A Geometric Study of Commutator Subgroups

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To my parents

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Abstract

Let G be a group and G' its commutator subgroup. Commutator length (cl) and stable commutator length (scl) are naturally defined concepts for elements of G'. We study cl and scl for two classes of groups. First, we compute scl in generalized Thompson's groups and their central extensions. As a consequence, we find examples of finitely presented groups in which scl takes irrational (in fact, transcendental) values. Second, we study large scale geometry of the Cayley graph $C_S(G')$ of a commutator subgroup G' with respect to the canonical generating set S of all commutators. When G is a non-elementary δ-hyperbolic group, we prove that there exists a quasi-isometrically embedded $\mathbb{Z}^n$ in $C_S(G')$, for each $n \in \mathbb{Z}_+$. Thus this graph is not δ-hyperbolic, has infinite asymptotic dimension, and has only one end. For a general finitely presented group, we show that this graph $C_S(G')$ is large scale simply connected.

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Chapter 1

Introduction

Let G be a group and $G' = [G, G]$ its commutator subgroup. Elements of G' are products of commutators. The *commutator length* (denoted cl) of an element $g \in G'$ is defined to be the least number of commutators whose products equals g. The *stable commutator length* (denoted scl) of g is the stabilized commutator length, i.e., the limit of $\mathrm{cl}(g^n)/n$, as $n \to \infty$. cl and scl have been studied not only in group theory, but also in topology, usually as genus norms. In the later case, there are two geometric approaches. The first is to use the topological definitions of cl and scl directly. In this approach, one essentially studies maps of surfaces (with boundaries) into spaces and tries to find the "simplest" one among them. The second approach comes from the deep connection of scl with bounded group cohomology. The main tool in this direction is the concept of (homogeneous) quasimorphisms. Quasimorphisms are homomorphisms up to bounded errors, and, surprisingly, many important invariants from geometry and dynamical systems can be regarded as quasimorphisms. The connection between scl and quasimorphisms comes from *Bavard's Duality Theorem*, which states that the set of all homogeneous quasimorphisms determines scl. One can also obtain nontrivial estimates of cl from quasimorphisms.

The purpose of this paper is to study arithmetic and geometric properties of cl and scl. We adopt the second approach via quasimorphisms. We study two classes of finitely generated groups which have different spaces of homogeneous quasimorphisms. The first class of groups is *generalized Thompson's groups* and their central extensions. Elements in these groups can be interpreted as automorphisms of the unit circle or the real line. We prove that the spaces of homogeneous quasimorphisms of these groups have finite dimensions (in fact 0 or 1), and when the dimension is 1, the only nontrivial (normalized) homogeneous quasimorphism is given by the *rotation quasimorphism*. Thus the computation of scl is reduced to that of rotation numbers. As a consequence, we have the following irrationality theorem:

Theorem A ([60]). *There are finitely presented groups, in which* scl *takes irrational (in fact, transcendental) values.*

In contrast to this irrationality theorem, D. Calegari [12] shows that scl takes only rational values in free groups. In general geometric settings, scl, viewed as a relative genus norm, is expected to take only rational values. Calegari's computation in free groups and Thurston's norm in 3-dimensional topology give important evidence of this. Our examples from generalized Thompson's groups display a totally different phenomenon.

The second class of groups are hyperbolic groups. In contrast to the first class of groups considered above, the spaces of homogeneous quasimorphisms of these groups are infinite dimensional, which has many geometric interpretations. One is through the notion of bounded cohomology, which says that the 2nd bounded cohomology (with $\mathbb{R}$-coefficients) of a hyperbolic group is infinite dimensional. In this paper, we give another interpretation through the *large scale geometry* of commutator subgroups. Let S be the set of all commutators, which form a canonical generating set for G'. Let $C_S(G')$ be the *Cayley graph* of G' with respect to S, meaning the vertices of $C_S(G')$ are elements in G', and two elements g_1 and g_2 are connected by an edge if $g_1^{-1}g_2$ is in S. By identifying each edge with the unit interval (with length 1), $C_S(G')$ becomes a metric space, on which G' acts by isometries. Then cl is the path metric in this graph and scl equals the translation length of this action. Cayley graphs are the most studied objects in geometric group theory, and we are interested in the large scale geometry of these graphs, i.e., those properties invariant under *quasi-isometries*. Roughly speaking, under quasi-isometries, we throw away local structures and only focus on large scale (or long range) properties of metric spaces. In the case of the Cayley graph of a commutator subgroup, we prove that

Theorem B. *Let G be a non-elementary word-hyperbolic group and $\mathbb{Z}^n$ the integral lattice in $\mathbb{R}^n$ with the induced metric. Then, for any $n \in \mathbb{Z}_+$, we have a map $\rho_n \colon \mathbb{Z}^n \to C_S(G')$, which is a quasi-isometric embedding.*

The proof implies that the geometry of $C_S(G')$ should be non-negatively curved and shows the existence of flats (zero curved subsets) of arbitrarily large dimensions. It uses *counting quasimorphisms*, constructed by R. Brooks [6] in free groups and Epstein-Fujiwara [27] in general hyperbolic groups. As a corollary of this theorem, we have

Corollary C. *Let G be a non-elementary word-hyperbolic group. Then we have*

1. *$C_S(G')$ is not δ-hyperbolic;*
2. $\mathrm{asdim}(C_S(G')) = \infty$*;*
3. *$C_S(G')$ is one-ended, i.e., $C_S(G')$ is connected at infinity.*

For a general group, we study the large scale topology of this graph and show that

Theorem D ([15]). *Let G be a finitely presented group. Then $C_S(G')$ is large scale simply connected.*

Theorem B, Corollary C and Theorem D are joint work with D. Calegari.

The next two preliminary chapters summarize the basic theory of cl and scl. Here we emphasize the geometric nature of these two notions. We give topological definitions and show the connection with group cohomology and quasimorphisms, which leads to a sketch of the proof of Bavard's Duality Theorem. In *Chapter* 4, we study generalized Thompson's groups. Two main ingredients here are M. Stein's work [58] on the homology of generalized Thompson's groups and I. Liousse's work [51] on the values of rotation numbers in these groups. We give detailed accounts of these works and deduce useful information for the computation of scl. Theorem A is proved at the end of this chapter. In the last chapter, we study the large scale geometry of the Cayley graph of a commutator subgroup. We give a brief overview of the large scale geometry of metric spaces, word-hyperbolic groups and counting quasimorphisms, and prove Theorem B, Corollary C and Theorem D. Both Theorem B and Corollary C can be extended to more general classes of groups, including mapping class groups of oriented surfaces. We state the corresponding theorems at the end.

Chapter 2

Commutator Length

In this chapter, we introduce the notion of commutator length. Commutator length is an algebraic invariant of elements in groups. It's related to the topological concept of the genus of a surface . We are going to explore this connection through the theory of group (co)homology. At the end, we give examples, exemplifying the computations of commutator length in various groups, which are important in geometry and dynamical systems.

2.1 Definitions of Commutator Length

Let G be a group. An element $a \in G$ is a commutator if there exist $b, c \in G$, such that $a = [b, c] = bcb^{-1}c^{-1}$. Let $G' = [G, G]$ denote the normal subgroup of G which is generated by commutators. We call G' the commutator subgroup of G and they fit into a short exact sequence

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G/G' \longrightarrow 0.$$

The quotient group G/G' is, by its construction, the largest abelian quotient group of G and when G is finitely generated, this abelian quotient group is well understood by the classification theorem of finitely generated abelian groups. So in principle, to study G, we only need to understand the commutator subgroup G' and G' contains all the information lost in the quotient process. A natural measure of complexity for the elements in G' is the notion of *commutator length*.

Definition 2.1.1. Let G be a group and $a \in G'$. The *commutator length* of a, denoted cl(a), is the smallest number of commutators whose product is equal to a, i.e.,

$$\text{cl}(a) = \min\{\ n \mid a = [b_1, c_1] \cdots [b_n, c_n], b_i, c_i \in G\ \}.$$

Set cl$(a) = \infty$ if a is not an element in G'.

Commutator length could also be defined topologically. Let X be a topological space and $G = \pi_1(X, *)$

($*$ is the base point). An element $\gamma \in \pi_1(X,*)$ can be represented by a map $f\colon (S^1,*) \to (X,*)$. Since commutator length takes the same value in a conjugacy class, we only need to consider the conjugacy class of γ, which can be interpreted as the free homotopy class of the map $f\colon S^1 \to X$. From now on, we say γ is represented by a loop l_γ, the image of the map f in X without the base point. If γ is an element in the commutator subgroup G', we can write

$$\gamma = [\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_g, \beta_g].$$

Let S be an oriented surface of genus g with one boundary component. S is obtained from a $(4g+1)$-gon P by identifying edges in pairs and the edges of P are labelled by $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c^{-1}$. Choose the loops in X representing $\gamma, \alpha_i, \beta_i, 1 \leq i \leq g$ and let $h\colon \partial P \to X$ be defined by sending edges of P to those loops in X by $a_i \to \alpha_i$, $b_i \to \beta_i$ and the free edge c to γ. By the construction, h factors through the quotient map $\partial P \to S$ induced by gluing up all but one of the edges. Moreover, by hypothesis, $h(\partial P)$ represents $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma^{-1} = 1$ in $\pi_1(X)$. Hence h can be extended to a map $h\colon S \to X$, sending ∂S to γ. Therefore a loop, corresponding to an element in $[\pi_1(X), \pi_1(X)]$, bounds a map of an oriented surface into X and the number of commutators needed in the product is the genus of the surface. In this language, commutator length has the following equivalent definition

Definition 2.1.2. Let X be a topological space and $G = \pi_1(X)$. Given $\gamma \in [\pi_1(X), \pi_1(X)]$, we have

$$\mathrm{cl}(\gamma) = \min_{S \in \Lambda} \{ \text{ genus}(S) \ \},$$

where $\Lambda = \{h\colon S \to X\}$ and S is an oriented surface with one boundary component such that $h(\partial S) \subset \gamma$ and $[h(\partial S)] = \pm[\gamma]$ in $H_1(\gamma, \mathbb{Z})$.

If a loop γ bounds an oriented surface in X, then $[\gamma]$, regarded as a dimension-1 homological class, represents the trivial element in $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$. The commutator length of γ measures the complexity of this triviality on the level of homology.

2.2 Group Homology and Commutator Length

In this section, we study the connection between commutator length and the theory of group (co)homology. The (co)homology theory of groups arose from both topological and algebraic sources. We briefly introduce the theory from both points of view and give a very rough interpretation of commutator length as a norm related to the homology of a group.

Definition 2.2.1. Let G be a group. A CW-complex Y is called an *Eilenberg-Maclane complex* of type $(G, 1)$ if Y satisfies the following conditions:

1. Y is connected;

2. $\pi_1(Y) \cong G$;

3. The universal cover $\widetilde{Y}$ of Y is contractible. Or equivalently $H_i(\widetilde{Y}) = 0$ for $i \geq 2$, or $\pi_i(\widetilde{Y}) = 0$ for $i \geq 2$.

By Hurewicz's theorem, the homotopy type of Y is determined by $G = \pi_1(Y)$, and we denote it by $K(G,1)$. For any group G, we can construct such a complex. Thus we have

Definition 2.2.2. Let G be a group. The homology of G with $\mathbb{Z}$-coefficients is defined to be the homology of the corresponding $K(G,1)$, i.e.,

$$H_*(G,\mathbb{Z}) = H_*(K(G,1),\mathbb{Z}).$$

The homology of a group can also be defined using the *bar complex*.

Definition 2.2.3. Let G be a group. The *bar complex* $C_*(G)$ is the complex generated in dimension n by n-tuples $(g_1, g_2, \ldots, g_n)$ with $g_i \in G$. The boundary map ∂ is defined by the formula

$$\partial(g_1,\ldots,g_n) = (g_2,\ldots,g_n) + \sum_{i=1}^{n-1}(-1)^i(g_1,\ldots,g_ig_{i+1},\ldots,g_n) + (-1)^n(g_1,\ldots,g_{n-1}).$$

With a coefficient group R ($= \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$), define the homology of the group G with coefficients in R to be $H_*(C_*(G) \otimes R)$.

All n-tuples $(g_1, g_2, \ldots, g_n)$, $g_i \in G$ form a canonical basis for the n-dimensional chain group $C_n(G)$, and we have the canonical inclusions $C_*(G,\mathbb{Z}) \hookrightarrow C_*(G,\mathbb{Q}) \hookrightarrow C_*(G,\mathbb{R})$. From now on, we'll only use $\mathbb{R}$ as the coefficient group, and the elements in $C_*(G,\mathbb{Z})$ or $C_*(G,\mathbb{Q})$ will be called integral or rational chains.

Let $[c]$ be a homology class in $H_i(G,\mathbb{R})$. We can write $c = \sum r_i\sigma_i \in C_i(G,\mathbb{R}), r_i \in \mathbb{R}$, as a chain representative of $[c]$. Define

$$\| c \|_1 = \sum_i |r_i|.$$

Definition 2.2.4. The (Gromov) L^1-norm of $[c] \in H_i(G)$ is defined by

$$\| [c] \|_1 = \inf_c \| c \|_1,$$

where c ranges over all chain representatives of $[c]$ in $C_i(G)$.

Denote the cycles and the boundaries with $\mathbb{R}$-coefficients by $Z_*(G)$ and $B_*(G)$ respectively. Then we have, in dimension 2, a short exact sequence

$$0 \longrightarrow Z_2(G) \xrightarrow{i} C_2(G) \xrightarrow{\partial} B_1(G) \longrightarrow 0.$$

The usual L^1-norm on $C_2(G)$ induces a quotient norm on $B_1(G)$.

Definition 2.2.5. Let $a \in B_1(G)$. The (Gersten) boundary norm of a, denoted $\| a \|_B$, is defined by

$$\| a \|_B = \inf_{A \in C_2(G), \partial A = a} \| A \|_1 .$$

The group G includes as a canonical basis in $C_1(G)$. If a is an element in G', then the image of a in $C_1(G)$ lies in $B_1(G)$. In fact, $a \in G'$ implies that a, thought of as a loop, bounds an oriented surface with one boundary component. A one-vertex triangulation of this surface, with the only vertex on the boundary, gives an expression of a as an element in $B_1(G)$. For example, if $a = [x, y]$, we have $\partial((xyx-1, x) + ([x, y], y) - (x, y)) = [x, y]$. Recall the topological definition of commutator length, which gives an interpretation of cl as a measure of complexity among all the surfaces with a as the only boundary component. It's not difficult to see, through counting the number of triangles, that if $a \in [G, G]$,

$$\| a \|_B \leq 4\mathrm{cl}(a) - 1.$$

It's not clear whether there exists an inequality in the opposite direction. And one way to overcome this difficulty is to "stabilize" both the boundary norm and commutator length, which will give an equality between them. Roughly speaking, we need to identify $\frac{g^n}{n}$ and g for any g and consider the boundary norm under this identification. In *Chapter* 2, we will explore this idea and study stable commutator length.

2.3 Computations of Commutator Length

In this section, we do computations of commutator length. Some of the groups are the special examples of more general classes of groups we are going to study in the following chapters.

2.3.1 Commutator length in free groups. Commutator length in free groups has been studied by many people. C. C. Edmunds in [22] [23] first showed that there exists an effective procedure for computing commutator length in free groups. M. Culler, using surface theory, also worked out an algorithm for computing commutator length, which we will describe below. See [17] for more details.

Let T_n be an orientable surface of genus n with one boundary component. Let Γ_r be the wedge product of r circles, then the fundamental group of Γ_r is free on r generators. So questions about commutator length in a free group translate into questions about maps from T_n to Γ_r. M. Culler shows that if $w \in [F_r, F_r]$ has commutator length n and $f\colon T_n \to \Gamma_r$ is any map such that $f(\partial T_n)$ represents w, then f is homotopic to a "tight" map. Suppose w is written as a reduced word, then each "tight" map gives a "pairing" of the letters in w. Conversely, given a "pairing" of the letters in w, we can construct a unique (up to homotopy) "tight" map and the genus of the surface can be read from the combinatorial information of the "pairing".

Example 2.3.1.

1. If $a_1, b_1, \cdots, a_n, b_n$ are elements of a basis of a free group, then we have

$$\text{cl}([a_1, b_1] \cdots [a_n, b_n]) = n.$$

2. If a, b are basis elements in a free group, then

$$\text{cl}([a, b]^n) = \lfloor \frac{n}{2} \rfloor + 1.$$

And we also obtain interesting commutator identities, for example

$$[a, b]^3 = [aba^{-1}, b^{-1}aba^{-2}][b^{-1}ab, b^2].$$

2.3.2 Commutator length in $\text{Homeo}^+(S^1)$.

Let $\text{Homeo}^+(S^1)$ be the group of orientation-preserving homeomorphisms of the circle. Every element in $\text{Homeo}^+(S^1)$ can be written as a product of two elements both of which have a fixed point. And a homeomorphism in $\text{Homeo}^+(S^1)$ with a fixed point can be written as a commutator. (In fact, such an element is conjugate to its square.) So cl ≤ 2 in $\text{Homeo}^+(S^1)$. Furthermore, one can show that cl ≤ 1 in $\text{Homeo}^+(S^1)$. See [25] for more details.

2.3.3 Knots in 3-sphere.

A knot γ is an embedding: $\gamma \colon S^1 \to S^3$. A *Seifert surface* for a knot γ is a connected, two-sided, compact embedded surface $\Sigma \subseteq S^3$ with $\partial\Sigma = \gamma$. Define the *genus* of a knot γ, denoted $g(\gamma)$, to be the least genus of all its Seifert surfaces. It follows from a deep theorem of D. Gabai [33] that $g(\gamma) = \text{cl}(\gamma)$, where γ is regarded as an element in $\pi_1(S^3 \backslash N(\gamma))$ and $N(\gamma)$ is an open neighborhood of γ. Genus of a knot is a very important knot invariant.

2.3.4 Commutator length in mapping class groups.

Mapping class group is a fundamental object in 2-dimensional topology. See [4] and [28] for more details.

Definition 2.3.2. Let S be an oriented surface (possibly punctured). The *mapping class group* of S, denoted $\text{MCG}(S)$, is the group of isotopy classes of orientation-preserving self-homeomorphisms of S.

$\text{MCG}(S)$ is finitely presentable and its generating set can be chosen from a special class of elements, called Dehn twists.

Definition 2.3.3. Let γ be an essential simple closed curve in S. A right-handed *Dehn twist* in γ is the map $t_\gamma \colon S \to S$ supported on an annulus neighborhood $\gamma \times [0, 1]$ which takes each curve $\gamma \times t$ to itself by a positive twist through a fraction t of its length. If the annulus is parameterized as $\mathbb{R}/\mathbb{Z} \times [0, 1]$, then in coordinates, the map is given by $(\theta, t) \to (\theta + t, t)$.

M. Korkmaz has the following interesting computation about commutator length in $\text{MCG}(S)$.

Theorem 2.3.4 ([47]). *Let* $a \in \mathrm{MCG}(S)$ *be a Dehn twist in a nonseparating closed curve. Then* a^{10} *can be written as a product of two commutators, i.e.,* $\mathrm{cl}(a^{10}) \leq 2$.

Chapter 3

Stable Commutator Length

In this chapter, we continue the study of group (co)homology and define stable commutator length both algebraically and topogically. A very important tool in the study of stable commutator length is the notion of (homogeneous) quasimorphisms. We sketch the proof of Bavard's theorem (3.2.10), which shows the duality between stable commutator length and homogeneous quasimorphisms. At the end, we give the explicit constructions of quasimorphisms in $\text{Homeo}^+(S^1)$ and free groups. These quasimorphisms will play important roles in the next two chapters.

3.1 Definitions of Stable Commutator Length

Definition 3.1.1. Let G be a group and $a \in G$. The *stable commutator length* of a, denoted $\text{scl}(a)$, is the following limit

$$\text{scl}(a) = \lim_{n\to\infty} \frac{\text{cl}(a^n)}{n}.$$

Set $\text{scl}(a) = \infty$, if no power of a is in $[G, G]$.

Commutator length cl clearly has the subadditive property, i.e., $\text{cl}(a^{m+n}) \leq \text{cl}(a^m) + \text{cl}(a^n)$. Then the existence of the limit in the definition follows from the lemma below.

Lemma 3.1.2. *If $a_{m+n} \leq a_m + a_n + L$, for all $m, n \in \mathbb{N}$ and some fixed L, then $\lim_{n\to\infty} a_n/n \in \mathbb{R} \cup \{-\infty\}$ exists.*

Proof. Suppose

$$\liminf_{n\to\infty} \frac{a_n}{n} < b < c,$$

then there exists n, $n > \frac{2L}{c-b}$ such that $\frac{a_n}{n} < b$. For sufficiently large l, $l > n$, $l(c-b) > 2\max_{r<n} a_n$, write $l = nk + r$, $0 < r < n$, and

$$\frac{a_l}{l} \leq \frac{ka_n + a_r + kL}{l} \leq \frac{a_n}{n} + \frac{a_r}{l} + \frac{L}{n} \leq b + \frac{c-b}{2} + \frac{c-b}{2} = c.$$

Thus $\limsup_{n\to\infty} a_n/n = \liminf_{n\to\infty} a_n/n$, and the limit exists. $\square$

Remark 3.1.3. cl and scl are invariant under the action of Aut(G), where Aut(G) is the group of automorphisms of G. In particular, they are conjugacy invariant.

Stable commutator length also has a topological description. Let $G \cong \pi_1(X)$ and γ is a loop in X, representing the conjugacy class of an element $a \in G$. One easily sees, from the topological definition of cl, that scl(a) equals the infimum of genus$(S)/n(S)$, where S is a connected, oriented surface with only one boundary component and S admits a map into X with ∂S wrapping around γ $n(S)$ times. One deficiency of this definition is that this infimum will never be achieved. For any such surface S with one boundary component, we can pass to finite covers of S and genus$(S)/n(S)$ can be reduced to $\frac{-\chi(S)}{2n(S)}$, where $\chi(S)$ is the Euler characteristic of S. Therefore we have the following alternative topological definition for stable commutator length.

Let G, X, a, γ be as above. Given a compact, oriented, not necessarily connected surface S, define $-\chi^-(S)$ to be the sum of $\max(-\chi, 0)$ over all components of S. Given a map $f\colon S \to X$, taking $\partial S \to \gamma$, define $n(s)$ to be the sum, over all components of ∂S, of the degree of the map $f|_{\partial S}$, i.e., $f_*[\partial S] = n(S)[\gamma]$, where $[\gamma]$ is the generator of $H_1(\gamma, \mathbb{Z})$.

Proposition and Definition 3.1.4 ([9]). *Let $G = \pi_1(X)$ and $\gamma \subset X$ a loop representing the conjugacy class of $a \in G$. Then*

$$\text{scl}(a) = \inf_S \frac{-\chi^-(S)}{2n(S)},$$

where the infimum is taken over all maps $f\colon (S, \partial S) \to (X, \gamma)$.

Definition 3.1.5. A surface S, admitting a map $f\colon (S, \partial S) \to (X, \gamma)$ which realizes the infimum of $\frac{-\chi^-(S)}{2n(S)}$, is said to be extremal.

Example 3.1.6.

1. If a, b are basis elements in a free group, by Example (2.3.1), we have

$$\text{scl}([a, b]) = \lim_{n\to\infty} \frac{\lfloor \frac{n}{2} \rfloor + 1}{n} = \frac{1}{2}.$$

2. In Homeo$^+(S^1)$, cl is bounded (≤ 1). So by definition, scl $\equiv 0$.

3. In mapping class groups, interesting lower bounds of scl can be obtained using gauge theory.

   Theorem 3.1.7 *(Endo-Kotschick [26], Kotschick [48]). Let S be a closed orientable surface of genus $g \geq 2$. If $a \in$ MCG(S) is the product of k right-handed Dehn twists along essential disjoint simple closed curves $\gamma_1, \cdots, \gamma_k$, then*

$$\text{scl}(a) \geq \frac{k}{6(3g-1)}.$$

Theorem 3.1.8 *(Kotschick). If t is a Dehn twist along a non-separating curve in a closed orientable surface of genus g, there is an estimate* $\mathrm{scl}(t) = O(\frac{1}{g})$.

3.2 Quasimorphisms and Bavard's Duality Theorem

In this section, we introduce the notion of quasimorphisms. Quasimorphisms are related to our study of stable commutator length by Bavard's duality theorem. In fact the content of this paper is the application of quasimorphisms and Bavard's theorem to the study of commutator subgroups, and this section is the foundation of the whole paper.

3.2.1 Definition of Quasimorphisms

Definition 3.2.1. Let G be a group. A *quasimorphism* on G is a function

$$\phi\colon G \to \mathbb{R},$$

for which there is a constant $D(\phi) \geq 0$ such that for any $a, b \in G$, we have an inequality

$$|\phi(a) + \phi(b) - \phi(ab)| \leq D(\phi).$$

In other words, a quasimorphism is like a homomorphism up to a bounded error. The least constant $D(\phi)$ with this property is called the *defect* of ϕ.

Definition 3.2.2. A quasimorphism is *homogeneous* if it satisfies the additional property

$$\phi(a^n) = n\phi(a)$$

for all $a \in G$ and $n \in \mathbb{Z}$.

Remark 3.2.3. $D(\phi) = 0$ if and only if ϕ is a homomorphism, i.e., $\phi \in \mathrm{Hom}(G, \mathbb{R})$. And it's not difficult to see that a homogeneous quasimorphism is a class function.

Denote the sets of quasimorphisms and homogeneous quasimorphisms by $\widehat{Q}(G)$ and $Q(G)$ respectively. Given $\phi \in \widehat{Q}(G)$, we can homogenize it to obtain a homogeneous quasimorphism.

Lemma 3.2.4 ([9]). $\phi \in \widehat{Q}(G)$ *with* $D(\phi)$*. Then for any* $a \in G$*, the limit*

$$\overline{\phi}(a) = \lim_{n\to\infty} \frac{\phi(a^n)}{n}$$

exists, and thus defines a homogeneous quasimorphism. Furthermore, we have $D(\overline{\phi}) \leq 4D(\phi)$.

3.2.2 Bounded Cohomology and Bavard's Duality Theorem

We defined bar complex $C_*(G, \mathbb{R})$ and used it to define the homology of a group G. Let $C^*(G, \mathbb{R}) = \mathrm{Hom}(C_*(G), \mathbb{R})$ be the dual chain complex and δ the adjoint of ∂. The homology group of $(C^*(G, \mathbb{R}), \delta)$ is called the cohomology group of G with coefficients in $\mathbb{R}$ and is denoted $H^*(G, \mathbb{R})$.

The chain group $C_*(G)$ has a canonical basis, consisting of all n-tuples $(g_1, \cdots, g_n)$, $g_i \in G$, in dimension n. A cochain $\alpha \in C^n(G, \mathbb{R})$ is called bounded if

$$\sup |\alpha(g_1, \cdots, g_n)| < \infty,$$

where the supremum is taken over all n-tuples. This supremum is called the L^∞-norm of α, and is denoted $\| \alpha \|_\infty$. The set of all bounded cochains forms a subcomplex $C_b^*(G, \mathbb{R})$ and its homology is the so-called *bounded cohomology* of G and is denoted $H_b^*(G)$. $\| \cdot \|_\infty$ induces a (pseudo)norm on $H_b^*(G)$ defined as follows: if $[\alpha] \in H_b^*(G)$ is a bounded cohomology class, set $\| [\alpha] \|_\infty = \inf \| \sigma \|_\infty$, where the infimum is taken over all bounded cocycles σ in the class of $[\alpha]$.

Let's see what these definitions mean in low dimensions. A dimension-1 cochain $\phi \in C^1(G)$ is just a real-valued function from G to $\mathbb{R}$ and ϕ is a cocycle if and only if $\delta\phi = 0$. By the definition of the coboundary map,

$$\delta\phi(a, b) = \phi(a) + \phi(b) - \phi(ab).$$

Thus ϕ is a cocycle if and only if ϕ is a homomorphism and $H^1(G, \mathbb{R})$ can be identified with $\mathrm{Hom}(G, \mathbb{R})$. Since any nontrivial homomorphism from G to $\mathbb{R}$ is unbounded, it's immediate that $H_b^1(G, \mathbb{R}) = 0$ for any group G.

Suppose ϕ is a quasimorphism defined above, then

$$|\delta\phi(a, b)| = |\phi(a) + \phi(b) - \phi(ab)| \leq D(\phi),$$

for any $a, b \in G$. Thus $\delta\phi$ is by definition a bounded 2-cochain, i.e., $\delta\phi \in C_b^2(G, \mathbb{R})$ and $\| \delta\phi \|_\infty = D(\phi)$. Since $\delta\phi$ is obviously a cocycle, we get that the image of the coboundary map of a quasimorphism is a bounded dimension-2 cocycle. Furthermore, we have

Theorem 3.2.5 ([9]). *There is an exact sequence*

$$0 \longrightarrow H^1(G, \mathbb{R}) \longrightarrow Q(G) \xrightarrow{\delta} H_b^2(G, \mathbb{R}) \to H^2(G, \mathbb{R}).$$

Proof. Consider the short exact sequence of cochain complexes

$$0 \longrightarrow C_b^* \longrightarrow C^* \longrightarrow C^*/C_b^* \longrightarrow 0,$$

and the associated long exact sequence of cohomology groups. We get an exact sequence

$$0 = H_b^1(G, \mathbb{R}) \longrightarrow H^1(G, \mathbb{R}) \longrightarrow H^1(C^*/C_b^*) \longrightarrow H_b^2(G, \mathbb{R}) \longrightarrow H^2(G, \mathbb{R}).$$

And $H^1(C^*/C_b^*) = \widehat{Q}/C_b^1 \cong Q$. We are done. □

Recall that in *Chapter* 1, we defined (Gersten) boundary norm $\| \cdot \|_B$ on $B_1(G, \mathbb{R})$, the dimension-1 boundary group and we tried to explore the relation between $\| a \|_B$ and $\mathrm{cl}(a)$ for an element $a \in [G, G] \subset B_1(G, \mathbb{R})$. In the following, we stabilize both of the notions and obtain an equality between them.

Proposition 3.2.6 ([9]). *Let $a \in [G, G]$, so that $a \in B_1(G)$ as a cycle. Then*

$$\| a \|_B = \sup_{\phi \in \widehat{Q}(G)/H^1(G,\mathbb{R})} \frac{|\phi(a)|}{D(\phi)}.$$

Proof. The dual space of $B_1(G)$ with respect to the $\| \cdot \|_B$ norm is $\widehat{Q}(G)/H^1(G, \mathbb{R})$ and the operator norm on the dual is equal to $D(\cdot) = \| \delta \cdot \|_\infty$. Then the equality follows from Hahn-Banach Theorem. □

Definition 3.2.7. Let G, a be as above. Define the *filling norm*, denoted $\mathrm{fill}(a)$, to be the homogenization of $\| a \|_\infty$. That is

$$\mathrm{fill}(a) = \lim_{n \to \infty} \frac{\| a^n \|_B}{n}.$$

Remark 3.2.8. $\mathrm{fill}(a)$ is the stabilized (Gersten) bounded norm.

Proposition 3.2.9 ([9]). *Let G be a group and $a \in G'$. There is an equality*

$$\mathrm{scl}(a) = \frac{1}{4}\mathrm{fill}(a).$$

Proof. We have known that for an element $a \in [G, G]$ and any $n \in \mathbb{Z}^+$,

$$\| a^n \|_B \leq 4 \cdot \mathrm{cl}(a^n) - 1.$$

Divide both sides by n, take the limit as $n \to \infty$, and we get the inequality

$$\mathrm{fill}(a) \leq 4 \cdot \mathrm{scl}(a).$$

Conversely, assume $G = \pi_1(X)$ and γ is a loop in X, representing the conjugacy class of a. let A be a chain with $\partial A = a$ and $\| A \|_1$ is close to $\| a \|_B$. WLOG, we assume that A is a rational chain. After scaling by some integer, we can assume that A is an integral chain and $\partial A = na$ for which the ratio $\| A \|_1 / n \| a \|_B$ is very close to 1. Write $A = \sum_i n_i \sigma_i$, where each $n_i \in \mathbb{Z}$, and each σ_i is a singular 2-simplex, i.e., a map $\sigma_i \colon \Delta^2 \to X$. We could group edges of σ_i's in pairs, except for those edges with images in γ. This pairing gives us an orientable surface S and a map $\Phi\colon S \to X$, such that $\Phi_*([S,\partial S]) = A$, where $[S,\partial S]$ is a chain representing the fundamental class of $(S,\partial S)$. By the construction, $\| A \|_1 = \| [S,\partial S] \|_1$. About the Gromov L^1-norm of a surface, we have the following inequality

$$\| [S,\partial S] \|_1 \geq -2\chi(S).$$

Dividing both sides by n, we get

$$\text{fill}(a) \geq 4 \cdot \text{scl}(a).$$

Putting this with the earlier inequality, we are done. □

Now combine Proposition (3.2.6) and Proposition (3.2.9) together, we get (see [1] or [9] for a proof)

Theorem 3.2.10 (Bavard's Duality Theorem [1]). *Let G be a group. Then for any $a \in [G,G]$, we have an equality*

$$\text{scl}(a) = \frac{1}{2} \sup_{\phi \in Q(G)/H^1(G,\mathbb{R})} \frac{|\phi(a)|}{D(\phi)}.$$

Remark 3.2.11.

1. Bavard's theorem reflects the duality between scl and homogeneous quasimorphisms and the duality (i.e., Hahn-Banach theorem) is contained in Proposition (3.2.6).

2. In principle, given a group G, we only need to work out the set of homogeneous quasimorphisms, and then we can compute scl by Bavard's theorem. This approach is especially fruitful when $Q(G)$ has small dimension. In *Chapter* 3, we are going to study several classes of finitely presented groups, of which the sets of homogeneous quasimorphisms are 1-dimensional.

3. There are many groups of which the sets of homogeneous quasimorphisms are infinite dimensional. In *Chapter* 4, we will see that this is a common phenomenon in the groups related to hyperbolic geometry.

Example 3.2.12. Let G be a group. Recall that a *mean* on G is a linear functional on $L^\infty(G)$ which maps the constant function $f(g) \equiv 1$ to 1, and maps non-negative functions to non-negative numbers.

Definition 3.2.13. A group G is *amenable* if there is a G-invariant mean $\pi\colon L^\infty(G) \to \mathbb{R}$ where G acts on $L^\infty(G)$ by $g \cdot f(h) = f(g^{-1}h)$, for all $g, h \in G$ and $f \in L^\infty(G)$.

Examples of amenable groups are finite groups, solvable groups (including abelian groups), and Grigorchuk's groups of intermediate growth.

Theorem 3.2.14 *(Johnson, Trauber, Gromov). If G is amenable, then $H_b^*(G,\mathbb{R}) = 0$.*

As a corollary of Theorem (3.2.5) and the theorem above, we have

Corollary 3.2.15. *If G is amenable, then $Q(G) = H^1(G,\mathbb{R}) = \text{Hom}(G,\mathbb{R})$.*

3.3 Further Properties and Constructions of Quasimorphisms

The importance of quasimorphisms has been displayed by Bavard's duality theorem (3.2.10). And in general, homogeneous quasimorphisms are easier to work with than ordinary quasimorphisms, but ordinary quasimorphisms are easier to construct. They are related by the homogenization procedure (Lemma 3.2.4). In this section, we describe some important constructions of quasimorphisms in two classes of groups. Before that, we first mention a lemma about defect estimation. Suppose ϕ is a homogeneous quasimorphism, then for any commutator $[a,b] \in G$, we have an inequality

$$|\phi([a,b])| \leq D(\phi).$$

And the following lemma says that this inequality is always sharp.

Lemma 3.3.1 (Bavard [1]). *Let ϕ be a homogeneous quasimorphism on G. Then there is an equality*

$$\sup_{a,b\in G} |\phi([a,b])| = D(\phi).$$

3.3.1 Rotation Number

Let $S^1 = [0,1]/\{0,1\}$ be the unit circle and $\pi\colon \mathbb{R} \to S^1$ the covering projection. Let $\mathrm{Homeo}^+(S^1)$ be the group of orientation-preserving homeomorphisms of S^1. Define $\widetilde{\mathrm{Homeo}}^+(S^1) = \{f \in \mathrm{Homeo}^+(\mathbb{R}) \mid f(x+1) = f(x)+1\}$. It's the subgroup of $\mathrm{Homeo}^+(\mathbb{R})$, consisting of all the possible lifts of elements in $\mathrm{Homeo}^+(S^1)$ under the covering projection π.We have the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}}^+(S^1) \xrightarrow{p} \mathrm{Homeo}^+(S^1) \longrightarrow 1,$$

where $\mathbb{Z}$ is generated by the unit translation and $p\colon \widetilde{\mathrm{Homeo}}^+(S^1) \to \mathrm{Homeo}^+(S^1)$ is the natural projection.

Definition 3.3.2. For $g \in \widetilde{\mathrm{Homeo}}^+(S^1)$, define the *rotation number* of g, denoted $\mathrm{rot}(g)$, to be

$$\mathrm{rot}(g) = \lim_{n\to\infty} \frac{g^n(0)}{n}.$$

Remark 3.3.3. Usually, rotation number is defined for elements in $\mathrm{Homeo}^+(S^1)$. For $f \in \mathrm{Homeo}^+(S^1)$, choose an arbitrary lift $\tilde{f} \in \widetilde{\mathrm{Homeo}}^+(S^1)$ and the usual rotation number of f is $\mathrm{rot}(\tilde{f}) \pmod{\mathbb{Z}}$, which is a value in $\mathbb{R}/\mathbb{Z}$.

Rotation number is a very important dynamical invariant in $\mathrm{Homeo}^+(S^1)$. We put together some well-known properties of rotation number in the following proposition. (See [45] for further discussions.)

Proposition 3.3.4.

1. $\mathrm{rot}(\cdot)$ *is continuous in C^0 topology.*

2. $\text{rot}(g) \in \mathbb{Q}$ *if and only if* $p(g) \in \text{Homeo}^+(S^1)$ *has a periodic point.*

3. *If* $\text{rot}(g)$ *is irrational and* $p(g) \in \text{Homeo}^+(S^1)$ *acts transitively on* S^1*, then* $p(g)$ *is conjugate to the rotation through the angle* $\text{rot}(g)(\text{mod } \mathbb{Z})$.

The following proposition shows that rotation number, as a function on $\widetilde{\text{Homeo}}^+(S^1)$, is a homogeneous quasimorphism.

Proposition 3.3.5 ([60]). $\text{rot}\colon \widetilde{\text{Homeo}}^+(S^1) \to \mathbb{R}$ *is a homogeneous quasimorphism and its defect* $D(\text{rot}) = 1$.

Proof. First let $f, g \in \widetilde{\text{Homeo}}^+(S^1)$. Without loss of generality, we assume that $0 \leq f(0), g(0) < 1$. So $0 \leq f \circ g(0) < 2$. And $0 \leq \text{rot}(f) \leq 1$, $0 \leq \text{rot}(g) \leq 1$, and $0 \leq \text{rot}(f \circ g) \leq 2$. Thus we have $|\text{rot}(f \circ g) - \text{rot}(f) - \text{rot}(g)| \leq 2$, and rot is a quasimorphism. rot being homogeneous is clear from its definition.

Second we show that $D(\text{rot}) = 1$ by using Lemma 3.3.1. Take any $f, g \in \widetilde{\text{Homeo}}^+(S^1)$. We want to compute $\text{rot}([f, g])$. We can still assume that $0 \leq f(0), g(0) < 1$. Suppose $0 \leq g(0) \leq f(0) < 1$, then we have, by the fact that f, g are both increasing functions:

$$g(f(0)) < g(1) = g(0) + 1 \leq f(0) + 1 \leq f(g(0)) + 1.$$

So

$$f(g(0)) - g(f(0)) > -1$$

We have two cases:

(i) If we also have $f(g(0)) - g(f(0)) \leq 1$, then

$$-1 \leq f(g(0)) - g(f(0)) \leq 1,$$

$$g(f(-1)) = g(f(0)) - 1 \leq f(g(0)) \leq g(f(0)) + 1 = g(f(1)),$$

which implies

$$-1 \leq f^{-1}g^{-1}fg(0) \leq 1 \Longrightarrow |\text{rot}([f, g])| \leq 1.$$

(ii) If instead we have $f(g(0)) > g(f(0)) + 1$, then

$$g(f(0)) < f(g(0)) - 1 = f(g(0) - 1) < f(0).$$

Consider $H(x) = f^{-1}g^{-1}fg(x) - 1 - x$, for $x \in [0,1]$. $H(0) = f^{-1}g^{-1}fg(0) - 1 > 0$ by assumption.

$$\begin{aligned} H(f(0)) &= f^{-1}g^{-1}fg(f(0)) - 1 - f(0) \\ &< f^{-1}g^{-1}f(f(0)) - 1 - f(0) \\ &= f^{-1}g^{-1}f^2(0) - 1 - f(0). \end{aligned}$$

We want to show that $H(f(0)) < 0$, which can be deduced from the inequality below

$$f^{-1}g^{-1}f^2(0) < 1 + f(0),$$

which is equivalent to

$$f^2(0) < g(f^2(0)) + 1.$$

This is always true since $x < g(x) + 1$, for any $x \in \mathbb{R}$.

So we have $H(0) > 0$ and $H(f(0)) < 0$, here $0 < f(0) < 1$. There must be a point $y \in (0, f(0))$ such that

$$H(y) = f^{-1}g^{-1}fg(y) - 1 - y = 0.$$

That is

$$f^{-1}g^{-1}fg(y) = 1 + y.$$

So

$$\text{rot}[f,g] = \lim_{n\to\infty} \frac{[f,g]^n(y) - y}{n} = 1.$$

The proof for the case $0 \leq f(0) \leq g(0) < 1$ is the same. Put all together, and we get $D(\text{rot}) = 1$. □

3.3.2 Counting Quasimorphisms

Counting quasimorphisms were introduced by R. Brooks [6] in the study of bounded cohomology of free groups. Later on, this construction was generalized to word-hyperbolic groups by Epstein-Fujiwara [27] and more general classes of groups by Fujiwara [30] [31] and Bestvina-Fujiwara [2]. An immediate result of these constructions is that the 2nd bounded cohomology of these groups are infinite dimensional. In this section, we focus on free groups and give Brooks' construction.

Let F be a free group with a finite free generating set S. Then any element in F has a unique reduced form, written as a word in $S \cup S^{-1}$. Let w be a reduced word and $g \in F$. The *big counting function* $C_w(g)$ is defined by

$$C_w(g) = \text{number of copies of } w \text{ in the reduced representative of } g,$$

and the *little counting function* $c_w(g)$ is defined by

$$c_w(g) = \text{maximal number of disjoint copies of } w \text{ in the reduced representative of } g.$$

Definition 3.3.6. A *big counting quasimorphism* is a function of the form

$$H_w(g) = C_w(g) - C_{w^{-1}}(g).$$

And a *little counting quasimorphism* is a function of the form

$$h_w(g) = c_w(g) - c_{w^{-1}}(g).$$

Example 3.3.7. Let $F = F_2 = \langle a, b \rangle$.

1. Let $w = a$ or b. It's clear that $H_w = h_w$ in this case and they are both homomorphisms. In fact they are the only cases in which H_w or h_w could be a homomorphism.

2. Let $w = aba$, then $H_w(ababa) = 2$, but $h_w(ababa) = 1$.

It's not difficult to see that both $H_w(\cdot)$ and $h_w(\cdot)$ are quasimorphisms with defects $\leq O(|w|)$, where $|w|$ is the word length of w. It turns out that for a big quasimorphism $H_w(\cdot)$, its defect $D(H_w)$ depends on $|w|$, but for a little quasimorphism $h_w(\cdot)$, we have the following uniform bound for all $w \in F$.

Theorem 3.3.8 (D. Calegari [9]). *Let F be a free group and $w \in F$ a reduced word. Let $h_w(\cdot)$ be the little counting quasimorphism. Then we have $D(h_w) \leq 2$. More precisely, we have*

1. $D(h_w) = 0$ *if and only if* $|w| = 1$*;*

2. $D(h_w) = 2$ *if and only if* w *is of the form* $w = w_1 w_2 w_1^{-1}$, $w = w_1 w_2 w_1^{-1} w_3$ *or* $w = w_1 w_2 w_3 w_2^{-1}$ *as reduced expressions;*

3. $D(h_w) = 1$ *otherwise.*

Remark 3.3.9.

1. The proof involves a careful analysis of the appearance of w or w^{-1} in the junction where 2 reduced words are concatenated.

2. We know from the discussion before Theorem (3.2.5) that $[\delta h_w] \in H_b^2(F, \mathbb{R})$. A careful choice of a sequence of w_i's with $|w_i| \to \infty$ will give h_w's such that the cohomology class $[\delta h_{w_i}]$'s are linearly independent, implying that $\dim_{\mathbb{R}} H_b^2(F, \mathbb{R}) = \infty$.

3. In *Chapter* 4, we will introduce the generalizations of the little counting quasimorphisms to word-hyperbolic groups, and their defects also have a uniform bound.

Chapter 4

SCL in Generalized Thompson's Groups

In this chapter, we study scl in generalized Thompson's groups. Generalized Thompson's groups are very important objects in many branches of mathematics. They can be realized as subgroups of $\text{Homeo}^+(S^1)$, and show a lot of interesting properties, similar to those of $\text{Homeo}^+(S^1)$. For several classes of generalized Thompson's groups, we prove that

1. For any group T in these classes, the space of homogeneous quasimorphisms $Q(T) = \{0\}$. As a consequence, $Q(\widetilde{T})$ has dimension 1 and is generated by the rotation quasimorphism. Here $\widetilde{T}$ is the central extension of T by $\mathbb{Z}$.

2. Thus by Bavard's Duality Theorem, we can compute scl in $\widetilde{T}$, and there are elements in $\widetilde{T}$, whose scl's take irrational (in fact, transcendental) values.

In contrast to bulletin 2 above, D. Calegari studies scl in free groups [12] (more generally, free product of abelian groups [13]) and shows that scl takes only rational values in them. And in general geometric settings, (relative) genus norms are expected to take rational values, like Thurston's norm in 3-dimensional topology. M. Gromov (in [36] $6.C_2$) asked the question of whether such a stable norm, or in our content, the stable commutator length in a finitely presented group, is always rational, or more generally, algebraic. And our computations give the negative answer.

4.1 Thompson's Groups and Generalized Thompson's Groups

Thompson's groups were first defined by Richard Thompson in the study of logic. They were used to construct finitely presented groups with unsolvable word problems [54]. Later, these groups were rediscovered by homotopy theorists in the work on homotopy idempotents [20][21][29][19]. Nowadays, Thompson's groups are still the main objects in many researches related to (geometric) group theory. Many important concepts and constructions have been applied to these groups, which has raised a lot of interesting problems. In this section, we briefly recall the definitions and basic properties of Thompson's groups and their generalizations.

Let F be the set of piecewise linear homeomorphisms from the closed unit interval $[0, 1]$ to itself that are differentiable except at finitely many dyadic rational numbers (i.e., numbers of the form $p \cdot 2^q, p, q \in \mathbb{Z}$) and such that on intervals of differentiability, the derivatives (slopes) are powers of 2 (i.e., numbers of the form $2^m, m \in \mathbb{Z}$). It's easy to verify that F is a group and is called Thompson's group F.

Example 4.1.1. Two elements x_0 and x_1 in F, which are generators of F.

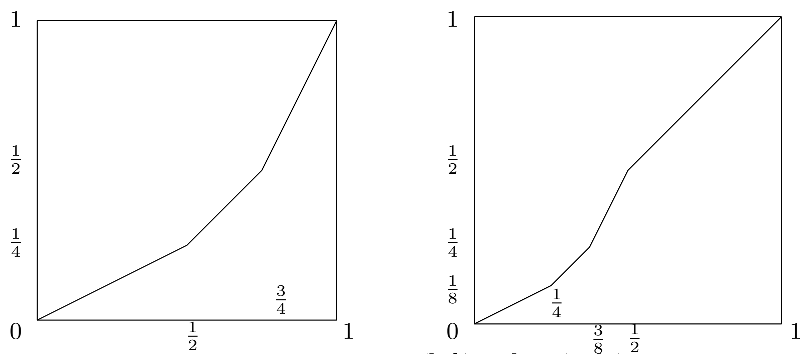

Figure 4.1: x_0 (left) and x_1 (right).

The main properties of F are contained in the following theorem. See [16] for proofs and further references.

Theorem 4.1.2.

1. *F is finitely presented and in fact* FP_∞*, i.e., there is an Eilenberg-Maclane complex $K(F, 1)$ with finite number of cells in each dimension.*

2. *The commutator subgroup $[F, F]$ of F consists of all elements that are trivial in the neighborhoods of 0 and 1, i.e., $[F, F] = \ker\rho$, where ρ is the following homomorphism*

$$\begin{aligned} \rho \colon \ F &\longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ f &\longmapsto (\log_2 f'(0+), \log_2 f'(1-)). \end{aligned}$$

   Thus $F/[F, F] \cong \mathbb{Z} \oplus \mathbb{Z}$.

3. *$[F, F]$ is a simple group.*

Next we define Thompson's group T. Consider S^1 as the unit interval $[0, 1]$ with the endpoints identified. Then T is the set of piecewise linear homeomorphisms from $S^1 = [0, 1]/\{0, 1\}$ to itself that map dyadic rational numbers to dyadic rational numbers and that are differentiable except at finitely many dyadic rational numbers and on intervals of differentiability, the derivatives (slopes) are powers of 2. T is a group and called Thompson's group T.

Theorem 4.1.3 (Brown-Geoghegan). *T is a finitely presented, infinite simple group and is* FP_∞.

There are several ways to generalize the definitions of Thompson's groups. The one we are going to study is due to M. Stein [58]. Let P be a multiplicative subgroup of the positive real numbers and let A be a $\mathbb{Z}P$-submodule of the reals with $PA = A$. Choose a number $l \in A, l > 0$. Let $F(l, A, P)$ be the group of piecewise linear homeomorphisms of $[0, l]$ with finitely many break points, all in A, having slopes only in P. Similarly define $T(l, A, P)$ to be the group of piecewise linear homeomorphisms of $[0, l]/\{0, l\}$ (the circle formed by identifying endpoints of the closed interval $[0, l]$) with finitely many break points in A and slopes in P, with the additional requirement that the homeomorphisms send $A \cap [0, l]$ to itself. It's clear that $F(l, A, P) \subset T(l, A, P)$. If we let $P = \langle 2 \rangle$ and $A = \mathbb{Z}[\frac{1}{2}]$, Thompson's groups are $F = F(1, \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle)$ and $T = T(1, \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle)$.

We are interested in the case that P is generated by integers, i.e., $P = \langle n_1, n_2, \dots, n_k \rangle$ and $A = \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}, \dots, \frac{1}{n_k}]$. P is a free abelian group and we can assume that $\{n_1, n_2, \dots, n_k\}$ forms a basis for P, i.e., $\log n_1, \dots, \log n_k$ are $\mathbb{Q}$-independent, and k is the rank of P. Let $d = gcd(n_1 - 1, \dots, n_k - 1)$ and $IP \cdot A$ the submodule of A generated by elements of the form $(1 - p)a$, where $a \in A$ and $p \in P$. An important theorem in studying generalized Thompson's groups is the following Bieri-Strebel criterion.

Theorem 4.1.4 (R. Bieri and R. Strebel [3]). *Let a, c, a', c' be elements of A with $a < c$ and $a' < c'$. Then there exists f, a piecewise linear homeomorphism of $\mathbb{R}$, with slopes in P and finitely many break points, all in A, mapping $[a, c]$ onto $[a', c']$ if and only if $c' - a'$ is congruent to $c - a$ modulo $IP \cdot A$.*

Proof. Assume first that such an f exists. Let $a = b_0, b_1, \dots, b_{n-1}, b_n = c$ be an increasing sequence of elements of A such that f is linear on $[b_{i-1}, b_i]$ with slope p_i, for all i. Then $c' - a' = \sum_{i=1}^{n} p_i(b_i - b_{i-1})$. But $c - a = \sum_{i=1}^{n}(b_i - b_{i-1})$, so $(c' - a') - (c - a) \in IP \cdot A$.

Conversely, suppose there exist $a_1, \dots, a_n \in A$ and $p_1, \dots, p_n \in P$ such that

$$(c' - a') = (c - a) + \sum_{i=1}^{n}(1 - p_i)a_i.$$

Set $b' = c' - a'$ and $b = c - a$. If we could find f mapping $[0, b]$ to $[0, b']$, composing f with translation by $-c$ on the right and c' on the left gives the desired map. Now there exists a permutation π of $\{1, 2, \dots, n\}$ such that the partial sum $b_j = b + \sum_{i=1}^{j}(1 - p_{\pi(i)})a_{\pi(i)}$ are positive for $j = 0, 1, 2, \dots, n$. If $f_1, f_2, \dots, f_n$ are piecewise linear homeomorphisms of $\mathbb{R}$ with slopes in P and break points in A such that $f_i([0, b_{j-1}]) = [0, b_j]$, then $f_n \circ \cdots \circ f_1$ is the desired f. Therefore it is sufficient to prove the claim for $n = 1$. Moreover, as $(1 - p_1)a = (1 - p_1^{-1})(-p_1 a)$, we may assume that $p_1 > 1$. So we need to construct f mapping $[0, b]$ to $[0, b + (p - 1)a]$, where $p > 1$, $a \neq 0$(if $a = 0$, the identity works), and b and $b + (p - 1)a$ are both positive.

Suppose first that $a > 0$. Choose a number k with $a < p^k b$, and set $a' = p^{-k}a$. Define $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_1(t) = \begin{cases} t & \text{if } t \leq b - a', \\ p(t - (b - a')) + (b - a') & \text{if } b - a' < t \leq b, \\ t + (p - 1)a' & \text{if } b < t. \end{cases}$$

$$f_2(t) = \begin{cases} t & \text{if } t \le b, \\ p^k(t-b)+b & \text{if } b < t \le b+(p-1)a', \\ t+(p-1)(a-a') & \text{if } b+(p-1)a' < t. \end{cases}$$

Then f_1 maps $[0,b]$ to $[0,b+(p-1)a']$ and f_2 maps $[0,b+(p-1)a']$ to $[0,b+(p-1)a]$, so $f = f_2 \circ f_1$ is the desired map.

On the other hand, if $a < 0$ then there exists f taking $[0,b+(p-1)a]$ to $[0,b+(p-1)a+(p-1)(-a)] = [0,b]$, so f^{-1} will be the map we needed. □

Remark 4.1.5. If $d = gcd(n_1 - 1, \ldots, n_k - 1) = 1$, $IP \cdot A = P(d\mathbb{Z}) = P\mathbb{Z} = A$, and the conclusion in Theorem (4.1.4) is vacuously satisfied. Therefore in this case, we have that for any $a < c$, $a' < c'$, $a, c, a', c' \in A$, there always exists an f with slopes in P and break points in A, such that f maps $[a,c]$ to $[a',c']$.

Theorem 4.1.6 (G. Higman [42]). *Let Ω be a totally ordered set and let B be a two-fold transitive, ordered permutation group on Ω, consisting of bounded elements. Then $[B,B]$ is a nontrivial simple group.*

Here "two-fold transitive" means that if $a < c$, $a' < c'$, $a, c, a', c' \in \Omega$, then there is an element of B taking a to a' and c to c'. Let $\Omega = IP \cdot A \cap [0,l]$, then Theorem (4.1.4) tells us that $F(l,A,P)$ is two-fold transitive on Ω, so we have

Corollary 4.1.7. *For any choice of P, A and l, $F = F(l,A,P)$ has a simple commutator subgroup.*

M. Stein also proves similar simplicity result for the group T.

Theorem 4.1.8 ([58]). *For any choice of P, A and l, $T(l,A,P)$ has a simple second commutator subgroup.*

4.2 Homology of Generalized Thompson's Groups

To study scl on generalized Thompson's groups, we need to know the perfectness of some subgroups of $F = F(l,A,P)$, which can be deduced from the information on the dimension-1 homology group of F. M. Stein constructs a contractible CW-complex K, on which F acts freely. So K/F is a $K(F,1)$ and $H_*(F,\mathbb{Z}) \cong H_*(K/F,\mathbb{Z})$. In the following, we give details of this construction, from which the desired property of $H_1(F)$ easily follows. Most of the materials in this section are taken from M. Stein's paper [58].

4.2.1 Construction of X

The X was first constructed by K. S. Brown [7], and Brown used it to show that $F = F(l,A,P)$ is finitely presented and of type FP_∞. Its costruction comes from a poset on which the group F acts. Let's describe the poset first.

From now on, we'll fix $P = \langle n_1, n_2, \dots, n_k \rangle$, $A = \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}, \dots, \frac{1}{n_k}]$, where n_i's $\in \mathbb{Z}^+$ form a basis for P and $l \in \mathbb{Z}^+ \cap A$. An element of the poset is a piecewise linear homeomorphism from $[0, a]$ to $[0, l]$, where $a \in \mathbb{Z}$, $a \equiv l$ (mod$d$), which has finitely many break points, all in A, and slopes in P. The "mod d" condition comes from Theorem (4.1.4), by which such a homeomorphism exists if and only if $a - l \in IP \cdot A$. And $A/IP \cdot A \cong \mathbb{Z}/d\mathbb{Z}$, thus we can have such a map if and only if $a - l \equiv 0$(mod d). Given $f_1 \colon [0, a] \to [0, l]$, we say f_2 is a simple expansion of f_1 if for some i, there exists $s \colon [0, a + n_i - 1] \to [0, a]$ such that $f_2 = f_1 \circ s$, where s is a homeomorphism, which has slope 1 everywhere except on some interval $[x, x + n_i]$, $x \in \{0, 1, \dots, a - 1\}$, on which it has slope $\frac{1}{n_i}$. We think of these expansion maps s as expanding the domain of f_1 by dividing some unit subinterval of the domain into n_i equal pieces and expanding each one to an interval of length one in the domain of f_2. See Figure (4.2) for an expansion when $a = 3, n_1 = 2$.

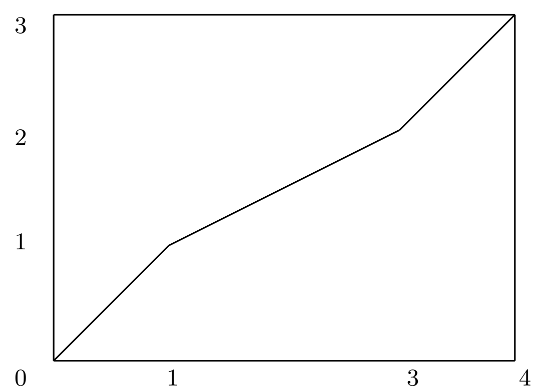

Figure 4.2: An expansion $(a = 3, n_1 = 2)$.

We extend this to a partial order by saying that $f_1 < f_2$ if f_2 can be obtained from f_1 by doing finitely many simple expansions. Then $F = F(l, A, P)$ acts on the poset by composition: given $f \in F$ and g in the poset, $f(g)$ is the map $f \circ g$. Since the group acts on the range of a poset element, and expansions take place in the domain, this action preserves the partial order.

This poset is directed. That is given any two elements $f \colon [0, a] \to [0, l]$ and $g \colon [0, b] \to [0, l]$, we can find $h \colon [0, c] \to [0, l]$ such that h is an expansion of both f and g. Now let X be the simplicial complex associated to the poset, i.e., X has an n-simplex for each linearly ordered $(n+1)$-tuple $f_0 < f_1 < \cdots < f_n$ in the poset. The poset being directed guarantees that $\pi_1(X) = 1$ and $H_n(X) = 0$ for any $n \geq 1$. By Whitehead's and Hurewicz's theorems, X is contractible. Since F acts freely on X, X/F is a $K(F, 1)$, the Eilenberg-Maclane space for F.

Let's take a close look at the action. If an element (or a vertex in X) f has domain $[0, a]$, we say that f is a basis of size a. It's not difficult to see that if v_1 and v_2 are two bases of the same size, there exists a unique group element f such that $f(v_1) = v_2$. So the bases of the same size form an orbit of the group action and in X/F, there is one vertex corresponding to each $a \in \mathbb{Z}^+$, $a \equiv l$(mod d). We refer to the basis which is just the identity map on $[0, l]$ as the standard basis and call the subposet of all expansions of the standard basis the standard subposet. By the same argument above, the translations of the standard subposet by group elements cover the whole poset, so is true for the subcomplex constructed from the standard subposet in X. Therefore, to study the cell structure in X/F, we only need to keep track of the relations within the standard subposet. One way to do this is to associate a *forest* to each basis. To the standard basis, we associate a row of l dots. Then if f is a basis in the standard subposet, we can inductively represent the simple expansion of f, obtained by expanding the ith interval into n_j, by drawing the forest for f, and then drawing n_j new leaves descending from the ith leaf in the forest for f. As an example, the following picture (Figure (4.3)) is for the expansion obtained from the standard basis ($l = 1$) by dividing the root interval in thirds, and then each interval of the result in halves.

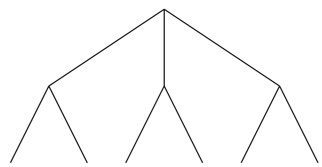

Figure 4.3: A forest.

Remark 4.2.1. The same construction with minor modifications also works for the group T.

4.2.2 Costruction of N

We write $[f, g]$ for the closed interval in the poset, i.e., the subposet $\{h \mid f \leq h \leq g\}$. Similarly, we write (f, g) for the open interval. $|[f, g]|$ and $|(f, g)|$ stand for the subcomplex of X spanned by these intervals. Since group elements act only on the range of bases, up to the group action, we are only concerned with which expansion maps take you from f to g, rather than the particular basis f. So we will be looking at the domain of f and how it is divided when expanding to g. Given a basis $f\colon [0, a] \to [0, l]$, we will refer to the intervals $[i, i+1]$, $i = 0, 1, \cdots, a-1$ of the domain of f as f-interval.

Now we are ready to study X. It turns out that many simplices of X are inessential in the sense of homotopy equivalence. Let's consider an arbitrary interval $[f, g]$. Let $\{f_i\}$ be the set of simple expansions of f which are in the interval, i.e., $f_i < g$. Let h be the least upper bound of f_i's.

Definition 4.2.2 ([58]). If $h = g$, $[f, g]$ (or (f, g)) is an elementary interval. If $h < g$, the interval is nonelementary. Furthermore, a simplex $(f_0, f_1, \cdots, f_n)$ is elementary if $[f_i, f_j]$ is elementary for any $i < j$. Let $N \subset X$ be the union of all elementary simplices.

Another way to describe an elementary interval is: $[f_i, f_j]$ is elementary if in expanding f_i to f_j, each f_i-interval is divided into n equal pieces, where n is some (possible empty) product of the n_i's in which each n_i appears at most once. Here are two examples for the case $F = F(1, \mathbb{Z}[\frac{1}{2}, \frac{1}{3}], \langle 2, 3\rangle)$. See Figure (4.4).

N is clearly an F-invariant subcomplex of X. Furthermore, we have

Theorem 4.2.3 ([58]). *The inclusion $i\colon N \hookrightarrow X$ is a homotopy equivalence. Thus N is contractible and N/F is a $K(F, 1)$.*

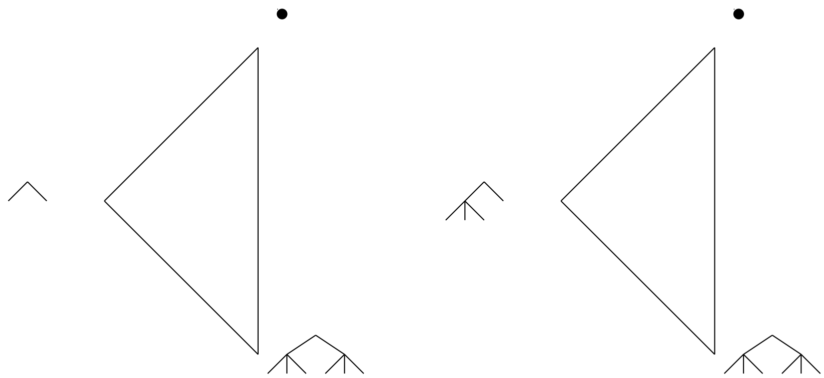

Figure 4.4: Elementary (left) and non-elementary (right) simplexes.

The proof is almost contained in the following lemma

Lemma 4.2.4 ([58]). *Let $\{f_i\}$, $1 \leq i \leq n$, be the simple expansions of f in (f,g). Let $(f,g)_N$ be the subposet consisting of all the least upper bounds of any subset of $\{f_i\}$, as long as these lub's are less than g. Then the inclusion of $|(f,g)_N|$ into $|(f,g)|$ is a homotopy equivalence. In particular, (f,g) is homotopy equivalent to S^{n-2} if the interval is elementary, and is contractible if the interval is nonelementary.*

We can build X in the following way. Let the *height* of $|(f,g)|$ be $b-a$, where the sizes of f and g are a and b respectively. We build X by starting with all of the vertices, then adjoining intervals of height 1, then all the intervals of height 2, etc. By Lemma (4.2.4), we only need to adjoin elementary intervals. In fact, instead of adjoining the full interval, we only need to adjoin $|[f,g]_N|$.

Let's see what $|[f,g]_N|$ is as a subcomplex. Suppose $[f,g]$ is elementary and $\{f_i\}_{i=1}^n$ is the simple expansions of f in $[f,g]$. Take n copies of $\{0,1\}$, viewed as a poset with $0 < 1$. Denote the poset of the n-tuples of 0's and 1's by C, then the geometric realization of C, denoted $|C|$, is just a triangulated n-cube (Figure (4.5)).

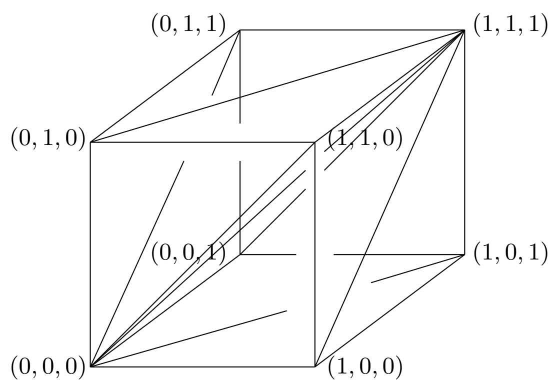

Figure 4.5: A triangulated 3-cube.

We make a poset isomorphism between C and $[f,g]_N$, by sending $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ with $\varepsilon_i \in \{0,1\}$ to the least upper bound of $\{f_i \mid \varepsilon_i = 1, i = 1, 2, \cdots, n\}$. This isomorphism reveals $|[f,g]_N|$ to be a triangulated n-cube. Now if P, the slope group, has rank 1 as a free abelian group, any two of these cubes intersect in a common face, so that the cubes give a CW-complex structure to N. Thus in the case of an F group with rank one slope group, the cubical chain complex for N can be used directly to compute the homology of F.

However, if P has rank ≥ 2, faces of these cubes may be attached to the interior of the same or higher dimensional cubes.

Example 4.2.5. In Figure (4.6), we have a square (the bigger one on the right) in the complex for $F = F(1, \mathbb{Z}[\frac{1}{2}, \frac{1}{3}], \langle 2, 3\rangle)$.

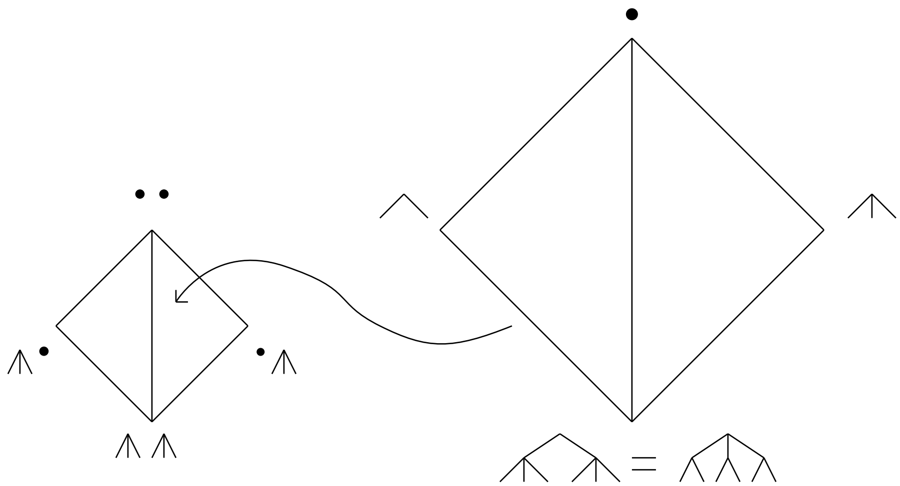

Figure 4.6: A non-cubic structure.

Notice that the bottom two sides of the square are diagonals of a 2-dimensional cube and a 3-dimensional cube respectively.

Therefore we can't just write down a chain complex with the cubes as generators. But for the purpose of obtaining a presentation for F, the cubical structure has already given enough information. The reason is that all the diagonals can be homotopied to the edges of the corresponding cube, thus don't need to be considered as elements in a generating set.

Example 4.2.6. Let $F = F(1, \mathbb{Z}[\frac{1}{2}], \langle 2\rangle)$, the Thompson's group F. In this case, N is actually a cubical complex. We lift a maximal tree in N/F to N by lifting the vertex of size one in N/F to the standard basis, and then making a tree by taking successive expansions at the leftmost interval (Figure (4.7)).

$x_0, x_1, \cdots$ can be taken to be the generators for F, and the relations among them come from 2-dimensional cubes. Here is an example of a nontrivial relation: $x_0 x_2 = x_1 x_0$ (Figure (4.8)).

If we do this analysis systematically, we could obtain the following presentation for F:

$$F(1, \mathbb{Z}[\frac{1}{2}], \langle 2\rangle) = \langle\, x_0, x_1, x_2, \cdots \mid x_i x_{j+1} = x_j x_i, \forall\, i < j \,\rangle.$$

And x_0 and x_1 are the two elements in Example (4.1.1). This is the well-known presentation for the Thompson's group F, which says that F has a universal conjugacy idempotent.

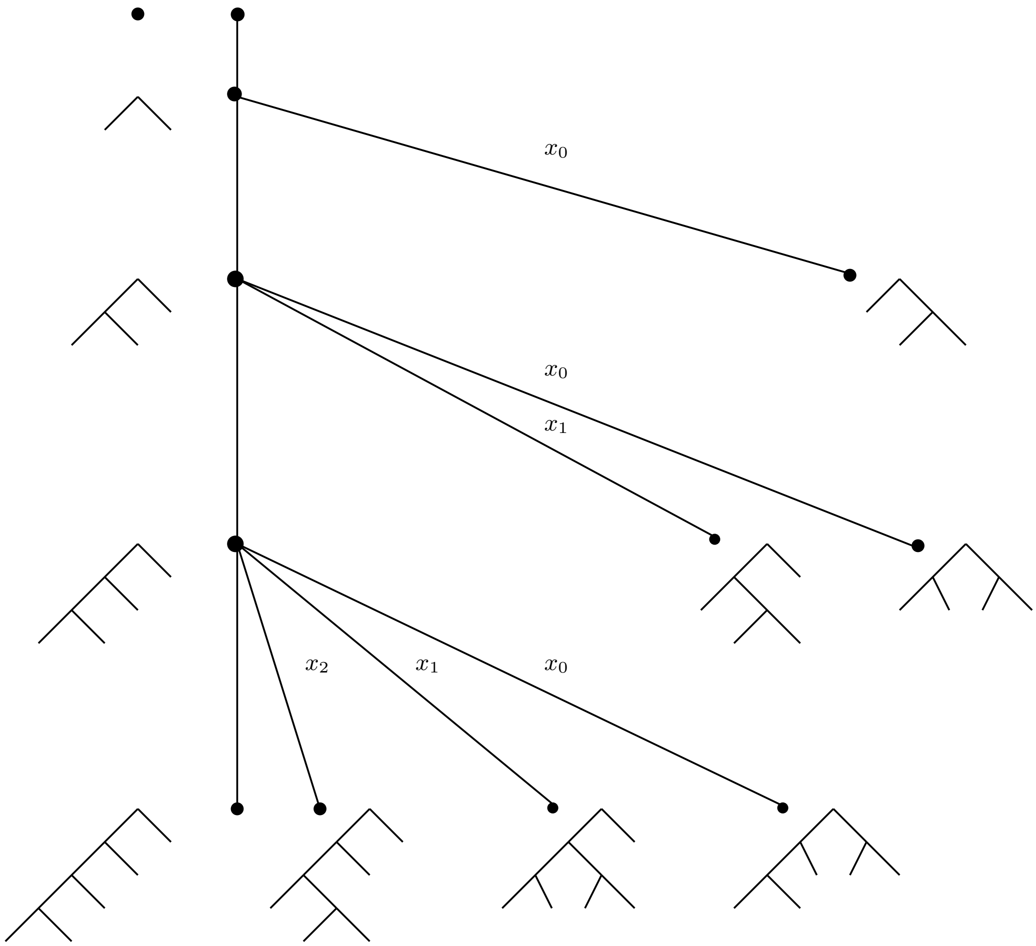

Figure 4.7: Part of the maximal tree.

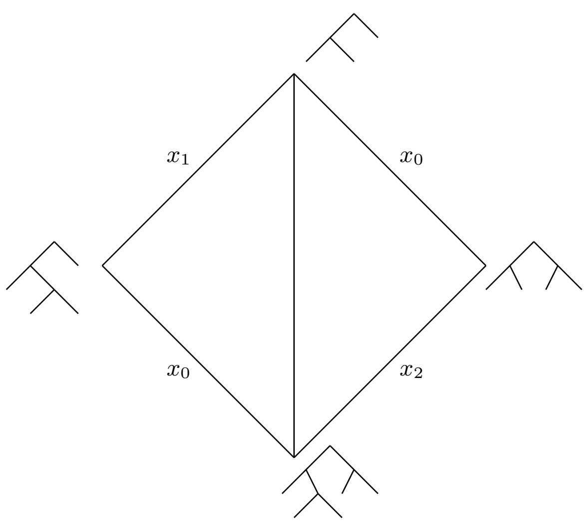

Figure 4.8: A relation from a 2-cube.

Theoretically, we are able to retrieve information on the dimension-1 homology of the group F from this presentation. But since the complex N is infinite in each dimension and for higher rank slope groups, the construction will be much more complicated. We need to do more collapsings on N to make the computation accessible.

4.2.3 Collapsing to a Complex of Finite Type

We view N as a "cell complex" with the cubes $|[f,g]_N|$ as "cells". We have seen that in the higher rank case, it's not a CW-complex structure. This means we can't build N by adjoining cubes in order of dimensions. There is a natural concept, called *degree*, which replaces the notion of dimension. For any cube $|[f,g]_N|$, each f-interval is expanded into p pieces, where p is a product of n_i's in which each n_i appears at most once. We say that $|[f,g]_N|$ has degree $(a_1, a_2, \cdots, a_k)$ (k is the rank), where a_j is the number of f-intervals which are divided into p pieces and p is a product of $k-j+1$ of the n_i's. Then the dimension of a cube of degree $(a_1, a_2, \cdots, a_k)$ is $ka_1 + (k-1)a_2 + \cdots + 2a_{k-1} + a_k$. The degrees are ordered lexicographically. Note that each face of $|[f,g]_N|$ is contained in a cube of smaller degree, even though it may have larger dimension. Then we build N by adjoining cubes in the order of degrees. And the action of F is cellular and preserves degrees.

Let's look at this action and give each cube in the quotient complex N/F a symbol. First notice that there is one zero cube for each natural number $s \equiv l$ (mod d) (corresponding to all bases of size s). We give the zero cube, corresponding to s, the symbol $v^s = v \cdots v$, a string of s v's. Now for each n-cube σ in the quotient, we may choose a lift $[f,g]_N$ in N and g is obtained by expanding certain f-intervals into pieces. We encode this information in a symbol of length s, where s is the number of f-intervals (i.e., the size of f). Our symbol has one letter for each f-interval, with the leftmost letter corresponding to the leftmost f-interval, etc. We put v if there is no expansion at an interval, and x_J if the interval is divided into $n_{i_1} n_{i_2} \cdots n_{i_r}$ equal pieces, where $J = \{i_1, i_2, \cdots, i_r\}$.

Example 4.2.7. $v^2 x_1 x_{1,2} v$ is a 3-cube of N/F such that any lift of it to N, written as $[f,g]_N$, has f, a basis of size 5. When expanding from f to g, the third interval is divided into n_1 equal pieces, and the forth interval is divided into $n_1 \cdot n_2$ equal pieces (Figure (4.9)).

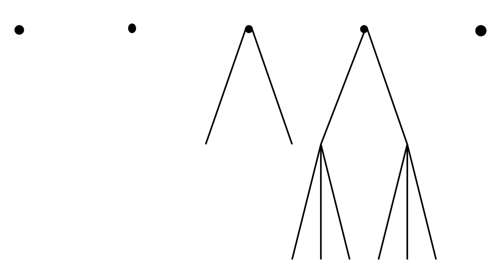

Figure 4.9: A forest representing $v^2 x_1 x_{1,2} v$ ($n_1 = 2$ and $n_2 = 3$).

It's clear that this symbol is independent of the choice of the lift of σ.

Remark 4.2.8.

1. These symbols are just another way of writing the forest symbols introduced in the previous sections.
2. These symbols can also be regarded as a product of cubes. For example, $v^2x_1x_{1,2}v$ is a product of two 0-cubes, one 1-cube, one 2-cube and the last 0-cube. This product structure will be used in the computation of the homology group.

With these symbols, we now define the *face operators*. An n-cube σ has $2n$ faces, $A_i(\sigma)$ and $B_i(\sigma)$ for $i \in \{1, 2, \cdots, n\}$. Suppose $\sigma = Sx_JR$, where S is an s-cube and $J = \{i_1, i_2, \cdots, i_r\}$, with $r \geq 1$ and $i_1 < i_2 < \cdots < i_r$. We may think of v as $x_\emptyset$. Choose j with $1 \leq j \leq r$, and let $J' = J\backslash\{i_j\}$. We define

$$A_{s+j}(\sigma) = Sx_{J'}R \qquad\qquad B_{s+j}(\sigma) = S\underbrace{x_{J'}\cdots x_{J'}}_{n_{i_j}}R$$

If we consider a cube $|C|$ as the geometric realization of the poset C of n-tuples of 0's and 1's (Figure (4.5)), the face operators defined above give exactly the $2n$ geometric faces of $|C|$ in pairs. But in the complex N, given an n-cube σ written as a symbol, the faces defined above may not be the actual geometric faces. In face, the A_i-faces of a cube defined above are precisely the geometric A_i-faces, whereas the B_i-faces are the geometric B_i-faces only if $r = 1$. Otherwise, the geometric B_i-faces of the cube are diagonals of the B_i-faces defined above.

Definition 4.2.9 ([58]). A collapsible patten is an x_1 not preceded by v. A redundant patten is n_1 v's in a row. A cube (symbol) is essential if it contains no collapsible or redundant patten. An inessential cube (symbol) is collapsible if the first (always from the left) such patten is collapsible, and redundant if the first such patten is redundant.

Let σ be a redundant cube of degree a, then $\sigma = Rv^{n_1}S$, where R is essential and doesn't end in v. Let $c(\sigma) = Rx_1S$. One can verify that c gives a bijection between the set of redundant cubes of degree $a = (a_1, \cdots, a_k)$ and the set of collapsible cubes of degree $(a_1, \cdots, a_k + 1)$, such that σ is a geometric face of $c(\sigma)$. We then build our complex N by first adjoining all essential cubes of a given degree a and then attaching each redundant cube σ of degree a along with $c(\sigma)$. In fact there is some order in which to adjoin the redundant cubes of a given degree such that when we adjoin σ, all other faces of the collapsible cube $c(\sigma)$ will have been already adjoined. Then this adjunction will just be an elementary expansion (in homotopy theory), and will not change the homotopy type. We then move on to the next greatest degree and repeat the process. This yields a quotient complex Y of N/F with one cube for each essential symbol of N/F, and the quotient map is a homotopy equivalence.

Checking the definition of essential symbols, we see that the number of v's in an essential symbol of a given dimension is bounded, so there are only finitely many cubes in each dimension, establishing

Theorem 4.2.10 ([58]). *$F = F(l, Z[\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}], \langle n_1, n_2, \ldots, n_k\rangle)$ is of type FP_∞.*

Since in rank $= 1$ case, the cubical structure is actually a CW-complex structure, we can use Y to compute the homology of F.

Theorem 4.2.11 ([58]). *If* $F = F(l, Z[\frac{1}{n}], \langle n \rangle)$*, then* $H_k(F) = \mathbb{Z}^{n(n-1)^{k-1}}$, $k \geq 1$.

And Y also gives a finite presentation for F.

Example 4.2.12. For $F = F(l, Z[\frac{1}{2}], \langle 2 \rangle)$, the Thompson's group F, we have

$$F = \langle x_0, x_1 \mid x_0 x_3 = x_2 x_0, x_1 x_4 = x_3 x_1 \rangle,$$

where $x_{i-2}^{-1} x_{i-1} x_{i-2} = x_i$, for any $i \geq 2$, and x_0 and x_1 are the two elements in Example (4.1.1).

4.2.4 More Collapsings and Construction of a CW-complex

The cubical structure of Y can't be used to compute homology group directly in the higher rank case, and the reason is, in the language of the face operators above, that some B_i-faces of an n-cube σ may not be the actual geometric B_i-faces of σ. If not, the geometric B_i-faces appear as the diagonals of the B_i-faces. So when we attach the n-cube σ, its boundary (or faces) doesn't go into lower dimensional cubes.

To make full use of the cubical structure, M. Stein constructed a new CW-complex K, which is homotopy equivalent to N. The idea of the construction is very simple. Since the problem lies in those cubes whose faces may be the diagonals of another higher dimensional cubes, we only need to find a way to push those diagonals into lower dimensional cubes, then we are done.

Example 4.2.13. In dimension 2, we have the following map: $d_2 \colon I \to I \times I$, where

$$d_2(t) = \begin{cases} (0, 2t) & \text{if } t \in [0, \frac{1}{2}] \\ (2t-1, 1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

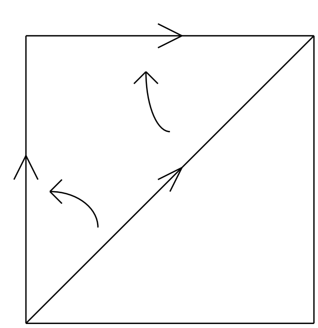

Figure 4.10: The pushing defined by d_2.

One systematic retraction from diagonals to sides of a cube is given by the Alexander-Whitney map, which has good functorial properties. And this gives us a method of constructing a CW-complex as follows: Corresponding to each cube in N, we have a cell of the same dimension. We start with 0-cells and then 1-cells, 2-cells, etc. At each adjoining, whenever we have a cell, corresponding to a cube σ (in N), where the diagonal phenomenon occurs, we push the diagonals (geometric faces of σ) into lower dimensional cells, which we have already adjoined in previous steps, thus we can attach the cell and obtain a CW-complex eventually. Denote it by K. By the construction, there is a $1-1$ correspondence between cells of K and cubes of N and K has a more complicated boundary maps (face operators) because of the pushing operations.

Theorem 4.2.14 ([58]). *K is homotopy equivalent to N.*

The cubical action of F on N induces a cellular action of F on K, which is still free, thus K/F is a $K(F,1)$. From now on, we won't distinguish the cubes in N from the cells in K. And we have a symbol for each cell. One can check that the collapsings we did on N in the previous section can be carried out on K, thus we have Y_K, homotopy equivalent to K/F and with only cells corresponding to essential symbols. In fact, we can do more collapsings in Y_K.

Definition 4.2.15 ([58]). A collapsible patten is an x_J not preceded by v, where $1 \in J$. A redundant patten is $n_1 - 1 = r_1$ v's followed by x_J, where $1 \notin J$. A cell (symbol) is essential if it contains no collapsible or redundant patten. An inessential cell (symbol) is collapsible if the first (always from the left) such patten is collapsible, and redundant if the first such patten is redundant.

Let σ be a redundant m-cube, then $\sigma = Rv^{r_1}x_JS$, where R is essential and $1 \notin J$. Let $c(\sigma) = Rx_{J\cup\{1\}}S$. Then the same argument, as we did in collapsing N, can be carried out exactly as before and we collapse Y_K to a complex Z with one cell for each essential symbol in Y_K.

Example 4.2.16. Let $P = \langle 2, n_2, \cdots, n_k\rangle$, $A = \mathbb{Z}[\frac{1}{2n_2\cdots n_k}]$, where $n_i \in \mathbb{Z}^+$ and $2 = n_1 < n_2 < \cdots < n_k$ form a basis for P. Then $d = gcd(n_1-1, \ldots, n_k-1) = 1$ and $r_1 = n_1 - 1 = 1$. Let $F = F(1, A, P)$, and let's check the ranks of low dimensional chain complex of Z.

In dimension 0, $C_0(Z)$ is generated by only one symbol v. Thus $\text{rank}_{\mathbb{Z}}(C_0(Z)) = 1$.

In dimension 1, $C_1(Z)$ is generated by the following essential symbols:

$$vx_1,\ vx_1v,\ x_2,\ x_2v,\ \cdots,\ x_k,\ x_kv.$$

Thus $\text{rank}_{\mathbb{Z}}(C_1(Z)) = 2k$.

In dimension 2, $C_2(Z)$ is generated by the following essential symbols:

$$\begin{array}{ccc} vx_{1l}, & vx_{1l}v, & 2 \leq l \leq k \\ x_{ij}, & x_{ij}v, & 2 \leq i < j \leq k \\ vx_1vx_1, & vx_1vx_1v, & \\ vx_1x_l, & vx_1x_lv, & 2 \leq l \leq k \\ x_lvx_1, & x_lvx_1v, & 2 \leq l \leq k \\ x_ix_j, & x_ix_jv, & 2 \leq i < j \leq k. \end{array}$$

Thus $\mathrm{rank}_{\mathbb{Z}}(C_2(Z)) = 3k^2 - k$.

In the example above, it's clear that $\partial_1 \colon C_1(Z) \to C_0(Z)$ must be a zero map since $H_0(Z) \cong \mathbb{Z}$ and it can be verified that $\partial_2 \colon C_2(Z) \to C_1(Z)$ is also a zero map, but we'll show this fact by other method below. In general, $\partial_* \colon C_*(Z) \to C_*(Z)$ could be nontrivial and the computation is very complicated.

In [58], M. Stein did the computation in the case where the slope group P has rank 2 and obtained

Theorem 4.2.17 ([58]). *Let $F = F(l, Z[\frac{1}{n_1}, \frac{1}{n_2}], \langle n_1, n_2 \rangle)$, where n_1, n_2 form a basis for the slope group. Let $d = gcd(n_1 - 1, n_2 - 1)$. Then $H_*(F)$ is a free abelian group, and if we set $h_j(F) = \mathrm{rank}_{\mathbb{Z}}(H_j(F))$, then they are given by $h_0(F) = 1$, $h_1(F) = 2(d+1)$, $h_2(F) = (1+4d)(d+1)$ and*

$$h_j(F) = dh_{j-2}(F) + 2dh_{j-1}(F), \; \forall \, j > 2.$$

Corollary 4.2.18. *Let $F = F(l, \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}], \langle n_1, n_2 \rangle)$, and $d = gcd(n_1 - 1, n_2 - 1) = 1$, then $\mathrm{rank}_{\mathbb{Z}}(H_1(F)) = 2(d+1) = 4$.*

Let B be the subgroup of $F = F(1, Z[\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}], \langle n_1, n_2, \ldots, n_k \rangle)$ consisting of homeomorphisms which are the identity in the neighborhoods of 0 and 1, i.e., B is the kernel of the following homomorphism

$$\begin{aligned} \rho \colon \; F &\longrightarrow P \times P \\ f &\longmapsto (f'(0+), f'(1-)). \end{aligned}$$

Then we have the following theorem

Theorem 4.2.19 (K. S. Brown). $H_*(F) \cong H_*(B) \otimes H_*(P \times P)$.

Proof. We have a split exact sequence

$$0 \longrightarrow B \xrightarrow{i} F \xrightarrow{\rho} P \times P \longrightarrow 0,$$

and a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & F & \longrightarrow & P\times P & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle m|_B} & & \Big\downarrow{\scriptstyle (m,\rho)} & & \Big\downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & B & \longrightarrow & B\times(P\times P) & \longrightarrow & P\times P & \longrightarrow & 0
\end{array}$$

where m is defined as follows: choose a piecewise linear homeomorphism φ with slopes in P and break points in A, taking $[0,1]$ to $[0+\varepsilon_1, 1-\varepsilon_2]$. If $h \in F$, define

$$m(h)(x) = \begin{cases} x & \text{if } x \in [0,\varepsilon_1] \cup [1-\varepsilon_2, 1] \\ \varphi \circ h \circ \varphi^{-1} & \text{if } x \in [\varepsilon_1, 1-\varepsilon_2]. \end{cases}$$

It's not difficult to see that $m|_B \colon B \to B$ induces the identity on $H_*(B)$. Thus the commutative diagram induces the identity maps between E^2 page of the spectral sequence for the two exact sequences. So (m,ρ) in the middle must induce an isomorphism on the homology and we have

$$H_*(F) \cong H_*(B \times P \times P) \cong H_*(B) \otimes H_*(P \times P).$$

□

Combine Theorem (4.2.19) and Example (4.2.16) together, we get

Proposition 4.2.20. *Let $P = \langle 2, n_2, \cdots, n_k\rangle$, $A = \mathbb{Z}[\frac{1}{2n_2\cdots n_k}]$, where $n_i \in \mathbb{Z}^+$ and $2 = n_1 < n_2 < \cdots < n_k$ form a basis for P. Let $F = F(1, A, P)$, then $H_1(F) \cong \mathbb{Z}^{2k}$.*

Proof. We computed the rank of the corresponding dimension-1 chain complex for F in Example (4.2.16) and $\text{rank}_{\mathbb{Z}}(C_1(Z)) = 2k$, therefore we must have that $\text{rank}_{\mathbb{Z}}(H_1(Z)) \leq \text{rank}_{\mathbb{Z}}(C_1(Z)) = 2k$. By Theorem (4.2.19) above, we get $H_1(F) \cong H_1(B) \oplus H_1(P \times P)$, where $P \times P \cong \mathbb{Z}^k \times \mathbb{Z}^k \cong \mathbb{Z}^{2k}$, thus $\text{rank}_{\mathbb{Z}}(H_1(Z)) \geq \text{rank}_{\mathbb{Z}}(H_1(P \times P)) = 2k$. Combining the two inequality together, we get $H_1(F) \cong \mathbb{Z}^{2k}$. □

The complex Z also gives much smaller finite presentations for generalized Thompson's groups.

Example 4.2.21. For $F = F(1, \mathbb{Z}[\frac{1}{6}], \langle 2,3\rangle)$, we have the following finite presentation.

$$\begin{aligned}
\langle x_0,\ x_1,\ y_0,\ y_1 \mid\ & x_0x_2 = x_3x_0, x_1x_3 = x_4x_1, x_0y_2 = y_3x_0, \\
& y_1y_3 = y_4y_1, y_0^+x_1 = x_3y_0^+, y_1^+x_2 = x_4y_1^+, \\
& y_0^+y_1 = y_3y_0^+, y_1^+y_2 = y_4y_1^+, \\
& y_0^+y_1^+x_0 = x_0x_1x_2y_0^+, y_1^+y_2^+x_1 = x_1x_2x_3y_1^+\rangle,
\end{aligned}$$

where $x_{i-2}^{-1}x_{i-1}x_{i-2} = x_i$, $x_{i-2}^{-1}y_{i-1}x_{i-2} = y_i$, $\forall i \geq 2$ and $y_i^+ = x_i x_{i+1} y_i y_{i+1}^{-1}$ $\forall i$. y_0 and y_0^+ are the following two elements:

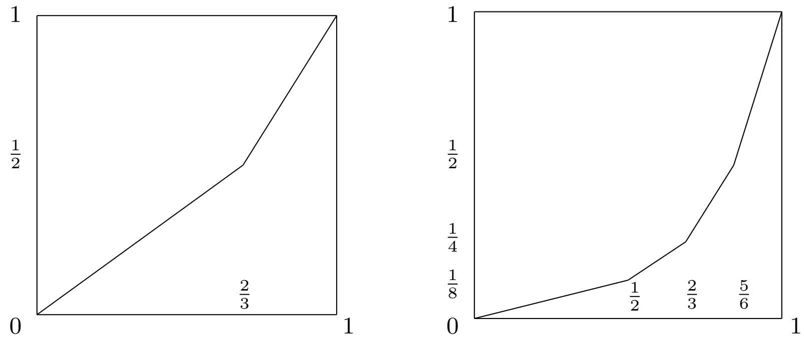

Figure 4.11: y_0 (left) and y_0^+ (right).

4.3 SCL in Generalized Thompson's Groups

4.3.1 Main Theorem

Recall that $T = T(1, A, P)$ is the group of piecewise linear homeomorphisms of the circle $S^1 = [0,1]/\{0,1\}$, with finitely many break points in A, slopes in P, and sending $A \cap [0,1]$ to itself. T is a subgroup of $\mathrm{Homeo}^+(S^1)$. Let $\widetilde{T}$ be the central extension of T by $\mathbb{Z}$, i.e., $\widetilde{T} = p^{-1}(T) \subset \widetilde{\mathrm{Homeo}}^+(S^1)$, and we have the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \widetilde{T} \xrightarrow{p} T \longrightarrow 1,$$

where $\mathbb{Z} = \langle t \rangle$ is generated by the unit translation t on $\mathbb{R}$.

In section (2.3.1), we defined rotation quasimorphism rot on $\widetilde{\mathrm{Homeo}}^+(S^1)$ and computed its defect $D(\mathrm{rot}) = 1$. It's clear that $\mathrm{rot}|_{\widetilde{T}}$ is still a homogeneous quasimorphism and we'll see that $\mathrm{rot}|_{\widetilde{T}}$ is essentially the only homogeneous (normalized) quasimorphism on $\widetilde{T}$. Let's compute $D(\mathrm{rot}|_{\widetilde{T}})$ first.

Lemma 4.3.1 ([60]). *Let $T = T(1, A, P)$, where $P = \langle n_1, n_2, \ldots, n_k \rangle$, $A = \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}]$, and $0 < n_1 < \cdots < n_k$, $n_i \in \mathbb{Z}$, form a basis for P. Then T is dense in* $\mathrm{Homeo}^+(S^1)$ *with the C^0-topology. Therefore $\widetilde{T}$ is dense in* $\widetilde{\mathrm{Homeo}}^+(S^1)$*, and* $D(\mathrm{rot}|_{\widetilde{T}}) = 1$.

Proof. Take an arbitrary homeomorphism $f \in \mathrm{Homeo}^+(S^1)$ and any $\epsilon > 0$. Since f is uniformly continuous, we can choose $0 = x_0 < x_1 < \cdots < x_l < 1$, x_i's lying in the dense subset $IP \cdot A \subset [0,1]$, such that $|f(x_i) - f(x_{i-1})| < \frac{\epsilon}{3}$. Since $IP \cdot A$ is dense in $[0,1]$, we can find $0 \leq y_0 < y_1 < \cdots < y_l < 1$, with $y_i \in IP \cdot A$ and $|y_i - f(x_i)| < \frac{\epsilon}{3}$. By Theorem (4.1.4), there exists $g \in T$ such that $g(x_i) = y_i$. From the choice of x_i's and y_i's, it's easy to see that $\|f - g\|_{C^0} < \epsilon$. So T is a dense subgroup of $\mathrm{Homeo}^+(S^1)$, and so is $\widetilde{T}$ in $\widetilde{\mathrm{Homeo}}^+(S^1)$.

On the other hand, by Proposition (3.3.4), the function $\mathrm{rot}\colon \widetilde{\mathrm{Homeo}}^+(S^1) \to \mathbb{R}$ is continuous in C^0-topology, so we must have $D(\mathrm{rot}|_{\widetilde{T}}) = D(\mathrm{rot}) = 1$. $\square$

We compute scl on the group T first.

Lemma 4.3.2 ([60]). *Let $T = T(1, A, P)$, where $P = \langle n_1, n_2, \ldots, n_k\rangle$, $A = \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}]$ and $d = gcd(n_1 - 1, \ldots, n_k - 1) = 1$. For any $f \in T$, there exist $g_1, g_2 \in T$ such that*

1. *$f = g_1 \circ g_2$;*
2. *Each g_i fixes an open arc of the circle, i.e., there exists an open interval $\alpha \subset [0, 1]/\{0, 1\}$, such that $g_i|_\alpha$ is the identity.*

Proof. Since $d = 1$, Theorem (4.1.4) is vacuously satisfied. For any $f \in T$, choose points $a < b$ and $c < d$, $a, b, c, d \in A \cap [0, 1]$, satisfying $[c, d] \cap ([a, b] \cup f([a, b])) = \emptyset$. Denote $[a_1, b_1] = f([a, b])$. Such points exist because A is dense in $[0, 1]$ and we only need to choose a, b such that $[a, b]$ is a very small interval. Since a_1, b_1 are also points in A, there exist piecewise linear homeomorphisms g_1 and g_2, with slopes in P and break points in A, such that g_1 sends $[b, c]$ to $[b_1, c]$ and g_2 sends $[d, a]$ to $[d, a_1]$.

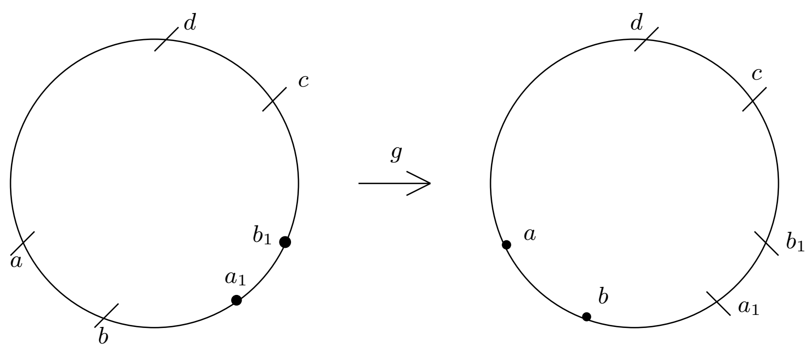

Figure 4.12: The homeomorphism g.

Define (see Figure (4.12))

$$g = \begin{cases} f & \text{if } x \in [a, b] \\ g_1 & \text{if } x \in [b, c] \\ id & \text{if } x \in [c, d] \\ g_2 & \text{if } x \in [d, a]. \end{cases}$$

Then $f = (f \circ g^{-1}) \circ g$. Clearly g fixes the interval (c, d) and $f \circ g^{-1}$ fixes the interval (a, b). $\square$

Theorem 4.3.3 ([60]). *Let $P = \langle n_1, n_2, \ldots, n_k\rangle$ and $A = \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}]$. $F = F(1, A, P)$ and $T = T(1, A, P)$. Suppose $d = gcd(n_1 - 1, \ldots, n_k - 1) = 1$ and $H_1(F)$ is a free abelian group of* rank $2k$. *Then*

1. *T is a simple group;*
2. *$Q(T) = \{0\}$, or equivalently* scl $\equiv 0$ *on T.*

Proof. For an arbitrary homeomorphism $f \in T$, by Lemma (4.3.2) above, we can write $f = g_1 \circ g_2$, where g_i fixes some proper open arc α_i in the circle. Take a point $\theta_i \in \alpha_i \cap A$ (since A is dense). Then R_{θ_i}, the rotation of the circle through angle θ_i, is an element in T. It's easy to see that supp$(R_{\theta_i} \circ g_i \circ R_{\theta_i}^{-1}) \subset (0, 1)$, i.e., $R_{\theta_i} \circ g_i \circ R_{\theta_i}^{-1}$ fixes a neighborhood of $0 = 1$. Write $h_i = R_{\theta_i} \circ g_i \circ R_{\theta_i}^{-1}$., which is the identity in the neighborhood $[0, \varepsilon)$ and $(1-\varepsilon, 1]$, so h_i is an element of the subgroup B, where B is the subgroup of F consisting of elements which are identity in the neighborhood of 0 and 1.

By Theorem (4.2.19), $H_1(F) = H_1(B) \oplus H_1(P \times P)$. We have $P \times P \cong \mathbb{Z}^k \oplus \mathbb{Z}^k \cong \mathbb{Z}^{2k}$, and the assumption says that $H_1(F) \cong \mathbb{Z}^{2k}$. Thus we must have $H_1(B) = 0$, i.e., $B = B' = [B, B]$, B is a perfect group. Therefore each h_i can be written as a product of commutators, so is g_i since g_i is conjugate to h_i. Therefore $f = g_1 \circ g_2$ also lies in the commutator subgroup. Since f is chosen arbitrarily, we get that T is perfect, i.e., $T = T'$. We know, by Theorem (4.1.8), that T'' is a simple group, so $T = T' = T''$ is also simple. Claim 1 is proved.

Let $\phi \in Q(T)$ be an arbitrary homogeneous quasimorphism on T. As we did above, for any $f \in T$, we write $f = g_1 \circ g_2$ and $h_i = R_{\theta_i} \circ g_i \circ R_{\theta_i}^{-1}$, $h_i \in B = B'$. Since h_1 and h_2 both are elements in B', write $h_i = [s_1^i, t_1^i] \cdots [s_{m_i}^i, t_{m_i}^i]$, where $i = 1, 2$ and s_j^i, t_j^i, $1 \le j \le m_i$, are homeomorphisms in B. Let $J = \bigcup_{i=1}^2 \bigcup_{j=1}^{m_i} (\text{supp}(s_j^i) \bigcup \text{supp}(t_j^i))$, then it's clear that J is a proper closed subset of $(0, 1)$. Assume $J \subset (\alpha_1, \beta_1) \subsetneqq (0, 1)$, where $\alpha_1, \beta_1 \in A$. We can choose $\alpha_i, \beta_i \in A \cap [0, 1]$, $i \ge 2$, such that

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 < \cdots < 1.$$

By Theorem (4.1.4), for any $l \in \mathbb{Z}^+$, we can construct $\gamma_l \in F$, a piecewise linear homeomorphism with slopes in P and break points in A, such that $\gamma_l([\alpha_i, \beta_i]) = [\alpha_{i+1}, \beta_{i+1}]$, $1 \le i \le l$. Define

$$\Delta_l(h_i) = \prod_{k=0}^{l-1} \gamma_l^k \circ h_i \circ (\gamma_l^k)^{-1}$$
$$\Delta_l(s_j^i) = \prod_{k=0}^{l-1} \gamma_l^k \circ s_j^i \circ (\gamma_l^k)^{-1}$$
$$\Delta_l(t_j^i) = \prod_{k=0}^{l-1} \gamma_l^k \circ t_j^i \circ (\gamma_l^k)^{-1}.$$

Since any two items in the products have disjoint supports ($\text{supp}(\gamma_l^k \circ h_i \circ (\gamma_l^k)^{-1}) \subseteq [\alpha_{k+1}, \beta_{k+1}]$), we still have that

$$\Delta_l(h_i) = [\Delta_l(s_1^i), \Delta_l(t_1^i)] \cdots [\Delta_l(s_{m_i}^i), \Delta_l(t_{m_i}^i)]$$

Define

$$h_i' = \prod_{k=0}^{l-1} \gamma_l^k \circ h_i^{k+1} \circ (\gamma_l^k)^{-1},$$

and we see, by direct calculation, that h_i, h_i' and γ_i satisfy the following equality

$$[\gamma_l,\ h_i'] \ = \ (\gamma_l h_i' \gamma_l^{-1}) \circ {h_i'}^{-1} \ = \ \Delta_l(h_i^{-1}) \circ (\gamma_l^l h_i^l \gamma_l^{-l}).$$

So we get

$$h_i^l \;=\; \gamma_l^{-l}\,\Delta_l(h_i)[\gamma_l, h_i']\,\gamma_l^l.$$

Then we have the following estimate of commutator length

$$\mathrm{cl}(h_i^l) \;\leq\; 1 + \mathrm{cl}(\Delta_l(h_i)) \;\leq\; 1 + m_i,$$

and

$$\frac{\mathrm{cl}(h_i^l)}{l} \;\leq\; \frac{1+m_i}{l}.$$

Since l can be made arbitrarily large, we get

$$\mathrm{scl}(h_i) \;=\; \lim_{l\to\infty} \frac{\mathrm{cl}(h_i^l)}{l} \;=\; 0,$$

for each $i = 1, 2$.

Then by Bavard's Duality Theorem (3.2.10), $\phi(h_i) = 0$, for any $\phi \in Q(T)$. Since g_i is conjugate to h_i and ϕ is homogeneous, we have $\phi(g_i) = \phi(h_i) = 0$.

For any $n \in \mathbb{Z}^+$, write $f^n = g_{1n} \circ g_{2n}$, such that $\phi(g_{1n}) = \phi(g_{2n}) = 0$. Then we have

$$|n\phi(f)| = |\phi(f^n)| = |\phi(g_{1n}g_{2n})| = |\phi(g_{1n}g_{2n}) - \phi(g_{1n}) - \phi(g_{2n})| \leq D(\phi),$$

so

$$|\phi(f)| \leq \frac{D(\phi)}{n},$$

for any $n \in \mathbb{Z}^+$. Let $n \to \infty$, we get $\phi(f) = 0$. That scl $\equiv 0$ follows directly from Bavard's Theorem (3.2.10). Claim 2 is proved. □

Remark 4.3.4. Part of the proof is derived from the proof of a more general theorem due to D. Calegari:

Theorem 4.3.5 *([11]). Let G be a subgroup of* $\mathrm{PL}^+(I)$*, then* scl *of every element in* $[G, G]$ *is zero.*

Now we are ready to compute scl in $\widetilde{T}$.

Theorem 4.3.6 ([60]). *Let $P = \langle n_1, n_2, \ldots, n_k\rangle$ and $A = \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}]$, where $0 < n_1 < n_2 < \cdots < n_k$ form a basis for P. $F = F(1, A, P)$, $T = T(1, A, P)$, and $\widetilde{T}$ is the central extension of T. Suppose $d = gcd(n_1 - 1, \ldots, n_k - 1) = 1$ and $H_1(F) \cong \mathbb{Z}^{2k}$, then* $\mathrm{rot}|_{\widetilde{T}}$ *is the unique homogeneous quasimorphism which sends the unit translation to* 1.

Thus by Bavard's Duality Theorem (3.2.10), we have for any $g \in \widetilde{T}$,

$$\mathrm{scl}(g) \;=\; \frac{|\mathrm{rot}(g)|}{2D(\mathrm{rot}|_{\widetilde{T}})} \;=\; \frac{|\mathrm{rot}(g)|}{2}.$$

Proof. Clearly $\mathrm{rot}|_{\widetilde{T}} \in Q(\widetilde{T})$ and sends the unit translation to 1. Suppose $\tau \in Q(\widetilde{T})$ is another such homogeneous quasimorphism. We consider their difference

$$\mathrm{rot} - \tau \colon \ \widetilde{T} \longrightarrow \mathbb{R},$$

which is still a homogeneous quasimorphism.

For any element $f \in T$, let f_1 and f_2 be two arbitrary lifts of f in $\widetilde{T}$. Then there is an m $\in \mathbb{Z}$ such that $f_2 = f_1 +$ m. Both f_1 and f_2 are elements of the subgroup $p^{-1}(f) \subset \widetilde{T}$, which is generated by f_1 and t, the unit translation. This subgroup is an abelian group. Thus by Corollary (3.2.15), rot $- \tau$, restricted to it, is a homomorphism. We have $(\mathrm{rot} - \tau)(f_2) = (\mathrm{rot} - \tau)(f_1 + m) = (\mathrm{rot} - \tau)(f_1) + (\mathrm{rot} - \tau)(m)$. The normalization assumption tells us that $\mathrm{rot}(m) = \tau(m)$, so $(\mathrm{rot} - \tau)(f_1) = (\mathrm{rot} - \tau)(f_2)$. Therefore rot $- \tau$ induces a function on T and it's easy to see that the induced function is still a homogeneous quasimorphism. By Theorem (4.3.3), we must have rot $= \tau$.

Thus $Q(\widetilde{T})$ is one dimensional and generated by $\mathrm{rot}|_{\widetilde{T}}$. And the last claim follows from Bavard's Duality Theorem and the defect estimation in Lemma (4.3.1). □

The last thing is to find generalized Thompson's groups satisfying those assumptions in Theorem (4.3.6) above. The computation of homology of generalized Thompson's groups in the previous section provides us with a lot of such groups. Let $P = \langle 2, n_2, \cdots, n_k \rangle$, $A = \mathbb{Z}[\frac{1}{2n_2 \cdots n_k}]$ or $P = \langle n_1, n_2 \rangle$, $A = \mathbb{Z}[\frac{1}{n_1 n_2}]$, where $gcd(n_1 - 1, n_2 - 1) = 1$. Let $F = F(1, A, P)$, $T = T(1, A, P)$, and $\widetilde{T}$ the central extension of T. In either case, Proposition (4.2.20) and Corollary (4.2.18) say that these groups satisfy the assumptions in Theorem (4.3.6), thus we have for any $g \in \widetilde{T}$,

$$\mathrm{scl}(g) \ = \ \frac{|\mathrm{rot}(g)|}{2}.$$

4.3.2 Examples

Theorem (4.3.6) leaves the computation of scl to the computation of rotation numbers in generalized Thompson's groups. There have been a lot of researches on rotation numbers in these groups, and more generally of piecewise linear homeomorphisms of S^1 with various assumptions on slopes and break points. In this section, we compile some known results and give concrete examples, in which the computation of rotation number is possible.

In [34], E. Ghys and V. Sergiescu studied the Thompson's group $T = T(1, \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle)$. They built a smooth action of T on S^1, i.e., a representation $\phi \colon T \to \mathrm{Diff}^{\infty}(S^1)$, with an exceptional minimal set. Then by a theorem of Denjoy, which says that every C^2 diffeomorphism of S^1 with an irrational rotation number has dense orbits, they deduced that

Theorem 4.3.7 ([34]). *Every element in T has a rational rotation number, and furthermore, every rational* *number can be realized as the rotation number of some element in T.*

Thus scl on $\widetilde{T} = \widetilde{T}(1, \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle)$ takes rational values, and we also get a finitely presented, infinite simple group, on which scl can take any rational number as its value.

Remark 4.3.8. A generalization of Theorem (4.3.7) is proved later by I. Liousse [51] and D. Calegari [10].

In [50] and [51], I. Liousse studied generalized Thompson's groups. She modified a construction of M. Boshernitzan and obtained

Theorem 4.3.9. *Let $P = \langle n_1, n_2, \ldots, n_k \rangle$, $A = \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}]$, where $0 < n_1, < \cdots < n_k$ form a basis for P. $T = T(1, A, P)$. Assume the* rank $k \geq 2$, *then there are elements in T with irrational rotation numbers. Furthermore, there are $(k-1)$ commuting homeomorphisms $\beta_1, \beta_2, \cdots, \beta_{k-1}$ in T, with irrational rotation numbers ρ_i, $1 \leq i \leq k-1$, such that 1 and the ρ_i's are $\mathbb{Q}$-independent.*

In the following, we give two examples of the computations of rotation numbers in generalized Thompson's groups. Both of them are taken from [50].

Example 4.3.10 *([50]).* Let $T = T(1, \mathbb{Z}[\frac{1}{6}], \langle 2, 3 \rangle)$. Let f be the following homeomorphism in T

$$f = \begin{cases} \frac{2}{3}x + \frac{2}{3} & \text{if } x \in [0, \frac{1}{2}] \\ \frac{4}{3}x - \frac{2}{3} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

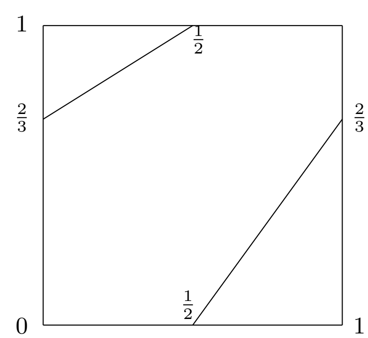

Figure 4.13: Graph of f.

The transformation $t(x) := 2 - 2^{1-x}$, $x \in [0, 1]$ conjugates f to the rotation by ρ, where $\rho = \frac{\log 3}{\log 2} - 1 \approx 0.58496250072 \cdots$. Thus any lift of f has the rotation number $\frac{\log 3}{\log 2} + n$, $n \in \mathbb{Z}$. By the theorem of Gel'fand-Schneider [49], the rotation number ρ is transcendental.

In general, we can't find the conjugacy homeomorphism and we need to work out rotation numbers indirectly. Here is such an example:

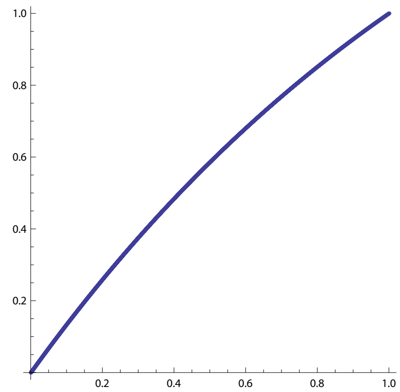

Figure 4.14: Graph of t.

Example 4.3.11 *([50]).* $T = T(1, \mathbb{Z}[\frac{1}{6}], \langle 2, 3\rangle)$, and $g \in T$ is the following homeomorphism

$$g = \begin{cases} x + \frac{8}{9} & \text{if } x \in [0, \frac{1}{9}] \\ \frac{1}{2}x - \frac{1}{18} & \text{if } x \in [\frac{1}{9}, \frac{5}{9}] \\ x - \frac{3}{9} & \text{if } x \in [\frac{5}{9}, \frac{8}{9}] \\ 3x - \frac{19}{9} & \text{if } x \in [\frac{8}{9}, 1]. \end{cases}$$

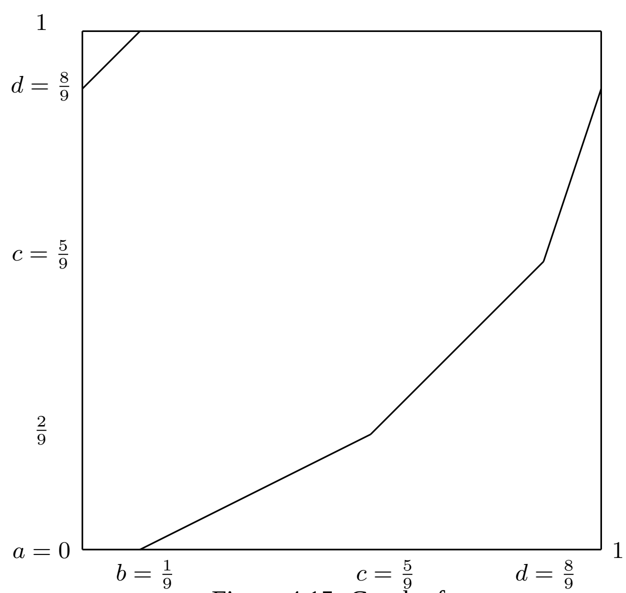

Figure 4.15: Graph of g.

The map g satisfies *Property D*, which, roughly speaking, says that the product of the g-jump (ratio of derivatives from two sides) on each orbit is trivial. In fact any piecewise linear homeomorphism with all break points in one orbit satisfies Property D. For such a homeomorphism, there exists $h \in \text{Homeo}^+(S^1)$, such that $h \circ g \circ h^{-1} \in \text{SO}(2)$. So there is a measure μ on S^1, which is invariant under the iteration of g. Let

Dg be the derivative function of g. Then we have the following equality

$$\int \log(Dg)d\mu = 0.$$

This equality will enable us to compute the rotation number of g.

Let's denote the break points to be $a = 0$, $b = \frac{1}{9}$, $c = \frac{5}{9}$ and $d = \frac{8}{9}$. They all lie in the same orbit of g: $g : b \mapsto a = 0 \mapsto d \mapsto c$. Suppose h is the homeomorphism, conjugating g to a rotation homeomorphism through angle ρ, where $0 \leq \rho < 1$. We want to find the images of a, b, c and d under h. WLOG, we can assume that $h(0) = h(a) = 0$. Since $b = g^{-1}(a)$, $d = g(a)$ and $c = g^2(a)$, we must have $h(b) = -\rho + n_b$, $h(c) = 2\rho + n_c$, and $h(d) = \rho + n_d$, where n_b, n_c and n_d are some proper integers. Combining this with the facts that h is an orientation-preserving homeomorphism, and $h(b), h(c)$, and $h(d)$ are points in $(0, 1)$, we get that $h(b) = 1 - \rho$, $h(c) = 2\rho - 1$, and $h(d) = \rho$. And on $[a, b]$, $Dg \equiv \lambda_1 = 1$. On $[b, c]$, $Dg \equiv \lambda_2 = \frac{1}{2}$. On $[c, d]$, $Dg \equiv \lambda_3 = 1$. On $[d, a]$, $Dg \equiv \lambda_4 = 3$. The integral equality gives a linear equation

$$\log \lambda_1 \cdot (1 - \rho) + \log \lambda_2 \cdot (3\rho - 2) + \log \lambda_3 \cdot (1 - \rho) + \log \lambda_4 \cdot (1 - \rho) = 0.$$

Substitute λ_i's, and we get

$$\rho = \frac{2 \log 2 + \log 3}{3 \log 2 + \log 3}.$$

Numerically, we can draw the graph of the conjugacy function h as follows:

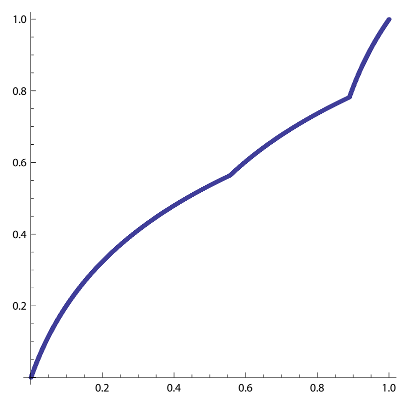

Figure 4.16: Graph of h.

As a corollary of Theorem (4.3.6), Theorem (4.3.9) and the computations in Example (4.3.10) and Example (4.3.11), we have

Corollary 4.3.12 ([60]). *There exist finitely presented groups, on which* scl *takes irrational (in fact, transcendental) values.*

Theorem (4.3.6) also reflects the deep connection between scl and dynamical properties of these groups. One series of groups in our list has very rigid dynamical properties. Let $P = \langle 2, n_2, \cdots, n_k \rangle$, $A =$

$\mathbb{Z}[\frac{1}{2n_2\cdots n_k}]$, $k \geq 2$ and $2 < n_2 < \cdots < n_k$ forms the basis for P. Let $T = T(1, A, P)$, then we have the following theorem due to I. Liousse.

Theorem 4.3.13 ([51]).

1. *Each nontrivial representation* ϕ *from* T *into* $\mathrm{Diff}^2(S^1)$ *is topologically conjugate to the standard representation in* $\mathrm{PL}(S^1)$.

2. *Each nontrivial representation* ϕ *from* T *into* $\mathrm{PL}(S^1)$ *is* PL*-conjugate to the standard representation in* $\mathrm{PL}(S^1)$.

3. *There exists* $l \geq 2$*, depending on the* $\log n_i$*'s diophantine coefficients, such that any representation from* T *into* $\mathrm{Diff}^l(S^1)$ *has finite images. In particular,* T *is not realizable in* $\mathrm{Diff}^\infty(S^1)$.

Chapter 5

Large Scale Geometry of Commutator Subgroups

In this chapter, we study commutator subgroup as a geometric object: its Cayley graph with respect to the canonical generating set of all commutators. With the path metric, cl and scl can both be interpreted as important geometric quantities on it. We are interested in the large scale geometry of this graph. First we prove that, for any finitely presented group, the graph is large scale simply connected. Then we focus on a very important class of groups in geometric group theory: hyperbolic groups. In contrast to generalized Thompson's groups, the space of homogeneous quasimorphisms of a hyperbolic group is infinite dimensional. We study the geometric implication of this infinite dimensional phenomenon, and prove that for a non-elementary hyperbolic group, the corresponding Cayley graph of the commutator subgroup contains a quasi-isometrically embedded $\mathbb{Z}^n$, for any $n \in \mathbb{Z}_+$. As corollaries, the graph is not δ-hyperbolic, has infinite asymptotic dimension and is one-ended.

5.1 Commutator Subgroup as a Metric Space

Let G be a group and $G' = [G, G]$ its commutator subgroup. The subgroup G' has a canonical generating set S consisting of all commutators. With G' and S, we can construct a graph $C_S(G')$ as follows: each element in G' gives a vertex in $C_S(G')$, and $g, h \in G'$ are connected by an edge if and only if $g^{-1}h \in S$, i.e., $g^{-1}h$ is a commutator. We call $C_S(G')$ the *Cayley graph* of G' with respect to the generating set S. By assigning each edge with length 1, $C_S(G')$ becomes a metric space, where the distance is defined by the path metric.

Proposition 5.1.1.

1. *The distance in $C_S(G')$ equals commutator length, i.e., for any $g, h \in G'$, $d(g, h) = \mathrm{cl}(g^{-1}h)$;*

2. *The semidirect product $G' \rtimes \mathrm{Aut}(G)$ acts on $C_S(G')$ by isometries;*

3. *The metric on G' inherited as a subset of $C_S(G')$ is both left- and right- invariant;*

4. *Simplicial loops in $C_S(G')$ through the origin 1 correspond to (marked) homotopy classes of maps of closed surfaces into a $K(G,1)$.*

Proof.

1. By the definition of the path metric, for any $g, h \in G'$,

$$d(g,h) = \min\{ \text{ length of } L \mid L \text{ is a simplicial path from } g \text{ to } h \text{ in } C_S(G')\}.$$

   By the definition of edges in $C_S(G')$, any such path L gives the following equality in G'

$$g[a_1, b_1] \cdots [a_n, b_n] = h,$$

   where $n =$ length of L. So the minimal length path represents a shortest expression of $g^{-1}h$ as a product of commutators, which is the commutator length of $g^{-1}h$. Thus we have $d(g,h) = \text{cl}(g^{-1}h)$ and in particular $d(1,g) = \text{cl}(g)$.

2. The set S of commutators, is characteristic, i.e., S is invariant under any automorphism of G. So Aut(G) acts on G' which induces an action on $C_S(G')$. The action is obviously by isometries.

   G' acts on itself by *left multiplication*, i.e., any $g \in G'$ corresponds to a bijection $L_g \colon G' \to G'$, where $L_g(h) = gh$. Then for any $h_1, h_2 \in G'$,

$$d(L_g(h_1), L_g(h_2)) = d(gh_1, gh_2) = \text{cl}((gh_1)^{-1}gh_2) = \text{cl}(h_1^{-1}h_2) = d(h_1, h_2).$$

   Thus G' acts on $C_S(G')$ by isometries. Combine these two actions together, we get the bulletin (2).

3. We only need to verify "right invariance". For any $g, h_1, h_2 \in G'$, we have

$$d(h_1 g, h_2 g) = \text{cl}((h_1 g)^{-1} h_2 g) = \text{cl}(g^{-1}(h_1^{-1}h_2)g) = \text{cl}(h_1^{-1}h_2) = d(h_1, h_2).$$

   The 3rd equality comes from the fact that the conjugate of a commutator is still a commutator.

4. A simplicial loop through 1 gives the following equality in G':

$$[a_1, b_1][a_2, b_2] \cdots [a_n, b_n] = 1.$$

   Suppose $X = K(G,1)$, then the discussion in *Chapter* 1 gives a map from a closed oriented surface S of genus n into X. The mark on S is the 1-skeleton from the standard polygon representation of S.

□

Definition 5.1.2. Given a metric space (X, d) and an isometry h of X, the *translation length* of h on X, denoted $\tau(h)$, is defined by the formula

$$\tau(h) = \lim_{n\to\infty} \frac{d(p, h^n(p))}{n},$$

where $p \in X$ is an arbitrary base point.

Remark 5.1.3.

1. The limit exists by Lemma (3.1.2), and the limit does not depend on the choice of the base point p by triangular inequality.

2. Let M^n be a dimension-n Riemannian manifold with negative section curvature and $\widetilde{M}^n$ its universal covering. Then $\widetilde{M}^n$ is homeomorphically $\mathbb{R}^n$ with a negatively curved Riemannian metric. We can define the corresponding path metric on $\widetilde{M}^n$. The fundamental group $\pi_1(M^n)$ acts on $\widetilde{M}^n$ isometrically, and the set of values of translation lengths corresponding to this action equals the *length spectrum* of M^n, which is the set of lengths of closed geodesics in M^n. Thus the set of translation lengths reflects very important geometric and dynamical properties of the underlying spaces and groups.

Proposition 5.1.4. *Let $g \in G'$ act on $C_S(G')$ by left multiplication. Then there is an equality*

$$\tau(g) = \mathrm{scl}(g).$$

Proof. We can choose $p = 1$ in $C_S(G')$. Then we have

$$\begin{aligned}\tau(g) &= \lim_{n\to\infty} \frac{\mathrm{d}\left(1, L_g^n(1)\right)}{n} \\ &= \lim_{n\to\infty} \frac{\mathrm{d}\left(1, g^n\right)}{n} \\ &= \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n} \;=\; \mathrm{scl}(g).\end{aligned}$$

□

One can obtain lower bounds on $\tau(g)$ by constructing a Lipschitz function on X which grows linearly on the orbit of a point under powers of g. One important class of Lipschitz functions on $C_S(G')$ are quasimorphisms.

By repeated application of the defining property of quasimorphisms and triangular inequality, one can estimate that

$$|\phi(f[g, h]) - \phi(f)| \le 7D(\phi),$$

where $\phi \in \hat{Q}(G)$ and $f, g, h \in G$. This immediately gives

Proposition 5.1.5. *A quasimorphism ϕ, restricted to G', is a $7D(\phi)$-Lipshitz map in the metric inherited from $C_S(G')$, and we have*

$$\mathrm{cl}(g) \geq \frac{|\phi(g) - \phi(1)|}{7D(\phi)},$$

for any $g \in G'$ and $\phi \in \hat{Q}(G)$.

So we have translated cl and scl into geometric notions on the metric space $C_S(G')$, cl as the path metric and scl as the translation length. In the following, we will study this metric space through our knowledge of cl and scl, and vice versa.

5.2 Large Scale Geometry of Metric Spaces

One entry of geometric ideas into group theory is through Cayley graphs and word lengths. The Cayley graph $C_S(G')$ for a commutator subgroup is one such example. In general, given a finitely generated group Γ and a finite symmetric generating set T, the corresponding Cayley graph $C_T(\Gamma)$ is the graph with vertex set Γ in which two vertices γ_1 and γ_2 are the ends of an edge if and only if $d_T(\gamma_1, \gamma_2) = 1$, i.e., $\gamma_1^{-1}\gamma_2 \in T$. Γ acts on $C_T(\Gamma)$ by left multiplications, and this action is obviously transitive. Each edge of $C_T(\Gamma)$ can be made a metric space isometric to the segment $[0, 1]$ of the real line, then one defines naturally the length of a path between two points (not necessarily two vertices) of the graph, and the distance between two points is defined to be the infimum of the appropriate path lengths. With this path metric, the left action of Γ on $C_T(\Gamma)$ is by isometries. (Γ also has a natural right action on $C_T(\Gamma)$, but in general this right action is not by isometries.) Moreover, $C_T(\Gamma)$ is a proper, geodesic metric space.

Definition 5.2.1.

1. A metric space (X, d) is proper if its closed balls of finite radius are compact.

2. A geodesic in (X, d) is a map $\sigma\colon \mathrm{I} \to X$ defined on an interval I of $\mathbb{R}$ such that

$$d(\sigma(t_1), \sigma(t_2)) = |\, t_1 - t_2|,$$

for any $t_1, t_2 \in$ I. X is said to be geodesic if any two points in X can be joined by a geodesic segment (not necessarily to be unique).

The Cayley graph of a commutator subgroup is generally not proper (since the set S of commutators is generally infinite.), but still geodesic.

One deficiency of the construction of the Cayley graph is that, for an arbitrarily finitely generated group, we don't have a canonical finite generating set. Thus each choice of a generating set gives a Cayley graph, and these graphs are usually not (even locally) isometric. Thus to obtain useful information from these graphs, we need to find a way to identify them and study the properties invariant under the identifications. Here is one such identification due to M. Gromov.

Definition 5.2.2. Let (X, d_X), (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is a (λ, c)-*quasi-isometric embedding* if there exist constants $\lambda \geq 1$, $c \geq 0$ such that

$$\frac{1}{\lambda}\, d_X(x,y) - c \;\leq\; d_Y(f(x), f(y)) \;\leq\; \lambda\, d_X(x,y) + c,$$

for all $x, y \in X$.

f is a *quasi-isometry* if there exists moreover a constant $D \geq 0$ such that any point of Y is within distance D from some point of $f(X)$. The two spaces X and Y are then quasi-isometric.

Remark 5.2.3.

1. Being Quasi-isometric is an equivalence relation.

2. A metric space is of finite diameter (bounded) if and only if it is quasi-isometric to the space of a point. This is the reason why the geometry of quasi-isometries are called large scale geometry.

3. let Γ be a group with two finite symmetric generating sets T_1 and T_2. Let d_1 and d_2 denote the corresponding word metrics (or path metrics on the Cayley graphs). The identity map of Γ, viewed as a map $(\Gamma, d_1) \to (\Gamma, d_2)$ is a $(\lambda, 0)$-quasi-isometry, i.e.,

$$\frac{1}{\lambda}\, d_1(\gamma_1, \gamma_2) \;\leq\; d_2(\gamma_1, \gamma_2) \;\leq\; \lambda\, d_1(\gamma_1, \gamma_2),$$

   for any $\gamma_1, \gamma_2 \in \Gamma$ and λ is determined by T_1 and T_2. Therefore, the quasi-isometric equivalent class of $C_T(\Gamma)$ is canonically associated with the group Γ itself.

4. It seems, from the bulletin (3), that the constant c in the definition is unnecessary. In fact, M. Gromov introduces this constant to study a broader class of spaces in geometry naturally related to group actions. We have the following theorem due to Efremovich [24], Švarc [59] and Milnor [55]: Let X be a metric space which is geodesic and proper. Let Γ be a group and $\Gamma \times X \to X$ an action by isometries (say from the left). Assume the action is proper and that the quotient $\Gamma \setminus X$ is compact. Then the group Γ is finitely generated and quasi-isometric to X. (The metric on Γ is the word metric with respect to some finite generating set.)

For the commutator subgroup G' of G, the generating set S of commutators is uniquely determined and thus there is no such quasi-isometric issue for $C_S(G')$. Nevertheless, we are still interested in studying properties of $C_S(G')$ that are invariant under quasi-isometries. The justification is that the study of finitely generated groups as geometric objects has proven to be very fruitful and it is standard to expect that the large scale geometry (invariant under quasi-isometries) of a Cayley graph will reveal useful information about a group.

In the following, we are going to introduce some concepts on metric spaces, which are quasi-isometrically invariant and they are generalizations of similar notions from topology, geometry and analysis.

Definition 5.2.4. A *thickening* Y of a metric space X is an isometric inclusion $X \to Y$ with the property that there is a constant C so that every point in Y is within distance C of some point in X.

Definition 5.2.5. A metric space X is *large scale* k*-connected* if for every thickening $X \subset Y$ there is a thickening $Y \subset Z$ which is k-connected in the usual sense, i.e., Z is path-connected, and $\pi_i(Z) = 0$ for $i \leq k$.

Large scale k-connectivity is a quasi-isometrically invariant property (or large scale property). For G a finitely generated group with a generating set T, Gromov outlines a proof ([36], $1.C_2$) that the Cayley graph $C_T(G)$ is large scale 1-connected if and only if G is finitely presented, and $C_T(G)$ is large scale k-connected if and only if there exists a proper simplicial action of G on a $(k+1)$-dimensional k-connected simplicial complex X with compact quotient X/G.

For T an infinite generating set, large scale simple connectivity is equivalent to the assertion that G admits a presentation $G = \langle\, T \mid R\, \rangle$ where all elements in R have *uniformly bounded length* as words in T, i.e., all relations in G are consequences of relations of bounded length.

Next we define the notion of connectivity at infinity.

Definition 5.2.6. A metric space (X, d) is called *disconnected at infinity* if for any $k > 0$, there exist two subsets X_1 and X_2 in X, such that

1. X_1 and X_2 are both unbounded;
2. X_1 and X_2 cover almost all X, i.e., the complement $X \setminus (X_1 \cup X_2)$ is bounded;
3. $d(X_1, X_2) \geq k$, i.e., $d(x_1, x_2) \geq k$ for all $x_i \in X_i$, $i = 1, 2$.

Then X is called *connected at infinity* if for some k, the above X_1, X_2 don't exist.

Similarly, we define the number of ends of X at infinity as follows

Definition 5.2.7. Let (X, d) be a metric space, the *number of ends* of X at infinity is the maximal $L \in \mathbb{Z}_+$ such that for any $k > 0$, there exist subsets $X_1, \cdots, X_L$, satisfying

1. $X_1, \cdots, X_L$ are all unbounded;
2. $X \setminus (\bigcup_{i=1}^{L} X_i)$ is bounded;
3. $d(X_i, X_j) \geq k$, for any $i \neq j$.

If X is a path connected metric space, the *space of ends* of X can be defined as follows. Fix a base point p in X. Let $B_i(p)$ be the ball of radius i centered at p. $\pi_0(X \setminus B_i)$ is the set of path components of $X \setminus B_i$. Then we have an inverse system:

$$\pi_0(X \setminus B_1) \longleftarrow \pi_0(X \setminus B_2) \longleftarrow \cdots\cdots \longleftarrow \pi_0(X \setminus B_n) \longleftarrow \cdots\cdots .$$

Let

$$E = \varprojlim \pi_0(X \setminus B_i).$$

E is called the space of ends and it's clear that the number of ends equals the cardinality of E. Number of ends is quasi-isometrically invariant. And the number of ends of the Cayley graph of a finitely generated group agrees with the usual definition of ends of a group.

Example 5.2.8.

1. Let $F_2 = \langle a, b\rangle$ be the free group of rank 2. Let $T = \{a, b, a^{-1}, b^{-1}\}$, then $C_T(F)$ is a 4-valence regular tree. Thus F_2 has infinite ends, and the space of ends is a Cantor set.

2. Let $G = \pi_1(S_g)$, where S_g is a closed oriented surface of genus $g \geq 2$. Then the Cayley graph of G is quasi-isometric to $\mathbb{H}^2$, the Poincaré disk with the hyperbolic metric. Thus G has only one end and is connected at infinity.

At last, we define the notion of asymptotic dimension.

Definition 5.2.9. Let X be a metric space and $X = \cup_i U_i$ a covering by subsets. For given $D \geq 0$, the *D-multiplicity* of the covering is at most n if for any $x \in X$, the closed D-ball centered at x intersects at most n of the U_i.

Definition 5.2.10. A metric space X has *asymptotic dimension at most n* if for every $D \geq 0$ there is a covering $X = \cup_i U_i$ for which the diameters of the U_i are uniformly bounded, and the D-multiplicity of the covering is at most $n + 1$. The least such n is the *asymptotic dimension* of X, and we write

$$\text{asdim}(X) = n$$

The definition of asymptotic dimension is a generalization of the usual topological (covering) dimension and they have similar properties. See [8] for other equivalent definitions and further properties.

Proposition 5.2.11.

1. $\text{asdim}(X)$ *is quasi-isometrically invariant;*

2. *monotonicity* $X' \subseteq X \Rightarrow \text{asdim}(X') \leq \text{asdim}(X)$*;*

3. *product* $\text{asdim}(X_1 \times X_2) \leq \text{asdim}(X_1) + \text{asdim}(X_2)$*;*

4. *finite union* $X = A \cup B$, *then* $\text{asdim}(X) = \max\{\text{asdim}(A), \text{asdim}(B)\}$.

Example 5.2.12.

1. $\text{asdim}(X) = 0 \Leftrightarrow X$ is bounded $\Leftrightarrow X \simeq_{q.i.}$ a point.

2. $\text{asdim}(\mathbb{R}^n) = n$ as expected. Write $\mathbb{Z}^n$ to be the free abelian group of rank n. The the Cayley graph of $\mathbb{Z}^n$ with respect to the standard free generating set is the integral lattice in $\mathbb{R}^n$, and thus $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$. So we get $\text{asdim}(\mathbb{Z}^n) = n$.

Remark 5.2.13. All definitions in this section are from M. Gromov's seminal paper "Asymptotic Invariants of Infinite Groups" ([36]), which has been the main source of ideas for the development of geometric group theory since its publication.

5.3 Large Scale Simple Connectivity

In this section, we are going to prove that for any finitely presented group G, the Cayley graph of the commutator subgroup $C_S(G')$ is large scale 1-connected (simply connected). This is a joint work with D. Calegari [15].

By the definition (5.2.5) and the discussion there, to show that $C_S(G')$ is large scale 1-connected, it suffices to show that there is a constant K so that for every simplicial loop γ in $C_S(G')$ there are a sequence of loops $\gamma = \gamma_0, \gamma_1, \cdots, \gamma_n$ where γ_n is the trivial loop, and each γ_i is obtained from γ_{i-1} by cutting out a subpath $\sigma_{i-1} \subset \gamma_{i-1}$ and replacing it by a subpath $\sigma_i \subset \gamma_i$ with the same endpoints, so that $|\sigma_{i-1}| + |\sigma_i| \leq K$.

More generally, we call the operation of cutting out a subpath σ and replacing it by a subpath σ' with the same endpoints where $|\sigma| + |\sigma'| \leq K$ a K*-move*.

Definition 5.3.1. Two loops γ and γ' are K*-equivalent* if there is a finite sequence of K-moves which begins at γ, and ends at γ'.

K-equivalence is (as the name suggests) an equivalence relation. The statement that $C_S(G')$ is large scale 1-connected is equivalent to the statement that there is a constant K such that every two loops in $C_S(G')$ are K-equivalent.

First we establish large scale simple connectivity in the case of a free group.

Lemma 5.3.2. *Let F be a finitely generated free group. Then $C_S(F')$ is large scale simply connected.*

Proof. Let γ be a loop in $C_S(F')$. After acting on γ by left translation, we may assume that γ passes through 1, so we may think of γ as a simplicial path in $C_S(F')$ which starts and ends at id. If $s_i \in S$ corresponds to the ith segment of γ, we obtain an expression

$$s_1 s_2 \cdots s_n = 1$$

in F, where each s_i is a commutator. For each i, let $a_i, b_i \in F$ be elements with $[a_i, b_i] = s_i$ (note that a_i, b_i with this property are not necessarily unique). Let Σ be a surface of genus n, and let α_i, β_i for $i \leq n$ be a standard basis for $\pi_1(\Sigma)$; see Figure 5.1.

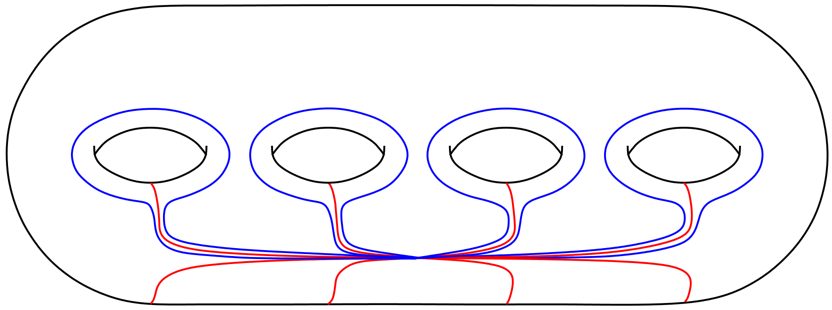

Figure 5.1: A standard basis for $\pi_1(\Sigma)$ where Σ has genus 4. The α_i curves are in red, and the β_i curves are in blue.

Let X be a wedge of circles corresponding to free generators for F, so that $\pi_1(X) = F$. We can construct a basepoint preserving map $f\colon \Sigma \to X$ with $f_*(\alpha_i) = a_i$ and $f_*(\beta_i) = b_i$ for each i. Since X is a $K(F, 1)$, the homotopy class of f is uniquely determined by the a_i, b_i.

Let ϕ be a (basepoint preserving) self-homeomorphism of Σ. The map $f \circ \phi\colon \Sigma \to X$ determines a new loop in $C_S(F')$ (also passing through 1) which we denote $\phi_*(\gamma)$ (despite the notation, this image does not depend only on γ, but on the choice of elements a_i, b_i as above).

Sublemma 5.3.3. *There is a universal constant K independent of γ or of ϕ (or even of F) so that after composing ϕ by an inner automorphism of $\pi_1(\Sigma)$ if necessary, γ and $\phi_*(\gamma)$ as above are K-equivalent.*

Proof. Suppose we can express ϕ as a product of (basepoint preserving) automorphisms

$$\phi = \phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_1$$

such that if α_i^j, β_i^j denote the images of α_i, β_i under $\phi_j \circ \phi_{j-1} \circ \cdots \circ \phi_1$, then ϕ_{j+1} fixes all but K consecutive pairs α_i^j, β_i^j up to (basepoint preserving) homotopy. Let $s_i^j = [f_*\alpha_i^j, f_*\beta_i^j]$, and let γ^j be the loop in $C_S(F')$ corresponding to the identity $s_1^j s_2^j \cdots s_n^j = \text{id}$ in F.

For each j, let supp_{j+1} denote the *support* of ϕ_{j+1}, i.e., the set of indices i such that $\phi_{j+1}(\alpha_i^j) \neq \alpha_i^j$ or $\phi_{j+1}(\beta_i^j) \neq \beta_i^j$. By hypothesis, supp_{j+1} consists of at most K indices for each j.

Because it is just the marking on Σ which has been changed and not the map f, if $k \leq i \leq k + K - 1$ is a maximal consecutive string of indices in supp_{j+1}, then there is an equality of products

$$s_k^j s_{k+1}^j \cdots s_{k+K-1}^j = s_k^{j+1} s_{k+1}^{j+1} \cdots s_{k+K-1}^{j+1}$$

as elements of F. This can be seen geometrically as follows. The expression on the left is the image under

f_* of an element represented by a certain embedded based loop in Σ, while the expression on the right is its image under $f_* \circ \phi_{j+1}$. The automorphism ϕ_{j+1} is represented by a homeomorphism of Σ whose support is contained in regions bounded by such loops. Hence the expressions are equal. It follows that γ^j and γ^{j+1} are $2K$-equivalent.

So to prove the Sublemma it suffices to show that any automorphism of S can be expressed (up to inner automorphism) as a product of automorphisms ϕ_i with the property above.

The hypothesis that we may compose ϕ by an inner automorphism means that we need only consider the image of ϕ in the mapping class group of Σ. It is well-known since Dehn [18] that the mapping class group of a closed oriented surface Σ of genus g is generated by twists in a finite standard set of curves, each of which intersects at most two of the α_i, β_i essentially; see Figure 5.2.

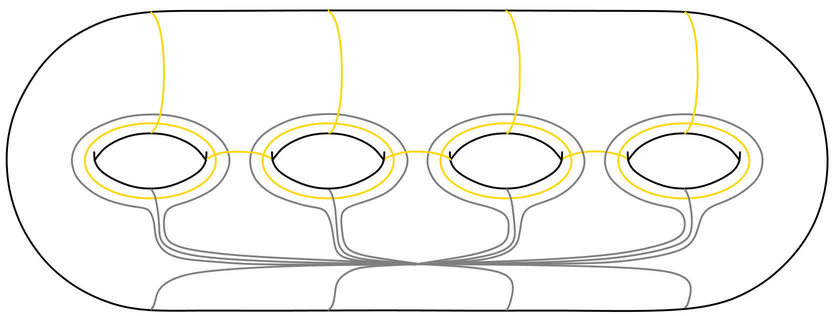

Figure 5.2: A standard set of $3g - 1$ simple curves, in yellow. Dehn twists in these curves generate the mapping class group of Σ.

So write $\phi = \tau_1\tau_2 \cdots \tau_m$ where the τ_i are all standard generators. Now define

$$\phi_j = \tau_1\tau_2 \cdots \tau_{j-1}\tau_j\tau_{j-1}^{-1} \cdots \tau_1^{-1}$$

We have

$$\phi_j\phi_{j-1} \cdots \phi_1 = \tau_1\tau_2 \cdots \tau_j$$

Moreover, each ϕ_j is a Dehn twist in a curve which is the image of a standard curve under $\phi_{j-1} \cdots \phi_1$, and therefore intersects $\alpha_i^{j-1}, \beta_i^{j-1}$ essentially for at most 2 (consecutive) indices i. This completes the proof of the Sublemma (and shows, in fact, that we can take $K = 4$). □

We now complete the proof of the Lemma. As observed by Stallings (see e.g. [57]), a nontrivial map $f : \Sigma \to X$ from a closed, oriented surface to a wedge of circles factors (up to homotopy) through a *pinch* in the following sense. Make f transverse to some edge e of X, and look at the preimage Γ of a regular value of f in e. After homotoping inessential loops of Γ off e, we may assume that for some edge e and some regular value, the preimage Γ contains an embedded essential loop δ.

There are two cases to consider. In the first case, δ is nonseparating. In this case, let ϕ be an automorphism which takes α_1 to the free homotopy class of δ. Then γ and $\phi_*(\gamma)$ are K-equivalent by the Sublemma.

However, since $f(\delta)$ is homotopically trivial in X, there is an identity $[\phi_*\alpha_1, \phi_*\beta_1] = \text{id}$ and therefore $\phi_*(\gamma)$ has length 1 shorter than γ.

In the second case, ϕ is separating, and we can let ϕ be an automorphism which takes the free homotopy class of $[\alpha_1, \beta_1] \cdots [\alpha_j, \beta_j]$ to δ. Again, by the Sublemma, γ and $\phi_*(\gamma)$ are K-equivalent. But now $\phi_*(\gamma)$ contains a subarc of length j with both endpoints at id, so we may write it as a product of two loops at id, each of length shorter than that of γ.

By induction, γ is K-equivalent to the trivial loop, and we are done. □

We are now in a position to prove our first main theorem.

Theorem 5.3.4. *Let G be a finitely presented group. Then $C_S(G')$ is large scale simply connected.*

Proof. Let W be a smooth 4-manifold (with boundary) satisfying $\pi_1(W) = G$. If $G = \langle T \mid R \rangle$ is a finite presentation, we can build W as a handlebody, with one 0-handle, one 1-handle for every generator in T, and one 2-handle for every relation in R. If $r_i \in R$ is a relation, let D_i be the cocore of the corresponding 2-handle, so that D_i is a properly embedded disk in W. Let $V \subset W$ be the union of the 0-handle and the 1-handles. Topologically, V is homotopy equivalent to a wedge of circles. By the definition of cocores, the complement of $\cup_i D_i$ in W deformation retracts to V. See e.g. [46], Chapter 1 for an introduction to handle decompositions of 4-manifolds.

Given γ a loop in $C_S(G')$, translate it by left multiplication so that it passes through 1. As before, let Σ be a closed oriented marked surface, and $f : \Sigma \to W$ a map representing γ.

Since G is finitely presented, $H_2(G; \mathbb{Z})$ is finitely generated. Choose finitely many closed oriented surfaces $S_1, \cdots, S_r$ in W which generate $H_2(G; \mathbb{Z})$. Let K' be the supremum of the genus of the S_i. We can choose a basepoint on each S_i, and maps to W which are basepoint preserving. By tubing Σ repeatedly to copies of the S_i with either orientation, we obtain a new surface and map $f' : \Sigma' \to W$ representing a loop γ' such that $f'(\Sigma')$ is null-homologous in W, and γ' is K'-equivalent to γ (note that K' depends on G but not on γ).

Put f' in general position with respect to the D_i by a homotopy. Since $f'(\Sigma')$ is null-homologous, for each proper disk D_i, the signed intersection number vanishes: $D_i \cap f'(\Sigma') = 0$. Hence $f'(\Sigma) \cap D_i = P_i$ is a finite, even number of points which can be partitioned into two sets of equal size corresponding to the local intersection number of $f'(\Sigma')$ with D_i at $p \in P_i$.

Let $p, q \in P_i$ have opposite signs, and let μ be an embedded path in D_i from $f'(p)$ to $f'(q)$. Identifying p and q implicitly with their preimages in Σ', let α and β be arcs in Σ' from the basepoint to $(f')^{-1}p$ and $(f')^{-1}q$. Since μ is contractible, there is a neighborhood of μ in D_i on which the normal bundle is trivializable. Hence, since $f'(\Sigma')$ and D_i are transverse, we can find a neighborhood U of μ in W disjoint from the other D_j, and co-ordinates on U satisfying

1. $D_i \cap U$ is the plane $(x, y, 0, 0)$;

2. $\mu \cap U$ is the interval $(t, 0, 0, 0)$ for $t \in [0, 1]$;

3. $f'(\Sigma') \cap U$ is the union of the planes $(0, 0, z, w)$ and $(1, 0, z, w)$.

Let A be the annulus consisting of points $(t, 0, \cos(\theta), \sin(\theta))$ where $t \in [0, 1]$. Then A is disjoint from D_i and all the other D_j, and we can tube $f'(\Sigma')$ with A to reduce the number of intersection points of $f'(\Sigma')$ with $\cup_i D_i$, at the cost of raising the genus by 1. Technically, we remove the disks $(f')^{-1}(0, 0, s\cos(\theta), s\sin(\theta))$ and $(f')^{-1}(1, 0, s\cos(\theta), s\sin(\theta))$ for $s \in [0, 1]$ from Σ', and sew in a new annulus which we map homeomorphically to A. The result is $f'' \colon \Sigma'' \to W$ with two fewer intersection points with $\cup_i D_i$. This has the effect of adding a new (trivial) edge to the start of γ', which is the commutator of the elements represented by the core of A and the loop $f'(\alpha) * \mu * f'(\beta)$. Let γ'' denote this resulting loop, and observe that γ'' is 1-equivalent to γ'. After finitely many operations of this kind, we obtain $f''' \colon \Sigma''' \to W$ corresponding to a loop γ''' which is $\max(1, K')$-equivalent to γ, such that $f'''(\Sigma''')$ is disjoint from $\cup_i D_i$.

After composing with a deformation retraction, we may assume f''' maps Σ''' into V. Let $F = \pi_1(V)$, and let $\rho \colon F \to G$ be the homomorphism induced by the inclusion $V \to W$. There is a loop γ^F in $C_S(F')$ corresponding to f''' such that $\rho_*(\gamma^F) = \gamma'''$ under the obvious simplicial map $\rho_* \colon C_S(F') \to C_S(G')$. By Lemma (5.3.2), the loop γ^F is K-equivalent to a trivial loop in $C_S(F')$. Pushing forward the sequence of intermediate loops by ρ_* shows that γ''' is K-equivalent to a trivial loop in $C_S(G')$. Since γ was arbitrary, we are done. □

Remark 5.3.5. A similar, though perhaps more combinatorial argument could be made working directly with 2-complexes in place of 4-manifolds.

In words, Theorem (5.3.4) says that for G a finitely presented group, all relations amongst the commutators of G are consequences of relations involving only boundedly many commutators.

The next example shows that the size of this bound depends on G:

Example 5.3.6. Let Σ be a closed surface of genus g, and $G = \pi_1(\Sigma)$. If γ is a loop in $C_S(G)$ through the origin, and $f \colon \Sigma' \to \Sigma$ is a corresponding map of a closed surface, then the homology class of Σ' is trivial unless the genus of Σ' is at least as big as that of Σ. Hence the loop in $C_S(G)$ of length g corresponding to the relation in the "standard" presentation of $\pi_1(\Sigma)$ is not K-equivalent to the trivial loop whenever $K < g$.

5.4 Hyperbolicity and Large Scale Geometry

In this section, we specialize to the case when the group G is word-hyperbolic, or more generally, G admits an action on a hyperbolic graph. This hyperbolicity implies that the space of homogeneous quasimorphisms on these groups is infinite dimensional, and quasimorphisms can be used to separate elements in the commutator subgroup G'. As a consequence, we show that in the Cayley graph $C_S(G')$, there exists a quasi-isometrically embedded $\mathbb{Z}^n$, for any $n \in \mathbb{Z}_+$. Therefore, the graph $C_S(G')$ is no longer hyperbolic, with only one end and its asymptotic dimension is infinite.

5.4.1 Hyperbolic Groups

M. Gromov introduced the definition of δ-hyperbolic spaces in [35]. This definition of δ-hyperbolicity is so robust that it encapsulates many of the global features of the geometry of complete, simply connected manifolds of negative curvature.

Definition 5.4.1 (Slim Triangle). Let $\delta > 0$. A geodesic triangle in a metric space is said to be δ-*slim* if each of its sides is contained in the δ-neighborhood of the union of the other two sides. A geodesic metric space X is said to be δ-hyperbolic if every geodesic triangle in X is δ-slim.

Definition 5.4.2. A group G with a finite symmetric generating set T is δ-hyperbolic if the corresponding Cayley graph $C_T(G)$, with the path metric, is δ-hyperbolic. A group G is *word-hyperbolic* if there is some $\delta > 0$ and a finite symmetric generating set T for which $C_T(G)$ is δ-hyperbolic.

Example 5.4.3.

1. Finitely generated free groups are word-hyperbolic. A geodesic metric space is 0-hyperbolic if and only if it is an $\mathbb{R}$-tree. Thus G is 0-hyperbolic if and only if G is a free group of finite rank.

2. Let M be a closed Riemannian manifold with section curvature uniformly bounded above by a negative number. Then $\pi_1(M)$ is hyperbolic. In particular, the fundamental groups of compact surfaces with $\chi < 0$ are hyperbolic.

Word-hyperbolic groups must be finitely presentable and the converse is almost true (in some probability sense). Thus hyperbolic groups represent a very large class of groups, interesting to geometer.

A very important object in the study of hyperbolic spaces is the notion of quasi-geodesics.

Definition 5.4.4. Let X be a metric space. A (λ, k)-*quasi-geodesic* is a (λ, k)-quasi-isometric embedding $\sigma\colon\ \mathrm{I} \to X$, where I is an interval of $\mathbb{R}$ or of $\mathbb{Z}$. In the case when $\mathrm{I} \subseteq \mathbb{Z}$, we say that we have a quasi-geodesic sequence.

We summarize some of the main properties of quasi-geodesics in hyperbolic spaces below (see [5] or [35] for details):

Theorem 5.4.5. *Let X be a δ-hyperbolic geodesic metric space.*

1. *Morse Lemma. For every λ, k, there is a universal constant $C(\delta, \lambda, k)$ such that every (λ, k)-quasi-geodesic segment with endpoints $p, q \in X$ lies in the C-neighborhood of any geodesic joining p to q.*

2. *Quasi-geodesic is local. For every λ, k there is a universal constant $C(\delta, \lambda, k)$ such that every map $\phi\colon \mathbb{R} \to X$ which restricts on each segment of length C to a (λ, k)-quasi-geodesic is a (globally) $(2\lambda, 2k)$-quasi-geodesic.*

3. *Ideal Boundary.* *There is an ideal boundary ∂X functorially associated to X, whose points consist of quasi-geodesic rays up to the equivalence relation of being a finite Hausdorff distance apart. There is a natural topology on ∂X for which it is metrizable. If X is proper, ∂X is compact. Moreover, any quasi-isometric embedding $X \to Y$ between hyperbolic spaces induces a continuous map $\partial X \to \partial Y$.*

If G is hyperbolic, we denote the ideal boundary of its Cayley graph by ∂G. As a topological space, this does not depend on the choice of a generating set, so we call it the *ideal boundary* (or just the *boundary*) of G. The left action of G on itself induces an action of G on ∂G by homeomorphisms. Every element $g \in G$ is either finite order (i.e., is elliptic), or fixes two points $p^{\pm}$ in ∂G with "source-sink" dynamics (i.e is hyperbolic).

A hyperbolic group G is called *non-elementary* if ∂G contains more than two (then uncountably infinitely many) points. Klein's ping-pong argument applied to the action of G on ∂G shows that in this case G contains many (quasi-isometrically embedded and quasi-convex) nonabelian free groups of arbitrary finite rank. On the contrary, a hyperbolic group is *elementary* if and only if it is virtually cyclic.

G acts on its Cayley graph on the left by isometries. If X is a geodesic metric space, and g fixes some geodesic l and acts on it as a translation, then the translation length of g, $\tau(g) = d_X(q, g(q))$ for any $q \in l$. For hyperbolic groups, we have the following Lemma:

Lemma 5.4.6 (Axes in hyperbolic Cayley graphs). *Let G be δ-hyperbolic with respect to the generating set T. Then there is a positive constant $C(\delta, |T|)$ such that every $g \in G$ either has finite order, or there is some $n \leq C$ such that g^n fixes some bi-infinite geodesic axis l_g and acts on it by translation.*

For a proof, see Theorem 5.1 from [27], or [5].

5.4.2 Generalized Counting Quasimorphisms

In this section, we introduce the generalizations of Brooks' counting quasimorphisms, due to Epstein-Fujiwara [27] and Fujiwara [30] [31] in general.

Let G be a group acting simplicially on a δ-hyperbolic complex X (not assumed to be locally finite).

Definition 5.4.7. Let σ be a finite oriented simplicial path in X, and let σ^{-1} denote the same path with the opposite orientation. A *copy* of σ is a translate $a \cdot \sigma$ where $a \in G$.

Definition 5.4.8. Let σ be a finite oriented simplicial path in X, and let $p \in X$ be a base vertex. For any oriented simplicial path γ in X, let $|\gamma|_\sigma$ denote the maximal number of disjoint copies of σ contained in γ. Given $a \in G$, define

$$c_\sigma(a) = d(p, a(p)) - \inf_\gamma(\text{length}(\gamma) - |\gamma|_\sigma),$$

where the infimum is taken over all oriented simplicial paths γ in X from p to $a(p)$.

Define the (small) *counting quasimorphism* h_σ by the formula

$$h_\sigma(a) = c_\sigma(a) - c_{\sigma^{-1}}(a).$$

For fixed p and a, a path γ is a *realizing path* if it realizes the infimum of $\text{length}(\gamma) - |\gamma|_\sigma$. Since the value of this function on any γ is an integer, realizing paths always exist. Realizing paths have the following universal geometric property.

Lemma 5.4.9 (Fujiwara[30]). *Suppose* $\text{length}(\sigma) \geq 2$*. Any realizing path for* c_σ *is a* $(2,4)$*-quasi-geodesic.*

By bullet (1) from Theorem (5.4.5) (i.e., the "Morse Lemma"), there is a constant $C(\delta)$ such that any realizing path for c_σ from p to $a(p)$ must be contained in the C-neighborhood of any geodesic between these two points. In particular, we have the following consequence:

Lemma 5.4.10. *There is a constant* $C(\delta)$ *such that for any path* σ *in* X *of length at least* 2*, and for any* $a \in G$*, if the* C*-neighborhood of any geodesic from* p *to* $a(p)$ *does not contain a copy of* σ*, then* $c_\sigma(a) = 0$*.*

Finally, the defect of h_σ is independent of the choice of σ:

Lemma 5.4.11 (Fujiwara[30]). *Let* σ *be a path of length at least* 2*. Then there is a constant* $C(\delta)$ *such that* $D(h_\sigma) \leq C$*.*

5.4.3 Quasi-isometrically Embedded $\mathbb{Z}^n$

In this section, we prove the following theorem concerning the existence of (high-dimensional) flats in the Cayley graph of a commutator subgroup. This is a joint work with D. Calegari.

Theorem 5.4.12. *Let* G *be a non-elementary word-hyperbolic group and* $C_S(G')$ *the Cayley graph of the commutator subgroup* G' *with respect to the set* S *of commutators. Let* $\mathbb{Z}^n$ *be the integral lattice in* $\mathbb{R}^n$ *with the induced metric. Then for any* $n \in \mathbb{Z}_+$*, we have a map* $\rho_n \colon \mathbb{Z}^n \to C_S(G')$*, which is a quasi-isometric embedding.*

This theorem can be regarded as an application of the following *separation theorem* about counting quasimorphisms.

Theorem 5.4.13 (Calegari-Fujiwara[14]). *Let* G *be a group which is* δ*-hyperbolic with respect to some symmetric generating set* T*. Let* a *be nontorsion, with no positive power conjugate to its inverse. Let* $a_i \in G$ *be a collection of elements with* $\tau \colon = \sup_i \tau(a_i)$ *finite. Suppose that for all nonzero integers* n, m *and all* $b \in G$ *and indices* i *we have an inequality*

$$a_i^m \neq b a^n b^{-1}.$$

Then there is a homogeneous quasimorphism $\phi \in Q(G)$ *such that*

1. *$\phi(a) = 1$ and $\phi(a_i) = 0$ for all i;*

2. *The defect satisfies $D(\phi) \le C(\delta, |T|)(\frac{\tau}{\tau(a)} + 1)$.*

Proof. By Lemma (5.4.6), after replacing each a_i by a fixed power whose size depends only on δ and $|T|$, we can assume that each a_i acts as translation on some geodesic axis l_i. Similarly, let l be a geodesic axis for a. Choose some big N (to be determined), and let σ be a fundamental domain for the action of a^N on l. The quasimorphism ϕ will be a multiple of the homogenization of h_σ, normalized to satisfy $\phi(a) = 1$. We need to show that if N is chosen sufficiently large, there are no copies of σ or σ^{-1} contained in the C-neighborhood of any l_i or l^{-1}, where C is as in Lemma (5.4.10).

Suppose for the sake of argument that there is such a copy, and let p be the midpoint of σ. The segment σ is contained in a translate $b(l)$. The translation length of a_i on l_i is $\tau(a_i) \le \tau$, and the translation length of bab^{-1} on $b(l)$ is $\tau(a)$ (the case of l^{-1} is similar and is omitted). For big N, we can assume the length of σ is large compared to $\tau(a)$ and $\tau(a_i)$. Then for each n which is small compared to N, the element $w_n := a_i b a^n b^{-1} a_i^{-1} b a^{-n} b^{-1}$ satisfies $d(p, w_n(p)) \le 4C$. Since there are less than $|T|^{4C}$ elements in the ball of radius $4C$ about any point, eventually we must have $w_n = w_m$ for distinct n, m. But this implies

$$a_i b a^n b^{-1} a_i^{-1} b a^{-n} b^{-1} = a_i b a^m b^{-1} a_i^{-1} b a^{-m} b^{-1}$$

and therefore a_i^{-1} and $ba^{n-m}b^{-1}$ commute. Since G is hyperbolic, commuting elements have powers which are equal, contrary to the hypothesis that no conjugate of a has a power equal to a power of a_i.

This contradiction implies that $\tau(a_i) + |T|^{4C}\tau(a) \ge N\tau(a)$. On the other hand, $D(h_\sigma)$ is uniformly bounded, by Lemma (5.4.11), and $h_\sigma$satisfies $h_\sigma(a^{Nn}) \ge n$. Homogenizing and scaling by the appropriate factor, we obtain the desired result. □

Proof of Theorem 5.4.12 . Let G be a non-elementary, δ-hyperbolic group with respect to some symmetric generating set T. Then we can find a sequence of elements $g_1,\ g_2,\ \cdots \in G'$ such that

1. g_i's are nontorsion;

2. $g_i^n \ne b g_i^{-m} b^{-1}$, for any nonzero positive m, n and $b \in G$;

3. $g_i^n \ne b g_j^m b^{-1}$, for any $i \ne j$, nonzero m, n and $b \in G$.

If G is a nonabelian free group, it's not difficult to see that such a sequence of elements exist in G'. In general, G contains quasi-isometrically embedded, quasi-convex nonabelian free groups, and such elements can be constructed accordingly. For more details, see Proposition 2 in [2].

For any fixed $n \in \mathbb{Z}_+$, define $\rho_n : \mathbb{Z}^n \to C_S(G')$ as follows:

$$\begin{aligned} \rho_n : \quad \mathbb{Z}^n \quad &\longrightarrow \quad C_S(G') \\ (k_1, \cdots, k_n) &\longmapsto \quad g_1^{k_1} \cdots g_n^{k_n}. \end{aligned}$$

Since g_i's $\in G'$, we have $g_1^{k_1} \cdots g_n^{k_n} \in G'$. Thus ρ_n is in fact a map from $\mathbb{Z}^n$ to the vertex set of the graph $C_S(G')$. We need to estimate the distances between the vertices in the image of ρ_n.

1. **(Upper Bound)** Pick any two points $p, q \in \mathbb{Z}^n$, say $p = (s_1, \cdots, s_n)$ and $q = (t_1, \cdots, t_n)$. The distance in $\mathbb{Z}^n$ can be written as

$$d_{\mathbb{Z}^n}(p,q) = |s_1 - t_1| + |s_2 - t_2| + \cdots + |s_n - t_n|,$$

which is the path metric of $\mathbb{Z}^n$ with respect to the standard free generating set. Write $d(\cdot,\cdot)$ for the metric in $C_S(G')$.

$$\begin{aligned} d(\rho_n(p), \rho_n(q)) &= d(g_1^{s_1} \cdots g_n^{s_n}, g_1^{t_1} \cdots g_n^{t^n}) \\ &= \text{cl}(g_n^{-s_n} \cdots g_1^{-s_1} g_1^{t_1} \cdots g_n^{t^n}) \end{aligned}$$

Claim.

$$g_n^{-s_n} \cdots g_1^{-s_1} g_1^{t_1} \cdots g_n^{t^n} = ({g_n}^{l_n})^{c_n} \cdots ({g_1}^{l_1})^{c_1}$$

where $({g_i}^{l_i})^{c_i} = c_i {g_i}^{l_i} c_i^{-1}$ *and* $|l_i| = |s_i - t_i|$.

The proof of the claim is a direct computation. We substitute this expression into the formula and get

$$\begin{aligned} d(\rho_n(p), \rho_n(q)) &= \text{cl}(({g_n}^{l_n})^{c_n} \cdots ({g_1}^{l_1})^{c_1}) \\ &\leq \text{cl}(({g_n}^{l_n})^{c_n}) + \cdots + \text{cl}(({g_1}^{l_1})^{c_1}) \\ &= \text{cl}({g_n}^{l_n}) + \cdots + \text{cl}({g_1}^{l_1}) \\ &= |l_n|\text{cl}(g_n) + \cdots + |l_1|\text{cl}(g_1) \\ &\leq A(|l_1| + \cdots + |l_n|) \\ &= A(|s_1 - t_1| + \cdots + |s_n - t_n|) \end{aligned}$$

where $A = \max_i \text{cl}(g_i)$, $1 \leq i \leq n$. So we have the up bound

$$d(\rho_n(p), \rho_n(q)) \ \leq \ A \, d_{\mathbb{Z}^n}(p,q).$$

2. **Lower Bound** By Separation Theorem (5.4.13), there exist homogeneous quasimorphisms ϕ_i, $1 \leq i \leq n$, such that $\phi_i(g_j) = \delta_{ij}$, where $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$. Write $D = \max_i D(\phi_i)$,

$1 \leq i \leq n$. Then we have

$$\begin{aligned} d(\rho_n(p), \rho_n(q)) &= \text{cl}(g_n^{-s_n} \cdots g_1^{-s_1} g_1^{t_1} \cdots g_n^{t^n}) \\ &= \text{cl}(({g_n}^{l_n})^{c_n} \cdots ({g_1}^{l_1})^{c_1}) \\ &\geq \frac{1}{7D(\phi_i)} |\ \phi_i(({g_n}^{l_n})^{c_n} \cdots ({g_1}^{l_1})^{c_1}) - \phi_i(1)\ | \\ &\geq \frac{1}{7D(\phi_i)} [\ |\phi_i(({g_1}^{l_1})^{c_1})| + \cdots + |\phi_i(({g_n}^{l_n})^{c_n})| - nD(\phi_i)\ ] \\ &= \frac{1}{7D(\phi_i)} [\ |l_1||\phi_i(g_1)| + \cdots + |l_n||\phi_i(g_n)| - nD(\phi_i)\ ] \\ &= \frac{|l_i| - nD(\phi_i)}{7D(\phi_i)} \\ &\geq \frac{|l_i| - nD}{7D} \end{aligned}$$

for any $1 \leq i \leq n$. Write $L = \max_i |l_i|$, $1 \leq i \leq n$. Then we have

$$nL \geq \sum_{i=1}^{n} |l_i| = \sum_{i=1}^{n} |s_i - t_i| = d_{\mathbb{Z}^n}(p, q).$$

So we obtain the lower bound

$$d(\rho_n(p), \rho_n(q)) \geq \frac{L - nD}{7D} \geq \frac{1}{7nD}\, d_{\mathbb{Z}^n}(p, q) - \frac{n}{7}.$$

Combining the two inequalities together, we have that for fixed $n \in \mathbb{Z}_+$ and any two points $p, q \in \mathbb{Z}^n$,

$$\frac{1}{7nD}\, d_{\mathbb{Z}^n}(p, q) - \frac{n}{7} \leq\ d(\rho_n(p), \rho_n(q)) \leq\ A\, d_{\mathbb{Z}^n}(p, q)$$

where D, n and A are constants independent of p and q. Thus $\rho_n\colon \mathbb{Z}^n \to C_S(G')$ is a quasi-isometric embedding. □

Theorem (5.4.12) tells us that in the graph $C_S(G')$, there exist a lot of flats with arbitrarily large dimensions and we immediately have the following corollaries.

Corollary 5.4.14. *Let G be a non-elementary, word-hyperbolic group and $C_S(G')$ the corresponding Cayley graph of the commutator subgroup. Then $C_S(G')$ is not (δ-)hyperbolic.*

Proof. $\mathbb{Z}^n$ with the standard path metric is clearly not δ-hyperbolics since there are "parallel" quasi-geodesics and the Hausdorff distances between them could be arbitrarily large, a contradiction to the bulletin (1) in Therem (5.4.5) . Theorem (5.4.12) transports these "parallel" quasi-geodesics into $C_S(G')$, thus $C_S(G')$ is not δ-hyperbolic. □

Corollary 5.4.15. *Let G and $C_S(G')$ be as above. Then* $\text{asdim}(C_S(G')) = \infty$.

Proof. By Example (5.2.12), asdim$(\mathbb{Z}^n) = n$. Theorem (5.4.12) gives quasi-isometrically embedded $\mathbb{Z}^n$ in $C_S(G')$ for any n, so by Proposition (5.2.11),

$$\text{asdim}(C_S(G')) \geq \text{asdim}(\rho_n(\mathbb{Z}^n)) = \text{asdim}(\mathbb{Z}^n) = n$$

for any n. Thus asdim$(C_S(G')) = \infty$. □

Theorem (5.4.12) is still true in a more general setting. To state the general case, we need to introduce some terminology first. All definitions below are from [2].

Let X be a path connected graph with the path metric d. Suppose (X, d) is δ-hyperbolic. Let G be a discrete group, acting on X simplicially and isometrically. An isometry $g \in G$ of X is called *hyperbolic* if it admits an invariant bi-infinite quasi-geodesic and we will refer to it as a *quasi-axis*.

Definition 5.4.16. We say the action of G on X satisfies *WPD* (weak proper discontinuity) if

1. G is not virtually cyclic;
2. G contains at least one element that acts on X as a hyperbolic isometry;
3. For every hyperbolic element $g \in G$, every $x \in X$, and every $c > 0$, there exists $N > 0$ such that the set
$$\{h \in G \mid d(x, h(x)) \; \leq \; c, \; d(g^N(x), hg^N(x)) \; \leq \; c\}$$
is finite.

Then we can state the general theorem.

Theorem 5.4.17. *Let G be a group, and G acts simplicially on a δ-hyperbolic graph X by isometries. Suppose the action satisfies WPD. Then for any $n \in \mathbb{Z}_+$, we have a map $\rho_n \colon \mathbb{Z}^n \to C_S(G')$, which is a quasi-isometric embedding.*

And similarly we have the corollary.

Corollary 5.4.18. *Let G be a group as above, then $C_S(G')$ is not δ-hyperbolic, and* asdim$(C_S(G')) = \infty$.

We omit the proofs since they are exactly the same as those in Theorem (5.4.12) and Corollary (5.4.14) and (5.4.15).

The main application of Theorem (5.4.17) is to the action of mapping class groups on curve complexes. Let S be a compact orientable surface of genus g and p punctures. We consider the associated mapping class group MCG(S) of S. This group acts on the curve complex $\mathcal{C}(S)$ of S defined by Harvey [41] and successfully used in the study of mapping class groups by Harer [40] [39] and by Ivanov [43] [44]. For our purpose, we restrict to the 1-skeleton of the curve complex, so that $\mathcal{C}(S)$ is a graph whose vertices are isotopy classes of essential, non-parallel, non-peripheral, pairwise disjoint simple closed curves in S and two distinct vertices are joined by an edge if the corresponding curve system can be realized simultaneously by pairwise disjoint curves. In certain sporadic cases, $\mathcal{C}(S)$ as defined above is 0-dimensional or empty (for example, when $g = 0, p \leq 4$ or $g = 1, p \leq 1$) and in the theorems below, these cases are excluded. The mapping class group $\mathrm{MCG}(S)$ acts on $\mathcal{C}(S)$ by $f \cdot a = f(a)$.

H. Masur and Y. Minsky [53] proved the following remarkable result.

Theorem 5.4.19. *The curve complex $\mathcal{C}(S)$ is δ-hyperbolic. An element of $\mathrm{MCG}(S)$ acts hyperbolically on $\mathcal{C}(S)$ if and only if it is pseudo-Anosov.*

In [2], Bestvina and Fujiwara study the action of $\mathrm{MCG}(s)$ on the curve complex $\mathcal{C}(S)$ and show that

Theorem 5.4.20. *Let S be a non-sporadic surface. Then the action of $\mathrm{MCG}(S)$ on the curve complex $\mathcal{C}(S)$ satisfies WPD.*

Combine 5.4.17, 5.4.18, 5.4.19 and 5.4.20 together, we have the following corollary.

Corollary 5.4.21. *Let S be a non-sporadic surface. Then for any $n \in \mathbb{Z}_+$, there exists quasi-isometrically embedded $\mathbb{Z}^n$ in the Cayley graph of the commutator subgroup $C_S(\mathrm{MCG}(S)')$. Thus $C_S(\mathrm{MCG}(S)')$ is not δ-hyperbolic and has infinite asymptotic dimension.*

Corolary (5.4.21) is especially interesting in the case when S is a closed orientable surface of genus $g \geq 3$. In this case, $\mathrm{MCG}(S)$ is perfect ([56]), i.e., $\mathrm{MCG}(S)' = \mathrm{MCG}(S)$. So $C_S(\mathrm{MCG}(S)') = C_S(\mathrm{MCG}(S))$ is the Cayley graph of $\mathrm{MCG}(S)$ itself with respect to a canoncial infinite generating set.

5.4.4 Large Scale Connectivity at Infinity

In this last section, we continue our study of large scale geometry of a commutator subgroup and show that when G is a non-elementary word-hyperbolic group or admits an action on a hyperbolic graph, the corresponding Cayley graph of the commutator subgroup $C_S(G')$ has only one end. This is a joint work with D. Calegari [15].

Again we only state and prove the case when G is a non-elementary word-hyperbolic group and the proof for the general case is the same.

Theorem 5.4.22. *Let G be a non-elementary word-hyperbolic group. Then $C_S(G')$ is one-ended; i.e., for any $r > 0$ there is an $R \geq r$ such that any two points in $C_S(G')$ at distance at least R from 1 can be joined by a path which does not come closer than distance r to* id*.*

We can use Theorem (5.4.12) to give a heuristic proof. For any $g, h \in G'$, we construct an element $a \in G'$ with $\mathrm{cl}(a) \gg \mathrm{cl}(g), \mathrm{cl}(h)$ such that g and a are "independent", meaning they have no powers which are conjugate to each other. And so are h and a. Then by the proof of Theorem (5.4.12), we have quasi-isometrically embedded $\mathbb{Z}^2$'s, generated by $\{g, a\}$ and $\{h, a\}$ respectively. Use these two $\mathbb{Z}^2$'s to find paths from g to a and from a to h, which are far away from 1. Then we are done. In the following, we give a direct proof, and the main tool is still to use counting quasimorphisms to obtain lower bounds for commutator lengths. The proof also gives us the picture of the local structure of the lattice constructed in Theorem (5.4.12)

Lemma 5.4.23. *Let G be a non-elementary word-hyperbolic group. Let g_i be a finite collection of elements of G. There is a commutator $s \in G'$ and a quasimorphism ϕ on G with the following properties:*

1. $|\phi(g_i)| = 0$ *for all* i*;*
2. $|\phi(s^n) - n| \leq K_1$ *for all* n*, where* K_1 *is a constant which depends only on* G*;*
3. $D(\phi) \leq K_2$ *where* K_2 *is a constant which depends only on* G*.*

Proof. Fix a finite generating set T so that $C_T(G)$ is δ-hyperbolic. There is a constant N such that for any nonzero $g \in G$, the power g^N fixes an axis l_g (Lemma (5.4.6)). Since G is non-elementary, it contains quasi-isometrically embedded copies of free groups of any fixed rank. So we can find a commutator s whose translation length (in $C_T(G)$) is as big as desired. In particular, given $g_1, \cdots, g_j$, we choose s with $\tau(s) \gg \tau(g_i)$ for all i. Let l be a geodesic axis for s^N, and let σ be a fundamental domain for the action of s^N on l. Since $|\sigma| = N\tau(s) \gg \tau(g_i)$, Lemma (5.4.10) implies that there are no copies of σ or σ^{-1} in a realizing path for any g_i. Hence $h_\sigma(g_i) = 0$ for all i. By Lemma (5.4.11), $D(h_\sigma) \leq K(\delta)$. It remains to estimate $h_\sigma(s^n)$.

The argument of the Separation Theorem (5.4.13) shows that for N sufficiently large (depending only on G and not on s) no copies of σ^{-1} are contained in any realizing path for s^n with n positive, and therefore $|h_\sigma(s^n) - \lfloor n/N \rfloor|$ is bounded by a constant depending only on G. The quasimorphism $\phi = N \cdot h_\sigma$ has the desired properties. □

We now give the proof of Theorem (5.4.22).

Proof. Let $g, h \in G'$ have commutator length at least R. Let $g = s_1 s_2 \cdots s_n$ and $h = t_1 t_2 \cdots t_m$ where $n, m \geq R$ are equal to the commutator lengths of g and h respectively, and each s_i, t_i is a commutator in G. Let s be a commutator with the properties described in Lemma (5.4.23) with respect to the elements g, h; that is, we want s for which there is a quasimorphism ϕ with $\phi(g) = \phi(h) = 0$, with $|\phi(s^n) - n| \leq K_1$ for all n, and with $D(\phi) \leq K_2$. Let $N \gg R$ be very large. We build a path in $C_S(G')$ from g to h out of four segments, none of which come too close to id.

The first segment is

$$g, gs, gs^2, gs^3, \cdots, gs^N.$$

Since s is a commutator, $d(gs^i, \text{id}) \geq R - i$ for any i. On the other hand,

$$\phi(gs^i) \geq \phi(g) + \phi(s^i) - D(\phi) \geq i - K_2 - K_1$$

where K_1, K_2 are as in Lemma (5.4.23) (and do not depend on g, h, s). We can estimate

$$d(gs^i, \mathrm{id}) = \mathrm{cl}(gs^i) \geq \frac{\phi(gs^i)}{7D(\phi)} \geq \frac{i - K_2 - K_1}{7K_2}.$$

Hence $d(gs^i, \mathrm{id}) \geq R/14K_2 - (K_1 + K_2)/7K_2$ for all i, so providing $R \gg K_1, K_2$, the path gs^i never gets too close to id.

The second segment is

$$gs^N = s_1 s_2 \cdots s_n s^N, s_2 \cdots s_n s^N, \cdots, s^N.$$

Note that consecutive elements in this segment are distance 1 apart in $C_S(G')$. Since $d(gs^N, \mathrm{id}) \geq (N - K_2 - K_1)/7K_2 \gg R$ for N sufficiently large, we have

$$d(s_i \cdots s_n s^N, \mathrm{id}) \gg R$$

for all i.

The third segment is

$$s^N, t_m s^N, t_{m-1} t_m s^N, \cdots, t_1 t_2 \cdots t_m s^N = hs^N,$$

and the fourth is

$$hs^N, hs^{N-1}, \cdots, hs, h.$$

For the same reason as above, neither of these segments gets too close to id. This completes the proof of the theorem, taking $r = R/14K_2 - (K_1 + K_2)/7K_2$. □

Bibliography

[1] C. Bavard, *Longueur stable des commutateur*, L'Enseign. Math. **37** (1991), 109–150

[2] M. Bestvina and K. Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. **6** (2002), 69–89

[3] R. Bieri and R. Strebel, *On groups of PL homeomorphisms of the real line*, Preprint (1985)

[4] J. S. Birman, *Braids, links, and mapping class groups*, Annals of Mathematics Studies, no. 82, Princeton University Press, Princeton, N.J., 1974

[5] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999

[6] R. Brooks, *Some remarks on bounded cohomology*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)(Princeton, N.J.), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, 53–63

[7] K. S. Brown, *Finiteness properties of groups*, J. Pure Appl. Algebra **44** (1987), 45–75

[8] S. Buyalo and V. Schroeder, *Elements of asymptotic geometry*, EMS Monographs in Mathematics, European Mathematical Society Publishing House, Zürich, 2007

[9] D. Calegari, *scl*, monograph, to be published in MSJ monograph series; available from

`http://www.its.caltech.edu/~dannyc`

[10] D. Calegari, *Denominator bounds in Thompson-like groups and flows*, Groups, Geometry, Dynamics **1** (2007), no. 2, 101–109

[11] D. Calegari, *Stable commutator length in subgroups of* $\mathrm{PL}^+(I)$, Pacific J. Math. **232** (2007), no. 2, 257–262

[12] D. Calegari, *Stable commutator length is rational in free groups*, Jour. Amer. Math. Soc., available at arXiv:0802.1352

[13] D. Calegari, *Scl, sails and surgery*, preprint

[14] D. Calegari and K. Fujiwara, *Stable commutator length in word-hyperbolic groups*, Groups, Geometry, Dynamics, available at arXiv:math.GR/0611889

[15] D. Calegari and D. Zhuang, *Large Scale Geometry of Commutator Subgroups*, Algebr. Geom. Topol. **8** (2008), 2131–2146 (electronic)

[16] J. W. Cannon, W. J. Floyd and W. R. Parry, *Introductory notes on Richard Thompson's group*, L'Enseign. Math. (2) **42** (1996), no. 3-4, 215–256

[17] M. Culler, *Using surfaces to solve equations in free groups*, Topology, Vol. 20 (1981), 133–145

[18] M. Dehn, *The group of mapping classes*, in *Papers on group theory and topology*, pp. 135–206. translated by John Stillwell, Springer-Verlag, New York, 1987

[19] R. K. Dennis and L. N. Vaserstein, *Commutators in linear groups*, K-theory **2** (1989), 761–767

[20] J. Dydak, *A simple proof that pointed connected FANR-spaces are regular fundamental retracts of ANR's*, Bull. Polon. Acad. Sci. Ser. Sci. Math. Astronom. Phys. **25** (1977), 55–62

[21] J. Dydak, 1*-movable continua need not be pointed* 1*-movable*, Bull. Polon. Acad. Sci. Ser. Sci. Math. Astronom. Phys. **25** (1977), 485–488

[22] C. C. Edmunds, *On the endomorphism problem for free groups*, Comm. in Algebra **3** (1957), 1–20

[23] C. C. Edmunds, *On the endomorphism problem for free groups II*, Proc. London Math. Soc. **38** (3) (1979), 153–168

[24] V. A. Efremovic, *The proximity geometry of Riemannian manifolds*, Uspekhi Math. Nauk. **8** (1953), 189

[25] D. Eisenbud, U. Hirsch and W. Neumann *Transverse foliations of Seifert bundles and self-homoemorphisms of the circle*, Comment. Math. Helv. **56**, no. 4 (1981), 638–660

[26] H. Endo and D. Kotschick, *Bounded cohomology and non-uniform perfection of mapping class groups*, Invent. Math. **144** (2001), no. 1, 169–175

[27] D. Epstein and K. Fujiwara, *The second bounded cohomology of word-hyperbolic groups*, Topology **36** (1997), no. 6, 1275–1289

[28] B. Farb and D. Margalit, *A primer on mapping class groups*, preprint; available from the authors websites, 2007

[29] P. Freyd and A. Heller, *Splitting homotopy idempotents II*, J. Pure and Applied Algebra **89** (1993), 93–106

[30] K. Fujiwara, *The second bounded cohomology of a group acting on a Gromov-hyperbolic space*, Proc. London Math. Soc. (3) **76** (1998), no. 1, 70–94.

[31] K. Fujiwara, *The second bounded cohomology of an amalgamated free product of groups*, Trans. Amer. Math. Soc. **352** (2000), no. 3, 111–1129.

[32] K. Fujiwara and K. Whyte, *A note on spaces of asymptotic dimension one*, Algebr. Geom. Topol. **7** (2007), 1063–1070

[33] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geom. **18** (1983), no. 3, 445–503

[34] É. Ghys and V. Sergiescu, *Sur un groupe remarquable de difféomorphismes du cercle*, Comment. Math. Helv. **62** (1987), no. 2, 185–239

[35] M. Gromov, *Hyperbolic groups*, in *Essays in group theory* Math. Sci. Res. Inst. Publ., 8, *Springer, New York*, 1987; pp. 75–263

[36] M. Gromov, *Asymptotic invariants of infinite groups*, in *LMS lecture note series* **182**, (1993)

[37] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. No. 56 (1982), 5–99

[38] J. Groves, *Rewriting systems and homology of groups*, Lecture Notes in Mathematics 1456, Springer-Verlag, Berlin, 1990, 114–141

[39] J. L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. **121** (1985), 215–249

[40] J. L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986), 157–176

[41] W. J. Harvey, *Boundary structure of the modular group*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978) (Princeton, N.J.), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, pp. 245–251

[42] G. Higman, *On infinite simple permutation groups*, Publ. Math. Debrecen **3** (1954), 221–226

[43] N. V. Ivanov, *Complexes of curves and Teichmüller spaces*, Mat. Zametki **49** (1991), 158, 54–61

[44] N. V. Ivanov, *Automorphism of complexes of curves and of Teichmüller spaces*, Internat. Math. Res. Notices (1997), 651–666

[45] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, 1995

[46] R. Kirby, *The topology of* 4*-manifolds*, Lecture Notes in Mathematics 1374, Springer-Verlag, Berlin, 1989

[47] M. Korkmaz, *Stable commutator length of a Dehn twist*, Michigan Math. J. **52** (2004), no. 1, 23–31

[48] D. Kotschick, *Quasi-homomorphisms and stable lengths in mapping class groups*, Proc. Amer. Math. Soc. **132** (2004), no. 11, 3167–3175 (electronic)

[49] S. Lang, *Algebra*, revised third edition, Springer-Verlag, (2002)

[50] I. Liousse, *Nombre de rotation dans les groupes de Thompson généralisés, automorphismes*, preprint

[51] I. Liousse, *Rotation numbers in Thompson-Stein groups and applications*, Geometriae Dedicata, **131**, (2008), no. 1, 49–71

[52] J. Manning, *Geometry of pseudocharacters*, Geom. Topol. **9** (2005), 1147–1185

[53] H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103C149

[54] R. McKenzie and R. J. Thompson, *An elementary construction of unsolvable word problems in group theory*, Word Problems (W. W. Boone, F. B. Cannonito and R. C. Lyndon, eds.). Studies in Logic and the Foundations of Mathematics, vol. **71**. North-Holland, Amsterdam, (1973), 457–478

[55] J. Milnor, *A note on curvature and fundamental group*, J. Diff. Geom. **2** (1968), 1–7 [Collected Papers, vol. 1, pages 53 and 55–61]

[56] J. Powell, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. **68** (1978), no. 3, 347C350

[57] J. Stallings, *Group theory and three-dimensional manifolds*, Yale Mathematical Monographs, 4. Yale University Press, New Haven, 1971

[58] M. Stein, *Groups of piecewise linear homeomorphisms*, Trans. Amer. Math. Soc. **332** (1992), no. 2, 477–514

[59] A. S. Švarc, *A volume invariant of coverings* (in Russian), Dokl. Ak. Nauk. **105** (1955), 32–34

[60] D. Zhuang, *Irrational stable commutator length in finitely presented groups*, J. Modern Dynamics **2** (2008), no. 3, 499–507