Generalized Foulkes' Conjecture and Tableaux Construction

Thesis by

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Appendix A

Association between Tableaux Spaces and Irreducibles

Let $n = ab$ and $H = \mathcal{S}_b \wr \mathcal{S}_a$. Recall that $\mathcal{W}^{\lambda,a} = \{T|T$ a λ-tableau filled with 1 to a, each b times$\}$. We will explicitly show why the multiplicity of χ_λ in $1_H^{\mathcal{S}_n}$ equals the dimension of $\mathbb{C}\{\mathbf{q}_T|T \in \mathcal{W}^{\lambda,a}\}$.

View H as a subgroup of $\mathcal{S}_n$, where H acts on

$$1, 2, \ldots b|b+1, \ldots, 2b|\ldots|(a-1)b+1, \ldots, ab$$

by $\mathcal{S}_b$ on each block and by $\mathcal{S}_a$ permuting the blocks. The elements of H are the form $(\pi_1, \ldots \pi_a, \sigma)$ with $\pi_i \in \mathcal{S}_b$, $\sigma \in \mathcal{S}_a$. Now $\mathcal{S}_a \times \mathcal{S}_n$ acts on $\mathcal{W}^{\lambda,a}$ with $\mathcal{S}_a$ acting on the numbers 1 to a and $\mathcal{S}_n$ acting on the positions (corresponding to labelling across the rows).

Let $K = \{(\sigma^{-1}, (\pi_1, \ldots \pi_a, \sigma))|\pi_i \in \mathcal{S}_b,\ \sigma \in \mathcal{S}_a\}$. So $K \leq \mathcal{S}_a \times H \leq \mathcal{S}_a \times \mathcal{S}_n$. Let T be the λ-tableau filled across the rows with b 1's, then b 2's, etc. Then $\mathcal{S}_a \times \mathcal{S}_n$ acting on T gives $\mathcal{W}^{\lambda,a}$ and K fixes T. Specifically, $Stab_{\mathcal{S}_a \times \mathcal{S}_n}(T) = K$. Hence as $\mathcal{S}_a \times \mathcal{S}_n$ modules, $\mathcal{W}^{\lambda,a} \simeq 1_{Stab(T)}^{\mathcal{S}_a \times \mathcal{S}_n} = 1_K^{\mathcal{S}_a \times \mathcal{S}_n}$.

Proposition A.0.2. $\mathcal{W}^{\lambda,a} \simeq \sum_{\mu \vdash a} \varphi_{\mathcal{S}_a}(\mu) \otimes (\varphi_H(\mu))^{\mathcal{S}_n}$ where $\varphi_{\mathcal{S}_a}(\mu)$ is the irreducible of $\mathcal{S}_a$ indexed by μ and $\varphi_H(\mu)$ is the irreducible of $H/(\mathcal{S}_b \times \ldots \times \mathcal{S}_b) \simeq \mathcal{S}_a$ indexed by μ.

Proof. Since $1 \times (\mathcal{S}_b \times \ldots \times \mathcal{S}_b) \leq K$, it is in the kernel of 1_K. As $\mathcal{S}_b \times \ldots \times \mathcal{S}_b \trianglelefteq H$, it is in the kernel of $1_K^{\mathcal{S}_a \times H}$. So we can view $1_K^{\mathcal{S}_a \times H}$ as an $\mathcal{S}_a \times H/(\mathcal{S}_b \times \ldots \times \mathcal{S}_b) \simeq \mathcal{S}_a \times \mathcal{S}_a$ module. Let $D = \{(\sigma^{-1}, \sigma) | \sigma \in \mathcal{S}_a\}$ be the image of K in $\mathcal{S}_a \times H/(\mathcal{S}_b \times \ldots \times \mathcal{S}_b)$. Hence $1_K^{\mathcal{S}_a \times H} \simeq 1_D^{\mathcal{S}_a \times \mathcal{S}_a}$ as $\mathcal{S}_a \times H/(\mathcal{S}_b \times \ldots \times \mathcal{S}_b)$ modules. Thus we can write $1_D^{\mathcal{S}_a \times \mathcal{S}_a} = \sum a_{\mu,\nu} \phi_\mu \otimes \phi_\nu$ for $\mu, \nu \vdash a$ and some $a_{\mu,\nu}$, where ϕ is the corresponding irreducible of $\mathcal{S}_a$.

By Frobenius reciprocity $a_{\mu,\nu} = (\phi_\mu \otimes \phi_\nu, 1_D^{\mathcal{S}_a \times \mathcal{S}_a}) = (\phi_\mu \otimes \phi_\nu|_D, 1_D)_D$. Now $\phi_\mu \otimes \phi_\nu|_D = \phi_\mu \overline{\phi_\nu}$. So $(\phi_\mu \otimes \phi_\nu|_D, 1_D)_D = \frac{1}{|D|} \sum_{\sigma \in \mathcal{S}_a} \phi_\mu(\sigma) \overline{\phi_\nu(\sigma)} = \frac{|\mathcal{S}_a|}{|D|} \cdot (\phi_\mu, \phi_\nu)_{\mathcal{S}_a}$. Using row orthogonality and $|\mathcal{S}_a| = |D|$, we have $a_{\mu,\nu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ otherwise} \end{cases}$.

So $1_D^{\mathcal{S}_a \times \mathcal{S}_a} = \sum_{\mu \vdash a} \phi_\mu \otimes \phi_\mu$. If we lift back to the original module $1_K^{\mathcal{S}_a \times H}$ we have $1_K^{\mathcal{S}_a \times H} = \sum_\mu \phi_{\mathcal{S}_a}(\mu) \otimes \phi_H(\mu)$ where $\phi_{\mathcal{S}_a}(\mu)$ is the irreducible of $\mathcal{S}_a$ indexed by μ and $\phi_H(\mu)$ is the irreducible of $H/(\mathcal{S}_b \times \cdots \times \mathcal{S}_b) \simeq \mathcal{S}_a$ indexed by μ. Since $1_K^{\mathcal{S}_a \times \mathcal{S}_n} = (1_K^{\mathcal{S}_a \times H})^{\mathcal{S}_a \times \mathcal{S}_n}$ we get

$$1_K^{\mathcal{S}_a \times \mathcal{S}_n} = (\sum_\mu \phi_{\mathcal{S}_a}(\mu) \otimes \phi_H(\mu))^{\mathcal{S}_a \times \mathcal{S}_n} = \sum_\mu \phi_{\mathcal{S}_a}(\mu) \otimes (\phi_H(\mu))^{\mathcal{S}_n}.$$

□

By this proposition we have $\mathcal{W}^{\lambda,a} \simeq \sum_{\mu \vdash a} \varphi_{\mathcal{S}_a}(\mu) \otimes (\varphi_H(\mu))^{\mathcal{S}_n}$ as $\mathcal{S}_a \times \mathcal{S}_n$ modules. Consider the submodule on which $\mathcal{S}_a$ is trivial, that is, $\mu = (a)$. This corresponds to $1_{\mathcal{S}_a} \otimes (1_H)^{\mathcal{S}_n}$. If $1_H^{\mathcal{S}_n} = \sum_{\nu \vdash n} m_\nu \chi_\nu$, this module corresponds to $\sum_{\nu \vdash n} 1_{\mathcal{S}_a} \otimes m_\nu \chi_\nu$. Now $e_\lambda = \sum_{\sigma \in R_T} \sum_{\tau \in C_T} \epsilon(\tau) \sigma \tau$ is an idempotent of $\mathcal{S}_n$ on λ-tableau T. So the action of e_λ on $\sum 1 \otimes m_\nu \chi_\nu$ is the same as the action of $q_\lambda = \sum_{\pi \in \mathcal{S}_a} \pi e_\lambda$ on $\mathcal{W}^{\lambda,a}$. Then $q_\lambda \cdot \mathcal{W}^{\lambda,a} \simeq m_\lambda (e_\lambda \cdot \mathcal{S}^\lambda)$ as $\mathcal{S}_n$ modules, as $e_\lambda \cdot \mathcal{S}^\nu = 0$ for $\lambda \neq \nu$. Now $\mathcal{S}^\lambda$ is is a cyclic $\mathcal{S}_n$-module generated by $e_\lambda(T)$. (Correspondingly, the semi-standard tableaux which span $\mathcal{S}^\lambda$ are equivalent under the action of $\mathcal{S}_n$.) Therefore $\dim(e_\lambda \mathcal{S}^\lambda) = 1$ and $\dim(q_\lambda \mathcal{W}^\lambda) = m_\lambda$. Hence $\{\mathbf{q}_T | T \in \mathcal{W}^{\lambda,a}\}$, spans a module of dimension m_λ, the multiplicity of χ_λ in $1_{\mathcal{S}_b \wr \mathcal{S}_a}^{\mathcal{S}_n}$. This proof is due to Wales, [22].

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