ELEMENTARY SOLUTIONS FOR THE H_∞- GENERAL DISTANCE PROBLEM- EQUIVALENCE OF H_2 AND H_∞ OPTIMIZATION PROBLEMS

Thesis by

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To my parents, İsmet and Münevver Kavranoğlu

&

to my wife Gülser, and my son Muhammed İbrahim.

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Abstract

This thesis addresses the H_∞ optimal control theory. It is shown that SISO H_∞ optimal control problems are equivalent to weighted Wiener-Hopf optimization in the sense that there exists a weighting function such that the solution of the weighted H_2 optimization problem also solves the given H_∞ problem. The weight is identified as the maximum magnitude Hankel singular vector of a particular function in H_∞ constructed from the data of the problem at hand, and thus a state-space expression for it is obtained. An interpretation of the weight as the worst-case disturbance in an optimal disturbance rejection problem is discussed.

A simple approach to obtain all solutions for the Nehari extension problem for a given performance level γ is introduced. By a limit taking procedure we give a parameterization of all optimal solutions for the Nehari's problem.

Using an imbedding idea [12] , it is proven that four-block general distance problem can be treated as a one-block problem. Using this result an elementary method is introduced to find a parameterization for all solutions to the four-block problem for a performance level γ .

The set of optimal solutions for the four-block GDP is obtained by treating the problem as a one-block problem. Several possible kinds of optimality are identified and their solutions are obtained.

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Chapter 1

Introduction

The H_∞ optimization problem has attracted enormous attention in the 1980's after being introduced by Zames [34]. It has been shown that many frequency-domain, control-design problems can be meaningfully formulated as H_∞ optimization problems [4], [11], [27] . Examples of such problems include minimization of the sensitivity transfer function in a minimax sense and optimizing the robustness margins for unstructured uncertainty. Mathematically, the H_∞ optimal control problem is to minimize the weighted infinity norm of some closed loop transfer function or a combination of transfer functions over the set of controllers that satisfy the internal stability requirement. The set of stabilizing controllers is parameterized by Youla's Lemma [33]. Youla's Lemma provides a great simplification in the H_∞ optimization problem. Using Youla's parametrization and inner-outer factorizations, the H_∞ optimization problem is reduced to the General Distance Problem (GDP). In special cases, the general distance problem becomes the Nehari extension problem or the so-called one-block, general distance problem. To distinguish the general case from the special case, namely, from Nehari's extension problem, we call it the four-block general distance problem.

H_∞ optimization can be interpreted as a loop-shaping tool. In this sense, the controller is selected so as to shape the magnitude of certain closed-loop transfer functions. The infinity norm is a generalization of the magnitude of a complex number. H_2 optimal control or Wiener-Hopf design methods and their time-domain counterpart LQG have been proposed as loop-shaping techniques. The H_∞ optimal control makes a more precise tool than Wiener-Hopf or LQG since the closed-loop transfer function is optimized. Selecting weights in the H_∞ optimization problem is an important part of H_∞ optimal control that requires engineering judgement and experience.

Until recently, the only method available to solve the four-block general distance problem has been the one introduced by Doyle [5]. That method results in controllers with large orders compared to the plant order. A new method, recently introduced by Glover and Doyle [14], Doyle et al. [6] solves the problem in sub-optimal case and requires solving two Riccati equations for each γ level. An important property of this solution is that it gives controllers with the same degree as the plant. The solution we present in this thesis also requires solving two Riccati equations but the controllers we obtain have at least three times the degree of the plant. This is because state space realizations of the controllers are not minimal realizations. Further analysis is necessary to eliminate the nonminimal modes.

On the other hand, satisfactory methods were available to solve the one-block problem for the last few years (see for example [3], [4], [9], [12], [27]).

For the four-block GDP a complete solution for the optimal case was not available. We present the first complete solution to the problem in this thesis. It is a direct generalization of the one-block case and is obtained from the method we developed for the suboptimal case.

1.1 Contributions of the Thesis

There are four chief contributions of this thesis:

1. Proof of $H_2 - H_\infty$ Equivalence in the Single Input Single Output (SISO) case:

We proved that H_∞ optimization is equivalent to H_2 optimization, that is, Wiener-Hopf optimization, in the sense that there exists a weighting function, which is generically unique, such that weighted H_2 optimization results in the same compensator. The weighting function has an interesting interpretation as the worst-case signal for H_∞ optimization. This result means that the worst-case signals are not sinusoids, which do not belong to H_2, but that they are signals in H_2.

2. An elementary solution for Nehari's problem:

An elementary solution that is inspired by the $H_2 - H_\infty$ equivalence has been introduced for both optimal and suboptimal cases. This approach considerably simplifies the theory behind the one-block problem.

3. Equivalence of the four-block problem with the one-block problem:

We proved that the four-block problem can be transformed into an equivalent one-block problem by using a result by Glover [12] based on the Positive Real Lemma [1]. This fact reduces the difficulty involved in solving the four-block problem almost to the level of difficulty involved in the one-block problem. Using this result, we obtain a parameterization of all suboptimal solutions for the four-block GDP, which requires elementary mathematics.

4. Optimal solution for the four-block problem:

We present the first complete solution to the four-block GDP in the optimal case, using our idea of transforming the four-block problem into a one-block problem. This is one of the most useful results of the idea of transforming the one-block problem into a four-block problem.

1.2 Summary of the Thesis

Chapter 1. Introduction.

Chapter 2. Mathematical Background.

This chapter covers the notation we use, some information on normed spaces, linear time-invariant systems and stability theory. Also covered in this chapter is an explanation, in some detail, of the motivation for the H_∞ optimal control theory and basic theorems of H_∞. A statement of the Positive Real Lemma is also given.

Chapter 3. Equivalence of H_2 and H_∞ Optimization - The SISO Case.

An iterative procedure for the "optimal weight", which is inspired by Lawson's algorithm [19], is introduced. It is proven that this iteration converges and has essentially a unique limit. The limit of the algorithm is the optimal weight. Some connections with existing results are discussed.

Chapter 4. A Simple Solution to the H_∞ Optimization Problems: the One-Block Problem.

This chapter presents a new method to obtain a parameterization of all solutions for the one-block problem. First a parameterization of all solutions for the suboptimal problem is obtained, and then the solution for the optimal case is obtained by taking limit.

Chapter 5. A Simple Parameterization of All Solutions for the Four-Block Problem - The Suboptimal Case.

We prove in this chapter that the four-block problem is equivalent to the one-block problem. We transform the four-block problem into a one-block problem and solve the one-block problem. Finally, we identify a subset of the set of solutions to the one-block problem, which is a parameterization of all solutions to the four-block problem.

Chapter 6. A Simple Parameterization of All Optimal Solutions for the Four-Block GDP.

Optimal solutions of the four-block GDP are obtained using the approach of Chapter 5. Different types of optimality are discussed and solutions are given. Some examples are given to illustrate the different cases.

Chapter 7. Examples.

We give two examples to illustrate our method.

Chapter 8. Conclusion.

Chapter 2

Mathematical Background

In this chapter we give a review of the definitions and the results that are necessary to understand this thesis. Some of the good sources of references are [5], [9], [12].

2.1 Table of Symbols

R	the real numbers
$R^{n \times m}$	the set of $n \times m$ matrices with real elements
C	the complex numbers
C_+	the complex right half-plane
C_-	the complex left half-plane

$C^{n \times m}$	the set of $n \times m$ matrices with complex elements
A^T	transpose of a matrix A
A^*	complex conjugate transpose of matrix A
$trace(A)$	the sum of the diagonal elements of matrix A
$\lambda_i(A)$	the i'th eigenvalue of matrix A
$\sigma_i(A)$	the i'th singular value of matrix A
$\rho(A)$	the spectral radius of matrix A
$\|x\|$	Euclidean norm of vector x in R^n or C^n
$\|A\|$	induced Euclidean norm of matrix A in $R^{n \times m}$ or $C^{n \times m}$ $(\sup_{\|x\|=1} \|Ax\|)$
L_∞	space of functions of a complex variable that are bounded on the $j\omega$ axis
H_∞	space of functions of a complex variable that are analytic in C_+ and bounded on the $j\omega$ axis

H_∞^-	space of functions of a complex variable that are analytic in C_- and bounded on $j\omega$ axis
L_2	space of functions of a complex variable that are square integrable on the $j\omega$ axis
H_2	space of functions of a complex variable that are analytic in C_+ and square integrable on $j\omega$ axis
$H_2^\perp$	space of functions of a complex variable that are analytic in C_- and square integrable on $j\omega$ axis
$B_\gamma H_\infty$	the set of functions in H_∞ that satisfy a norm bound γ
R (as a prefix)	restricted to be rational (RH_∞, RH_2, etc.)
GDP	the General Distance Problem
SISO	Single Input Single Output
MIMO	Multiple Input Multiple Output
$\|\|G\|\|_\infty$	$\sup_{\omega\in R}\sigma_{max}(G(j\omega))$

$\|G\|_2$	$\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\,(G^*(j\omega)G(j\omega))\,d\omega\right]^{1/2}$
$F_l(J,Q)$	$J_{11} + J_{12}Q(I - J_{22}Q)^{-1}J_{21}$ (lower fractional transformation)
$F_u(J,Q)$	$J_{22} + J_{21}Q(I - J_{11}Q)^{-1}J_{12}$ (upper fractional transformation)
$G^{\sim}(s)$	$G^T(-s)$
σ_i	i'th Hankel singular value
Γ_G	the Hankel operator with symbol $G(s)$
$[G(s)]_+$	projection of $G(s) \in L_\infty$ to H_∞
$[G(s)]_-$	projection of $G(s) \in L_\infty$ to $H_2^\perp$
I_m	$m \times m$ identity matrix
$A^\dagger$	$\lim_{\epsilon \longrightarrow 0} \left(A^T A + \epsilon^2 I\right)^{-1} A^T$

2.2 Norms, Singular Values

For a vector in C^n or R^n, the Euclidean norm is defined by

$$\|x\| := < x, x >^{1/2} \tag{2.1}$$

where $< x, y >$, the inner product, is given by

$$< x, y > := x^* y. \tag{2.2}$$

For a matrix $A \in C^{m \times n}$ the induced 2-norm is given by

$$\|A\|_2 = \max_{\|x\| \leq 1} \|Ax\| \tag{2.3}$$

where $x \in C^n$. The singular values of a matrix $A \in C^{m \times n}$ are given by

$$\sigma_i = [\lambda_i(A^* A)]^{1/2} \tag{2.4}$$

where λ_i are the eigenvalues of $A^* A$ for $i = 1$ to $\min\{m, n\}$, and nonnegative squareroot is taken. It can be shown that

$$\|A\|_2 = \sigma_{max}(A). \tag{2.5}$$

The spectral radius of A is given by

$$\rho(A) = \max_i |\lambda_i(A)|. \tag{2.6}$$

For any $m \times n$ matrix A, there exists a singular value decomposition (SVD) given by

$$A = U \Sigma V^T \tag{2.7}$$

where Σ is an $m \times n$ matrix defined by

$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

and U and V are unitary matrices of dimension $m \times m$ and $n \times n$, respectively.

2.2.1 Time Domain Spaces; $L_2[-\infty,\infty]$, $L_2[0,\infty]$, $L_2[-\infty,0]$

Consider a signal $x(t)$ defined for all time, $-\infty < t < \infty$, and taking values in C^n. Consider the set of all $x(.)$ such that

$$\|x\|_2 := \left[\int_{-\infty}^{\infty} \|x(t)\|^2 dt\right]^{1/2} < \infty \tag{2.9}$$

where $\|.\|$ denotes the previously defined norm on C^n. The set of all such signals is the Lebesgue space $L_2[-\infty,\infty]$. This space is a Hilbert space with the inner product

$$< x, y > := \int_{-\infty}^{\infty} x^*(t)y(t)dt. \tag{2.10}$$

The set of all signals in $L_2[-\infty,\infty]$, which equal zero for almost all $t < 0$, is a closed subspace, denoted $L_2[0,\infty]$. Its orthogonal complement is denoted $L_2[-\infty,0]$.

2.2.2 Frequency Domain Spaces; L_2, H_2, $H_2^\perp$, L_∞, H_∞

Consider a function $G(j\omega)$ that is defined for all frequencies, $-\infty < \omega < \infty$, takes values in $C^{n\times m}$, and is square-integrable with respect to ω. The space of all such functions is denoted L_2 and is a Hilbert space under the inner product

$$< G, F > := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}[G^*(j\omega)F(j\omega)]\,d\omega. \tag{2.11}$$

The norm on L_2 will be denoted $\|G\|_2$.

H_2 is the space of all functions that are analytic in C_+, take values in $C^{n\times m}$, and satisfy the uniform square-integrability condition

$$\|G\|_2 := \left[\sup_{\epsilon>0} \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}[G^*(\epsilon + j\omega)G(\epsilon + j\omega)]\,d\omega\right]^{1/2} < \infty. \tag{2.12}$$

This makes H_2 a Banach space. It is customary to consider the limiting values of $G(s) \in H_2$ for $\epsilon \longrightarrow 0$. So henceforth we regard H_2 as a closed subspace of the Hilbert space L_2. The orthogonal complement complement $H_2^\perp$ of H_2 in L_2 is the space of functions $G(s)$ with the following properties: $G(s)$ is analytic in C_-, takes values in $C^{n\times m}$, and the supremum

$$\sup_{\epsilon<0}\int_{-\infty}^{\infty} \text{trace}[G^*(\epsilon + j\omega)G(\epsilon + j\omega)]\,d\omega \tag{2.13}$$

is finite. Again, we regard $H_2^\perp$ as a closed subspace of L_2. (More detailed explanation can be found in [5] or in [9].)

An $n \times m$ complex-valued matrix $F(j\omega)$ belongs to the space L_∞ if and only if $\sigma_{max}(F(j\omega))$ is essentially bounded (bounded except possibly on a set of measure zero). Then the L_∞ norm of F is defined to be

$$\|F\|_\infty := \text{ess}\ \sup_{\omega} \sigma_{max}(F(j\omega)). \tag{2.14}$$

This makes L_∞ a Banach space.

The space of H_∞ consists of functions $F(s)$ that are analytic in C_+, take value in $C^{n\times m}$, and are bounded in C_+ in the sense that

$$\sup_{s\in C_+} \sigma_{max}(F(s)) < \infty. \tag{2.15}$$

2.15 defines the H_∞ norm of F. We again consider the limit values of H_∞ as a closed subspace of the Banach space L_∞.

2.3 Linear Time-Invariant (LTI) Systems

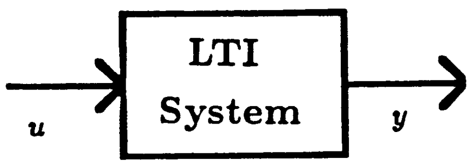
Throughout this thesis we work with linear time-invariant systems. One way of expressing a linear time-invariant system is the state-space representation:

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ x(0) &= x_0 \end{aligned}\right\} \tag{2.16}$$

where A, B, C, D are constant matrices. x is called "the state vector," u "the input vector," y "the output vector" and x_0 is "the initial state" of the system. Unless otherwise stated we take $x_0 = 0$ and we denote (2.16) by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2.17}$$

(2.17) also represents a linear mapping from u to y:



The representation (2.16) is not unique; for example, for any nonsingular matrix T there exists another representation of g:

$$\left.\begin{aligned} \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ y &= CTz + Du \end{aligned}\right\} \tag{2.18}$$

where the only difference between (2.16) and (2.18) is the state vector z; the input-output behavior of both systems is exactly the same. These two systems are called input-output equivalent systems.

The output, y, can be given in terms of u as follows:

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{2.19}$$

The degree of a system is the dimension of the state space, which is the same as the dimension of A. One natural question is: " Could we possibly find another representation with a lower dimensional A matrix such that we would have the same input-output behavior?" This question brings us to the concepts of controllability, observability and minimality of a representation [16]. $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a controllable realization iff there is no λ and $x \neq 0$ such that

$$x^T A = \lambda x^T \tag{2.20}$$

$$x^T B = 0 \tag{2.21}$$

are both satisfied (this is the PHB test for controllability [16]). If for some λ there exists $x \neq 0$ such that (2.20) and (2.21) are both satisfied, then λ is called an uncontrollable mode of the realization. If all of the uncontrollable modes of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ are stable (i.e., $Re[\lambda] < 0$), then the realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is called a stabilizable realization. Similarly, a quadruple $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is an observable realization iff there is no λ and $x \neq 0$ such that

$$Ax = \lambda x \tag{2.22}$$

$$Cx = 0 \tag{2.23}$$

are both satisfied. If for some λ there exists a $x \neq 0$ such that (2.22) and (2.23) are both satisfied, then λ is called an unobservable mode of the realization. If all of the unobservable modes of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ are stable, then the realization is called a detectable realization.

If a realization is both controllable and observable, then it is a minimal realization (i.e., a realization with a minimum number of states).

The transfer function of a system is given in terms of its realization as follows:

$$\begin{aligned} G(s) &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\ &= C(sI - A)^{-1}B + D. \end{aligned} \tag{2.24}$$

Remark 2.1 *In this thesis we use the terms "transfer function" and "transfer matrix," both meaning matrix-valued functions of a complex variable s.*

For an antistable system (i.e., all of the modes in C_+), the controllablility and the observability gramians are defined as follows:

$$L_c = \int_0^\infty e^{(-At)} BB^T e^{(-A^T t)} dt \tag{2.25}$$

$$L_o = \int_0^\infty e^{(-A^T t)} C^T C e^{(-At)} dt. \tag{2.26}$$

It is easy to verify that P and Q satisfy the following Lyapunov's equations:

$$AL_c + L_c A^T = BB^T \tag{2.27}$$

$$A^T L_o + L_o A = C^T C. \tag{2.28}$$

Lemma 2.1 *[12]*

If $Re[\lambda_i(A)] > 0 \ \forall i$, *then*

$L_c > 0$ *if and only if* (A, B) *is controllable,*

$L_o > 0$ *if and only if* (A, C) *is observable.*

For an antistable $G(s)$ the Hankel singular values of $G(s)$ are defined as

$$\sigma_i(G(s)) := \{\lambda_i(L_c L_o)\}^{1/2} \tag{2.29}$$

where by convention $\sigma_i(G(s)) \geq \sigma_{i+1}(G(s))$. We should note that the Hankel singular values of a transfer function are input-output invariant, i.e., independent of the realization.

A realization is called a balanced realization [21] if

$$L_c = L_o = \Sigma = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_n). \tag{2.30}$$

For an antistable transfer function $G(s)$, the largest Hankel singular value σ_1 is also called the Hankel norm of $G(s)$.

2.3.1 γ-Inner, γ-Co-Inner, γ-Anti-Inner, γ-Anti-Co-Inner Transfer Functions

Definition 2.1 *A stable transfer function $G(s)$ with no more columns than rows is γ-inner if $G^{\sim}(s)G(s) = \gamma^2 I$.*

Definition 2.2 *A stable transfer function $G(s)$ with no more rows than columns is γ-co-inner if $G(s)G^{\sim}(s) = \gamma^2 I$.*

Definition 2.3 *An antistable transfer function $G(s)$ is γ-anti-inner if $G(-s)$ is γ-inner.*

Definition 2.4 *An antistable transfer function $G(s)$ is γ-anti-co-inner if $G(-s)$ is γ-co-inner.*

Definition 2.5 *A square transfer function $G(s) \in RL_\infty$ is γ- all-pass if $G(s)G^{\sim}(s) = G^{\sim}(s)G(s) = \gamma^2 I$,*

The following lemma provides a state-space characterization of real, rational, inner transfer functions.

Lemma 2.2 *[5]*

An (anti) stable, rational transfer function $G(s)$ with a minimal realization $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is γ-(anti)-inner iff*

$$A^T L_o + L_o A = C^T C \tag{2.31}$$

$$D^T C - B^T L_o = 0 \tag{2.32}$$

$$D^T D = \gamma^2 I. \tag{2.33}$$

We have a dual lemma about γ-co-inner transfer functions.

Lemma 2.3 *[5]*

An (anti) stable, rational transfer function $G(s)$ with a minimal realization $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is γ-(anti)-co-inner iff*

$$A L_c + L_c A^T = B B^T \tag{2.34}$$

$$D B^T - C L_c = 0 \tag{2.35}$$

$$D D^T = \gamma^2 I. \tag{2.36}$$

If we use the fact that $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a minimal realization, we conclude the following:

1- $L_c < 0$ and $L_o < 0$ if $G(s)$ is γ-inner or γ-co-inner.

1- $L_c > 0$ and $L_o > 0$ if $G(s)$ is γ-anti-inner or γ-anti-co-inner.

We have a similar lemma for γ-all pass functions.

Lemma 2.4 *[12]*

A square transfer function $G(s) \in RL_\infty$ with a minimal realization $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

is γ-all pass if and only if

$$AL_c + L_cA^T = BB^T \tag{2.37}$$

$$A^TL_o + L_oA = C^TC \tag{2.38}$$

$$L_oL_c = \gamma^2 I \tag{2.39}$$

$$DB^T - CL_c = 0 \tag{2.40}$$

$$D^TC - B^TL_o = 0 \tag{2.41}$$

$$DD^T = D^TD = \gamma^2 I. \tag{2.42}$$

Proofs can be obtained by simple state-space manipulations and are given in [5] [12] (The notation γ-(anti)-inner and γ-(anti)-co-inner are new. When $\gamma = 1$ we simply say *(anti)-inner, (anti)-co-inner*).

Remark 2.2

If we multiply a function by an (anti-) inner, (anti-) co-inner function, the L_∞ norm and the L_2 norm do not increase.

Definition 2.6 *A stable transfer function is called outer if it has constant rank in C_+.*

Definition 2.7 *An antistable function is called anti-outer if it has constant rank in C_-.*

Remark 2.3 *An (anti)- outer function can have zeros on the $j\omega$- axis.*

Definition 2.8 *An inner-outer factorization of a function $G(s) \in RH_\infty$ is a factorization*

$$G(s) = G_i(s)G_o(s)$$

where $G_i(s)$ is inner and $G_0(s)$ is outer.

The following theorem is given in [9] and a full treatment of the problem can be found in [31].

Theorem 2.1 *Every transfer function in RH_∞ has an inner-outer factorization.*

The following corollary is an immediate result of Theorem 2.1.

Corollary 2.1 *Every transfer function in RH_∞^- has an anti-inner anti-outer factorization.*

We have introduced the above definitions to be able to perform inner-outer-like factorization for antistable functions, which we frequently use later in this thesis.

2.3.2 Rules of Manipulation in State Space

We summarize some of the basic state-space rules of manipulation [5], [9].

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI-A)^{-1}B + D$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right]$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{\sim} = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D \end{array}\right]$$

$$G_1(s)G_2(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A_1 & 0 & B_2 \\ B_1C_2 & A_2 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right]$$

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} A_1 & Q & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 - XB_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_1X + C_2 & D \end{array}\right] \tag{2.43}$$

where X solves

$$A_1X - XA_2 + Q = 0.$$

2.4 Linear Fractional Transformations

Consider the feedback system in Figure 2.1. $P(s)$ and $\Delta(s)$ are LTI transfer functions and $P(s)$ is partitioned as $P(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix}$. The transfer function from u to y is

$$F_u\left(P(s), \Delta(s)\right) := P_{22}(s) + P_{21}(s)\Delta(s)\left(I - P_{11}(s)\Delta(s)\right)^{-1} P_{12}(s) \tag{2.44}$$

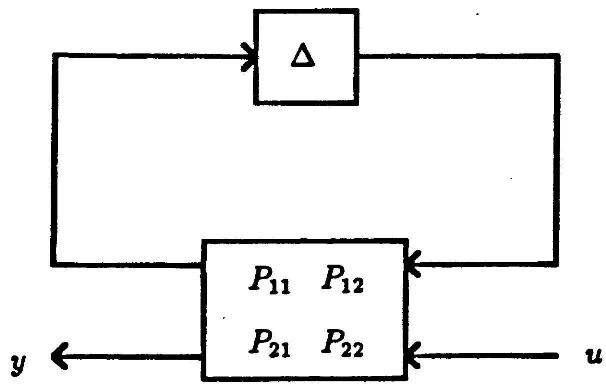


Figure 2.1: An upper linear fractional transformation.

where the subscript u stands for *upper*.

Similarly, consider Figure 2.2. The transfer function from u to y is

$$F_l(P(s), K(s)) := P_{11}(s) + P_{12}(s)K(s)\left(I - P_{22}(s)K(s)\right)^{-1} P_{21}(s) \tag{2.45}$$

where l stands for *lower*.

2.4.1 Interconnection of Linear Fractional Transformations

An important property of Linear Fractional Transformations, LFT, is that any interconnection of LFT's is again an LFT. Suppose $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$. Then Figure 2.3 represents the following algebraic relation:

$$F_l(P, F_l(J, Q)) = F_l(T, Q). \tag{2.46}$$

Similarly,

$$F_u(J, F_u(P, \Delta)) = F_u(T, \Delta) \tag{2.47}$$

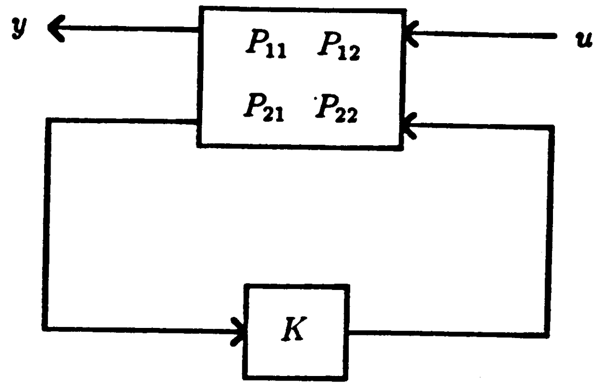


Figure 2.2: A lower linear fractional transformation.

where

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

$$= \begin{pmatrix} P_{11} + P_{12}J_{11}(I - P_{22}J_{11})^{-1} P_{21} & P_{12}(I - J_{11}P_{22})^{-1} J_{12} \\ J_{21}(I - P_{22}J_{11})^{-1} P_{21} & J_{22} + J_{21}P_{22}(I - J_{11}P_{22})^{-1} J_{12} \end{pmatrix}. \tag{2.48}$$

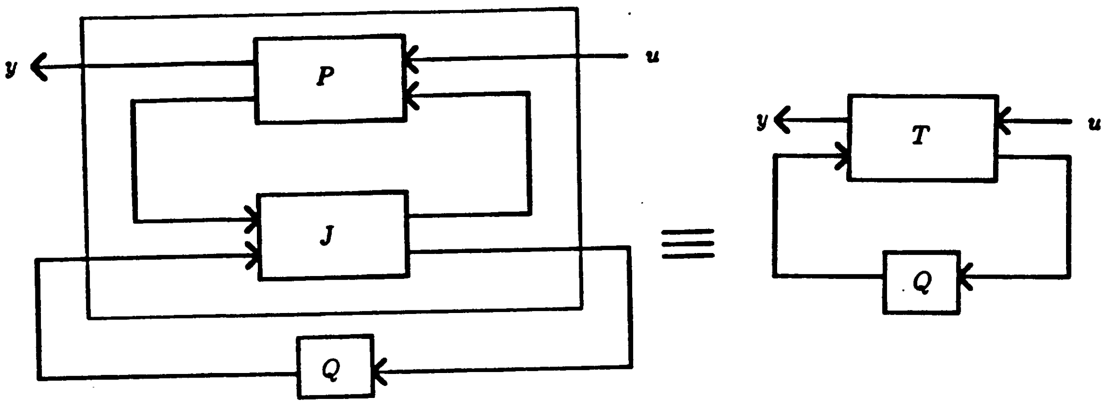


Figure 2.3: Interconnection of linear fractional transformations.

Given a unitary matrix U, then we have the following fact:

Lemma 2.5 *If $F(s) \in BH_\infty$, then*

$$K(s) = F_l(U, F(s)) \in BH_\infty. \tag{2.49}$$

Proof: First, we prove that $\|K\|_\infty \leq 1$ for all $F(s) \in BH_\infty$. We can represent (2.49) by the following equations:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = U \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.50}$$

$$v_2 = F(s)y_2. \tag{2.51}$$

Since U is unitary we have

$$\|v_1\|_2^2 + \|v_2\|_2^2 = \|y_1\|_2^2 + \|y_2\|_2^2. \tag{2.52}$$

Substitute (2.51) in (2.52) and use the fact that $\|Fy_2\|_2 \le \|F\|_\infty \|y_2\|_2$. We then have

$$\|v_1\|_2^2 + \|F\|_\infty^2 \|y_2\|_2^2 \ge \|y_1\|_2^2 + \|y_2\|_2^2. \tag{2.53}$$

From (2.53) we have

$$\|v_1\|_2^2 - \|y_1\|_2^2 \ge (1 - \|F\|_\infty^2)\|y_2\|_2^2 \ge 0, \text{ since } \|F\|_\infty \le 1. \tag{2.54}$$

We conclude that

$$\|y_1\|_2 \le \|v_1\|_2 \tag{2.55}$$

for all v_1. This implies that

$$\|F_l(U,F)\|_\infty := \sup_{v_1 \in H_2} \frac{\|y_1\|_2}{\|v_1\|_2} \le 1. \tag{2.56}$$

The fact that $K(s) \in H_\infty$ follows from the small gain theorem and from the fact that $K(s)$ does not have any poles on $j\omega$-axis, since $\|K\|_\infty \le 1$. ■

2.5 Internal Stability

In this section we briefly review the notion of internal stability [5], [22].

Consider the block diagram in Figure 2.2, which represents the two equations

$$\begin{pmatrix} e \\ y \end{pmatrix} = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}, \quad u = Ky. \tag{2.57}$$

The feedback system is *well-posed* if the transfer function from v to e exists and is proper.

Theorem 2.2 *[5]*

The feedback system in Figure 2.2 is well-posed if and only if $det(I + P_{22}(\infty)K(\infty)) \neq 0$.

Definition 2.9 *(Internal stability)*

Consider Figure 2.4 and the transfer function $T(s) : \begin{pmatrix} v_1^T & v_2^T \end{pmatrix} \longrightarrow \begin{pmatrix} e_1^T & e_2^T \end{pmatrix}$:

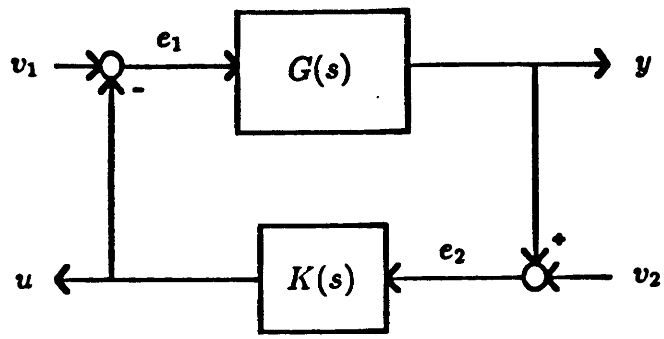


Figure 2.4: Internal stability.

$$T(s) = \begin{pmatrix} (I + P(s)K(s))^{-1} & -G(s)\,(I + K(s)P(s))^{-1} \\ K(s)\,(I - P(s)K(s))^{-1} & (I + K(s)P(s))^{-1} \end{pmatrix}; \tag{2.58}$$

the feedback system is called internally stable if and only if $T(s) \in H_\infty$.

Consider again Figure 2.2. We have the following theorem regarding internal stability of the system:

Theorem 2.3 *[5]*

Consider a minimal realization of the system $P(s)$*:*

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \tag{2.59}$$

There exists a feedback controller $K(s)$ *such that the feedback system in Figure 2.2 is internally stable if and only if* (A, B_2) *is stabilizable and* (C_2, A) *is detectable.*

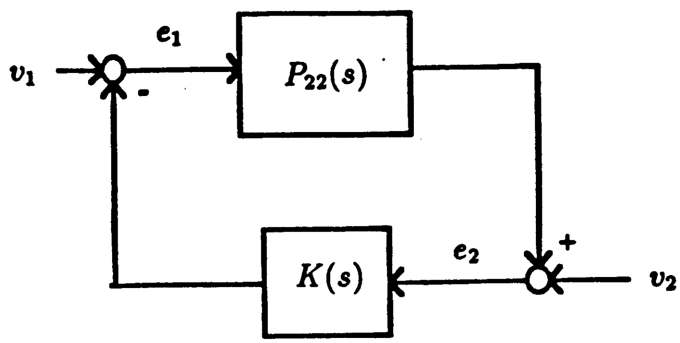


Figure 2.5: Internal stability of a linear fractional transformation.

Theorem 2.4 *[5]*

If (A, B_2) is stabilizable and (C_2, A) is detectable, then the feedback system in Figure 2.2 is internally stable if and only if the feedback system in Figure 2.5 is internally stable.

2.6 Parameterization of All Stabilizing Controllers

A pair of transfer functions $N(s)$, $M(s) \in H_\infty$ with the same number of columns is *right coprime* if there exists $X(s)$, $Y(s) \in H_\infty$ such that

$$X(s)M(s) + Y(s)N(s) = I. \tag{2.60}$$

Similarly, a pair of transfer functions $\hat{N}(s)$, $\hat{M}(s) \in H_\infty$ with the same number of rows is *left coprime* if there exists $\hat{X}(s)$, $\hat{Y}(s) \in H_\infty$ such that

$$\hat{M}(s)\hat{X}(s) + \hat{N}(s)\hat{Y}(s) = I. \tag{2.61}$$

Every real-rational transfer function $G(s)$ has a right/left coprime factorization; i.e., it can be expressed as $G(s) = N(s)M^{-1}(s) = \hat{M}^{-1}(s)\hat{N}(s)$ such that (2.60) and (2.61) are satisfied.

A parameterization of all stabilizing controllers, $K(s)$, for Figure 2.2 is given in the following theorem.

Theorem 2.5 *[5]*

Given right/left coprime factorizations of $P_{22}(s) = N(s)M^{-1}(s) = \hat{M}^{-1}(s)\hat{N}(s)$ *and a stabilizing controller* $K_0(s) := U_0(s)V_0^{-1}(s) = \hat{V}_0^{-1}(s)\hat{U}_0(s)$*, then the set of all proper controllers achieving internal stability for the feedback system (Figure 2.5) is given as follows:*

$$K(s) = (U_0(s) + M(s)Q(s))(V_0(s) + N(s)Q(s))^{-1} \tag{2.62}$$

$$= \left(\hat{V}_0(s) + Q(s)\hat{N}(s)\right)^{-1}\left(\hat{U}_0(s) + Q(s)\hat{M}(s)\right) \tag{2.63}$$

$$= F_l(J(s), Q(s)) \tag{2.64}$$

with

$$J(s) := \begin{pmatrix} K_0(s) & \hat{V}_0^{-1}(s) \\ V_0^{-1}(s) & -V_0^{-1}(s)N(s) \end{pmatrix} \tag{2.65}$$

where $Q(s) \in H_\infty$ *and* $det\left[\left(I + V_0^{-1}NQ\right)(\infty)\right] \neq 0$.

In the following theorem we give the parameterization of all stable transfer functions from u to y in terms of a free parameter $Q(s) \in H_\infty$.

Theorem 2.6 *[5]*

The set of all closed-loop transfer functions from u to y (Figure 2.2) achievable by an internally stabilizing proper controller is given by

$$H(s) := F_l(P(s), K(s)) = P_{11}(s) + P_{12}(s)K(s)(I - P_{22}(s)K(s))^{-1}, \tag{2.66}$$

where $K(s)$ is an internally stabilizing and proper controller

$$= T_{11}(s) + T_{12}(s)Q(s)T_{21}(s), \quad Q(s) \in H_\infty \text{ and } det(I + D_{22}Q(\infty)) \neq 0 \tag{2.67}$$

where

$$T_{11}(s) := P_{11}(s) + P_{12}(s)U_0(s)\hat{M}(s)P_{21}(s) \tag{2.68}$$

$$T_{12}(s) := P_{12}(s)M(s) \tag{2.69}$$

$$T_{21}(s) = \hat{M}(s)P_{21}(s). \tag{2.70}$$

In the following lemma we give a state-space solution for all the necessary parameters in Theorem 2.5 and Theorem 2.6.

Lemma 2.6 *[5], [24]*

Given F and H such that $A + B_2F$ and $A + HC_2$ are stable matrices, i.e., matrices with all the eigenvalues in C_-, then we have the following state-space solutions:

$$\begin{pmatrix} M(s) & U_0(s) \\ N(s) & V_0(s) \end{pmatrix} = \left[\begin{array}{c|cc} A + B_2F & B_2 & -H \\ \hline F & I & 0 \\ C_2 + D_{22}F & D_{22} & I \end{array}\right] \tag{2.71}$$

$$\begin{pmatrix} \hat{V}_0(s) & -\hat{U}_0(s) \\ -\hat{N}(s) & \hat{M}(s) \end{pmatrix} = \left[\begin{array}{c|cc} A + HC_2 & -(B_2 + HD_{22}) & H \\ \hline F & I & 0 \\ C_2 & -D_{22} & I \end{array}\right] \tag{2.72}$$

$$J(s) = \left[\begin{array}{c|cc} A + B_2F + HC_2 + HD_{22}F & -H & B_2 + HD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right] \tag{2.73}$$

Lemma 2.7 *[5]*

A state-space realization for $T(s)$ is

$$T(s) = \begin{pmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & 0 \end{pmatrix} \tag{2.74}$$

$$= \left[\begin{array}{cc|cc} A+B_2F & -HC_2 & -HD_{21} & B_2 \\ 0 & A+HC_2 & B_1+HD_{21} & 0 \\ \hline C_1+D_{12}F & C_1 & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right]. \tag{2.75}$$

Given a Hamiltonian matrix

$$H = \begin{pmatrix} A & -R \\ -Q & -A^T \end{pmatrix}, \tag{2.76}$$

where $R = R^T \geq 0$, $Q = Q^T$, the notation

$$X = Ric(H) \tag{2.77}$$

means that X is the unique stabilizing solution of the Algebraic Riccati Equation, (ARE), associated with H (i.e., $A - RX$ is a stable matrix) [5]. The ARE is given by

$$A^TX + XA - XRX + Q = 0. \tag{2.78}$$

If we pick F and H in a particular way, we obtain a very useful parameterization in which

$$N_{12}(s) := T_{12}(s)R_D^{-1/2}, \tag{2.79}$$

with $R_D := D_{12}^T D_{12}$, is inner and

$$N_{21}(s) := \hat{R}_D^{-1/2} T_{21}(s), \tag{2.80}$$

with $\hat{R}_D := D_{21}D_{21}^T$, is co-inner.

First, assume that $P_{12}(s)$ and $P_{21}(s)$ do not have transmission zeros on $j\omega$-axis (including ∞). This guarantees $R_D > 0$ and $\hat{R}_D > 0$. Define $D_\perp := (D_{12})_\perp$ and

$\hat{D}_\perp := (D_{21})_\perp$, where

$$D_\perp D_\perp^T = I - D_{12} R_D^{-1} D_{12}^T \tag{2.81}$$

$$\hat{D}_\perp^T \hat{D}_\perp = I - D_{21}^T \hat{R}_D^{-1} D_{21}. \tag{2.82}$$

If we pick

$$F = -R_D^{-1}(D_{12}^T C_1 + B_2^T X) \tag{2.83}$$

$$H = -(B_1 D_{21}^T + Y C_2^T)\hat{R}_D^{-1} \tag{2.84}$$

where

$$X = \text{Ric}\begin{bmatrix} A - B_2 R_D^{-1} D_{12}^T C_1 & -B_2 R_D^{-1} B_2^T \\ -C_1^T D_\perp D_\perp^T C_1 & -\left(A - B_2 R_D^{-1} D_{12}^T C_1\right)^T \end{bmatrix}, \tag{2.85}$$

$$Y = \text{Ric}\begin{bmatrix} \left(A - B_1 D_{21}^T \hat{R}_D^{-1} C_2\right)^T & -C_2^T \hat{R}_D^{-1} C_2 \\ -B_1^T \hat{D}_\perp^T \hat{D}_\perp B_1^T & -\left(A - B_1 D_{21}^T \hat{R}_D^{-1} C_2\right) \end{bmatrix}, \tag{2.86}$$

then $N_{12}(s)$ is inner and $N_{21}(s)$ is co-inner.

Finally, we have

$$\begin{aligned} F_l(P(s), K(s)) &= F_l(T(s), Q(s)) &(2.87)\\ &= T_{11}(s) + T_{12}(s)Q(s)T_{21}(s) &(2.88)\\ &= T_{11}(s) - N_{12}(s)\left(-R_D^{1/2} Q(s) \hat{R}_D^{1/2}\right) N_{21}(s) &(2.89)\\ &= T_{11}(s) - N_{12}(s)\hat{Q}(s)N_{21}(s) &(2.90)\end{aligned}$$

where

$$\hat{Q}(s) := -R_D^{1/2} Q(s) \hat{R}_D^{1/2} \in H_\infty. \tag{2.91}$$

Since both the $\|.\|_2$ and $\|.\|_\infty$ norms are unitarily invariant, we have the following equalities for any $\hat{Q}(s) \in H_\infty$ ($\alpha = 2$ or ∞):

$\|T_{11}(s) - N_{12}(s)\hat{Q}(s)N_{21}(s)\|_\alpha =$

$$= \left\| T_{11}(s) - \begin{pmatrix} N_{12}(s) & N_\perp(s) \end{pmatrix} \begin{pmatrix} \hat{Q}(s) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} N_{21}(s) \\ \hat{N}_\perp(s) \end{pmatrix} \right\|_\alpha$$

$$= \left\| \begin{pmatrix} N_{12}^\sim(s) \\ N_\perp^\sim(s) \end{pmatrix} T_{11}(s) \begin{pmatrix} N_{21}^\sim(s) & \hat{N}_\perp^\sim(s) \end{pmatrix} - \begin{pmatrix} \hat{Q}(s) & 0 \\ 0 & 0 \end{pmatrix} \right\|_\alpha$$

$$= \left\| \begin{matrix} R_{11}(s) - \hat{Q}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{matrix} \right\|_\alpha \tag{2.92}$$

where

$$R(s) = \begin{pmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix} = \begin{pmatrix} N_{12}^\sim(s) \\ N_\perp^\sim(s) \end{pmatrix} T_{11}(s) \begin{pmatrix} N_{21}^\sim(s) & \hat{N}_\perp^\sim(s) \end{pmatrix}. \tag{2.93}$$

$N_\perp(s)$ and $\hat{N}_\perp(s)$ are matrices such that $\begin{pmatrix} N_{12}(s) & N_\perp(s) \end{pmatrix}$ and $\begin{pmatrix} N_{21}(s) \\ \hat{N}_\perp(s) \end{pmatrix}$ are square and inner matrices.

The following theorem provides a state-space realization of $R(s)$ in terms of the original plant parameters. $R(s)$ is completely unstable; i.e., all the poles in the open right half plane.

Theorem 2.7 *[5]*

$$R(s) = R_1(s)R_2(s) \tag{2.94}$$

where

$$R_1(s) = \left[\begin{array}{c|cc} -(A + B_2F)^T & (C_1 + D_{12}F)^T & -XH \\ \hline -(B_2R_D^{-1/2})^T & (D_{12}R_D^{-1/2})^T & 0 \\ -D_\perp^T C_1 X^\dagger & D_\perp^T & 0 \end{array} \right] \tag{2.95}$$

$$R_2(s) = \left[\begin{array}{c|cc} -(A + HC_2)^T & -(\hat{R}_D^{-1/2}C_2)^T & Y^\dagger B_1\hat{D}_\perp^T \\ \hline C_1Y + D_{11}(B_1 + HD_{21})^T & D_{11}(\hat{R}_D^{-1/2}D_{21})^T & D_{11}\hat{D}_\perp^T \\ 0 & \hat{R}_D^{1/2} & 0 \end{array}\right]. \tag{2.96}$$

Remark 2.4 *If we assume $D_{11} = 0$ then it follows Theorem 2.7 that the "D" term of $R(s) = R_1(s)R_2(s)$ is equal to zero. Safonov and Limebeer [26] have shown that we can always achieve $D_{11} = 0$ (and more) via "loop shifting". Therefore, without loss of generality, we can assume $R(\infty) = 0$.*

2.7 Some Standard Feedback Control Problems and H_∞ Optimal Control

2.7.1 Nominal Stability

Consider the block diagram in Figure 2.6. We generally think of $P(s)$ as the nominal, i.e., unperturbed plant, while Δ represents the allowable perturbation or uncertainties in the plant. The word *plant* is used in a general fashion here, since it might as well include the feedback compensator. The objective of this analysis is to determine the effects of model uncertainity on the closed-loop feedback system after a compensator has been designed and connected to the open loop plant, so $P(s)$ will generally describe the nominal closed-loop system. Δ is assumed to be stable, $\Delta \in RH_\infty$, and norm-bounded by 1. The Δ generally has a block diagonal structure with many blocks. If the actual plant uncertainity $\hat{\Delta}$ is not equal in magnitude to 1 at all frequencies, we simply construct a frequency-dependent weighting function $W(s)$ such that $\sigma_{\max}(W(j\omega)\Delta) \leq 1$ for all frequencies, and absorb $W(s)$ into the nominal plant.

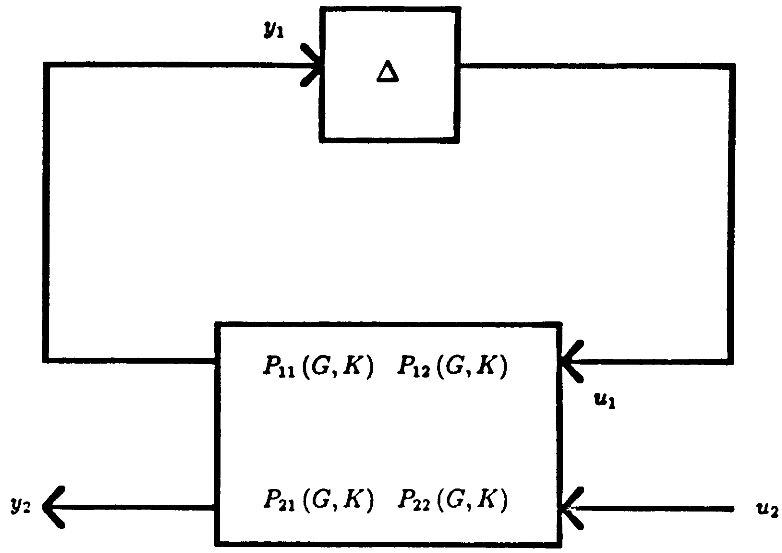


Figure 2.6: Generic Control System with Uncertainty.

If Δ has a block diagonal structure, assuming it as a full block one can lead to conservative results. In order to take advantage of the block diagonal structure, the structured singular value analysis [5] needs to be used.

The input-output behavior of the perturbed system with a particular perturbation Δ is simply $F_u(P(s), \Delta)$. To determine whether the nominal system is stable, we can check the stability of $F_u(P(s), 0)$. But $F_u(P(s), 0) = P_{22}(s)$; therefore,

$$\text{the closed-loop system is nominally stable} \Longleftrightarrow P_{22}(s) \text{ is stable.} \tag{2.97}$$

In Section 2.6 we explained how to find all controllers that internally stabilize a given nominal plant.

2.7.2 Robust Stability

A much more interesting problem is whether the feedback system in Figure 2.6 remains stable for all $\Delta \in H_\infty$ with $\|\Delta\|_\infty \leq 1$. The answer is the following: The feedback system in Figure 2.6 is stable for all $\Delta \in BH_\infty$ if and only if

$$\|P_{11}\|_\infty < 1. \tag{2.98}$$

In the synthesis of a robustly stabilizing controller we need to find a controller, $K(s)$, which nominally stabilizes the system and satisfies (2.98). The system is called robustly stabilizable if and only if

$$\min_{K(s),\ \text{stabilizing}} \|P_{11}\|_\infty < 1. \tag{2.99}$$

A $K(s)$ that achieves the minimum in (2.99) is called a maximally robust controller [13].

We observe that finding the optimally stabilizing controller is an H_∞ optimization problem.

2.7.3 Nominal Performance Problem

Good performance for a feedback regulator usually means that the change in the regulated output is small for large changes in the system input (the sensitivity minimization). In other words, the performance is good if the closed-loop transfer function is small at all frequencies. Many control problems can be stated in this way by appropriately defining the input and the output of the system (tracking, etc.) and by adding appropriate weights reflecting our goals for the design, which requires engineering judgment. In this thesis we do not deal with the problem of selecting the weights; we assume that the weights have been selected and the problem has been brought into the standard framework. In the nominal performance problem, we need to find a controller, $K(s)$, which internally stabilizes the system and satisfies

$$\|M\|_\infty < 1 \tag{2.100}$$

where $M(s)$ is an appropriately defined transfer function. We define the standard nominal performance problem in terms of Figure 2.6. In the nominal case (i.e., $\Delta = 0$) we have nominal performance if

$$\|P_{22}(K)\|_\infty < 1. \tag{2.101}$$

The H_∞ optimal control problem is to to find a controller, $K(s)$, to minimize $\|P_{22}(K)\|_\infty$; i.e.,

$$\min_{K(s),\ \text{stabilizing}} \|P_{22}(K)\|_\infty. \tag{2.102}$$

The robust performance problem is defined as follows: Find a stabilizing controller, $K(s)$, such that we have satisfactory performance for all possible Δ [5].

2.8 Hankel Operators, Nehari's Theorem

We define the Hankel operator in discrete time for convenience. T represents the unit circle on the complex plane.

Assume that $G(z)$ is bounded on the unit circle and has the power series expansion

$$\sum_{i=-\infty}^{\infty} G_i z^i. \tag{2.103}$$

We have the following definitions:

Definition 2.10 *(Multiplicative Laurent Operator)*

The multiplicative (Laurent) operator Λ_G generated by $G(z)$ is defined as

$$\Lambda_G : L_2(T) \longrightarrow L_2(T) \tag{2.104}$$

$$f \longrightarrow \Lambda_G f = Gf. \tag{2.105}$$

Definition 2.11 *(Hankel Operator)*

The Hankel operator Γ_G generated by $G(z)$, is defined as

$$\Gamma_G : H_2(T) \longrightarrow H_2^{\perp}(T) \tag{2.106}$$

$$f \longrightarrow \Gamma_G f = (P_{H_2^{\perp}} \Lambda_G) f = [Gf]_- . \tag{2.107}$$

A matrix representation of Λ_G is

$$H = \begin{pmatrix} G_{-1} & G_{-2} & G_{-3} & \cdot & \cdot & \cdot \\ G_{-2} & G_{-3} & G_{-4} & \cdot & \cdot & \cdot \\ G_{-3} & G_{-4} & G_{-5} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{2.108}$$

which is an infinite Hankel matrix.

It can be shown that [12] if $G(z)$ is a rational transfer function, then

$$rank(H) = \text{ McMillan degree of } G(z).$$

The following theorem is one of the fundamental theorems of H_∞ optimization.

Theorem 2.8 *[5], [9], [23] (Nehari's Theorem)*

Consider the following minimization problem,

$$\gamma_0 := \min_{Q(z) \in H_\infty(T)} \|G - Q\|_\infty . \tag{2.109}$$

Then

$$\gamma_0 = \|\Gamma_G\| \tag{2.110}$$

and the minimum is achieved by some $Q(z) \in H_\infty(T)$.

Remark 2.5 *If we use the bilinear transformation*

$$z = \frac{s-1}{s+1}$$

and define

$$F(s) := G\left(\frac{s-1}{s+1}\right),$$

then $G(z)$ and $F(s)$ have the same McMillan degree and the same Hankel singular values [12]; therefore, solving the H_∞ optimization problem in discrete time or continuous time does not make any difference. In this thesis, we work with continuous time systems.

Remark 2.6 *If $G(s) \in RH_\infty^-$, then $\|\Gamma_G\| = \rho^{1/2}(L_o L_c)$.*

2.9 Positive Real Functions, Spectral Factorization

In this section we give some basic facts about the Positive Real Lemma. Reference [1] is an excellent source for more details.

Definition 2.12 *An $m \times m$ matrix $Z(s)$ of real rational functions is positive real if*

1) All elements of $Z(s)$ are analytic in C_+.

2) $Z(s) + Z^T(\overline{s}) \geq 0$ in C_+.

Lemma 2.8 *(Youla's Spectral Factorization Lemma) [32] Let $Z(s)$ be an $m \times m$ positive real rational matrix, with no elements of $Z(s)$ possessing a pure imaginary pole. Then there exists an $r \times m$ matrix $W(.)$ of real rational functions of s satisfying*

$$Z(s) + Z^T(-s) = W^T(-s)W(s) \tag{2.111}$$

where r is the normal rank of $Z(s) + Z^T(-s)$, i.e., the rank almost everywhere. Furthermore, $W(s)$ has no elements with a pole in $Re(s) \geq 0$, $W(s)$ has constant rank in C_+, *and $W(s)$ is unique within left multiplication by an arbitrary, real, constant orthogonal matrix.*

Lemma 2.9 *(Positive Real Lemma) [1]*

Let $Z(.)$ be an $m \times m$ matrix of real rational functions of a complex variable s, with $Z(\infty) < \infty$. Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a realization of $Z(s)$ with (A, B) controllable. Then $Z(s)$ is positive real if and only if there exist real matrices P, L, W_0, with P positive semi-definite, symmetric, such that

$$A^T P + PA = -LL^T \tag{2.112}$$

$$PB = C^T - LW_0 \tag{2.113}$$

$$W_0^T W_0 = D + D^T. \tag{2.114}$$

(The number of rows of W_0 and number of columns of L are unspecified, while all other dimensions are automatically fixed.)

Lemma 2.10 *Existence of a Spectral Factor [1]*

Let $Z(s)$ be a positive real matrix of rational functions with $Z(\infty) < \infty$. Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a realization of $Z(s)$ with (A, B) controllable, and assume that all conditions in the statement of the positive real lemma hold. Then the transfer matrix

$$W(s) = \left[\begin{array}{c|c} A & B \\ \hline L^T & W_0 \end{array}\right] \tag{2.115}$$

is a spectral factor of $Z^T(-s) + Z(s)$ in the sense that

$$Z(s) + Z^T(-s) = W^T(-s) W(s).$$

Lemma 2.11 *(A Special Solution of the Positive Real Lemma Equations) [1]*

Let $Z(s)$ be a positive real matrix of rational functions of s, with $Z(\infty) < \infty$. Suppose that $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is a realization of $Z(s)$ with (A, B) controllable, and $R := D + D^T$ a nonsingular matrix. Then a solution to the spectral factorization problem*

$$Z(s) + Z^T(-s) = W^T(-s)W(s)$$

is given as follows:

$$W_0 = VR^{1/2} \tag{2.116}$$

$$L = \left(C^T + PB\right) R^{-1/2}V^T \tag{2.117}$$

where P is given by

$$P = Ric \begin{bmatrix} A - BR^{-1}C & -BR^{-1}B^T \\ -C^T R^{-1}C & -(A - BR^{-1}C)^T \end{bmatrix}. \tag{2.118}$$

If R is not invertible we have the following lemma to solve the Positive Real Equations, (2.112)- (2.114).

Lemma 2.12 *(A solution for the positive real equations when R is a singular matrix)[2]*
Let $Z(s)$ be a positive real matrix of rational functions of s, with $Z(\infty) < \infty$. Suppose that $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is a realization of $Z(s)$ with (A, B) controllable. Then a solution to the spectral factorization problem*

$$Z(s) + Z^T(-s) = W^T(-s)W(s) \tag{2.119}$$

is given by the following algorithm:

1) Transform the continuous time positive real function $Z(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *into a discrete time positive real function* $Z_d(z) = \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$ *via the bilinear transformation* $s = [\alpha(z-1)]/(z+1)$ *for a positive constant* α*. This results in*

$$A_d = (\alpha I - A)^{-1}(\alpha I + A) \tag{2.120}$$

$$B_d = \sqrt{2\alpha}(\alpha I - A)^{-1}B \tag{2.121}$$

$$C_d^T = \sqrt{2\alpha}(\alpha I - A^T)^{-1}C^T \tag{2.122}$$

$$D_d = D + C(\alpha I - A)^{-1}B. \tag{2.123}$$

2) Find

$$\Phi := \lim_{n \longrightarrow \infty} \Phi(n) \tag{2.124}$$

where $\Phi(n)$ *is determined recursively by*

$$\Phi(i+1) = A_d^T\Phi(i)A_d - [A_d^T\Phi(i)B_d + C_d^T][B_d^T\Phi(i)B_d + D_d + D_d^T]^{\dagger}[A_d^T\Phi(i)B + C_d^T]^T \tag{2.125}$$

initialized by $\Phi(0) = 0$*.*

3) Calculate N *from*

$$N^TN = B_d^T\Phi B_d + D_d + D_d^T. \tag{2.126}$$

4) Then a solution for the spectral factorization problem (2.119) is given as follows:

$$W(s) = \left[\begin{array}{c|c} A & B \\ \hline L^T & W_0 \end{array}\right] \tag{2.127}$$

where

$$L = \frac{1}{\sqrt{2\alpha}}(\alpha I - A)(A_d^T\Phi B_d + C_d^T)(B_d^T\Phi B_d + D_d + D_d^T)^{\dagger}N^T \tag{2.128}$$

$$W_0 = N - \frac{1}{\sqrt{2\alpha}}N(B_d^T\Phi B_d + D_d + D_d^T)^{\dagger}(A_d^T\Phi B_d + C_d^T)^T B. \tag{2.129}$$

Remark 2.7

1) It is not neccesary to have (C, A) observable in Lemma 2.12. If (C, A) is observable then Φ is also invertible.

2) Lemma 2.12 is valid for both R singular and nonsingular.

Next, we prove the following important theorem.

Theorem 2.9 *Given any $G(s) \in B_\gamma RH_\infty$, then the spectral factorization problem*

$$Z_O^\sim(s) Z_O(s) = \gamma^2 I - G^\sim(s) G(s) \tag{2.130}$$

always has a solution. The solution is given as follows:

$$Z_O(s) = \left[\begin{array}{c|c} A & B \\ \hline L^T & W_0 \end{array}\right] \tag{2.131}$$

where L and W_0 satisfy

$$W_0^T W_0 = \gamma^2 I - D^T D \tag{2.132}$$

$$A^T P_1 + P_1 A = -LL^T \tag{2.133}$$

$$P_1 B = L_o B - C^T D - L W_0 \tag{2.134}$$

with

$$A^T L_o + L_o A = C^T C.$$

Proof: Let us take a realization $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with (A, B) controllable and A a matrix with all the eigenvalues in C_-. Then

$$\gamma^2 I - G^\sim(s) G(s) = Z(s) + Z^\sim(s) \tag{2.135}$$

where

$$Z(s) = \left[\begin{array}{c|c} A & B \\ \hline B^T L_o - D^T C & \left(\gamma^2 I - D^T D\right)/2 \end{array}\right]. \tag{2.136}$$

$Z(s)$ is obviously a positive real function [1]. From Positive Real Lemma we obtain the equations (2.132)- (2.134). ■

In case W_0 in (2.132) is invertible the spectral factor, $Z_O(s)$, is given by the following lemma:

Lemma 2.13 *[1]*

If W_0 is invertible then the L that corresponds to a spectral factor is obtained from

$$L = -(PB + C^T D)W_0^{-1} \tag{2.137}$$

where

$$P = Ric\left[\begin{array}{cc} A - B(W_0^T W_0)^{-1} D^T C & -B(W_0^T W_0)^{-1} B^T \\ -C^T[I + D(W_0^T W_0)^{-1} D^T]C & -\left(A - B(W_0^T W_0)^{-1} D^T C\right)^T \end{array}\right]. \tag{2.138}$$

The following remark provides a method to solve the spectral factorization problem when W_0 is singular.

Remark 2.8 *If W_0 in (2.132) is not invertible we obtain the solution via Lemma 2.12. The calculation goes as follows:*

1) Given $G(s) \in B_\gamma H_\infty$ with a controllable realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, we first calculate $Z(s)$ as given by (2.136).

2) Apply Lemma 2.12 to $Z(s)$ (i.e., replace C by $B^T L_o - D^T C$) to get the solution.

Chapter 3

Equivalence of H_2 and H_∞ Optimization- The SISO Case

Consider the scalar H_∞ optimization problem

$$\min_{q(s)\in RH_\infty} \|r-q\|_\infty \tag{3.1}$$

where $r(s)$ is a given scalar function in RH_∞^-. We have indicated in Section 2.7 that many interesting control problems can be reduced to problem (3.1). In the following paragraphs we give a short review of the solution for the scalar case, which can be found in [9].

First assume that we are given a minimal realization for $r(s) \in RH_2^\perp$

$$r(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]. \tag{3.2}$$

(Without loss of generality we can assume that $r(s)$ does not have any factor in RH_∞; if it has any such factor, we can add it to $q(s)$.)

Theorem 3.1 *[9]*

The following algorithm provides the unique solution, $q(s)$, for problem (3.1).

1) Solve the equations

$$AL_c + L_cA^T = BB^T \tag{3.3}$$

$$A^T L_o + L_o A = C^T C. \tag{3.4}$$

2) Find the maximum eigenvalue σ_1^2 of $L_c L_o$ and the corresponding eigenvector w.

3) Define

$$f(s) = \left[\begin{array}{c|c} A & w \\ \hline C & 0 \end{array}\right] \tag{3.5}$$

$$g(s) = \left[\begin{array}{c|c} -A^T & \sigma_1^{-1} L_o w \\ \hline B^T & 0 \end{array}\right]. \tag{3.6}$$

4) Calculate

$$q(s) = r(s) - \sigma_1 f(s)/g(s). \tag{3.7}$$

Note that $(f(s), g(s))$ is called a Schmidt pair for the Hankel operator with symbol $r(s)$, and we have the following equations [12]

$$\Gamma_r g = \sigma_1 f \tag{3.8}$$

$$\Gamma_r^* f = \sigma_1 g \tag{3.9}$$

$$\Gamma_r^* \Gamma_r g = \sigma_1^2 g. \tag{3.10}$$

From Nehari's theorem we know that the minimum achievable error in (3.1) is $\|\Gamma_r\| = \sigma_1$. The following theorem gives an important property of the solution:

Theorem 3.2 *[9]*

The solution of problem (3.1), $q(s)$, is unique and for this unique solution the error $E(s) = r(s) - q(s)$ is σ_1 all-pass.

So much for the review of problem (3.1); in the next section we calculate the weight that achieves H_2- H_∞ equivalence.

3.1 Solving for the Optimal Weight

We now consider the weighted H_2 optimization problem

$$\min_{q(s)\in RH_\infty} \|W\,[r-q]\,\|_2 \tag{3.11}$$

where $W(s) \in RL_\infty$ and $r(s) \in RH_2^\perp$ are given functions. Problem (3.11) has a very simple solution since L_2 is a Hilbert space; namely, we obtain the solution by projection. The calculation goes as follows:

$$\begin{aligned}
&\|W(s)\,[r(s)-q(s)]\,\|_2^2 = \|W_o(s)\,[r(s)-q(s)]\,\|_2^2 = \\
&= \|[W_o(s)r(s)]_- + [W_o(s)r(s)]_+ - W_o(s)q(s)\|_2^2 \\
&= \|[W_o(s)r(s)]_-\|_2^2 + \|[W_o(s)r(s)]_+ - W_o(s)q(s)\|_2^2.
\end{aligned} \tag{3.12}$$

Therefore,

$$\min_{q(s)\in RH_\infty} \|W(s)[r(s)-q(s)]\|_2 \geq \|[W_o(s)r(s)]_-\|_2 \tag{3.13}$$

and the lower bound is achieved in (3.13) if we pick

$$[W_o(s)r(s)]_+ = W_o(s)q(s) \tag{3.14}$$

or

$$q(s) = [W_o(s)r(s)]_+ \,/W_o(s) \tag{3.15}$$

where $W_o(s)$ is the outer, i.e., the stable and minimum phase, part of $W(s)$. Note that every $W(s) \in RL_\infty$ has an all-pass outer factorization

$$\begin{aligned}
W(s) &= \frac{n_-(s)n_+(s)}{d_-(s)d_+(s)} = \\
&= \left(\frac{n_-(s)}{n_-^\sim(s)}\frac{d_-^\sim(s)}{d_-(s)}\right)\left(\frac{n_-^\sim(s)}{d_-^\sim(s)}\frac{n_+(s)}{d_+(s)}\right) = \\
&= W_a(s)W_o(s)
\end{aligned}$$

where subscript (-) indicates a polynomial with zeros in C_+ and the subscript (+) indicates a polynomial in C_-, i.e., a Hurwitz polynomial. $W_a(s)$ is the all-pass part of $W(s)$ and $W_o(s)$ is the outer part of $W(s)$ [10].

Note that the $\|.\|_\infty$ and $\|.\|_2$ are invariant under multiplication by an all-pass function.

In this chapter we show that given an H_∞ optimization problem (3.1), there exists a unique weight $W(s)$ such that the solution of the weighted H_2 problem (3.11) also solves problem (3.1) , and furthermore, we calculate the optimal weight (in [28] we treated the same problem with a different approach).

The motivation for the H_2- H_∞ equivalence comes from Lawson's algorithm [7], [19], [20] which basically states that: in a finite set Z that consists of N distinct points, there exist a sequence of best weighted least-squares approximations, which, under suitable conditions, converge to the best Chebysev approximation to f on Z.

We generalize the idea behind Lawson's algorithm to our case. We propose the following algorithm for the solution of the problem.

1) Pick any initial outer weight $W_0(s) \in RH_2$.

2) Update the weight according to the rule

$$W_{k+1}(s) = [W_k(s)r(s)]_-^{\sim} . \tag{3.16}$$

3) <u>If</u> $W_{k+1}(s)$ *is outer, <u>then</u> continue <u>else</u> goto Step 2.*

4) <u>If</u> $W_{k+1}(s) = \lambda W_k(s)$ *<u>then</u> the optimal weight is* $W_k(s)$ *and the optimal solution is*

$$q(s) = [W(s)r(s)]_+ / W(s) \tag{3.17}$$

<u>else</u> goto step 2

<u>end</u>.

Some Remarks about the algorithm

1) If we multiply $W_k(s)$ by a scalar, it does not affect the solution, $q(s)$. In a numerical procedure we need to do some scaling in Step 2.

2) In some cases, in the initial steps of the algorithm we obtain weights that have right half-plane zeros; this is the reason why we have Step 3, so that we can avoid taking the outer part in Step 4.

3) The algorithm always converges, as we prove, to a generically unique limit, which is an outer weight.

4) The denominator polynomial of $W_k(s)$ is completely determined after the first iteration; namely, it is the mirror image of the denominator polynomial of $r(s)$. We can see this fact from Step 2 as follows: Pick the initial weight $W_0(s) = \frac{n_0(s)}{d_0(s)}$, where $n_0(s)$ and $d_0(s)$ are Hurwitz polynomials. We also have $r(s) = \frac{n_r(s)}{d_r(s)}$, where $d_r^{\sim}(s)$ is a Hurwitz polynomial. Then calculate $[W_0(s)r(s)]_- = \left[\frac{n_0(s)n_r(s)}{d_0(s)d_r(s)}\right]_- = \frac{\alpha(s)}{d_r(s)}$, where $\alpha(s)$ is a polynomial with a degree less than the degree of $d_r(s)$. Then we have

$$W_1(s) = [W_0(s)r(s)]_-^{\sim} = \frac{\alpha(-s)}{d_r(-s)}. \tag{3.18}$$

From (3.18) the proof of the claim is apparent.

From this observation we conclude that we need only to determine the numerator polynomial of the weight; i.e., the iteration is performed over the numerator coefficients after the first iteration. If the degree of $d_r(s) = n$, then we need to determine only n parameters to find out what $W(s)$ is. We use this observation and make an educated guess for the weight in state-space representation of $W(s)$ and we pick

$$W_k(s) = \left[\begin{array}{c|c} -A & B \\ \hline w_k^T & 0 \end{array}\right] \tag{3.19}$$

where w_k is a column vector that needs to be determined in order to fully determine $W(s)$, the optimal weight. w_k has n parameters, the same as the degree of freedom left in the optimal weight after we have fixed the denominator polynomial. Also note that all of the poles of (3.19) are at the mirror images of the poles of $r(s)$.

5) For the limiting weight (later we will call it the optimal weight), the error $E(s) = r(s) - q(s)$ is λ-all-pass. Namely,

$$E(s) = r(s) - q(s) = \frac{[W(s)r(s)]_-}{W(s)} = \frac{\lambda W^\sim(s)}{W(s)}, \tag{3.20}$$

which follows from Step 4 of the algorithm. The error is obviously λ-all-pass (for scalar functions the necessary and sufficient condition for a function to be all-pass is that the numerator and the denominator polynomials must be mirror images of each other.

3.2 The Limit of the Algorithm

We start with Step 4 of the algorithm. Assuming that we have a w_k such that $W_k(s)$, which is given by (3.19), has all the zeros in C_-.

We now implement the algorithm; the calculation is given as follows:

$$\begin{aligned} W_k(s)r(s) &= \left[\begin{array}{c|c} -A & B \\ \hline w_k^T & 0 \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -A & BC & 0 \\ 0 & A & B \\ \hline \omega_k^T & 0 & 0 \end{array}\right] \end{aligned}$$

$$= \left[\begin{array}{c|c} -A & LB \\ \hline w_k^T & 0 \end{array}\right] + \left[\begin{array}{c|c} A & B \\ \hline -w_k^T L & 0 \end{array}\right]$$

then we have

$$[W_k(s)r(s)]_- = \left[\begin{array}{c|c} A & B \\ \hline -w_k^T L & 0 \end{array}\right] \tag{3.21}$$

and from (3.21) we have

$$W_{k+1}(s) = [W_k(s)r(s)]_-^\sim = \left[\begin{array}{c|c} -A & B \\ \hline w_k^T L & 0 \end{array}\right] \tag{3.22}$$

where L solves

$$AL + LA + BC = 0. \tag{3.23}$$

From (3.23) we obtain

$$w_{k+1}^T = w_k^T L. \tag{3.24}$$

If we repeat the iteration l times we get

$$w_{k+l}^T = w_k^T L^l. \tag{3.25}$$

We realize that (3.25) is the power iteration that is used to calculate the largest eigenvalue and corresponding eigenvector of the matrix L. Therefore, we generically have

$$w_{k+l} \longrightarrow w \tag{3.26}$$

where w is the eigenvector corresponding to the largest eigenvalue of L. (The algorithm would not converge if the largest eigenvalue were complex, which is not the case as we will prove shortly; the algorithm would not converge to the largest eigenvalue if the orthogonal projection of the initial vector w_k to the subspace generated by the largest eigenvalue, i.e. by the corresponding eigenvector, is zero).

We sum up these observations in the following lemma.

Lemma 3.1

The limiting weight of the iteration is given by

$$W(s) = \left[\begin{array}{c|c} A & B \\ \hline w^T & 0 \end{array}\right] \tag{3.27}$$

where w is the eigenvector of L corresponding to the largest eigenvalue and L is given by (3.23). The limiting weight is unique if the largest eigenvalue of L has multiplicity one.

Lemma 3.2

In the SISO case we have

$$L_c L_o = L^2. \tag{3.28}$$

Proof: First note that if we make a coordinate change in state space by $x = Tz$, we have $\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = [T^{-1}AT, T^{-1}B, CT, 0]$, $L_c L_o \longrightarrow T^{-1}L_c L_o T$, $L \longrightarrow T^{-1}LT$. These relations imply that if (3.28) is satisfied in some coordinate system, then it will be satisfied in any coordinate system. We prove it for the balanced realization. First note that in the case of balanced realizations, we have $L_c = L_o = \Sigma$ =diag$(\sigma_1, \sigma_2, ..., \sigma_n)$. In the SISO case for balanced realizations we have the following equalities [30]

$$C = B^T R \tag{3.29}$$

$$A = RA^T R \tag{3.30}$$

where R is a diagonal matrix having $+1$ or -1 as diagonal entries. We now use relations (3.29) and(3.30) in (3.23). We get the following equation,

$$AL + LRA^T R + BB^T R = 0 \tag{3.31}$$

and from (3.31) we get

$$ALR + LRA^T + BB^T = 0. \tag{3.32}$$

If we compare (3.3) and (3.32), we conclude that in case of balanced realization, in the SISO case, we have $L_c = \Sigma = -LR = L_o$. This implies that

$$L = -\Sigma R = -R\Sigma. \tag{3.33}$$

From (3.33) we conclude the result. ∎

The following lemma gives the relation between the Schmidt pair and the limiting weight:

Lemma 3.3

$$g(s) = W(s) \tag{3.34}$$

$$f(s) = -sign(\lambda_{max})W^{\sim}(s) \tag{3.35}$$

where λ_{max} is the largest eigenvalue of L.

Proof: We have $g(s) = g^T(s)$ since $g(s)$ is a scalar function; then we have

$$g(s) = \left[\begin{array}{c|c} -A & B \\ \hline w^T\Sigma\sigma_1^{-1} & 0 \end{array}\right]. \tag{3.36}$$

Consider balanced realizations as before. Since w has to satisfy $\Sigma^2 w = \sigma_1^2 w$, then we conclude that $w = (1, 0, 0, ..0)^T$. Therefore, $w^T\Sigma\sigma_1^{-1} = w^T$ and we finally have

$$g(s) = \left[\begin{array}{c|c} -A & B \\ \hline w^T & 0 \end{array}\right] = W(s). \tag{3.37}$$

Next, consider $f(s)$

$$f^{\sim}(s) = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline w^T & 0 \end{array}\right]. \tag{3.38}$$

If we use the the relations (3.29) and (3.30) we get

$$f^{\sim}(s) = \left[\begin{array}{c|c} -RAR & -RB \\ \hline w^T & 0 \end{array}\right] \tag{3.39}$$

$$= \left[\begin{array}{c|c} -A & B \\ \hline -w^T R & 0 \end{array}\right] \tag{3.40}$$

$$= \sigma_1^{-1} \left[\begin{array}{c|c} -A & B \\ \hline -w^T \Sigma R & 0 \end{array}\right] \tag{3.41}$$

$$= \sigma_1^{-1} \left[\begin{array}{c|c} -A & B \\ \hline w^T L & 0 \end{array}\right] \tag{3.42}$$

$$= \sigma_1^{-1} \left[\begin{array}{c|c} -A & B \\ \hline \lambda w^T & 0 \end{array}\right] \tag{3.43}$$

$$= \frac{\lambda}{\sigma_1} \left[\begin{array}{c|c} -A & B \\ \hline w^T & 0 \end{array}\right] \tag{3.44}$$

$$= \frac{\lambda}{\sigma_1} W(s). \ \blacksquare \tag{3.45}$$

An immediate result of Lemma 3.3 is the following corollary:

Corollary 3.1 $$\sigma_1 \frac{f(s)}{g(s)} = \lambda \frac{W^{\sim}(s)}{W(s)}$$ (3

and therefore,

$$q(s) = r(s) - \lambda \frac{W^{\sim}(s)}{W(s)} \tag{3.47}$$

is the optimal solution of the problem (3.1).

This completes the proof of the claim that the algorithm we proposed converges to the optimal solution to (3.1). Next we will state an obvious fact as a lemma.

Lemma 3.4 *For a solution $q(s)$ the optimal weight is unique.*

Proof: Assume that there are two weights that give the same solution, $q(s)$. We then have

$$E(s) = \lambda \frac{W_1^{\sim}(s)}{W_1(s)} = \lambda \frac{W_2^{\sim}(s)}{W_2(s)}; \tag{3.48}$$

from this we obtain

$$\frac{W_1^{\sim}(s)}{W_2^{\sim}(s)} = \frac{W_1(s)}{W_2(s)}. \tag{3.49}$$

Since $W_1(s)$ and $W_2(s)$ are both stable and minimum phase, then $\frac{W_1(s)}{W_2(s)}$ is a stable function, i.e., analytic in C_+. Similarly, $\frac{W_1(s)}{W_2(s)}$ is analytic in C_-. Then we conclude that (3.49) can be satisfied if and only if $W_1(s) = kW_2(s)$. ■

3.2.1 Interpretation of the Optimal Weight

Consider the following block diagram:

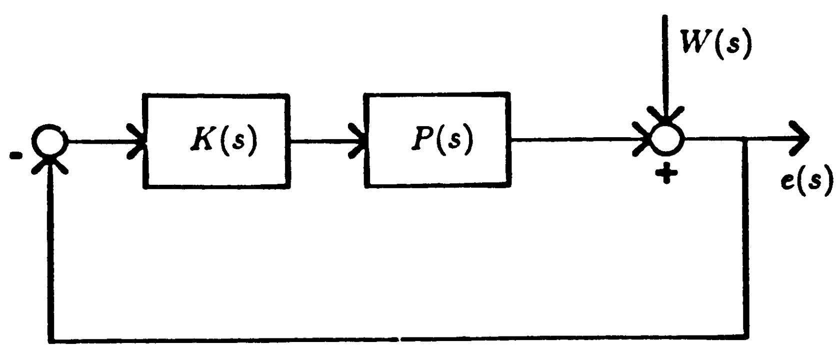


Figure 3.1: Disturbance rejection problem.

For a given $W(s) \in RH_2$,

$$\min_{K(s),\text{stabilizing}} \|e\|_2 \tag{3.50}$$

is a weighted H_2- optimization problem. We define

$$e(s) := (1 + P(s)K(s))^{-1}\, W(s) \tag{3.51}$$

where $K(s)$ internally stabilizes the feedback system in Figure 3.1. If we use Youla's parameterization lemma we get

$$e(s) = (A(s) - B(s)q(s))\, W(s) \tag{3.52}$$

where $A(s)$ and $B(s)$ are functions determined by $P(s)$. Problem (3.50) can be transformed into a problem as in (3.11), and the basic steps are as follows:

$$\min_{K(s),\text{ stabilizing}} \|e\|_2 = \min_{q(s)\in RH_\infty} \|(A - Bq)\, W\|_2 =$$

$$\min_{q(s)\in RH_\infty} \|(AB_i^{\sim} - B_o q)\, W\|_2 = \min_{\hat{q}(s)\in RH_\infty} \|(r - \hat{q})\, W\|_2$$

where $B_i(s)$ and $B_o(s)$ are the inner and outer factors of $B(s)$, respectively, and $\hat{q}(s) := B_o(s)q(s)$.

Now consider the following problem,

$$\min_{K(s),\text{stabilizing}} \left\{ \sup_{W(s)\in RH_2} \frac{\|e\|_2}{\|W\|_2} \right\}. \tag{3.53}$$

Using the same arguments as before we get

$$\min_{K(s),\text{stabilizing}} \left\{ \sup_{W(s)\in RH_2} \frac{\|e\|_2}{\|W\|_2} \right\} = \min_{\hat{q}(s)\in RH_\infty} \left\{ \sup_{W\in RH_2} \frac{\|(r - \hat{q})\, W\|_2}{\|W\|_2} \right\}. \tag{3.54}$$

By definition of the L_∞ norm we have

$$\sup_{W(s)\in RH_2} \frac{\|(r - \hat{q})\, W\|_2}{\|W\|_2} = \|r - \hat{q}\|_\infty. \tag{3.55}$$

Therefore, we conclude that the optimal weight is the worst-case signal for problem (3.54). This shows that the worst-case signals for H_∞ problems are not necessarily sinusoids, but they are signals which belong to RH_2.

3.2.2 An Example

Consider Figure 3.1 with

$$P(s) = \frac{s-1}{s-0.2}$$

and suppose that we want to minimize the peak of the sensitivity transfer function

$$S(s) := \frac{1}{1+P(s)K(s)},$$

which is the transfer function from W to e, over the set of stabilizing controllers $K(s)$. It is easily verified that all admissible $S(s)$, the set of $S(s)$ parameterized by stabilizing controllers, can be parameterized as

$$S(s) = \frac{2.5(s-.2)}{s+1} - \frac{(s-1)(s-0.2)}{(s+1)^2}q(s) \tag{3.56}$$

for $q(s) \in H_\infty$. Minimizing the peak of the sensitivity transfer function is then equivalent with

$$\min_{Q(s)\in RH_\infty} \|r - Q\|_\infty \tag{3.57}$$

where

$$r(s) := \frac{2.5(s+0.2)}{s-1}$$

and

$$Q(s) := \frac{s+0.2}{s+1}q(s).$$

A minimal state space realization for $r(s)$ is

$$r(s) = \left[\begin{array}{c|c} 1 & 1 \\ \hline 3 & 2.5 \end{array}\right].$$

This gives $L = -1.5$. Therefore, the minimum achievable error is $\sigma_1 = 1.5$. Since $r(s)$ is a first-degree transfer function $W(s)$ does not have any finite zeros and it is

$$w(s) = \frac{1}{s+1}.$$

Therefore, the optimal sensitivity can be found from

$$\|wS\|_2 \tag{3.58}$$

where minimization is over the set of all stabilizing controllers. Note that according to the interpretation of the optimal weight following Section 3.2.1 results, $w(s) = 1/(s+1)$ is the worst input signal for the system in Figure 3.1 for the input-output pair (W, e).

Chapter 4

A Simple Solution to H_∞ Optimization Problems: the One-Block Problem

When we attempt to generalize the result of Chapter 3, namely, that there exists a $W(s) \in RH_2$ such that $\min_{Q(s)\in RH_\infty} \|X - Q\|_\infty$ and that $\min_{Q(s)\in RH_\infty} \|W(X-Q)\|_2$ has the same solution and the error is all-pass, we fail, because, it turns out that maximum eigenvalue of L_cL_o does not always have enough multiplicity. As we will see in the following section, if we give up the idea of having a strictly proper H_2 weight and try to generalize all-passness of the error, we get a solution.

4.1 A Solution for the One-Block-Suboptimal Problem

Consider the antistable rational $m \times m$ system

$$R(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{4.1}$$

such that (A, B) is controllable and (C, A) is observable. The *one-block* problem is defined as follows : *Given a γ, find a $Q \in RH_\infty$ such that*

$$\|R - Q\|_\infty \leq \gamma. \tag{4.2}$$

From Nehari's theorem we have $\gamma \geq \|\Gamma_R\|$.

Throughout this chapter we will consider the suboptimal case; i.e., $\gamma > \|\Gamma_R\|$. Taking $R(s)$ to be square does not entail any loss of generality. This is because adding zero rows or columns to $R(s)$ does not change the Hankel norm of $R(s)$, and the solution is later obtained by a compression, which does not increase the norm. Now consider some $W(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & I \end{array}\right] \in RH_\infty$ with $W^{-1}(s) \in RH_\infty$, which will be specified later. We obtain:

$$RW = [RW]_- + [RW]_+ \tag{4.3}$$

$$RW = \{[RW]_- + D_1\} + \{[RW]_+ - D_1\} \tag{4.4}$$

where D_1 will be determined later. Define

$$U := [RW]_- + D_1 \tag{4.5}$$

$$V := [RW]_+ - D_1. \tag{4.6}$$

From (4.4), it follows that

$$R(s) = E(s) + Q(s) \tag{4.7}$$

where

$$E(s) := U(s)W^{-1}(s) \tag{4.8}$$

$$Q(s) := V(s)W^{-1}(s). \tag{4.9}$$

Now we have the error and the solution of (4.2) in terms of $W(s)$ and D_1. Let us require $E(s)$ to be γ-all-pass; i.e.

$$E^{\sim}(s)E(s) = E(s)E^{\sim}(s) = \gamma^2 I \tag{4.10}$$

and that $Q(s) \in RH_\infty$. At this point it is clear that we need the factor D_1 for $E(s)$ to be γ-all-pass.

If we use the definition of $E(s)$ given by (4.8) in (4.10), we get

$$W^{-\sim}(s)U^{\sim}(s)U(s)W^{-1}(s) = \gamma^2 I. \tag{4.11}$$

This is satisfied if and only if

$$U^{\sim}(s)U(s) = \gamma^2 W^{\sim}(s)W(s). \tag{4.12}$$

We first calculate $U(s)$ and $V(s)$ as follows:

$$\begin{aligned} RW &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & I \end{array}\right] \\ &= \left[\begin{array}{cc|c} A & 0 & B - X_1 B_1 \\ 0 & A_1 & B_1 \\ \hline C & CX_1 + DC_1 & D \end{array}\right], \end{aligned} \tag{4.13}$$

where the second equality is obtained via the similarity transformation

$$T_1 = \begin{pmatrix} I & X_1 \\ 0 & I \end{pmatrix}, \tag{4.14}$$

where X_1 satisfies

$$AX_1 - X_1 A_1 + BC_1 = 0. \tag{4.15}$$

From (4.13) we obtain

$$U = \left[\begin{array}{c|c} A & B - X_1 B_1 \\ \hline C & D_1 \end{array}\right] \tag{4.16}$$

$$V = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline CX_1 + DC_1 & D - D_1 \end{array}\right]. \tag{4.17}$$

We next calculate the quantities $W^{\sim}(s)W(s)$ and $U^{\sim}(s)U(s)$.

$$\begin{aligned} W^{\sim}(s)W(s) &= \left[\begin{array}{c|c} -A_1^T & -C_1^T \\ \hline B_1^T & I \end{array}\right] \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & I \end{array}\right] \\ &= \left[\begin{array}{cc|c} -A_1^T & -C_1^T C_1 & -C_1^T \\ 0 & A_1^T & B_1 \\ \hline B_1^T & C_1 & I \end{array}\right] \\ &= \left[\begin{array}{cc|c} -A_1^T & 0 & -C_1^T - X_2 B_1 \\ 0 & A_1^T & B_1 \\ \hline B_1^T & B_1^T X_2 + C_1 & I \end{array}\right]. \end{aligned} \tag{4.18}$$

The second equality is obtained through the similarity transformation

$$T_2 = \begin{pmatrix} I & X_2 \\ 0 & I \end{pmatrix}, \tag{4.19}$$

where X_2 satisfies

$$A_1^T X_2 + X_2 A_1 + {C_1}^T C_1 = 0. \tag{4.20}$$

From (4.18) we obtain

$$W^{\sim}(s)W(s) = \left[\begin{array}{c|c} -A_1^T & -C_1^T - X_2 B_1 \\ \hline B_1^T & 0 \end{array}\right] +$$

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline B_1^T X_2 + C_1 & I \end{array}\right]. \tag{4.21}$$

A similar calculation gives

$$U^{\sim}(s)U(s) = \left[\begin{array}{c|c} -A^T & -C^T D_1 - X_3(B - X_1 B_1) \\ \hline (B - X_1 B_1)^T & D_1^T D_1 \end{array}\right] + \left[\begin{array}{c|c} A & B - X_1 B_1 \\ \hline (D_1^T C + (B - X_1 B_1)^T X_3) & 0 \end{array}\right], \tag{4.22}$$

where X_3 satisfies

$$A^T X_3 + X_3 A + C^T C = 0. \tag{4.23}$$

Now we impose the condition (4.12), which is satisfied if and only if (since the only freedom left is in similarity transformations),

$$A_1 = -A^T \tag{4.24}$$

$$\gamma^2 B_1 = -C^T D_1 - X_3 B + X_3 X_1 B_1 \tag{4.25}$$

$$B_1{}^T X_2 + C_1 = B^T - B_1{}^T X_1{}^T \tag{4.26}$$

$$D_1{}^T D_1 = \gamma^2 I. \tag{4.27}$$

(4.27) implies

$$D_1 := \gamma U_0 \tag{4.28}$$

where U_0 is a unitary matrix. If we combine (4.26) with (4.15) and (4.20), we conclude

$$C_1 = B^T \tag{4.29}$$

$$X_2 = -X_1 = L_c \tag{4.30}$$

and also, by definition,

$$X_3 = -L_o. \tag{4.31}$$

From (4.25) we get

$$B_1 = -N(\gamma)[C^T D_1 - L_o B] \tag{4.32}$$

where

$$N(\gamma) := \left(\gamma^2 I - L_o L_c\right)^{-1}. \tag{4.33}$$

Summarizing, we have

$$W = \left[\begin{array}{c|c} -A^T & -N(\gamma)(C^T D_1 - L_o B) \\ \hline B^T & I \end{array}\right] \tag{4.34}$$

$$U = \left[\begin{array}{c|c} A & B - L_c N(\gamma)(C^T D_1 - L_o B) \\ \hline C & D_1 \end{array}\right] \tag{4.35}$$

$$V = \left[\begin{array}{c|c} -A^T & -N(\gamma)(C^T D_1 - L_o B) \\ \hline -C L_c + D B^T & D - D_1 \end{array}\right] \tag{4.36}$$

where D_1 is defined through (4.28). From (4.34) we can easily get

$$W^{-1}(s) = \left[\begin{array}{c|c} -(A - \Delta)^T & N(\gamma)(C^T D_1 - L_o B) \\ \hline B^T & I \end{array}\right] \tag{4.37}$$

where Δ is defined as

$$\Delta := B\left[{D_1}^T C - B^T L_o\right] N^T(\gamma). \tag{4.38}$$

We can now calculate $Q(s)$ as:

$$Q(s) = V(s) W^{-1}(s). \tag{4.39}$$

From (4.39)

$$Q = \left[\begin{array}{cc|c} -A^T & -\Delta^T & N(\gamma)(L_oB - C^TD_1) \\ 0 & -(A-\Delta)^T & -N(\gamma)(L_oB - C^TD_1) \\ \hline DB^T - CL_c & (D-D_1)B^T & D - D_1 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -A^T & 0 & 0 \\ 0 & -(A-\Delta)^T & -N(\gamma)(L_oB - C^TD_1) \\ \hline DB^T - CL_c & CL_c - D_1B^T & D - D_1 \end{array}\right] \tag{4.40}$$

where the second equality follows from the similarity transformation

$$T = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}. \tag{4.41}$$

A realization for $Q(s)$ is then

$$Q(s) = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] :=$$

$$= \left[\begin{array}{c|c} -(A-\Delta)^T & N(\gamma)(C^TD_1 - L_oB) \\ \hline CL_c - D_1B^T & D - D_1 \end{array}\right], \tag{4.42}$$

with parameters A_Q, B_Q, C_Q, D_Q defined in an obvious way. $E(s)$ is then γ-all-pass by construction. We need to check to see if $Q(s)$ is stable. To this end, we prove the following lemma.

Lemma 4.1 *For any D_1 with $\sigma_{max}(D_1) \leq \gamma$, $Q(s)$ given by*

$$Q(s) = \left[\begin{array}{c|c} -A^T + N(\gamma)(C^TD_1 - L_oB)B^T & N(\gamma)(C^TD_1 - L_oB) \\ \hline CL_c - D_1B^T & D - D_1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] \tag{4.43}$$

is stable.

Proof: A_Q satisfies the following Riccati equation :

$$A_Q(N(\gamma)L_o) + (N(\gamma)L_o)A_Q^T + B_Q B_Q^T + N(\gamma)C^T(\gamma^2 I - D_1 D_1^T)CN^T(\gamma) = 0. \quad (4.44)$$

Since $\gamma > \rho(L_o L_c)$and $N(\gamma)L_o > 0$, we conclude that A_Q has all its eigenvalues in the closed left half plane (see [12], Theorem 3.3). Therefore, the only kind of unstable modes we may have are on the $j\omega$ axis.

Claim 1: There are no uncontrollable modes on $j\omega$ axis.

We prove this claim as follows: Assume that (A_Q, B_Q) has an uncontrollable mode $\lambda = j\omega$; then we have

$$x^T A_Q = j\omega x^T \quad (4.45)$$

$$x^T B_Q = 0. \quad (4.46)$$

If we use the definition of A_Q and (4.46)in (4.45), we conclude that

$$x^T(-A^T) = j\omega x^T, \quad (4.47)$$

but this contradicts the fact that A has all of its eigenvalues in the right half plane. We conclude that $Q(s)$ does not have any uncontrollable modes on the $j\omega$ axis.

Claim 2: There are no controllable modes on $j\omega$ axis.

Proof of Claim 2: Assume that A_Q has an eigenvalue on the $j\omega$ axis; then we have

$$x^T A_Q = j\omega x^T \quad (4.48)$$

$$A_Q^T \overline{x} = -j\omega \overline{x} \quad (4.49)$$

since A_Q is a matrix with real elements. Multiply (4.44) by x^T from left and $\overline{x}$ from right; then we get

$$x^T\left(B_Q B_Q^T\right)\overline{x} + x^T\left(N(\gamma)C^T(\gamma^2 I - D_1 D_1^T)CN^T(\gamma)\right)\overline{x} = 0. \tag{4.50}$$

From Equation (4.50) we conclude that

$$x^T B_Q = 0. \tag{4.51}$$

(4.48) together with (4.51) implies that there is an uncontrollable eigenvalue on the $j\omega$ axis, which contradicts Claim 1.

Claim 1 and Claim 2 together imply that A_Q does not have any modes on the $j\omega$ axis. Therefore, A_Q has all of its eigenvalues in left half plane. ■

4.2 Parameterization of All Solutions to the One-Block Problem

In [9] a parameterization of all the solutions to (4.2) is given. We use the same notation as in [9] for comparison. In the previous section we obtained a solution to (4.2), which gives a γ all-pass error,

$$E(s) = U(s)W^{-1}(s) \tag{4.52}$$

where $W(s)$ and $U(s)$ are given by (4.34) and (4.35). If we appropriately group the expressions for $U(s)$, $W(s)$ and $V(s)$, we get

$$U = L_1(\gamma, s)D_1 + L_2(\gamma, s) \tag{4.53}$$

$$W = L_3(\gamma, s)D_1 + L_4(\gamma, s) \tag{4.54}$$

$$V = -L_5(\gamma, s)D_1 + L_6(\gamma, s) \tag{4.55}$$

where $D_1 = \gamma U$, U any unitary matrix and

$$L_1(\gamma, s) := \left[\begin{array}{c|c} A & -L_c N(\gamma) C^T \\ \hline C & I \end{array}\right] \tag{4.56}$$

$$L_2(\gamma, s) := \left[\begin{array}{c|c} A & \gamma^2 N^T(\gamma) B \\ \hline C & 0 \end{array}\right] \tag{4.57}$$

$$L_3(\gamma, s) := \left[\begin{array}{c|c} -A^T & N(\gamma) C^T \\ \hline -B^T & 0 \end{array}\right] \tag{4.58}$$

$$L_4(\gamma, s) := \left[\begin{array}{c|c} -A^T & N(\gamma) L_o B \\ \hline B^T & I \end{array}\right] \tag{4.59}$$

$$L_5(\gamma, s) := \left[\begin{array}{c|c} -A^T & N(\gamma) C^T \\ \hline DB^T - CL_c & I \end{array}\right] \tag{4.60}$$

$$L_6(\gamma, s) := \left[\begin{array}{c|c} -A^T & N(\gamma) L_o B \\ \hline DB^T - CL_c & D \end{array}\right]. \tag{4.61}$$

From here on we will drop the γ and s in the notation of $L_i(\gamma, s)$.

Definition 4.1 *A given error function $E(s)$ is called "admissible", for $R(s)$, if and only if for some $Q(s) \in H_\infty$, $E(s) = R(s) - Q(s)$ satisfies $\|E\|_\infty \leq \gamma$.*

In this section we will parameterize all possible solutions for (4.2). The parameterization is easily obtained from the solutions we obtained in Section 4.1. Let a given admissible error function $E(s)$ be expressed as

$$E(s) = U(s)W^{-1}(s) = [L_1 Y(s) + L_2][L_3 Y(s) + L_4]^{-1} \tag{4.62}$$

for some $Y(s) \in H_\infty$. Then we obtain

$$Q(s) = V(s)W^{-1}(s) = [-L_5 Y(s) + L_6][L_3 Y(s) + L_4]^{-1} \tag{4.63}$$

from $Q(s) = R(s) - E(s)$ and straightforward algebra.

We will need the following result in the sequel:

Lemma 4.2 $[L_1Y(s) + L_2][L_3Y(s) + L_4]^{-1}$ *and* $[-L_5Y(s) + L_6][L_3Y(s) + L_4]^{-1}$ *do not have any right half-plane pole, zero cancellation for any* $Y(s)$, *i.e. if* $L_3Y(s) + L_4$ *has a right half-plane zero it will be a right-half plane pole for* $Q(s)$ *and* $E(s)$.

Proof: Assume the contrary; i.e., for some $Y(s)$ (not necessarily stable) $L_3Y(s) + L_4$ has a zero at some s_0 with $Re\{s_0\} \geq 0$, but $Q(s)$ is stable; i.e., there is a pole zero cancellation for $Q(s)$ at $s = s_0$. Then there exist a nonzero vector v such that

$$[L_3Y(s_0) + L_4]\, v = 0. \tag{4.64}$$

Since $Q(s_0)$ is a finite matrix, we have

$$Q(s_0)\, [L_3Y(s_0) + L_4]\, v = 0. \tag{4.65}$$

Finally, from (4.63) we obtain

$$[-L_5Y(s_0) + L_6]\, v = 0. \tag{4.66}$$

Note that $z := Y(s_0)v \neq 0$; otherwise, (4.64) implies $L_4v = 0$, which contradicts the fact that L_4^{-1} is stable. (The fact that $L_4^{-1}(s)$ is stable follows from Lemma 4.1; i.e., take $D_1 = 0$ and consider the fact that $L_4^{-1}(s)$ and $Q(s)$ in (4.42) have the same "A".)

Then we have

$$L_3z + L_4v = 0, \quad s = s_0 \tag{4.67}$$

$$-L_5z + L_6v = 0, \quad s = s_0. \tag{4.68}$$

Next we solve for v from (4.67) and substitute in (4.68). We get the following necessary condition for (4.64) to be true:

$$\left[L_5 + L_6 L_4^{-1} L_3\right] z = 0, \text{ for some } s = s_0, \text{ with } Re\{s_0\} \geq 0. \tag{4.69}$$

Define

$$H(s) := L_5 + L_6 L_4^{-1} L_3. \tag{4.70}$$

In (4.69), again we made use of the fact that L_4^{-1} is stable. After a straightforward calculation we get the following state-space realization for $H(s)$:

$$H(s) = \left[\begin{array}{c|c} -A^T - N(\gamma)L_o BB^T & -N(\gamma)C^T \\ \hline CL_c & I \end{array}\right]. \tag{4.71}$$

$H(s)$ has a right half-plane zero if and only if $H^{-1}(s)$ is unstable. From (4.71) we obtain

$$H^{-1}(s) = \left[\begin{array}{c|c} -A^T - N(\gamma)L_o BB^T + N(\gamma)C^T CL_c & N(\gamma)C^T \\ \hline CL_c & I \end{array}\right]. \tag{4.72}$$

By using (2.27) and (2.28) in (4.72) we get

$$\begin{aligned} H^{-1}(s) &= \left[\begin{array}{c|c} -N(\gamma)A^T N^{-1}(\gamma) & N(\gamma)C^T \\ \hline CL_c & I \end{array}\right] \\ &= \left[\begin{array}{c|c} -A^T & C^T \\ \hline CL_cN(\gamma) & I \end{array}\right] \\ &= L_1^{\sim}(s). \end{aligned} \tag{4.73}$$

We conclude that $H^{-1}(s)$ is stable, which contradicts (4.69) and the original assumption that $v \neq 0$ and that there exists an s_0, with $Re\{s_0\} \geq 0$.

Similarly, we can show that $L_1 Y(s) + L_2$ and $L_3 Y(s) + L_4$ do not have any right half-plane, pole zero cancellation. ■

We will need the following lemma for the next theorem.

Lemma 4.3

$$L_1^{\sim} L_1 - \gamma^2 L_3^{\sim} L_3 = I \tag{4.74}$$

$$L_1^{\sim} L_2 = \gamma^2 L_3^{\sim} L_4 \tag{4.75}$$

$$L_4^{\sim} L_4 - (1/\gamma^2) L_2^{\sim} L_2 = I \tag{4.76}$$

Proof: The proof of Lemma 4.3 follows by direct verification. ■

Theorem 4.1 *$\|E\|_\infty \leq \gamma$ if and only if $\|Y\|_\infty \leq \gamma$ and $\|E\|_\infty = \gamma$ if and only if $\|Y\|_\infty = \gamma$.*

Proof:

Assume $det\,(L_3 Y(j\omega) + L_4) \neq 0, \forall\omega$. Otherwise $\|E\|_\infty = \infty$. Then we have the following equivalent statements:

$\|E\|_\infty \leq \gamma$

$\iff$

$\gamma^2 I - E^{\sim}(j\omega)E(j\omega) \geq 0 \;, \forall\omega.$

$\iff$

$\gamma^2 I - \;[(L_3 Y + L_4)^{-1}]^{\sim}\,(L_1 Y + L_2)^{\sim}\,(L_1 Y + L_2)\,(L_3 Y + L_4)^{-1} \geq 0, \forall\omega.$

$\iff$

$[(L_3 Y + L_4)^{\sim}(L_3 Y + L_4)\gamma^2 - (L_1 Y + L_2)^{\sim}(L_1 Y + L_2)] \geq 0 \;, \forall\omega.$

$\iff$

$-Y^{\sim}(L_1^{\sim} L_1 - \gamma^2 L_3^{\sim} L_3)Y - Y^{\sim}(L_1^{\sim} L_2 - \gamma^2 L_3^{\sim} L_4) + (L_2^{\sim} L_1 - \gamma^2 L_4^{\sim} L_3)Y + (\gamma^2 L_4^{\sim} L_4 - L_2^{\sim} L_2) \geq 0 \;, \forall\omega.$

$\iff$

By using (4.74), (4.75) and (4.76)

$\gamma^2 I - Y^{\sim}(j\omega)Y(j\omega) \geq 0 \;, \forall\omega.$

$\Longleftrightarrow$

$\|Y\|_\infty \leq \gamma$. ■

It is immediate from the proof of Theorem 4.1 that

Corollary 4.1 $$E^\sim(s)E(s) = \gamma^2 I \Longleftrightarrow Y^\sim(s)Y(s) = \gamma^2 I. \quad (4$$

We next prove that for any $Y \in B_\gamma H_\infty$, (4.63) gives a stable solution. Later we will prove that actually this set covers all solutions of (4.2). First, we need the following lemma:

Lemma 4.4 $\|L_4^{-1}L_3\|_\infty < 1/\gamma$.

Proof: From (4.63) with $Y(s) = D_1$ and (4.43) we get:

$$Q(s) \text{ is stable if and only if } \quad [L_3D_1 + L_4]^{-1} \quad \text{is stable.} \tag{4.78}$$

Stability of $Q(s)$ is guaranteed for all D_1 by Lemma 4.1 . Then we have :

$$\left[L_4^{-1}L_3D_1 + I\right]^{-1} \tag{4.79}$$

is stable for all D_1 such that $\sigma_{max}(D_1) \leq \gamma$, and the result follows. ■

Theorem 4.2 *For* $\|Y\|_\infty \leq \gamma$, $Q(s)$ *given by*

$$Q(s) = V(s)W^{-1}(s) = [-L_5Y(s) + L_6][L_3Y(s) + L_4]^{-1} \tag{4.80}$$

is stable if and only if $Y(s) \in H_\infty$.

Proof: From (4.63) and Lemma 4.2 we have that $Q(s)$ is stable iff $W^{-1}(s)$ is stable.

$$W^{-1}(s) = [L_3Y(s) + L_4]^{-1} \tag{4.81}$$

is stable if and only if

$$\left[L_4^{-1}L_3Y(s) + I\right]^{-1} \tag{4.82}$$

is stable, noting that L_4^{-1} is stable. Since $\|Y\|_\infty \leq \gamma$, $\left[L_4^{-1}L_3Y(s) + I\right]^{-1}$ is stable if and only if $Y(s) \in H_\infty$. This last statement follows from the Nyquist stability criterion and from Lemma 4.4. ■

The only remaining task is to prove that (4.63) actually gives all admissible solutions.

Theorem 4.3 *All admissible $E(s)$ are obtained from:*

$$E(s) = [L_1Y(s) + L_2][L_3Y(s) + L_4]^{-1} \tag{4.83}$$

for $Y(s) \in B_\gamma H_\infty$.

Proof: For a given admissible $E(s)$, we solve for $Y(s)$ from Equation (4.83). The result is

$$Y(s) = -[L_1 - E(s)L_3]^{-1}[L_2 - E(s)L_4]. \tag{4.84}$$

$\|E\|_\infty \leq \gamma$ since $E(s)$ is admissible, and from Theorem 4.1 we conclude that $\|Y\|_\infty \leq \gamma$. From Theorem 4.2, $Q(s)$ is stable if and only if $Y(s)$ is stable. ■

We summarize the results in the following theorem:

Theorem 4.4 *Given a $G(s) \in RH_\infty^-$ with a minimal realization $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, a parameterization of all solutions, $Q(s)$, satisfying the norm bound that*

$$\|G - Q\|_\infty \leq \gamma \tag{4.85}$$

is given by

$$Q(s) = (-L_5Y(s) + L_6)(L_3Y(s) + L_4)^{-1} \tag{4.86}$$

with $Y(s) \in B_\gamma H_\infty$.

The following lemma gives the parameterization of all solutions as a usual linear fractional transformation.

Theorem 4.5 *The parameterization given by Theorem 4.4 can also be represented by the linear fractional transformation*

$$Q(s) = F_l\left(H(s), Y(s)\right) \tag{4.87}$$

where

$$H(s) = \left[\begin{array}{c|cc} -N(\gamma)\left(\gamma^2 A^T + L_o A L_c\right) & -N(\gamma) L_o B & N(\gamma) C^T \\ \hline C L_c & D & -I \\ B^T & I & 0 \end{array}\right]. \tag{4.88}$$

Proof: From (4.86), we have

$$Q(s) = \left(-L_5 Y(s) + L_6\right)\left(L_3 Y(s) + L_4\right). \tag{4.89}$$

By using the matrix inversion lemma, we obtain:

$$\left(L_3 Y(s) + L_4\right)^{-1} = L_4^{-1} - L_4^{-1} L_3 \left(I + Y(s) L_4^{-1} L_3\right)^{-1} Y(s) L_4^{-1}. \tag{4.90}$$

By substituting (4.90) in (4.89), we get

$$Q(s) = L_6 L_4^{-1} - \left(L_5 + L_6 L_4^{-1} L_3\right) Y(s) \left(I - \left(-L_4^{-1} L_3\right) Y(s)\right)^{-1} L_4^{-1}. \tag{4.91}$$

Next it is a matter of straightforward algebra to show:

$$\begin{pmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{pmatrix} = \begin{pmatrix} L_6 L_4^{-1} & -\left(L_5 + L_6 L_4^{-1} L_3\right) \\ L_4^{-1} & -L_4^{-1} L_3 \end{pmatrix}. \tag{4.92}$$

By calculating the quantities in (4.92), the result follows. ■

4.3 Parameterization of All Optimal Solutions to the One-Block Problem: the Optimal Case

In this section we obtain a parameterization of all optimal solutions to the one-block problem by calculating the limiting value for $\gamma \longrightarrow \sigma_1$ of the suboptimal solution set.

When we take the limit $\gamma \longrightarrow \sigma_1$, the only difficulty appears because $(\sigma_1^2 I - L_o L_c)$ becomes singular.

Consider the equations corresponding to (4.88):

$$\dot{x} = -N(\gamma)(\gamma^2 A^T + L_o A L_c)x + N(\gamma)\begin{pmatrix} -L_o B & C^T \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{4.93}$$

$$y = \begin{pmatrix} C L_c \\ B^T \end{pmatrix} x + \begin{pmatrix} D & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.94}$$

After we multiply both sides of Equation (4.93) by $N^{-1}(\gamma)$ we get

$$N^{-1}(\gamma)\dot{x} = -(\gamma^2 A^T + L_o A L_c)x + \begin{pmatrix} -L_o B & C^T \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.95}$$

Equation (4.95) does not have any singularity for $\gamma = \sigma_1$ any more. This is the main idea behind the derivation of all the optimal solutions for Nehari's extension problem. In this section we use the balanced realization and related partition given as follows:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \sigma_1 I_r & 0 \\ 0 & \Sigma_1 \end{pmatrix} + \begin{pmatrix} \sigma_1 I_r & 0 \\ 0 & \Sigma_1 \end{pmatrix}\begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix} = \begin{pmatrix} B_1 B_1^T & B_1 B_2^T \\ B_2 B_1^T & B_2 B_2^T \end{pmatrix} \tag{4.96}$$

$$\begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix}\begin{pmatrix} \sigma_1 I_r & 0 \\ 0 & \Sigma_1 \end{pmatrix} + \begin{pmatrix} \sigma_1 I_r & 0 \\ 0 & \Sigma_1 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} C_1^T C_1 & C_1^T C_2 \\ C_2^T C_1 & C_2^T C_2 \end{pmatrix}. \tag{4.97}$$

In this coordinate system

$$N(\gamma) = \begin{pmatrix} \frac{1}{\gamma^2 - \sigma_1^2} I_r & 0 \\ 0 & \Gamma_\gamma^{-1} \end{pmatrix} \tag{4.98}$$

where

$$\Gamma_\gamma := \gamma^2 I - \Sigma_1^2, \tag{4.99}$$

$$\Gamma := \Gamma_{\sigma_1}. \tag{4.100}$$

The following lemma provides a parameterization of all solutions to the Nehari's extension problem.

Theorem 4.6 *A parameterization of all solutions for Nehari's extension problem is given by*

$$Q(s) = F_l(H_0(s), Y(s)) \tag{4.101}$$

where $Y(s) \in B_{\sigma_1} H_\infty$ *and where*

$$H_0(s) = \left[\begin{array}{c|cc} A_0 & \Gamma^{-1}\left(\sigma_1 C_2^T U_0 - \Sigma_1 B_2\right) & \Gamma^{-1} C_2^T M_1 \\ \hline C_2\Sigma_1 - \sigma_1 U_0 B_2^T & D - \sigma_1 U_0 & -M_1 \\ M_2 B_2^T & M_2 & \frac{1}{\sigma_1} U_0^T \end{array}\right], \tag{4.102}$$

and

$$U_0 := C_1 (B_1^T)^\dagger \tag{4.103}$$

$$M_1 := I - C_1 C_1^\dagger \tag{4.104}$$

$$M_2 := I - B_1^\dagger B_1 \tag{4.105}$$

$$A_0 := -\Gamma^{-1}\left(\sigma_1^2 A_{22}^T + \Sigma_1 A_{22} \Sigma_1 - \sigma_1 C_2^T U_0 B_2^T\right). \tag{4.106}$$

Proof: This result is obtained by using the balanced realization in (4.95) and (4.94). For $\gamma = \sigma_1$ from (4.95) we get

$$\begin{pmatrix} 0 \\ \Gamma \dot{x}_2 \end{pmatrix} = -\begin{pmatrix} \sigma_1^2 (A_{11}^T + A_{11}) & \sigma_1\left(\sigma_1 A_{21}^T + A_{12}\Sigma_1\right) \\ \sigma_1\left(\sigma_1 A_{12}^T + \Sigma_1 A_{21}\right) & \gamma^2 A_{22}^T + \Sigma_1 A_{22} \Sigma_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + $$
$$+ \begin{pmatrix} -\sigma_1 B_1 & C_1^T \\ -\Sigma_1 B_2 & C_2^T \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.107}$$

From (4.96) and (4.97) we can observe that

$$\sigma_1\left(A_{11}^T + A_{11}\right) = B_1 B_1^T = C_1^T C_1 \tag{4.108}$$

$$\sigma_1 A_{21}^T + A_{12}\Sigma_1 = B_1 B_2^T \tag{4.109}$$

$$\sigma_1 A_{12}^T + \Sigma_1 A_{21} = C_2^T C_1. \tag{4.110}$$

If we substitute (4.108)- (4.110) in (4.107), we obtain

$$\begin{pmatrix} 0 \\ \Gamma\dot{x}_2 \end{pmatrix} = -\begin{pmatrix} \sigma_1 B_1 B_1^T & \sigma_1 B_1 B_2^T \\ \sigma_1 C_2^T C_1 & s\Gamma + \sigma_1^2 A_{22}^T + \Sigma_1 A_{22}\Sigma_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \\ + \begin{pmatrix} -\sigma_1 B_1 & C_1^T \\ -\Sigma_1 B_2 & C_2^T \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.111}$$

From (4.111) we obtain

$$\begin{pmatrix} 0 \\ \dot{x}_2 \end{pmatrix} = -\begin{pmatrix} \sigma_1 B_1 B_1^T & \sigma_1 B_1 B_2^T \\ \sigma_1 \Gamma^{-1} C_2^T C_1 & sI + \Gamma^{-1}\left(\sigma_1^2 A_{22}^T + \Sigma_1 A_{22}\Sigma_1\right) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \\ + \begin{pmatrix} -\sigma_1 B_1 & C_1^T \\ -\Gamma^{-1}\Sigma_1 B_2 & \Gamma^{-1} C_2^T \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.112}$$

Equation (4.112) consists of one algebraic and one dynamic equation in the unknowns x_1 and x_2. First, we solve for x_1 in terms of x_2, u_1 and u_2 from the algebraic equation and then substitute the solution in the dynamic equation. We make the following modification to take care of problems that could appear when $B_1 B_1^T$ is not invertible. Define

$$\alpha_\epsilon := \epsilon^2 I + B_1 B_1^T. \tag{4.113}$$

We replace the term $B_1 B_1^T$ in (4.112) by α_ϵ; then we have $\hat{x}_1$ and $\hat{x}_2$ instead of x_1 and x_2. Note that if we take the limit as $\epsilon \longrightarrow 0$, we have $\hat{x}_1 \longrightarrow x_1$ and $\hat{x}_2 \longrightarrow x_2$. After some straightforward but lengthy calculation we get

$$H_\epsilon(s) = \left[\begin{array}{c|cc} A_\epsilon & \Gamma^{-1}\left(\sigma_1 C_2^T \alpha_\epsilon^{-1} B_1 - \Sigma_1 B_2\right) & \Gamma^{-1} C_2^T\left(I - C_1 \alpha_\epsilon^{-1} C_1^T\right) \\ \hline C_2\Sigma_1 - \sigma_1 \alpha_\epsilon^{-1} B_1 B_2^T & D - \sigma_1 \alpha_\epsilon^{-1} B_1 & -\left(I - C_1 \alpha_\epsilon^{-1} C_1^T\right) \\ B_2^T - B_1^T \alpha_\epsilon^{-1} B_1 B_2^T & I - B_1^T \alpha_\epsilon^{-1} B_1 & \frac{1}{\sigma_1} B_1^T \alpha_\epsilon^{-1} C_1^T \end{array}\right] \tag{4.114}$$

where

$$A_\epsilon := -\Gamma^{-1}\left(\sigma_1^2 A_{22}^T + \Sigma_1 A_{22}\Sigma_1 - \sigma_1 C_2^T C_1 \alpha_\epsilon^{-1} B_1 B_2^T\right). \tag{4.115}$$

Observe that

$$\lim_{\epsilon \longrightarrow 0} \alpha_\epsilon^{-1} B_1 = (B_1^T)^\dagger \tag{4.116}$$

$$\lim_{\epsilon \longrightarrow 0} \alpha_\epsilon^{-1} C_1^T = C_1^\dagger. \tag{4.117}$$

After we use (4.116) and (4.117), we conclude the result. ■

Let us calculate U_0, M_1, and M_2 in terms of SVD's of B_1 and C_1. From 4.108 it follows that B_1 and C_1 have SVD's given as follows:

$$B_1 = U \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} V_B^T \tag{4.118}$$

$$C_1^T = U \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} V_C^T. \tag{4.119}$$

It is easy to prove that

$$B_1^\dagger = V_B \begin{pmatrix} \Sigma_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T \tag{4.120}$$

$$C_1^\dagger = U \begin{pmatrix} \Sigma_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} V_C^T. \tag{4.121}$$

We then have

$$\begin{aligned} U_0 = C_1(B_1^T)^\dagger = C_1(B_1^\dagger)^T &= V_C \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} U^T U \begin{pmatrix} \Sigma_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} V_B^T \\ &= V_C \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} V_B^T \end{aligned} \tag{4.122}$$

Similarly

$$M_1 = I - C_1 C_1^{\dagger} = V_C \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} V_C^T \tag{4.123}$$

$$M_2 = I - B_1^{\dagger} B_1 = V_B \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} V_B^T. \tag{4.124}$$

Lemma 4.5 $$U = \begin{pmatrix} U_0 & M_1 \\ M_2 & U_0^T \end{pmatrix}$$
is an orthogonal matrix.

Proof: Substitute (4.122), (4.123), (4.124) in the definition of U. We get

$$\begin{aligned} U &= \begin{pmatrix} V_C \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} V_B^T & V_C \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} V_C^T \\ V_B \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} V_B^T & V_B \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} V_C^T \end{pmatrix} \\ &= \begin{pmatrix} V_C & 0 \\ 0 & V_B \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \begin{pmatrix} V_B^T & 0 \\ 0 & V_C^T \end{pmatrix}. \end{aligned} \tag{4.125}$$

We see that U is obtained by multiplying three orthogonal matrices, which means that U itself is an orthogonal matrix. ■

Lemma 4.6 *$K(s) = \sigma_1 F_l(U, Y(s)/\sigma_1)$ satisfies the following relations for every $Y(s) \in B_{\sigma_1} H_\infty$:*

$$\|K\|_\infty = \sigma_1 \tag{4.126}$$

$$\sigma_1 B_1 = C_1^T K(s) \tag{4.127}$$

$$\sigma_1 C_1 = K(s) B_1^T. \tag{4.128}$$

Proof: From the definition of $K(s)$ we have

$$\| K \|_\infty = \sigma_1 \| F_l(U, Y(s)/\sigma_1) \|_\infty . \tag{4.129}$$

From Lemma 2.5 it follows that $F_l(U, Y(s)/\sigma_1) \in BH_\infty$ since $Y(s) \in B_{\sigma_1} H_\infty$ and U is unitary. Therefore, we have $K(s) \in B_{\sigma_1} H_\infty$.

If we substitute $K(s)$ in (4.127), we get

$$\begin{aligned} C_1^T K(s) &= \sigma_1 C_1^T \left[U_0 + M_1 Y(s)/\sigma_1 (I - U_0^T Y(s)/\sigma_1)^{-1} M_2 \right] \\ &= \sigma_1 \left[C_1^T U_0 + C_1^T M_1 Y(s)/\sigma_1 (I - U_0^T Y(s)/\sigma_1)^{-1} M_2 \right]. \end{aligned} \tag{4.130}$$

Let us calculate $C_1^T U_0$ and $C_1^T M_1$ using SVD's:

$$\begin{aligned} C_1^T U_0 &= U \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} V_C^T V_C \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} V_B^T \\ &= U \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} V_B^T \\ &= B_1 \end{aligned} \tag{4.131}$$

$$C_1^T M_1 = U \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} V_C^T V_C \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} V_C^T = 0. \tag{4.132}$$

Therefore, we conclude that

$$\sigma_1 B_1 = C_1^T K(s). \tag{4.133}$$

Similarly, we first show that $U_0 B_1^T = C_1$ and $M_2 B_1^T = 0$ to obtain $\sigma_1 C_1 = K(s) B_1^T$.

■

Lemma 4.7 *The following two sets are equal*

Set 1:

$$i) K(s) \in B_{\sigma_1} H_\infty \tag{4.134}$$

$$ii) \sigma_1 B_1 = C_1^T K(s) \tag{4.135}$$

$$iii) \sigma_1 C_1 = K(s) B_1^T \tag{4.136}$$

Set 2:

$$K(s) = \sigma_1 F_l(U, Y(s)/\sigma_1) \text{ with } Y(s) \in B_{\sigma_1} H_\infty. \tag{4.137}$$

Proof: We show that for every $K(s)$ in Set 1 there exists a $Y(s)$ in Set 2; then together with Lemma 4.6 we conclude the result.

Use SVD's in (4.135)

$$\sigma_1 U \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} V_B^T = U \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} V_C^T K(s). \tag{4.138}$$

From (4.138) we get

$$\begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} (\sigma_1 I - V_C^T K(s) V_B) = 0. \tag{4.139}$$

Define

$$\hat{K}(s) = \begin{pmatrix} \hat{K}_{11}(s) & \hat{K}_{12}(s) \\ \hat{K}_{21}(s) & \hat{K}_{22}(s) \end{pmatrix} = V_C^T K(s) V_B. \tag{4.140}$$

After we substitute (4.140) in (4.139), we get

$$\begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 I_{r_1} - \hat{K}_{11}(s) & -\hat{K}_{12}(s) \\ -\hat{K}_{21}(s) & \sigma_1 I_{r_2} - \hat{K}_{22}(s) \end{pmatrix} = 0. \tag{4.141}$$

.

From (4.141) we conclude that $\hat{K}_{11}(s) = \sigma_1 I_{r_1}$ and that $\hat{K}_{12}(s) = 0$. Similarly, from (4.136) we conclude that $\hat{K}_{21}(s) = 0$.

Therefore (4.134), (4.135) and (4.136) are satisfied if and only if

$$K(s) = V_C \begin{pmatrix} \sigma_1 I_{r_1} & 0 \\ 0 & F(s) \end{pmatrix} V_B^T \tag{4.142}$$

where $F(s) \in B_{\sigma_1} H_\infty$. On the other hand, we have

$$\begin{aligned}
\sigma_1 F_l(U, Y(s)/\sigma_1) &= \sigma_1 \left(U_0 + M_1 \frac{Y(s)}{\sigma_1} (I - U_0^T Y(s)/\sigma_1)^{-1} M_2 \right) \\
&= \sigma_1 V_C \begin{pmatrix} I & 0 \\ 0 & L_{22}(s) + L_{21}(s)(I - L_{11}(s))^{-1} L_{12}(s) \end{pmatrix} V_B^T \\
&= \sigma_1 V_C \begin{pmatrix} I & 0 \\ 0 & F_u(L(s), I) \end{pmatrix} V_B^T \qquad (4.143)
\end{aligned}$$

where

$$L(s) = \begin{pmatrix} L_{11}(s) & L_{12}(s) \\ L_{21}(s) & L_{22}(s) \end{pmatrix} := V_C^T \frac{Y(s)}{\sigma_1} V_B. \qquad (4.144)$$

It is easy to verify that $F_u(L(s), I) = F_l \left(\left(\begin{array}{c|cc} 0 & 0 & I \\ \hline 0 & I & 0 \\ I & 0 & 0 \end{array} \right), L(s) \right)$. From Lemma 2.5 we conclude that $F_u(L(s), I) \in BH_\infty$. Therefore, every element of Set 2 is an element of Set 1. Conversely, from (4.142) and (4.143) we conclude that for every element in Set 1 we can find an element in Set 2 (elements of Set 1 are parameterized by $F(s) \in B_{\sigma_1} H_\infty$; for every $F(s)$ pick $L_{22}(s) = F(s)/\sigma_1$, $L_{12}(s) = 0$, $L_{21}(s) = 0$ and $L_{11}(s) = I$).

Theorem 4.7 *An alternative representation of the optimal solution is given as follows:*

$$Q(s) = F_l(T_0(s), K(s)) \qquad (4.145)$$

with

$$T_0(s) = \left[\begin{array}{c|cc} -\Gamma^{-1} \left(\sigma_1^2 A_{22}^T + \Sigma_1 A_{22} \Sigma_1 \right) & -\Gamma^{-1} \Sigma_1 B_2 & \Gamma^{-1} C_2^T \\ \hline C_2 \Sigma_1 & D & -I \\ B_2^T & I & 0 \end{array} \right] \qquad (4.146)$$

where

$$K(s) \in B_{\sigma_1} H_\infty \tag{4.147}$$

$$\sigma_1 B_1 = C_1^T K(s). \tag{4.148}$$

Proof: It is obtained by direct verification; namely,

$$Q(s) = F_l\left(H_0(s), Y(s)\right) = F_l\left(T_0(s), F_l\left(J, Y(s)\right)\right). \tag{4.149}$$

■

Corollary 4.2 *If we pick*

$$Y(s) = V_C \begin{pmatrix} 0 & 0 \\ 0 & F(s) \end{pmatrix} V_B^T \tag{4.150}$$

or

$$Y(s) = V_C \begin{pmatrix} \sigma_1 I_{r_1} & 0 \\ 0 & F(s) \end{pmatrix} V_B^T \tag{4.151}$$

with $\| F \|_\infty \leq \sigma_1$,

we cover all the possible optimal solutions. In particular, if we take

$$Y(s) = V_C \begin{pmatrix} \sigma_1 I_{r_1} & 0 \\ 0 & F(s) \end{pmatrix} V_B^T, \tag{4.152}$$

we have

$$K(s) = Y(s) = \sigma_1 F_l(U, Y(s)/\sigma_1). \tag{4.153}$$

Proof: Immediate from the previous lemma. ■

Remark 4.1 *The solution* $Q(s) = F_l(T_0(s), K(s))$ *together with conditions* $\| K \|_\infty \leq \sigma_1$ *and* $\sigma_1 B_1 = C_1^T K(s)$ *is exactly the same solution given in Glover [12].*

Chapter 5

Parameterization of All Solutions for the Four-Block GDP: the Suboptimal Case

The GDP is defined as follows: *Given a minimal realization for* $R(s) \in RH_\infty^-$

$$R(s) = \begin{matrix} & m_1 & m_2 \\ n_1 & \multicolumn{2}{c}{\multirow{2}{*}{}} \end{matrix} \begin{pmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{5.1}$$

and a γ*, find a* $Q(s) \in H_\infty$ *such that*

$$\left\| \begin{matrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{matrix} \right\|_\infty \leq \gamma \tag{5.2}$$

is satisfied.

Without loss of generality we take $n_1 = m_1$. If $n_1 \neq m_1$ then we add some 0 rows or columns to $R(s)$ to get a $\hat{R}(s)$ that satisfies $n_1 = m_1$. Note that adding 0 columns or rows does not change the norm. We solve the problem for $\hat{R}(s)$ and get the solution to the original problem by compression.

We choose a γ such that

$$\gamma > \gamma_R := \max \left\{ \left\| \begin{array}{c} R_{12} \\ R_{22} \end{array} \right\|_\infty , \| \; R_{21} \quad R_{22} \; \|_\infty \right\}. \tag{5.3}$$

Necessity of (5.3) is obvious from (5.2).

5.1 Equivalence of the Four-Block Problem with the One-Block Problem

In this section we prove that obtaining a parameterization of all solutions, $Q(s)$, for the four-block problem (5.2) is equivalent to solving a special one-block problem of larger dimension. This will be clear by the end of this section.

First, we quote a theorem from [12], the proof of which is based on the Positive Real Lemma.

Theorem 5.1 (see [12], Thm. 5.2) *Given a rational $p \times m$, transfer function $G(s) \in RL_\infty$ of McMillan degree n with a minimal realization* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ *such that*

$$\|G\|_\infty \leq \gamma; \tag{5.4}$$

then there exist

$$\hat{D} := \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D \end{pmatrix}_{(p+m)\times(p+m)} \tag{5.5}$$

$$\hat{B} := \begin{pmatrix} B_0 & B \end{pmatrix}_{n\times(p+m)} \tag{5.6}$$

$$\hat{C} := \begin{pmatrix} C_0 \\ C \end{pmatrix}_{(p+m)\times n} \tag{5.7}$$

such that, given

$$H(s) := \hat{D} + \hat{C}\,(sI - A)^{-1}\,\hat{B}, \tag{5.8}$$

then $H(s)$ is γ-all-pass; i.e. $H^{\sim}(s)H(s) = \gamma^2 I$.

Lemma 5.1 *Given* $\left\| \begin{matrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{matrix} \right\|_{\infty} \leq \gamma$, *we can always obtain a square matrix $E(s)$:*

$$E(s) = \left[\begin{array}{cc|cccc} A & 0 & B_x & B_y & B_1 & B_2 \\ 0 & A_Q & B_{q_1} & B_{q_2} & B_Q & 0 \\ \hline C_x & -C_{q_1} & -d_{11} & -d_{12} & -d_{13} & d_{14} \\ C_y & -C_{q_2} & -d_{21} & -d_{22} & -d_{23} & d_{24} \\ C_1 & -C_Q & -d_{31} & -d_{32} & D_{11} - D_Q & D_{12} \\ C_2 & 0 & d_{41} & d_{42} & D_{21} & D_{22} \end{array}\right] \tag{5.9}$$

such that E is γ- all-pass. (The partition is defined such that d_{24} and d_{42} are square matrices.)

Proof: If (5.2) is satisfied for a given $Q(s) = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]$, then we have

$$\begin{pmatrix} R_{11}(s) - Q(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix} = \left[\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ 0 & A_Q & B_Q & 0 \\ \hline C_1 & -C_Q & D_{11} - D_Q & D_{12} \\ C_2 & 0 & D_{21} & D_{22} \end{array}\right]. \tag{5.10}$$

By applying Theorem 5.1 on $\begin{pmatrix} R_{11}(s) - Q(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix}$, we get

$$E(s) = \left[\begin{array}{cc|cccc} A & 0 & B_x & B_y & B_1 & B_2 \\ 0 & A_Q & B_{q_1} & B_{q_2} & B_Q & 0 \\ \hline C_x & -C_{q_1} & -d_{11} & -d_{12} & -d_{13} & d_{14} \\ C_y & -C_{q_2} & -d_{21} & -d_{22} & -d_{23} & d_{24} \\ C_1 & -C_Q & -d_{31} & -d_{32} & D_{11} - D_Q & D_{12} \\ C_2 & 0 & d_{41} & d_{42} & D_{21} & D_{22} \end{array}\right]$$

$$= \begin{pmatrix} Z_{11}(s) - Q_{11}(s) & Z_{12}(s) - Q_{12}(s) & Z_{13}(s) - Q_{13}(s) & Z_{14}(s) \\ Z_{21}(s) - Q_{21}(s) & Z_{22}(s) - Q_{22}(s) & Z_{23}(s) - Q_{23}(s) & Z_{24}(s) \\ Z_{31}(s) - Q_{31}(s) & Z_{32}(s) - Q_{32}(s) & R_{11}(s) - Q(s) & R_{12}(s) \\ Z_{41}(s) & Z_{42}(s) & R_{21}(s) & R_{22}(s) \end{pmatrix}. \tag{5.11}$$

■

We now assume that for a given $R(s)$ and $Q(s)$, $E(s)$, which is given by 5.11, has already been calculated according to Lemma 5.1.

First consider $E^{\sim}(s)$. Note that $E(s)$ is γ-all-pass if and only if $E^{\sim}(s)$ is γ-all-pass.

Next, we calculate the outer factors of $\begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \end{pmatrix}^{\sim}$ and $\begin{pmatrix} Z_{41}(s) & Z_{42}(s) \end{pmatrix}^{\sim}$. We know that every transfer matrix has an inner-outer factorization [31] . The outer factors are defined as follows:

$$Z_{1O}(s) := \text{outer part of } \begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \end{pmatrix}^{\sim} = \text{spectral factor of} \left\{ \begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \end{pmatrix}^{\sim} \begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \end{pmatrix} \right\} \tag{5.12}$$

$$\begin{aligned} Z_{2O}(s) &:= \text{outer part of} \begin{pmatrix} Z_{41}(s) & Z_{42}(s) \end{pmatrix}^{\sim} \\ &= \text{spectral factor of} \left\{ \begin{pmatrix} Z_{41}(s) & Z_{42}(s) \end{pmatrix} \begin{pmatrix} Z_{41}(s) & Z_{42}(s) \end{pmatrix}^{\sim} \right\}. \end{aligned} \tag{5.13}$$

Since $E^{\sim}(s)$ is γ- all-pass then we have

$$\begin{pmatrix} Z_{14}^{\sim}(s) & Z_{24}^{\sim}(s) & R_{12}^{\sim}(s) & R_{22}^{\sim}(s) \end{pmatrix} \begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \\ R_{12}(s) \\ R_{22}(s) \end{pmatrix} = \gamma^2 I \tag{5.14}$$

and

$$\begin{pmatrix} Z_{41}(s) & Z_{42}(s) & R_{21}(s) & R_{22}(s) \end{pmatrix} \begin{pmatrix} Z_{41}^{\sim}(s) \\ Z_{42}^{\sim}(s) \\ R_{21}^{\sim}(s) \\ R_{22}^{\sim}(s) \end{pmatrix} = \gamma^2 I. \tag{5.15}$$

From (5.14) we obtain

$$Z_{1O}^{\sim} Z_{1O}(s) = \begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \end{pmatrix}^{\sim} \begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \end{pmatrix} = \gamma^2 I - \begin{pmatrix} R_{12}(s) \\ R_{22}(s) \end{pmatrix}^{\sim} \begin{pmatrix} R_{12}(s) \\ R_{22}(s) \end{pmatrix}. \tag{5.16}$$

(5.16) is a spectral factorization problem and Z_{1O} is the spectral factor.

Another way of looking at (5.16) is as follows: (5.16) is equivalent to

$$\begin{pmatrix} Z_{1O}^{\sim}(s) \\ R_{12}(s) \\ R_{22}(s) \end{pmatrix}^{\sim} \begin{pmatrix} Z_{1O}^{\sim}(s) \\ R_{12}(s) \\ R_{22}(s) \end{pmatrix} = \gamma^2 I. \tag{5.17}$$

This means that $G_{12}(s) := \begin{pmatrix} Z_{1O}^{\sim}(s) \\ R_{12}(s) \\ R_{22}(s) \end{pmatrix}$ is γ-anti-inner. In state-space,

$$G_{12}(s) = \left[\begin{array}{c|c} A & B_2 \\ \hline C_0 & u_{13} \\ C_1 & D_{12} \\ C_2 & D_{22} \end{array}\right] \tag{5.18}$$

where

$$X_{13}(s) := Z_{1O}^{\sim}(s) = \left[\begin{array}{c|c} A & B_2 \\ \hline C_0 & u_{13} \end{array}\right]. \tag{5.19}$$

Similarly, for $X_{31}(s)$ we obtain the condition that $G_{21}(s) := \begin{pmatrix} X_{31}(s) & R_{21}(s) & R_{22}(s) \end{pmatrix}$ is γ-anti-co-inner. In state-space

$$G_{21}(s) = \left[\begin{array}{c|ccc} A & B_0 & B_1 & B_2 \\ \hline C_2 & U_{31} & D_{21} & D_{22} \end{array}\right] \tag{5.20}$$

where

$$X_{31}(s) = \left[\begin{array}{c|c} A & B_0 \\ \hline C_2 & u_{31} \end{array}\right]. \tag{5.21}$$

Remark 5.1 *Since $Z_{1O}(s)$ is a spectral factor, then $X_{13}(s) = Z_{1O}^{\sim}(s)$ is an anti-outer function. Similarly, $X_{31}(s)$ is an anti-outer function.*

We use Lemma 2.2 and Lemma 2.3 to calculate B_0, C_0, u_{31} and u_{13} so that $G_{12}(s)$ is γ-anti-inner and $G_{21}(s)$ is γ-anti-co-inner. First we apply Lemma 2.2 on $G_{12}(s)$ given by (5.18) and solve for $X_{13}(s)$:

From (2.33)

$$u_{13}^T u_{13} + D_{12}^T D_{12} + D_{22}^T D_{22} = \gamma^2 I. \tag{5.22}$$

From (2.31)

$$A^T L_o + L_o A = C_0^T C_0 + C_1^T C_1 + C_2^T C_2. \tag{5.23}$$

From (2.32)

$$\begin{pmatrix} u_{13}^T & D_{12}^T & D_{22}^T \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix} - B_2^T L_o = 0. \tag{5.24}$$

From (5.22) we solve for u_{13}:

$$u_{13}^T u_{13} = \Theta := \gamma^2 I - D_{12}^T D_{12} - D_{22}^T D_{22}. \tag{5.25}$$

Note that $\Theta > 0$ from (5.3). From (5.24) we solve for C_0:

$$C_0 = -u_{13}^{-T}(D_{12}^T C_1 + D_{22}^T C_2 - B_2^T L_o). \tag{5.26}$$

If we substitute C_0 in (5.23), after rearrangement, we obtain the following Riccati equation for L_o:

$$\left[A + B_2\Theta^{-1}\left(D_{12}^T C_1 + D_{22}^T C_2\right)\right]^T L_o + L_o\left[A + B_2\Theta^{-1}\left(D_{12}^T C_1 + D_{22}^T C_2\right)\right]$$

$$-L_o B_2 \Theta^{-1} B_2^T L_o - \begin{pmatrix} C_1^T & C_2^T \end{pmatrix} \left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} D_{12} \\ D_{22} \end{pmatrix} \Theta^{-1} \begin{pmatrix} D_{12}^T & D_{22}^T \end{pmatrix} \right\} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0. \tag{5.27}$$

We need to take a symmetric, positive-definite solution of (5.27) such that $A + B_2\Theta^{-1}(D_{12}^T C_1 + D_{22}^T C_2) - B_2\Theta^{-1}B_2^T L_o$ is a matrix with all the eigenvalues in C_+. This condition is equivalent to having $X_{13}(s)$ anti-outer (i.e., $Z_{1O}(s)$ outer). (We get exactly the same solution if we apply the spectral factorization theorem on (5.16) [1], [9].)

Similarly, to calculate $X_{31}(s)$ we apply Lemma 2.3 on $G_{21}(s)$ given by (5.20). Similarly from (2.34), we obtain

$$AL_c + L_cA^T = B_0B_0^T + B_1B_1^T + B_2B_2^T. \tag{5.28}$$

From (2.35), we obtain

$$C_2L_c - \begin{pmatrix} u_{31} & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} B_0^T \\ B_1^T \\ B_2^T \end{pmatrix} = 0. \tag{5.29}$$

From (2.36), we get

$$u_{31}u_{31}^T = \Phi := \gamma^2 I - D_{21}D_{21}^T - D_{22}D_{22}^T, \tag{5.30}$$

where $\Phi > 0$ by (5.3). Also B_0 is given by

$$B_0 = -\left(B_1D_{21}^T + B_2D_{22}^T - L_cC_2^T\right) u_{31}^{-T}, \tag{5.31}$$

where L_c is a symmetric, positive definite solution of another Riccati equation:

$$\left[A + \left(B_1D_{21}^T + B_2D_{22}^T\right)\Phi^{-1}C_2\right] L_c + L_c\left[A + \left(B_1D_{21}^T + B_2D_{22}^T\right)\Phi^{-1}C_2\right]^T$$

$$-L_cC_2^T\Phi^{-1}C_2L_c - (B_1\ B_2)\left\{\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} D_{21}^T \\ D_{22}^T \end{pmatrix}\Phi^{-1}(D_{21}\ D_{22})\right\}\begin{pmatrix} B_1^T \\ B_2^T \end{pmatrix} = 0. \tag{5.32}$$

such that $A + (B_1D_{21}^T + B_2D_{22}^T)\Phi C_2 - L_cC_2^T\Phi C_2$ is a matrix with all the eigenvalues in C_+. This condition is equivalent to having $X_{31}^{-1}(s)$ anti-stable.

Remark 5.2 *Notice that the anti-outer factors $X_{13}(s)$ and $X_{31}(s)$ are independent of the solution, $Q(s)$. They depend only on γ, the optimality level, and $R(s)$. This is an essential observation for the results that follow.*

Define the anti-inner factors as follows:

$$\begin{pmatrix} Z_{1i} \\ Z_{2i} \end{pmatrix} := \text{inner factor of } \begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \end{pmatrix}, \tag{5.33}$$

and

$$\begin{pmatrix} Y_{1i} & Y_{2i} \end{pmatrix} := \text{co-inner factor of } \begin{pmatrix} Z_{41}(s) & Z_{42}(s) \end{pmatrix}, \tag{5.34}$$

where both $\begin{pmatrix} Z_{1i} \\ Z_{2i} \end{pmatrix}$ and $\begin{pmatrix} Y_{1i} & Y_{2i} \end{pmatrix}$ are antistable functions. We have the following anti-inner-outer factorizations:

$$\begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \end{pmatrix} = \begin{pmatrix} Z_{1i} \\ Z_{2i} \end{pmatrix} X_{13}(s) \tag{5.35}$$

$$\begin{pmatrix} Z_{41}(s) & Z_{42}(s) \end{pmatrix} = X_{31}(s) \begin{pmatrix} Y_{1i} & Y_{2i} \end{pmatrix}. \tag{5.36}$$

We are now ready to state one of the most important theorems of this thesis:

Theorem 5.2 *Given $R(s)$ and $Q(s)$ with minimal realizations*

$$R(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{5.37}$$

$$Q(s) = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] \tag{5.38}$$

satisfying

$$\left\| \begin{matrix} R_{11}(s) - Q(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{matrix} \right\|_\infty \leq \gamma, \tag{5.39}$$

we can always find $\hat{Q}_{11}(s)$, $\hat{Q}_{12}(s)$ and $\hat{Q}_{21}(s)$ such that

$$\left\| X - \hat{Q} \right\|_\infty = \gamma \tag{5.40}$$

where

$$X(s) = \left[\begin{array}{c|cc|c} A & B_0 & B_1 & B_2 \\ \hline C_0 & 0 & 0 & u_{13} \\ C_1 & 0 & D_{11} & D_{12} \\ \hline C_2 & u_{31} & D_{21} & D_{22} \end{array}\right] = \left(\begin{array}{cc|c} X_{11}(s) & X_{12}(s) & X_{13}(s) \\ X_{21}(s) & R_{11}(s) & R_{12}(s) \\ \hline X_{31}(s) & R_{21}(s) & R_{22}(s) \end{array}\right) \tag{5.41}$$

$$\hat{Q}(s) = \begin{pmatrix} \hat{Q}_{11}(s) & \hat{Q}_{12}(s) & 0 \\ \hat{Q}_{21}(s) & Q(s) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.42}$$

Proof: The proof is obtained starting from $E(s)$ in (5.11). Instead of $\begin{pmatrix} Z_{14}(s) \\ Z_{24}(s) \end{pmatrix}$ substitute (5.35) and instead of $\begin{pmatrix} Z_{41}(s) & Z_{42}(s) \end{pmatrix}$ substitute (5.36).

Define

$$V_L(s) = \begin{pmatrix} Z_{1i}^{\sim}(s) & Z_{2i}^{\sim}(s) & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \tag{5.43}$$

$$V_R(s) = \begin{pmatrix} Y_{1i}^{\sim}(s) & 0 & 0 \\ Y_{2i}^{\sim}(s) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}. \tag{5.44}$$

and note that $V_L V_L^* = I$, $V_R^* V_R = I$. Then, $\|V_L E V_R\|_\infty \leq \|V_L\|_\infty \|E\|_\infty \|V_R\|_\infty = \|E\|_\infty = \gamma$. If we study $V_L(s)E(s)V_R(s)$, we conclude that the last block-column and the last block rows have norm equal to γ.

It is easy to see that

$$X(s) = [V_L(s)E(s)V_R(s)]_- + \left(\begin{array}{cc|c} 0 & 0 & u_{13} \\ 0 & D_{11} & D_{12} \\ \hline u_{31} & D_{21} & D_{22} \end{array}\right) = \left[\begin{array}{c|cc|c} A & B_0 & B_1 & B_2 \\ \hline C_0 & 0 & 0 & u_{13} \\ C_1 & 0 & D_{11} & D_{12} \\ \cline{1-1}\cline{2-3} C_2 & u_{31} & D_{21} & D_{22} \end{array}\right]. \tag{5.45}$$

Define

$$\hat{Q}(s) = [V_L(s)E(s)V_R(s)]_+ - \left(\begin{array}{cc|c} 0 & 0 & u_{13} \\ 0 & D_{11} & D_{12} \\ \hline u_{31} & D_{21} & D_{22} \end{array}\right) = \begin{pmatrix} \hat{Q}_1(s) & 0 \\ 0 & 0 \end{pmatrix} \tag{5.46}$$

and observe that $Q(s) = \begin{pmatrix} 0 & I \end{pmatrix} \hat{Q}_1(s) \begin{pmatrix} 0 \\ I \end{pmatrix}$. ■

An immediate result of Theorem 5.2 is the following result:

Theorem 5.3 *Given $R(s)$ with minimal realization*

$$R(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \tag{5.47}$$

all the solutions satisfying $\left\| \begin{matrix} R_{11}(s) - Q(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{matrix} \right\|_\infty \leq \gamma$ *for any given* γ *can be obtained by the following steps:*

i) Given $R(s)$ and γ, calculate $X(s)$.

ii) Find all the solutions satisfying $\|X - Q'\|_\infty \leq \gamma$.

iii) Find the subset of the set Q', which has the form

$$\hat{Q}(s) = \begin{pmatrix} \hat{Q}_1(s) & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.48}$$

Then

$$Q(s) = \begin{pmatrix} 0 & I & 0 \end{pmatrix} \hat{Q}(s) \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix}, \tag{5.49}$$

where all the partitions are made to conform with the partition on $R(s)$.

5.2 All Solutions to the Four-Block GDP

In this section, we characterize the set of all solutions of the Nehari extension problem (5.40), which are of the form (5.48). According to Theorem 5.3, this set of solutions gives the set of all finite dimensional solutions for the four-block GDP.

Note that L_o and L_c solving the Riccati Equations (5.27) and (5.32), respectively, also give the observability and controllability grammians of $G(s)$ (see (5.23) and (5.28)).

A solution for (5.39) exists if and only if both of the following inequalities hold:

$$\gamma \geq \gamma_1 := \rho^{1/2}[L_o(\gamma)L_c(\gamma)] \tag{5.50}$$

$$\gamma \geq \gamma_R. \tag{5.51}$$

It can be shown that γ_1 is a nonincreasing function of γ [8]. Thus, if (5.50) is not satisfied, γ is increased until this is so. Note that γ_0, the minimum achievable error in (5.39), is obtained when (5.50) and/or (5.51) are satisfied as an equality. γ_0 is computed by means of a bisection procedure. Each time we update γ we need to solve the two Riccati Equations (5.27) and (5.32) again since they also depend on γ. In this chapter we solve the case when $\gamma > \gamma_0$, the suboptimal case. When $\gamma > \gamma_1$ is satisfied, we get the parameterization of all solutions for (5.40) from (4.63); i.e., the problem can then be treated as a one-block problem. It is necessary, however, to identify in this parameterization the solutions of the form in (5.48).

Theorem 5.4 *Given appropriately constructed, i.e., as in Theorem 5.2,*

$$X(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{c|cc|c} A & B_0 & B_1 & B_2 \\ \hline C_0 & 0 & 0 & u_{13} \\ C_1 & 0 & D_{11} & D_{12} \\ \hline C_2 & u_{31} & D_{21} & D_{22} \end{array}\right], \tag{5.52}$$

the solutions of

$$\|X - \hat{Q}\|_\infty \leq \gamma \tag{5.53}$$

which are of the form

$$\hat{Q}(s) = \begin{pmatrix} \hat{Q}_1(s) & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.54}$$

are given by (4.63) when and only when $Y(s) \in B_\gamma H_\infty$ is of the form

$$Y(s) = \begin{pmatrix} y_{11}(s) & y_{12}(s) & u_{13} \\ y_{21}(s) & y_{22}(s) & D_{12} \\ u_{31} & D_{21} & D_{22} \end{pmatrix}. \tag{5.55}$$

Theorem 5.4 provides a parameterization of all solutions to the four-block GDP in terms of $Y_1(s) := \begin{pmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{pmatrix}$.

Proof: First partition $Y(s)$, $W(s)$ and $V(s)$ conformally with $G(s)$:

$$Y(s) =: \begin{pmatrix} Y_{11}(s) & Y_{12}(s) \\ Y_{21}(s) & Y_{22}(s) \end{pmatrix}, \tag{5.56}$$

$$W(s) =: \begin{pmatrix} W_{11}(s) & W_{12}(s) \\ W_{21}(s) & W_{22}(s) \end{pmatrix} \text{ and} \tag{5.57}$$

$$V(s) =: \begin{pmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{pmatrix}, \tag{5.58}$$

where from (4.54) and (4.55) $W(s)$ and $V(s)$ are given by

$$\begin{aligned} W(s) &= L_3Y(s) + L_4 \\ &= B^T\left(sI + A^T\right)^{-1} N(\gamma)\left(L_oB - C^TY(s)\right) + I \end{aligned} \tag{5.59}$$

$$\begin{aligned} V(s) &= -L_5Y(s) + L_6 \\ &= (DB^T - CL_c)\left(sI + A^T\right)^{-1} N(\gamma)\left(L_oB - C^TY(s)\right) + D - Y(s). \end{aligned} \tag{5.60}$$

From (5.29) and (5.60) we obtain $V_{21}(s) = \begin{pmatrix} u_{31} & D_{21} \end{pmatrix} - Y_{21}(s)$ and $V_{22}(s) = D_{22} - Y_{22}(s)$. From (4.63) we have

$$\hat{Q}(s)W(s) = V(s). \tag{5.61}$$

By making use of the fact that $\hat{Q}(s)$ has to be as in (5.54), we get the following identity:

$$\begin{pmatrix} Q_{11}(s)W_{11}(s) & Q_{11}(s)W_{12}(s) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{pmatrix}. \tag{5.62}$$

From (5.62) we conclude that $V_{21}(s) = \begin{pmatrix} u_{31} & D_{21} \end{pmatrix} - Y_{21}(s) = 0$ and $V_{22}(s) = D_{22} - Y_{22}(s) = 0$. We then have $Y_{21}(s) = \begin{pmatrix} u_{31} & D_{21} \end{pmatrix}$, $Y_{22}(s) = D_{22}$, and $Y(s)$ is of the form:

$$Y(s) = \begin{pmatrix} Y_{11}(s) & Y_{12}(s) \\ u_{31} \;\; D_{21} & D_{22} \end{pmatrix}. \tag{5.63}$$

We next determine $Y_{12}(s)$. From (5.61) together with (5.59) and (5.60), we get

$$Q(s)B^T\left(sI+A^T\right)^{-1}N(\gamma)\left(L_oB-C^TY(s)\right)+Q(s)=$$
$$=(DB^T-CL_c)\left(sI+A^T\right)^{-1}N(\gamma)\left(L_oB-C^TY(s)\right)+D-Y(s). \tag{5.64}$$

If we calculate the last columns of both sides of this equality and use (5.24), we get the following equation:

$$-Q(s)B^T(sI+A^T)^{-1}N(\gamma)C^TY_0(s)=-(DB^T-CL_c)(sI+A^T)^{-1}N(\gamma)C^TY_0(s)-Y_0(s), \tag{5.65}$$

where $Y_0(s)$ is defined by

$$Y_0(s):=\begin{pmatrix} Y_{12}(s)-\begin{pmatrix} u_{13} \\ D_{12} \end{pmatrix} \\ 0 \end{pmatrix}, \tag{5.66}$$

and from this we get

$$\left(Q(s)L_3(s)+L_5(s)\right)Y_0=0 \quad \forall s. \tag{5.67}$$

For (5.67) to be true, we must have either $Y_0(s)=0$, $\forall s$ or $H(s)=Q(s)L_3(s)+L_5(s)=0$, $\forall s$. Since $H(\infty)=I$, we have to have $Y_0(s)=0$, $\forall s$. Then, from (5.66) we conclude that

$$Y_{12}(s)=\begin{pmatrix} u_{13} \\ D_{12} \end{pmatrix}. \tag{5.68}$$

Therefore, we showed that $Y(s)$ necessarily has the form in (5.55). Next we show that if $Y(s)$ is as in (5.55), then $Q(s)$ from (4.63) is of the form (5.54). From (5.59) together with (5.24), we get

$$W(s)=\begin{pmatrix} W_{11}(s) & 0 \\ W_{21}(s) & I \end{pmatrix}. \tag{5.69}$$

From (5.60), (5.24) together with (5.29), we get

$$V(s) = \begin{pmatrix} V_{11}(s) & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.70}$$

Then from (5.69), we get

$$W^{-1}(s) = \begin{pmatrix} W_{11}^{-1}(s) & 0 \\ -W_{21}(s)W_{11}^{-1}(s) & I \end{pmatrix}. \tag{5.71}$$

From (5.70) and (5.71), we conclude that

$$\hat{Q}(s) = V(s)W^{-1}(s) = \begin{pmatrix} V_{11}(s)W_{11}^{-1}(s) & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.72}$$

which has the desired form. ■

Once we have $\hat{Q}(s)$, we calculate a solution to the original problem $Q(s)$, by taking the subblock of $\hat{Q}(s)$ corresponding to $R_{11}(s)$ in $X(s)$. From Theorem 5.3 it follows that all such finite dimensional solutions are covered by the parameterization given by Theorem 5.4.

The following lemma gives a complete parameterization for $Y_1(s)$ so that $Y(s)$ in (5.55) is as required in Theorem 5. First we define γ_D and $\gamma_{D_{22}}$ as follows:

$$\gamma_D := max \left\{ \left\| \begin{matrix} D_{12} \\ D_{22} \end{matrix} \right\|_\infty, \left\| \begin{matrix} D_{21} & D_{22} \end{matrix} \right\|_\infty \right\}. \tag{5.73}$$

$$\gamma_{D_{22}} := \sigma_{\max}(D_{22}) \tag{5.74}$$

Lemma 5.2 *With the parameters defined as in Section 5.2, for a given $\gamma > \gamma_D$, the complete set of* $\begin{pmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{pmatrix} \in H_\infty$ *satisfying*

$$\left\| \begin{matrix} y_{11}(s) & y_{12}(s) & u_{13} \\ y_{21}(s) & y_{22}(s) & D_{12} \\ u_{31} & D_{21} & D_{22} \end{matrix} \right\|_\infty = \gamma \tag{5.75}$$

is given by

$$\begin{pmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{pmatrix} = U(\gamma) \begin{pmatrix} -D_{22}^T & 0 \\ 0 & m(s) \end{pmatrix} V^T(\gamma) \tag{5.76}$$

where $m(s) \in H_\infty$ *with* $\|m\|_\infty \leq \gamma$. $U(\gamma)$ *and* $V(\gamma)$ *are defined through the following singular-value decompositions:*

$$Z_1 := \begin{pmatrix} u_{13} \\ D_{12} \end{pmatrix} \left(\gamma^2 I - D_{22}^T D_{22}\right)^{-1/2} \tag{5.77}$$

$$= U(\gamma) \begin{pmatrix} I_{m_2} \\ 0 \end{pmatrix}, \tag{5.78}$$

$$Z_2 := \left(\gamma^2 I - D_{22} D_{22}^T\right)^{-1/2} \begin{pmatrix} u_{31} & D_{21} \end{pmatrix} \tag{5.79}$$

$$= \begin{pmatrix} I_{n_2} & 0 \end{pmatrix} V^T(\gamma) \tag{5.80}$$

where

$$U(\gamma) = \begin{pmatrix} u_{13}\left(\gamma^2 I - D_{22}^T D_{22}\right)^{-1/2} & -u_{13}^{-T} D_{12}^T \left(I + D_{12}(u_{13}^T u_{13})^{-1} D_{12}^T\right)^{-1/2} \\ D_{12}\left(\gamma^2 I - D_{22}^T D_{22}\right)^{-1/2} & \left(I + D_{12}(u_{13}^T u_{13})^{-1} D_{12}^T\right)^{-1/2} \end{pmatrix} \tag{5.81}$$

$$V(\gamma) = \begin{pmatrix} u_{31}^T\left(\gamma^2 I - D_{22} D_{22}^T\right)^{-1/2} & -u_{31}^{-1} D_{21} \left(I + D_{21}^T(u_{31} u_{31}^T)^{-1} D_{21}\right)^{-1/2} \\ D_{21}^T\left(\gamma^2 I - D_{22} D_{22}^T\right)^{-1/2} & \left(I + D_{21}^T(u_{31} u_{31}^T)^{-1} D_{21}\right)^{-1/2} \end{pmatrix}. \tag{5.82}$$

Proof: The solution can be obtained as

$$\begin{pmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{pmatrix} = -Z_1 D_{22}^T Z_2 + \left(I - Z_1 Z_1^T\right)^{1/2} M(s) \left(I - Z_2^T Z_2\right)^{1/2} \tag{5.83}$$

where $M(s) \in H_\infty$ with $\|M\|_\infty \le \gamma$. Z_1 and Z_2 are given by (5.77) and (5.79) (for proof of this fact see, for example, [5]). The Equations (5.78) and (5.80) follow from the fact that

$$Z_1^T Z_1 = I_{m_2} \tag{5.84}$$

$$Z_2 Z_2^T = I_{n_2}. \tag{5.85}$$

The result is simply obtained by using (5.78) and (5.80) in (5.83). We obtained (5.81) and (5.82) by inspection. ■

Lemma 5.3 *Parameterization of all $Y(s)$ is given as follows:*

$$Y(s) = T_l(\gamma) Y_0(s) T_r(\gamma) \tag{5.86}$$

where

$$Y_0(s) := \left(\begin{array}{cc|c} -D_{22}^T & 0 & \left(\gamma^2 I - D_{22}^T D_{22}\right)^{1/2} \\ 0 & m(s) & 0 \\ \hline \left(\gamma^2 I - D_{22} D_{22}^T\right)^{1/2} & 0 & D_{22} \end{array}\right) \tag{5.87}$$

$$T_l(\gamma) := \begin{pmatrix} U(\gamma) & 0 \\ 0 & I \end{pmatrix} \tag{5.88}$$

$$T_r(\gamma) := \begin{pmatrix} V^T(\gamma) & 0 \\ 0 & I \end{pmatrix}. \tag{5.89}$$

Proof: From (5.77) and (5.78) we obtain

$$\begin{pmatrix} u_{13} \\ D_{12} \end{pmatrix} = U(\gamma) \begin{pmatrix} \left(\gamma^2 I - D_{22}^T D_{22}\right)^{1/2} \\ 0 \end{pmatrix}. \tag{5.90}$$

From (5.79) and (5.80), we obtain

$$\begin{pmatrix} u_{31} & D_{21} \end{pmatrix} = \begin{pmatrix} \left(\gamma^2 I - D_{22}D_{22}^T\right)^{1/2} & 0 \end{pmatrix} V^T(\gamma). \tag{5.91}$$

We then substitute (5.76), (5.90) and (5.91) in (5.75) to get (5.86). ■

Lemma 5.4 *An alternative solution of the set of* $Y = \begin{pmatrix} y_{11}(s) & y_{12}(s) & u_{13} \\ y_{21}(s) & y_{22}(s) & D_{12} \\ u_{31} & D_{21}(s) & D_{22} \end{pmatrix} \in$ $B_\gamma H_\infty$ *is given as follows:*

$$Y(s) = U_1 + U_2 M(s) U_3 \tag{5.92}$$

where $M(s) \in B_\gamma H_\infty$. *We have the following definitions:*

$$U_1 = \begin{pmatrix} -u_{13}D_{22}^T(\gamma^2 I - D_{22}D_{22}^T)^{-1}u_{31} & -u_{13}D_{22}^T(\gamma^2 I - D_{22}D_{22}^T)^{-1}D_{21} & u_{13} \\ -D_{12}D_{22}^T(\gamma^2 I - D_{22}D_{22}^T)^{-1}u_{31} & -D_{12}D_{22}^T(\gamma^2 I - D_{22}D_{22}^T)^{-1}D_{21} & D_{12} \\ u_{31} & D_{21} & D_{22} \end{pmatrix} \tag{5.93}$$

$$U_2 = \begin{pmatrix} I - u_{13}(\gamma^2 I - D_{22}^T D_{22})^{-1}u_{13}^T & -u_{13}(\gamma^2 I - D_{22}^T D_{22})^{-1}D_{12}^T \\ -D_{12}(\gamma^2 I - D_{22}^T D_{22})^{-1}u_{13}^T & I - D_{12}(\gamma^2 I - D_{22}^T D_{22})^{-1}D_{12}^T \\ 0 & 0 \end{pmatrix} \tag{5.94}$$

$$U_3 = \begin{pmatrix} I - u_{31}^T(\gamma^2 I - D_{22}D_{22}^T)^{-1}u_{31} & -u_{31}^T(\gamma^2 I - D_{22}D_{22}^T)^{-1}D_{21} & 0 \\ -D_{21}^T(\gamma^2 I - D_{22}D_{22}^T)^{-1}u_{31} & I - D_{21}^T(\gamma^2 I - D_{22}D_{22}^T)^{-1}D_{21} & 0 \end{pmatrix}. \tag{5.95}$$

Proof: From (5.83), we have

$$Y = \begin{pmatrix} -Z_1 D_{22}^T Z_2 + (I - Z_1 Z_1^T)^{1/2} M(s) (I - Z_2^T Z_2)^{1/2} & \begin{matrix} u_{13} \\ D_{12} \end{matrix} \\ u_{31} \quad D_{21} & D_{22} \end{pmatrix}. \tag{5.96}$$

For $M(s) = 0$ we have

$$U_1 = \begin{pmatrix} -Z_1 D_{22}^T Z_2 & \begin{matrix} u_{13} \\ D_{12} \end{matrix} \\ u_{31} \quad D_{21} & D_{22} \end{pmatrix}$$

(5.97)

$$U_2 M(s) U_3 = \begin{pmatrix} (I - Z_1 Z_1^T)^{1/2} \\ 0 \end{pmatrix} M(s) \begin{pmatrix} (I - Z_2^T Z_2)^{1/2} & 0 \end{pmatrix} \tag{5.98}$$

$$U_2 = \begin{pmatrix} (I - Z_1 Z_1^T)^{1/2} \\ 0 \end{pmatrix} \tag{5.99}$$

$$U_3 = \begin{pmatrix} (I - Z_2^T Z_2)^{1/2} & 0 \end{pmatrix}. \tag{5.100}$$

Substitute the definition of Z_1 and Z_2

$$\begin{aligned} Z_1 D_{22}^T Z_2 &= \begin{pmatrix} u_{13} \\ D_{12} \end{pmatrix} (\gamma^2 I - D_{22}^T D_{22})^{-1/2} D_{22}^T (\gamma^2 I - D_{22} D_{22}^T)^{-1/2} \begin{pmatrix} u_{31} & D_{21} \end{pmatrix} \\ &= \begin{pmatrix} u_{13} \\ D_{12} \end{pmatrix} D_{22}^T (\gamma^2 I - D_{22} D_{22}^T)^{-1} \begin{pmatrix} u_{31} & D_{21} \end{pmatrix}. \end{aligned} \tag{5.101}$$

Substitute 5.101 in 5.97 to get U_1. From 5.101, we obtain 5.93.

$$(I - Z_1 Z_1^T)(I - Z_1 Z_1^T) = I - 2 Z_1 Z_1^T + Z_1 Z_1^T Z_1 Z_1^T \tag{5.102}$$

From (5.102), it follows that

$$(I - Z_1 Z_1^T)(I - Z_1 Z_1^T) = I - 2 Z_1 Z_1^T + Z_1 Z_1^T = I - Z_1 Z_1^T. \tag{5.103}$$

Therefore

$$(I - Z_1 Z_1^T)^{1/2} = I - Z_1 Z_1^T. \tag{5.104}$$

Similarly

$$(I - Z_2^T Z_2)^{1/2} = I - Z_2^T Z_2. \tag{5.105}$$

The rest of the proof is obtained by simply substituting the definitions of Z_1 and Z_2 into 5.99 and 5.100. ■

Remark 5.3 *The parameterization given by Lemma 5.4 does not break down when u_{13} or/and u_{31} are singular, but it does break down if we have $\gamma_0 = \gamma_R = \gamma_D = \sigma_{max}(D_{22})$. The parameterization given by Lemma 5.3 makes use of the fact that u_{13} and u_{31} are nonsingular and extracts the redundancy in $M(s)$.*

The following lemma connects $Q(s)$ to $m(s)$ as a usual lower fractional transformation.

Theorem 5.5 *The solutions for the four-block problem, $Q(s)$, are given by a linear fractional transformation on $m(s)$ as follows:*

$$Q(s) = F_l\left(T(s), m(s)\right) \tag{5.106}$$

where

$$T(s) := \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right] \tag{5.107}$$

with

$$A_T := -N(\gamma)\left(\gamma^2 A^T + L_o A L_c - \tilde{C}^T M_0 \tilde{B}^T\right) \tag{5.108}$$

$$B_T := N(\gamma)\left(\begin{array}{cc} \left(-\tilde{C}_0^T D_{22}^T + C_2^T\left(\gamma^2 I - D_{22}D_{22}^T\right)^{1/2}\right) V_{21}^T(\gamma) - L_o B_1 & \tilde{C}_1^T \end{array}\right) \tag{5.109}$$

$$C_T := \left(\begin{array}{c} C_1 L_c - U_{21}(\gamma)\left(-D_{22}^T \tilde{B}_0^T + \left(\gamma^2 I - D_{22}^T D_{22}\right)^{1/2} B_2^T\right) \\ \tilde{B}_1^T \end{array}\right) \tag{5.110}$$

$$D_T := \begin{pmatrix} D_{11} + U_{21}(\gamma)D_{22}^T V_{21}^T(\gamma) & -U_{22}(\gamma) \\ V_{22}^T(\gamma) & 0 \end{pmatrix}. \tag{5.111}$$

We have the following definitions:

$$\begin{pmatrix} \tilde{B}_0 & \tilde{B}_1 \end{pmatrix} := \begin{pmatrix} B_0 & B_1 \end{pmatrix} V(\gamma) \tag{5.112}$$

$$\begin{pmatrix} \tilde{C}_0^T & \tilde{C}_1^T \end{pmatrix} := \begin{pmatrix} C_0^T & C_1^T \end{pmatrix} U(\gamma) \tag{5.113}$$

$$\tilde{B} := \begin{pmatrix} \tilde{B}_0 & \tilde{B}_1 & B_2 \end{pmatrix} \tag{5.114}$$

$$\tilde{C}^T := \begin{pmatrix} \tilde{C}_0^T & \tilde{C}_1^T & C_2^T \end{pmatrix} \tag{5.115}$$

$$M_0 := \left(\begin{array}{cc|c} -D_{22}^T & 0 & \left(\gamma^2 I - D_{22}^T D_{22}\right)^{1/2} \\ 0 & 0 & 0 \\ \hline \left(\gamma^2 I - D_{22} D_{22}^T\right)^{1/2} & 0 & D_{22} \end{array}\right). \tag{5.116}$$

Proof: The proof follows from the following observations:

$$Q(s) = \begin{pmatrix} 0 & I & 0 \end{pmatrix} \hat{Q}(s) \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix}, \tag{5.117}$$

$$\hat{Q}(s) = F_l\left(H(s), Y(s)\right), \tag{5.118}$$

$$Y(s) = T_l(\gamma) Y_0(s) T_r(\gamma), \tag{5.119}$$

$$Y_0(s) = M_0 + \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix} m(s) \begin{pmatrix} 0 & I & 0 \end{pmatrix}. \tag{5.120}$$

We see that (5.117)-(5.120) are linear fractional transformations. We obtain the result by using the rules to calculate

$$F_l\left(P(s), F_l\left(J(s), Q(s)\right)\right) = F_l\left(T(s), Q(s)\right). \tag{5.121}$$

The calculations are omitted being straightforward but lengthy. ■

Chapter 6

Parameterization of All Optimal Solutions for the Four-Block GDP

In this chapter we deal with the optimal solutions of the four-block GDP. Given $R(s)$, obtaining the optimal solutions consists of three steps: First, we calculate the optimal γ, γ_0. Secondly, we need to calculate u_{13}, u_{31}, C_0, and B_0, which completely determine $X(s)$ in (5.41) for a given $\gamma = \gamma_0$. Finally, we calculate the parameterization of all optimal solutions for the one-block problem

$$\min_{\hat{Q}(s) \in H_\infty} \|X - \hat{Q}\|_\infty \tag{6.1}$$

and in the parameterization we restrict $Y(s)$ as in (5.53).

Let us now address each step in more detail. γ_0 is calculated by a numerical search. We start the search by taking $\gamma = \gamma_R$; γ_R is defined by (5.3). A summary of the different cases that can arise follows.

I) Optimality of the inertia type: If we have $\gamma_0 > \gamma_R$, we call this case optimality of the inertia type since optimality is related to the Hankel norm of $X(s)$ as in the one-block case. This case is very much like the one-block problem (Theorem 4.5) and no extra complications appear.

II) Optimality of the Parrot type: If we have $\gamma_R \geq \rho^{1/2}(L_o(\gamma_R)L_c(\gamma_R))$, then $\gamma_0 = \gamma_R$, this case is called optimality of the Parrot type since the lower bound for Parrot's theorem is achieved. There are two possible cases:

IIa) We can have strict inequality, $\gamma_R > \rho^{1/2}(L_o(\gamma_R)L_c(\gamma_R))$. In this case obtaining the optimal solutions do not require a special treatment since $N(\gamma_R)$ is not singular; therefore, optimal solutions do not drop in degree.

IIb) We can have equality, $\gamma_R = \rho^{1/2}(L_o(\gamma_R)L_c(\gamma_R))$. Calculation of the solutions is similar to case I (optimality of the inertia type).

Calculation of u_{13}, u_{31}, C_0, and B_0 involves solving two spectral factorization problems:

$$X_{13}^{\sim}(s)X_{13}(s) = \gamma^2 I - R_{12}^{\sim}(s)R_{12}(s) - R_{22}^{\sim}(s)R_{22}(s) \geq 0 \tag{6.2}$$

$$X_{31}(s)X_{31}^{\sim}(s) = \gamma^2 I - R_{21}(s)R_{21}^{\sim}(s) - R_{22}(s)R_{22}^{\sim}(s) \geq 0 \tag{6.3}$$

such that $X_{13}^{\sim}(s)$ and $X_{31}^{\sim}(s)$ are minimum phase; i.e., they have constant rank in C_+, and also (5.24) and (5.29) are satisfied. When $\gamma_0 > \gamma_D$ the desired $X_{13}(s)$ follows from (5.25)- (5.27) and $X_{31}(s)$ follows from (5.30)- (5.32). If $\gamma_0 = \gamma_D$ existence of the desired $X_{13}(s)$ and/or $X_{31}(s)$ is not clear. A sufficient condition for existence of the desired $X_{13}(s)$ is that (A, B_2) is controllable. Similarly, a sufficient condition for existence of the desired $X_{31}(s)$ is that (C_2, A) is observable. If these sufficient conditions are satisfied then the solution is guaranteed by Theorem 2.9 and a solution can be obtained as it is explained in the following lemma.

Lemma 6.1
If $\gamma \geq \left\| \begin{matrix} R_{12} \\ R_{22} \end{matrix} \right\|_{\infty}$ *and* (A, B_2) *is controllable then the spectral factorization problem (6.2) always has a solution* $X_{13}(s) = \left[\begin{array}{c|c} A & B_2 \\ \hline C_0 & u_{13} \end{array}\right]$ *such that (5.24) is satisfied.*

Proof: We use Remark 2.8 to obtain the solution. Define

$$G(s) = \begin{pmatrix} R_{12}(s) \\ R_{22}(s) \end{pmatrix} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \\ C_2 & D_{22} \end{array}\right],$$

which is a controllable realization by assumption. We then apply the steps in Remark 2.8.

■

$X_{31}(s)$ can be obtained by a dual argument if (C_2, A) is observable. Alternatively, by taking the transpose of (6.3) we can reduce it to an equation of the form (6.2).

Remark 6.1 *From Remark 2.4, we know that without loss of generality we can assume* $R(\infty) = 0$, *which implies* $\gamma_D = 0$. *Therefore, we can always avoid having* $\gamma_0 = \gamma_D$ *via loop shifting [26]. Therefore, we do not have to deal with the problem of having* $\gamma_0 = \gamma_D$ *since it can always be avoided.*

The problem of parameterizing all $Y(s) = \begin{pmatrix} y_{11}(s) & y_{12}(s) & u_{13} \\ y_{21}(s) & y_{22}(s) & D_{12} \\ u_{31} & D_{21}(s) & D_{22} \end{pmatrix} \in B_{\gamma}H_{\infty}$ takes a different nature in optimality of the Parrot type. There are three different cases (with the definitions of γ_0, γ_R, γ_D and $\gamma_{D_{22}}$ as in Chapter 5):

1) $\gamma_0 = \gamma_R > \gamma_D$: In this case we can apply Lemma 5.3 or Lemma 5.4 to obtain the solution. For this reason this case is very much like the inertia-type optimality.

2) $\gamma_0 = \gamma_R = \gamma_D > \gamma_{D_{22}}$: In this case Lemma 5.3 is not valid any more since u_{13} and/or u_{31} are singular. Lemma 5.4 can be used to obtain the solution.

3) $\gamma_0 = \gamma_R = \gamma_D = \gamma_{D_{22}}$: Neither Lemma 5.3 nor Lemma 5.4 can be used to obtain the solution in this case. A new method needs to be introduced, taking into account the fact that $(\gamma_0 I - D_{22}D_{22}^T)$ is singular.

In the rest of the chapter we assume that the spectral factorization problems (6.2) and (6.3) are solved (therefore, B_0, C_0, u_{13} and u_{31} are known).

6.1 Optimality of Inertia Type

As we mentioned before this case is similar to the one-block problem.

Consider $X(s)$, which is given by (5.41) and related equations. From Section 4 we know how to obtain the set of optimal solutions such that $\|X - \hat{Q}\|_\infty$ is minimized. The set of solutions is given by Theorem 4.6 (or Theorem 4.7) in Section 4.3. We will adopt Theorem 4.7 for the optimal case for convenience. In this section we assume that $\gamma_0 > \gamma_R$ has already been calculated, $X(s)$ has been obtained, and a balanced realization for $X(s)$ has been obtained. We define the following partition on $G(s)$ corresponding to the balanced realization for the "optimal case" conforming with the notation of Chapter 4:

$$A =: \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{6.4}$$

$$B = \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \end{pmatrix} := \begin{pmatrix} B_{01} & B_{11} & B_{21} \\ B_{02} & B_{12} & B_{22} \end{pmatrix} = \begin{pmatrix} B_0 & B_1 & B_2 \end{pmatrix} \tag{6.5}$$

$$C = \begin{pmatrix} \hat{C}_1 & \hat{C}_2 \end{pmatrix} := \begin{pmatrix} C_{01} & C_{02} \\ C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix} \tag{6.6}$$

$$\hat{D} := \left(\begin{array}{cc|c} 0 & 0 & u_{13} \\ 0 & D_{11} & D_{12} \\ \hline u_{31} & D_{21} & D_{22} \end{array}\right). \tag{6.7}$$

We need to find the set of $m(s)$, such that $Y(s)$ given by (5.86), satisfies:

$$\sigma_1 \hat{B}_1 = \hat{C}_1^T Y(s) \tag{6.8}$$

$$\sigma_1 \hat{C}_1 = Y(s) \hat{B}_1^T. \tag{6.9}$$

The following lemma charecterizes such $m(s)$.

Lemma 6.2

$$\sigma_1 \hat{B}_1 = \hat{C}_1^T Y(s) \tag{6.10}$$

is satisfied if and only if

$$\sigma_1 \tilde{B}_{11} = \tilde{C}_{11}^T m(s) \tag{6.11}$$

is satisfied, where

$$\begin{pmatrix} \tilde{B}_{01} & \tilde{B}_{11} \end{pmatrix} := \begin{pmatrix} B_{01} & B_{11} \end{pmatrix} V(\sigma_1) \tag{6.12}$$

$$\begin{pmatrix} \tilde{C}_{01}^T & \tilde{C}_{11}^T \end{pmatrix} := \begin{pmatrix} C_{01}^T & C_{11}^T \end{pmatrix} U(\sigma_1). \tag{6.13}$$

Proof: If we substitute (5.86) in (6.10) we get

$$\sigma_1 \left(\tilde{B}_{01}\ \tilde{B}_{11}\ B_{21} \right) = \left(\tilde{C}_{01}^T\ \tilde{C}_{11}^T\ C_{21}^T \right) \left(\begin{array}{cc|c} -D_{22}^T & 0 & \left(\sigma_1^2 I - D_{22}^T D_{22}\right)^{1/2} \\ 0 & m(s) & 0 \\ \hline \left(\sigma_1^2 I - D_{22} D_{22}^T\right)^{1/2} & 0 & D_{22} \end{array} \right) . \quad (6.14)$$

Equation (6.14) implies that (6.10) is satisfied if and only if

$$\sigma_1 \tilde{B}_{01} = -\tilde{C}_{01}^T D_{22}^T + C_{21}^T \left(\sigma_1^2 I - D_{22} D_{22}^T\right)^{1/2}, \quad (6.15)$$

$$\sigma_1 \tilde{B}_{11} = \tilde{C}_{11}^T m(s), \quad (6.16)$$

and

$$\sigma_1 B_{21} = \tilde{C}_{01}^T \left(\sigma_1^2 I - D_{22}^T D_{22}\right)^{1/2} + C_{21}^T D_{22}. \quad (6.17)$$

In the following, we prove that (6.15) and (6.17), which are independent of $m(s)$, are always true. We can then conclude that (6.10) is equivalent to (6.11).

From (5.29) we have

$$\sigma_1 C_{21} - \left(\begin{array}{ccc} u_{31} & D_{21} & D_{22} \end{array} \right) \left(\begin{array}{c} B_{01}^T \\ B_{11}^T \\ B_{21}^T \end{array} \right) = 0 \quad (6.18)$$

and from (5.24) we get

$$\sigma_1 B_{21} - \left(\begin{array}{ccc} C_{01}^T & C_{11}^T & C_{21}^T \end{array} \right) \left(\begin{array}{c} u_{13} \\ D_{12} \\ D_{22} \end{array} \right) = 0. \quad (6.19)$$

If we utilize (5.90) in (6.19), we obtain (6.17). Equation (6.15) is proved as follows: (6.15) is satisfied if and only if

$$\sigma_1 \tilde{B}_{01} \left(\sigma_1^2 I - D_{22} D_{22}^T\right)^{1/2} = -\tilde{C}_{01}^T D_{22}^T \left(\sigma_1^2 I - D_{22} D_{22}^T\right)^{1/2} + C_{21}^T \left(\sigma_1^2 I - D_{22} D_{22}^T\right). \quad (6.20)$$

Equation (6.20), in turn, is satisfied if and only if

$$\sigma_1 \tilde{B}_{01} \left(\sigma_1^2 I - D_{22} D_{22}^T\right)^{1/2} = -\tilde{C}_{01}^T \left(\gamma^2 I - D_{22}^T D_{22}\right)^{1/2} D_{22}^T + C_{21}^T \left(\sigma_1^2 I - D_{22} D_{22}^T\right). \quad (6.21)$$

From (6.17) we have

$$\tilde{C}_{01}^T = \left(\sigma_1 B_{21} - C_{21}^T D_{22}\right) \left(\sigma_1^2 I - D_{22}^T D_{22}\right)^{-1/2}. \quad (6.22)$$

Similarly from (6.12) and (6.18) we conclude that

$$\tilde{B}_{01} = \left(\sigma_1 C_{21}^T - B_{21} D_{22}^T\right) \left(\sigma_1^2 I - D_{22} D_{22}^T\right)^{-1/2}. \quad (6.23)$$

If we use (6.22) and (6.23) in (6.21), we get

$$\sigma_1 \left(\sigma_1 C_{21}^T - B_{21} D_{22}^T\right) = -\left(\sigma_1 B_{21} - C_{21}^T D_{22}\right) D_{22}^T + C_{21}^T \left(\sigma_1^2 I - D_{22} D_{22}^T\right). \quad (6.24)$$

It is easy to see that (6.24) is satisfied, which implies that (6.15) is satisfied. ■

Lemma 6.3 $\tilde{B}_{11} \tilde{B}_{11}^T = \tilde{C}_{11}^T \tilde{C}_{11}$.

Proof: Since we are using balanced realization, we have

$$\hat{B}_1 \hat{B}_1^T = \hat{C}_1^T \hat{C}_1. \quad (6.25)$$

From (6.25) we have, by definition,

$$\begin{pmatrix} B_{01} & B_{11} & B_{21} \end{pmatrix} \begin{pmatrix} B_{01}^T \\ B_{11}^T \\ B_{21}^T \end{pmatrix} = \begin{pmatrix} C_{01}^T & C_{11}^T & C_{21}^T \end{pmatrix} \begin{pmatrix} C_{01} \\ C_{11} \\ C_{21} \end{pmatrix}. \quad (6.26)$$

From (6.26), we have

$$\begin{pmatrix} B_{01} & B_{11} \end{pmatrix} \begin{pmatrix} B_{01}^T \\ B_{11}^T \end{pmatrix} = \begin{pmatrix} C_{01}^T & C_{11}^T \end{pmatrix} \begin{pmatrix} C_{01} \\ C_{11} \end{pmatrix} + C_{21}^T C_{21} - B_{21} B_{21}^T. \quad (6.27)$$

From (6.27), we have

$$\begin{pmatrix} B_{01} & B_{11} \end{pmatrix} V(\sigma_1)V^T(\sigma_1) \begin{pmatrix} B_{01}^T \\ B_{11}^T \end{pmatrix} = \begin{pmatrix} C_{01}^T & C_{11}^T \end{pmatrix} U(\sigma_1)U^T(\sigma_1) \begin{pmatrix} C_{01} \\ C_{11} \end{pmatrix} + \\ + C_{21}^T C_{21} - B_{21} B_{21}^T \quad (6.28)$$

Finally, from (6.28), we conclude

$$\tilde{B}_{11}\tilde{B}_{11}^T - \tilde{C}_{11}^T\tilde{C}_{11} = \tilde{C}_{01}^T\tilde{C}_{01} - \tilde{B}_{01}\tilde{B}_{01}^T + C_{21}^T C_{21} - B_{21}B_{21}^T. \quad (6.29)$$

We substitute (6.22) and (6.23) in (6.29), after some straightforward calculation, we conclude that

$$\tilde{C}_{01}^T\tilde{C}_{01} - \tilde{B}_{01}\tilde{B}_{01}^T + C_{21}^T C_{21} - B_{21}B_{21}^T = 0, \quad (6.30)$$

which completes the proof. ■

Lemma 6.4 *$m(s) \in B_{\sigma_1}H_\infty$ satisfies*

$$\sigma_1 \tilde{B}_{11} = \tilde{C}_{11}^T m(s), \quad (6.31)$$

if and only if

$$m(s) = V_{\tilde{C}_{11}} \begin{pmatrix} \sigma_1 I_{r_1} & 0 \\ 0 & F(s) \end{pmatrix} V_{\tilde{B}_{11}}^T, \quad (6.32)$$

with $F(s) \in B_{\sigma_1}H_\infty$. $V_{\tilde{C}_{11}}$ and $V_{\tilde{B}_{11}}$ are defined through the following SVD's:

$$\tilde{B}_{11} =: U_0 \begin{pmatrix} \Sigma_{r_1\times r_1} & 0 \\ 0 & 0 \end{pmatrix} V_{\tilde{B}_{11}}^T \quad (6.33)$$

$$\tilde{C}_{11}^T =: U_0 \begin{pmatrix} \Sigma_{r_1\times r_1} & 0 \\ 0 & 0 \end{pmatrix} V_{\tilde{C}_{11}}^T \quad (6.34)$$

where $r_1 := rank\{\tilde{B}_{11}\} = rank\{\tilde{C}_{11}\}$.

Proof: Expressions (6.33) and (6.34) are justified by Lemma 6.3. If we use (6.33) and (6.34) together with the condition $\|m\|_\infty \leq \sigma_1$ in (6.31) we conclude the result. ■

The solution is given by the following theorem:

Theorem 6.1 *The set of all solutions, $Q(s)$, for the four-block problem is given as a linear fractional transformation on $m(s)$ as follows:*

$$Q(s) = F_l\left(J(s), m(s)\right) \tag{6.35}$$

such that $m(s) \in B_{\sigma_1} H_\infty$ satisfies

$$\sigma_1 \tilde{B}_{11} = \tilde{C}_{11}^T m(s), \tag{6.36}$$

where

$$J(s) := \left[\begin{array}{c|c} A_J & B_J \\ \hline C_J & D_J \end{array}\right] \tag{6.37}$$

with

$$A_J := -\Gamma^{-1}\left(\sigma_1^2 A_{22}^T + \Sigma_1 A_{22}\Sigma_1 - \tilde{\tilde{C}}_2^T M_0(\sigma_1)\tilde{\tilde{B}}_2^T\right) \tag{6.38}$$

$$B_J := \Gamma^{-1}\left(\begin{array}{cc} \left(-\tilde{C}_{02}^T D_{22}^T + C_{22}^T\left(\sigma_1^2 I - D_{22}D_{22}^T\right)^{1/2}\right)V_{21}^T(\sigma_1) - \Sigma_1 B_{12} & \tilde{C}_{12}^T \end{array}\right) \tag{6.39}$$

$$C_J := \left(\begin{array}{c} C_{12}\Sigma_1 + U_{21}(\sigma_1)\left(D_{22}^T\tilde{B}_{02}^T - \left(\sigma_1^2 I - D_{22}^T D_{22}\right)^{1/2} B_{22}^T\right) \\ \tilde{B}_{12}^T \end{array}\right) \tag{6.40}$$

$$D_J := \left(\begin{array}{cc} D_{11} + U_{21}(\sigma_1)D_{22}^T V_{21}^T(\sigma_1) & -U_{22}(\sigma_1) \\ V_{22}^T(\sigma_1) & 0 \end{array}\right). \tag{6.41}$$

We have the following definitions:

$$\left(\begin{array}{cc} \tilde{B}_{02} & \tilde{B}_{12} \end{array}\right) := \left(\begin{array}{cc} B_{02} & B_{12} \end{array}\right) V(\sigma_1) \tag{6.42}$$

$$\begin{pmatrix} \tilde{C}_{02}^T & \tilde{C}_{12}^T \end{pmatrix} := \begin{pmatrix} C_{02}^T & C_{12}^T \end{pmatrix} U(\sigma_1) \tag{6.43}$$

$$\tilde{\tilde{B}}_2 := \begin{pmatrix} \tilde{B}_{02} & \tilde{B}_{12} & B_{22} \end{pmatrix} \tag{6.44}$$

$$\tilde{\tilde{C}}_2^T := \begin{pmatrix} \tilde{C}_{02}^T & \tilde{C}_{12}^T & C_{22}^T \end{pmatrix}. \tag{6.45}$$

Proof: The proof follows from the following observations:

$$Q(s) = \begin{pmatrix} 0 & I & 0 \end{pmatrix} \hat{Q}(s) \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix}, \tag{6.46}$$

$$\hat{Q}(s) = F_l\left(T_0(s), Y(s)\right), \tag{6.47}$$

where

$$T_0(s) = \left[\begin{array}{c|cc} -\Gamma^{-1}\left(\sigma_1^2 A_{22}^T + \Sigma_1 A_{22}\Sigma_1\right) & -\Gamma^{-1}\Sigma_1\hat{B}_2 & \Gamma^{-1}\hat{C}_2^T \\ \hline \hat{C}_2\Sigma_1 & D & -I \\ \hat{B}_2^T & I & 0 \end{array}\right], \tag{6.48}$$

which is obtained by applying Theorem 4.7 to the system given by (6.4), (6.5), (6.6) and (6.7) as a one-block problem. We also have

$$Y(s) = T_l(\gamma)Y_0(s,\sigma_1)T_r(\gamma), \tag{6.49}$$

$$Y_0(s,\sigma_1) = M_0(\sigma_1) + \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix} m(s) \begin{pmatrix} 0 & I & 0 \end{pmatrix}. \tag{6.50}$$

We see that (6.46), (6.47), (6.49) and (6.50) are linear fractional transformations. We obtain the result by using the rules to calculate

$$F_l(P(s), F_l(T(s), Q(s))) = F_l(J(s), Q(s)). \tag{6.51}$$

The calculations are omitted being straightforward but lengthy.

6.2 Optimality of the Parrot Type

As we explained before we can have two different cases.

1) If $\gamma_R > \rho^{1/2}(L_o(\gamma_R)L_c(\gamma_R))$, then $N(\gamma_R)$ is not singular; therefore, to obtain the set of optimal solutions, we use Theorem 4.5.

2) If $\gamma_R = \rho^{1/2}(L_o(\gamma_R)L_c(\gamma_R))$, then $N(\gamma_R)$ is singular and we apply Theorem 4.7 to get the optimal solution for the problem.

After we obtain the solution for the one-block problem, we need to identify the subset of the solutions elements of which have the form $\begin{pmatrix} \hat{Q}_1(s) & 0 \\ 0 & 0 \end{pmatrix}$. As we explained before, identifying such a subset amounts to finding the parameterization of all $Y_1(s) = \begin{pmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{pmatrix} \in H_\infty$ such that

$$Y(s) = \begin{pmatrix} y_{11}(s) & y_{12}(s) & u_{13} \\ y_{21}(s) & y_{22}(s) & D_{12} \\ u_{31} & D_{21}(s) & D_{22} \end{pmatrix} \in B_{\gamma_R} H_\infty. \tag{6.52}$$

There are three different cases, depending on the values of D_{12}, D_{21} and D_{22}, that lead to three different kinds of solutions:

1) $\gamma_0 = \gamma_R > \gamma_D$: In this case parameterization is simple and is given by Lemma 5.3 or Lemma 5.4.

2) $\gamma_0 = \gamma_R = \gamma_D > \gamma_{D_{22}}$: In this case parameterization can be obtained by Lemma 5.4.

3) $\gamma_0 = \gamma_R = \gamma_D = \gamma_{D_{22}}$: In this case Lemma 5.3 and Lemma 5.4 both fail, and a special solution is necessary. In the following lemma we solve this problem.

6.2.1 Parameterization of All Solutions, $\gamma_0 = \gamma_{D_{22}}$

From (5.77), (5.83) and (5.79) we have the following solution:

$$\begin{pmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{pmatrix} = -Z_1 D_{22}^T Z_2 + (I - Z_1 Z_1^T)^{1/2} M(s) (I - Z_2^T Z_2)^{1/2} \tag{6.53}$$

where Z_1 and Z_2 satisfy

$$Z_1(\gamma^2 I - D_{22}^T D_{22})^{1/2} = \begin{pmatrix} u_{13} \\ D_{12} \end{pmatrix} \tag{6.54}$$

$$(\gamma^2 I - D_{22} D_{22}^T)^{1/2} Z_2 = \begin{pmatrix} u_{31} & D_{21} \end{pmatrix}. \tag{6.55}$$

For simplicity assume, without loss of generality, that $D_{22} = \begin{pmatrix} \Sigma_D & 0 \\ 0 & \gamma_0 I_s \end{pmatrix}$;

Otherwise find the SVD of D_{22}; $D_{22} = U_D \begin{pmatrix} \Sigma_D & 0 \\ 0 & \gamma_0 I_s \end{pmatrix} V_D^T$.

Lemma 6.5 *If* $D_{22} = \begin{pmatrix} \Sigma_D & 0 \\ 0 & \gamma_0 I_s \end{pmatrix}$,

$$u_{13}^T u_{13} = \gamma^2 I - D_{12}^T D_{12} - D_{22}^T D_{22} \geq 0 \tag{6.56}$$

and

$$u_{31} u_{13}^T = \gamma^2 I - D_{21} D_{21}^T - D_{22} D_{22}^T \geq 0 \tag{6.57}$$

are all true, then we necessarily have

$$\begin{pmatrix} u_{13} \\ D_{12} \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ d_{11} & 0 \end{pmatrix} \tag{6.58}$$

and

$$\begin{pmatrix} u_{31} & D_{21} \end{pmatrix} = \begin{pmatrix} u_2 & d_{21} \\ 0 & 0 \end{pmatrix}. \tag{6.59}$$

Proof: Follows by direct calculation from (6.56) and (6.57). ■

After we use Lemma 6.5, the problem reduces to finding $\begin{pmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{pmatrix}$ such that

$$\left\| \begin{matrix} y_{11} & y_{12} & u_1 & 0 \\ y_{21} & y_{22} & d_{11} & 0 \\ u_2 & d_{21} & \Sigma_D & 0 \\ 0 & 0 & 0 & \gamma_0 I_s \end{matrix} \right\|_\infty \leq \gamma_0 \tag{6.60}$$

and this problem reduces to solving

$$\left\| \begin{matrix} y_{11} & y_{12} & u_1 \\ y_{21} & y_{22} & d_{11} \\ u_2 & d_{21} & \Sigma_D \end{matrix} \right\|_\infty \leq \gamma_0. \tag{6.61}$$

Since $\left(\gamma_0^2 I - \Sigma_D \Sigma_D^T\right)$ is not singular, we can use Lemma 5.4 to get the solution. Consequently we have the parameterization of all solutions as in cases 1 and 2.

Chapter 7

Examples

In this chapter we give two examples to illustrate some points.

7.1 Example One

The example below illustrates how we get the sub-optimal case and the several different kinds of optimal cases. Example is designed to be fairly simple to make the point clear.

Consider

$$R(s) = \begin{pmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix} = \left[\begin{array}{c|cc} A & B & 0 \\ \hline C & D & 0 \\ 0 & 0 & \alpha I \end{array}\right] \tag{7.62}$$

such that $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a minimal realization for $R_{11}(s)$.

After we calculate the imbedding, we get

$$A^T L_o + L_o A = C^T C$$

$$A L_c + L_c A^T = B B^T$$

$$u_{31} = u_{13} = \sqrt{\gamma^2 - \alpha^2 I}$$

$$C_0 = 0 \text{ and } B_0 = 0.$$

Notice that L_o and L_c are independent of γ. We have

$$\gamma_R = |\alpha|$$

$$\rho^{1/2}(L_o L_c) = \|\Gamma_{R_{11}}\|.$$

The minimum achievable norm is given by

$$\gamma_0 = \max\{\|\Gamma_{R_{11}}\|\ ,\ |\alpha|\}.$$

In the sub-optimal case we have $\gamma > \gamma_0$. There are three possible kinds of optimality:

1) $\gamma_0 = \|\Gamma_{R_{11}}\| > |\alpha|$ (Optimality of inertia type),

2) $\gamma_0 = |\alpha| = \|\Gamma_{R_{11}}\|$ (Optimality of the Parrot type),

3) $\gamma_0 = |\alpha| > \|\Gamma_{R_{11}}\|$ (Optimality of the Parrot type).

Notice that the solution to this problem can be obtained by inspection.

Remark 7.1 *This example shows that we do not need to have (A, B_2) controllable and/or (C_2, A) observable to obtain the solution using our method.*

7.2 Example Two

In this section we give a nontrivial, but still workable by hand example to illustrate computational flow of the solution.

Consider the following four-block problem:

$$R(s) = \begin{pmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix} = \left[\begin{array}{c|cc} 1 & -1 & 1 \\ \hline 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]. \tag{7.63}$$

We want to find the optimal solution(s) to the problem.

Solution: First we need to find the optimal error. We have

$$\gamma_R = \gamma_D = 1$$

Therefore, 1 is a lower bound for the optimal error. We next calculate u_{13} and u_{31}: We have

$$\Phi = u_{31}u_{31}^T = \gamma^2 - 1$$

$$\Theta = u_{13}^T u_{13} = \gamma^2 - 1,$$

which results in

$$u_{13} = u_{31} = \sqrt{\gamma^2 - 1}. \tag{7.64}$$

From equation (5.27), we get

$$L_o^2 - 2\gamma^2 L_o + \gamma^2 = 0. \tag{7.65}$$

From (7.65), the following desired solution is obtained:

$$L_o = \gamma^2 - \gamma\sqrt{\gamma^2 - 1}. \tag{7.66}$$

From equation (5.32), we obtain

$$2L_c - \frac{2\gamma^2 - 1}{\gamma^2 - 1} = 0. \tag{7.67}$$

From (7.67), we get

$$L_c = \frac{\gamma^2 - 0.5}{\gamma^2 - 1}. \tag{7.68}$$

The optimal value for γ is obtained if conditions $\gamma_0^2 - L_o(\gamma_0)L_c(\gamma_0) = 0$ and $\gamma_0 \geq 1$ are both satisfied. In order to determine the optimal value, we need to solve

$$L_o L_c = \left(\gamma_0^2 - \sqrt{\gamma_0^4 - \gamma_0^2}\right) \frac{\gamma_0^2 - 0.5}{\gamma_0^2 - 1} = \gamma_0^2. \tag{7.69}$$

The solution of (7.69), γ_0, is approximately equal to $\gamma_0 = 1.19149$, which is greater than 1. Therefore the minimum norm is

$$\gamma_0 = 1.1945. \tag{7.70}$$

We next calculate B_0 and C_0 to complete the imbedding. We easily obtain

$$B_0 = \frac{1}{\sqrt{\gamma_0^2 - 1}}, \tag{7.71}$$

$$C_0 = \sqrt{\gamma_0^2 - 1} - \gamma_0. \tag{7.72}$$

The new imbedded plant is given below:

$$X(s) = \left[\begin{array}{c|ccc} 1 & \frac{1}{\sqrt{\gamma_0^2-1}} & -1 & 1 \\ \hline \sqrt{\gamma_0^2 - 1} - \gamma_0 & 0 & 0 & \sqrt{\gamma_0^2 - 1} \\ 1 & 0 & 0 & 1 \\ 0 & \sqrt{\gamma_0^2 - 1} & 1 & 0 \end{array}\right]. \tag{7.73}$$

Next we find a balanced realization. Finding a balanced realization is equivalent to finding a scalar t such that the following realization of $X(s)$ has $\hat{L}_c = \hat{L}_o = L$:

$$X(s) = \left[\begin{array}{c|ccc} 1 & \left(\frac{1}{\sqrt{\gamma_0^2-1}}\right. & -1 & \left. 1\right)\frac{1}{t} \\ \hline t\begin{pmatrix} \sqrt{\gamma_0^2 - 1} - \gamma_0 \\ 1 \\ 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ \sqrt{\gamma_0^2 - 1} \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & \begin{matrix} \sqrt{\gamma_0^2 - 1} \\ 1 \\ 0 \end{matrix} \end{array}\right]. \tag{7.74}$$

We obtain

$$t = \sqrt[4]{\frac{2\gamma_0^2 - 1}{(\gamma_0^2 - 1)[1 + (\sqrt{\gamma_0^2 - 1} - \gamma_0)^2]}}. \tag{7.75}$$

We have

$$L = \sqrt{\frac{(2\gamma_0^2 - 1)[1 + (\sqrt{\gamma_0^2 - 1} - \gamma_0)^2]}{\gamma_0^2 - 1}}. \tag{7.76}$$

From (5.81) and (5.82) we obtain

$$U(\gamma) = V(\gamma) = \begin{pmatrix} \sqrt{\frac{\gamma^2-1}{\gamma}} & -\frac{1}{\gamma} \\ \frac{1}{\gamma} & \sqrt{\frac{\gamma^2-1}{\gamma}} \end{pmatrix}. \tag{7.77}$$

We then have

$$\tilde{B}_{11} = -\frac{\gamma_0}{t\sqrt{\gamma_0^2 - 1}} \tag{7.78}$$

$$\tilde{C}_{11} = t. \tag{7.79}$$

The value of the free parameter satisfying the condition

$$\gamma_0 \tilde{B}_{11} = \tilde{C}_{11}^T m \tag{7.80}$$

is obtained as

$$m = -\frac{\gamma_0^2}{t^2\sqrt{\gamma_0^2 - 1}}. \tag{7.81}$$

Finally the optimal solution is

$$Q(s) = F_l(J(s), m(s)) = F_l(D_J, m) = constant \tag{7.82}$$

as expected. A little calculation results:

$$Q(s) = (\gamma_0^2 - 1)\sqrt{\frac{1 + (\sqrt{\gamma_0^2 - 1} - \gamma_0)^2}{2\gamma_0^2 - 1}} = 0.3522. \tag{7.83}$$

Chapter 8

Conclusion

In this thesis it is shown that SISO H_∞ optimization problems are equivalent with weighted H_2 optimization problems. The optimal weight has been calculated and is proven to be unique. To obtain the solution, an iterative procedure is introduced and its limit calculated. This result established a direct link between the H_∞ optimal control and Wiener-Hopf control theories. The weight can be interpreted as the worst case signal for the H_∞ optimization problem.

An elementary method is introduced to obtain a parameterization of all solutions to the one-block, general distance problem.

We prove that the four-block problem can be transformed into a one-block problem in a very simple way. We give a parameterization of all solutions to the four-block problem for both optimal and suboptimal cases.

Doyle et al. [6] has given a state space solution for the H_∞ optimization problem which does not involve the Youla parameterization. They show that some of the solutions in the set they characterize have the same degree as the plant itself. In our solution, since we address the problem in the frequency domain, the controllers we get seem to have at least three times the degree of the plant. Obviously, since both methods solve the same problem , some of the controllers obtained by our approach must be non-minimal. Minimality property of the controllers obtained via our method remains to be investigated and is the subject of current research.

In the case of Parrot type optimality, we face a singular spectral factorization problem. Good numerical algorithms are needed to solve this special, spectral factorization problem.

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