GENERALIZED TRANSLATION OPERATORS.

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ABSTRACT

A study is made of generalized translation operators of the Delsarte-Levitan-Povzner type. After reviewing the method of associating such operators with linear second order differential equations, an abstract theory is developed with the aim of constructing an L_1-convolution algebra. The chief novelty is a device of comparing one family of translation operators with another "known" family. The Plancherel theorem and Bochner's theorem on positive definite functions are derived by the Krein-Godement method of locally compact group theory. An application to the classical Sturm-Liouville problem is discussed.

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CHAPTER 1

INTRODUCTION

If f is a function defined on the real line, and y is a real number, then the function whose value at x is $f(x+y)$ can be thought of as the result of applying a translation operator T^y to f. Thus

$$(T^y f)(x) = f(x + y).$$

More general translation operators can be obtained in a rather trivial way by replacing the additive group of the real line by other groups. In 1938 Delsarte [1], [2]* formulated an entirely new generalization of the notion of translation operator.

Delsarte had been interested in finding a formal generalization of the Taylor expansion formula,

$$f(x+y) = \sum_{n=o}^{\infty} \frac{y^n}{n!} \left(\frac{d^n}{dx^n} f\right)(x).$$

He regarded the functions $\phi_n(y) = \frac{y^n}{n!}$ as being related to the differential operator $L = \frac{d}{dx}$ in a very special way. The solution $\phi(x,\lambda)$ of

$$L\phi = \lambda \phi$$

which satisfies $\phi(0,\lambda) = 1$, is $\phi(x,\lambda) = e^{\lambda x}$. For each real x this is an entire function of λ and

$$\phi(x, \lambda) = \sum_{n=o}^{\infty} \phi_n(x) \lambda^n.$$

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* Numbers in square brackets refer to the bibliography at the end of the paper.

The functions ϕ_n satisfy

$$L\phi_o = 0, \qquad \phi_o(1) = 1,$$
$$L\phi_n = \phi_{n-1}, \qquad \phi_n(0) = 0, \qquad n = 1,2, \ldots .$$

The expansion formula is

$$f(x + y) = \sum_{n=o}^{\infty} \phi_n(y)\ (L^n f)(x).$$

Delsarte proposed to replace L by some other operator. It will be of particular interest to consider the case when

$$L = p(x) - \frac{d^2}{dx^2}$$

p being a suitably smooth function defined on the positive real axis. Let α be a given (complex) constant and let $\phi(x,\lambda)$ be the solution of

$$L\phi = \lambda\phi,$$
$$\phi(0,\lambda) = \sin\alpha,$$
$$\phi'(0,\lambda) = \cos\alpha.$$

Then

$$\phi(x,\lambda) = \sum_{n=o}^{\infty} \phi_n(x)\lambda^n,$$

where the functions ϕ_n satisfy

$$L\phi_o = 0, \quad \phi_o(0) = \sin\alpha, \quad \phi_o'(0) = \cos\alpha,$$
$$L\phi_n = \phi_{n-1}, \ \phi_n(0) = \phi_n'(0) = 0, \qquad n = 1,2, \ldots .$$

Delsarte's generalization of Taylor's formula for this case is

$$(T^y f)(x) = \sum_{n=o}^{\infty} \phi_n(y)(L^n f)(x).$$

The function $\phi(x,\lambda)$ considered as a function of x with λ fixed will be denoted by ϕ_λ. If f is a finite linear combination of the functions ϕ_λ, say

$$f = a_1\phi_{\lambda_1} + \cdots + a_n\phi_{\lambda_n}$$

then

$$(T^y f)(x) = a_1\phi_{\lambda_1}(y)\phi_{\lambda_1}(x) + \cdots + a_n\phi_{\lambda_n}(y)\phi_{\lambda_n}(x).$$

If $u(x,y) = (T^y f)(x)$ then u satisfies the hyperbolic equation

$$p(x)\, u(x,y) - u_{xx}(x,y) = p(y)\, u(x,y) - u_{yy}(x,y),$$

and the conditions

$$u(x,0) = f(x) \sin \alpha,$$

$$u_y(x,0) = f(x) \cos \alpha,$$

and is symmetric

$$u(y,x) = u(x,y).$$

Povzner [3] made systematic use of Riemann's method for solving the hyperbolic equation. This method leads to a representation of the solution u in the form

$$u(x,y) = \frac{\sin \alpha}{2} [f(x+y) + f(|x-y|)] + \int_{|x-y|}^{x+y} k(x,y,t)\, f(t)\, dt,$$

where k is continuous, and hence makes it possible to define the translation operators over a very large family of functions f.

Another generalization of the idea of translation operator is obtained from a system of orthogonal functions $\{\phi_n\}$ which have a multiplication table

$$\phi_m(x)\ \phi_n(x) = \sum_k c_{m,n,k}\ \phi_k(x).$$

For a suitable function f defined on the integers the translated function $T^m f$ with "displacement" m is defined by

$$(T^m f)\ (n) = \sum_k c_{m,n,k}\ f(k).$$

Translation operators of this kind were first studied by Levitan [4], although related earlier work had been done by Haar [5].

Levitan investigated the notions of positive definite function and almost periodic function in connection with systems of generalized translation operators [4], [6], [7], [8]. He formulated an axiomatic description of systems of generalized translation operators (see especially [8]) which was sufficiently general to include at the same time the translation operators of a (not necessarily abelian) locally compact group, the translation operators arising from a large class of linear second order differential equations, and translation operators arising from the multiplication tables of certain types of orthogonal function systems. In this general setting he developed a Plancherel theory based on the spectral theory of bounded normal operators on a Hilbert space.

Povzner investigated [3],[9],[10] the translation operators arising from linear differential equations on a half axis $g''(x) - p(x)\ g(x) = 0$ with the boundary condition $g'(0) = 0$, when the function p satisfies conditions such as $p(x) = O(x^{-n-\varepsilon})$ as $x \to \infty$, $n = 2,3,$. In these cases he constructed an analogue of the

L_1-algebra of a locally compact abelian group, with convolution defined by

$$(f*g)(x) = \int_0^\infty f(y)\ (T^y g)(x)\ dy.$$

He also gives a method for constructing a commutative normed ring (of operators rather than functions) which works, for example, when p is bounded and absolutely integrable over $(0,\infty)$.

From the work of Levitan it becomes clear that the generalized translation operators bear the same relation to a hypercomplex system as do ordinary translation operators to a group.

Let G be a locally compact group and let δ_x denote the measure consisting of one unit of mass located at the group element x. For any continuous function f with compact support

$$\delta_x(f) = \int f(t)\ d\,\delta_x(t) = f(x),$$

and for any Baire measure μ on G

$$\mu(f) = \int f(x)\ d\mu(x) = \int \delta_x(f)\ d\mu(x).$$

Thus μ is represented, in a certain sense, by

$$\mu = \int \delta_x\ d\mu(x)$$

as a linear combination of the base elements $\{\delta_x\}$. The convolution product of two base elements is

$$\delta_x * \delta_y = \delta_{x*y}$$

where $x*y$ is the group product. The group translation operators are defined by

$$f(x*y) = (T^y f)(x) = \int f(t)\, d(\delta_x * \delta_y)(t).$$

A generalized system of translation operators is represented in the form

$$(T^y f)(x) = \int f(t)\, d\mu_{x,y}(t)$$

where $\mu_{x,y}$ is a measure depending on x and y. Such a representation is provided, for example, by the Riemann solution of a hyperbolic equation, or by the multiplication constants $c_{x,y,t}$ of an orthogonal function system. The underlying space, over which the variable x ranges, is now to be thought of as the basis of a hypercomplex system. Each element of the hypercomplex system is a measure, that is, a linear combination of the basis elements δ_x:

$$\mu = \int \delta_x \, d\mu(x).$$

The multiplication table for the basis elements is

$$\delta_x * \delta_y = \mu_{x,y} = \int \delta_z \, d\mu_{x,y}(z).$$

Thus there emerges the idea of a hypercomplex system with a locally compact basis. Berezanskii and S.G. Krein in two papers [11],[12] discuss abelian hypercomplex systems with compact bases, in which the measures $\mu_{x,y} = \delta_x * \delta_y$ are all positive. In a third paper [13] Berezanskii studies the abelian discrete basis case.

The present paper contains five chapters. The first chapter is meant to provide an introduction to the subject. The second chapter contains reference material on Riemann's method of solving hyperbolic equations. In the third chapter the translation operators associated with a differential equation $\frac{d^2g}{dx^2} - p(x)\ g = 0$ on the positive axis $x \geq 0$ are constructed, and their general properties are derived. The main results here are not new, but lemma 3.4.2 is new and is used to provide a new proof for theorems 3.4.1 and 3.4.2. The only previous proofs of these theorems, due to Povzner [9] (see also Levitan [8]) were based on certain results of the classical Sturm-Liouville theory. Finally in section 3.5 a simple class of important examples, which does not appear to have been noted previously, is discussed.

The fourth chapter is by far the largest. The starting point here is a system of postulates which describe a family of translation operators. These postulates are satisfied by the translation operators described in chapter 3, by the translation operators associated with the classical Sturm-Liouville problem, and, in a large class of examples, by the family of translation operators generated by the multiplication table of a system of orthogonal polynomials. Those postulates which are purely algebraic are obtained by specializing the postulates of Levitan [8]. The very restrictive condition (4.1.6) is not postulated by Levitan, but on the other hand he makes assumptions concerning boundedness of the translation operators on the Lebesgue measure L_1-space which are not made here.

When the postulates are satisfied the space of continuous functions with compact supports can be made into a convolution algebra. Under additional assumptions this algebra can be enlarged and there is a measure whose L_1-space becomes a Banach algebra. In section 4.3 such a Banach algebra is constructed by a method, used by Berezanskii and S.G. Krein [11] for the compact case, and in section 4.4 the scope of the method is extended by a comparison technique which is new.

Under still further assumptions a "unitary" representation of the algebra on the Lebesgue measure L_2-space is constructed, and the Plancherel theorem and Bochner's theorem on positive definite functions are proved. The proofs of these theorems are adaptations of classical proofs used in locally compact group theory.

In chapter 5 the theory of chapter 4 is applied to the classical Sturm-Liouville problem.

CHAPTER 2

THE EQUATION $v_{st} - qv = 0$.

In this chapter a few well known results concerning the hyperbolic equation $v_{st} - qv = 0$ have been gathered together for convenient reference. These results are stated in the form of two theorems, the proofs of which are omitted.

2.1. S denotes the plane,

$$S = \{(s,t);\ -\infty < s,t < \infty\},$$

and the set of all functions f which are in $C^{(1)}(S)$ and for which the mixed second derivative $f_{s,t}$ exists and is continuous on S, is denoted by H. If E is a subset of S then $H(E)$ is defined similarly.

It is assumed throughout that q is a given function in $C(S)$. For some results additional smoothness properties will be assumed for q.

<u>Definition</u>: A function v is called a <u>solution</u> of the differential equation

$$v_{st} - qv = 0 \tag{2.1.1}$$

on a domain E provided $v \in H(E)$ and satisfies (2.1.1) on E.

<u>THEOREM 2.1.1</u>: For each point (s_o,t_o) in S there is a unique solution R of (2.1.1) on S which assumes the value one everywhere along the lines $s = s_o$ and $t = t_o$. The value of R at the point (s,t) is written as $R(s,t;\ s_o,t_o)$. Considered as a function of the four variables, R is called the <u>Riemann</u> <u>function</u> of (2.1.1).

The Riemann function is symmetric in (s,t) and (s_o,t_o), that is,

$$R(s,t;\ s_o,t_o) = R(s_o,t_o;\ s,t). \tag{2.1.2}$$

The function R, together with its derivatives R_s and R_t, is continuous on $S \times S$, and R satisfies the differential equation

$$R_{st}(s,t;\ s_o,t_o) - q(s,t)\, R(s,t;\ s_o,t_o) = 0 \tag{2.1.3}$$

and the boundary conditions

$$R(s_o,t;\ s_o,t_o) = 1, \qquad R(s,t_o;\ s_o,t_o) = 1. \tag{2.1.4}$$

Furthermore, R satisfies the integral equation

$$R(s,t;\ s_o,t_o) = 1 - \int_{s_o}^{s} d\sigma \int_{t}^{t_o} R(\sigma,\tau;\ s_o,t_o)\, q(\sigma,\tau)\, d\tau. \tag{2.1.5}$$

The value $R(s,t;\ s_o,t_o)$ is completely determined provided q is known only in the rectangle with vertices at (s,t), (s_o,t), (s_o,t_o) and (s,t_o). In fact

$$R(s,t;\ s_o,t_o) = \sum_{k=o}^{\infty} (-1)^k\, Q_k(s,t;\ s_o,t_o) \tag{2.1.6}$$

where

(2.1.7)

$$Q_o(s,t;\ s_o,t_o) = 1,$$

$$Q_n(s,t;\ s_o,t_o) = \int_{s_o}^{s} d\sigma \int_{t}^{t_o} Q_{n-1}(\sigma,\tau;s_o,t_o)\, q(\sigma,\tau)\, d\tau,$$

$$n \geq 1.$$

The series (2.1.6) converges uniformly on every compact set in $S \times S$, and so also do the series for R_s and R_t obtained by termwise differentiation of (2.1.6).

If $q \in C^{(1)}(S)$ then $R \in C^{(2)}(S \times S)$ and the functions R_s and R_t are twice differentiable with respect to the variables s_o, t_o, the derivatives being continuous on $S \times S$. If $q \in C^{(3)}(S)$ then $R \in C^{(4)}(S \times S)$.

Finally, R is continuously dependent on q. More precisely, let $\{q^{(n)}\}$, n = 0, 1, 2, ... be a sequence of coefficient functions and $\{R^{(n)}\}$ the corresponding sequence of Riemann functions. Suppose that $q^{(n)}$ converges to $q^{(0)}$ uniformly on every compact set as $n \to \infty$. Then $R^{(n)} \to R^{(0)}$, $R_s^{(n)} \to R_s^{(0)}$ and $R_t^{(n)} \to R_t^{(0)}$ as $n \to \infty$, the convergence of each sequence being uniform on every compact set in $S \times S$.

<u>THEOREM 2.1.2</u>: Let I be any interval, open, closed, or partly open, possibly infinite, on the real line, and let

$$E = \{(s,t);\ t \geq s,\ s \in I,\ t \in I\}.$$

Let $F \in C^{(1)}(I)$ and $G \in C(I)$. Then (2.1.1) has a unique solution v on E such that

$$v(r,r) = F(r), \qquad r \in I,$$

$$v_t(r,r) - v_s(r,r) = G(r), \qquad r \in I.$$

The solution is given by

$$(2.1.8) \qquad v(s_o,t_o) = \frac{1}{2}[F(t_o) + F(s_o)] + \frac{1}{2}\int_{s_o}^{t_o} [R_s(r,r;\ s_o,t_o) - R_t(r,r;s_o,t_o)]\ F(r)\ dr + \frac{1}{2}\int_{s_o}^{t_o} R(r,r;\ s_o,t_o)\ G(r)\ dr.$$

CHAPTER 3

TRANSLATION OPERATORS ON A HALF AXIS.

In this chapter the translation operators associated with the differential operator $-p(x) + \frac{d^2}{dx^2}$ on $0 \leq x < \infty$, and a homogeneous boundary condition at $x = 0$, are constructed, and their general properties studied.

3.1. The half line will be denoted by Ω, thus $\Omega = \{x;\ 0 \leq x < \infty\}$. The space $C(\Omega)$ will be denoted simply by C and the set of all functions f which belong to C and have compact supports will be denoted by C_o.

D_α is defined as the set of all functions f in $C^{(2)}(\Omega)$ which satisfy

$$f(0) \cos \alpha - f'(0) \sin \alpha = 0.$$

For the remainder of the chapter it is assumed that α is a given constant.

If $f \varepsilon C^{(2)}(\Omega)$ then Lf is the continuous function

$$(Lf)(x) = f''(x) - p(x)\ f(x). \tag{3.1.1}$$

It is assumed henceforth that p is a given continuous complex valued function on Ω. The following well known fact will be referred to as Green's theorem:

If f and g are in D_α and at least one of these functions has compact support then

$$\int_o^\infty f(x)\ (Lg)\ (x)\ dx = \int_o^\infty (Lf)\ (x)\ g(x)\ dx.$$

3.2. As indicated in Chapter 1, if $\{T^y f\}$ is the family of translates of a suitable function f and $u(x,y) = (T^y f)(x)$ then u will satisfy

$$(3.2.1) \qquad u_{xx} - u_{yy} - [p(x) - p(y)]\, u = 0,$$

together with

$$(3.2.2) \qquad \begin{cases} u(x,0) = f(x) \sin \alpha, \\ u_y(x,0) = f(x) \cos \alpha, \end{cases}$$

and the symmetry condition

$$(3.2.3) \qquad u(y,x) = u(x,y).$$

Introducting the characteristic coordinates

$$s = x - y$$
$$t = x + y$$

and writing $u(x,y) = v(s,t)$, (3.2.1) is transformed to

$$(3.2.4) \qquad v_{st} - qv = 0,$$

where

$$(3.2.5) \qquad q(s,t) = \frac{1}{4}\left[p\left(\frac{t+s}{2}\right) - p\left(\frac{t-s}{2}\right)\right].$$

The domain $\Omega \times \Omega$ is mapped onto the domain

$$(3.2.6) \qquad \Sigma = \{(s,t);\ t \geq 0, \quad -t \leq s \leq t\}.$$

The correspondence between $C^{(2)}(\Omega \times \Omega)$ solutions of (3.2.1) and $C^{(2)}(\Sigma)$ solutions of (3.2.4) is bi-unique. The initial conditions (3.2.2) become conditions on the line $s = t$;

$$(3.2.7)\qquad \begin{aligned} v(r,r) &= f(r)\sin\alpha, \\ v_t(r,r) - v_s(r,r) &= f(r)\cos\alpha, \end{aligned}$$

and the symmetry condition becomes

$$(3.2.8)\qquad v(-s,t) = v(s,t).$$

It is convenient to imagine that p has been extended to be a continuous function on the whole line $-\infty < x < \infty$, and that q is then defined by (3.2.5) over the whole (s,t)-plane S. If occasionally it is assumed that $p \varepsilon C^{(1)}(\Omega)$, then it will be understood that the extension has been made so that p remains of class $C^{(1)}$ on the whole line. It will be seen later that the translation operators will not be affected by the particular way in which this extension is made.

<u>LEMMA 3.2.1</u>: Assume that $p \varepsilon C^{(1)}(\Omega)$. Then for each $f \varepsilon D_\alpha$ there is a unique $v \varepsilon C^{(2)}(\Sigma')$, where

$$\Sigma' = \{ (s,t);\ \ t \geq 0,\ \ 0 \leq s \leq t \},$$

which satisfies (3.2.4) on Σ' and (3.2.7) for $r \geq 0$. This function is given by

$$(3.2.9)\qquad v(s_o,t_o) = \frac{\sin\alpha}{2}[f(t_o) + f(s_o)] + \frac{\sin\alpha}{2}\int_{s_o}^{t_o}[R_s(r,r;\ s_o,t_o) - R_t(r,r;s_o,t_o)]\,f(r)\,dr + \frac{\cos\alpha}{2}\int_{s_o}^{t_o} R(r,r;s_o,t_o)\,f(r)\,dr,$$

where R is the Riemann function of (3.2.4).

Furthermore when the domain of v is extended to Σ by means of relation (3.2.8), the extended function is of class $C^{(2)}(\Sigma)$ and satisfies (3.2.4) on Σ.

PROOF: By theorem (2.1.2), for any particular extension of p, (3.2.9) defines the unique solution of (3.2.4) on Σ' which satisfies (3.2.7) for $r \geq 0$. The values of the Riemann function and its derivatives involved in (3.2.9) do not depend on the way in which p has been extended. Since $p \in C^{(1)}(\Omega)$ and hence $q \in C^{(1)}(S)$, it follows from the differentiability properties of the Riemann function that (3.2.9) defines a function $v \in C^{(2)}(\Sigma')$. Thus v exists and is unique.

Since v is of class $C^{(2)}$ on the closed domain Σ', to show that it becomes of class $C^{(2)}(\Sigma)$ when extended from Σ' to Σ by relation (3.2.8), it is sufficient to show that the normal derivative of v on the boundary line $s = 0$ of Σ' is everywhere zero. This normal derivative (computed in Σ') is $v_s(0+,t)$, and is given by

$$v_s(0+,t) = v_s(0+,0) + \int_0^t v_{st}(0+,r)\, dr.$$

Since v satisfies (3.2.4) on Σ' and since $q(0,r) = 0$ for all $r \geq 0$, the integral vanishes and

$$v_s(0+,t) = v_s(0+,0).$$

Computing $v_s(0+,0)$ from (3.2.9), using the fact that $R(0,0;0,0) = 1$, one has

$$v_s(0+,0) = -\frac{1}{2}\,[f(0)\cos\alpha - f'(0)\sin\alpha]$$
$$- f(0)\,\frac{\sin\alpha}{2}\,[R_s(0,0;0,0) - R_t(0,0;0,0)].$$

The first term vanishes because $f \varepsilon D_\alpha$. It follows easily from (2,1.5) that $R_s(0,0;0,0) = R_t(0,0;0,0) = 0$. Consequently, the extended function is of class $C^{(2)}(\Sigma)$.

Since $q(-s,t) = q(s,t)$ the extended function satisfies (3.2.4) on Σ. With this the lemma is established.

THEOREM 3.2.1: Assume $p \varepsilon C^{(1)}(\Omega)$. Let $f \varepsilon D_\alpha$ and let

(3.2.10) $$E = \{ (x,y); \quad x \geq 0, \quad 0 \leq y \leq x \}.$$

Then there is a unique function $u \varepsilon C^{(2)}(E)$ which satisfies (3.2.1) on E and satisfies (3.2.2) for $x \geq 0$. This function is given by

(3.2.11) $$u(x,y) = \frac{\sin \alpha}{2} [f(x + y) + f(x - y)]$$

$$+ \frac{\sin \alpha}{2} \int_{x-y}^{x+y} [R_s(z,z;x-y,x+y) - R_t(z,z;x-y,x+y)] f(z) dz$$

$$+ \frac{\cos \alpha}{2} \int_{x-y}^{x+y} R(z,z;x-y,x+y) \; f(z) \; dz$$

where R is the Riemann function of (3.2.4).

Furthermore, when the domain of u is extended to $\Omega \times \Omega$ by the symmetry relation (3.2.3), the extended function is of class $C^{(2)}(\Omega \times \Omega)$ and satisfies (3.2.1) on $\Omega \times \Omega$.

PROOF: In view of the correspondence between $C^{(2)}$ solutions of (3.2.1) and $C^{(2)}$ solutions of (3.2.4), the theorem is merely a restatement of the preceding lemma.

3.3. Let

$$(3.3.1)\quad K(\xi,\eta,\zeta) = \frac{\sin\alpha}{2}\left[R_s(\zeta,\zeta;\xi,\eta) - R_t(\zeta,\zeta;\xi,\eta)\right] + \frac{\cos\alpha}{2} R(\zeta,\zeta;\xi,\eta)$$

where R is the Riemann function of (3.2.4). K is defined and continuous over the entire 3-dimensional space. If $0 \le \xi \le \zeta \le \eta$ then the value $K(\xi,\eta,\zeta)$ is completely determined by the behavior of $q(s,t)$ on the rectangle with vertices at (ξ,η), (ξ,ζ), (ζ,ζ), (ζ,η), and hence is completely determined by the behavior of $p(x)$ on the range $0 \le x \le \eta$.

<u>Definition</u>: The family of translation operators $\{T^y\}$, $y \in \Omega$, is defined on the space C as follows. For each $f \in C$, $T^y f$ is the function

$$(3.3.2)\quad (T^y f)(x) = \frac{\sin\alpha}{2}\left[f(x+y) + f(|x-y|)\right] + \int_{|x-y|}^{x+y} K(|x-y|, x+y, z)\, f(z)\, dz.$$

The preceding remarks concerning the function K imply that the translation operators are completely and uniquely determined when the function p on Ω and the constant α are known.

Certain properties of the operators are immediate consequences of the definition. For example:

(3.3.3) If $f \in C$ and $u(x,y) = (T^y f)(x)$ then $u \in C(\Omega \times \Omega)$. In particular $T^y f \in C$ for every $y \in \Omega$. If $f \in C_o$, say $f(x) = 0$ for $x \ge b$, then $T^y f \in C_o$, in fact $(T^y f)(x) = 0$ for $x \ge b + y$.

If $f, g \in C$, λ, μ are complex numbers and $x, y \in \Omega$ then

(3.3.4) $$(T^y f)(x) = (T^x f)(y),$$

(3.3.5) $$(T^o f)(x) = f(x) \sin \alpha,$$

(3.3.6) $$T^y(\lambda f + \mu g) = \lambda T^y f + \mu T^y g.$$

(3.3.7) If the functions f_n, $n = 0, 1, 2, \ldots$ all belong to C, and converge to f_o as $n \to \infty$, uniformly on every compact set in Ω, and if $u_n(x,y) = (T^y f_n)(x)$, then u_n converges to u_o uniformly on every compact set in $\Omega \times \Omega$. If $\sin \alpha = 0$, and if the functions f_n of C are uniformly bounded on $0 \leq x \leq 1$ and converge to f_o uniformly on every interval $\varepsilon \leq x \leq \frac{1}{\varepsilon}$, $\varepsilon > 0$, then u_n converges to u_o uniformly on every compact set in $\Omega \times \Omega$.

(3.3.8) Let $\{p_n\}$, $n = 0, 1, 2, \ldots$ be a sequence of coefficient functions and let $\{T_n^y\}$ be the corresponding families of translation operators. Suppose that p_n converges to p_o as $n \to \infty$, uniformly on every compact set in Ω. Let $f \in C$ and let $u_n(x,y) = (T_n^y f)(x)$. Then u_n converges to u_o, uniformly on every compact set in $\Omega \times \Omega$.

<u>Remarks</u>: Since K is continuous, (3.3.7) follows from (3.3.2) by Lebesgue's bounded convergence theorem. (3.3.8) expresses the fact that the Riemann function is continuously dependent on q (theorem 2.1.1).

(3.3.9) Assume $p \in C^{(1)}(\Omega)$. Let $f \in D_\alpha$, and let $u(x,y) = (T^y f)(x)$. Then u is of class $C^{(2)}(\Omega \times \Omega)$, satisfies (3.2.1) on $\Omega \times \Omega$ and satisfies (3.2.2) for $x \geq 0$. Moreover, if E is defined by (3.2.10) then any function of class $C^{(2)}(E)$ which satisfies (3.2.1) on E

and satisfies (3.2.2) for $x \geq 0$, coincides with u on E. The fact that $(T^y f)(x)$ satisfies (3.2.1) will sometimes be expressed by writing

$$\left[\frac{\partial^2}{\partial y^2} - p(y)\right] \cdot [(T^y f)(x)] = (LT^y f)(x).$$

3.4. In this section two additional properties of the operators $\{T^y\}$ are deduced. These properties give rise to the following two theorems:

<u>THEOREM 3.4.1</u>: (Self adjointness) If $f, g \varepsilon C$ and at least one of these functions has compact support then

$$\int_0^\infty f(x)\ (T^y g)(x)\ dx = \int_0^\infty (T^y f)(x)\ g(x)\ dx$$

for every $y \varepsilon \Omega$.

<u>THEOREM 3.4.2</u>: (Commutativity) For every $f \varepsilon C$ and all $y, z \varepsilon \Omega$

$$T^y T^z f = T^z T^y f.$$

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The theorems will be proved by means of several lemmas.

<u>LEMMA 3.4.1</u>: Let $g, h \varepsilon C$ and suppose at least one of these functions has compact support. Let

$$u(x,y) = \int_0^\infty (T^x g)(z)\ (T^y h)(z)\ dz.$$

Then $u \varepsilon C(\Omega \times \Omega)$.

If in addition it is assumed that $g,h \in D_\alpha$ then $u \in C^{(2)}(\Omega \times \Omega)$ and the partial derivatives of u can be computed by differentiation under the integral sign.

PROOF: There is a positive number b such that either $g(x) = 0$ for $x \geq b$ or $h(x) = 0$ for $x \geq b$. Let a be any positive number. Then when $x \leq a$ and $y \leq a$

$$(T^x g)(z)\ (T^y h)(z) = 0 \quad \text{for} \quad z \geq a + b,$$

and hence

$$u(x,y) = \int_0^{a+b} (T^x g)(z)\ (T^y h)(z)\ dz.$$

On account of (3.3.3) it follows that u is continuous on the square

$$\left\{ (x,y);\ 0 \leq x \leq a,\ 0 \leq y \leq a \right\}.$$

Since a is arbitary, $u \in C(\Omega \times \Omega)$.

The rest of the lemma follows by a similar argument from (3.3.9).

LEMMA 3.4.2: If $h \in C$ and $(T^y h)(y) = 0$ for all $y \in \Omega$ then $h = 0$.

PROOF: From (3.3.2)

$$\frac{\sin \alpha}{2} [h(2y) + h(0)] + \int_0^{2y} K(0,2y,z)\ h(z)\ dz = 0$$

for all $y \geq 0$. Putting $y = 0$ gives $\sin \alpha\ h(0) = 0$. Hence h satisfies the integral equation

$$\sin \alpha \cdot h(y) + 2\int_0^{y} K(0,y,z)\ h(z)\ dz = 0.$$

If $\sin\alpha \neq 0$, since K is continuous, this imples that $h = 0$. If $\sin\alpha = 0$ then

$$K(0,y,z) = \frac{\cos\alpha}{2} R(z,z;0,y)$$

and it follows by differentiating with respect to y and using $R(y,y;0,y) = 1$, that h satisfies the integral equation

$$h(y) + \int_0^y R_{t_0}(z,z;0,y)\, h(z)\, dz = 0.$$

Since R_{t_0} is continuous it follows that $h = 0$ in this case also. Thus the lemma is true.

LEMMA 3.4.3: Assume the $p \in C^{(1)}(\Omega)$. Let $f \in D_\alpha$ and $g \in D_\alpha \cap C_0$. Let

$$h(x) = \int_0^\infty [(T^x f)(z)\, g(z) - f(z)(T^x g)(z)]\, dz,$$

$$u(x,y) = \int_0^\infty [(T^x f)(z)(T^y g)(z) - (T^y f)(z)(T^x g)(z)]\, dz.$$

Then $h(x) = 0$ and $u(x,y) = 0$ for all x and y in Ω.

PROOF: According to lemma (3.4.1), $h \in D_\alpha$, $u \in C^{(2)}(\Omega \times \Omega)$, and the partial derivatives of u can be computed by differentiation under the integral sign. In particular

$$\left.\begin{aligned} u(x,0) &= h(x) \sin\alpha \\ u_y(x,0) &= h(x) \cos\alpha \end{aligned}\right\} x \geq 0,$$

and, using (3.3.9),

$$u_{xx}(x,y) - p(x)\, u(x,y) = \int_0^\infty [(LT^x f)(z)(T^y g)(z)$$

$$- (T^y f)(z)(LT^x g)(z)]\, dz,$$

and

$$u_{yy}(x,y) - p(y)\, u(x,y) = \int_0^\infty [(T^x f)(z)(LT^y g)(z)$$

$$- (LT^y f)(z)(T^x g)(z)]\, dz.$$

By Green's theorem, the right-hand sides of the last two equations are identical. Hence u satisfies (3.2.1) on $\Omega \times \Omega$. It follows from (3.3.9) that

$$u(x,y) = (T^y h)(x) \quad \text{for} \quad 0 \leq y \leq x.$$

But it is clear that $u(y,y) = 0$ for all $y \geq 0$. Hence $(T^y h)(y) = 0$ for all y in Ω and $h = 0$ by the preceding lemma. Therefore, $u(x,y) = 0$ for $y \leq x$. But from the definition of u it is clear that

$$u(y,x) = - u(x,y).$$

Hence $u = 0$ and the proof is complete.

PROOF OF THEOREM 3.4.1: If $p \in C^{(1)}(\Omega)$ and $f,g \in D_\alpha$, the asserted equality is valid by virtue of lemma 3.4.3. Its validity in the general case will be established by approximation arguments.

First suppose $p \in C^{(1)}(\Omega)$. If $\sin \alpha \neq 0$ then there are sequences $\{f_n\}$ and $\{g_n\}$ in D_α which converge to f and g respectively, uniformly on every compact set in Ω. If $\sin \alpha = 0$

then every function in D_α vanishes at $x = 0$, but it is true that there are sequences $\{f_n\}$ and $\{g_n\}$ in D_α which converge to f and g respectively on $0 < x < \infty$, uniformly on every interval $\varepsilon \leq x \leq \frac{1}{\varepsilon}$, $\varepsilon > 0$, each sequence being uniformly bounded on $0 \leq x \leq 1$. Furthermore, since g has compact support, it can be assumed that all the functions g_n vanish outside of some fixed compact set. It follows from (3.3.7) and lemma (3.4.3) that

$$\int_0^\infty f(x)\ (T^y g)(x)\ dx - \int_0^\infty (T^y f)(x)\ g(x)\ dx$$

$$= \lim_{n\to\infty} \left[\int_0^\infty f_n(x)(T^y g_n)(x)\ dx - \int_0^\infty (T^y f_n)(x)\ g_n(x)\ dx \right]$$

$$= 0.$$

This proves the theorem for the case when $p \in C^{(1)}(\Omega)$.

Since each of the integrals is really over a finite range (for any given y), the result for the case when p is merely continuous now follows from (3.3.8). This completes the proof.

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The following lemma, which is easily deduced from theorem 3.4.1, is recorded here for later use.

<u>LEMMA 3.4.4</u>: If $f \in D_\alpha$ and $p \in C^{(1)}(\Omega)$ then

$$T^y Lf = LT^y f$$

for each $y \in \Omega$.

<u>PROOF</u>: Let $g \in D_\alpha \cap C_o$. Using (3.3.9) and theorem (3.4.1) one has

$$\int_0^\infty (LT^y f)(x)\ g(x)\ dx = \left[\frac{d^2}{dy^2} - p(y)\right] \cdot \int_0^\infty (T^y f)(x)\ g(x)\ dx$$

$$= \left[\frac{d^2}{dy^2} - p(y)\right] \cdot \int_0^\infty f(x)\ (T^y g)(x)\ dx$$

$$= \int_0^\infty f(x)\ (LT^y g)(x)\ dx.$$

By Green's theorem the right-hand side is equal to

$$\int_0^\infty (Lf)(x)(T^y g)(x)\ dx$$

which, using theorem 3.4.1 again, is equal to

$$\int_0^\infty (T^y Lf)(x)\ g(x)\ dx.$$

Hence,

$$\int_0^\infty [(LT^y f)(x) - (T^y Lf)(x)]\ g(x)\ dx = 0.$$

Since g is any element of $C_o \cap D_\alpha$, it follows that

$$(LT^y f)(x) - (T^y Lf)(x) = 0,$$

for almost all x, and therefore, by continuity, for all x. This proves the lemma.

PROOF OF THEOREM 3.4.2: First suppose $p \in C^{(1)}(\Omega)$. Let $f_n \in D_\alpha$ and $g \in D_\alpha \cap C_o$. From lemma 3.4.3,

$$\int_0^\infty (T^z f_n)(x)(T^y g)(x)dx = \int_0^\infty (T^y f_n)(x)(T^z g)(x)dx.$$

For fixed y and z these integrals are both really over a finite range. Hence by virtue of (3.3.8) the equality is valid even if p is merely continuous.

Now let $f_n \to f$ where f is any given element of C, the convergence being either uniform on every compact set in case $\sin \alpha \neq 0$, or else bounded on $0 \leq x \leq 1$ and uniform on every interval $\varepsilon \leq x \leq \frac{1}{\varepsilon}$, $\varepsilon > 0$, in case $\sin \alpha = 0$. According to (3.3.7), it is permitted to pass to the limit under the integral signs in the above equality, obtaining

$$\int_0^\infty (T^z f)(x)(T^y g)(x)\ dx = \int_0^\infty (T^y f)(x)(T^z g)(x)\ dx.$$

Applying theorem 3.3.1 to each member of this equation gives

$$\int_0^\infty (T^y T^z f)(x)\ g(x)\ dx = \int_0^\infty (T^z T^y f)(x)\ g(x)\ dx.$$

Since g is an arbitrary element of $C_o \cap D_\alpha$, it follows that

$$(T^y T^z f)(x) = (T^z T^y f)(x)$$

for almost all x, and by continuity, for all x. This proves the theorem.

3.5. A simple class of important examples will be discussed.

If $q(s,t) = \frac{1}{4}\left[p(\frac{t+s}{2}) - p(\frac{t-s}{2})\right]$ splits into a product

$$q(s,t) = \Theta(s)\Psi(t), \qquad 0 \leq s \leq \Psi \tag{3.5.1}$$

then the Riemann function assumes a particularly simple form. The functions Q_n of (2.1.7) are given by

$$Q_o(s,t;s_o,t_o) = 1,$$

$$(3.5.2) \quad Q_1(s,t;s_o,t_o) = \int_{s_o}^{s} d\sigma \int_{t}^{t_o} \theta(6)\, \Psi(\tau)\, d\tau ,$$

and

$$(3.5.3) \quad Q_n(s,t;s_o,t_o) = \frac{1}{(n!)^2} [Q_1(s,t;s_o,t_o)]^n$$

for $n = 2, 3, \ldots$, and hence

$$(3.5.4) \quad R(s,t;s_o,t_o) = \sum_{k=o}^{\infty} (-1)^k \frac{1}{(k!)^2} [Q_1(s,t;s_o,t_o)]^k ,$$

which can be expressed in terms of Bessel functions. If p is monotone increasing then $q(s,t)$ is non-negative for $0 \leq s \leq t$ and hence $Q_1(r,r;s_o,t_o)$ is non-negative for $0 \leq s_o \leq t_o$ and $s_o \leq r \leq t_o$. In this case the series (3.5.4) is conveniently compared with the Bessel function

$$(3.5.5) \quad J_o(z) = \sum_{k=o}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} .$$

Thus

$$(3.5.6) \quad R(r,r;s_o,t_o) = J_o(2\sqrt{Q_1(r,r;s_o,t_o)}\ ).$$

Similarly if p is monotone decreasing then $Q_1(r,r;s_o,t_o)$ is non-positive for $0 \leq s_o \leq r \leq t_o$, and in terms of

$$(3.5.7) \quad I_o(z) = \sum_{k=o}^{\infty} \frac{1}{(k!)^2} \left(\frac{z}{2}\right)^{2k} ,$$

one has

$$(3.5.8) \quad R(r,r;s_o,t_o) = I_o(2\sqrt{-Q_1(r,r;s_o,t_o)}\ ).$$

The simplest example of this type arises when p is a constant function, in which case the Riemann function is identically equal to one. Although there are not many other examples of this type, they are all examples of importance. The next lemma determines the possible forms of Θ and Ψ in (3.5.1).

<u>LEMMA 3.5.1</u>: Let h be a non-constant function defined and continuous on Ω and suppose there are finite valued functions Θ, Ψ defined on Ω such that

(3.5.9) $$h(t+s) - h(t-s) = \Theta(t)\,\Psi(s)$$

whenever $0 \leq s \leq t$. Then $\Psi \in C^{(2)}(\Omega)$ and $\Theta \in C^{(2)}(t > 0)$, and there is a complex number λ such that

$$\Psi''(s) + \lambda\Psi(s) = 0, \qquad s \geq 0,$$
$$\Theta''(t) + \lambda\Theta(t) = 0, \qquad t > 0.$$

<u>PROOF</u>: Since h is continuous and not constant, given $\varepsilon > 0$ one can find numbers x,y such that $0 < y < x$, $x - y < \varepsilon$, and $h(x) \neq h(y)$. Hence

$$0 \neq h(x) - h(y) = \Theta\left(\frac{x+y}{2}\right)\,\Psi\left(\frac{x-y}{2}\right),$$

which shows that neither Θ nor Ψ is identically zero and that Ψ assumes non-zero values in every neighborhood of $s = 0$. Now if $s_o > 0$ and $\Psi(s_o) \neq 0$, then (3.5.9) shows that Θ is continuous on the interval $t \geq s_o$. Hence it has been proved that Θ is continuous on $t > 0$.

Next it will be shown that h is not constant on any interval $x \geq b$. Suppose there is a finite $b > 0$ such that $h(x) = h(b)$ for all $x \geq b$. Choose $\varepsilon > 0$, $0 < \varepsilon < b$ so that $\Psi(\varepsilon) \neq 0$. Then

$$0 = h(b + 2\varepsilon) - h(b) = \theta(b + \varepsilon)\ \Psi(\varepsilon),$$

and hence $\theta(b + \varepsilon) = 0$. Since ε can be chosen arbitrarily small, it follows by continuity of θ that $\theta(b) = 0$. Now if $0 \leq s \leq b$ then

$$\begin{aligned} 0 = \theta(b)\ \Psi(s) &= h(b + s) - h(b - s) \\ &= h(b) - h(b - s) \end{aligned}$$

showing that h is a constant function. This contradicts the hypotheses, and hence h is not constant on any interval $x \geq b$.

Since

$$h(2t) = h(0) + \theta(t)\ \Psi(t)$$

it follows that $\theta(t)$ is different from zero for arbitrarily large values of t. This, together with (3.5.9) shows that $\Psi \varepsilon C(\Omega)$.

Now there are arbitrarily small positive numbers a such that

$$\int_0^a \Psi(s)\ ds \neq 0.$$

If $t \geq a$, then

$$\begin{aligned} \theta(t) \int_0^a \Psi(s)\ ds &= \int_0^a [h(t + s) - h(t - s)]\ ds \\ &= \int_t^{t+a} h(x)\ dx + \int_t^{t-a} h(x)\ dx, \end{aligned}$$

and hence $\theta \varepsilon C^{(1)}$ $(t > 0)$. Furthermore, if $t \geq a$ then

$$\begin{aligned} \theta'(t) \int_0^a \Psi(s)\ ds &= [h(t + a) - h(t)] - [h(t) - h(t - a)] \\ &= \theta(t + \tfrac{a}{2})\ \Psi(\tfrac{a}{2}) - \theta(t - \tfrac{a}{2})\ \Psi(\tfrac{a}{2}). \end{aligned}$$

Consequently $\theta \varepsilon C^{(\infty)}$ $(t > 0)$.

Since θ assumes non-zero values on every interval $t \geq b$, for any $b > 0$ there is a number $c > b$ such that

$$\int_b^c \theta(t)\, dt \neq 0.$$

If $0 \leq s \leq b$ then

$$\text{(3.5.10)} \quad \Psi(s) \int_b^c \theta(t)\, dt = \int_{b+s}^{c+s} h(x)\, dx - \int_{b-s}^{c-s} h(x)\, dx.$$

Consequently $\Psi \in C^{(1)}(\Omega)$. It now follows from

$$h(2t) = h(0) + \theta(t)\, \Psi(t)$$

that $h \in C^{(1)}$ $(t > 0)$. Hence (3.5.10) shows that $\Psi \in C^{(2)}(\Omega)$.

Now if $0 \leq s \leq t$, $t > 0$ then

$$\theta''(t)\, \Psi(s) = h''(t+s) - h''(t-s) = \theta(t)\, \Psi''(s)$$

and since $\theta(t_o) \neq 0$ for some t_o, the conclusion follows with

$$\lambda = -\frac{\theta''(t_o)}{\theta(t_o)}.$$

<u>Remarks</u>: The converse of the lemma is rather trivial. The function Ψ vanishes at $s = 0$ but does not vanish identically. Hence $\Psi'(0) \neq 0$, and without loss of generality it can be assumed that $\Psi'(0) = 1$. Thus

$$\Psi(s) = \begin{cases} \dfrac{\sin \sqrt{\lambda}\, s}{\sqrt{\lambda}}, & \text{if } \lambda \neq 0 \\ s, & \text{if } \lambda = 0. \end{cases}$$

With this choice of Ψ, it is an easy matter to verify that if θ is any non-zero solution of

$$\theta'' + \lambda\theta = 0$$

then $h(t) = A + \theta(\frac{t}{2})\,\Psi(\frac{t}{2})$ defines a non-constant continuous solution of (3.5.9), A being an arbitrary constant.

The following examples are particularly noted:

$$p(x) = \pm x,\ \ \pm x^2,\ \ \pm e^{Bx},\ \ \pm \cos Bx,$$

$$\pm \sin Bx,\ (\pm \cosh Bx,\ \pm \sinh Bx),$$

where B is any complex number.

CHAPTER 4

THE ABSTRACT THEORY

In this chapter a space Ω and an associated family of translation operators $\{T^y\}$ are described axiomatically. The space Ω of the previous chapter and its family of translation operators $\{T^y\}$ form a special instance of the abstract structure so described. Other applications are discussed later.

The problem of finding a measure on Ω such that the L_1-space of this measure can be made into a convolution algebra is then considered and a method depending on the comparison of one system of translation operators with a "known" system is developed. Once such an algebra has been constructed it is possible, under certain fairly general circumstances, to prove various theorems analogous to theorems in the theory of harmonic analysis on locally compact abelian groups. Analogues of Bochner's theorem on positive definite functions and the Plancherel theorem are proved, assuming that the operators $\{T^y\}$ can be represented in a suitable way as bounded operators on the space $L_2(\Omega)$ (Lebesgue measure).

The reader is assumed to be familiar with the material of chapter X in Halmos' book [14] on measure theory, where functions and measures on locally compact Hausdorff spaces are discussed. In particular the following result is frequently useful.

PROPOSITION: Let K be a compact set and F a closed set disjoint from K, in a locally compact Hausdorff space Ω . Then there is a real valued continuous function f on Ω such that

$$f(x) = 0 \quad \text{if} \quad x \in F,$$

$$f(x) = 1 \quad \text{if} \quad x \in K,$$

and

$$0 \leq f(x) \leq 1 \quad \text{for all} \quad x.$$

The reader should also be familiar with the theory of Banach algebras as set forth, for example, in the book of Loomis [15], chapters IV and V.

4.1. It is assumed that a locally compact separable Hausdorff space Ω is given and that a non-negative Baire measure on Ω is prescribed. This measure will be called the Lebesgue measure on Ω and the integral of a function f with respect to it will be denoted by

$$\int f(x)\, dx.$$

The assumption that Ω is separable implies that every Borel set is a Baire set and that the space Ω is a countable union of compact sets.

It is assumed that

(4.1.1) Ω consists of more than one point.

Any non-negative Baire measure on a locally compact Hausdorff space has the properties that

(4.1.2) every compact Baire set is of finite measure

and

(4.1.3) the measure is regular. That is the measure of any Baire set E is the common value of the inf of the measures of the open Baire sets containing E and the sup of the measures of the compact Baire sets contained in E.

It is assumed that

(4.1.4) every non-void open set in Ω has non-zero Lebesgue measure.

The set of all continuous complex valued functions on Ω is denoted by C, the set of all functions in C with compact supports is denoted by C_o. The set of all regular Baire measures on Ω is denoted by $\mathcal{M}$.

Finally it is assumed that a mapping $T^y : f \to T^y f$ of C into itself is defined with the following properties.

(4.1.5) If $f \varepsilon C$ and $u(x,y) = (T^y f)(x)$ then $u \varepsilon C(\Omega x \Omega)$.

(4.1.6) If $f \varepsilon C_o$ and K is any compact set in Ω then there is a compact set K_1 such that for each $y \varepsilon K$

$$(T^y f)(x) = 0$$

whenever x is outside K_1.

(4.1.7) $(T^y f)(x) = (T^x f)(y)$

for all $f \varepsilon C$ and all $x,y \varepsilon \Omega$.

(4.1.8) There is a special point in Ω, denoted by 0, and a constant α such that

$$(T^0 f)(x) = f(x) \sin \alpha$$

for every $f \varepsilon C$ and all $x \varepsilon \Omega$.

(4.1.9) For all $f,g \varepsilon C$, all complex numbers λ, μ and all $y \varepsilon \Omega$,

$$T^y(\lambda f + \mu g) = \lambda T^y f + \mu T^y g.$$

(4.1.10) $$T^y T^z f = T^z T^y f$$

for all $f \in C$ and all $y, z \in$.

(4.1.11) $\quad f(x)\ (T^y g)(x)\ dx = \quad (T^y f)(x)\ g(x)\ dx$

for all $y \in \Omega$ provided $f \in C$, $g \in C_o$.

(4.1.12) If the functions f_n in C converge to zero, uniformly on every compact set, then for each fixed y the functions $T^y f_n$ converge to zero, uniformly on every compact set.

Additional assumptions are made in later sections.

4.2. If f and g are in C and at least one of these functions has compact support, then by (4.1.6), for each fixed x,

$$f(y)\ (T^y g)(x) = f(y)(T^x g)(x)$$

vanishes when y is outside of some compact set. Consequently, the integral

(4.2.1) $\quad (f * g)(x) = \quad f(y)(T^y g)(x)\ dy$

is finite.

<u>Definition</u>: If f and g are in C and at least one of these functions is in C_o, the <u>convolution product</u> $f * g$ is the function given by (4.2.1).

Because of (4.1.11),

(4.2.2) $\quad f * g = g * f.$

<u>LEMMA 4.2.1</u>: $f * g \in C$ for all $f \in C_o$, $g \in C$.

PROOF: By (4.1.5) $(T^y g)(x)$ is jointly continuous in x and y. The result follows by a standard compactness argument.

If f and g are both in C_o it follows from (4.1.6) and (4.2.1) that $f * g$ vanishes outside some compact set and hence $f * g \varepsilon C_o$.

LEMMA 4.2.2: Let $f, g \varepsilon C$ and suppose one of these functions is in C_o. Then

$$T^y(f * g) = f * T^y g$$

for all $y \varepsilon$.

PROOF: Let $h \varepsilon C_o$. By (4.1.11)

$$\int (T^y(f * g))(x)\, h(x)\, dx = \int (f * g)(x)(T^y h)(x)\, dx$$

$$= \int (T^y h)(x)\left[\int f(z)(T^z g)(x) dz\right] dx.$$

Now $T^y h$ has a compact support, say K. Either because f has compact support or because of (4.1.6) together with the fact that g has compact support, there is a compact set K_1 such that for every $x \varepsilon K$,

$$f(z)(T^z g)(x) = 0$$

whenever z is outside K_1. Consequently the iterated integral above is equal to a double integral over the compact set $K \times K_1$. First using Fubini's theorem and then (4.1.11) and (4.1.10), one obtains

$$\int (T^y(f*g))(x) h(x)\, dx = \int f(z)\left[\int (T^z g)(x)(T^y h)(x)\, dx\right] dz$$

$$= \int f(z)\left[\int (T^z T^y g)(x)\, h(x)\, dx\right] dz.$$

Using Fubini's theorem again

$$\int (T^y(f*g))(x)\ h(x)\ dx = \int h(x)[\int f(z)(T^zT^yg)(x)\ dz]\ dx$$

$$= \int h(x)(F*T^yg)(x)\ dx.$$

Since h is an arbitrary element of C_o, it follows that the continuous function

$$T^y(f*g) - f*T^yg$$

vanishes almost everywhere on Ω. Because of (4.1.4) this implies that the function is zero, and the lemma is established.

<u>LEMMA 4.2.3</u>: Let $f,g \in C_o$ and $h \in C$. Then

$$f*(g*h) = (f*g)*h,$$

each term being defined since $g*h \in C$ and $f*g \in C_o$.

<u>PROOF</u>: Using lemma 4.2.2 and (4.2.2),

$$(f*(g*h))(x) = (f*(h*g))(x)$$

$$= \int f(y)(T^y(h*g))(x)\ dy$$

$$= \int f(y)[\int (T^yg)(z)(T^zh)(x)\ dz]\ dy.$$

Now f has compact support say K, and there is a compact set K_1 such that the support of T^yg is contained in K_1 for every $y \in K$. Hence the iterated integral is equal to a double integral over the compact set $K \times K_1$. Hence by Fubini's theorem

$$(f*(g*h))(x) = \int (T^zh)(x)[\int f(y)(T^yg)(z)\ dy]\ dz$$

$$= \int (T^z h)(x)(f*g)(z)\, dz$$

$$= ((f*g)*h)(x),$$

and this proves the lemma.

THEOREM 4.2.1: The linear space C_o, provided with the multiplication $f*g$, becomes a commutative complex algebra.

PROOF: It has been seen that $f*g$ is in C_o when f and g are in C_o, and that $f*g = g*f$. The associativity of multiplication follows from lemma 4.2.3. It is clear that if λ is a complex number and $f, g, h \in C_o$ then

$$f*(g + h) = f*g + f*h$$

and

$$f*\lambda g = \lambda(f*g).$$

An element of $\mathcal{M}$, that is, a regular Baire measure μ on Ω, besides being regarded as a set function can be regarded as a complex linear functional on C_o. The value of the functional at an element f of C_o is

$$\mu(f) = (f,\mu) \equiv \int f(x)\, d\mu(x).$$

By a compactness argument similar to that used in the proof of lemma 4.2.1 it can be shown that for each μ in $\mathcal{M}$ and each f in C_o, $(T^y f, \mu)$ is a continuous function of y on Ω.

LEMMA 4.2.4: Let σ be a non-zero element of $\mathcal{M}$ such that

(4.2.3) $\quad (f*g,\sigma) = (f,\sigma)(g,\sigma) \qquad$ for all $f,g \in C_o$.

Then σ is absolutely continuous with respect to the Lebesgue measure of Ω, and the density function $\phi(x) = \frac{d\sigma(x)}{dx}$ is (essentially)* continuous and satisfies

$$(4.2.4) \qquad (g,\sigma)\phi(x) = (T^x g,\sigma) = (\phi * g)(x), \qquad \text{all } x \in \Omega,$$

for every $g \in C_o$. Furthermore

$$(4.2.5) \qquad T^y\phi = \phi(y)\phi \qquad \text{all } y \in \Omega.$$

Conversely every non-trivial continuous solution ϕ of (4.2.5) is the density function of a non-zero measure which satisfies (4.2.3), σ being given by

$$(4.2.6) \qquad (f,\sigma) = \int f(x)\phi(x)\, dx.$$

<u>PROOF</u>: Let $f, g \in C_o$. Then

$$\begin{aligned} ((g,\sigma)f,\sigma) &= (f,\sigma)(g,\sigma) \\ &= (f*g,\sigma) \\ &= \int [\int f(x)(T^x g)(y)\, dx]\, d\sigma(y). \end{aligned}$$

Since $f(x)(T^x g)(y)$ vanishes outside a compact set in the (x,y)-plane, the order of integration can be inverted, giving

$$((g,\sigma)f,\sigma) = \int f(x)(T^x g,\sigma)\, dx.$$

Since $\sigma \neq o$, there is an element $g_o \in C_o$ such that $(g_o,\sigma) = 1$. Putting $g = g_o$ in the above formula, it follows from the fact that the

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* That is, amongst the family of functions any one of which can be taken as the density function, and any two of which are equal almost everywhere, there is a continuous function.

formula is valid for every $f \in C_o$ that σ is absolutely continuous with respect to the Lebesgue measure of Ω and that

$$\phi(x) \equiv \frac{d\sigma(x)}{dx} = (T^x g_o, \sigma).$$

This function is (essentially) continuous. It now follows that for arbitrary f and g in C_o

$$\int (g,\sigma)\ f(x)\ \phi(x)\ dx = \int f(x)\ (T^x g,\sigma)\ dx.$$

Hence $(g,\sigma)\ \phi(x) = (T^x g,\sigma)$ for almost all x, and by continuity, for all x. Moreover

$$(T^x g,\sigma) = \int (T^x g)(z)\ \phi(z)\ dz = (\phi * g)(x).$$

Consequently, $\phi(x) = (\phi * g_o)(x)$ and by lemma 4.2.2

$$\begin{aligned}(T^y\phi)(x) &= (\phi * T^y g_o)(x)\\ &= (T^x T^y g_o,\sigma)\\ &= (T^y g_o,\sigma)\ \phi(x)\\ &= (g_o,\sigma)\ \phi(y)\ \phi(x)\\ &= \phi(y)\ \phi(x).\end{aligned}$$

Now suppose ϕ is any continuous solution of (4.2.5). Then (4.2.6) defines a measure σ, and for any $f,g \in C_o$

$$(f*g,\sigma) = \int [\ \int f(y)(T^y g)(x)\ dy]\ \phi(x)\ dx.$$

First using Fubini's theorem, then (4.1.11), then (4.3.3) one finds

$$(f*g,\sigma) = \int f(y)[\int (T^y g)(x)\ \phi(x)\ dx]\ dy$$

$$= \int f(y)[\int g(x)\ (T^y\phi)(x)\ dx]\ dy$$

$$= (f,\sigma)(g,\sigma).$$

This completes the proof.

<u>Definition</u>: Every non-trivial continuous solution of (4.2.5) will be called an <u>eigenfunction</u>.

The next theorem deals with the special situation of the previous chapter.

<u>THEOREM 4.2.2</u>: Let the symbols Ω, T^y, C, and C_0 have the meanings assigned to them in chapter 3. Then there is a one-one correspondence between the set of all eigenfunctions and the set of all complex numbers, such that if ϕ_λ is the eigenfunction corresponding to the complex number λ then ϕ_λ is the unique solution of

(4.2.7) $\quad L\phi_\lambda + \lambda\phi_\lambda = 0, \qquad \phi_\lambda \varepsilon\, C^{(2)}(\Omega),$

such that

(4.2.8) $$\begin{cases} \phi_\lambda(0) = \sin\alpha, \\ \phi'_\lambda(0) = \cos\alpha. \end{cases}$$

<u>PROOF</u>: First suppose ϕ_λ satisfies (4.2.7) and (4.2.8). Let $u(x,y) = \phi_\lambda(x)\ \phi_\lambda(y)$ and $v(s,t) = u(\frac{t+s}{2}, \frac{t-s}{2})$. Then $v \varepsilon\, C^{(2)}(\Sigma)$ and v satisfies (3.2.4), (3.2.7) with $f = \phi_\lambda$, for $r \geq 0$, and (3.2.8). From the uniqueness part of theorem (2.1.2) and from the definition (3.3.2), it follows that

$$u(x,y) = (T^y\phi_\lambda)(x).$$

Hence ϕ_λ satisfies (4.2.5), and therefore is an eigenfunction.

Now suppose ϕ is any eigenfunction. First assume that $p \in C^{(1)}(\Omega)$. Since ϕ is continuous and not everywhere zero there is a function $g_o \in C_o \cap D_\alpha$ such that

$$\int \phi(x)\ g_o(x)\ dx = 1.$$

If σ is defined in terms of ϕ by (4.2.6) then $(g_o,\sigma) = 1$ and by (4.3.2)

$$(4.2.9) \quad \phi(x) = (\phi * g_o)(x) = \int \phi(y)(T^x g_o)(y)\ dy.$$

As in the proof of lemma (3.4.1) it follows that $\phi \in C^{(2)}(\Omega)$ and the derivatives of ϕ can be computed by differentiation of (4.2.9) under the integral sign. In particular $\phi(0) = \sin \alpha$ and $\phi'(0) = \cos \alpha$. Using lemma 3.4.4, one finds that

$$\begin{aligned}(L\phi)(x) &= \int \phi(y)(T^x L g_o)(y)\ dy \\ &= (T^x L g_o, \sigma) \\ &= (L g_o, \sigma)\phi(x).\end{aligned}$$

Hence $L\phi + \lambda\phi = 0$ with $\lambda = -(Lg_o,\sigma)$. This completes the proof for the case $p \in C^{(1)}(\Omega)$.

Next assume that p is any element of $C(\Omega)$. Let $\{p_n\}$ be a sequence of functions in $C^{(1)}(\Omega)$ which converge to p uniformly on every compact set. Defining g_o as before one has

$$\phi(x) = \lim_{n \to \infty} \int \phi(y)(T^x_{(n)} g_o)(y)\ dy$$

where $\{T^y_{(n)}\}$ are the translation operators corresponding to p_n.

The convergence is uniform on every compact set. Let

$$\phi_n(x) = \int \phi(y)(T^x_{(n)}g_o)(y)\, dy.$$

Each function ϕ_n is in $C^{(2)}(\Omega)$ and

$$(L\phi_n)(x) = \int \phi(y)(T^x_{(n)}Lg_o)(y)\, dy,$$

which converges as $n \to \infty$, to $\int \phi(y)\ (T^x Lg_o)(y)\, dy$ uniformly on every compact set. Consequently the second derivatives

$$\phi_n'' = L\phi_n + p_n\phi_n$$

converge uniformly on every compact set. Since $\phi_n(0) = \sin\alpha$ and $\phi_n'(0) = \cos\alpha$ for every n, it follows that the functions ϕ_n converge uniformly on every compact set to a function of class $C^{(2)}(\Omega)$. Thus $\phi \in C^{(2)}(\Omega)$. It also follows that $\phi(0) = \sin\alpha$, $\phi'(0) = \cos\alpha$ and

$$\begin{aligned}(L\phi)(x) = \lim_{n\to\infty} (L\phi_n)(x) &= \int \phi(y)\ (T^x Lg_o)(y)\, dy \\ &= (T^x Lg_o, \sigma) \\ &= (Lg_o, \sigma)\ \phi(x).\end{aligned}$$

Thus in this case also, $L\phi + \lambda\phi = 0$ with $\lambda = -(Lg_o, \sigma)$. This completes the proof.

4.3. This section is once again concerned with the abstract structure described in section 4.1.

The hypothesis (4.1.12) implies that for each fixed (x,y) in $\Omega \times \Omega$ the complex linear functional on C defined by the mapping

$$f \to (T^y f)(x)$$

is determined by a measure with compact support. That is, for each $(x,y) \in \Omega \times \Omega$ there is a measure $\mu_{x,y}$ in $\mathcal{M}$ which has compact support and is such that

$$(4.3.1) \qquad (T^y f)(x) = \int f(z)\, d\mu_{x,y}(z)$$

for every $f \in C$.

(In the examples of chapter 3 the measure $\mu_{x,y}$ is supported by the interval $|x - y| \le x \le x + y$. It has a mass of $\frac{\sin \alpha}{2}$ at each end of the interval and a continuous density on the interior of the interval.)

For the remainder of this section it is assumed that the measures $\mu_{x,y}$ are all non-negative. The assumption is dropped in the next section.

If f and g are real valued Lebesgue measurable functions on Ω, the notation

$$f \le g$$

will signify that $f(x) \le g(x)$, almost everywhere. If f is any complex valued function on Ω the symbol $|f|$ will denote the function

$$|f|(x) = |f(x)|.$$

From (4.3.1) it follows that

$$(4.3.2) \qquad |T^y f| \le T^y |f|$$

for every $f \in C$ and all $y \in \Omega$. Using this inequality in (4.2.1) gives

$$(4.3.3) \qquad |f*g| \le |f| * |g|$$

for all f,g in C when at least one function is in C_o.

<u>Definition</u>: A function r on Ω is called a <u>modulus</u> if

(i) r is continuous, non-negative, and $r(x) \neq 0$ for all $x \neq 0$, and

(ii) $(T^y r)(x) \leq r(x)\, r(y)$ for all $x, y \in \Omega$.

In particular any eigenfunction which is positive for $x \neq 0$ is a modulus.

Suppose a modulus r exists. For $f \in C_o$ let

$$(4.3.4) \qquad ||f|| = \int |f(x)|r(x)\, dx.$$

From (4.3.2), if $f \in C_o$, $y \in \Omega$ then

$$\begin{aligned} ||T^y f|| &= \int |(T^y f)(x)|r(x)\, dx \\ &\leq \int (T^y|f|)(x)\, r(x)\, dx \\ &= \int |f|(x)\, (T^y r)(x)\, dx \end{aligned}$$

and hence

$$(4.3.5) \qquad ||T^y f|| \leq r(y)\, ||f||.$$

With the aid of Fubini's theorem one obtains from (4.3.3),

$$\begin{aligned} ||f*g|| &\leq ||\ |f|*|g|\ || \\ &= \int [\int |f|(y)(T^y|g|)(x)dy]\, r(x)\, dx \\ &= \int |f|(y)\ [\int (T^y|g|)(x)\, r(x)\, dx]\, dy \end{aligned}$$

and hence

$$(4.3.6) \qquad ||f*g|| \leq ||f||\cdot||g||$$

for all $f, g \in C_o$.

Let A denote the complex Banach space of all functions f on Ω which are measurable and for which

$$||f|| = \int |f(x)|\ r(x)\ dx$$

is finite, two functions equal almost everywhere with respect to $r(x)dx$ being identified.

Since functions in C_o are dense in A, the operators $\{T^y\}$ can be extended uniquely to bounded linear transformations of A into itself such that (4.3.5) is valid for every $f \in A$ and all $y \in \Omega$. Similarly the multiplication $f*g$ can be extended to A in a unique way such that (4.3.6) remains valid for all f and $g \in A$. When this extension is made the rules

$$f*g = g*f,$$
$$\lambda(f*g) = f*\lambda g,$$
$$f*(g + h) = f*g + f*h$$
$$f*(g*h) = (f*g)*h$$

remain valid, and A becomes a commutative complex Banach algebra.

4.4. In this section the measures $\mu_{x,y}$ are no longer assumed to be non-negative, or even real. The measure which is the total variation of $\mu_{x,y}$ is denoted by $|\mu_{x,y}|$. Thus for any Baire set E,

$$|\mu_{x,y}|(E) = \sup \sum_j |\mu_{x,y}(E_j)|$$

where the sum is taken over a finite collection $E_1, \ldots, E_k$ of disjoint Baire subsets of E and the sup is taken over all such sums.

THEOREM 4.4.1: Suppose there is a second system $\{S^y\}$ of translation operators based on the space Ω and the same Lebesgue measure. The constant $\sin\alpha$ of (4.1.8) need not be the same for the two systems. Let $\{\nu_{x,y}\}$ denote the measures associated with the operators $\{S^y\}$ so that

$$(S^y f)(x) = \int f(t)\, d\nu_{x,y}(t).$$

Assume that

(i) the measures $\nu_{x,y}$ are all non-negative,

(ii) the system $\{S^y\}$ has a modulus s, and

(iii) there is a constant m such that

$$|\mu_{x,y}|(E) \leq m\,\nu_{x,y}(E)$$

for all x,y and every Baire set E. Let $r(x) = ms(x)$ and let A denote the Banach space of all measurable functions f on Ω such that

$$(4.4.1) \qquad ||f|| = \int |f(x)|r(x)\, dx.$$

Then $A \cap C$ is dense in A and the operators $\{T^y\}$ have unique extensions to bounded linear transformations of A into itself with norms

$$(4.4.2) \qquad ||T^y|| \leq r(y).$$

Moreover the multiplication $f*g$ can be extended to A in a unique way so that

$$(4.4.3) \qquad ||f*g|| \leq ||f||\ ||g||$$

for all $f,g \in A$. Provided with this multiplication, A becomes a commutative complex Banach algebra.

PROOF: For any $f \in C$

$$(T^y f)(x) \leq \int |f(t)| \, m d\nu_{x,y}(t) = m(S^y|f|)(x).$$

From the results of the previous section it follows that $f \in A \cap C$ implies $T^y f \in A \cap C$ and

$$||T^y f|| \leq r(y) \; ||f||.$$

This proves the assertion concerning extension of the operators $\{T^y\}$.

If $f, g \in A \cap C$ and at least one of these functions is in C_o then

$$\begin{aligned} |(f*g)(x)| &\leq \int |f(y)| \, |(T^y g)(x)| \, dy \\ &\leq \int |f|(y) \; m(S^y|g|)(x) \; dy \\ &= m(|f| \circledast |g|)(x) \end{aligned}$$

where $\circledast$ denotes convolution with respect to the system $\{S^y\}$. Hence, as in the previous section,

$$\begin{aligned} ||f*g|| &\leq m^2 \int (|f| \circledast |g|) \; (x) \; s(x) \; dx \\ &\leq m^2 \int |f|(x) \; s(x) \; dx \; \int |g|(y) \; s(y) \; dy \\ &= ||f|| \cdot ||g||. \end{aligned}$$

Consequently the multiplication extends to A in a unique way such that (4.4.3) remains valid for all $f, g \in A$. The final statement of the theorem follows.

The particular case of the theorem when $m = 1$ and $S^y = T^y$ for each y gives the results of the previous section.

The constant m could be set equal to one without any great loss but is convenient to have available for applications.

It should be noted that the function r is continuous, non-negative, and different from zero except possibly at $x = 0$. The phrase "almost everywhere" will always mean "almost everywhere with respect to Lebesgue measure", which may not be the same as "almost everywhere with respect to the measure $r(x)\,dx$". The discrepancy arises if and only if $r(0) = 0$ and at the same time the point $x = 0$ carries positive Lebesgue measure.

The formula

$$(4.4.4) \qquad \theta(f) = \int f(x)\,\theta(x)\,dx$$

establishes a correspondence between the set of all bounded linear functionals on A, and the set of all measurable functions θ on Ω which satisfy an inequality

$$|\theta(x)| \leq Mr(x)$$

almost everywhere. The smallest admissible constant M is the norm of the functional θ. Two different measurable functions correspond to the same functional if and only if they are equal almost everywhere.

If the functional θ of (4.4.4) is a multiplicative functional of the Banach algebra A then the measure $d\sigma(x) = \theta(x)\,dx$ satisfies (4.2.3) for all $f, g \in C_0$ and hence there is an eigenfunction ϕ such that

$$\int f(x)\,\theta(x)\,dx = \int f(x)\,\phi(x)\,dx$$

for all f in C_o, and consequently

$$\theta(x) = \phi(x)$$

almost everywhere. Therefore ϕ determines a bounded linear functional on A and $\phi = \theta$ is an equality for the functionals. Conversely if ϕ is any eigenfunction which satisfies an inequality

$$|\phi(x)| \leq Mr(x)$$

almost everywhere, then

$$\phi(f) = \int f(x)\, \phi(x)\, dx$$

defines a multiplicative linear functional on A. For applications it is helpful to observe that since any multiplicative linear functional on a Banach algebra is automatically continuous and of norm ≤ 1, it follows that any eigenfunction which satisfies an inequality

$$|\phi(x)| \leq Mr(x)$$

for all x, satisfies the inequality with $M = 1$.

According to the Gelfand theory of Banach algebras, the set of all non-zero multiplicative linear functionals of A, provided with its relative w*-topology considered as a subset of the conjugate space of A, forms a locally compact Hausdorff space called the <u>spectrum</u> of A. This space will be denoted by Λ. If λ is a point of Λ the corresponding multiplicative functional will be denoted by ϕ_λ and the same symbol ϕ_λ will be used to denote the corresponding eigenfunction. For each $f \in A$ the symbol $\hat{f}$ will denote the function on Λ defined by

$$(4.4.12) \quad \hat{f}(\lambda) = (f, \phi_\lambda) = \int f(x)\, \phi_\lambda(x)\, dx.$$

Each $\hat{f}$ is continuous, vanishes at infinity and

(4.4.13) $\widehat{f*g} = \hat{f}\,\hat{g}$ (pointwise product).

The spectral radius of f is

(4.4.14) $||\hat{f}||_\infty = \sup \{ |\hat{f}(\lambda)|; \ \lambda \varepsilon \Lambda \}$,

and satisfies

$$||\hat{f}||_\infty \leq ||f||.$$

Furthermore,

(4.4.15) $||\hat{f}||_\infty = \lim_{n \to \infty} ||f^n||^{1/n}$,

f^n denoting the convolution of n copies of f. The elements f of the radical of A are characterized by

$$\lim_{n \to \infty} ||f^n||^{1/n} = 0.$$

A is called semi-simple if its radical consists of the zero element only.

Now consider the special examples mentioned at the end of chapter 3. When $p(x)$ is constant the Riemann function is identically one. Hence in this case $\mu_{x,y}$ has mass $\frac{\sin \alpha}{2}$ at each end of the supporting interval and the constant density $\frac{\cos \alpha}{2}$ on the interior of the interval. Thus $\mu_{x,y}$ is non-negative if $\sin \alpha \geq 0$, $\cos \alpha \geq 0$. When these conditions are satisfied there are non-negative eigenfunctions and they do not vanish, except perhaps at $x = 0$; consequently any one of them is a modulus. In particular the Fourier sine and cosine transforms are of this type.

For the Fourier cosine case,

$$(T^y f)(x) = \frac{1}{2} [f(x + y) + f(|x - y|)]$$

and $r(x) \equiv 1$ is a modulus. For any $a > 0$, $\cosh ax$ is a modulus.

For the Fourier sine case

$$(T^y f)(x) = \frac{1}{2} \int_{|x-y|}^{x+y} f(z)\, dz$$

and $r(x) \equiv x$ is a modulus. For any $a > 0$, $\frac{\sinh ax}{a}$ is a modulus.

For the mixed case

$$(T^y f)(x) = \frac{\sin \alpha}{2} [f(x + y) + f(|x - y|)] + \frac{\cos \alpha}{2} \int_{|x-y|}^{x+y} f(z)dz$$

and $r(x) = \sin \alpha + x \cos \alpha$ is a modulus. For any $a > 0$,

$$\sin \alpha \cosh ax + \cos \alpha \frac{\sinh ax}{a}$$

is a modulus.

When p is a constant but $\sin \alpha$ and $\cos \alpha$ are not both non-negative, a comparison can be made with the system $\{S^y\}$ which is obtained when p is constant and the angle is α_o where $\sin \alpha_o = |\sin \alpha|$ and $\cos \alpha_o = |\cos \alpha|$.

Again consider the Fourier sine transform. If $r(x) = x$ is chosen as a modulus then the multiplicative functionals of A are the eigenfunctions ϕ such that $|\phi(x)| \leq x$. Hence in this case the spectrum Λ can be identified with the positive real axis $\lambda \geq 0$ of the complex plane (there is a question of two topologies here). On the other hand if $r(x) = \frac{\sinh ax}{a}$, $a > 0$, is chosen as a modulus then the multiplicative functionals of A are the eigenfunctions ϕ which satisfy

$$|\phi(x)| \leq \frac{\sinh ax}{a}, \qquad x \geq 0$$

and these correspond to the set of complex numbers such that

$$\left|\frac{\sin\sqrt{\lambda}\ x}{\sqrt{\lambda}}\right| \leq \frac{\sinh ax}{a}, \qquad x \geq 0,$$

or what is the same thing, to the set of complex numbers $\lambda = \mu + i\nu$ such that

$$\nu^2 \leq 4a^2(a^2 + \mu).$$

Thus in this case Λ can be identified with the points inside and on a certain parabola in the complex plane.

Finally consider the case when $p(x) = x^2$ and $\sin \alpha = 0$, $\cos \alpha = 1$. The translation operators are given by

$$(T^y f)(x) = \frac{1}{2}\int_{|x-y|}^{x+y} J_o\left(\left(\frac{[z^2-(x-y)^2][(x+y)^2-z^2]}{2}\right)^{1/2}\right) f(z)dz.$$

Although the Bessel function is not positive it is bounded by one and the operators $\{S^y\}$ of the Fourier sine transform serve as a comparison system, and $r(x) = x$ is a modulus. The asymptotic behavior of the eigenfunctions is well-known and it can be shown that the only eigenfunctions which are in modulus $\leq x$ for all $x > 0$ are the odd Hermite functions. Thus Λ in this case is a discrete space.

4.5. It is henceforth assumed that the conditions of section 4.4 are satisfied, and the Banach algebra A has been constructed.

<u>LEMMA 4.5.1</u>: The mapping $y \to T^y$ is a strongly continuous mapping of Ω into bounded operators on A.

PROOF: Let $f \in A$, let $y_o \in \Omega$, and let $\varepsilon > 0$ be given. If $g \in C_o$ and $y \in \Omega$ then

$$||T^y f - T^{y_o} f|| \leq ||T^y(f-g)|| + ||T^y g - T^{y_o} g|| + ||T^{y_o}(g-f)||$$

$$\leq [r(y) + r(y_o)] \, ||f - g|| + ||T^y g - T^{y_o} g||.$$

Now r is continuous and therefore bounded near y_o, while $||f-g||$ can be made small by choice of g. Hence for suitable g

$$[r(y) + r(y_o)] \, ||f-g|| < \frac{1}{2} \varepsilon$$

when y is in a certain neighborhood U of y_o. Now with g fixed, (4.1.5) assures the existence of a neighborhood V of y_o such that

$$||T^y g - T^{y_o} g|| < \frac{1}{2} \varepsilon$$

when $y \in V$. Hence $||T^y f - T^{y_o} f|| < \varepsilon$ if $y \in U \cap V$, which proves the lemma.

LEMMA 4.5.2: Let $f \in A$, $y \in \Omega$, and assuming that Λ is not void, let $\lambda \in \Lambda$. Then

(4.5.1) $\widehat{T^y f}(\lambda) = \phi_\lambda(y) \hat{f}(\lambda)$.

If Θ and Φ are defined by

$$\Theta(x, \lambda) = (\widehat{T^x f})(\lambda),$$

$$\Phi(x, \lambda) = \phi_\lambda(x)$$

then Θ and Φ are continuous on $\Omega \times \Lambda$.

<u>PROOF</u>: Choose $g \in C_o$ so that $\hat{g}(\lambda) \neq 0$. Then

$$\begin{aligned}
\widehat{T^y f}(\lambda)\, \hat{g}(\lambda) &= ((T^y f)*g, \phi_\lambda) \\
&= (f*T^y g, \phi_\lambda) \\
&= \hat{f}(\lambda) \int (T^y g)(x)\, \phi_\lambda(x)\, dx \\
&= \hat{f}(\lambda) \int g(x)\, (T^y \phi_\lambda)(x)\, dx \\
&= \hat{f}(\lambda)\, \phi_\lambda(y)\, \hat{g}(\lambda),
\end{aligned}$$

for which (4.5.1) follows.

Now let $(x, \lambda) \in \Omega \times \Lambda$, and let $\varepsilon > 0$ be given. If $y \in \Omega$, $\lambda' \in \Lambda$ then

$$\begin{aligned}
|\theta(y, \lambda') - \theta(x, \lambda)| &\leq |\widehat{T^y f}(\lambda') - \widehat{T^x f}(\lambda')| \\
&\quad + |\widehat{T^x f}(\lambda') - \widehat{T^x f}(\lambda)| \\
&\leq \|T^y f - T^x f\| \\
&\quad + |\widehat{T^x f}(\lambda') - \widehat{T^x f}(\lambda)|.
\end{aligned}$$

According to the lemma 4.5.1, the first term on the right can be made less than $\frac{1}{2}\varepsilon$ by choosing y to be in a suitable neighborhood U of x. Since $\widehat{T^x f}$ is continuous on Λ, the second term is less than $\frac{1}{2}\varepsilon$ when λ' is in some neighborhood V of λ. Hence the sum is less than ε when $(y, \lambda') \in U \times V$, proving that θ is continuous.

Given any point (x, λ_o), choose $f \in A$ so that $\hat{f}(\lambda_o) \neq 0$. Then $\hat{f}$ is non-zero throughout some neighborhood of λ_o. Since

$$\Phi(x, \lambda) = \frac{\widehat{T^x f}(\lambda)}{\hat{f}(\lambda)}$$

for any x if λ is near λ_o, it follows that Φ is continuous at every point of $\Omega \times \Lambda$, and hence is continuous.

4.6. In this section a new basic assumption is made, namely that

(4.6.1) there is a non-zero constant c such that

$$\lim_{x \to 0} \frac{(T^y g)(x)}{r(x)} = cg(y)$$

for each g in C. Here, if $x = 0$ is not an isolated point of Ω the limit is understood to be uniform with respect to y on each compact set, while if $x = 0$ is isolated it is assumed that $r(0) \neq 0$ and the limit is to be interpreted as

$$\frac{(T^y g)(0)}{r(0)} .$$

As a matter of fact, if $\sin \alpha \neq 0$, then this assumption is a consequence of (4.1.5), (4.1.7) and (4.1.8). To see this, first observe that for any $x \neq 0$,

$$r(x) \ |\sin \alpha| = |(T^o r)(x)| \leq r(x) \ r(0)$$

and $r(x) \neq 0$ so

$$0 < |\sin \alpha| \leq r(0).$$

Thus $r(0) \neq 0$ and the above assumption follows.

On the other hand if the assumption holds and $r(0) \neq 0$ then $\sin \alpha \neq 0$.

If $r(0) = 0$ then $\sin \alpha = 0$, and it should be noted that even though the point $x = 0$ may carry positive Lebesgue measure, the contribution from $y = 0$ to an integral of the form

$$\int f(y) \ (T^y g)(x) \ dy$$

is zero because $(T^{o}g)(x) = g(x) \sin \alpha = 0$.

In group theory there is a method (see Krein [16], Godement [17], Loomis [15] Chapter V) for developing the theory of positive definite functions and the Plancherel theorem by studying the ideal J, in the L_1-algebra of the group, formed by the bounded, absolutely integrable continuous functions. On this ideal, there is defined a positive continuous functional Γ: $\Gamma(f) = f(e)$ where e is the identity element of the group. This functional is of central importance in the theory.

It is proposed to construct a suitable ideal J in the algebra A, so that the functional

$$\Gamma(f) = \frac{1}{c} \lim_{x \to 0} \frac{f(x)}{r(x)}$$

can be defined on J. The remainder of this section is devoted to the ideal J and auxiliary objects.

Let $g \in C$. The function $\frac{g}{r}$ will be called continuous, on Ω if either $r(0) \neq 0$ so that the function is defined and continuous in the usual sense, or if $r(0) = 0$ and

$$\lim_{x \to 0} \frac{g(x)}{r(x)}$$

exists, and is finite. If $\frac{g}{r}$ is continuous on Ω then

$$\lim_{x \to 0} \frac{g(x)}{r(x)}$$

will be interpreted in the usual way if $x = 0$ is not an isolated point of Ω, and as $\frac{g(0)}{r(0)}$ otherwise.

If $\frac{g}{r}$ is continuous and also bounded on Ω then $\left\|\frac{g}{r}\right\|_\infty$ is defined to be $\sup \left\{ \left|\frac{g(x)}{r(x)}\right| ; x \in \Omega \right\}$ if $r(0) \neq 0$, and to be $\sup \left\{ \left|\frac{g(x)}{r(x)}\right| ; x \in \Omega - (0) \right\}$ if $r(0) = 0$.

<u>Definition</u>: J is the set of all $g \in C$ such that $g \in A$ and $\frac{g}{r}$ is continuous and bounded on Ω. J_o is the set of all functions in J with compact supports.

<u>LEMMA 4.6.1</u>: J_o is dense in A.

<u>PROOF</u>: If $r(0) \neq 0$ then $J_o = C_o$ and the result is true. Suppose $r(0) = 0$. Let $f \in C_o$ and let $\varepsilon > 0$ be given. There is a neighborhood V of $x = 0$, with compact closure $\overline{V}$, such that

$$\int_{\overline{V}} |f(x)| r(x) \, dx < \frac{1}{2} \varepsilon.$$

Since the measure is regular there is a closed subset F of $\Omega - \overline{V}$ such that

$$\int_{\Omega - F} |f(x)| \, r(x) \, dx < \varepsilon.$$

Now there is a function g in C such that

$$0 \leq g(x) \leq 1 \quad \text{for all } x,$$
$$g(x) = 0 \quad \text{if } x \in \overline{V}$$
$$g(x) = 1 \quad \text{if } x \in F.$$

Let $h(x) = f(x)\, g(x)$. Then $h \in J_o$ and $h(x) = f(x)$ on F, while $|h(x)| \leq |f(x)|$ on $\Omega - F$. Therefore $||h - f|| < 2\varepsilon$. Thus J_o is A-dense in C_o and hence dense in A.

<u>LEMMA 4.6.2</u>: J is a dense ideal in A, and if $g \in J$ then $T^y g \in J$ for each y, and

(4.6.2) $$\left|\left|\frac{T^y g}{r}\right|\right|_\infty \le r(y)\left|\left|\frac{g}{r}\right|\right|_\infty .$$

If $f \in A$ and $g \in J$ then

(4.6.3) $$\left|\left|\frac{f*g}{r}\right|\right|_\infty \le ||f||\ \left|\left|\frac{g}{r}\right|\right|_\infty .$$

PROOF: Let $g \in J$. By (4.6.1), $\frac{T^y g}{r}$ is continuous on Ω. Further

$$|(T^y g)(x)| \le m(S^y|g|)(x) = m^2 \int \left|\frac{g(t)}{r(t)}\right| s(t)\ d\nu_{x,y}(t)$$
$$\le r(x)\ r(y)\ \left|\left|\frac{g}{r}\right|\right|_\infty .$$

Hence $T^y g \in J$ and (4.6.2) holds. If $f \in A$ and $g \in J$ then

$$|(f*g)(x)| \le \int |f(y)|\cdot|(T^x g)(y)|\ dy$$
$$= \int |f(y)|\ r(x)\ r(y)\ \left|\left|\frac{g}{r}\right|\right|_\infty dy$$
$$= r(x)\ ||f||\ \left|\left|\frac{g}{r}\right|\right|_\infty .$$

If $r(0) \neq 0$ then $\frac{f*g}{r}$ is clearly continuous, and by the above inequality is in J and satisfies (4.6.3). Suppose $r(0) = 0$. For $x \neq 0$

$$\frac{(f*g)(x)}{r(x)} = \int \frac{(T^y g)(x)}{r(x)r(y)}\ r(y)\ f(y)\ dy,$$

(remember the integral gets no contribution at $y = 0$) and here $r(y)\ f(y)$ is integrable while $\frac{(T^y g)(x)}{r(x)r(y)}$ is bounded as $x \to 0$, uniformly in y. Hence by dominated convergence

$$\lim_{x \to 0} \frac{(f*g)(x)}{r(x)}$$

exists and is equal to

$$\int r(y)\, f(y)\, \frac{cg(y)}{r(y)}\, dy = c \int f(y)\, g(y)\, dy.$$

Thus in this case also $f*g$ is in J and satisfies (4.6.3). Since J is a linear subspace of A it follows that J is an ideal. By the preceding lemma J is dense in A, which completes the proof.

Corollary: If $f \varepsilon A$ and $g \varepsilon J$ then

$$\lim_{x \to 0} \frac{(f*g)(x)}{r(x)} = c \int f(x)\, g(x)\, dx.$$

At this point it is convenient to discuss the existence of identity elements in A.

Let U_n, $n = 1,2, \ldots$ be a complete system of neighborhoods of $x = 0$ which have compact closures, and form a decreasing sequence

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \supseteq \cdots .$$

For each n let u_n be a continuous function which vanishes outside U_n and satisfies

$$\int u_n(x)\, r(x)\, dx = \frac{1}{c},$$

where c is the constant of (4.6.1). Without loss of generality it can be assumed that u_n is a multiple of a real non-negative function, so that

$$||u_n|| = \frac{1}{c} \qquad \text{for all } n.$$

It can also be assumed that the functions u_n all belong to the ideal J.

If the measures $\mu_{x,y}$ are all real then the constant c is real and the functions u_n can be assumed to be real.

Now suppose $f \varepsilon C_o$. Then

$$(u_n*f)(x) - f(x) = \int u_n(y)(T^y f)(x)\, dy - f(x)$$

$$= \int u_n(y)\, r(y) \left[\frac{(T^x f)(y)}{r(y)} - cf(x) \right] dy.$$

According to (4.1.6) and (4.1.7) there is a compact set K such that

$$\frac{(T^x f)(y)}{r(y)} - cf(x) = 0$$

for every y in U_1 whenever x is outside K, and according to (4.6.1)

$$\frac{(T^x f)(y)}{r(y)} - cf(x)$$

converges to zero uniformly on K as $y \to 0$. Hence the functions

$$u_n*f - f$$

all vanish outside K and converge to zero uniformly on K as $n \to \infty$. Hence

$$u_n*f \to f \qquad \text{in } A$$

as $n \to \infty$. Since $||u_n||$ is uniformly bounded, the usual approximation argument shows that

$$||u_n*f - f|| \to 0$$

as $n \to \infty$, for every $f \varepsilon A$.

In order that A have an identity element it is necessary and sufficient that $r(0)$ be non-zero and at the same time the point $x = 0$ have non-zero Lebesgue measure. For on the one hand if these conditions are satisfied then it is clear that a certain multiple of the function

$$g(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

is an identity element for A. On the other hand, if A has an identity element e then e can be represented as a Baire function. If u_n and U_n have the meanings assigned to them above then

$$u_n = u_n * e \to e \qquad \text{in } A,$$

and hence

$$\int_E |e(x) - u_n(x)|\, dx \to 0 \qquad \text{as } n \to \infty,$$

for every Baire set E. Let $E_k = \Omega - U_k$. Then u_n vanishes in E_k for large n and

$$\int_{E_k} |e(x)|\, dx = \int_{E_k} |e(x) - u_n(x)|dx \to 0 \qquad \text{as } n \to \infty.$$

Therefore $e(x) = 0$ for almost all x in E_k and consequently for almost all x in

$$\sum_{k=1}^{\infty} E_k = \Omega - (0).$$

Hence $||e|| = e(0)\cdot r(0)\cdot b$ where b is the Lebesgue measure of $x = 0$. By (4.1.1), A contains non-zero elements, so $||e|| \neq 0$, showing that $r(0) \neq 0$ and $b \neq 0$.

4.7. The Hilbert space of all functions defined on Ω and measurable and square integrable with respect to Lebesgue measure will be denoted by $\mathcal{H}$. The Hilbert space norm of a function f will be denoted by $||f||_2$, so that

$$(4.7.1) \qquad ||f||_2 = \left(\int |f(x)|^2 \, dx\right)^{1/2}$$

and the scalar product will be denoted by $\langle f,g\rangle$, so that

$$(4.7.2) \qquad \langle f,g\rangle = \int f(x) \, g^*(x) \, dx$$

where $g^*(x)$ is the complex conjugate of $g(x)$.

The functions in J are also in $\mathcal{H}$, in fact if $f \varepsilon J$ then

$$\int |f(x)|^2 \, dx = \int |f(x) \, r(x)| \cdot \left|\frac{f(x)}{r(x)}\right| dx$$

and hence

$$(4.7.3) \qquad ||f||_2 \leq \left(||f|| \quad \left|\left|\frac{f}{r}\right|\right|_\infty \right)^{1/2} \qquad \text{for } f \varepsilon J.$$

If the measures $\mu_{x,y}$ are all real then for each $f \varepsilon C$ and each $y \varepsilon \Omega$

$$(4.7.4) \qquad T^y(f*) = (T^y f)*$$

where the * denotes complex conjugation. Furthermore if f and g are in J then, $T^y f$ and $T^y g$ being in J by lemma 4.5.1, one has

$$\begin{aligned} \langle T^y f,g\rangle &= \int (T^y f)(x) \, g^*(x) \, dx \\ &= \int f(x) \, (T^y g)^*(x) \, dx, \end{aligned}$$

or

(4.7.5) $\langle T^y f, g\rangle = \langle f, T^y g\rangle$.

Since the operators T^y map J into itself, they can be regarded as linear transformations of $\mathcal{H}$ into itself, defined over the subspace J. It will be important to know when these transformations are bounded on J in the Hilbert space sense, and also to know under what circumstances J is dense in $\mathcal{H}$.

The latter question is easy to answer. J_o is dense in $\mathcal{H}$ except when $x = 0$ has positive Lebesgue measure and at the same time $r(0) = 0$, and when these two conditions hold, J is not dense in $\mathcal{H}$. To prove this, first suppose the two conditions hold. Since $r(0) = 0$, every function in J vanishes at $x = 0$, and hence the function

$$h(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ 0 & \text{otherwise} \end{cases}$$

which is an element of h of non-zero norm, cannot be approximated in $\mathcal{H}$ by functions in J. In fact if $f \in J$ then

$$\begin{aligned} ||f - h||_2^2 &= \int |f(x) - h(x)|^2 \, dx \\ &= ||f||_2^2 + ||h||_2^2. \end{aligned}$$

On the other hand suppose one of the two conditions does not hold. If $r(0) \neq 0$ then $C_o = J_o$ is dense in $\mathcal{H}$. If $x = 0$ has Lebesgue measure zero then, by means of the argument used to prove lemma 4.6.1, it can be shown that functions in $\mathcal{H}$ can be approximated in $\mathcal{H}$ by functions in C_o which vanish near $x = 0$.

The following lemma provides a sufficient condition in order that the operators T^y be not merely $\mathcal{H}$-bounded on J but satisfy an inequality

$$(4.7.6) \qquad ||T^y g||_2 \leq kr(y) \; ||g||_2$$

for every g in J, where k is a constant not depending on g or on y. This guarantees that the operation of convolution by an element f of A is an $\mathcal{H}$-bounded operator on J.

<u>LEMMA 4.7.1:</u> Suppose there is a constant k such that

$$(4.7.7) \qquad (S^y 1)(x) \leq ks(y) \qquad \text{for all } x,y \in \Omega,$$

where 1 denotes the constant function with value one. Then (4.7.6) holds for all $g \in J$.

<u>PROOF</u>: Let $g \in J_o$, $h \in C$ and $f(x) = g(x)\, h(x)$. By Holder's inequality

$$|(T^y f)(x)|^2 = \left| \int g(t)\, h(t)\, d \quad_{x,y}(t) \right|^2$$

$$\leq m^2 \int |g(t)|^2\, d\nu_{x,y}(t) \int |h(t)|^2\, d\nu_{x,y}(t)$$

$$= m^2 \, (S^y |h|^2)(x) \; (S^y |g|^2)(x).$$

Now let $h = 1$ so that $f = g$, and

$$|(T^y g)(x)|^2 \leq m^2 (S^y 1)(x) (S^y |g|^2)(x)$$

$$\leq m^2 ks(y) (S^y |g|^2)(x).$$

Since $g \in J_o$, $S^y|g|^2 \in C_o$ and

$$\int |(T^y g)(x)|^2 \, dx \leq m^2 ks(y) \int (S^y|g|^2)(x) \, dx$$
$$= m^2 ks(y) \int |g|^2(x)(S^y 1)(x) \, dx$$
$$\leq k^2 r^2(y) \, ||g||_2^2,$$

from which (4.7.6) for $g \in J_o$ follows.

Now suppose $g \in J$. Let $\{g_n\}$ be a sequence of functions in J_o which converge to g uniformly on every compact set and which, as elements of $\mathcal{H}$, converge to g in $\mathcal{H}$. Then

$$||T^y(g_m - g_n)||_2 \leq kr(y)||g_m - g_n||_2,$$

so the sequence $T^y g_n$ converges in $\mathcal{H}$. But the functions $T^y g_n$ converges to $T^y g$ uniformly on every compact set. Hence $T^y g_n$ converges to $T^y g$ in $\mathcal{H}$. Consequently

$$||T^y g||_2 = \lim_{n\to\infty} ||T^y g_n||_2 \leq \limsup_{n \to \infty} kr(y) \, ||g_n||_2$$
$$= kr(y) \, ||g||_2.$$

This proves the lemma.

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For each $f \in A$ let B_f be the linear transformation of $\mathcal{H}$ with domain J defined by

$$B_f h = f * h, \qquad h \in J.$$

These operators map J into itself. The mapping $f \to B_f$ is linear and is in fact a homomorphism, that is $B_{f*g} = B_f B_g$, for $f, g \in A$.

If the measures $\mu_{x,y}$ are all real then the complex conjugation $f \to f^*$ is a *-operation for A. That is, if f and g are elements of A and λ and μ are complex numbers then

$$(\lambda f + \mu g)^* = \bar{\lambda} f^* + \bar{\mu} g^*,$$

$$(f^*)^* = f,$$

and

$$(f*g)^* = f^* * g^*.$$

THEOREM 4.7.1: Assume that

(i) either $r(0) \neq 0$, or else $x = 0$ has Lebesgue measure zero;

(ii) the measures $\mu_{x,y}$ are all real;

(iii) there is a constant k such that (4.7.6) holds for all $g \in J$.

Then the operators T^y can be extended uniquely to bounded operators defined everywhere on $\mathcal{H}$, and the operators B_f, $f \in A$, have unique extensions to bounded linear operators defined everywhere on $\mathcal{H}$. The extended operators satisfy

(4.7.8) $||T^y|| \leq kr(y)$,

and

(4.7.9) $||B_f|| \leq k||f||$.

The mapping $f \to B_f$ is a continuous *-isomorphism of A into the algebra of all bounded operators of $\mathcal{H}$.

<u>PROOF</u>: Let $f \varepsilon A$, $g \varepsilon J$. If h is any element of C_o then

$$<f*g,h> = \int [\int f(y)(T^y g)(x)\, dy]\, h*(x)\, dx.$$

Now

$$|f(y)(T^y g)(x)| \leq |f(y)r(y)r(x) \left|\left|\frac{g}{r}\right|\right|_\infty h*(x)|,$$

from which it is clear that Fubini's theorem can be applied to the above integral. One obtains

$$<f*g,h> = \int f(y) <T^y g,h>\, dy.$$

Hence, using hypothesis (iii),

$$|<f*g,h>| \leq k||f|| \cdot ||g||_2 \cdot ||h||_2.$$

Since h is an arbitrary element of C_o, and since C_o is dense in $\mathcal{H}$, this shows that

$$||B_f g||_2 \equiv ||f*g||_2 \leq k||f|| \cdot ||g||_2$$

for $f \varepsilon J$.

Hypothesis (i) implies that J is dense in $\mathcal{H}$. It follows at once that the operators T^y and B_f extend uniquely to bounded operators on all of $\mathcal{H}$, and that the extensions satisfy (4.7.8) and (4.7.9).

It hs now been proved that the mapping $f \to B_f$ is a continuous homomorphism. Suppose $f \varepsilon A$, $g,h \varepsilon J_o$. Then as above

$$\begin{aligned} <f*g,h> &= \int f(y) <T^y g,h>\, dy \\ &= \int f(y) <g,T^y h>\, dy \\ &= \int f(y)[\int g(x)(T^y h)^*(x)\, dx]\, dy. \end{aligned}$$

Using Fubini's theorem,

$$\begin{aligned} \langle f*g,h\rangle &= \int g(x)\left[\int f(y)(T^y h)^*(x)\,dy\right]dx \\ &= \int g(x)(f^* * h)^*\;(x)\,dx \\ &= \langle g, f^* * h\rangle \end{aligned}$$

or

$$\langle B_f g, h\rangle = \langle g, B_{f*} h\rangle.$$

Since J_o is dense in $\mathcal{H}$ it follows that

$$B_{f*} = (B_f)^*$$

where the right side denotes the adjoint operator of B_f. Hence the mapping is a $*$-homomorphism.

It remains to show that the mapping is isomorphic. If f is the zero element of A then $f*g = 0$ for $g \varepsilon J$ so $B_f = 0$. Suppose on the other hand that $B_f = 0$. Then for any $g \varepsilon J$ the continuous function $f * g$ is the zero element of $\mathcal{H}$ and hence vanishes almost everywhere with respect to Lebesgue measure, and hence everywhere. Therefore

$$c \int f(y)\; g(y)\; dy = \lim_{x \to 0} \frac{(f*g)(x)}{r(x)} = 0$$

and $c \neq 0$. This is true if, in particular, g is a continuous function with compact support and vanishes near $x = 0$. Hence $f(x) = 0$ almost everywhere on $\Omega - (0)$. This imples that $f \varepsilon \mathcal{H}$. But f is orthogonal to the dense subspace J of $\mathcal{H}$, so $f(x) = 0$ for almost all x in the sense of Lebesgue measure. By virtue of hypothesis (i), f is the zero element of A. Thus the mapping is isomorphic.

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Corollary: Under the conditions of the theorem, A is semi-simple.

PROOF: Recall that for any bounded operator T of a Hilbert space, the equality $||T^*T|| = ||T||^2$ is valid. Suppose f is in the radical of A. Let $g = f*f^*$. Then $g = g^*$, $B_g = (B_g)^*$,

$$||B_{g^2}|| = ||B_g(B_g)^*|| = ||B_g||^2,$$

$$||B_{g^4}|| = ||B_{g^2}(B_{g^2})^*|| = ||B_g||^4,$$

and in general,

$$||B_{g^{2^n}}|| = ||B_g||^{2^n}, \qquad n = 1,2, \ldots .$$

Hence

$$||B_g|| = ||B_{g^{2^n}}||^{2^{-n}} \leq (k||g^{2^n}||)^{2^{-n}}$$

which $\to 0$ as $n \to \infty$ since g is in the radical of A. Consequently $||B_g|| = 0$. But

$$||B_g|| = ||B_f||^2$$

so $B_f = 0$ and $f = 0$, which proves the corollary.

4.8. For the remainder of the chapter it is assumed that the hypotheses of theorem 4.7.1 are valid.

It should be recalled that the hypothesis that the measures $\mu_{x,y}$ be real implies that the constant c of (4.6.1) is real and that the approximate identity $\{u_n\}$ can be chosen to be self adjoint, that is $u_n^* = u_n$ for each n.

The results of the previous sections will be used to establish the Plancherel theorem and the Bochner representation theorem for positive definite functions. The proofs are adaptations to the present situation of those given in section 26 of Loomis' book [15]. These proofs are due to Godement [17] and M.G. Krein [16].

By a C*-algebra is meant an algebra of bounded operators of a Hilbert space which contains the identity operator, and contains with an operator T also its adjoint T^*, and which is closed in the uniform topology of operators.

For an abelian C*-algebra $\mathfrak{A}$, the Gelfand transformation, which maps the elements T of $\mathfrak{A}$ onto the functions $\hat{T}$ defined on the (compact) spectrum M of $\mathfrak{A}$, is an isometric *-isomorphism of $\mathfrak{A}$ onto the algebra $C(M)$ of all complex valued continuous functions on M.

The particular abelian C*-algebra of interest here is constructed by first forming $\mathfrak{A}'$ the closure in the uniform operator topology of the set of all B_f, $f \in A$, and then $\mathfrak{A}$ is the set of all operators $\mu I + B$ where $B \in \mathfrak{A}'$ and μ is a complex number. Of course it may happen that $\mathfrak{A} = \mathfrak{A}'$.

Let M denote the spectrum of $\mathfrak{A}$ and M' that of $\mathfrak{A}'$. If $\mathfrak{A}'$ contains the identity operator then $\mathfrak{A}' = \mathfrak{A}$ and $M' = M$. Otherwise $\mathfrak{A}' \neq \mathfrak{A}$ and M is the one-point compactification of M'.

The mapping $f \to B_f$ is a *-homomorphism of A onto a dense subset of $\mathfrak{A}'$. The adjoint mapping maps each multiplicative functional ϕ of $\mathfrak{A}'$ onto the multiplicative functional $\phi(f) = \phi(B_f)$ of A. This adjoint mapping is a homeomorphism of M' onto a closed subset of the spectrum Λ of A (see for example Loomis [15], section 24B).

It will be assumed that M' has been identified in this way with a closed subset of Λ.

The algebra $\tilde{A}$ is defined to be A in case A has an identity e and to be the algebra obtained by adjoining an identity e to A if A has no identity. The general element of $\tilde{A}$ may be denoted by

$$h = \mu e + f$$

where $f \in A$ and μ is a complex number. The norm of this element is defined as

$$||h|| = |\mu| + ||f||$$

when $\tilde{A} \neq A$.

<u>LEMMA 4.8.1</u>: Let θ be a linear functional on A which is positive in the sense that

(4.8.1) $\theta(f*f^*) \geq 0$ for all $f \in A$.

Then θ is continuous if and only if

(4.8.2) $\theta(f*) = \overline{\theta(f)}$,

and there is a constant K such that

(4.8.3) $|\theta(f)|^2 \leq K\theta(f*f^*)$.

<u>PROOF</u>: If f and $g \in A$ and γ is any complex number then

$$0 \leq \theta((f+\gamma g)*(f+\gamma g)^*)$$

$$= \theta(f*f^*) + |\gamma|^2 \theta(g*g^*) + \gamma\theta(g*f^*) + \overline{\gamma}\,\theta(f*g^*).$$

Hence $\gamma\theta(g*f^*) + \overline{\gamma}\,\theta(f*g^*)$ is real for every complex number γ.

This implies that

$$(4.8.4) \quad \theta(g*f^*) = \overline{\theta(f*g^*)} \; .$$

Consequently the bilinear form $[f,g] = \theta(f*g^*)$ has all the properties of a scalar product except that perhaps $[f,f] = 0$ may happen when $f \neq 0$. Hence the Schwarz inequality

$$(4.8.5) \quad |\theta(f*g^*)|^2 \leq \theta(f*f^*) \; \theta(g*g^*)$$

is valid.

Now suppose that θ is continuous. Let $\{u_n\}$ be the self adjoint approximate identity of A. Putting $g = u_n$ in (4.8.4) and letting $n \to \infty$ one obtains (4.8.2). Putting $g = u_n$ in (4.8.5) and noting that

$$\theta(u_n*u_n^*) \leq ||\theta|| \cdot ||u_n||^2 = \frac{||\theta||}{c^2}$$

one obtains (4.8.3) with $K = \frac{||\theta||}{c^2}$.

On the other hand suppose the two conditions hold. If $\tilde{A} \neq A$ extend θ to $\tilde{A}$ by defining

$$\theta(\mu e + f) = \quad K + \theta(f).$$

The extended functional is positive because

$$\begin{aligned} \theta((\mu e + f)*(\mu e + f)^*) &= |\mu|^2 \cdot K + \mu\theta(f^*) + \overline{\mu}\theta(f) + \theta(f*f^*) \\ &\geq |\mu|^2 \cdot K - 2|\mu| \; K^{1/2}\theta^{1/2}(f*f^*) + \theta(f*f^*) \\ &= [|\mu| \cdot K^{1/2} - \theta^{1/2}(f*f^*)]^2 \\ &\geq 0. \end{aligned}$$

If h is a self adjoint element of $\tilde{A}$ with $||h|| < 1$ then $e - h$ has a square root k which can be computed by the power series for

$(1 - t)^{1/2}$. Since the involution is continuous, $k^* = k$. Therefore $0 \leq \theta(e - h) = \theta(k*k^*)$, so that $\theta(h) \leq \theta(e)$. Similarly $\theta(-h) \leq \theta(e)$. Hence

$$|\theta(h)| \leq \theta(e) = K$$

when h is self adjoint and $||h|| < 1$. Now let h be any element of $\tilde{A}$ with $||h|| < 1$. Then $h_1 = \frac{1}{2}(h + h*)$ and $h_2 = \frac{1}{2i}(h - h*)$ are self adjoint and of norm less than one. Hence

$$|\theta(h)| = |\theta(h_1) + i\theta(h_2)| \leq \sqrt{2} \cdot K,$$

which shows that θ is continuous. This proves the lemma.

<u>Definition</u>: Let P denote the set of all continuous positive functionals on A. An element θ of P is called a <u>positive definite functional</u> and any one of the measurable functions θ on Ω such that

$$\theta(f) = \int f(x)\, \theta(x)\, dx$$

for all $f \in A$ is called a <u>positive definite function</u>.

If $\theta_1, \theta_2 \in P$ and γ is a non-negative number then $\theta_1 + \theta_2$ and $\gamma\theta_1$ are in P. Thus P is a cone.

<u>THEOREM 4.8.1</u>: There is a one-one correspondence $\theta \longleftrightarrow \mu_\theta$ between the set P and the set of all finite positive Baire measures on M', such that for every $f \in A$

$$(4.8.6) \quad \theta(f) = \int \hat{f}(\lambda)\, d\mu_\theta(\lambda).$$

PROOF: First let μ_θ be a finite positive Baire measure on M', and let θ be the linear functional on A defined by (4.8.6). Since $\mathfrak{A}$ is an abelian C^*-algebra, $\widehat{B^*}(\lambda) = \overline{\widehat{B}(\lambda)}$ for every $\lambda \varepsilon M$, $B \varepsilon \mathfrak{A}$. In particular if $\lambda \varepsilon M'$ and $f \varepsilon A$ then $\widehat{f^*}(\lambda) = \overline{\widehat{f}(\lambda)}$. Consequently,

$$\theta(f*f^*) = \int |\hat{f}(\lambda)|^2 \, d\mu_\theta \geq 0,$$

$$\theta(f^*) = \int \overline{\hat{f}(\lambda)} \, d\mu_\theta = \overline{\theta(f)}$$

and

$$|\theta(f)|^2 = \left| \int \hat{f}(\lambda) \, d\mu_\theta \right|^2 \leq \int |\hat{f}(\lambda)|^2 \, d\mu_\theta \cdot \int d\mu_\theta$$

$$= ||\mu_\theta|| \cdot \theta(f*f^*).$$

By lemma 4.8.1 it follows that $\theta \varepsilon P$. Since A is dense in $\mathfrak{A}'$, the functions $\hat{f}$, $f \varepsilon A$, viewed as functions on M', are dense in the space of all continuous functions vanishing at infinity on M'. Hence two distinct measures μ_θ give rise to different functionals θ. It is clear that different functionals θ arise from different measures.

Now suppose $\theta \varepsilon P$. Then θ can be viewed as a positive linear functional on the set of all B_f, and in fact, as in the proof of lemma 4.8.1, θ can be extended to be a positive linear functional on the set of all operators $\mu I + B_f$ where I is the identity operator. Hence θ can be regarded as a positive linear functional defined on a dense subalgebra of the algebra $C(M)$ of all continuous functions on M. There is a unique extension to a positive linear functional θ' defined on all of $C(M)$. By the Riesz-Markoff theorem there is a unique finite Baire measure μ'_θ on M such that

$$\theta'(\hat{B}) = \int \hat{B}(\lambda)\, d\mu'_{\theta}(\lambda)$$

for all $B \in \mathfrak{A}$. In particular if $f \in A$ then

(4.8.7) $\quad \theta(f) = \theta'(\hat{B}_f) = \int \hat{f}(\lambda)\, d\mu'_{\theta}(\lambda)$

($\hat{B}_f$ vanishes at infinity). Since all the functions $\hat{f}$ vanish at infinity, the measure μ'_{θ} can be restricted to a Baire measure on M', without altering (4.8.7), and (4.8.6) results

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<u>THEOREM 4.8.2:</u> Every positive definite function θ is essentially continuous and is given by

(4.8.8) $\quad \theta(x) = \int \phi_{\lambda}(x)\, d\mu_{\theta}(\lambda)$

where μ_{θ} is the corresponding Baire measure on M'. This formula sets up a bicontinuous isomorphism between the cone of all finite positive Baire measures on M' and the cone P.

<u>PROOF:</u> Let θ be a positive definite function. For each $f \in A$

$$\int f(x)\, \theta(x)\, dx = \int \hat{f}(\lambda)\, d\mu_{\theta}(\lambda)$$
$$= \int [\int f(x)\, \phi_{\lambda}(x)\, dx]\, d\mu_{\theta}(\lambda),$$

where μ_{θ} is the corresponding Baire measure on M'. Since $\phi_{\lambda}(x)$ is jointly continuous in x and λ and $|\phi_{\lambda}(x)| \le r(x)$ for all $\lambda \in M'$, Fubini's theorem applies and one finds that

$$\int f(x)\, \theta(x)\, dx = \int f(x)\, [\int \phi_{\lambda}(x)\, d\mu_{\theta}(\lambda)]\, dx.$$

Hence (4.8.8) is valid for almost all x.

To show that the integral (4.8.8) is continuous, let y be any point in Ω. There is a neighborhood U of y such that $r(x) \leq 1 + r(y)$ for all $x \in U$. Given $\varepsilon > 0$ there is a compact set K in M' such that

$$\mu_\theta(M' - K) < \frac{\varepsilon}{1 + r(y)} .$$

Hence for $x \in U$,

$$|\theta(x) - \theta(y)| < \int_K |\phi_\lambda(x) - \phi_\lambda(y)| \, d\mu_\theta + 2\varepsilon.$$

A compactness argument shows that the first term on the right can be made less than ε by restriction of U. Thus the integral is continuous.

The norm of θ as a functional on A is

$$||\theta|| = \left|\left|\frac{\theta}{r}\right|\right|_\infty$$

where $\left|\left|\frac{\theta}{r}\right|\right|_\infty$ is defined as in section 4.6. Since $|\phi_\lambda(x)| \leq r(x)$ for all $\lambda \in M'$, (4.8.8) shows that

$$|\theta(x)| \leq r(x) \; ||\mu_\theta||$$

where θ now denotes the continuous integral (4.8.8). Hence

$$\left|\left|\frac{\theta}{r}\right|\right|_\infty \leq ||\mu_\theta||.$$

On the other hand $\lim\limits_{x \to 0} \dfrac{\phi_\lambda(x)}{r(x)} = c$ for each $\lambda \in M'$. Therefore, by dominated convergence $\lim\limits_{x \to 0} \dfrac{\theta(x)}{r(x)} = c||\mu_\theta||$ and

$$\left|\left|\frac{\theta}{r}\right|\right|_\infty \geq c||\mu_\theta||.$$

This shows that the mapping is bicontinuous. If $c = 1$ it is actually norm preserving.

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Let E^+ be the set of all positive definite functions which are also functions in $A \cap C$, and let E be the set of all finite complex linear combinations of elements of E^+.

<u>LEMMA 4.8.2</u>: If $g \varepsilon J$ then $g*g^* \varepsilon E^+$.

<u>PROOF</u>: Let θ denote the functional defined on A by

$$\theta(f) = \int f(x)\ (g*g^*)(x)\ dx.$$

Clearly θ is continuous. According to the corollary of lemma 4.6.2, for any $f \varepsilon A$

$$\begin{aligned}\theta(f*f^*) &= \int (f*f^*)(x)(g*g^*)(x)\ dx \\ &= \frac{1}{c} \lim_{x \to 0} \frac{((f*f^*)*(g*g^*))(x)}{r(x)} \\ &= \frac{1}{c} \lim_{x \to 0} \frac{((f*g)*(f^**g^*))(x)}{r(x)}.\end{aligned}$$

Since $f*g \varepsilon A$ and $f^**g^* \varepsilon J$ the corollary can be used again, giving

$$\theta(f*f^*) = \int (f*g)(x)(f^**g^*)(x)\ dx = ||f*g||_2^2.$$

Hence θ is a positive functional. Since $g*g^* \varepsilon C$ the lemma is established.

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<u>LEMMA 4.8.3</u>: Let F be any continuous non-negative function on M' which vanishes at infinity, and let $\varepsilon > 0$ be given. Then there is a function $f \varepsilon E^+$ such that

$$|F(\lambda) - \hat{f}(\lambda)| < \varepsilon \qquad \text{for all} \quad \lambda \varepsilon M'.$$

<u>PROOF</u>: As noted in the proof of theorem 4.8.1, the functions $\hat{g}$, $g \varepsilon A$, are dense in the space of continuous functions which vanish at infinity on M'. For any positive b there is an element $g \varepsilon A$ such that

$$|\sqrt{F(\lambda)} - \hat{g}(\lambda)| < b\varepsilon \qquad \text{for all} \quad \lambda \varepsilon M'.$$

It can be assumed that $g = g^*$ (otherwise replace g by $\frac{1}{2}(g + g^*)$ and the inequality improves), and it can be assumed that

$$|\hat{g}(\lambda)| \leq 1 + \sqrt{F(\lambda)} \qquad \text{for all} \quad \lambda \varepsilon M'.$$

Now choose $h \varepsilon J$ so that $h = h^*$ and

$$||g - h|| < b\varepsilon.$$

Then for all $\lambda \varepsilon M'$

$$|\sqrt{F(\lambda)} - \hat{h}(\lambda)| \leq |\sqrt{F(\lambda)} - \hat{g}(\lambda)| + ||g - h||$$
$$< 2\, b\varepsilon$$

and

$$|\hat{h}(\lambda)| \leq |\hat{g}(\lambda)| + |\hat{h}(\lambda) - \hat{g}(\lambda)|$$
$$\leq 1 + b\varepsilon + \sqrt{F(\lambda)}.$$

Hence

$$|F(\lambda) - \widehat{h*h}(\lambda)| = |\sqrt{F(\lambda)} - \hat{h}(\lambda)| \cdot |\sqrt{F(\lambda)} + \hat{h}(\lambda)|$$
$$< b\varepsilon(1 + b\varepsilon + 2\sqrt{F(\lambda)}),$$

and this is less than ε for suitable b. But by lemma 4.8.2, $f = h*h = h*h^*$ is in E^+, and this proves the result.

<u>THEOREM 4.8.3</u>: There is a unique regular Baire measure m on M' such that $\mathcal{H}$ is isometric to $L_2(M',m)$ under a unitary mapping $f \to Uf$ which coincides, for elements of E, with the mapping $f \to \hat{f}$.

<u>PROOF:</u> Let $g \in E^+$. Then $g \in P$ and there is a unique finite positive Baire measure μ_g on M' such that

$$\int g(x)\, f(x)\, dx = \int \hat{f}(\lambda)\, d\mu_g(\lambda)$$

for all $f \in A$. If $g,h \in E^+$ and μ_g, μ_h are the corresponding measures, then for each $f \in A$, since both g and h are in the ideal J,

$$\int \hat{f}(\lambda)\, \hat{h}(\lambda)\, d\mu_g(\lambda) = \int (f*h)(x)\, g(x)\, dx$$

$$= \frac{1}{c} \lim_{x \to 0} \frac{((f*h)*g)(x)}{r(x)}$$

$$= \frac{1}{c} \lim_{x \to 0} \frac{(h*(f*g))(x)}{r(x)}$$

$$= \int h(x)\, (f*g)(x)\, dx$$

$$= \int \hat{f}(\lambda)\, \hat{g}(\lambda)\, d\mu_h(\lambda).$$

Consequently,

(4.8.9) $$\int G(\lambda)\, \hat{h}(\lambda)\, d\mu_g(\lambda) = \int G(\lambda)\, \hat{g}(\lambda)\, d\mu_h(\lambda)$$

for every continuous function G on M' which vanishes at infinity.

Suppose F is a continuous function on M' with compact support, say K. For each $\lambda \varepsilon K$ there is an element $g \varepsilon J$ such that $\hat{g}(\lambda) \neq 0$. By compactness of K it follows from lemma 4.8.2 that there is an element of E^+ of the form

$$g = g_1 * g_1^* + \cdots + g_n * g_n^*$$

such that $\hat{g}$ is bounded away from zero on K. Let g,h be any two elements of E^+ such that $\hat{g},\hat{h}$ are bounded away from zero on K. Then

$$\int \frac{F(\lambda)}{\hat{g}(\lambda)} \, d\mu_g(\lambda) = \int \frac{F(\lambda)}{\hat{g}(\lambda)\hat{h}(\lambda)} \hat{h}(\lambda) \, d\mu_g(\lambda).$$

Using (4.8.9) with

$$G(\lambda) = \frac{F(\lambda)}{\hat{g}(\lambda)\hat{h}(\lambda)}$$

gives

$$\int \frac{F(\lambda)}{\hat{g}(\lambda)} \, d\mu_g(\lambda) = \int \frac{F(\lambda)}{\hat{h}(\lambda)} \, d\mu_h(\lambda).$$

Consequently, the common value $I(F)$ of these two integrals is a well defined functional, clearly linear and positive, on the space of continuous functions on M' with compact supports. By the Riesz-Markoff theorem there is a unique positive Baire measure m on M' such that

$$I(F) = \int F(\lambda) \, dm(\lambda)$$

for every continuous function F with compact support.

Now let F be a continuous function on M' with compact support, let $g \in E^+$, and let h be any element of E^+ such that $\hat{h}$ is bounded away from zero on the support of F. Then

$$\int F(\lambda)\, \hat{g}(\lambda)\, dm(\lambda) = \int \frac{F(\lambda)}{\hat{h}(\lambda)}\, \hat{g}(\lambda)\, d\mu_h(\lambda)$$

$$= \int \frac{F(\lambda)}{\hat{h}(\lambda)}\, \hat{h}(\lambda)\, d\mu_g(\lambda)$$

by (4.8.9). Hence

(4.8.10) $$\int F(\lambda)\, \hat{g}(\lambda)\, dm(\lambda) = \int F(\lambda)\, d\mu_g(\lambda).$$

Consequently,

$$\mu_g(K) = \int_K \hat{g}(\lambda)\, dm(\lambda)$$

for every compact Baire set K. Therefore since $\hat{g} \geq 0$, it follows that $\hat{g} \in L_1(M',m)$,

$$\int \hat{g}(\lambda)\, dm(\lambda) = ||\mu_g||$$

and

$$\int G(\lambda)\, \hat{g}(\lambda)\, dm(\lambda) = \int G(\lambda)\, d\mu_g(\lambda)$$

for every continuous function G which vanishes at infinity. If $f \in A$ then

(4.8.11) $$\int f(x)\, g(x)\, dx = \int \hat{f}(\lambda)\, d\mu_g(\lambda)$$

$$= \int \hat{f}(\lambda)\, \hat{g}(\lambda)\, dm(\lambda).$$

The scalar product of two functions F and G in $L_2(M',m)$ will be denoted by $\langle F,G \rangle$. Thus

$$\langle F,G \rangle = \int F(\lambda)\, \overline{G(\lambda)}\, dm(\lambda).$$

If $g \in E^+$ then $\hat{g} \in L_1(M',m)$ and $\hat{g}$ is also a continuous function vanishing at infinity, so that $\hat{g} \in L_2(M',m)$. Hence $\hat{g} \in L_2(M',m)$ whenever $g \in E$. From (4.8.11) it follows easily that

$$\langle g,f \rangle = \langle \hat{g},\hat{f} \rangle$$

whenever f and g are both in E. Hence $f \to \hat{f}$ induces a unitary mapping U' of E into $L_2(M',m)$.

Next it is shown that E is dense in $\mathcal{H}$. It is sufficient to show that every self adjoint element f of J_o can be approximated in $\mathcal{H}$ by elements of E, because J_o is dense in $\mathcal{H}$ and each element of J_o is a linear combination of two self adjoint elements.

Let f be a self adjoint element of J_o. Then lemma 4.8.2 shows that if $u_n \in J$ then

$$u_n*f = \frac{1}{4}\,[(f + u_n)*(f + u_n) - (f - u_n)*(f - u_n)]$$

is a difference of two elements of E^+. Now let $\{u_n\}$ be the self adjoint approximate identity for A, with $u_n \in J$, discussed in section 4.6. Then the functions u_n*f all vanish outside a fixed compact set, and converge uniformly to f as $n \to \infty$. Hence $u_n*f \to f$ in $\mathcal{H}$. This proves that E is dense in $\mathcal{H}$.

It follows that the unitary transformation U' can be extended in one and only one way to a unitary transformation U of $\mathcal{H}$ into $L_2(M',m)$. The range of U is a closed linear subspace of $L_2(M',m)$. It remains to show that the range of U is all of $L_2(M',m)$.

Let F be a continuous non-negative function on M' with a compact support K. Let ε and θ be positive numbers. By lemma 4.8.3 there is an element g_ε in E^+ such that

$$|F(\lambda) - \hat{g}_\varepsilon(\lambda)| < \varepsilon \qquad \text{for all} \quad \lambda \varepsilon M'.$$

Again by lemma 4.8.3 there is an element h_θ in E^+ such that

$$1 < \hat{h}_\theta(\lambda) < 1 + \theta \qquad \text{if} \quad \lambda \varepsilon K$$

and

$$0 \leq \hat{h}_\theta(\lambda) \leq 1 + \theta \qquad \text{for all} \quad \lambda \varepsilon M'.$$

Let $f = g_\varepsilon * h_\theta$. Then $f \varepsilon J$ and $\hat{f}(\lambda) = \hat{g}_\varepsilon(\lambda)\hat{h}_\theta(\lambda) \geq 0$, which implies that $f \varepsilon E^+$. Hence $Uf = \hat{g}_\varepsilon \cdot \hat{h}_\theta$. If $\lambda \varepsilon M' - K$ then

$$|F(\lambda) - \hat{g}_\varepsilon(\lambda)\hat{h}_\theta(\lambda)| = |\hat{g}_\varepsilon(\lambda)\hat{h}_\theta(\lambda)| < \varepsilon\hat{h}_\theta(\lambda).$$

If $\lambda \varepsilon K$ then

$$\begin{aligned} |F(\lambda) - \hat{g}_\varepsilon(\lambda)\hat{h}_\theta(\lambda)| &\leq |F(\lambda) - \hat{g}_\varepsilon(\lambda)| + |\hat{g}_\varepsilon(\lambda)[1 - \hat{h}_\theta(\lambda)]| \\ &< \varepsilon + \theta|\hat{g}_\varepsilon(\lambda)| \\ &< \varepsilon + \varepsilon\theta + \theta F(\lambda). \end{aligned}$$

Therefore,

$$||F - Uf||_2 \leq \varepsilon||\hat{h}_\theta||_2 + (\varepsilon + \varepsilon\theta)\sqrt{m(K)} + \theta||F||_2$$

The last term can be made small by choice of θ, and then with θ fixed the first two terms can be made small by choice of ε. This proves that F is in the (closed) range of U. It follows that the range of U is the entire space $L_2(M', m)$, and the theorem is proved.

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CHAPTER 5

THE CLASSICAL STURM-LIOUVILLE PROBLEM

In this chapter the application of the theory of the preceding chapter to the classical Sturm-Liouville problem is briefly described.

5.1. The space Ω is now the finite interval $0 \leq x \leq b$ on the real axis, with its usual Lebesgue measure. Since Ω is compact $C = C_o$.

The differential operator L is defined on the space $C^{(2)}(\Omega)$ by

$$(5.1.1) \qquad (Lf)(x) = f''(x) - p(x)\, f(x)$$

where p is a <u>real valued</u> continuous function on Ω. Two boundary conditions are prescribed, in terms of two real constants α and β. A function $f \in C^{(1)}(\Omega)$ satisfies the boundary conditions if

$$(5.1.2) \qquad \begin{aligned} f(0) \cos\alpha - f'(0) \sin\alpha &= 0, \\ f(b) \cos\beta - f'(b) \sin\beta &= 0. \end{aligned}$$

The set of all functions $f \in C^{(2)}(\Omega)$ which satisfy the boundary conditions is denoted by $D_{\alpha,\beta}$.

$\Omega \times \Omega$ is the square $\{ (x,y);\ 0 \leq x,y \leq b \}$. It is convenient to distinguish two triangles

$$\Delta_1 = \{ (x,y);\ 0 \leq y \leq x;\ \ x + y \leq b \} ,$$

and

$$\Delta_2 = \{ (x,y);\ 0 \leq y \leq x \leq b;\ \ b \leq x + y \} .$$

The fundamental theorem is the following analogue of theorem 3.2.1.

THEOREM 5.1.1: Assume that $p \in C^{(3)}(\Omega)$. Assume also that either $\sin\alpha = \sin\beta = 0$ or else $\sin\beta \neq 0$. Then for each $f \in D_{\alpha,\beta}$ there is a unique function $u \in C^{(2)}(\Delta_1 \cup \Delta_2)$ which satisfies

(5.1.3) $$u_{xx} - u_{yy} - [p(x) - p(y)]\,u = 0 \quad \text{on} \quad \Delta_1 \cup \Delta_2,$$

(5.1.4) $$\left.\begin{aligned} u(x,0) &= f(x)\sin\alpha, \\ u_y(x,0) &= f(x)\cos\alpha \end{aligned}\right\} \quad \text{for} \quad x \in \Omega,$$

and

(5.1.5) $$u(b,y)\cos\beta - u_x(b,y)\sin\beta = 0 \quad \text{for} \quad x \in \Omega.$$

The solution u is given by

(5.1.6) $$u(x,y) = \frac{\sin\alpha}{2}[f(x+y) + f(x-y)]$$
$$+ \frac{\sin\alpha}{2}\int_{x-y}^{x+y}[R_s(z,z;x-y,x+y) - R_t(z,z;x-y,x+y)]f(z)\,dz$$
$$+ \frac{\cos\alpha}{2}\int_{x-y}^{x+y} R(z,z;x-y,x+y)\,f(z)\,dz$$

for any $(x,y) \in \Delta_1$,

(5.1.7) $$u(x,y) = \frac{\sin\beta}{2}[g(y+(b-x)) + g(y-(b-x))]$$
$$+ \frac{\sin\beta}{2}\int_{y-(b-x)}^{y+(b-x)}[R_s(b-z,b+z;x-y,x+y)$$
$$+ R_t(b-z,b+z;x-y,x+y)]\,g(z)\,dz$$
$$- \frac{\cos\beta}{2}\int_{y-(b-x)}^{y+(b-x)} R(b-z,b+z;x-y,x+y)g(z)\,dz$$

for $(x,y) \in \Delta_2$, where g is the unique continuous solution of the integral equation

$$(5.1.8)\quad \frac{\sin\beta}{2} g(y) + \frac{\sin\beta}{2}\int_0^y [R_s(b-z,b+z;b-y,b)+R_t(b-z,b+z;b-y,b)]g(z)dz$$

$$-\frac{\cos\beta}{2}\int_0^y R(b-z,b+z;b-y,b)\ g(z)dz$$

$$= \frac{\sin\alpha}{2} f(b-y)$$

$$+\frac{\sin\alpha}{2}\int_{b-y}^b [R_s(z,z;b-y,b)-R_t(z,z;b-y,b)]f(z)\ dz$$

$$+\frac{\cos\alpha}{2}\int_{b-y}^b R(z,z;b-y,b)\ f(z)dz,$$

for $y \in \Omega$, and where R is the Riemann function constructed from the function

$$(5.1.9)\quad q(s,t) = \frac{1}{4}\left[p\left(\frac{t+s}{2}\right) - p\left(\frac{t-s}{2}\right)\right]$$

according to formulas (2.1.6) and (2.1.7). Furthermore the solution u may be extended to be a $C^{(2)}$ solution on $\Omega \times \Omega$ of (5.1.1) by means of the symmetry relation

$$(5.1.10)\qquad u(y,x) = u(x,y).$$

Remarks: (1) The proof is long, and only the general method will be indicated. If the function u is assumed to exist then there is a function $g \in D_{\alpha,\beta}$ such that

$$(5.1.11)\quad \left.\begin{aligned} u(b,y) &= g(y)\sin\beta\ , \\ u_x(b,y) &= g(y)\cos\beta\ , \end{aligned}\right\}\qquad \text{for } y \in \Omega.$$

(5.1.6) and (5.1.7) give the unique solutions of (5.1.3) on Δ_1 and Δ_2 respectively, which satisfy (5.1.4) and (5.1.11) respectively. The condition that these solutions coincide along the common boundary of Δ_1 and Δ_2 is that g satisfy (5.1.8). Now (5.1.8) has a unique continuous solution - the assumptions concerning $\sin\alpha$ and $\sin\beta$ guarantee that this is still true when f is merely continuous - and when g is taken to be this solution it can be verified that the composite function defined by (5.1.6) on Δ_1 and (5.1.7) on Δ_2 satisfies the conditions of the theorem. The verification is extremely long and tedious.

(2) A case with $\sin\beta = 0$ and $\sin\alpha \neq 0$ can of course be transformed to the case $\sin\alpha = 0$, $\sin\beta \neq 0$.

(3) The differentiability restrictions on p are probably unnecessarily strong, but this point has not been properly investigated.

When $\sin\beta \neq 0$, the integral equation (5.1.8) can be written in the form

(5.1.12) $$g(y) = \int_0^y H(y,z)g(z)dz + \frac{\sin\alpha}{\sin\beta} f(b-y) + \int_{b-y}^{b} K(y,z)f(z)\,dz,$$

where

(5.1.13) $$H(y,z) = \cot\beta \; R(b-z,b+z;b-y,b) \\ -[R_s(b-z,b+z;b-y,b) + R_t(b-z,b+z;b-y,b)]$$

and

(5.1.14) $$K(y,z) = \frac{\cos\alpha}{\cos\beta} \; R(z,z;b-y,b) \\ + \frac{\sin\alpha}{\sin\beta}\,[R_s(z,z;b-y,b) - R_t(z,z;b-y,b)].$$

The so-called resolvent kernel [18] for (5.1.12) is

$$(5.1.15)\quad h(y,z) = \sum_{n=1}^{\infty} H_n(y,z)$$

where

$$(5.1.16)\quad \begin{cases} H_1(y,z) = H(y,z), \\ H_n(y,z) = \int_z^y H_1(y,t)\, H_{n-1}(t,z)dt, \quad n = 2,3,\ldots \end{cases}$$

The solution of (5.1.12) is

$$g(y) = \frac{\sin \alpha}{\sin \beta} \quad f(b-y) + \int_{b-y}^{b} K(y,z)\, f(z)\, dz$$

$$+ \int_0^y h(y,z)\, [\frac{\sin \alpha}{\sin \beta} f(b-z) + \int_{b-z}^{b} K(z,t) f(t)\, dt]\, dz$$

or

$$(5.1.17)\quad g(y) = \frac{\sin \alpha}{\sin \beta} f(b-y) + \int_{b-y}^{b} N(y,t)\, f(t)\, dt$$

where

$$(5.1.18)\quad N(y,t) = K(y,t) + \frac{\sin \alpha}{\sin \beta}\, h(y,b-t) + \int_{b-t}^{b} h(y,z)\, K(z,t)dz.$$

In the special case when $p(x) \equiv 0$ so that

$$H = \cot \beta \;, \qquad K = \frac{\cos \alpha}{\sin \beta}$$

one finds that

$$N(y,t) = \frac{\sin(\alpha + \beta)}{\sin^2 \beta} \exp\, [(y - (b - t)) \cot \beta\,].$$

When $\sin \beta = 0$ and hence also $\sin \alpha = 0$, the integral equation for g assumes the form

$$(5.1.12') \quad g(y) = \int_0^y H_*(y,z)\, g(z)dz - \frac{\cos\alpha}{\cos\beta} f(b-y) + \int_{b-y}^{b} K_*(y,z)f(z)dz,$$

where

$$(5.1.13') \quad H_*(y,z) = R_{s_o}(b-z,b+z;b-y,b)$$

and

$$(5.1.14') \quad K_*(y,z) = R_{s_o}(z,z;b-y,b)\frac{\cos\alpha}{\cos\beta} \quad .$$

The resolvent kernel is

$$(5.1.15') \quad h_*(y,z) = \sum_{n=1}^{\infty} H_{*,n}(y,z)$$

where

$$(5.1.16') \quad \begin{aligned} H_{*,1}(y,z) &= H_*(y,z) \\ H_{*,n}(y,z) &= \int_z^y H_{*,1}(y,t)\, H_{*,n-1}(t,z)\, dt, \end{aligned}$$

$$n = 2,3,4,\ldots$$

The solution of (5.1.12') is

$$(5.1.17') \quad g(y) = -\frac{\cos\alpha}{\cos\beta} f(b-y) + \int_{b-y}^{b} N_*(y,t)\, f(t)\, dt$$

where

$$(5.1.18') \quad N_*(y,t) = K_*(y,t) - \frac{\cos\alpha}{\cos\beta} h_*(y,b-t) + \int_{b-t}^{y} h_*(y,z)K_*(z,t)dz.$$

In the special case when $p(x) \equiv 0$ so that $H_* = 0$, $K_* = 0$, one finds that $N_* = 0$ and

$$g(y) = -\frac{\cos\alpha}{\cos\beta} f(b-y).$$

The Riemann function R is well defined even when p is merely continuous, and in terms of R, the function N (or N_*) can be defined by (5.1.18). The translation operators are defined by

$$(T^y f)(x) = u(x,y) \qquad \text{for each } f \in C,$$

where u is the function on $\Omega \times \Omega$ determined by equations (5.1.6), (5.1.7), (5.1.10) and (5.1.17). The general properties of the operators can be derived by methods very similar to those used in chapter 3, and it is found that the conditions of section 4.1 are fulfilled. The validity of these conditions will be assumed and the easily verified fact that a continuous function ϕ on Ω is an eigenfunction if and only if it is in $D_{\alpha,\beta}$ and satisfies

$$\phi''(x) - p(x)\,\phi(x) + \lambda\phi = 0$$

for some complex number λ, and the conditions $\phi(0) = \sin\alpha$, $\phi''(0) = \cos\alpha$, will be also assumed.

A comparison can be made with a family of operators $\{S^y\}$ belonging to the case $p(x) = 0$ with suitably chosen values α_o, β_o of the constants which determine the boundary conditions. More precisely

(i) when $\sin\alpha \neq 0$, the values $\alpha_o = \frac{\pi}{4}$, $\beta_o = \frac{\pi}{2}$ may be used;

(ii) when $\sin\alpha = 0$, the values $\alpha_o = 0$, $\beta_o = \frac{\pi}{2}$ may be used.

Suppose $\sin\beta \neq 0$. Then $(T^y f)(x) = u(x,y)$ is given by (5.1.6) for $(x,y) \in \Delta_1$, while if $(x,y) \in \Delta_2$ one finds that

$$(5.1.19)\quad (T^y f)(x) = \frac{\sin\alpha}{2}\,[f((b-y)+(b-x)) + f((b-y)-(b-x))]$$

$$+ \frac{\sin\beta}{2}\int_{(b-y)+(b-x)}^{b} N(y-(b-x),t)\,f(t)\,dt$$

$$+ \frac{\sin\beta}{2}\int_{(b-y)-(b-x)}^{b} N(y+(b-x),t)\,f(t)\,dt$$

$$+ \frac{\sin\alpha}{\sin\beta}\int_{(b-y)-(b-x)}^{(b-y)+(b-x)} J(x,y,b-t)f(t)\,dt$$

$$+ \int_{(b-y)-(b-x)}^{b}\Big[\int_{\max[y-(b-x),b-t)]}^{y+(b-x)} J(x,y,z)N(z,t)dz\Big]f(t)dt,$$

where

$$J(x,y,z) = \frac{\sin\beta}{2}\,[R_s(b-z,b+z;x-y,x+y)+R_t(b-z,b+z;x-y,x+y)]$$

$$- \frac{\cos\beta}{2}\, R(b-z,b+z;x-y,x+y).$$

On the other hand if $\{S^y\}$ are the operators associated with the case $p(x) \equiv 0$, for the values $\alpha = \alpha_o$ and $\beta = \frac{\pi}{2}$, α_o being for the moment undetermined, then

$$(5.1.20)\quad (S^y f)(x) = \frac{\sin\alpha_o}{2}\,[f(x+y) + f(x-y)] + \frac{\cos\alpha_o}{2}\int_{x-y}^{x+y} f(t)dt$$

for $(x,y) \in \Delta_1$, and

$$(5.1.21)\quad (S^y f)(x) = \frac{\sin\alpha_o}{2}\,[f((b-y)+(b-x)) + f((b-y)-(b-x))]$$

$$+ \frac{\sin(\alpha_o + \frac{\pi}{2})}{2}\int_{(b-y)+(b-x)}^{b} f(t)\,dt$$

$$+ \frac{\sin(\alpha_o + \frac{\pi}{2})}{2}\int_{(b-y)-(b-x)}^{b} f(t)\,dt$$

for $(x,y) \varepsilon \Delta_1$. Consequently, for any $(x,y) \varepsilon \Omega \times \Omega$

$$(S^y f)(x) = \int f(t) \, d\nu_{x,y}(t)$$

where the measures $\nu_{x,y}$ are non-negative if $0 \leq \alpha_o \leq \frac{\pi}{2}$. An inspection of formulas (5.1.6), (5.1.19), (5.1.20) and (5.1.21) shows that the comparisons can be made as asserted in (i) and (ii) above when $\sin \beta \neq 0$. A similar discussion can be carried out when $\sin \beta = 0$.

From these comparisons it can be concluded that

(i) a multiple of $r(x) = 1$ is a modulus when $\sin \alpha \neq 0$;

(ii) a multiple of $r(x) = x$ is a modulus when $\sin \alpha = 0$.

When $\sin \alpha = \sin \beta = 0$, it can be shown by a comparison with the case $p(x) = \varepsilon x$, where ε is positive and small, $\alpha_o = 0$, $\beta_o = \pi$, that the function $r(x) = x(b - x)$ can be used in place of a modulus (since this function vanishes at $x = b$ it cannot be a modulus in the strict sense).

In these examples, the special circumstance that the eigenfunctions are in the algebra A makes it quite easy to prove a Wiener type Tauberian theorem, which states that an element f of A is contained in a proper closed ideal if and only if $\hat{f}(\lambda) = 0$ for some $\lambda \varepsilon \Lambda$.

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