{z}}$	$\textsc{a} = \textsc{b}$, $\mathsf{x} = 0$
$(1,5,4,3,2)$	$(-1)^{\textsc{z}+\textsc{b}+\textsc{a}}$	$\textsc{a} = \textsc{b}$, $\mathsf{x} = 0$
$(1,5,4,2,3)$	$(-1)^{2\textsc{z}+\textsc{a}}$	$\textsc{a} = \textsc{b}$, $\mathsf{x} = 0$
$(3,4,1,2,5)$	$(-1)^{d+\textsc{z}+\textsc{a}}$	$\textsc{b} = \textsc{z}$
$(3,5,1,2,4)$	$(-1)^{d}$	$\textsc{b} = \textsc{z}$
$(3,4,1,5,2)$	$(-1)^{d}$	$\textsc{b} = \textsc{z}$

For $\textsc{b} < \textsc{z}$ and $\textsc{a} \ne \textsc{b} + \mathsf{x}$, we get a weight sum of $1 + (-1)^{\textsc{a}+\textsc{b}} + (-1)^{\textsc{z}+\textsc{a}}$ which is non-zero. When $\textsc{a} = \textsc{b}$, $\mathsf{x} = 0$, we get a weight sum of $1+1+(-1)^{\textsc{z}+\textsc{a}}+(-1)^{\textsc{z}}+(-1)^{\textsc{z}}+$

$(-1)^{\mathrm{A}}$. This is non-zero by the parity constraints which imply Z even. Similarly, if $\mathrm{A} < \mathrm{B} = \mathrm{Z}$, then the weight sum is $1+(-1)^{\mathrm{A}+\mathrm{Z}}+(-1)^{\mathrm{Z}+\mathrm{A}}+(-1)^{d+\mathrm{Z}+\mathrm{A}}+(-1)^d+(-1)^d$, which is also non-zero by the parity constraints.

When $\mathrm{A} = \mathrm{B} = \mathrm{Z}$ the tableau has many symmetries. Since the weight of rows two and three are the same, interchanging rows yields a new weight assignment with a difference in sign of $(-1)^{d+\mathrm{A}+\mathrm{B}} = 1$. Hence we will not count those weight assignments which are inversions of rows two and three. First consider the possible element pairs (x, y) that can have $\omega_3(x, y) = (d, 2\mathrm{Z})$. These are $(1, 2)$, $(1, 5)$, $(3, 4)$, $(3, 5)$, $(5, 3)$, and $(5, 1)$, along with $(2, 1)$, $(2, 4)$, $(4, 3)$, $(4, 2)$ when $\mathsf{x} = 0$. Two of these pairs make up a weight assignment (not counting row inversion). If $\mathsf{x} \neq 0$ the possible pair assignments are: $(1, 2)(3, 4)$; $(1, 2)(3, 5)$; $(1, 2)(5, 3)$; $(1, 5)(3, 4)$; and $(3, 4)(5, 1)$. Since there is an odd number, this weight sum in non-zero. If $\mathsf{x} = 0$, the condition d even, implies Z is even. Hence all the column blocks are even. Since any valid weight assignment moves full column blocks, all weight assignments are positive and thus R_1 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau R_2

$$R_2 = \begin{array}{c|cccccc} \mathsf{x}+\mathrm{Z} & \mathrm{Z} & \mathrm{Z} & \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{C} \\ \hline 2 & 4 & 4 & 5 & 4 & 4 & 2 \\ 3 & 5 & 3 & 3 & 5 & 3 & 5 \\ 1 & 1 & 1 & 2 & 2 & & \end{array} \qquad \begin{array}{l} 0 \le \mathrm{A} \le \mathrm{Z} - \mathrm{C} \\ 0 \le \mathrm{B} \le \mathrm{Z} - \mathrm{C} \\ 0 < \mathrm{C} \\ \mathrm{B} \le \mathrm{A} \end{array} \qquad \begin{array}{l} \mathrm{Z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ d \text{ even, if } \mathrm{B} = \mathrm{A} + \mathsf{x} \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & 0 & 2\mathrm{Z}+\mathrm{A}+\mathrm{C}+\mathsf{x} & 0 & \mathrm{Z}+\mathrm{B}+\mathrm{C} \\ d & \mathrm{B}+\mathrm{A} & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [3d - 2\mathrm{A} - 2\mathrm{B} - 2\mathrm{C}, d + \mathrm{A} + \mathrm{B} + 2\mathrm{C}, d + \mathrm{A} + \mathrm{B}]$$

$$r = 2d - 3\mathrm{A} - 3\mathrm{B} - 4\mathrm{C},\ s = 2\mathrm{C},\ t = d + \mathrm{A} + \mathrm{B}$$

A valid weight assignment must have $\omega_3(1) = d$ since $\mathrm{C} > 0$. Then we must have $\omega_3(2 \text{ or } 5) = \mathrm{A} + \mathrm{B}$, or if $\mathrm{B} = 0$, then $\omega_2(3) = \mathrm{A}$ is also possible. There are not enough 2's to have $\omega_2(2) = 2\mathrm{Z} + \mathrm{A} + \mathrm{C} + \mathsf{x}$. If we have $\omega_2(3) = 2\mathrm{Z} + \mathrm{A} + \mathrm{C} + \mathsf{x}$, then

$\omega_2(5) = \text{Z} + \text{B} + \text{C}$. If we have $\omega_2(4) = 2\text{Z} + \text{A} + \text{C} + \mathsf{x}$, then $\omega_2(2) = \text{Z} + \text{B} + \text{C}$, and $\text{B} = \text{A}$, $\mathsf{x} = 0$. There are not enough 5's to have $\omega_2(5) = 2\text{Z} + \text{A} + \text{C} + \mathsf{x}$.

Hence there are two possible weight assignments: $(1, 2, 3, 4, 5)$ with sign 1 and $(1, 5, 4, 3, 2)$ with sign $(-1)^d$, which occurs only when $\text{B} = \text{A}$, $\mathsf{x} = 0$. By our parity constraint, this sum is positive. If $\text{B} = 0$, we also have the weight assignment $(1, 3, 4, 5, 2)$ when $\text{A} = \text{B}$, $\mathsf{x} = 0$. The weight sum is odd and hence the tableau is non-zero.

Maximality: The tableau is maximal by Lemma 7.1.5. To see this, note that when row three has the maximum weight d, the largest possible weights for row two are: 0 for the element 1, $\text{Z} + \mathsf{x} + \text{A} + \text{B} + \text{C}$ for the element 2, $2\text{Z} + \mathsf{x} + \text{A} + \text{C}$ for the element 3, $2\text{Z} + \text{B} + \text{C}$ for the element 4, $\text{Z} + \text{A} + \text{B} + \text{C}$ for the element 5. Since $\text{Z} - \text{C} \geq \text{A} \geq \text{B}$, the weight $2\text{Z} + \mathsf{x} + \text{A} + \text{C}$ is the largest. Given this, Lemma 7.1.4 shows that row two is maximal. Moreover rows two and three cannot have elements in common, given the columns remaining after the largest weight elements have been assigned. Hence the tableau is maximal.

Tableau R_3

$$R_3 = \begin{array}{c|ccccccc} \text{Z}+\mathsf{x} & \text{Z} & \text{Z} & \text{B} & \text{A} & & & \\ \hline 5 & 2 & 5 & 4 & 5 & 2 & 5 & 2 \\ 3 & 3 & 4 & 3 & 4 & 3 & 4 & 4 \\ 1 & 1 & 1 & 2 & 2 & & & \end{array} \qquad 0 \leq \text{A} < \text{B} \leq \text{Z} - 1 \qquad \begin{array}{l} \text{Z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & 0 & 2\text{Z}+\mathsf{x}+\text{B}+1 & \text{Z}+\text{A}+2 & 0 \\ d & \text{A}+\text{B} & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [3d - 2\text{A} - 2\text{B} - 3, d + \text{A} + \text{B} + 3, d + \text{A} + \text{B}]$$

$$r = 2d - 3\text{A} - 3\text{B} - 6,\ s = 3,\ t = d + \text{A} + \text{B},$$

We must have $\omega_3(1) = d$. Since $\text{A} < \text{B}$, only the element 3 may have $\omega_2 = 2\text{Z} + \mathsf{x} + \text{B} + 1$. Then $\omega_2(4) = \text{Z} + \text{A} + 2$. Hence there are no other valid weight assignments. Thus R_3 is non-zero.

Maximality: Row two is maximal by Lemma 7.1.2 and row three is maximal by Lemma 7.1.4. Inspection shows that it is not possible to have the non-d weights assigned to the same element. Hence the tableau is maximal by Lemma 7.1.5.

Tableau R_4

$$R_4 = \begin{array}{cccccccccc} \mathsf{x} & \textsc{z}-1 & \textsc{z}{+}1 & \textsc{z} & \textsc{z}-\textsc{a} & \textsc{z}-\textsc{a} & \textsc{a}-1 & \textsc{b} & \textsc{a}{+}1 & \textsc{c} \\ \hline 2 & 2 & 5 & 5 & 4 & 5 & 5 & 2 & 2 & 5 \\ 3 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 3 & 4 \\ 1 & 1 & 1 & 1 & 2 & 2 & & & & \end{array}$$

$1 \le \textsc{a} \le \textsc{z}$, $\textsc{b} \le \textsc{a}$, $\textsc{c} \le \mathsf{x}$

$d + \textsc{a} + \textsc{c}$ even, if $\textsc{a} = \textsc{b}$

$\textsc{a} + \textsc{b}$ even, if $\textsc{c} = \mathsf{x}$

$\textsc{z} = \frac{d-\mathsf{x}}{3}$

$d \equiv \mathsf{x} \pmod 3$

$\textsc{z} \ge 2$

$\omega_{2,3} = \begin{pmatrix} 0 & 0 & d & 2\textsc{z}{+}\textsc{b}{+}\textsc{c} & 0 \\ d & 2\textsc{z}{-}2\textsc{a} & 0 & 0 & 0 \end{pmatrix}$

$\lambda = [3d - 4\textsc{z} - \textsc{b} - \textsc{c} + 2\textsc{a}, d + 2\textsc{z} + \textsc{b} + \textsc{c}, d + 2\textsc{z} - 2\textsc{a}]$

$r = 2d - 6\textsc{z} + 2\textsc{a} - 2\textsc{b} - 2\textsc{c}$, $s = 2\textsc{a} + \textsc{b} + \textsc{c}$, $t = d + 2\textsc{z} - 2\textsc{a}$,

By construction, only the element 1 may have $\omega_3(1) = d$. This means any valid weight assignment must have $\omega_3(2 \text{ or } 3) = 2\textsc{z} - 2\textsc{a}$ (unless $\textsc{a} = \textsc{z}$ in which case the weight assignments are equivalent to those using 2 or 3). Furthermore, the elements appearing in row two must either be both 3 and 4 or both 2 and 5. Now we apply weight-set counting.

Assignment	Sign	Condition
$(1, 2, 3, 4, 5)$	1	
$(1, 2, 4, 3, 5)$	$(-1)^{\textsc{z}-\textsc{a}}$	$\textsc{b} = \textsc{a}$, $\textsc{c} = \mathsf{x}$
$(1, 3, 2, 5, 4)$	$(-1)^{d+\textsc{b}+\textsc{c}}$	$\textsc{b} = \textsc{a}$
$(1, 3, 5, 2, 4)$	$(-1)^{\mathsf{x}+\textsc{a}+\textsc{b}+\textsc{c}}$	$\textsc{c} = \mathsf{x}$

By our parity constraints, all these terms are positive. Hence R_4 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. Inspection shows that it is not possible to have the non-d weights assigned to the same element. Hence the tableau is maximal.

Tableau R_5

$$R_5 = \begin{array}{cccccccc} \mathrm{Y}+\mathrm{W}-\mathrm{A} & \mathrm{Y}-\mathrm{A} & \mathrm{A} & \mathrm{A} & \mathrm{B} & \mathrm{Y} & \mathrm{Y} & \mathrm{W} \\ \hline 4 & 5 & 1 & 4 & 5 & 5 & 4 & 5 \\ 3 & 2 & 3 & 2 & 1 & 3 & 2 & 2 \\ 1 & 1 & & & & & & \end{array} \qquad \begin{array}{ll} \mathrm{A} \text{ even, if } \mathrm{A}=\mathrm{B} & \mathrm{Y} = \frac{d^*}{2} \\ 0 \le \mathrm{B} \le \mathrm{A} < \mathrm{Y} & 2\mathrm{Y}+\mathrm{W} = d \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} \mathrm{B} & d & d & 0 & 0 \\ d-2\mathrm{A} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [2d-\mathrm{B}+2\mathrm{A}, 2d+\mathrm{B}, d-2\mathrm{A}]$$

$$r = 2\mathrm{A}-2\mathrm{B},\ s = d+2\mathrm{A}+\mathrm{B},\ t = d-2\mathrm{A}$$

Since $\mathrm{Y}-\mathrm{A} > 0$, any weight assignment must have $\omega_3(1) = d-2\mathrm{A}$. If $\omega_2(3) = d$, then $\omega_2(2) = d$ and vice versa. Similarly for the elements 4 and 5, however we may only have $\omega_2(5) = d$ if $\mathrm{A} = \mathrm{B}$. So unless $\mathrm{A} = \mathrm{B}$ there is only one valid weight assignment. When $\mathrm{A} = \mathrm{B}$, we also have $(1,4,5,3,2)$ which has sign $(-1)^{2d+\mathrm{A}}$. Since A is even in this case, the sum is non-zero. Therefore R_5 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_6

$$R_6 = \begin{array}{cccccccccc} \mathsf{x} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z}-\mathrm{A} & \mathrm{Z}-\mathrm{A}-1 & \mathrm{A} & \mathrm{B} & \mathrm{A}+1 & \mathrm{C} \\ \hline 2 & 2 & 5 & 5 & 4 & 5 & 5 & 2 & 2 & 5\ 2\ 5\ 4 \\ 3 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 3 & 4 \\ 1 & 1 & 1 & 1 & 2 & 2 & & & & \end{array} \qquad \begin{array}{ll} 1 \le \mathrm{A} \le \mathrm{Z}-2 & \mathrm{Z} = \frac{d-\mathsf{x}}{3} \\ 0 \le \mathrm{B} < \mathrm{A} & d \equiv \mathsf{x} \pmod 3 \\ \mathrm{C} \le \mathsf{x} & \mathrm{Z} \ge 3 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & 0 & d & 2\mathrm{Z}+\mathrm{B}+\mathrm{C} & 0 \\ d & 2\mathrm{Z}-2\mathrm{A}-1 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [3d-4\mathrm{Z}-\mathrm{B}+2\mathrm{A}+1, d+2\mathrm{Z}+\mathrm{B}+\mathsf{x}, d+2\mathrm{Z}-2\mathrm{A}-1]$$

$$r = 2d-6\mathrm{Z}+2\mathrm{A}-2\mathrm{B}-2\mathrm{C}+1,\ s = 2\mathrm{A}+\mathrm{B}+\mathrm{C}+1,\ t = d+2\mathrm{Z}-2\mathrm{A}-1$$

Only the element 1 can have $\omega_3 = d$ and then any weight assignment must have $\omega_2(3) = d$. This forces $\omega_2(4) = 2\mathrm{Z}+\mathrm{A}+1+\mathsf{x}$. Hence $\omega_3(4) = 0$ which implies $\omega_3(2) > 0$. Thus there are no other valid weight-assignments and R_6 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. Inspection shows that it is non possible to have the non-d weights assigned to the same element. Hence the tableau is maximal.

Tableau R_7

$$R_7 = \begin{array}{ccccccccc} \textsc{y}-\textsc{a} & \textsc{y}-\textsc{a} & \textsc{a}{+}1 & \textsc{a} & \textsc{a}-1 & \textsc{y} & \textsc{y}{+}1 & & \\ \hline 4 & 5 & 1 & 4 & 5 & 4 & 5 & 1\,4\,5 \\ 3 & 2 & 3 & 2 & 1 & 3 & 2 & \\ 1 & 1 & & & & & & \end{array} \qquad \begin{array}{l} 1 \le \textsc{a} < \textsc{y} \\ \textsc{y} = \frac{d-1}{2} \\ d \text{ odd} \\ d \ge 5 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} \textsc{a}-1 & d & d & 0 & 0 \\ d-1-2\textsc{a} & 0 & 0 & 0 & 0 \end{smallmatrix} \right)$$

$$\lambda = [2d + \textsc{a} + 2, 2d + \textsc{a} - 1, d - 2\textsc{a} - 1]$$

$$r = 3,\ s = d + 3\textsc{a},\ t = d - 2\textsc{a} - 1$$

As $\textsc{y} - \textsc{a} > 0$, any weight assignment must have $\omega_3(1) = d - 1 - 2\textsc{a}$. Moreover, only the elements 2 and 3 can have $\omega_2 = d$. Hence there are no other valid weight assignments, so R_7 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_8

$$R_8 = \begin{array}{cccccccc} \textsc{y}-\textsc{a}-1 & \textsc{y}-\textsc{a} & \textsc{a}{+}1 & \textsc{a} & \textsc{a}-1 & \textsc{y} & \textsc{y} & \\ \hline 4 & 5 & 1 & 4 & 5 & 4\,5\,1\,4\,5 \\ 3 & 2 & 3 & 2 & 1 & 3\,2 \\ 1 & 1 & & & & \end{array} \qquad \begin{array}{l} 1 \le \textsc{a} \le \textsc{y} - 2 \\ \textsc{y} = \frac{d}{2} \\ d \text{ even} \\ d \ge 6 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} \textsc{a}-1 & d & d & 0 & 0 \\ d-1-2\textsc{a} & 0 & 0 & 0 & 0 \end{smallmatrix} \right)$$

$$\lambda = [2d + \textsc{a} + 2, 2d + \textsc{a} - 1, d - 2\textsc{a} - 1]$$

$$r = 3,\ s = d + 3\textsc{a},\ t = d - 2\textsc{a} - 1$$

Since $\textsc{a} < \textsc{y} - 1$, any weight assignment must have $\omega_3(1)$=$d - 1 - 2\textsc{a}$. Moreover, the only elements 2 and 3 can both have $\omega_2 = d$. Therefore there are no other valid weight assignments, so R_8 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_9

$$R_9 = \begin{array}{cccccccccc} \mathsf{x} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z} & \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \\ \hline 3 & 4 & 4 & 3 & 2 & 4 & 3 & 4 & 4 & 4 \\ 2 & 2 & 5 & 5 & 5 & 2 & 2 & 2 & 3 & 5 \\ 1 & 1 & 1 & 1 & 3 & 3 \end{array} \qquad \begin{array}{l} 0 \leq \mathrm{A} < \mathrm{Z} \\ \mathrm{B}+\mathrm{C},\ \mathrm{B}+\mathrm{D} < \mathrm{Z}-\mathrm{A} \\ \mathrm{C}+\mathrm{D}+\mathrm{E} < \mathrm{Z}-\mathrm{A}+\mathsf{x} \\ 0 \leq \mathrm{E} \leq \mathsf{x} \\ 0 \leq \mathrm{D} < \mathsf{x}+\mathrm{B} \end{array} \qquad \begin{array}{l} \mathrm{Z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ \mathsf{x} \neq 0 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & \mathrm{Z}+\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathsf{x} & \mathrm{D} & 0 & 3\mathrm{Z}+\mathrm{E} \\ d & 0 & \mathrm{Z}+\mathrm{A} & 0 & 0 \end{pmatrix}$$

$$\lambda = [3d-2\mathrm{A}-\mathrm{B}-\mathrm{C}-\mathrm{D}-\mathrm{E}-2\mathrm{Z},\ d+\mathrm{Z}+\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E},\ d+\mathrm{Z}+\mathrm{A}]$$

$$r = 2d-3\mathrm{A}-2\mathrm{B}-2\mathrm{C}-2\mathrm{D}-2\mathrm{E}-3\mathrm{Z},\ s = \mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E},\ t = d+\mathrm{Z}+\mathrm{A}$$

Any valid weight assignment must have $\omega_3(1) = d$. Only the elements 1 and 5 may have $\omega_2 = 3\mathrm{Z}+\mathrm{E}$. If $\omega_2(5) = 3\mathrm{Z}+\mathrm{E}$, then only the element 2 may have $\omega_2 = \mathsf{x}+\mathrm{Z}+\mathrm{A}+\mathrm{B}+\mathrm{C}$ due to the condition $\mathrm{D} < \mathrm{B}+\mathsf{x}$.

Examining the tableau in light of these constraints shows that $(1, 2, 3, 4, 5)$ is the only valid weight assignment. Thus the tableau is non-zero.

Maximality: As the discussion above shows, the tableau weights are as large as possible. Hence by Lemma 7.1.5, this tableau is maximal.

Tableau R_{10}

$$R_{10} = \begin{array}{ccccccccccc} \mathsf{x} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z-A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathsf{x} \\ \hline 3 & 4 & 4 & 3 & 3 & 2 & 4 & 3 & 4 & 3 & 4 \\ 2 & 2 & 5 & 5 & 5 & 5 & 3 & 2 & 2 & 5 & 5 \\ 1 & 1 & 1 & 1 & 2 & & & & & & \end{array}$$

$1 \leq \mathrm{A} \leq \mathrm{Z}$
$\mathrm{B} \leq \lfloor \frac{\mathrm{A}}{2} \rfloor$
$\mathrm{C}, \mathrm{E} \leq \lfloor \frac{\mathrm{Z}}{2} \rfloor$
$\mathrm{D} \leq \lceil \frac{\mathrm{Z}}{2} \rceil + \lfloor \frac{\mathrm{A}}{2} \rfloor$
$\mathrm{F} \leq \lceil \frac{\mathrm{A}}{2} \rceil$
C even

$\mathrm{D} > \mathrm{C}$
$\mathrm{Z} + \mathrm{B} > \mathrm{C} + \mathrm{D}$
$\mathrm{Z} - \mathrm{A} + \mathrm{B} + \mathrm{F} > \mathrm{C} + \mathrm{E}$
$\mathrm{Z} + \mathrm{F} > \mathrm{D} + \mathrm{E}$
$\mathrm{Z} = \frac{d - \mathsf{x}}{3}$
$d \equiv \mathsf{x} \pmod 3$
$\mathrm{Z} \geq 2$

$$\omega_{2,3} = \begin{pmatrix} 0 & \mathsf{x}+\mathrm{Z}+\mathrm{D}+\mathrm{E} & \mathrm{C} & 0 & 3\mathrm{Z}-\mathrm{A}+\mathrm{B}+\mathrm{F}+\mathsf{x} \\ d & \mathrm{Z}-\mathrm{A} & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [3d - 2\mathrm{Z} + 2\mathrm{A} - \mathrm{B} - \mathrm{C} - \mathrm{D} - \mathrm{E} - \mathrm{F} - \mathsf{x}, d + \mathrm{Z} - \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + \mathsf{x}, d + \mathrm{Z} - \mathrm{A}]$$

$$r = d + 3\mathrm{A} - 2\mathrm{B} - 2\mathrm{C} - 2\mathrm{D} - 2\mathrm{E} - 2\mathrm{F} - \mathsf{x},\ s = \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + \mathsf{x},\ t = d + \mathrm{Z} - \mathrm{A}$$

Any valid weight assignment must have $\omega_3(1) = d$. Given the bounding parameters, only the element 5 can have $\omega_2 = 2\mathrm{Z} + \mathrm{A} + \mathrm{B} + \mathrm{F} + \mathsf{x}$. When $\omega_2(5)$ is maximal, only the element 2 may have $\omega_2 = \mathsf{x} + \mathrm{Z} + \mathrm{D} + \mathrm{E}$.

From this we find that the valid weight assignments are $(1, 2, 3, 4, 5)$ with sign 1, and $(1, 2, 4, 3, 5)$ with sign $(-1)^{\mathrm{C}} = 1$.

Maximality: The discussion above shows that the row two weights are maximized. Hence by Lemma 7.1.5, this tableau is maximal.

Tableau R_{11}

$$R_{11} = \begin{array}{ccccccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & & & \\ \hline 3 & 5 & 3 & 5 & 1 & 5 & 3 & 1 & 5 \\ 4 & 2 & 4 & 2 & 2 & 1 & 4 & 2 & 4 \\ 1 & 1 & & & & & & & \end{array}$$

$0 < \mathrm{A}, \mathrm{B}$
$\mathrm{C} \leq d - \mathrm{A} - 2$
$\mathrm{D} \leq d - \mathrm{B} - \mathrm{E} - 1$
$0 < \mathrm{F} \leq \mathrm{E}$

$\mathrm{E} + \mathrm{F} \leq d - \mathrm{A} - \mathrm{B} - 1$
$\mathrm{B} + \mathrm{D} > \mathrm{F}$
$\mathrm{A} + \mathrm{C} \geq \mathrm{E} + \mathrm{F}$

$$\omega_{2,3} = \begin{pmatrix} \mathrm{F} & \mathrm{B}+\mathrm{D}+\mathrm{E}+1 & \mathrm{A}+\mathrm{C}+2 & 0 & 0 \\ \mathrm{A}+\mathrm{B} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = [5d - 2\text{A} - 2\text{B} - \text{C} - \text{D} - \text{E} - \text{F} - 3, \text{A} + \text{B} + \text{C} + \text{D} + \text{E} + \text{F} + 3, \text{A} + \text{B}]$$

$$r = 5d - 3\text{A} - 3\text{B} - 2\text{C} - 2\text{D} - 2\text{E} - 2\text{F} - 6,\ s = \text{C} + \text{D} + \text{E} + \text{F} + 3,\ t = \text{A} + \text{B}$$

Any valid weight assignment must have $\omega_3(1) = \text{A} + \text{B}$ and hence $\omega_2(1) = \text{F}$. This implies $\omega_2(2) > 0$. But to have exactly three non-zero weights in row two, we then must have $\omega_2(4) > 0$. Hence no other weight assignments are possible and hence the tableau is non-zero.

Maximality: The possible row two weights, given the maximization of row three, are $\text{E} + \text{F} + 1$ for the element 1, $\text{B} + \text{D} + \text{E} + 1$ for the element 2, $\text{A} + \text{C} + 1$ for the element 3, $\text{A} + \text{C} + 2$ for the element 4, and $\text{B} + \text{D} + \text{F} + 1$ for the element 5. Given the conditions on these parameters we see that the weights for 2 and 4 are the largest. Hence by Lemma 7.1.5, this tableau is maximal.

Tableau R_{12}

$$R_{12} = \begin{array}{ccccccc} \multicolumn{2}{c}{\text{Y} - 1 + \text{W}} & \text{Y} & \text{Y} & \text{Y} & \text{A} & \text{W} \\ \hline 1 & 1 & 5 & 4 & 1 & 5 & 5 \\ 3 & 3 & 3 & 2 & 2 & 4 & 2 \\ 4 & & & & & & \end{array} \qquad \begin{array}{ll} 0 < \text{A} \leq d - \text{Y} - 1 & \text{Y} = \frac{d^*}{2} \\ \text{Y even, if Y} = \text{A} & 2\text{Y} + \text{W} = d \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & d & \text{A} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d - \text{A} - 1, 2d + \text{A}, 1]$$

$$r = d - 2\text{A} - 1,\ s = 2d + \text{A} - 1,\ t = 1$$

Any valid weight assignment must have two of the elements 1, 3, and 4 having non-zero weight in row two. Hence at least one of them must have $\omega_2 = d$. Moreover, we can not have a row two weight of d for both of these elements. If $\omega_2(1) = d$ we may not have $\omega_2(2) = d$, so the only option is $\omega_2(5) = d$ when $\text{A} = \text{Y}$. This corresponds to a weight assignment of $(3, 1, 5, 4, 2)$ with sign $(-1)^{2d+\text{A}}$ which equals 1 due to the parity constraint. If $\omega_2(3) = d$ we may not have $\omega_2(5) = d$, so the only option is $\omega_2(2) = d$. This corresponds to a weight assignment of $(1, 2, 3, 4, 5)$ with sign 1. When $\omega_2(4) = d$, neither the element 2 nor the element 5 may have $\omega_2 = d$, so there is no weight assignment with this option. Hence the weight sum is always positive and thus R_{12} is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_{13}

$$R_{13} = \begin{array}{ccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \\ \hline 1 & 3 & 1 & 3 & 1 \\ 2 & 4 & 4 & 2 & 2 \\ 5 & 5 \end{array}$$

$\text{A} + \text{D} + \text{E} \leq d$	$\text{D} \geq \text{C}$
$\text{A} + \text{B} \leq d$	$\text{A} + \text{E} \geq \text{B}$
$\text{B} + \text{D} \leq d$	If $\text{C} = \text{D}$ then $\text{A} + \text{B} + \text{C} + \text{D} + \text{E}$ even
$\text{A}, \text{B}, \text{C}, \text{D}, \text{E} > 0$	If $\text{B} = \text{A} + \text{E}$ then $\text{A} + \text{B} + \text{C} + \text{D} + \text{E}$ even

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A+D+E} & 0 & \text{B+C} & 0 \\ 0 & 0 & 0 & 0 & \text{A+B} \end{smallmatrix}\right)$

$\lambda = [5d - 2\text{A} - 2\text{B} - \text{C} - \text{D} - \text{E}, \text{A} + \text{B} + \text{C} + \text{D} + \text{E}, \text{A} + \text{B}]$

$r = 5d - 3\text{A} - 3\text{B} - 2\text{C} - 2\text{D} - 2\text{E},$, $s = \text{C} + \text{D} + \text{E}$, $t = \text{A} + \text{B}$

Since $\text{A}, \text{B} > 0$ any valid weight assignment must have $\omega_3(5) = \text{A} + \text{B}$. We must have $\omega_2(1 \text{ or } 2) > 0$. If $\omega_2(2) > 0$, then $\omega_2(4) > 0$; if $\omega_2(1) > 0$, then $\omega_2(3) > 0$.

This shows the only valid weight assignments are $(1, 2, 3, 4, 5)$ and $(2, 1, 4, 3, 5)$. The second one only occurs when $\text{D} = \text{C}$ or $\text{B} = \text{A} + \text{E}$. If that happens, the sign is $(-1)^{\text{A+B+C+D+E}} = 1$. Hence R_{13} is non-zero.

Maximality: The possible maximal weights for row two are: $\text{A} + \text{C} + \text{E}$ for the element 1; $\text{A} + \text{D} + \text{E}$ for the element 2; $\text{B} + \text{D}$ for the element 3; $\text{B} + \text{C}$ for the element 4; and 0 for the element 5. By our parameter conditions, $\text{A} + \text{D} + \text{E}$ is the largest. Hence by Lemma 7.1.5 the tableau is maximal.

Tableau R_{14}

$$R_{14} = \begin{array}{cccccc} d-3 & 2 & d-4 & & & \\ \hline 1 & 4 & 1 & 5 & 1\,5\,1 \\ 2 & 2 & 2 & 3 & 3\,4\,4 \\ 3 \end{array} \qquad d \geq 5$$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & d-3 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [3d, 2d-1, 1]$

$r = d+1,\ s = 2d-2,\ t = 1$

For $d > 5$, any valid weight assignment must have $\omega_2(2) = d$. This forces $\omega_2(3) = d-3$ and $\omega_2(4) = 2$. Hence there are no other valid weight assignments. When $d = 5$, we may have $\omega_2(1) = d$ or $\omega_2(1) = d-3$ as well. Then the possible weight assignments are $(1, 2, 3, 4, 5)$, $(3, 2, 1, 5, 4,)$, and $(3, 1, 2, 5, 4)$. Since there is an odd number, this sum is non-zero and thus so is R_{14}.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau R_{15}

$$R_{15} = \begin{array}{c|ccccc} \multicolumn{6}{l}{\mathsf{z}+\mathsf{x}\ \ \mathsf{z}\ \ \mathsf{z}} \\ \hline 2 & 5 & 5 & 2 & 2 & 2 \\ 3 & 4 & 3 & 4 & 3 & 3 \\ 1 & 1 & 1 & & & \end{array} \qquad \begin{array}{l} \mathsf{z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ \mathsf{z} \geq 2 \end{array}$$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & 2\mathsf{z}+\mathsf{x}+2 & \mathsf{z}+1 & 0 \\ d & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [3d-3, d+3, d]$

$r = 2d-6,\ s = 3,\ t = d$

Since only the element 1 may have $\omega_3 = d$ and only the element 3 may have $\omega_2 = 2\mathsf{z}+\mathsf{x}+2$, there are no other valid weight assignments. Hence the tableau is non-zero.

Maximality: Given the row three weight of d, the row two weights are as large as possible. Thus by Lemma 7.1.5 this tableau is maximal.

Tableau R_{16}

$$R_{16} = \begin{array}{l|l} d-2 & \text{A} \\ \hline 1 & 5\ 5\ 5\ 3\ 1 \\ 2 & 4\ 4\ 2\ 4\ 2 \\ 3 & 3 \end{array}$$

$1 \leq \text{A} \leq d-4$

$d \geq 5$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & \text{A}+2 & 0 \\ 0 & 0 & d-1 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [3d - \text{A} - 1, d + \text{A} + 2, d - 1]$

$r = 2d - 2\text{A} - 3,\ s = \text{A} + 3,\ t = d - 1$

Any valid weight assignment must have $\omega_3(3) = d - 1$ and $\omega_2(2) = d$. This forces $\omega_2(4) = \text{A}+2$. Hence there are no other valid weight assignments and R_{16} is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_{17}

$$R_{17} = \begin{array}{l|l|l} \mathsf{x}\ \text{Z}\ \text{Z}\ \text{Z} & \text{Z}-1 & \text{A}\ \text{Z}\ \text{B} \\ \hline 2\ 2\ 5\ 5\ 2 & 5 & 2\ 2\ 5\ 2\ 5\ 4 \\ 3\ 4\ 4\ 3\ 3 & 3 & 4\ 3\ 4 \\ 1\ 1\ 1\ 1\ 4 & & \end{array}$$

$0 \leq \text{A} \leq \text{Z} - 2$

$\text{B} \leq \mathsf{x}$

$\text{Z} = \frac{d-\mathsf{x}}{3}$

$d \equiv \mathsf{x} \pmod 3$

$\text{Z} \geq 2$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & d & 2\text{Z}+\text{A}+\text{B} & 0 \\ d & 0 & 0 & 1 & 0 \end{smallmatrix}\right)$

$\lambda = [3d - 2\text{Z} - \text{A} - 1, d + 2\text{Z} + \text{A} + \mathsf{x}, d + 1]$

$r = 2d - 4\text{Z} - 2\text{A} - 2\text{B} - 1,\ s = 2\text{Z} + \text{A} + \text{B} - 1,\ t = d + 1$

Only the element 1 can have $\omega_3 = d$ and then any weight assignment must have $\omega_2(3) = d$. This forces $\omega_2(4) = 2\text{Z}+\text{A} + 1 + \mathsf{x}$. Thus there are no other valid weight assignments and R_{17} is non-zero.

Maximality: The tableau is maximal by Lemma 7.1.3.

Tableau R_{18}

$$R_{18} = \begin{array}{ccccc} \mathsf{x} & \mathrm{z} & \mathrm{z} & \mathrm{z} & \mathrm{z} \\ \hline 5 & 2 & 4 & 4 & 2 \\ 3 & 3 & 3 & 5 & 3 \\ 1 & 1 & 1 & 1 & 5 \end{array} \qquad \mathsf{x} \text{ even}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & d & 0 & \mathrm{z} \\ d & 0 & 0 & 0 & \mathrm{z} \end{smallmatrix}\right)$$

$$\lambda = [3d - 2\mathrm{z}, d + \mathrm{z}, d + \mathrm{z}]$$

$$r = 2d - 3\mathrm{z},\ s = 0,\ t = d + \mathrm{z}$$

Only the elements 1 and 3 may have a weight of d. Hence the only valid weight assignments are $(1, 2, 3, 4, 5)$ and $(3, 2, 1, 4, 5)$, both of which have sign 1 since x is even. Hence R_{18} is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau R_{19}

$$R_{19} = \begin{array}{cccccc} d-2 & & & & d-3 & \\ \hline 1 & 5 & 5 & 3 & 5 & 1 \\ 2 & 4 & 2 & 2 & 4 & 4 \\ 3 & 3 & & & & \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & d-1 \\ 0 & 0 & d-1 & 0 \end{smallmatrix}\right)$$

$$\lambda = [2d + 2, 2d - 1, d - 1]$$

$$r = 3,\ s = d,\ t = d - 1$$

Any valid weight assignment must have $\omega_3(3) = d$. Then $\omega_2(2) = d$. Hence there are no other weight assignments possible. Thus R_{19} is non-zero. **Maximality:** This tableau is maximal by Lemma 7.1.3.

7.6 Tableaux for $c = 6$

Tableau S_1

$$S_1 = \begin{array}{cccccc} \mathsf{x}+\mathrm{Z} & \mathsf{x}+\mathrm{Z} & \mathrm{Z} & \mathrm{Z} & \mathrm{A} & \mathrm{A} \\ \hline 5 & 6 & 5 & 6 & 5 & 6 \\ 4 & 3 & 3 & 4 & 3 & 4 \\ 1 & 2 & 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{l} 0 \leq \mathrm{A} \leq \mathrm{Z} \\ \mathrm{Z} + \mathsf{x} \text{ even, if } \mathrm{A} = 0 \end{array} \qquad \begin{array}{l} \mathrm{Z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & 2\mathrm{Z}+\mathrm{A}+\mathsf{x} & 2\mathrm{Z}+\mathrm{A}+\mathsf{x} & 0 & 0 \\ d & \mathrm{Z}+2\mathrm{A}+\mathsf{x} & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [4d - 2\mathrm{Z} - 2\mathsf{x} - 4\mathrm{A}, d + \mathrm{Z} + 2\mathrm{A} + \mathsf{x}, d + \mathrm{Z} + 2\mathrm{A} + \mathsf{x}]$$

$$r = 2d - 2\mathsf{x} - 6\mathrm{A},\ s = 0,\ t = d + 2\mathrm{A} + \mathrm{Z} + \mathsf{x}$$

The construction of S_1 means that any valid weight assignment with $\mathrm{A} > 0$ must have $\omega_i(j) > 0$ if and only if $\omega_i(k) > 0$ for the pairs $(j,k) = (1,2)$, $(3,4)$, or $(5,6)$. These constraints show that the only valid weight assignments are those that interchange complete rows in the body of S_1. Since the length of these rows $t = 4\mathrm{Z}+2\mathsf{x}+2\mathrm{A}$ is even, all valid weight assignments are positive and hence S_1 is non-zero.

When $\mathrm{A} = 0$ we must have $\omega_3(1) = d$ and $\omega_2(3,4) > 0$ or $\omega_2(5,6) > 0$. Hence the valid weight assignments are: $(1,2,3,4,5,6)$ with sign 1; $(1,6,3,4,5,2)$ with sign $(-1)^{\mathrm{Z}+\mathsf{x}}$; $(1,2,5,6,3,4)$ with sign $(-1)^{4\mathrm{Z}+2\mathsf{x}}$; and $(1,3,5,6,2,4)$ with sign $(-1)^{5\mathrm{Z}+3\mathsf{x}}$. Since $\mathrm{Z} + \mathsf{x}$ is even, these assignments are all positive. Hence the tableau is non-zero.

Maximality: By Lemma 7.1.5 this tableau is maximal.

Tableau S_2

$$S_2 = \begin{array}{c|cccccc} d-1 & \mathrm{A} & & & & & \\ \hline 4 & 2 & 2 & 2 & 4 & 2 & 3 \\ 5 & 1 & 5 & 1 & 1 & 1 & 1 \\ 6 & 3 & 3 & 6 & 3 & & \end{array} \qquad \begin{array}{l} 0 \leq \mathrm{A} \leq d - 4 \\ \mathrm{A} \equiv d \pmod 2 \\ d \geq 4 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} \mathrm{A}+4 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & \mathrm{A}+2 & 0 & 0 & d \end{smallmatrix}\right)$$

$$\lambda = [4d - 2\mathrm{A} - 6, d + \mathrm{A} + 4, d + \mathrm{A} + 2]$$

$$r = 3d - 3\mathrm{A} - 10,\ s = 2,\ t = d + \mathrm{A} + 2$$

Any valid weight assignment must have $\omega_2(1) = \text{A}+4$ since if $\omega_2(1) = 0$ then we must have $\omega_2(2,3) = (\text{A}+4, d)$ which is not possible. If $\omega_2(1) = \text{A}+4$, then $\omega_2(5) = d$. Thus the only valid weight assignments are $(1,2,3,4,5,6)$ with sign 1 and $(1,3,2,6,5,4)$ with sign $(-1)^{d+\text{A}+2}$. Since $\text{A} \equiv d \pmod 2$ this sum is positive. Hence S_2 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. Since it is not possible for the non-d element of row three to have a row two weight of $\text{A}+2$, the tableau is maximal.

<u>Tableau S_3</u>

$$S_3 = \begin{array}{c|cccccccc} d-4 & \text{A} & 2 & 2 & & & & & \\ \hline 5 & 1 & 1 & 3 & 5 & 5 & 5 & 1 & 1 \\ 2 & 4 & 2 & 4 & 4 & 2 & 4 & 2 & 4 \\ 6 & 3 & 6 & 6 & 3 & 3 & & & \end{array} \qquad \begin{array}{l} 0 \le \text{A} \le d-5 \\ d \ge 5 \end{array}$$

$$\omega_{2,3} = \begin{pmatrix} 0 & d & 0 & \text{A}+5 & 0 & 0 \\ 0 & 0 & \text{A}+2 & 0 & 0 & d \end{pmatrix}$$

$$\lambda = [4d - 2\text{A} - 7, d + \text{A} + 5, d + \text{A} + 2]$$

$$r = 3d - 3\text{A} - 12,\ s = 3,\ t = d + \text{A} + 2$$

Any valid weight assignment must have $\omega_3(6) = d$ and $\omega_3(3) = \text{A}+2$ (or $\omega_3(5) = 2$ if $\text{A} = 0$). Then only the element 2 may have $\omega_2 = d$ (or 4 if $\text{A} = d-5$). Moreover if $\omega_2(2) = d$, then $\omega_2(4) = \text{A}+5$. Hence the only valid weight assignments are $(1,2,3,4,5,6)$ with sign 1 and, when $\text{A} = 0$, $(1,2,5,4,3,6)$ with sign $(-1)^2$. Therefore the weight sum is positive and S_3 is non-zero.

Maximality: Rows two and three are maximal by Lemma 7.1.2. Since the non-d element of row two has weight $\text{A}+5$ and there are only two copies of the element 3 available for row two, the tableau is maximal.

Tableau S_4

$$S_4 = \begin{array}{c|c|c} d-3 & \text{A} & 2 \\ \hline 5 & 1\,5\,3\,5\,1\,1\,1\,1\,1 & \\ 2 & 4\,2\,4\,4\,2\,2\,3\,4\,4 & \\ 6 & 3\,6\,6\,3\,3 & \end{array} \qquad \begin{array}{l} 0 \le \text{A} \le d-5 \\ d \ge 5 \end{array}$$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 1 & \text{A}+5 & 0 & 0 \\ 0 & 0 & \text{A}+2 & 0 & 0 & d \end{smallmatrix}\right)$

$\lambda = [4d - 2\text{A} - 8, d + \text{A} + 6, d + \text{A} + 2]$

$r = 3d - 3\text{A} - 14,\ s = 4,\ t = d + \text{A} + 2$

Any valid weight assignment must have $\omega_3(6) = d$. Now $\omega_3(6) = d$ implies $\omega_3(3) = \text{A}+2$ and $\omega_2(3) = 1$. Unless $\text{A} = d-5$, we must have $\omega_2(2) = d$ and so $\omega_2(4) = \text{A}+5$. When $\text{A} = d-5$ we may have $\omega_2(1) = d$ or $\omega_2(4) = d$. However, if $\omega_2(1) = d$, then there is no element with $\omega_2 = \text{A}+5$. If $\omega_2(4) = d$, then $\omega_2(2) = \text{A}+5$. This shows the only weight assignment is $(1, 2, 3, 4, 5, 6)$. Hence S_4 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5 and the discussion above.

Tableau S_5

$$S_5 = \begin{array}{c|c|c|c|c|c|c|c|c} \text{x}+\text{z}' & \text{x}+\text{z}' & \text{z}' & \text{z}' & \text{z}' & \text{z}' & \text{A} & \text{B} & \text{C} \\ \hline 5 & 6 & 5 & 6 & 5 & 6 & 6 & 5 & 1 \\ 4 & 3 & 3 & 4 & 3 & 4 & 2 & 4 & 3 \\ 1 & 2 & 1 & 1 & 2 & 2 & & & \end{array}$$

$0 < \text{A}, \text{B}, \text{C} \le d - e$

A even

C even if $\text{A} = \text{B}$

B even if $\text{A} = \text{C}$

$\text{B}, \text{C} \ge \text{A}$

$\text{z}' = \frac{e-\text{x}}{3}$

$e \equiv \text{x} \pmod 3$

$3 \le e < d$

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A} & e+\text{C} & e+\text{B} & 0 & 0 \\ e & e & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [6d - 4e - \text{A} - \text{B} - \text{C}, 2e + \text{A} + \text{B} + \text{C}, 2e]$

$r = 6d - 6e - 2\text{A} - 2\text{B} - 2\text{C},\ s = \text{A} + \text{B} + \text{C},\ t = 2e$

The construction of S_5 forces the following constraints on any valid weight assignment.

- $\omega_3(1) > 0 \iff \omega_3(2) > 0$.
- $\omega_3(3) > 0 \iff \omega_3(4) > 0$.
- $\omega_3(5) > 0 \iff \omega_3(6) > 0$.
- If $\omega_2(j) = \text{A}$, then $\omega_3(j) > 0$.
- If $\omega_2(1) = \text{A}$, then $\omega_2(5,6) > 0$ and $\text{A} = \text{C}$.
- If $\omega_2(2) = \text{A}$, then $\omega_2(3,4) > 0$.
- If $\omega_2(3) = \text{A}$, then $\omega_2(5,6) > 0$ and $\text{A} = \text{C}$.
- If $\omega_2(4) = \text{A}$, then $\omega_2(1,2) > 0$ and $\text{A} = \text{B}$.
- If $\omega_2(5) = \text{A}$, then $\omega_2(1,2) > 0$ and $\text{A} = \text{B}$.
- If $\omega_2(6) = \text{A}$, then $\omega_2(3,4) > 0$.

From this we can derive a signed weight table.

Assignment	Sign	Condition
$(1,2,3,4,5,6)$	1	
$(2,1,6,5,4,3)$	$(-1)^{2e+\text{A}+\text{B}+\text{C}}$	$\text{A} = \text{C}$
$(3,4,1,2,5,6)$	$(-1)^{2e+\text{C}}$	$\text{A} = \text{B}$
$(4,3,6,5,1,2)$	$(-1)^{4e+\text{A}+\text{B}}$	$\text{A} = \text{C}$
$(5,6,3,4,1,2)$	$(-1)^{2e+\text{A}}$	
$(6,5,1,2,3,4)$	$(-1)^{4e+\text{B}+\text{C}}$	$\text{A} = \text{B}$

Computing the weight sum we obtain $3+3(-1)^{\text{A}} = 6$ when $\text{A} = \text{B} = \text{C}$ as A is always even. For $\text{A} = \text{B} \neq \text{C}$, the sum is $2 + 2(-1)^{\text{C}}$. This is non-zero as C is even when $\text{A} = \text{B}$. For $\text{A} = \text{C} \neq \text{B}$, we have $2 + 2(-1)^{\text{B}}$. This is non-zero as B even when $\text{A} = \text{C}$. Finally if $\text{A} \neq \text{B}$, C the sum is $1 + (-1)^{\text{A}}$ which is non-zero. Hence S_5 is non-zero.

Maximality: By Lemma 7.1.4, row three is maximal. Since B, C $\geq$ A, row two is maximal by Lemma 7.1.4. As the number of row three elements available for row two is bounded by A, this is the largest weight and hence the tableau is maximal.

Tableau S_6

$$S_6 = \begin{array}{cccccccc} \text{X}+\text{Z}' & \text{X}+\text{Z}' & \text{Z}' & \text{Z}' & \text{Z}' & \text{Z}'-1 & \text{A} & \text{B} & \text{C} \\ \hline 3 & 4 & 4 & 3 & 3 & 4 & 2 & 3 & 4 \\ 5 & 6 & 5 & 6 & 5 & 6 & 6 & 5 & 1 \\ 1 & 2 & 1 & 1 & 2 & 2 & & & \end{array}$$

$0 < \text{A},\ \text{B},\ \text{C} \leq d - e$ $\qquad$ $\text{Z}' = \frac{e-\text{X}}{3}$

$\text{A},\ \text{B} \geq \text{C}$ $\qquad$ $e \equiv \text{X} \pmod 3$

$\text{B} \geq \text{A} - 1$ $\qquad$ $3 \leq e \leq d$

$\omega_{2,3} = \left(\begin{smallmatrix} \text{C} & 0 & e+\text{B} & e+\text{A}-1 & 0 & 0 \\ e & e-1 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$

$\lambda = [6d - 4e - \text{A} - \text{B} - \text{C} + 2, 2e + \text{A} + \text{B} + \text{C} - 1, 2e - 1]$

$r = 6d - 6e - 2\text{A} - 2\text{B} - 2\text{C} + 3$, $s = \text{A} + \text{B} + \text{C}$, $t = 2e - 1$

For any valid weight assignment we can have $\omega_3(j) = e$ only for $j \in \{1, 3, 5\}$. Moreover, $\omega_3(j) = e$ if and only if $\omega_3(j+1) = e - 1$. If $\omega_3(j) = e$ then $\omega_2(j) = \text{C}$. If $\omega_2(1) = \text{C}$, then $\omega_2(5, 6) > 0$. If $\omega_2(3) = \text{C}$, then $\omega_2(1, 2) > 0$ and $\text{B} = \text{C}$. If $\omega_2(5) = \text{C}$, then $\omega_2(1, 2) > 0$ and $\text{B} = \text{C}$.

This means $(1, 2, 3, 4, 5, 6)$ is the only valid weight assignment when $\text{B} \neq \text{C}$. If $\text{B} = \text{C}$, then we additionally have weight assignments $(3, 4, 1, 2, 5, 6)$ and $(5, 6, 1, 2, 3, 4)$. In either case the weight sum is odd and hence non-zero. Thus S_6 is non-zero.

Maximality: By Lemma 7.1.4 row three is maximal. Since A, B $\geq$ C, row two is maximal by Lemma 7.1.4. As the number of row three elements available for row two is bounded by C, this is the largest weight and the tableau is maximal.

Tableau S_7

$$S_7 = \begin{array}{cccccccc} \text{A} & \text{B} & \text{C} & \text{D} & & & & \text{E} \\ \hline 1 & 4 & 3 & 1 & 1 & 1 & 3 & 5 \\ 2 & 2 & 2 & 4 & 2 & 2 & 4 & 6 \\ 3 & 3 & & & & & & \end{array}$$

$\text{A} + \text{B} + \text{C} = d - 2$ $\qquad$ $\text{A}, \text{B}, \text{D} > 0$

$\text{B} + \text{D} \leq d - 2$ $\qquad$ $0 \leq \text{E} \leq d$

$\text{A} + \text{D} < d - 2$ $\qquad$ E even

$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & \text{D}+1 & 0 & \text{E} \\ 0 & 0 & \text{A}+\text{B} & 0 & 0 & 0 \end{smallmatrix}\right)$

$$\lambda = [5d - \mathrm{A} - \mathrm{B} - \mathrm{D} - \mathrm{E} - 1, d + \mathrm{D} + \mathrm{E} + 1, \mathrm{A} + \mathrm{B}]$$

$$r = 4d - \mathrm{A} - \mathrm{B} - 2\mathrm{D} - 2\mathrm{E} - 2,\ s = \mathrm{C} + \mathrm{D} + \mathrm{E} + 3,\ t = \mathrm{A} + \mathrm{B}$$

Any valid weight assignment must have $\omega_2(5 \text{ or } 6) = \mathrm{E}$ and $\omega_3(3) = \mathrm{A} + \mathrm{B}$. Since $\mathrm{A} + \mathrm{D} < d - 2$, we must have $\omega_2(2) = d$. Then, as $\mathrm{D} > 0$ we must have $\omega_2(4) = \mathrm{D} + 1$. Hence the only valid weight assignments are $(1, 2, 3, 4, 5, 6)$ with sign 1 and $(1, 2, 3, 4, 6, 5)$ with sign $(-1)^{\mathrm{E}}$. Since is E is even, this weight sum is positive. Hence S_7 is non-zero.

Maximality: Inspection shows that rows two and three are maximal. Since $\mathrm{D} > 0$ we cannot have any 3's in a maximal row two. Thus S_7 is maximal by Lemma 7.1.5.

<u>Tableau S_8</u>

$$S_8 = \begin{array}{ccccccccccc} & & d-4 & & d-2 & d-4 & & 2 & & & \\ \hline 6 & 6 & 6 & 3 & 2 & 1 & 2 & 3 & 6 & 6 & 2 \\ 3 & 4 & 3 & 4 & 4 & 5 & 5 & 5 & 1 & 5 & 1 \\ 1 & 1 & & & & & & & & & \end{array} \qquad d \geq 5$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 2 & 0 & d-3 & d & d & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d - 1, 3d - 1, 2]$$

$$r = 0,\ s = 3d - 3,\ t = 2$$

Examining the tableau shows that we can only have $\omega_2(j, k) = (d, d)$ for $(j, k) = (2, 3)$, $(2, 6)$ or $(4, 5)$. Also, we must have $\omega_{2,3}(1 \text{ or } 6) > 0$, so $\omega_2(2, 6) = (d, d)$ and $\omega_2(1) = 2$. This is possible only when $d = 6$. Hence the valid weight assignments are: $(1, 2, 3, 4, 5, 6)$ with sign 1; $(6, 4, 1, 2, 3, 5)$ with sign $(-1)^{2d+4}$; $(1, 4, 3, 2, 6, 5)$ with sign $(-1)^{3d-1}$ when $d = 6$; and $(6, 2, 1, 4, 5, 3)$ with sign $(-1)^3$ when $d = 5$. In all cases, the weight sum is positive and hence S_8 is non-zero.

Maximality: This tableau is not maximal since $\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & d-1 & d & d & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$. However, this weight is zero in $\mathbf{q}_T$. This tableau cannot be put in maximal form, hence we will need to prove directly that it is disjoint from the requisite tableaux. This will be done in Section 8.7.

Tableau S_9

$$S_9 = \begin{array}{ccccccccc} \scriptstyle\mathrm{A} & \scriptstyle\mathrm{B} & \scriptstyle\mathrm{C} & \scriptstyle\mathrm{D} & \scriptstyle\mathrm{E} & \scriptstyle\mathrm{F} & \scriptstyle\mathrm{G} & & \\ \hline 1 & 5 & 1 & 5 & 3 & 6 & 6 & 3 & 6 \\ 2 & 4 & 2 & 2 & 5 & 4 & 5 & 4 & 2 \\ 3 & 3 & & & & & & & \end{array}$$

$$\mathrm{A} + \mathrm{C} + \mathrm{D} = d - 1$$
$$\mathrm{B} + \mathrm{F} = d - 1$$
$$\mathrm{F} + \mathrm{G} \leq d - 1$$
$$\mathrm{A} + \mathrm{B} + \mathrm{E} \leq d - 1$$
$$\mathrm{B} + \mathrm{D} + \mathrm{E} + \mathrm{G} \leq d - 1$$
$$\mathrm{A},\ \mathrm{B} > 0$$
$$\mathrm{E},\ \mathrm{G} > 0 \text{ or } \mathrm{E} = 0 \text{ and } \mathrm{G} \text{ even}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & 0 & d & \mathrm{E}+\mathrm{G} & 0 \\ 0 & 0 & \mathrm{A}+\mathrm{B} & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [4d - \mathrm{A} - \mathrm{B} - \mathrm{E} - \mathrm{G}, 2d + \mathrm{E} + \mathrm{G}, \mathrm{A} + \mathrm{B}]$$

$$r = 2d - \mathrm{A} - \mathrm{B} - 2\mathrm{E} - 2\mathrm{G},\ s = \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + \mathrm{G} + 2,\ t = \mathrm{A} + \mathrm{B}$$

Any valid weight assignment must have $\omega_2(3) = \mathrm{A}+\mathrm{B}$ because $\mathrm{A}, \mathrm{B} > 0$. Only the elements 2 and 4 may simultaneously have $\omega_2 = d$. Then $\omega_2(5) = \mathrm{E} + \mathrm{G}$ unless E or G is 0. If $\mathrm{E} = 0$ we may also have $\omega_2(6) = \mathrm{G}$. Hence the weight assignments are are $(1, 2, 3, 4, 5, 6)$ with sign 1 and, if $\mathrm{E} = 0$, $(1, 2, 3, 4, 6, 5)$ with sign $(-1)^{\mathrm{G}} = 1$. Thus S_9 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5 since $\mathrm{G} > 0$.

Tableau S_{10}

$$S_{10} = \begin{array}{cccccc} \scriptstyle 2 & & & & \scriptstyle\mathrm{A} & \\ \hline 1 & 5 & 5 & 1 & 6 & 5 \\ 2 & 4 & 2 & 2 & 4 & 3 \\ 3 & 3 & & & & \end{array}$$

$$2 \leq \mathrm{A} \leq 3$$
$$d = 4$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 4 & 1 & \mathrm{A}+1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [15 - \mathrm{A}, \mathrm{A} + 6, 3]$$

$$r = 9 - 2\mathrm{A},\ s = \mathrm{A} + 3,\ t = 3$$

Any valid weight assignment must have $\omega_3(3) = 3$. Other than 3, the only elements that can have $\omega_2 = d$ is 2, or 4 if $\text{A} = 3$. Thus there are no other valid weight assignments possible and the tableau is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

Tableau S_{11}

$$S_{11} = \begin{array}{cccccc} d-1 & & d-3 & d-1 & & \\ \hline 1 & 5 & 6 & 1 & 6 & 5 \\ 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & & & & & \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & d & d-2 & d & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [3d+1, 3d-2, 1]$$

$$r = 3,\ s = 3d-3,\ t = 1$$

Any valid weight assignment must have two of the elements 1, 2, and 3 with $\omega_2 > 0$. Since there are not enough 1's in the body for this to happen, we must have $\omega_2(2) = d$ and $\omega_2(3) = d-2$. This force $\omega_2(4) = d$. Hence there are no other valid weight assignments. Thus the tableau is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

Tableau S_{12}

$$S_{12} = \begin{array}{ccccccc} \text{A} & \text{B} & & & & & \text{C} \\ \hline 3 & 3 & 1 & 1 & 1 & 3 & 5 \\ 2 & 2 & 4 & 2 & 2 & 4 & 6 \\ 4 & & & & & & \end{array} \qquad \begin{array}{l} 0 \le \text{A} \le d-3 \\ 0 \le \text{B} \le d-2 \\ \text{A} \ge \text{B} \\ 0 \le \text{C} \le d \\ \text{C even} \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & \text{A}+3 & 0 & \text{B}+1 & 0 & \text{C} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [6d - \text{A} - \text{B} - \text{C} - 5, \text{A} + \text{B} + \text{C} + 4, 1]$$

$$r = 6d - 2\text{A} - 2\text{B} - 2\text{C} - 9,\ s = \text{A} + \text{B} + \text{C} + 3,\ t = 1$$

Any valid weight assignment must have $\omega_2(5 \text{ or } 6) = \text{C}$. We also must have

$\omega_2(2) = \text{A} + 3$ since $\text{A} \geq \text{B}$. Then $\omega_2(4) = \text{B} + 1$ unless $\text{B} = 0$, in which case $\omega_2(3) = 1$ is possible. Hence the weight assignments are $(1, 2, 3, 4, 5, 6)$ with sign 1 and $(1, 2, 3, 4, 6, 5)$ with sign $(-1)^{\text{C}}$. When $\text{B} = 0$ we also have $(1, 2, 4, 3, 5, 6)$ with sign $(-1)^2$, and $(1, 2, 4, 3, 6, 5)$ with sign $(-1)^{\text{C}+2}$. Since C is even, this sum is positive. Hence S_{12} is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.5.

7.7 Tableaux for $c = 7$ and $c = 8$

For $c = 7$ and 8, nearly all the required tableaux can be obtained by joining those tableaux already constructed. This is demonstrated in Chapter 8. However, we do need to construct one additional tableau, which is listed below.

<u>Tableau W_1</u>

$$S_1 = \frac{\begin{matrix} \mathsf{x} & \mathsf{x} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{Z} & \text{A} \end{matrix}}{\begin{matrix} 7 & 8 & 6 & 7 & 8 & 8 & 3 & 7 & 8 & 7 & 3 \\ 4 & 5 & 4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 \\ 1 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & \end{matrix}} \qquad \begin{array}{l} \text{A} \leq \mathsf{x} \\ \text{Z} = \frac{d-\mathsf{x}}{3} \\ d \equiv \mathsf{x} \pmod 3 \\ \text{A} = 0, 2 \end{array}$$

$$\omega_{2,3} = \left(\begin{smallmatrix} 0 & 0 & 0 & d & d & 2\text{Z}+\text{A} & 0 & 0 & \\ d & d & 2\text{Z} & 0 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$\lambda = [4d - 4\text{Z} - \text{A}, 2d + 2\text{Z} + \text{A}, 2d + 2\text{Z}]$$

$$r = 2d - 6\text{Z} - 2\text{A},\ s = \text{A},\ t = 2d + 2\text{Z}$$

Only the triples $(1, 2, 3)$ and $(4, 5, 6)$ may have $\omega_i = (d, d, 2\text{Z})$ or larger. Hence the only valid weight assignments are $(1, 2, 3, 4, 5, 6, 7, 8)$ with sign 1, and $(4, 5, 6, 1, 2, 3, 7, 8)$ with sign $(-1)^{2d+2\text{Z}+\text{A}}$. Since A is even, the weight sum in positive. Hence W_1 is non-zero.

Maximality: This tableau is maximal by Lemma 7.1.3.

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