Level-raising for GSp(4)

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Abstract

This thesis provides congruences between unstable and stable automorphic forms for the symplectic similitude group GSp(4). More precisely, we raise the level of certain CAP representations Π of Saito-Kurokawa type, arising from classical modular forms $f \in S_4(\Gamma_0(N))$ of square-free level and root number $\epsilon_f = -1$. We first transfer Π to a suitable inner form G such that $G(\mathbb{R})$ is compact modulo its center. This is achieved by viewing G as a similitude spin group of a definite quadratic form in five variables, and then θ-lifting the whole Waldspurger packet for $\widetilde{\mathrm{SL}}(2)$ determined by f. Thereby we obtain an automorphic representation π of G. For the inner form we prove a precise level-raising result, inspired by the work of Bellaiche and Clozel, and relying on computations of Schmidt. Thus we obtain a $\tilde{\pi}$ congruent to π, with a local component that is irreducibly induced from an unramified twist of the Steinberg representation of the Klingen Levi subgroup. To transfer $\tilde{\pi}$ back to GSp(4), we use Arthur's stable trace formula and the exhaustive work of Hales on Shalika germs and the fundamental lemma in this case. Since $\tilde{\pi}$ has a local component of the above type, all endoscopic error terms vanish. Indeed, by Weissauer, we only need to show that such a component does not participate in the θ-correspondence with any GO(4). This is an exercise in using Kudla's filtration of the Jacquet modules of the Weil representation. Thus we get a cuspidal automorphic representation $\tilde{\Pi}$ of GSp(4) congruent to Π, which is neither CAP nor endoscopic. In particular, its Galois representations are irreducible by work of Ramakrishnan. It is crucial for our application that we can arrange for $\tilde{\Pi}$ to have vectors fixed by the non-special maximal compact subgroups at all primes dividing N. Since G is necessarily ramified at some prime r, we have to show a non-special analogue of the fundamental lemma at r. Fortunately, by work of Kottwitz we can compare the involved orbital integrals to twisted orbital integrals over the unramified quadratic extension of $\mathbb{Q}_r$. The inner form G splits over this extension, and the comparison of the twisted orbital integrals can be done by hand. Finally we give an application of our main result to the Bloch-Kato conjecture. Assuming a conjecture of Skinner and Urban on the rank of the monodromy operators at the primes dividing N, we construct a torsion class in the Selmer group of the motive $M_f(2)$.

Contents

Chapter 1

Introduction

We fix a prime r, and let D be the quaternion algebra over $\mathbb{Q}$ with ramification locus $S = \{\infty, r\}$. Then let G be the unitary similitude group of D^2, where we take the hermitian form to be the identity matrix I. Thus, for example, the group of $\mathbb{Q}$-points is the following:

$$G(\mathbb{Q}) = \{x \in \mathrm{GL}(2, D)\colon\ x^*x = c(x)I,\ c(x) \in \mathbb{Q}^*\}.$$

Then G is an inner form of GSp(4) such that $G(\mathbb{R})$ is compact modulo its center. More precisely, its adjoint group $G^{\mathrm{ad}}(\mathbb{R})$ is anisotropic SO(5). Similarly, $G^{\mathrm{ad}}(\mathbb{Q}_r)$ is the special orthogonal group of a quadratic form in 5 variables over $\mathbb{Q}_r$ with Witt index 1. There is another description of $G(\mathbb{Q}_r)$ in section 5.3 below. At all other primes p the group G is split, and hence $G(\mathbb{Q}_p)$ can be identified with $\mathrm{GSp}(4, \mathbb{Q}_p)$. Let $\mathbb{A}_f$ denote the finite part of the ring of rational adeles $\mathbb{A}$.

The compact open subgroups K in $G(\mathbb{A}_f)$ form a directed set by opposite inclusion. Let $\mathcal{H}_{K,\mathbb{Z}}$ denote the natural $\mathbb{Z}$-structure in the Hecke algebra of K-biinvariant compactly supported functions on $G(\mathbb{A}_f)$. As K varies, the centers $Z(\mathcal{H}_{K,\mathbb{Z}})$ form an inverse system of algebras with respect to the canonical maps $Z(\mathcal{H}_{J,\mathbb{Z}}) \to Z(\mathcal{H}_{K,\mathbb{Z}})$ given by $\phi \mapsto e_K \star \phi$ for $J \subset K$. Consider the inverse limit

$$\mathcal{Z} = \varprojlim Z(\mathcal{H}_{K,\mathbb{Z}}).$$

This makes sense locally, and then $\mathcal{Z} = \otimes_{p<\infty} \mathcal{Z}_p$, where $\mathcal{Z}_p$ is obtained by the analogous construction at p. If π is an irreducible admissible representation of $G(\mathbb{A})$, there is a unique character $\eta_\pi : \mathcal{Z} \to \mathbb{C}$ such that $\eta_\pi = \eta_{\pi_f^K} \circ \mathrm{pr}_K$ whenever $\pi_f^K \neq 0$. Similarly, we have characters η_{π_p} locally and then $\eta_\pi = \otimes_{p<\infty} \eta_{\pi_p}$ under the isomorphism above. If π is automorphic and $\pi_\infty = \mathbf{1}$, the values of η_π are algebraic integers. Throughout, we will use the following definition of being congruent:

Definition 1. *Let $\tilde{\pi}$ and π be automorphic representations of $G(\mathbb{A})$, both trivial at infinity, and let*

λ be a finite place of $\bar{\mathbb{Q}}$. Then we define $\tilde{\pi}$ and π to be congruent modulo λ when the congruence

$$\eta_{\tilde{\pi}}(\phi) \equiv \eta_{\pi}(\phi) \ (mod\ \lambda)$$

holds for all $\phi \in \mathcal{Z}$. In this case, we simply write $\tilde{\pi} \equiv \pi$ (mod λ).

Analogously, it makes sense to say the local components $\tilde{\pi}_p$ and π_p are congruent. Then $\tilde{\pi} \equiv \pi$ (mod λ) if and only if $\tilde{\pi}_p \equiv \pi_p$ (mod λ) for all $p < \infty$. If $\tilde{\pi}_p$ and π_p are both unramified, $\tilde{\pi}_p \equiv \pi_p$ (mod λ) simply means their Satake parameters are congruent.

The following definition gives the analogue of the notion Eisenstein modulo λ in Clozel's paper [Clo].

Definition 2. *Let π be an automorphic representation of $G(\mathbb{A})$ with $\pi_\infty = \mathbf{1}$, and let λ be a finite place of $\bar{\mathbb{Q}}$. We say that π is abelian modulo λ if there exists an automorphic character χ of $G(\mathbb{A})$ with infinity type $\chi_\infty = \mathbf{1}$ such that $\pi \equiv \chi$ (mod λ).*

We prefer the terminology abelian modulo λ since the group G has no $\mathbb{Q}$-parabolics. We note that there exists non-abelian π exactly because $G(\mathbb{R})$ is assumed to be compact modulo its center.

For the next theorem we fix a good small compact open subgroup $K = \prod K_p$ (see section 2.1.4 for the precise definition of good small). Let N be an integer such that $p \nmid N$ implies that K_p is hyperspecial. Then we have the following level-raising result for the inner form G.

Theorem A *Let $\lambda | \ell$ be a finite place of $\bar{\mathbb{Q}}$, with ℓ outside a finite set determined by K. Let π be an automorphic representation of $G(\mathbb{A})$, with ω_π and π_∞ trivial, such that $\pi_f^K \neq 0$. Assume π occurs with multiplicity one, and that π is non-abelian modulo λ. Suppose $q \nmid N\ell$ is a prime number with $q^i \neq 1$ (mod ℓ) for $i = 1, \ldots, 4$, such that modulo the Weyl-action we have the congruence*

$$\mathfrak{t}_{\pi_q \otimes |c|^{-3/2}} \equiv \begin{pmatrix} 1 & & & \\ & q & & \\ & & q^2 & \\ & & & q^3 \end{pmatrix} \ (mod\ \lambda).$$

Here $\mathfrak{t}$ denotes the Satake parameter of the lift to $GL(4)$. Then there exists an automorphic repre-

sentation $\tilde{\pi} \equiv \pi \pmod{\lambda}$ of $G(\mathbb{A})$, with $\omega_{\tilde{\pi}}$ and $\tilde{\pi}_\infty$ trivial, such that $\tilde{\pi}_f^{K^q} \neq 0$ and

$$\tilde{\pi}_q \text{ is of type } \begin{cases} IIIa \\ IIa \end{cases} \quad \text{when } \pi_q \text{ is of type } \begin{cases} IIb \\ IIIb \end{cases}.$$

In the remaining case where π_q is generic, one can choose $\tilde{\pi}_q$ to be of type IIa or IIIa.

The finite set of primes ℓ that we have to discard, are those dividing the discriminant of the Hecke algebra of K. See section 2.1.4 below for more details.

We use the classification of [Sch], reproduced in Appendix A and B. Note that the two types IIb and IIIb are the typical unramified local components of CAP representations. The representations of type IIa are of the form $\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ (induced from the Siegel parabolic), while those of type IIIa are $\chi \rtimes \sigma \mathrm{St}_{\mathrm{GL}(2)}$ (induced from the Klingen parabolic). Both are ramified, generic, Klingen- and Siegel-spherical. Moreover, a representation of type IIIa is tempered if and only if it is unitary. In the generic case one can choose between the types IIa and IIIa depending on the application one has in mind. Representations of type IIa are expected to transfer to the inner form over $\mathbb{Q}_q$, while those of type IIIa cannot occur in endoscopic lifts. We will prove this below.

The proof of the above theorem is inspired by the work of Bellaiche [Bel] and Clozel [Clo]. They were both dealing with a unitary group U(3), split over some imaginary quadratic extension $E/\mathbb{Q}$. Clozel considered the case where q is inert in E. Here the semisimple rank is one, and he obtained a $\tilde{\pi}$ with a Steinberg component at q. In his thesis, Bellaiche dealt with the case where q is split in E. Here the semisimple rank is two and this makes things more complicated. In this case one gets a $\tilde{\pi}$ with $\tilde{\pi}_q$ ramified but having fixed vectors under any maximal parahoric in GL(3). This in turn implies that $\tilde{\pi}_q = \chi \mathrm{St}_{\mathrm{GL}(2)} \times \sigma$ by the classification of Iwahori-spherical representations of GL(3) in my paper [Sor]. For GSp(4) this classification is much more complicated, but fortunately it has been done by R. Schmidt [Sch]. To really utilize the tables in [Sch] we need to modify Bellaiche's argument a bit. For example, we incorporate the action of the Bernstein center at q, and we get a precise condition on what characteristics ℓ we need to discard. At a crucial point we rely on results of Lazarus [Laz] describing the structure of universal modules.

The approach in [Sor] is different. There, we are using the arguments of Taylor [Tay] in a more general setup. In the special case of an inner form of GSp(4), the main result has weaker assumptions (no multiplicity one or banality is needed, and it works for arbitrary ℓ) but also a weaker conclusion (one can only say that $\tilde{\pi}_q$ is of type I, IIa, or IIIa).

Next we prove a purely local result at the prime r, which will be crucial later on for our application of the trace formula. Now D_r is the division quaternion $\mathbb{Q}_r$-algebra, and we fix an unramified quadratic subfield E. Let θ be the generator for $\mathrm{Gal}(E/\mathbb{Q}_r)$. From now on, let G' denote $\mathrm{GSp}(4)$. We view G as the non-split inner form of G' over $\mathbb{Q}_r$. It comes with a class of inner twistings,

$$\psi : G \to G'.$$

That is, ψ is an isomorphism over $\bar{\mathbb{Q}}_r$ such that $\sigma\psi \circ \psi^{-1}$ is an inner automorphism of G' for all σ in the Galois group of $\mathbb{Q}_r$. We fix a ψ defined over E. Since G^{der} is simply connected, stable conjugacy is just $G(\bar{\mathbb{Q}}_r)$-conjugacy. Similarly for G'. Then ψ defines an injection from the semisimple stable conjugacy classes in $G(\mathbb{Q}_r)$ to the semisimple stable conjugacy classes in $G'(\mathbb{Q}_r)$.

Let $\gamma \in G(\mathbb{Q}_r)$ be a semisimple element, and let $G_\gamma(\mathbb{Q}_r)$ denote its extended centralizer:

$$G_\gamma(\mathbb{Q}_r) = \{x \in G(\mathbb{Q}_r) : x^{-1}\gamma x \in Z(\mathbb{Q}_r)\gamma\}.$$

After choosing Haar measures on $G(\mathbb{Q}_r)$ and $G_\gamma(\mathbb{Q}_r)$, we consider the orbital integral

$$O_\gamma(f) = \int_{G_\gamma(\mathbb{Q}_r)\backslash G(\mathbb{Q}_r)} f(x^{-1}\gamma x)dx$$

of a function $f \in C_c^\infty(G^{\mathrm{ad}}(\mathbb{Q}_r))$. Now let $\{\tilde{\gamma}\}$ be a set of representatives for the conjugacy classes within the stable conjugacy class of γ, modulo $Z(\mathbb{Q}_r)$. Then $G_{\tilde{\gamma}}$ is an inner form of G_γ, and we choose compatible measures. Let $e(G_{\tilde{\gamma}})$ denote the Kottwitz sign [K], and form the stable orbital integral

$$SO_\gamma(f) = \sum\nolimits_{\tilde{\gamma}} e(G_{\tilde{\gamma}})O_{\tilde{\gamma}}(f).$$

The definitions for G' are completely analogous. Now consider two functions $f \in C_c^\infty(G^{\mathrm{ad}}(\mathbb{Q}_r))$ and $f' \in C_c^\infty(G'^{\mathrm{ad}}(\mathbb{Q}_r))$. They have matching orbital integrals if, for all semisimple $\gamma' \in G'(\mathbb{Q}_r)$,

$$SO_{\gamma'}(f') = \begin{cases} SO_\gamma(f) & \text{if } \gamma' \text{ belongs to } \psi(\gamma) \bmod Z(\mathbb{Q}_r), \\ 0 & \text{if } \gamma' \text{ does not come from } G(\mathbb{Q}_r). \end{cases}$$

Here we use compatible Haar measures on both sides. We note that Waldspurger has shown in [Wa] (using results of Langlands and Shelstad [LS]) that one can always find a function f' matching a given f. We will take f to be the idempotent of a maximal compact subgroup in $G(\mathbb{Q}_r)$ and show that we can take f' to be biinvariant under a corresponding maximal compact subgroup in $G'(\mathbb{Q}_r)$.

The semisimple $\mathbb{Q}_r$-rank of G' is two, and the reduced building is covered by two-dimensional apartments. Each apartment is tessellated by equilateral right-angled triangles. The vertices at the right angles are obviously non-special, whereas all the other vertices are hyperspecial. Correspondingly, the group $G'(\mathbb{Q}_r)$ has two conjugacy classes of maximal compact subgroups. The hyperspecial ones and the (non-special) paramodular ones.

The group G has semisimple $\mathbb{Q}_r$-rank one, so its reduced building is an inhomogeneous tree. In fact, for $r = 2$ there is a picture of it on page 48 in the article of Tits [Tt]. All its vertices are special. Each edge has one vertex of order r^2+1 and one vertex of order $r+1$. The former maps to a non-special vertex in the building over E, whereas the latter maps to the midpoint of a long edge. The stabilizer of a vertex of order r^2+1 is also called paramodular.

Let K' be a paramodular group in $G'(\mathbb{Q}_r)$. Concretely, one can take K' to be the subgroup generated by the Klingen parahoric and the matrix

$$\begin{pmatrix} & & & -r^{-1} \\ & 1 & & \\ & & 1 & \\ r & & & \end{pmatrix}.$$

Besides $Z(\mathbb{Q}_r)$ and K' itself, its normalizer contains an element η called the Atkin-Lehner element in the paper of Ralf Schmidt [Sch]. It has the following form:

$$\eta = \begin{pmatrix} & & 1 & \\ & & & 1 \\ r & & & \\ & r & & \end{pmatrix}.$$

Note that it satisfies the identity $\eta^2 = r \cdot I$.

Theorem B *Let K and K' be arbitrary paramodular subgroups in $G(\mathbb{Q}_r)$ and $G'(\mathbb{Q}_r)$, respectively. Then the centralized characteristic functions e_K and $e_{\eta K'}$ have matching orbital integrals.*

Here e_K denotes the characteristic function of $Z(\mathbb{Q}_r)K$. The main ingredient of the proof is a slight modification of the results obtained by Kottwitz in [Kot]. First, since G splits over E, we may compare the stable orbital integrals of e_K and $e_{\eta K'}$ to stable twisted orbital integrals on $G(E)$ and

$G'(E)$. In turn, these integrals can be compared explicitly by hand via the inner twisting.

The next result is a special case of the Langlands functoriality conjecture. More precisely, it is an analogue for GSp(4) of the Jacquet-Langlands correspondence between the spectra of GL(2) and its inner forms. It allows us to transfer the $\tilde{\pi}$ from Theorem A to $G'(\mathbb{A})$ in the cases we are interested in. The notion of being endoscopic is made precise in section 4.1.4 below. Here we note that a cuspidal automorphic representation Π of $G'(\mathbb{A})$ is said to be CAP with respect to a $\mathbb{Q}$-parabolic P, with Levi component M, if there exists a cuspidal automorphic representation τ of $M(\mathbb{A})$ such that Π is weakly equivalent to the constituents of the induced representation of τ to $G'(\mathbb{A})$. Recall that weakly equivalent means isomorphic at all but finitely many places.

Theorem C *Let π be an automorphic representation of $G(\mathbb{A})$, with ω_π and π_∞ trivial. Suppose there exists a prime $q \notin S$ such that π_q is of type IIIa of the form $\chi \rtimes \sigma St_{GL(2)}$ where $\chi^2 \neq \mathbf{1}$. Pick a cohomological discrete series representation $\Pi_{\mathbf{1}}$ of $G'(\mathbb{R})$, holomorphic or generic. Then there exists a cuspidal automorphic representation Π of $G'(\mathbb{A})$, with ω_Π trivial and $\Pi_\infty = \Pi_{\mathbf{1}}$, such that $\Pi_p = \pi_p$ for all $p \notin S$. Any such Π is neither CAP nor endoscopic. Moreover, if π_r is para-spherical (that is, has vectors fixed by a paramodular group), there exists a Π as above with Π_r para-spherical and ramified.*

Let us briefly sketch the ideas of the proof. The main tool is Arthur's stable trace formula. The point is that the endoscopic group $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ for PGSp(4) has no endoscopy itself, and we therefore only need the standard fundamental lemma proved by Hales (not the weighted version). Hales has also computed the Shalika germs for GSp(4) and its inner forms. Then, from the general results of Langlands and Shelstad on descent for transfer factors, one immediately deduces the transfer conjecture in our cases. In fact, more recently, Waldspurger has shown in general that the fundamental lemma implies the transfer conjecture. Intuitively, this enables us to match the geometric sides of the trace formulas for G and G'. Consequently, the spectral sides match and we can compare the spectra. There is a serious problem to overcome though. The distribution defined by the trace formula is unstable. One makes it stable by subtracting suitable endoscopic error terms. To show that these error terms vanish in our situation, we invoke results of Weissauer describing endoscopic lifts in terms of the θ-correspondence with $\mathrm{GO}(X)$ for X four-dimensional. It remains to show that type IIIa representations do not participate in these correspondences. For this purpose, we use Kudla's filtration of the Jacquet modules of the Weil representation. Roughly, this filtration reveals that the Weil representation is compatible with parabolic induction. This way, we are reduced to showing that the Steinberg representation $\mathrm{St}_{\mathrm{SL}(2)}$ does not occur in the θ-correspondence with split O(2). This is a well-known fact. A standard argument, based on the linear independence of characters, then gives a discrete automorphic representation Π with $\Pi^S = \pi^S$. It is actually cuspidal: By the theory of Eisenstein series we rule out that it occurs in the residual spectrum since it has a tempered component. To see it is not CAP we use work of Piatetski-Shapiro and Soudry. To make sure the component Π_∞ lies in the expected L-packet, and that we can indeed choose a specific member, we rely on the exhaustive work of Shelstad in the archimedean case.

To get the paramodular refinement at r, we appeal to Theorem B. In fact, we then get a Π such that the Atkin-Lehner operator on the paramodular invariants of Π_r has a positive trace. Using work of Weissauer [W3] on the Ramanujan conjecture we show that Π_r is in fact also ramified. Then, by the computations of Schmidt, Π_r must be of type IIa, Vb, Vc, or VIc. We expect that Π_r is necessarily tempered. If so, it is of type IIa of the form $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ with $\chi\sigma$ non-trivial quadratic.

We note that slight modifications of the above trace formula argument, combined with Weissauer's work on weak endoscopic lifts, in fact prove the existence of weak transfer in general.

The foregoing discussion culminates in the following main result, which provides congruences between unstable and stable automorphic forms for GSp(4). Let $f \in S_4(\Gamma_0(N))$ be a newform of weight 4 and square-free level N (and trivial character). We assume that f has root number

$$\epsilon_f = -1.$$

In other words, the L-function $L(s, f)$ vanishes to an odd order at $s = 2$. For example, such a newform f exists for $N = 13$. By the sign condition, we may lift f to a Saito-Kurokawa form SK(f) on GSp(4). This is a CAP representation, holomorphic at infinity, having Galois representation

$$\rho_{\mathrm{SK}(f),\lambda} \simeq \rho_{f,\lambda} \oplus \omega_\ell^{-1} \oplus \omega_\ell^{-2}.$$

Here ω_ℓ is the ℓ-adic cyclotomic character, and $\rho_{f,\lambda}$ is the system of Galois representations attached to f by Deligne [Del]. More recently, Laumon [Lau] and Weissauer [W3] have attached Galois representations to any cuspidal automorphic representation of GSp(4), which is a discrete series at infinity. We produce congruences between SK(f) and certain stable forms of small level:

Theorem D *With notation as above, let $\lambda|\ell$ be a finite place of $\bar{\mathbb{Q}}$, with ℓ outside a finite set of primes determined by N, such that $\bar{\rho}_{f,\lambda}$ is irreducible. Suppose $q \nmid N\ell$ is a prime such that*

- *$q^i \not\equiv 1$ (mod ℓ) for $i = 1, \ldots, 4$,*

- *$\bar{\rho}_{f,\lambda}(Frob_q)$ has a fixed vector.*

Then there exists a cuspidal automorphic representation $\Pi \equiv SK(f)$ (mod λ) of $PGSp(4)$, neither CAP nor endoscopic, having the following properties: Π_∞ is the cohomological holomorphic discrete series representation, Π_p is unramified and tempered for $p \nmid Nq$,

- *The Galois representation $\rho_{\Pi,\lambda}$ is irreducible,*

- *Π_q is of type IIIa (hence tempered, generic, and ramified),*

- *Π_p is para-spherical for all primes p dividing N.*

Moreover, if f is not CM, there exists a positive density of primes q satisfying the conditions above.

The proof is a combination of all our previous results. We outline the main ideas. Since f has trivial character, it generates a cuspidal automorphic representation τ of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character, and with τ_∞ being the holomorphic discrete series of weight 4, so that we have the equality between L-functions $L(s - 3/2, \tau_f) = L(s, f)$. We may, and we will, view τ as a cuspidal automorphic representation of $\mathrm{PGL}(2)$. Choose a prime r such that τ_r is the Steinberg representation (and not its unramified quadratic twist). Let G be the definite inner form of $\mathrm{GSp}(4)$ with ramification locus $\{\infty, r\}$. Its adjoint group G^{ad} is the special orthogonal group of a definite quadratic form in five variables over $\mathbb{Q}$. Let A_τ be the global Waldspurger packet for the metaplectic group $\widetilde{\mathrm{SL}}(2)$ determined by τ. Then $\mathrm{SK}(f) = \theta(\sigma)$ for some $\sigma \in A_\tau$. We consider the reflection $\breve{\sigma} \in A_\tau$ and its lifting $\theta(\breve{\sigma})$ to the inner form G. This turns out to be para-spherical at all primes dividing N. By Theorem A we can raise the level: Since $\mathrm{SK}(f)$ has local components of type IIb outside N, we get a $\pi \equiv \theta(\breve{\sigma})$ (mod λ) with π_q of type IIIa. Then, by Theorem C we can transfer π to an automorphic representation Π of $\mathrm{GSp}(4)$ agreeing with π outside of $\{\infty, r\}$. The irreducibility of $\rho_{\Pi,\lambda}$ was essentially proved by Ramakrishnan in [Ram].

Finally, we give an application of Theorem D to prove new cases of the Bloch-Kato conjecture for classical modular forms, assuming a conjecture of Skinner and Urban. Before we state our result, we briefly recall the definition of Selmer groups. Let V be a continuous representation of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, with coefficients in a finite extension $L/\mathbb{Q}_\ell$. Choose a lattice Λ and define W by

$$0 \to \Lambda \xrightarrow{i} V \xrightarrow{\mathrm{pr}} W \to 0.$$

Let λ be the maximal ideal in the ring of integers of L. Then we identify the reduction $\Lambda/\lambda\Lambda$ with the λ-torsion in W. For each prime p, let I_p be the inertia group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{nr}})$. Let B_{cris} be Fontaine's crystalline Barsotti-Tate ring, which is a $\mathbb{Q}_\ell$-algebra of periods [BK]. Then we define the finite part of the Galois cohomology:

$$H^1_f(\mathbb{Q}_p, V) = \begin{cases} \ker\{H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p^{\mathrm{nr}}, V)\}, & \text{for } p \neq \ell, \\ \ker\{H^1(\mathbb{Q}_\ell, V) \to H^1(\mathbb{Q}_\ell, B_{\mathrm{cris}} \otimes V)\}, & \text{for } p = \ell. \end{cases}$$

The Selmer group $H^1_f(\mathbb{Q}, V)$ is then the subgroup of $H^1(\mathbb{Q}, V)$ cut out by these local conditions:

$$H^1_f(\mathbb{Q}, V) = \ker\{H^1(\mathbb{Q}, V) \to \prod_p H^1(\mathbb{Q}_p, V)/H^1_f(\mathbb{Q}_p, V)\}.$$

Using the maps on cohomology induced by i and pr, we then define the finite parts for Λ and W,

$$H^1_f(\mathbb{Q}_p, \Lambda) = i_*^{-1} H^1_f(\mathbb{Q}_p, V), \quad H^1_f(\mathbb{Q}_p, W) = \mathrm{pr}_* H^1_f(\mathbb{Q}_p, V).$$

The Selmer groups $H^1_f(\mathbb{Q}, \Lambda)$ and $H^1_f(\mathbb{Q}, W)$ are then defined as above. If V is the ℓ-adic realization of a motive, the latter group is sometimes called the ℓ-part of the Selmer group of the motive. It sits in a short exact sequence, where the quotient is a conjecturally finite ℓ-group,

$$0 \to \mathrm{pr}_* H^1_f(\mathbb{Q}, V) \to H^1_f(\mathbb{Q}, W) \to \text{Ш}(\mathbb{Q}, W) \to 0.$$

This quotient $\text{Ш}(\mathbb{Q}, W)$ is called the ℓ-part of the Tate-Shafarevich group. Momentarily, let $\bar{\Lambda}$ denote the reduction $\Lambda/\lambda\Lambda$. Then the finite part $H^1_f(\mathbb{Q}_p, \bar{\Lambda})$ is defined to be the image of $H^1_f(\mathbb{Q}_p, \Lambda)$ under the natural map. See page 17 in [Rub]. The Selmer group $H^1_f(\mathbb{Q}, \bar{\Lambda})$ is then defined as before. In the situations we are eventually interested in below, it can be identified with the λ-torsion in $H^1_f(\mathbb{Q}, W)$.

The classes in $H^1_f(\mathbb{Q}, V)$ correspond to equivalence classes of certain good extensions of the trivial representation $\mathbf{1}$ by V. To be precise, consider an extension of ℓ-adic $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-modules,

$$0 \to V \to X \to \mathbf{1} \to 0.$$

It is said to have good reduction at $p \neq \ell$ if the sequence remains exact after taking I_p-invariants. In particular, if V is unramified at p, this simply means that X is unramified at p. Similarly, the extension is said to have good reduction at ℓ if the sequence remains exact after applying the functor

$$D_{\mathrm{cris}}(V) = H^0(\mathbb{Q}_\ell, B_{\mathrm{cris}} \otimes V).$$

This is a filtered module of dimension at most $\dim_{\mathbb{Q}_\ell}(V)$. If the dimensions are equal, V is called crystalline. In this case, X is required to be crystalline. An extension X with good reduction everywhere gives rise to a cohomology class in $H^1_f(\mathbb{Q}, V)$ via the connecting homomorphism. This defines a bijection. The other Selmer groups described above have similar interpretations. For example, the finite part $H^1_f(\mathbb{Q}_\ell, \bar{\Lambda})$ is connected to the notion of being Fontaine-Laffaille [FL].

Here we are content with formulating the Bloch-Kato conjecture [BK] for classical modular forms. At first, we consider an arbitrary newform $f \in S_{2\kappa}(\Gamma_0(N))$. We take V above to be the κ-th Tate twist $\rho_{f,\lambda}(\kappa)$ of the Galois representation attached to f. Then conjecturally we have the relation

$$\mathrm{ord}_{s=\kappa} L(s, f) \stackrel{?}{=} \dim_{\mathbb{Q}_\ell} H^1_f(\mathbb{Q}, \rho_{f,\lambda}(\kappa)).$$

If $\epsilon_f = -1$ the L-function vanishes at the point $s = \kappa$. Then the conjecture predicts that the pertinent Selmer group is non-trivial. This was proved by Skinner and Urban in [SU] under the assumption that f is ordinary at λ (meaning that the Hecke eigenvalue $a_\ell(f)$ is a λ-adic unit). Their proof relies on the deep results of Kato [Ka]. However, in the square-free case they give a different argument, bypassing the work of Kato, but instead relying on Conjecture 1 below.

Let ρ be a continuous representation of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on a finite-dimensional vector space V over the ℓ-adic field L. Assume $p \neq \ell$. Then by a famous result of Grothendieck, ρ is potentially semistable. This means there exists a nilpotent endomorphism $N : V \to V$ such that

$$\rho(\sigma) = \exp(t_\ell(\sigma) N)$$

for σ in a finite index subgroup of I_p. Here $t_\ell : I_p \twoheadrightarrow \mathbb{Z}_\ell$ is a homomorphism intertwining the natural actions of the Weil group at p. The endomorphism N is called the monodromy operator.

The following is basically Conjecture 3.1.7 on page 41 in the paper of Skinner and Urban [SU].

Conjecture 1. *Let Π be a cuspidal automorphic representation of GSp(4), neither CAP nor endoscopic, with Π_∞ cohomological. Suppose the local component Π_p has non-zero vectors fixed by the paramodular group. Then the corresponding monodromy operator at p has rank at most one.*

As explained later in this introduction, this conjecture follows from the expected compatibility between the local and global Langlands correspondence for GSp(4). Our application to the Bloch-Kato conjecture is contingent on Conjecture 1.

Theorem E *Let $f \in S_4(\Gamma_0(N))$ be a newform of square-free level N with root number $\epsilon_f = -1$. Assume f is not of CM type. Let $\lambda|\ell$ be a finite place of $\bar{\mathbb{Q}}$, with ℓ outside a finite set, such that $\bar{\rho}_{f,\lambda}$ is irreducible. Assume the above Conjecture 1. Then the Selmer group*

$$H^1_f(\mathbb{Q}, \bar{\rho}_{f,\lambda}(2)) \neq 0,$$

as predicted by the Bloch-Kato conjecture since the L-function $L(s,f)$ vanishes at $s=2$.

By the result of Jordan and Livne on level-lowering for modular forms of higher weight [JL], we may assume that $\bar{\rho}_{f,\lambda}$ is ramified at all primes $p|N$. Indeed congruent eigenforms have equal root numbers. This minimality assumption turns out to be crucial for the proof.

It follows immediately from Theorem E that the Selmer group of $\rho_{f,\lambda}(2)$ is non-trivial, assuming that the ℓ-part of the Tate-Shafarevich group is trivial. This should always be the case according to the Tamagawa number conjecture. See Conjecture 5.15 on page 376 in [BK].

We outline the main ideas of the proof of Theorem E. By Theorem D we obtain a prime q and an automorphic representation Π. First we choose a lattice Λ in the space of $\rho_{\Pi,\lambda}$ such that $\bar{\Lambda}$ has $\bar{\rho}_{f,\lambda}$ as its unique irreducible quotient. The goal is then to show that $\bar{\omega}_\ell^{-2}$ embeds into $\bar{\Lambda}$. If not, $\bar{\omega}_\ell^{-1}$ is the unique irreducible subrepresentation of $\bar{\Lambda}$. Thus we get two non-split extensions

$$0 \to \bar{\omega}_\ell^{-1} \to X \to \bar{\omega}_\ell^{-2} \to 0 \quad \text{and} \quad 0 \to \bar{\omega}_\ell^{-2} \to Y \to \bar{\rho}_{f,\lambda} \to 0.$$

Both X and Y are subquotients of the etale intersection cohomology (for the middle perversity):

$$IH^3_{\text{et}}(\bar{S}_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \bar{\mathbb{Q}}_\ell).$$

Here K is paramodular at primes dividing N, Klingen at q, and hyperspecial outside Nq. We denote by $\bar{S}_K$ the Satake compactification of the Siegel threefold S_K. Obviously X and Y are then both unramified outside Nq. In addition they are both Fontaine-Laffaille. To show that X and Y both have good reduction at primes dividing N, we use Conjecture 1 and our minimality assumption. At q the monodromy operator has order two by a result of Genestier and Tilouine. This allows us to show that X or Y is unramified at q. This is a contradiction. Indeed, by Kummer theory $H^1_f(\mathbb{Q}, \bar{\omega}_\ell) = 0$, and by Kato's paper [Ka] we also have $H^1_f(\mathbb{Q}, \bar{\rho}_{f,\lambda}(1)) = 0$. Therefore $\bar{\omega}_\ell^{-2}$ does embed into $\bar{\Lambda}$, and we get a non-split extension with good reduction everywhere:

$$0 \to \bar{\omega}_\ell^{-1} \to Z \to \bar{\rho}_{f,\lambda} \to 0.$$

Taking the dual extension, and a suitable Tate twist, we obtain the desired class in $H^1_f(\mathbb{Q}, \bar{\rho}_{f,\lambda}(2))$.

We end this introduction by giving a heuristic argument for Conjecture 1. Let Π be a cuspidal automorphic representation of GSp(4), neither CAP nor endoscopic, with Π_∞ cohomological. Assume moreover that Π_p is para-spherical, and that Π^p has vectors fixed by some compact open subgroup K^p in $G(\mathbb{A}_f^p)$. Then we consider the Siegel threefold S_K of level K, where K_p is paramodular. This is a quasi-projective variety over $\mathbb{Q}$. Its base change to $\mathbb{Q}_p$ has an integral model, also denoted by S_K, representing the following moduli problem: For a $\mathbb{Z}_p$-scheme S consider triples $(A, \lambda, \bar{\eta})$ where

- A is an abelian S-scheme of relative dimension two,

- $\lambda : A \to A^\vee$ is a polarization of degree p^2,

- $\bar{\eta} : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} H_1(A, \mathbb{A}_f^p)$ modulo K^p is a K^p-level structure.

The Galois representation $\rho_{\Pi,\lambda}$ is given by the Π_f^K-isotypic component of the interior cohomology of the generic fiber $S_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}$. This is the image of the cohomology of compact support. It is expected that, by finding a suitably nice toroidal compactification of S_K, there is a natural isomorphism

$$H^3_c(S_K \times_{\mathbb{Z}_p} \bar{\mathbb{F}}_p, R\Psi(\bar{\mathbb{Q}}_\ell)) \xrightarrow{\sim} H^3_c(S_K \times_{\mathbb{Z}_p} \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell).$$

Here $R\Psi(\bar{\mathbb{Q}}_\ell)$ is the sheaf of nearby cycles on the special fiber. This is really an object in some limit of derived categories, but we will not need the precise definition in our discussion.

To understand the geometry of the special fiber $S_K \times_{\mathbb{Z}_p} \bar{\mathbb{F}}_p$ we should write down equations for the Rapoport-Zink local model. This has been done for the other parahorics. When K_p is hyperspecial, S_K has good reduction. The Iwahori case was studied by de Jong [JO]. The Siegel case was treated by Chai and Norman [CN], and Genestier and Tilouine [GT] dealt with the Klingen case.

Given the above isomorphism, since Π is not endoscopic, it seems likely that the techniques of Haines and Ngo [HN] can be adapted to show the following important identity:

$$L^{ss}(s-3/2, \Pi_p, \mathrm{spin}) = \det{}^{ss}(1 - \mathrm{Frob}_p \cdot p^{-s} : \rho_{\Pi,\lambda}^{I_p})^{-1}.$$

Here the ss means we are considering the semisimple L-factors of Rapoport [Rap]. We expect that Π_p is necessarily tempered if Π is tempered almost everywhere. Combining this with a special case of the weight-monodromy conjecture, it should then follow that the above identity holds without the ss. To define the L-factor on the left, we use the parameter (ϱ, N) attached to Π_p by Kazhdan and Lusztig [KL]. Thus $\varrho : W_{\mathbb{Q}_p} \to \mathrm{GSp}_4(\mathbb{C})$ is a homomorphism whose image consists of semisimple elements, and $N \in \mathfrak{gsp}_4(\mathbb{C})$ is a nilpotent operator satisfying the following relation:

$$\varrho(\mathrm{Frob}_p^{-1}) \cdot N \cdot \varrho(\mathrm{Frob}_p) = p \cdot N.$$

In Conjecture 1 we may clearly assume that Π_p is ramified. If Π_p is also tempered, it follows from the appendices that Π_p is actually of type IIa. To be exact, it has the form $\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ (induced from the Siegel parabolic) for two unitary characters χ and σ. It is the unique irreducible subrepresentation of the principal series $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$. Then ϱ is the Satake parameter of the unramified quotient $\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$. There is only one non-trivial conjugacy class of nilpotent N satisfying the above relation. Namely, the class containing

$$N \sim \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

In particular $\dim \ker N = 3$. We then deduce from the above equality of L-factors that $\dim \rho_{\Pi,\lambda}^{I_p} = 3$.

Chapter 2

Level-raising

2.1 Algebraic Modular Forms

2.1.1 The Complex Case

Let G be an inner form of GSp(4) over $\mathbb{Q}$ such that $G(\mathbb{R})$ is compact modulo its center. Concretely, G is the unitary similitude group of D^2, for some definite quaternion algebra D over $\mathbb{Q}$, where we take the hermitian form to be the identity I. Throughout, we assume that D has ramification locus $S = \{\infty, r\}$ for some prime r. Let $c : G \to \mathbb{G}_m$ denote the similitude, and let $Z \simeq \mathbb{G}_m$ be the center. Then we consider the space of automorphic forms,

$$\mathcal{A} = \{\text{smooth } f : Z(\mathbb{A}_f)G(\mathbb{Q})\backslash G(\mathbb{A}_f) \to \mathbb{C}\}.$$

Here $\mathbb{A}_f$ is the finite part of the adele ring $\mathbb{A}$. There is an admissible representation r of $G(\mathbb{A}_f)$ on this space given by right translations. In turn, the Hecke algebra $\mathcal{H}$ of compactly supported smooth functions on $G(\mathbb{A}_f)$ also acts. We equip $\mathcal{A}$ with the pairing defined by the following integral,

$$\langle f, f' \rangle = \int_{G(\mathbb{Q})\backslash G(\mathbb{A}_f)} f(x)f'(x)dx.$$

This is well-defined since $G(\mathbb{Q})$ is a discrete cocompact subgroup of $G(\mathbb{A}_f)$. As one easily verifies,

$$\langle r(\phi)f, f' \rangle = \langle f, r(\phi^\vee)f' \rangle,$$

where the anti-involution $\phi \mapsto \phi^\vee$ of $\mathcal{H}$ is defined by $\phi^\vee(x) = \phi(x^{-1})$. It reflects contragredients. Now let K be a compact open subgroup of $G(\mathbb{A}_f)$, and let $\mathcal{A}_K$ be the space of K-invariants. The

Hecke algebra $\mathcal{H}_K$ of K-biinvariant compactly supported functions on $G(\mathbb{A}_f)$ then acts semisimply:

$$\mathcal{A}_K \simeq \bigoplus\nolimits_{\Pi} m(\Pi)\Pi_f^K,$$

where Π varies over the automorphic representations of $G(\mathbb{A})$, with trivial central character, such that Π_∞ is trivial and $\Pi_f^K \neq 0$. Let us choose representatives $\{x_i\}$ for the finite set of cardinality h,

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K.$$

Then $f \mapsto (f(x_i))$ identifies $\mathcal{A}_K$ with a subspace of $\mathbb{C}^h$. We introduce the following finite groups,

$$\Gamma_i = G(\mathbb{Q}) \cap x_i K x_i^{-1}.$$

These are all trivial if K is sufficiently small. To be precise, if the projection of K to some $G(\mathbb{Q}_p)$ does not contain non-trivial elements of finite order. Finally, let us consider the pairing on $\mathcal{A}$ restricted to $\mathcal{A}_K$. A straightforward calculation shows that we have the following formula for $f, f' \in \mathcal{A}_K$,

$$\langle f, f' \rangle = \sum_i f(x_i) f'(x_i) \# \Gamma_i^{-1},$$

up to normalization. In particular, it follows immediately that the above pairing is non-degenerate.

2.1.2 Models Over Number Fields

Let $\mathcal{H}_{K,\mathbb{Z}}$ denote the natural $\mathbb{Z}$-structure in the algebra $\mathcal{H}_K$. It preserves the lattice $\mathcal{A}_{K,\mathbb{Z}}$ consisting of $\mathbb{Z}$-valued functions in $\mathcal{A}_K$. If L is a number field, we then define $\mathcal{H}_{K,L}$ and $\mathcal{A}_{K,L}$ by extension of scalars. We choose L so that $\mathrm{Aut}(\mathbb{C}/L)$ fixes the simple $\mathcal{H}_K$-submodules of $\mathcal{A}_K$. Then

$$\Pi_f^K \simeq \mathbb{C} \otimes_L \Pi_f^K(L), \quad \text{where} \quad \Pi_f^K(L) = \Pi_f^K \cap \mathcal{A}_{K,L}.$$

Moreover, the L-model $\Pi_f^K(L)$ is unique up to complex scalars. We retain the decomposition of $\mathcal{A}_{K,L}$ into a sum of the various $\Pi_f^K(L)$, where Π runs through the usual set of automorphic representations.

2.1.3 Integral Models

Let λ be a finite place of L above ℓ, and let $\mathcal{O}$ denote the ring of integers in the completion L_λ. Then $\mathcal{A}_{K,L_\lambda}$ is the sum of the $\Pi_f^K(L_\lambda)$ obtained from $\Pi_f^K(L)$ by tensoring with L_λ. Then

$$\Pi_f^K(L_\lambda) \simeq L_\lambda \otimes_{\mathcal{O}} \Pi_f^K(\mathcal{O}), \quad \text{where} \quad \Pi_f^K(\mathcal{O}) = \Pi_f^K(L_\lambda) \cap \mathcal{A}_{K,\mathcal{O}}.$$

However, the integral model $\Pi_f^K(\mathcal{O})$ need not be unique up to scalars. We also remark that their sum need not exhaust $\mathcal{A}_{K,\mathcal{O}}$. However, the corresponding quotient is at worst torsion.

2.1.4 Algebraic Modular Forms Mod ℓ

Let $\mathbb{F}$ be the residue field of $\mathcal{O}$. By the Brauer-Nesbitt principle, see page 80 in [Vig], the semisimplification is independent of the lattice (up to isomorphism) so that we have the usual decomposition

$$\mathcal{A}_{K,\mathbb{F}}^{\mathrm{ss}} \simeq \bigoplus\nolimits_{\Pi} m(\Pi)\Pi_f^K(\mathbb{F}),$$

where $\Pi_f^K(\mathbb{F})$ denotes the semisimplification of $\mathbb{F} \otimes_{\mathcal{O}} \Pi_f^K(\mathcal{O})$. Indeed we consider the reductions of the lattice $\mathcal{A}_{K,\mathcal{O}}$ and its sublattice

$$\bigoplus\nolimits_{\Pi} m(\Pi)\Pi_f^K(\mathcal{O}),$$

and then take their semisimplifications. We say that K is a good small subgroup if the Hecke-module Π_f^K determines the representation Π_f. For almost all ℓ we have semisimplicity:

Lemma 1. *Suppose K is a good small subgroup. For ℓ outside a finite set of primes determined by K the following holds. The $\mathcal{H}_{K,\mathbb{F}}$-module $\mathcal{A}_{K,\mathbb{F}}$ is semisimple, all the $\Pi_f^K(\mathbb{F})$ are simple submodules, and each $\Pi_f^K(\mathbb{F})$ occurs with multiplicity $m(\Pi)$.*

Proof. In this proof let $H_{K,\mathbb{Z}}$ denote the image of $\mathcal{H}_{K,\mathbb{Z}}$ in $\mathrm{End}\mathcal{A}_{K,\mathbb{Z}}$. The algebra $H_{K,\mathbb{Z}}$ comes endowed with a natural symmetric pairing given by the trace. We consider its discriminant,

$$\det\{\mathrm{tr}(T_i \circ T_j)\} \in \mathbb{Z} - \{0\},$$

where $\{T_i\}$ is a basis for $H_{K,\mathbb{Z}}$. To see that this is non-zero, let H_K denote the algebra acting faithfully on $\mathcal{A}_K$. It is semisimple, so the natural pairing on H_K is non-degenerate since its radical is contained in the Jacobson radical. Now let ℓ be a prime not dividing the discriminant. Then the extended pairing on $\mathbb{F} \otimes_{\mathbb{Z}} H_{K,\mathbb{Z}}$ is non-degenerate, and it is therefore semisimple. The surjection

$$\mathbb{F} \otimes_{\mathbb{Z}} H_{K,\mathbb{Z}} \twoheadrightarrow H_{K,\mathbb{F}}$$

is then an isomorphism since its kernel is nilpotent. Consequently $H_{K,\mathbb{F}}$ is semisimple. We now proceed to compute its dimension in two different ways. Decompose $\Pi_f^K(\mathbb{F})$ into simple submodules X with multiplicity $m_\Pi(X)$. By Wedderburn theory, H_K is a product of matrix algebras. Hence, by computing $\dim_{\mathbb{C}} H_K = \dim_{\mathbb{F}} H_{K,\mathbb{F}}$ in two ways, we obtain the following equality

$$\sum\nolimits_X (\dim X)^2 = \sum\nolimits_{\Pi,X} m_\Pi(X)^2 (\dim X)^2 + \text{mixed terms},$$

where the mixed terms are non-negative contributions coming from distinct types in $\Pi_f^K(\mathbb{F})$:

$$\sum\nolimits_{\Pi,X\neq X'} m_\Pi(X) m_\Pi(X')(\dim X)(\dim X').$$

We deduce that there are no mixed terms, and each X is a constituent of a unique $\Pi_f^K(\mathbb{F})$ with multiplicity one. Therefore the $\Pi_f^K(\mathbb{F})$ must be simple and inequivalent for varying Π. □

If ℓ does not divide the orders $\#\Gamma_i$ for all i, we can define a Hecke-compatible non-degenerate pairing on $\mathcal{A}_{K,\mathbb{F}}$ by the previous formula. This is automatic when ℓ is sufficiently large:

Lemma 2. *ℓ does not divide the $\#\Gamma_i$ if $\ell > 5$.*

Proof. Suppose $\ell > 5$ divides $\#\Gamma_i$. Then ℓ divides the pro-order of $\mathrm{GSp}(4, \mathbb{Z}_p)$ for almost all p.

$$\#\mathrm{GSp}(4, \mathbb{F}_p) = p^4(p-1)(p^2-1)(p^4-1),$$

so p has multiplicative order at most 4 mod ℓ. This contradicts Dirichlet's theorem. □

2.2 Generalized Eigenspaces

Let π be a fixed automorphic representation of $G(\mathbb{A})$, with trivial central character, such that π_∞ is trivial and $\pi_f^K \neq 0$. By Schur's lemma, the center $Z(\mathcal{H}_{K,\mathbb{Z}})$ acts on π_f^K by a character

$$\eta : Z(\mathcal{H}_{K,\mathbb{Z}}) \to L.$$

Here we may have to enlarge the field L. Since $Z(\mathcal{H}_{K,\mathbb{Z}})$ preserves $\mathcal{A}_{K,\mathbb{Z}}$, the values of η are in fact algebraic integers. We denote by $\bar{\eta}$ its reduction modulo λ, and look at its generalized eigenspace:

$$\mathcal{A}_{K,\mathbb{F}}(\bar{\eta}) = \{f \in \mathcal{A}_{K,\mathbb{F}} \text{ for which } \exists n \text{ such that } (r(\phi) - \bar{\eta}(\phi))^n f = 0, \forall \phi \in Z(\mathcal{H}_{K,\mathbb{F}})\}.$$

This subspace is preserved by $\mathcal{H}_{K,\mathbb{F}}$. Its semisimplification is given by the following lemma:

Lemma 3. $\mathcal{A}_{K,\mathbb{F}}(\bar{\eta})^{ss} \simeq \bigoplus_{\Pi:\ \Pi \equiv \pi\ (mod\ \lambda)} m(\Pi)\Pi_f^K(\mathbb{F})$.

Proof. In this proof let $\mathbb{T}$ denote the image of $Z(\mathcal{H}_{K,\mathcal{O}})$ in $\mathrm{End}\mathcal{A}_{K,\mathcal{O}}$. Since $\mathcal{O}$ is complete, $\mathbb{T}$ is the direct product of its localizations. Clearly $\bar{\eta}$ factors through $\mathbb{T}$, and we let $\mathfrak{m} = \ker(\bar{\eta})$ be the corresponding maximal ideal in $\mathbb{T}$. Then $(\mathcal{A}_{K,\mathcal{O}})_{\mathfrak{m}} = \mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{T}} \mathcal{A}_{K,\mathcal{O}}$ is a lattice in

$$(\mathcal{A}_{K,L_\lambda})_{\mathfrak{m}} = \bigoplus_{\eta' \equiv \eta \pmod{\lambda}} \mathcal{A}_{K,L_\lambda}(\eta') \simeq \bigoplus_{\Pi:\ \Pi \equiv \pi \pmod{\lambda}} m(\Pi)\Pi_f^K(L_\lambda).$$

Also, clearly $(\mathcal{A}_{K,\mathbb{F}})_{\mathfrak{m}} = \mathcal{A}_{K,\mathbb{F}}(\bar{\eta})$, so the conclusion follows from the Brauer-Nesbitt principle. □

We will assume that $\ell > 5$ from now on. We then prove the following crucial result:

Lemma 4. $\mathcal{A}_{K,\mathbb{F}}(\bar{\eta})$ *is selfdual.*

Proof. By an easy inductive argument based on the socle filtration, it follows that the pairing on $\mathcal{A}_{K,\mathbb{F}}$ restricts to a non-degenerate pairing between the generalized eigenspaces $\mathcal{A}_{K,\mathbb{F}}(\bar{\eta})$ and $\mathcal{A}_{K,\mathbb{F}}(\bar{\eta}^\vee)$. However, it is well-known that $\pi \simeq \pi^\vee$ (this is even true locally for any odd rank special orthogonal group) so $\bar{\eta} = \bar{\eta}^\vee$. The Hecke actions are intertwined by the compatibility relation. □

2.3 The Universal Module

In this section we fix a prime $q \neq r$. We let $G = G(\mathbb{Q}_q)$ and fix a hyperspecial subgroup K. Also, we fix a Borel subgroup B. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_\ell$ and consider the spherical Hecke algebra $\mathcal{H}_{K,\mathbb{F}}$. We look at the degree character, giving the Hecke-action on the trivial representation

$$\deg : \mathcal{H}_{K,\mathbb{F}} \to \mathbb{F}.$$

We define a category with objects (V, v), where V is a smooth G-representation over $\mathbb{F}$ and $v \in V$ is a K-fixed vector on which $\mathcal{H}_{K,\mathbb{F}}$ acts by deg, and with the obvious morphisms. This category has a universal initial object. An explicit construction realizes it as the following induced module,

$$\mathcal{M} = C_c(G/K) \otimes_{\mathcal{H}_{K,\mathbb{F}}} \mathbb{F}_{\deg}.$$

Obviously $\mathcal{M}^K$ is spanned by the class of e_K, the neutral element in $\mathcal{H}_{K,\mathbb{F}}$. Also observe that $\mathcal{M}$ is generated by its K-invariants, hence cyclic. We will need the following theorem of Lazarus [Laz]:

Theorem 1. *Suppose $q \neq \ell$ and $q^4 \neq 1 \pmod{\ell}$. Then $\mathcal{M}^\vee \simeq C^\infty(B\backslash G)$.*

Proof. Let δ_B denote the modulus character of B. Note that $C^\infty(B\backslash G)^\vee$ is nothing but the principal series of $\delta_B^{1/2}$. By the universal property of $\mathcal{M}$ there is canonical surjective G-map

$$\mathcal{M} \to C^\infty(B\backslash G)^\vee.$$

By assumption ℓ is banal for q, that is, $\ell \neq q$ does not divide $\#G(\mathbb{F}_q)$. Therefore, Theorem 1.0.3 in [Laz] applies. Consequently, the above map must be an isomorphism since the two representations have the same semisimplifications. □

We say that ℓ is banal for q if it satisfies the hypothesis of this theorem. Note that we must then have $\ell > 5$. The result allows us to write down a composition series for $\mathcal{M}^\vee$. There are two parabolic subgroups containing B. The Klingen parabolic P_α, and the Siegel parabolic P_β. The latter has abelian unipotent radical. Let us take B to be the subgroup of upper triangular matrices:

$$B = \{ \begin{pmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \}.$$

Then these two maximal parabolic subgroups have the following matrix realizations:

$$P_\alpha = \{ \begin{pmatrix} c & & \\ & g & \\ & & c^{-1}\det g \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \},$$

and

$$P_\beta = \{ \begin{pmatrix} g & \\ & c\,{}^\tau g^{-1} \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \},$$

where ${}^\tau g$ denotes the skew-transpose of g. That is, transpose with respect to the second diagonal. Now suppose $q^i \neq 1 \pmod{\ell}$ for $i = 1, \dots, 4$. By Theorem 4.7.2 in [Laz] the following filtration,

$$0 \subset \{\text{constants}\} \subset C^\infty(P_\alpha \backslash G) \subset C^\infty(P_\alpha \backslash G) + C^\infty(P_\beta \backslash G) \subset C^\infty(B \backslash G),$$

has irreducible subquotients $\mathbf{1}$, V_α, V_β, and St, all occurring with multiplicity one.

2.4 Existence of Certain Subquotients

Let π be as above, but assume that $m(\pi) = 1$. Moreover, suppose K is a good small subgroup and that ℓ lies outside a finite set of primes as in Lemma 1. Let $q \notin \{\ell, r\}$ be a prime such that

- $K = K_q K^q$ with K_q hyperspecial,

- $q^i \neq 1 \pmod{\ell}$ for $i = 1, \dots, 4$,

- $\pi_q \equiv \mathbf{1} \pmod{\lambda}$.

Then fix an Iwahori subgroup $I_q \subset K_q$ and let $I = I_q K^q$. By our assumptions on π and ℓ, we can identify $\pi_f^K(\mathbb{F})$ with a submodule of $\mathcal{A}_{K,\mathbb{F}}(\bar{\eta})$. We look at the Iwahori-modules they generate:

$$\mathcal{H}_{I,\mathbb{F}} \cdot \pi_f^K(\mathbb{F}) \subset \mathcal{H}_{I,\mathbb{F}} \cdot \mathcal{A}_{K,\mathbb{F}}(\bar{\eta}) \subset \mathcal{A}_{I,\mathbb{F}}(\bar{\eta}).$$

Here we abuse notation a bit, and let $\mathcal{A}_{I,\mathbb{F}}(\bar{\eta})$ denote the generalized eigenspace for $\bar{\eta}$ composed with

$$Z(\mathcal{H}_{I,\mathbb{F}}) \to Z(\mathcal{H}_{K,\mathbb{F}}), \quad \phi \mapsto e_K \star \phi.$$

This is in fact an isomorphism, but we will not use that. The connection with $\mathcal{M}$ is given by:

Lemma 5. *The universal module* $\mathcal{M}$ *has a quotient* $\mathcal{N}$ *such that, as modules over* $\mathcal{H}_{I,\mathbb{F}}$,

$$\mathcal{N}^I \otimes \pi_f^K(\mathbb{F}) \simeq \mathcal{H}_{I,\mathbb{F}} \cdot \pi_f^K(\mathbb{F}).$$

Moreover, if the component π_q *is a full unramified principal series, we have* $\mathcal{M} = \mathcal{N}$.

Proof. Note that $\mathcal{H}_{I,\mathbb{F}} \cdot \pi_f^K(\mathbb{F})$ is a multiple of $\pi_f^K(\mathbb{F})$, viewed as a simple module over $\mathcal{H}_{K^q,\mathbb{F}}$. By Theorem 3.12 in [Vig], since ℓ is banal for q, there is a representation $\mathcal{N}$ of $G(\mathbb{Q}_q)$ such that

$$\mathcal{N}^I \simeq \mathrm{Hom}_{\mathcal{H}_{K^q,\mathbb{F}}}(\pi_f^K(\mathbb{F}), \mathcal{H}_{I,\mathbb{F}} \cdot \pi_f^K(\mathbb{F})).$$

Moreover, $\mathcal{N}$ is generated by its Iwahori-invariants. It remains to show that $\mathcal{N}$ is a quotient of $\mathcal{M}$. By the universal property of $\mathcal{M}$, there is a canonical surjective map of $\mathcal{H}_{I,\mathbb{F}}$-modules,

$$\mathcal{M}^I \otimes \pi_f^K(\mathbb{F}) \twoheadrightarrow \mathcal{H}_{I,\mathbb{F}} \cdot \pi_f^K(\mathbb{F}).$$

This in turn defines a surjective map $\mathcal{M}^I \twoheadrightarrow \mathcal{N}^I$; indeed it is enough to show surjectivity after tensoring with $\pi_f^K(\mathbb{F})$. By the result of Vigneras mentioned above, this map comes from a map of representations $\mathcal{M} \to \mathcal{N}$, which must be surjective since $\mathcal{N}^I$ generates $\mathcal{N}$.

Now let us assume that π_q is generic, and show that the above canonical map is injective. We do this by comparing dimensions. Obviously the source has dimension $8 \dim \pi_f^K$. Furthermore,

$$\mathcal{H}_{I,\mathbb{F}} \cdot \pi_f^K(\mathbb{F}) \simeq \mathbb{F} \otimes_{\mathcal{O}} \pi_f^I(\mathcal{O}).$$

Since π_q is generic, $\dim \pi_q^{I_q} = 8$, and therefore $\mathcal{H}_{I,\mathbb{F}} \cdot \pi_f^K(\mathbb{F})$ also has dimension $8 \dim \pi_f^K$. □

Lemma 6. $\mathcal{N}$ *is the trivial representation only if* π *is abelian modulo* λ.

Proof. Suppose $\mathcal{N} = \mathbf{1}$. Then $\mathcal{A}_{I,\mathbb{F}}(\bar{\eta})$ contains an eigenform f such that the Iwahori-Hecke algebra acts on $\mathbb{F} \cdot f$ by the degree character. Therefore, f is $G(\mathbb{Q}_q)$-invariant (on both sides). Note that

$$1 \to G^{\mathrm{der}}(\mathbb{A}_f) \to G(\mathbb{A}_f) \xrightarrow{c} \mathbb{A}_f^* \to 1$$

is exact since $H^1(\mathbb{Q}_p, G^{\mathrm{der}}) = 1$, as G^{der} is simple and simply connected. We claim that f factors through c. This follows easily from strong approximation, using that $G^{\mathrm{der}}(\mathbb{Q}_q)$ is non-compact. Thus $\bar{\eta}$ occurs in the space of $\mathbb{F}$-valued functions on the finite abelian group $\mathbb{A}_f^*/\mathbb{Q}_+^* c(I)$. By the

Deligne-Serre lifting lemma, Lemma 6.11 in [DS], there is a character $\eta' \equiv \eta \pmod{\lambda}$ occurring in the space $\mathbb{C}$-valued functions. The group characters form a basis for this space, so there is an automorphic character χ of $G(\mathbb{A}_f)$ such that $\eta' = \eta_\chi$. Hence π is abelian modulo λ. $\square$

In the following, we use the terminology of [Sch]. See Appendix A and B for the notation.

Lemma 7. *Assume π is non-abelian modulo λ. Then π_q must be of type I, IIb or IIIb. Then*

$$\mathcal{N}^\vee \simeq C^\infty(P\backslash G), \quad \text{where} \quad P = \begin{cases} B, & \text{if } \pi_q \text{ is of type I}, \\ P_\alpha, & \text{if } \pi_q \text{ is of type IIb}, \\ P_\beta, & \text{if } \pi_q \text{ is of type IIIb}. \end{cases}$$

Proof. The trivial representation is the unique irreducible quotient of $\mathcal{M}$, so it is also a quotient of $\mathcal{N}$. However, $\mathcal{N} \neq \mathbf{1}$ by the previous lemma. Write down a composition series for $\mathcal{N}^\vee$ of the form

$$0 \overset{\mathbf{1}}{\subset} \mathbf{1} \overset{V}{\subset} X \subset \cdots \subset \mathcal{N}^\vee,$$

with irreducible subquotients. Here X is a non-trivial extension of V by $\mathbf{1}$. Otherwise $V^\vee$ is a quotient of $\mathcal{M}$ and hence trivial. However, all constituents of $\mathcal{M}$ occur with multiplicity one. Thus

$$\mathrm{Ext}^1(\mathbf{1}, V) \neq 0$$

by selfduality. According to [Clo], the arguments in Casselman's paper [Cas] remain valid as long as ℓ is banal for q. Therefore, $V \simeq V_P$ for a maximal parabolic subgroup P. Moreover, there is an isomorphism

$$X \simeq C^\infty(P\backslash G).$$

Suppose $P = P_\alpha$. Then $C^\infty(P_\alpha\backslash G)^\vee$ is a quotient of $\mathcal{N}$. In turn, there is a surjective map

$$\mathcal{N}^J \otimes \pi_f^K(\mathbb{F}) \simeq \mathcal{H}_{J,\mathbb{F}} \cdot \pi_f^K(\mathbb{F}) \twoheadrightarrow C^\infty(P_\alpha\backslash G/J)^\vee \otimes \pi_f^K(\mathbb{F}),$$

for any J. If we take $J = J_\beta$ we deduce that $\dim \pi_q^{J_\beta}$ is at least 3. Since π_q is also unramified, it follows from Appendix B that it must be of type I or IIb. In the latter case note that $\dim \mathcal{N}^I$ and $\#P_\alpha\backslash G/I$ both equal 4. Similarly, if $P = P_\beta$ we conclude that π_q must be of type I or IIIb. $\square$

2.5 Proof of Theorem A

Let P be a maximal parabolic subgroup such that $\mathcal{N}^\vee$ contains $C^\infty(P\backslash G)$. Thus $P = P_\alpha$ if π_q is of type IIb, and $P = P_\beta$ if π_q is of type IIIb. When π_q is generic, P can be arbitrary.

Lemma 8. *The modules $V_P^I \otimes \pi_f^K(\mathbb{F})$ and $\mathbf{1}^I \otimes \pi_f^K(\mathbb{F})$ occur with multiplicity one in $\mathcal{H}_{I,\mathbb{F}} \cdot \mathcal{A}_{K,\mathbb{F}}(\bar{\eta})$.*

Proof. By the universality of $\mathcal{M}$, there is a canonical surjective map of $\mathcal{H}_{I,\mathbb{F}}$-modules,

$$\mathcal{M}^I \otimes \mathcal{A}_{K,\mathbb{F}}(\bar{\eta}) \twoheadrightarrow \mathcal{H}_{I,\mathbb{F}} \cdot \mathcal{A}_{K,\mathbb{F}}(\bar{\eta}).$$

Now recall that $\mathcal{M}$ satisfies multiplicity one, and $\pi_f^K(\mathbb{F})$ occurs only once in $\mathcal{A}_{K,\mathbb{F}}(\bar{\eta})$. □

Lemma 9. *The module $V_P^I \otimes \pi_f^K(\mathbb{F})$ occurs in the quotient $\mathcal{A}_{I,\mathbb{F}}(\bar{\eta})/\mathcal{H}_{I,\mathbb{F}} \cdot \mathcal{A}_{K,\mathbb{F}}(\bar{\eta})$.*

Proof. First we show that $V_P^I \otimes \pi_f^K(\mathbb{F})$ or $\mathbf{1}^I \otimes \pi_f^K(\mathbb{F})$ occurs in the quotient in the lemma. Then we rule out the latter. Otherwise, both modules occur with multiplicity one in $\mathcal{A}_{I,\mathbb{F}}(\bar{\eta})$ by the previous lemma. Now, $\mathcal{A}_{I,\mathbb{F}}(\bar{\eta})$ has a composition series where the constituent $V_P^I \otimes \pi_f^K(\mathbb{F})$ is the left neighbor of $\mathbf{1}^I \otimes \pi_f^K(\mathbb{F})$. Now we recall that $\mathcal{A}_{I,\mathbb{F}}(\bar{\eta})$ is selfdual by Lemma 4. Therefore,

$$C^\infty(P\backslash G/I) \otimes \pi_f^K(\mathbb{F})$$

is also a subquotient. In particular it has a composition series where the constituents form a subseries of the above composition series. By multiplicity one, we must have an exact sequence

$$0 \to V_P^I \otimes \pi_f^K(\mathbb{F}) \to C^\infty(P\backslash G/I) \otimes \pi_f^K(\mathbb{F}) \to \mathbf{1}^I \otimes \pi_f^K(\mathbb{F}) \to 0.$$

However, this is impossible since $\mathbf{1}$ is not a quotient of $C^\infty(P\backslash G)$. Suppose $\mathbf{1}^I \otimes \pi_f^K(\mathbb{F})$ occurs in the quotient. Then there exists an automorphic representation $\Pi \equiv \pi \pmod{\lambda}$, with $\Pi_f^K = 0$, such that $\mathbf{1}^I \otimes \pi_f^K(\mathbb{F})$ is a summand of $\Pi_f^I(\mathbb{F})$. Applying the idempotent e_K we reach a contradiction. □

We can now finish the proof of Theorem A as follows: Suppose $P = P_\alpha$. Then there exists an automorphic representation $\tilde{\pi} \equiv \pi \pmod{\lambda}$, with $\tilde{\pi}_f^K = 0$, such that $V_\alpha^I \otimes \pi_f^K(\mathbb{F})$ is a summand of $\tilde{\pi}_f^I(\mathbb{F})$. Applying the idempotent e_{J_β} we see from Appendix B that $\dim \tilde{\pi}_q^{J_\beta}$ is at least 2. Since $\tilde{\pi}_q$ is also ramified, we conclude (again using Appendix B) that it must be of type IIIa. The type IVb is immediately ruled out as it is not unitary. Analogously, if $P = P_\beta$ we deduce that $\tilde{\pi}_q$ is of type IIa.
□

Chapter 3

Matching Orbital Integrals

3.1 Twisted Orbital Integrals

Momentarily, we let G denote the non-split inner form of G' over $\mathbb{Q}_r$. It splits over the unramified quadratic extension E. Let θ be the generator of $\mathrm{Gal}(E/\mathbb{Q}_r)$, and fix an inner twisting ψ defined over E. We define a norm mapping $N : G(E) \to G(E)$ by setting $N\delta = \delta\theta(\delta)$. For $\delta \in G(E)$, we then define $\mathcal{N}\delta$ by intersecting the stable conjugacy class of $N\delta$ with $G(\mathbb{Q}_r)$. It may happen that $\mathcal{N}\delta$ is empty since G is not quasi-split. Otherwise, the stable twisted conjugacy class of δ is defined to be the fiber of $\mathcal{N}$ through δ. It is a finite union of twisted conjugacy classes.

We consider the $\mathbb{Q}_r$-group I obtained from G by restriction of scalars from E. Then θ defines an automorphism of I over $\mathbb{Q}_r$, again denoted by θ. Now let $\delta \in G(E)$ be an element such that $N\delta$ is semisimple. The extended twisted centralizer of δ is the $\mathbb{Q}_r$-group $I_{\delta\theta}$ with rational points

$$I_{\delta\theta}(\mathbb{Q}_r) = \{x \in G(E) : x^{-1}\delta\theta(x) \in Z(\mathbb{Q}_r)\delta\}.$$

After choosing measures on $G(E)$ and $I_{\delta\theta}(\mathbb{Q}_r)$, we consider the twisted orbital integral

$$O_{\delta\theta}(f_E) = \int_{I_{\delta\theta}(\mathbb{Q}_r)\backslash G(E)} f_E(x^{-1}\delta\theta(x))dx$$

of a function $f_E \in C_c^\infty(G^{\mathrm{ad}}(E))$. Now let $\{\tilde{\delta}\}$ be a set of representatives for the twisted conjugacy classes within the stable twisted conjugacy class of δ mod $Z(\mathbb{Q}_r)$. Then $I_{\tilde{\delta}\theta}$ is an inner form of $I_{\delta\theta}$ and we transform the measure as usual. Then define the stable twisted orbital integral of f_E to be

$$SO_{\delta\theta}(f_E) = \sum\nolimits_{\tilde{\delta}} e(I_{\tilde{\delta}\theta})O_{\tilde{\delta}\theta}(f_E).$$

To be precise, we put $SO_{\delta\theta}(f_E)=0$ if $\mathcal{N}\delta$ is empty. Now consider an $f\in C_c^\infty(G^{\mathrm{ad}}(\mathbb{Q}_r))$. We say that the functions f and f_E have matching orbital integrals if for all semisimple $\gamma\in G(\mathbb{Q}_r)$,

$$SO_\gamma(f)=\begin{cases} SO_{\delta\theta}(f_E) & \text{if } \gamma \text{ belongs to } \mathcal{N}\delta \bmod Z(\mathbb{Q}_r),\\ 0 & \text{if } \gamma \text{ does not come from } G(E).\end{cases}$$

We note that G_γ and $I_{\delta\theta}$ are inner forms if $\gamma\in\mathcal{N}\delta$, and we use compatible Haar measures on both sides. Also, the measures on $G(\mathbb{Q}_r)$ and $G(E)$ are fixed. In practice they will be normalized compatibly. Finally, we note that all these definitions carry over to G'. Indeed things are more well-behaved since G' is quasi-split. For example, $\mathcal{N}$ is defined everywhere.

3.2 Base Change for Idempotents

Let K be a paramodular subgroup of $G(\mathbb{Q}_r)$, and let x be the vertex in the tree fixed by K. Since $E/\mathbb{Q}_r$ is unramified, we may view x as a θ-invariant point in the building of G over E. Then let K_E be the parahoric subgroup of $G(E)$ fixing x. We choose measures on $G(\mathbb{Q}_r)$ and $G(E)$, such that K and K_E have the same measure. The following crucial result is due to Kottwitz [Kot].

Theorem 2. *The idempotents e_K and e_{K_E} have matching orbital integrals.*

Proof. Let L be the completion of the maximal unramified extension of $\mathbb{Q}_r$, and let σ be the Frobenius over $\mathbb{Q}_r$. We view x as a point in the building of G over L, and let K_L be the open bounded subgroup of $G(L)$ fixing x. We claim that K_L satisfies the three conditions on page 240 in [Kot]. Clearly K_L is fixed by σ. Secondly, to see that $k\mapsto k^{-1}\sigma(k)$ defines a surjective map from K_L to itself, we argue as in [Kot]. Namely, let $\mathcal{G}$ be the smooth affine group scheme over $\mathbb{Z}_r$ attached to x in Bruhat-Tits theory. It has generic fiber G, and $\mathcal{O}_L$-points K_L. Since G^{der} is simply connected, the special fiber $\bar{\mathcal{G}}$ is connected and we can refer to Proposition 3 in Greenberg [Gre]. Now the result follows by paraphrasing the arguments in [Kot] with our definition of orbital integrals. □

Similarly, let K' be a paramodular subgroup of $G'(\mathbb{Q}_r)$ fixing the vertex x' in the building. View x' as a point in the building of G' over E, and let K'_E be the parahoric subgroup of $G'(E)$ fixing x'.

Theorem 3. *The functions $e_{\eta K'}$ and $e_{\eta K'_E}$ have matching orbital integrals.*

Proof. As before, the idempotents $e_{K'}$ and $e_{K'_E}$ have matching orbital integrals. Now

$$e_{\langle\eta,K'\rangle}=e_{K'}+e_{\eta K'},$$

and similarly over E, so it remains to show that $e_{\langle \eta, K' \rangle}$ and $e_{\langle \eta, K'_E \rangle}$ match. Again, this follows from the arguments in [Kot]. However, this is not as straightforward as above since the group $\langle \eta, K'_L \rangle$ does not satisfy the conditions on page 240. Indeed the map $k \mapsto k^{-1}\sigma(k)$ only maps onto the subgroup K'_L. This is sufficient however. Indeed, using the notation of [Kot], it is still true that X mod center is identified with the set of fixed points of the Frobenius σ on X_L mod center. □

3.3 The Comparison Over E

It is well-known that G' has a unique inner form over $\mathbb{Q}_r$. Thus, by the inflation-restriction sequence,

$$H^1(E/\mathbb{Q}_r, G'^{\mathrm{ad}}(E)) \simeq H^1(\mathbb{Q}_r, G'^{\mathrm{ad}}) \simeq \{\pm 1\}.$$

The non-trivial cohomology class is represented by the cocycle $\theta \mapsto \eta$. We may therefore choose our twisting ψ such that $\theta\psi \circ \psi^{-1}$ is conjugation by η. Then the following integrals match.

Lemma 10. *$O_{\delta\theta}(e_{K_E}) = O_{\delta'\theta}(e_{\eta K'_E})$ where $\delta' = \psi(\delta)\eta^{-1}$.*

Proof. Obviously ψ restricts to an isomorphism between $I_{\delta\theta}$ and $I_{\delta'\theta}$. Moreover, we clearly have

$$O_{\delta\theta}(e_{K_E}) = O_{\delta'\theta}(e_{\psi(K_E)\eta^{-1}}).$$

The inverse image of K'_E under ψ is θ-invariant. Hence it stabilizes a conjugate of x, so that

$$\psi(K_E) = \xi K'_E \xi^{-1}$$

for some $\xi \in G'(E)$. It follows that $\theta(\xi)^{-1}\eta\xi$ normalizes K'_E, and then $\psi(K_E)\eta^{-1}$ is a θ-conjugate of $\eta K'_E$ mod center. Then the characteristic functions have the same twisted orbital integrals. □

Lemma 11. *$SO_{\delta\theta}(e_{K_E}) = SO_{\delta'\theta}(e_{\eta K'_E})$ where $\delta' = \psi(\delta)\eta^{-1}$.*

Proof. First we deal with the case where $\mathcal{N}\delta$ is non-empty. Let $\{\tilde{\delta}\}$ be a set of representatives for the twisted conjugacy classes within the stable twisted conjugacy class of δ mod $Z(\mathbb{Q}_r)$. It is straightforward to check that $\{\tilde{\delta}'\}$ is then an analogous set of representatives for δ'. Then the result follows from the previous lemma. If $\mathcal{N}\delta$ is empty, it remains to show that

$$SO_{\delta'\theta}(e_{\eta K'_E}) = 0.$$

Otherwise $O_{\tilde{\delta}'\theta}(e_{\eta K'_E})$ is non-zero for some $\tilde{\delta}'$. However, it equals $O_{\tilde{\delta}\theta}(e_{K_E})$ by the previous lemma. By the theorem on page 243 in [Kot], there is a corresponding $\tilde{\gamma} \in \mathcal{N}\tilde{\delta}$. Hence $\mathcal{N}\delta$ is non-empty. □

3.4 Proof of Theorem B

To prove Theorem B, let $\gamma' \in G'(\mathbb{Q}_r)$ be an arbitrary semisimple element. First we assume that γ' does not come from $G(\mathbb{Q}_r)$. Then, we must show that $SO_{\gamma'}(e_{\eta K'})$ vanishes. We may clearly assume that γ' belongs to $\mathcal{N}\delta'$ for some $\delta' \in G'(E)$. Write $\delta' = \psi(\delta)\eta^{-1}$. Then, from what we have shown,

$$SO_{\gamma'}(e_{\eta K'}) = SO_{\delta'\theta}(e_{\eta K'_E}) = SO_{\delta\theta}(e_{K_E}).$$

As a result, it suffices to show that $\mathcal{N}\delta$ is empty. Otherwise, there exists a $\gamma \in G(\mathbb{Q}_r)$ that is stably conjugate to $N\delta$ mod center. However, $\psi(N\delta) = rN\delta'$ so $\psi(\gamma)$ is then stably conjugate to γ' modulo the center. This contradicts our assumption that γ' does not come from $G(\mathbb{Q}_r)$. Next we assume that γ' is stably conjugate to $\psi(\gamma)$ for some $\gamma \in G(\mathbb{Q}_r)$. We must show that

$$SO_{\gamma'}(e_{\eta K'}) = SO_{\gamma}(e_K).$$

It is easy to check that $\gamma \in \mathcal{N}\delta$ mod center if and only if $\gamma' \in \mathcal{N}\delta'$ mod center. If this does not hold, both sides are zero. If it does hold, the first string of equalities can be extended by $SO_{\gamma}(e_K)$. □

Chapter 4

Functoriality

4.1 Endoscopy

4.1.1 The Endoscopic Group H

Up to equivalence, the inner class of G over $\mathbb{Q}$ admits a unique non-trivial elliptic endoscopic triple.

$$H = (\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1)$$

is the underlying group, where GL(1) is centrally embedded by $x \mapsto (x, x^{-1})$. Its dual group is

$$\hat{H} = \{(x, x') \in \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}) : \det x = \det x'\}.$$

There is a natural embedding into $\hat{G}$, identifying $\hat{H}$ with the centralizer of $\mathrm{diag}(1, -1, -1, 1)$.

4.1.2 Transfer and the Fundamental Lemma

Let p be a prime. A semisimple element $\delta \in H(\mathbb{Q}_p)$ is said to be (G, H)-regular if $\alpha(\delta) \neq 1$ for every root α of G that does not come from H. We have the following fundamental result in our case:

Theorem 4. *For every $f \in C_c^\infty(G^{ad}(\mathbb{Q}_p))$ there exists a function $f^H \in C_c^\infty(H^{ad}(\mathbb{Q}_p))$ such that*

$$SO_\delta(f^H) = \sum\nolimits_\gamma \Delta_{G,H}(\delta, \gamma) e(G_\gamma) O_\gamma(f)$$

for all (G, H)-regular semisimple $\delta \in H(\mathbb{Q}_p)$. Here the sum runs over a set of representatives for the conjugacy classes in $G(\mathbb{Q}_p)$ belonging to the stable conjugacy class associated to δ. We use compatible measures on both sides. The $\Delta_{G,H}(\delta, \gamma)$ are the Langlands-Shelstad transfer factors.

Proof. By a descent argument due to Langlands and Shelstad, see page 495 of [LS], it suffices to prove the theorem for G and its centralizers near the identity. Here we have Shalika germ expansions of the orbital integrals, and Hales computed and matched these germs in [H1]. $\square$

We remark that, since H has no endoscopy itself, SO_δ equals O_δ up to a sign. We also note that, by a more recent result of Waldspurger, see page 157 of [Wa], the previous theorem in fact follows from the following supplementary result known as the standard fundamental lemma:

Theorem 5. *Let $p \neq r$, and let K and K_H be hyperspecial subgroups of $G(\mathbb{Q}_p)$ and $H(\mathbb{Q}_p)$, respectively. Then, if f equals the idempotent e_K, we may take f^H above to be the idempotent e_{K_H}.*

Proof. This is due to Hales [H2]. $\square$

By [Wa], one can also transfer functions f on $G(\mathbb{Q}_p)$ to functions $f^{G'}$ on $G'(\mathbb{Q}_p)$ with matching orbital integrals. The archimedean case of Theorem 4 was proved by Shelstad in [S1] and [S2]. Finally we mention that, of course, all we have said is also true for G'.

4.1.3 Local Character Identities

Let ρ be an irreducible admissible representation of $H(\mathbb{Q}_p)$. It factors as $\rho_1 \otimes \rho_2$, where the ρ_i are representations of GL(2) with the same central character. Since H has no endoscopy, the character $\mathrm{tr}\rho$ is a stable distribution. By results of Arthur [A1] and Shelstad [S2], there is an expansion

$$\mathrm{tr}\rho(f^H) = \sum\nolimits_\pi \Delta_{G,H}(\rho,\pi)\mathrm{tr}\pi(f)$$

for any $f \in C_c^\infty(G(\mathbb{Q}_p))$. Here π runs over irreducible representations of $G(\mathbb{Q}_p)$, and the $\Delta_{G,H}(\rho,\pi)$ are spectral analogues of the Langlands-Shelstad transfer factors. There is a similar expansion of $\mathrm{tr}\rho$ in terms of representations of $G'(\mathbb{Q}_p)$. Using θ-correspondence, Weissauer has made this expansion explicit in [W1] and [W2]. We recall his results below. There are precisely two isomorphism classes of quaternary quadratic spaces X with discriminant one. Namely, the split space X^s and the anisotropic space X^a. Now, the key is that we have the following two identifications:

$$\mathrm{GSO}(X^s) \simeq H, \qquad \mathrm{GSO}(X^a) \simeq \breve{H} = (D^* \times D^*)/\mathrm{GL}(1).$$

Here D is the division quaternion algebra over $\mathbb{Q}_p$, and $\mathrm{GSO}(X)$ denotes the identity component of the orthogonal similitude group $\mathrm{GO}(X)$. Note that, by the Jacquet-Langlands correspondence, there is a one-to-one correspondence between irreducible representations $\breve{\rho}$ of $\breve{H}(\mathbb{Q}_p)$ and discrete series representations ρ of $H(\mathbb{Q}_p)$. We intend to transfer to GSp(4) using θ-correspondence.

Next we briefly review a result of Roberts [R1] on θ-correspondence for similitudes in our special case. Assume p is odd, and fix a non-trivial character of $\mathbb{Q}_p$. Correspondingly, we have the Weil representation ω of $\mathrm{Sp}(4) \times \mathrm{O}(X)$ on the Schwartz space of X^2. It extends to a representation of

$$\{(x, x') \in \mathrm{GSp}(4) \times \mathrm{GO}(X) : c(x) = c(x')\},$$

denoted by $\tilde{\omega}$. Let $\mathcal{R}_X(\mathrm{GSp}(4))$ denote the set of irreducible representations π of GSp(4) such that the restriction to Sp(4) is multiplicity-free and has a constituent that is a quotient of ω. Define the set $\mathcal{R}_4(\mathrm{GO}(X))$ similarly. Since $\mathrm{disc} X = 1$, the condition $\mathrm{Hom}(\tilde{\omega}, \pi \otimes \rho) \neq 0$ defines a bijection

$$\mathcal{R}_X(\mathrm{GSp}(4)) \leftrightarrow \mathcal{R}_4(\mathrm{GO}(X))$$

by [R1]. We denote this map and its inverse by $\pi \mapsto \theta(\pi)$ and $\rho \mapsto \theta(\rho)$, respectively.

Now let $\rho = \rho_1 \otimes \rho_2$ be a representation of GSO(X). It is said to be regular if $\rho_1 \neq \rho_2$. In this case, by Mackey theory, the induced representation of ρ to GO(X) is irreducible and we denote it by ρ^+. When $\rho_1 = \rho_2$ we say that ρ is invariant. If so, it has exactly two extensions to GO(X). However, by [R2] there is a unique extension ρ^+ occurring the θ-correspondence with GSp(4).

Theorem 6. *Let ρ be a discrete series representation of $H(\mathbb{Q}_p)$. Then we have the identity*

$$tr\rho(f^H) = tr\theta(\rho^+)(f) - tr\theta(\breve{\rho}^+)(f)$$

for any $f \in C_c^\infty(G(\mathbb{Q}_p))$. Here $\breve{\rho}$ is associated to ρ under the Jacquet-Langlands correspondence.

Proof. This is due to Weissauer. See Proposition 1 in [W2]. □

Weissauer makes the following supplementary remarks: The lift $\theta(\rho^+)$ is always generic, whereas $\theta(\breve{\rho}^+)$ is non-generic. If ρ is regular, both θ-lifts are discrete series representations (indeed $\theta(\breve{\rho}^+)$ is always supercuspidal). On the other hand, if ρ is invariant, the θ-lifts are limits of discrete series. When ρ is not a discrete series, one can still expand $\mathrm{tr}\rho$ using the compatibility properties described on page 4 in [W2]. This is done in great detail on page 93 in [W1].

4.1.4 Weak Endoscopic Lifts

One says that a cuspidal automorphic representation π of $G'(\mathbb{A})$ is endoscopic if there exist two cuspidal automorphic representations ρ_i of GL(2), with central character ω_π, such that

$$L(s,\pi_p,\mathrm{spin}) = L(s,\rho_{1,p})L(s,\rho_{2,p})$$

for almost all p. Then we also say that π is a weak endoscopic lift of $\rho = \rho_1 \otimes \rho_2$. Moreover, let us recall what it means for π to be CAP (cuspidal associated to parabolic): π is said to be CAP with respect to a parabolic P, with Levi component M, if there exists a cuspidal automorphic representation τ of $M(\mathbb{A})$ such that π is weakly equivalent to the constituents of the induced representation of τ to $G'(\mathbb{A})$. The CAP representations for $G'(\mathbb{A})$ are described completely in [PS] and [S].

Theorem 7. *Let π be a weak endoscopic lift of ρ above, which is not CAP. Then for all p,*

$$\Delta_{G',H}(\rho_p,\pi_p) \neq 0.$$

Proof. This is part 3 of the main theorem of [W2] on page 16. The main ingredient is a result of Kudla, Rallis, and Soudry [KRS], showing that any constituent of π restricted to Sp(4) is a global θ-lift from some $\mathrm{O}(X)$ since the degree 5 L-function of π has a simple pole at $s = 1$. $\square$

In addition, Weissauer shows that ρ has a weak endoscopic lift as above if and only if $\rho_1 \neq \rho_2$.

4.1.5 Representations of Type IIIa and θ-Correspondence

As we have shown in Theorem A, by raising the level of a suitable automorphic representation π of $G(\mathbb{A})$, we obtain a $\tilde{\pi} \equiv \pi \pmod{\lambda}$ with $\tilde{\pi}_q$ of type IIIa. This means precisely that $\tilde{\pi}_q$ is of the form

$$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GL}(2)}$$

for unramified characters χ and σ of $\mathbb{Q}_q^*$ such that $\chi \neq \mathbf{1}$ and $\chi \neq |\cdot|^{\pm 2}$. They are both unitary in our case. Throughout, we use the notation of [ST], so the above representation is induced from the Klingen-Levi. In our case it has trivial central character, that is, $\chi\sigma^2 = \mathbf{1}$. We note that $\chi^2 \neq \mathbf{1}$: Indeed $\tilde{\pi}_q$ is congruent (mod λ) to its unramified relative $\chi \rtimes \sigma \mathbf{1}_{\mathrm{GL}(2)}$, which has Satake parameters

$$\{q\alpha^{-1}, q^2\alpha^{-1}, q\alpha, q^2\alpha\}, \quad \alpha = \sigma(q),$$

after twisting by $|c|^{-3/2}$. Since $\tilde{\pi}_q \equiv \pi_q \equiv \mathbf{1} \pmod{\lambda}$, the above parameters are congruent to

$$\{1, q, q^2, q^3\}.$$

If $\alpha^4 = 1$ this can only happen if $q^4 \equiv 1 \pmod{\ell}$, contradicting banality. Therefore $\chi^2 \neq \mathbf{1}$, and by Theorem 5.2 part (iv) in [ST] the restriction of $\tilde{\pi}_q$ to Sp(4) remains irreducible.

Lemma 12. *Let $\chi \rtimes \sigma St_{GL(2)}$ be a unitary representation of $G'^{ad}(\mathbb{Q}_q)$ of type IIIa, where $\chi^2 \neq \mathbf{1}$. Let X be an even-dimensional quadratic space over $\mathbb{Q}_q$ of discriminant 1. The representation does not occur in the θ-correspondence with $GO(X)$ if X is anisotropic, or if $\dim X$ is at most 4.*

Proof. By Lemma 4.2 in [R1] it suffices to show that $\chi \rtimes \mathrm{St}_{\mathrm{SL}(2)}$ does not occur in the θ-correspondence with $\mathrm{O}(X)$. In other words, by Frobenius reciprocity, we need to show that

$$\mathrm{Hom}_{\mathrm{GL}(1)\times\mathrm{SL}(2)}(r(\omega), \chi \otimes \mathrm{St}_{\mathrm{SL}(2)}) = 0.$$

Here $r(\omega)$ is the Jacquet module for the Weil representation ω with respect to the Klingen parabolic in Sp(4). We will utilize Kudla's filtration of $r(\omega)$ as described in Theorem 8.1 in [K1]:

$$0 \to \mathrm{Ind}_P^{\mathrm{O}(X)}(\breve{\omega} \otimes \omega_{1,\breve{X}}) \to r(\omega) \to \chi_X \otimes \omega_{1,X} \to 0.$$

Here, up to a real twist, χ_X is a quadratic character. Of course, $\omega_{1,X}$ denotes the Weil representation for the pair $\mathrm{SL}(2)\times\mathrm{O}(X)$. The submodule of $r(\omega)$ is to be regarded as being trivial if X is anisotropic. Otherwise, P denotes the parabolic subgroup of $\mathrm{O}(X)$ with Levi component $\mathrm{GL}(1) \times \mathrm{O}(\breve{X})$, where $\breve{X}$ is the space in the Witt tower of X with index one less than that of X. Up to a twist, $\breve{\omega}$ is the representation of $\mathrm{GL}(1) \times \mathrm{GL}(1)$ on Schwartz functions on $\mathbb{Q}_q$ given by translation composed with multiplication. Let us first note that the following space vanishes:

$$\mathrm{Hom}_{\mathrm{GL}(1)\times\mathrm{SL}(2)}(\chi_X \otimes \omega_{1,X}, \chi \otimes \mathrm{St}_{\mathrm{SL}(2)}) = 0.$$

Otherwise, $\chi = \chi_X$. However, χ is unitary and non-quadratic. This proves the lemma when X is anisotropic. We may then assume that X is split of dimension 2 or 4. It remains to show that

$$\mathrm{Hom}_{\mathrm{GL}(1)\times\mathrm{SL}(2)}(\mathrm{Ind}_P^{\mathrm{O}(X)}(\breve{\omega} \otimes \omega_{1,\breve{X}}), \chi \otimes \mathrm{St}_{\mathrm{SL}(2)}) = 0.$$

Otherwise, it follows immediately from Lemma 9.4 in [GG] that $\chi \otimes \mathrm{St}_{\mathrm{SL}(2)}$ is also a quotient of the representation $\breve{\omega}\otimes\omega_{1,\breve{X}}$. Consequently, $\mathrm{St}_{\mathrm{SL}(2)}$ occurs in the θ-correspondence with $\mathrm{O}(\breve{X})$. However, it is well-known that $\mathrm{St}_{\mathrm{SL}(2)}$ does not come from split $\mathrm{O}(2)$. See the example on page 86 in [K1]. $\square$

Corollary 1. *Let π be a cuspidal automorphic representation of $G'^{ad}(\mathbb{A})$ having a local component of type IIIa of the form $\chi \rtimes \sigma St_{GL(2)}$, where $\chi^2 \neq \mathbf{1}$. Then π is neither CAP nor endoscopic.*

Proof. Suppose π is CAP with respect to the Siegel parabolic P_β or B. Note that PGSp(4) is the same as split SO(5). Then by [PS] it comes from $\widetilde{\mathrm{SL}}(2)$ via global θ-lifting. Locally, one can compute these θ-lifts and check that they are all non-generic. We will have more to say about this in the next chapter. However, type IIIa representations are generic. Now suppose π is CAP with respect to the Klingen parabolic P_α. By [S] there exist a two-dimensional anisotropic quadratic space X over $\mathbb{Q}$ such that π is a global θ-lift from $\mathrm{GO}(X)$. However, by Lemma 12, type IIIa representations do not occur in the θ-correspondence with any two-dimensional quadratic space. Finally, suppose π is a weak endoscopic lift of ρ. By Theorem 7, the local component $\pi_q = \chi \rtimes \sigma \mathrm{St}_{\mathrm{GL}(2)}$ occurs in the expansion of $\mathrm{tr}\rho_q$. If ρ_q is a discrete series, this is impossible by Theorem 6 and Lemma 12. Otherwise, its character expansion is given by a single representation. See page 94 in [W1]. This representation is irreducibly induced from P_β or B. Thus it cannot be of type IIIa. $\square$

4.2 Stability

4.2.1 Stabilization of the Trace Formula

The trace formula for G' is an equality between two expansions of a very complicated invariant distribution $I^{G'}$ on $G'(\mathbb{A})$. One expansion is in terms of geometric data such as conjugacy classes, Tamagawa numbers, and (weighted) orbital integrals. The other expansion is in terms of spectral data such as automorphic representations, multiplicities, and (weighted) characters. For our purpose, we are only interested in the terms occurring discretely in the trace formula. Their sum defines an invariant distribution denoted by $I^{G'}_{\mathrm{disc}}$. The main contribution comes from the trace on the discrete spectrum, but there are also terms coming from what Arthur refers to as surviving remnants of Eisenstein series. The distribution has an expansion of the following form:

$$I^{G'}_{\mathrm{disc}}(f') = \sum_{\Pi} a^{G'}_{\mathrm{disc}}(\Pi)\mathrm{tr}\Pi(f'),$$

for a smooth function f' on $G'(\mathbb{A})$. Here $a^{G'}_{\mathrm{disc}}(\Pi)$ is a complex number attached to the discrete automorphic representation Π. If Π is cuspidal, but not CAP, the number $a^{G'}_{\mathrm{disc}}(\Pi)$ is simply the multiplicity of Π. The distribution $I^{G'}_{\text{disc}}$ is unstable (recall that a distribution is said to be stable if it is supported on the stable orbital integrals). However, by the work of Arthur announced in [A2],

$$S^{G'}_{\text{disc}}(f') = I^{G'}_{\text{disc}}(f') - \frac{1}{4}I^{H}_{\text{disc}}(f'^{H})$$

does define a stable distribution. Here, if $f' = \otimes f'_p$ is a pure tensor, we may take the matching function to be $f'^H = \otimes f'^H_p$. Now we turn our attention to the trace formula for G. Since G is anisotropic modulo its center, the trace formula takes its simplest form. All terms occur discretely,

$$I^{G}_{\text{disc}}(f) = \sum_{\pi} a^{G}_{\text{disc}}(\pi)\text{tr}\pi(f),$$

for a smooth function f on $G(\mathbb{A})$. Here $a^{G}_{\text{disc}}(\pi)$ is always the multiplicity of π. Again, this distribution is unstable, but it can be rewritten in terms of stable distribution on the endsocopic groups:

$$I^{G}_{\text{disc}}(f) = S^{G'}_{\text{disc}}(f^{G'}) + \frac{1}{4}I^{H}_{\text{disc}}(f^{H}).$$

This was first proved by Kottwitz and Langlands, but it is also a very special case of the aforementioned work of Arthur. If $f = \otimes f_p$ is a tensor product, we may take $f^{G'} = \otimes f^{G'}_p$ as before.

4.2.2 A Semilocal Spectral Identity

As we have already observed, we have global transfer. For example, if f is a function on $G(\mathbb{A})$, there is a function f^H on $H(\mathbb{A})$ with matching orbital integrals. There is also a global character identity:

$$\text{tr}\rho(f^H) = \sum\nolimits_{\pi} \Delta_{G,H}(\rho,\pi)\text{tr}\pi(f)$$

for any $f \in C^{\infty}_{c}(G(\mathbb{A}))$, where ρ is an irreducible admissible representations of $H(\mathbb{A})$. In the sum, π runs over irreducible admissible representations of $G(\mathbb{A})$, and $\Delta_{G,H}(\rho,\pi)$ is the product of the local transfer factors $\Delta_{G,H}(\rho_p,\pi_p)$. If we insert this expansion in the stable trace formula, we see that

$$\sum\nolimits_{\pi}\{a^{G}_{\text{disc}}(\pi) - \frac{1}{4}\sum\nolimits_{\rho} a^{H}_{\text{disc}}(\rho)\Delta_{G,H}(\rho,\pi)\}\text{tr}\pi(f)$$

equals

$$\sum\nolimits_{\Pi}\{a^{G'}_{\text{disc}}(\Pi) - \frac{1}{4}\sum\nolimits_{\rho} a^{H}_{\text{disc}}(\rho)\Delta_{G',H}(\rho,\Pi)\}\text{tr}\Pi(f')$$

for any pair of matching functions f and f'. We want to refine this identity. The point is that G is split over $\mathbb{Q}_p$ for all $p \notin S$. Thus, if we fix an irreducible representation τ^S of the group $G(\mathbb{A}^S)$,

$$\sum\nolimits_{\pi_S}\{a^G_{\mathrm{disc}}(\pi_S \otimes \tau^S) - \frac{1}{4}\sum\nolimits_{\rho} a^H_{\mathrm{disc}}(\rho)\Delta_{G,H}(\rho, \pi_S \otimes \tau^S)\}\mathrm{tr}\pi_S(f_S)$$

equals

$$\sum\nolimits_{\Pi_S}\{a^{G'}_{\mathrm{disc}}(\Pi_S \otimes \tau^S) - \frac{1}{4}\sum\nolimits_{\rho} a^H_{\mathrm{disc}}(\rho)\Delta_{G',H}(\rho, \Pi_S \otimes \tau^S)\}\mathrm{tr}\Pi_S(f'_S)$$

for any pair of matching functions f_S and f'_S, by linear independence of characters for $G(\mathbb{A}^S)$. From now on we assume that τ^S comes from an automorphic representation τ of $G(\mathbb{A})$ such that

$$\Delta_{G,H}(\rho_p, \tau_p) = 0 \text{ for some } p \notin S,$$

for every discrete automorphic representation ρ of $H(\mathbb{A})$. This is true, for example, if τ^S has a local component of type IIIa as above. Under this assumption, the above identity simplifies immensely:

$$\sum\nolimits_{\pi_S} a^G_{\mathrm{disc}}(\pi_S \otimes \tau^S)\mathrm{tr}\pi_S(f_S) = \sum\nolimits_{\Pi_S} a^{G'}_{\mathrm{disc}}(\Pi_S \otimes \tau^S)\mathrm{tr}\Pi_S(f'_S)$$

for any pair of matching functions f_S and f'_S. Let us mention that if the above hypothesis on τ^S does not hold, then there exists a ρ such that $\Delta_{G,H}(\rho_p, \tau_p)$ is non-zero for all $p \notin S$. Then we may construct a weak transfer of τ to $G'(\mathbb{A})$ by looking at the global θ-lift of ρ as in [W2].

4.2.3 Incorporating Shelstad's Results at Infinity

For now, let us fix a pair of matching functions f_r and f'_r at r, and consider the distribution

$$T = \sum\nolimits_{\Pi_S} a^{G'}_{\mathrm{disc}}(\Pi_S \otimes \tau^S)\mathrm{tr}\Pi_r(f'_r)\mathrm{tr}\Pi_\infty$$

on $G'(\mathbb{R})$. From our previous considerations, this is clearly stable. Now recall that, by the Langlands classification, the irreducible admissible representations of $G'(\mathbb{R})$ are partitioned into finite L-packets Π_μ parameterized by admissible homomorphisms $\mu : W_{\mathbb{R}} \to {}^L G$. Then by [S2] T has an expansion

$$T = \sum\nolimits_{\mu} c_\mu \mathrm{tr}\Pi_\mu, \quad \mathrm{tr}\Pi_\mu = \sum\nolimits_{\Pi_\infty \in \Pi_\mu} \mathrm{tr}\Pi_\infty,$$

where μ varies over the tempered parameters; that is, μ such that the projection of $\mu(W_{\mathbb{R}})$ onto the neutral component of ${}^LG$ is bounded. We deduce that only Π_S with Π_∞ tempered occur in the sum defining T. Moreover, for every tempered L-parameter μ, the coefficient c_μ is given by

$$c_\mu = \sum\nolimits_{\Pi_r} a_{\text{disc}}^{G'}(\Pi_\infty \otimes \Pi_r \otimes \tau^S)\text{tr}\Pi_r(f_r')$$

for any choice $\Pi_\infty \in \Pi_\mu$. Now, since $G(\mathbb{R})$ is compact, its L-packets are singletons $\{\pi_\mu\}$. The finite-dimensional irreducible representations π_μ are parameterized by discrete L-parameters μ (that is, μ which does not map into a Levi subgroup). In this case, the L-packet Π_μ for $G'(\mathbb{R})$ consists of two classes of discrete series representations $\{\Pi_\mu^H, \Pi_\mu^W\}$ with the same central and infinitesimal characters as π_μ. The representation Π_μ^H is a holomorphic discrete series, whereas Π_μ^W is generic. We will now invoke the following character identity over $\mathbb{R}$ proved by Shelstad in [S1]:

$$\text{tr}\Pi_\mu(f_\infty') = \begin{cases} \text{tr}\pi_\mu(f_\infty) & \text{if } \mu \text{ is discrete,} \\ 0 & \text{otherwise,} \end{cases}$$

for matching functions f_∞ and f_∞'. Inserting this in the trace formula derived in the last section,

$$c_\mu = \sum\nolimits_{\pi_r} a_{\text{disc}}^{G}(\pi_\mu \otimes \pi_r \otimes \tau^S)\text{tr}\pi_r(f_r),$$

for any discrete μ. To elaborate on this, we compute $T(f_\infty')$ in two ways, and then use linear independence of characters for $G(\mathbb{R})$. Comparing this with the above, we obtain our key identity:

$$\sum\nolimits_{\pi_r} a_{\text{disc}}^{G}(\pi_\mu \otimes \pi_r \otimes \tau^S)\text{tr}\pi_r(f_r) = \sum\nolimits_{\Pi_r} a_{\text{disc}}^{G'}(\Pi_\infty \otimes \Pi_r \otimes \tau^S)\text{tr}\Pi_r(f_r')$$

valid for any discrete μ, any choice $\Pi_\infty \in \Pi_\mu$, and any pair of matching f_r and f_r' at r.

4.3 Proof of Theorem C

Let τ be an automorphic representation of $G(\mathbb{A})$ having a local component of type IIIa outside S. Suppose $\tau_\infty = \pi_\mu$. Then, by linear independence of characters for $G(\mathbb{Q}_r)$, there exists a function f_r such that the left-hand side of the key identity above is non-zero. Let f_r' be any matching function. Then the right-hand side is non-zero, and there exists a Π_r with $\text{tr}\Pi_r(f_r') \neq 0$ such that $\Pi_\infty \otimes \Pi_r \otimes \tau^S$ is a discrete automorphic representation of $G'(\mathbb{A})$. Call it Π. It has a tempered component (namely the one of type IIIa) so Π must in fact be cuspidal. This is a standard argument using the fact, proved by Langlands [La], that residual representations arise from residues of Eisenstein series for non-unitary parameters. The same argument is used on page 6 in a paper of Labesse and Muller [LM]. As we have shown, since Π has a component of type IIIa, it is neither CAP nor endoscopic. Finally, we note that our argument can be extended to allow central characters.

To conclude, we refine our argument to gain information at the prime r. Let τ be as above, but insist that $\omega_\tau = \mathbf{1}$ and that τ_r is para-spherical. This means $\tau_r^{K_r} \neq 0$ for a paramodular group K_r in $G(\mathbb{Q}_r)$. If we take $f_r = e_{K_r}$ the left-hand side of the key identity is positive. By our Theorem B we may then take $f_r' = e_{\eta K_r'}$. Hence there exists a cuspidal automorphic representation Π of $G'^{\mathrm{ad}}(\mathbb{A})$ with $\Pi_\infty \in \Pi_\mu$ and $\Pi^S = \tau^S$ such that the trace of η on $\Pi_r^{K_r'}$ is positive. In particular, Π_r is para-spherical. We claim that Π_r is also ramified: Otherwise, since Π is not CAP, Π_r is tempered by Theorem I in [W3]. Thus Π_r must be a full unramified principal series. However, then the Atkin-Lehner operator on $\Pi_r^{K_r'}$ is traceless by Table 3 in [Sch]. This is a contradiction. $\square$

Remark. The aforementioned table yields that Π_r must be of type IIa, Vb, Vc, or VIc. We suspect that Π_r is necessarily tempered. If this is true, we deduce that Π_r is of type IIa of the form $\chi\mathrm{St} \rtimes \sigma$ (induced from the Siegel parabolic) where $\chi\sigma$ is the non-trivial unramified quadratic character of $\mathbb{Q}_r^*$.

Chapter 5

Saito-Kurokawa Forms

5.1 Modular Forms and Root Numbers

Let $f \in S_4(\Gamma_0(N))$ be a newform of square-free level N, and consider its L-function given by

$$L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_{p|N} (1 - a_p p^{-s})^{-1} \cdot \prod_{p \nmid N} (1 - a_p p^{-s} + p^{3-2s})^{-1}.$$

Here the a_n are the Hecke eigenvalues. It admits analytic continuation to an entire function, and

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f) = \epsilon_f N^{2-s}\Lambda(4-s,f),$$

where the root number $\epsilon_f \in \{\pm 1\}$ is given by the parity of the order of vanishing of $L(s,f)$ at the point $s = 2$. Now, we wish to work in the context of automorphic representations. By an elementary construction, one associates to f a cuspidal automorphic representation τ of $\mathrm{PGL}(2,\mathbb{A})$. Namely, first one pulls back f to a function on $\mathrm{GL}(2,\mathbb{R})$. Then, by strong approximation, one views it as a function on $\mathrm{GL}(2,\mathbb{A})$ and τ is the representation it generates. It is uniquely determined by the following properties: τ_p is unramified for $p \nmid N$, and its Satake parameters $\{\alpha_p, \alpha_p^{-1}\}$ satisfy

$$a_p = p^{3/2}(\alpha_p + \alpha_p^{-1}).$$

Moreover, up to an appropriate twist, τ_∞ is the (holomorphic) discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with the same central and infinitesimal characters as $\mathrm{Sym}^2(\mathbb{C}^2)$. For p dividing N, the component τ_p is in fact an unramified quadratic twist of $\mathrm{St}_{\mathrm{GL}(2)}$ since N is assumed to be square-free. We note that the Jacquet-Langlands L-function $L(s,\tau)$ is simply $\Lambda(s+3/2,f)$. In addition, $\epsilon_f N^{1/2-s}$ equals the exponential function $\epsilon(s,\tau)$ in its functional equation. Thus $\epsilon(1/2,\tau) = \epsilon_f$.

5.2 The Saito-Kurokawa Lifting

Let f be a newform as above, but now assume that $\epsilon_f = -1$. We then lift τ to PGSp(4).

Proposition 1. *There exists a cuspidal automorphic representation* Π *of* $PGSp(4)$*, with* Π_∞ *being the cohomological holomorphic discrete series representation, such that for all primes* p *we have*

$$\Pi_p \simeq L(\nu^{1/2}\tau_p \rtimes \nu^{-1/2}).$$

Here ν *denotes the normalized absolute value, and* $L(-)$ *is the unique irreducible quotient. In particular,* Π_p *is of type IIb for* $p \nmid N$*. On the other hand,* Π_p *is of type VIc or Vb for* p *dividing* N *according to whether* τ_p *is* $St_{GL(2)}$ *or its non-trivial unramified quadratic twist* $\xi_0 St_{GL(2)}$*.*

Proof. Let $\widetilde{\mathrm{SL}}(2)$ denote the twofold metaplectic covering of SL(2). Throughout, we also fix a non-trivial additive unitary character $\psi = \otimes\psi_p$. Each τ_p is infinite-dimensional, so it determines a local Waldspurger packet A_{τ_p} of irreducible unitary representations of $\widetilde{\mathrm{SL}}(2)$ over $\mathbb{Q}_p$. This packet is a singleton $\{\sigma_{\tau_p}^+\}$ when τ_p is a principal series. Otherwise, when τ_p is a discrete series, we have

$$A_{\tau_p} = \{\sigma_{\tau_p}^+, \sigma_{\tau_p}^-\}.$$

Here $\sigma_{\tau_p}^+$ is ψ_p-generic, whereas $\sigma_{\tau_p}^-$ is not. Recall that PGL(2) is the same as split SO(3), and its inner form PD^* is anisotropic SO(3). Then $\sigma_{\tau_p}^+$ can be described as the θ-lift of τ_p. In the discrete series case, $\sigma_{\tau_p}^-$ is the θ-lift of the Jacquet-Langlands transfer $\check{\tau}_p$. Consider the tensor product:

$$A_\tau = \otimes A_{\tau_p} = \{\sigma = \otimes\sigma_{\tau_p}^{\epsilon_p} \text{ with } \epsilon_p = \pm \text{ and } \epsilon_p = + \text{ for almost all } p\}.$$

This is the global Waldspurger packet determined by τ. It is a finite set of irreducible unitary representations of $\widetilde{\mathrm{SL}}(2)$ over $\mathbb{A}$. They are not all automorphic. By a famous result of Waldspurger,

$$\sigma = \otimes\sigma_{\tau_p}^{\epsilon_p} \text{ is automorphic if and only if } \epsilon(1/2, \tau) = \prod \epsilon_p.$$

For example, in our case $\sigma = \sigma_{\tau_\infty}^- \otimes_{p<\infty} \sigma_{\tau_p}^+$ is automorphic since $\epsilon_f = -1$. Now we think of PGSp(4) as split SO(5), and look at the global θ-series lifting $\theta(\sigma)$. This is non-zero. Indeed we are in the stable range. We claim that $\theta(\sigma)$ is contained in the space of cusp forms. Otherwise, by the theory of towers due to Rallis, σ would have a cuspidal lift to PGL(2). However, a result of Waldspurger then implies that σ is generic. But σ_∞ is non-generic. By a short argument, see for example Proposition

2.12 and its proof in [G], it then follows that we have local-global compatibility. That is,

$$\Pi = \theta(\sigma) = \theta(\sigma^-_{\tau_\infty}) \otimes_{p<\infty} \theta(\sigma^+_{\tau_p}).$$

In particular $\theta(\sigma)$ is irreducible. We should mention that local Howe duality is known in this special situation. The case $p = 2$ can be checked by hand. It remains to compute the local lifts above. Using Proposition 4.1 in [K1] it is not hard to show that $\theta(\sigma^+_{\tau_p})$ is the Langlands quotient given in our proposition. $\theta(\sigma^-_{\tau_\infty})$ is the holomorphic discrete series with minimal K-type $(3, 3)$ by [Li]. $\Box$

It follows immediately that Π is of Saito-Kurokawa type (that is, CAP with respect to the Siegel parabolic). Moreover, it is non-tempered and para-spherical at all finite primes.

5.3 Transferring to an Inner Form

Let us now assume that our additive character $\psi = \otimes\psi_p$ has trivial conductor. Then we have

$$\epsilon(1/2, \mathrm{St}_{\mathrm{GL}(2)}, \psi_p) = -1, \quad \epsilon(1/2, \xi_0\mathrm{St}_{\mathrm{GL}(2)}, \psi_p) = +1.$$

We have $N > 1$, so we pick a prime r such that $\tau_r = \mathrm{St}_{\mathrm{GL}(2)}$. Then let D be the division quaternion algebra over $\mathbb{Q}$ with ramification locus $S = \{\infty, r\}$, and let G be the unitary similitude group of D^2. The reduced norm of D_r maps onto $\mathbb{Q}_r$, so all hermitian forms on D_r^2 are equivalent. Therefore,

$$G(\mathbb{Q}_r) = \{x \in \mathrm{GL}(2, D_r) : x^* \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} x = c(x) \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, c(x) \in \mathbb{Q}_r^*\}.$$

Consider the isotropic subspace $D_r \oplus 0$. Its stabilizer is the minimal parabolic subgroup over $\mathbb{Q}_r$:

$$P = \{\begin{pmatrix} a & \\ & c\bar{a}^{-1} \end{pmatrix} : a \in D_r^* \text{ and } c \in \mathbb{Q}_r^*\} \ltimes \{\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} : b + \bar{b} = 0\}.$$

It has Levi component $D_r^* \times \mathbb{Q}_r^*$ and abelian unipotent radical. The modulus character is

$$\delta_P : \begin{pmatrix} a & \\ & c\bar{a}^{-1} \end{pmatrix} \mapsto |N_{D_r/\mathbb{Q}_r}(a)|^3 \cdot |c|^{-3},$$

as shown by a standard calculation. We now transfer Π to G using θ-correspondence:

Proposition 2. *There exists an automorphic representation π of $G(\mathbb{A})$, with ω_π and π_∞ being trivial, agreeing with Π outside of S, such that the local component at the ramified prime r is*

$$\pi_r \simeq \nu^{1/2}\mathbf{1}_{D^*} \rtimes \nu^{-1/2}.$$

Proof. We use the notation from the proof of the previous proposition. Since τ_r is a discrete series representation, the global Waldspurger packet A_τ contains another automorphic member. Namely,

$$\breve{\sigma} = \sigma^+_{\tau_\infty} \otimes \sigma^-_{\tau_r} \otimes_{p \neq r} \sigma^+_{\tau_p}.$$

Now we realize the adjoint group G^{ad} as a certain anisotropic SO(5), and look at the global θ-lift to this group $\theta(\breve{\sigma})$. We are no longer in the stable range, so to make sure this is non-vanishing we appeal to the Rallis inner product formula. The case we need is reviewed on page 9 in [G]. Our quadratic space has dimension 5, so all special L-values in the inner product formula are non-zero. Consequently, $\theta(\breve{\sigma}) \neq 0$ if and only if all the local lifts $\theta(\breve{\sigma}_p)$ are non-vanishing. However,

$$\theta(\sigma^+_{\tau_\infty}) = \mathbf{1}, \quad \theta(\sigma^-_{\tau_r}) = L(\nu^{1/2}\mathbf{1}_{D^*} \rtimes \nu^{-1/2}).$$

The first identity is a consequence of Theorem 5.1 in [K1], and the second is easily derived from Proposition 4.1 in [K1]. As before, we have local-global compatibility, and we take $\pi = \theta(\breve{\sigma})$. It remains to show that the unramified principal series $\nu^{1/2}\mathbf{1}_{D^*} \rtimes \nu^{-1/2}$ is irreducible. However, this is an easy exercise using the expression for δ_P and results of Kato reviewed on page 144 in [Car].□

We note in passing that the unramified principal series $\nu^{1/2}\xi_0\mathbf{1}_{D^*} \rtimes \nu^{-1/2}$ is reducible. Therefore it is crucial that we pick r such that τ_r is the actual Steinberg representation $\mathrm{St}_{\mathrm{GL}(2)}$, and not its twist $\xi_0\mathrm{St}_{\mathrm{GL}(2)}$. The following lemma allows us to apply our Theorem A to raise the level of π.

Lemma 13. *The representation π occurs with multiplicity one in the spectrum of $G(\mathbb{A})$.*

Proof. We first recall that for $\widetilde{\mathrm{SL}}(2,\mathbb{A})$, Waldspurger proved multiplicity one. Essentially this follows from the multiplicity one theorem for $\mathrm{PGL}(2,\mathbb{A})$ using the θ-correspondence. We can therefore identify the abstract representation $\breve{\sigma}$ with a space of cusp forms on the metaplectic group. By a formal argument, for example as on page 8 in [G], the θ-correspondence preserves multiplicity. Thus the representation $\pi = \theta(\breve{\sigma})$ occurs with multiplicity one in the spectrum of $G(\mathbb{A})$. □

Gan proves a more general result in his preprint [G]. Analogous to work of Piatetski-Shapiro and

Sayag in the isotropic case, Gan characterizes (certain) CAP representations of an anisotropic inner form of GSp(4) as θ-lifts from the metaplectic group. As a corollary he deduces that all these CAP representations have multiplicity one. We will use this characterization later below.

Let us end this section with a few words about the Bruhat-Tits theory of $G(\mathbb{Q}_r)$. We denote by $\mathcal{O}_{D_r}$ the maximal compact subring of D_r, and let $\mathfrak{p}_{D_r}$ be its (bilateral) maximal ideal. We choose a uniformizing parameter ϖ_{D_r}. The order $\mathcal{O}_{D_r}$ defines an integral model for $G/\mathbb{Q}_r$ and we let

$$K = G(\mathbb{Z}_r) = \{x \in \mathrm{GL}(2, \mathcal{O}_{D_r}) : x^* \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} x = c(x) \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, c(x) \in \mathbb{Z}_r^*\}.$$

This is the special maximal compact subgroup of $G(\mathbb{Q}_r)$, which becomes the Siegel parahoric over the unramified quadratic extension of $\mathbb{Q}_r$. Inside K, there is the Iwahori subgroup I cut out by the condition of being upper triangular modulo $\mathfrak{p}_{D_r}$. Then $\tilde{K}$ is the subgroup generated by I and

$$\begin{pmatrix} & \varpi_{D_r}^{-1} \\ \varpi_{D_r} & \end{pmatrix}.$$

Here $\tilde{K}$ is the paramodular maximal compact subgroup. Both K and $\tilde{K}$ are special, so they fit in Iwasawa decompositions of G relative to P. Consequently π_r is both K-spherical and $\tilde{K}$-spherical.

5.4 Galois Representations

Let L_f be the number field generated by the Hecke eigenvalues of f. A classical construction due to Deligne [Del] provides a compatible system of continuous irreducible Galois representations

$$\rho_{f,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2, L_{f,\lambda}),$$

indexed by the finite places $\lambda|\ell$ of L_f, such that $\rho_{f,\lambda}$ is unramified at primes $p \nmid N\ell$ and

$$L_p(s,f) = \det(1 - \rho_{f,\lambda}(\mathrm{Frob}_p)p^{-s})^{-1}$$

for such p. Here Frob_p denotes a geometric Frobenius. This result has been generalized to GSp(4) by Laumon [Lau] and Weissauer [W3]. Namely, suppose Π is a cuspidal automorphic representation of GSp(4) with Π_∞ being a cohomological discrete series. Then there exists a number field L_Π and a compatible system of continuous semisimple four-dimensional Galois representations

$$\rho_{\Pi,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(4, L_{\Pi,\lambda}),$$

indexed by the finite places $\lambda|\ell$ of L_Π, such that $\rho_{\Pi,\lambda}$ is unramified at $p \neq \ell$ outside the ramification locus of Π. Moreover, for such primes p there is the following relation with the spinor L-factor:

$$L(s - w/2, \Pi_p, \mathrm{spin}) = \det(1 - \rho_{\Pi,\lambda}(\mathrm{Frob}_p)p^{-s})^{-1}.$$

Here $w = k_1 + k_2 - 3$, where (k_2, k_2) is the weight of Π_∞. We note that, when Π is not CAP, the representation $\rho_{\Pi,\lambda}$ can be shown [W3] to be pure of weight w. This means the eigenvalues of Frob_p have absolute value $p^{w/2}$. When Π is CAP or endsocopic, $\rho_{\Pi,\lambda}$ is reducible and essentially given by the above construction of Deligne. For example, for our Saito-Kurokawa form it is given by:

Lemma 14. *Let Π be the Saito-Kurokawa lift of f as in Proposition 1. Then we have*

$$\rho_{\Pi,\lambda} \simeq \rho_{f,\lambda} \oplus \omega_\ell^{-1} \oplus \omega_\ell^{-2}$$

for all $\lambda|\ell$, where ω_ℓ denotes the ℓ-adic cyclotomic character.

Proof. Suppose $p \nmid N\ell$ and τ_p is induced from the unramified character χ with $\alpha_p = \chi(p)$. Then

$$\Pi_p \simeq L(\nu^{1/2}\chi \times \nu^{1/2}\chi^{-1} \rtimes \nu^{-1/2}) \simeq \chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}.$$

This allows us to calculate the Satake parameters, and then write down its L-factor:

$$L(s, \Pi_p, \mathrm{spin}) = L(s, \tau_p)\zeta_p(s - 1/2)\zeta_p(s + 1/2).$$

In our case $k_1 = k_2 = 3$ so that $w = 3$. Now use that $\omega_\ell(\mathrm{Frob}_p) = p^{-1}$ for $p \neq \ell$. □

On the contrary, when Π is neither CAP nor endoscopic, the Galois representation $\rho_{\Pi,\lambda}$ is expected to be irreducible. For large ℓ there is the following precise result in this direction:

Theorem 8. *Let Π be a cuspidal automorphic representation of $GSp(4)$ with Π_∞ cohomological. Assume $\ell > 2w + 1$. Suppose $\rho_{\Pi,\lambda}$ is reducible and all its two-dimensional constituents are odd. Then Π is CAP or endoscopic.*

Proof. This is Theorem 3.2.1 in [SU]. It relies on work of Ramakrishnan [Ram]. □

5.5 Proof of Theorem D

Let $K = \prod K_p$ where K_p is hyperspecial for $p \nmid N$, and K_p is paramodular for $p|N$. We want to apply Theorem A to the automorphic representation π from Proposition 2. However, we cannot prove that K is a good small subgroup (in the sense that π_f^K determines π_f). Indeed the paramodular groups do not have Iwahori factorizations with respect to any parabolic. There is a way to circumvent this problem. All we need is that the module π_f^K has multiplicity one in $\mathcal{A}_K$. To see this, suppose

$$\pi_f'^K \simeq \pi_f^K$$

for an automorphic representation π' with $\omega_{\pi'}$ and π'_∞ trivial. We wish to show that $\pi' \simeq \pi$. Then our claim follows from Lemma 13. Clearly $\pi'_p \simeq \pi_p$ for $p \nmid N$. Thus π' is weakly equivalent to the CAP representation Π of $G'(\mathbb{A})$. Then, by Theorem 7.1 in [G], we have $\pi' = \theta(\sigma')$ for some automorphic representation σ' in the Waldspurger packet $A_{\tau'}$ for some τ'. Here τ' must be weakly equivalent to τ. Hence $\tau' \simeq \tau$ by strong multiplicity one for GL(2). Now consider a prime $p|N$. Then $\pi'_p = \theta(\sigma^{\pm}_{\tau_p})$. First, look at the case where $p \neq r$. Here $\theta(\sigma^{-}_{\tau_p})$ is supercuspidal or of type VIb, see for example Proposition 5.5 in [G]. Both are para-ramified, so $\pi'_p = \pi_p$. Finally, let $p = r$. Since τ_r is the Steinberg representation, $\theta(\sigma^{+}_{\tau_r}) = 0$ by Proposition 6.5 in [G]. Therefore $\pi'_r = \pi_r$.

Now we apply Theorem A to π. Let $\lambda|\ell$ be a finite place of $\bar{\mathbb{Q}}$, with ℓ not dividing the discriminant of $H_{K,\mathbb{Z}}$, such that $\bar{\rho}_{f,\lambda}$ is irreducible. Then π is non-abelian modulo λ (otherwise Π would be congruent to an automorphic character, and its Galois representation would be a sum of characters modulo λ, contradicting Lemma 14). Now suppose $q \nmid N\ell$ is a prime number satisfying

- $q^i \neq 1 \pmod{\ell}$ for $i = 1, \dots, 4$,

- $\bar{\rho}_{f,\lambda}(\mathrm{Frob}_q)$ has a fixed vector.

The Satake parameters of π_q are $\{\alpha_q, q^{-1/2}, q^{1/2}, \alpha_q^{-1}\}$. Since $\bar{\rho}_{f,\lambda}(\mathrm{Frob}_q)$ has eigenvalues $\{1, q^3\}$, the level-raising condition in Theorem A is satisfied. As a result, we find an automorphic representation $\tilde{\pi} \equiv \pi \pmod{\lambda}$ of $G(\mathbb{A})$, with $\omega_{\tilde{\pi}}$ and $\tilde{\pi}_\infty$ trivial, such that $\tilde{\pi}_f^{K^q} \neq 0$ and $\tilde{\pi}_q$ is of type IIIa.

Now we apply Theorem C to $\tilde{\pi}$. As we have seen earlier, $\tilde{\pi}_q$ must have the form $\chi \rtimes \sigma \mathrm{St}_{\mathrm{GL}(2)}$ with $\chi^2 \neq \mathbf{1}$. Pick a cohomological discrete series representation $\Pi_{\mathbf{1}}$ of $G'(\mathbb{R})$, holomorphic or generic of weight $(3,3)$. Then we find a cuspidal automorphic representation $\tilde{\Pi}$ of $G'^{\mathrm{ad}}(\mathbb{A})$, with $\tilde{\Pi}_\infty = \Pi_{\mathbf{1}}$, such that $\tilde{\Pi}_p = \tilde{\pi}_p$ for $p \neq r$. Moreover, $\tilde{\Pi}_r$ is para-spherical since $\tilde{\pi}_r$ is. Thus $\tilde{\Pi}_p$ is para-spherical for all $p|N$, unramified and tempered [W3] for $p \nmid Nq$, and of type IIIa for $p = q$.

Obviously $\tilde{\Pi}_p \equiv \Pi_p \pmod{\lambda}$ for almost all p. Therefore the Galois representations $\rho_{\tilde{\Pi},\lambda}$ and $\rho_{\Pi,\lambda}$ have the same semisimplifications modulo λ. In other words, by Lemma 14, we have an isomorphism

$$\rho_{\tilde{\Pi},\lambda} \simeq \bar{\rho}_{f,\lambda} \oplus \bar{\omega}_\ell^{-1} \oplus \bar{\omega}_\ell^{-2}$$

up to semisimplification. It remains to show that $\rho_{\tilde{\Pi},\lambda}$ is irreducible. Suppose it is reducible. Then it is a sum $\varrho \oplus \varrho'$ of a pair of two-dimensional representations. Interchanging the two, we may assume

$$\bar{\varrho} \simeq \bar{\rho}_{f,\lambda}, \quad \bar{\varrho}' \simeq \bar{\omega}_\ell^{-1} \oplus \bar{\omega}_\ell^{-2}.$$

Then clearly ϱ and ϱ' are both odd. Theorem 8 applies for $\ell > 7$. Hence $\tilde{\Pi}$ is CAP or endoscopic, contradicting Theorem C. This proves irreducibility of $\rho_{\tilde{\Pi},\lambda}$, and finishes the proof of Theorem D. □

5.6 Existence of Good Primes

To apply Theorem D, we need to know the existence of primes q where we can raise the level. Assume $\ell > 13$. Then we choose $g \in \mathbb{Z}$ prime to ℓ, which is a generator for $\mathbb{F}_\ell^*$ modulo ℓ. Thus

$$g^i \neq 1 \pmod{\ell}$$

for $i = 1, \ldots, 12$. Now assume f is not CM. Then in a suitable basis the image of $\rho_{f,\lambda}$ contains

$$\{x \in \mathrm{GL}(2, \mathbb{Z}_\ell) : \det x \in (\mathbb{Z}_\ell^*)^3\}$$

by Theorem 3.1 in Ribet's article [Rib]. In particular, the diagonal matrix with entries $\{1, g^3\}$ lies in the image of $\bar{\rho}_{f,\lambda}$. Then, by the Chebotarev density theorem, there exists a positive density of primes $q \nmid N\ell$ such that $\bar{\rho}_{f,\lambda}(\mathrm{Frob}_q)$ has eigenvalues $\{1, g^3\}$. The determinant is q^3 so we must have $g = \zeta q$ for some $\zeta \in \mathbb{F}_\ell$ with $\zeta^3 = 1$. If $q^i \equiv 1 \pmod{\ell}$ for some $i = 1, \ldots, 4$, then $g^{3i} \equiv 1 \pmod{\ell}$.

Chapter 6

The Bloch-Kato Conjecture

6.1 An Application of Theorem D

We continue to let $f \in S_4(\Gamma_0(N))$ be a newform of square-free level N, not of CM type, having root number $\epsilon_f = -1$. This sign condition implies that the L-function of f vanishes at the critical center $s = 2$ (under the classical normalization of the functional equation $s \mapsto 4 - s$). In this situation, a conjecture of Bloch-Kato ([BK], page 376) predicts that an associated Selmer group is positive dimensional. This expectation was proved for ℓ ordinary for f in [SU] (that is, $a_\ell(f)$ is an ℓ-adic unit), and our object in this section is to make progress on the conjecture when ℓ is supersingular.

Let $\lambda|\ell$ be a finite place of $\bar{\mathbb{Q}}$, with ℓ outside a finite set, such that $\bar{\rho}_{f,\lambda}$ is irreducible. We fix a prime $q \nmid N\ell$ such that the following two conditions hold:

- $q^i \not\equiv 1 \pmod{\ell}$ for $i = 1, \ldots, 4$,

- $\bar{\rho}_{f,\lambda}(\mathrm{Frob}_q)$ has a fixed vector.

Here Frob_q is a fixed geometric Frobenius in the Galois group of $\mathbb{Q}$. Then, by Theorem D, there exists a cuspidal automorphic representation Π of $\mathrm{PGSp}(4)$ such that Π_∞ is the cohomological holomorphic discrete series representation, Π_p is unramified and tempered for $p \nmid Nq$,

- $\rho_{\Pi,\lambda}$ is irreducible, but $\bar{\rho}_{\Pi,\lambda} \simeq \bar{\rho}_{f,\lambda} \oplus \bar{\omega}_\ell^{-1} \oplus \bar{\omega}_\ell^{-2}$,

- Π_q is of type IIIa (hence tempered, generic and ramified),

- Π_p is para-spherical for all primes p dividing N.

Recall that $\rho_{\Pi,\lambda}$ is the four-dimensional λ-adic representation associated to the form Π by Weissauer and Laumon, and that $\bar{\rho}_{\Pi,\lambda}$ is its reduction modulo λ.

In the following, we let V denote the space of $\rho_{\Pi,\lambda}$. This is a four-dimensional vector space over the ℓ-adic field L. We let $\mathcal{O}$ be the ring of integers in L. By abuse of notation, λ also denotes the maximal ideal it generates in $\mathcal{O}$. Moreover, $\mathbb{F}$ is the residue field $\mathcal{O}/\lambda$.

6.2 Choosing a Lattice

Under our assumptions, $\bar{\rho}_{\Pi,\lambda}(\mathrm{Frob}_q)$ has eigenvalues $\{1, q, q^2, q^3\}$. By banality they are all distinct, so by Hensel's lemma $\rho_{\Pi,\lambda}(\mathrm{Frob}_q)$ has eigenvalues $\{\alpha, \beta, \gamma, \delta\}$ reducing to $\{1, q, q^2, q^3\}$ modulo λ. We let $v \in V$ be an eigenvector for $\alpha \equiv 1 \pmod{\lambda}$. Then consider the module it generates,

$$\Lambda = \mathcal{O}[\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})] \cdot v.$$

This is a non-zero Galois stable cyclic $\mathcal{O}$-module. By the irreducibility of $\rho_{\Pi,\lambda}$, we must have $\Lambda \otimes L = V$, implying that Λ is a Galois-stable $\mathcal{O}$-lattice in V. We look at its reduction $\Lambda_{\mathbb{F}}$. This is cyclic, generated by the class of v. Hence $\bar{\rho}_{f,\lambda}$ is the unique irreducible quotient of $\Lambda_{\mathbb{F}}$.

6.3 Kummer Theory

It is known from Kummer theory that $H^1_f(\mathbb{Q}, \omega_\ell) = 0$ (as it should be since $\zeta(0)$ is nonzero). Here, as in Proposition 5.1 in [Bel], we observe that the analogous statement modulo ℓ is true.

Lemma 15. $H^1_f(\mathbb{Q}, \bar{\omega}_\ell) = 0$.

Proof. The connecting homomorphism for the Kummer sequence yields a canonical isomorphism

$$H^1(\mathbb{Q}, \bar{\omega}_\ell) \simeq \mathbb{Q}^*/\mathbb{Q}^{*\ell}.$$

Fix $a \in \mathbb{Q}^*$ and let $\delta(a)$ be the corresponding cohomology class. Clearly, $\delta(a)$ is unramified at $p \neq \ell$ if and only if ℓ divides $\mathrm{ord}_p(a)$. By the discussion on page 26 in Rubin's book [Rub] it is also true that $\delta(a)$ restricts to a class in $H^1_f(\mathbb{Q}_\ell, \bar{\omega}_\ell)$ if and only if ℓ divides $\mathrm{ord}_\ell(a)$. Therefore $H^1_f(\mathbb{Q}, \bar{\omega}_\ell) = 0$. $\square$

6.4 Kato's Result

Since $\rho_{f,\lambda}(1)$ has positive weight, the Bloch-Kato conjecture predicts that $H^1_f(\mathbb{Q}, \rho_{f,\lambda}(1)) = 0$. This was proved by Kato in [Ka]. Here we deduce the analogous result modulo ℓ from this.

Lemma 16. *$H^1_f(\mathbb{Q}, \bar{\rho}_{f,\lambda}(1)) = 0$, for almost all ℓ.*

Proof. In this proof let V be the space of $\rho_{f,\lambda}(1)$, and let Λ be a Galois-stable lattice in V. Let W denote the quotient V/Λ. By Lemma 1.5.4 on page 22 in [Rub], there is a natural surjection

$$H^1_f(\mathbb{Q}, \Lambda/\lambda\Lambda) \twoheadrightarrow H^1_f(\mathbb{Q}, W)[\lambda].$$

This is in fact an isomorphism, for almost all ℓ, since $H^0(\mathbb{Q}, W) = 0$ by Proposition 14.11 on page 241 in [Ka]. However, $H^1_f(\mathbb{Q}, W) = 0$ by Theorem 14.2 on page 235 in [Ka]. □

6.5 Existence of Certain Submodules

In this section we show that $\bar{\omega}_\ell^{-2}$ embeds in $\Lambda_{\mathbb{F}}$. Suppose it does not. Then $\bar{\omega}_\ell^{-1}$ is the unique irreducible subrepresentation of $\Lambda_{\mathbb{F}}$. Writing down a composition series, we get non-split extensions

$$0 \to \bar{\omega}_\ell \to X \to \mathbf{1} \to 0 \quad \text{and} \quad 0 \to \bar{\rho}_{f,\lambda}(1) \to Y \to \mathbf{1} \to 0.$$

Up to a Tate twist, X and $Y^\vee$ are subquotients of $\Lambda_{\mathbb{F}}$. By Lemma 15 and 16, to get a contradiction, it suffices to show that one of the corresponding cohomology classes lies in the Selmer group.

Lemma 17. *X and Y are both Fontaine-Laffaille at ℓ.*

Proof. Since Π is neither CAP nor endoscopic, it follows from [W3] that $\rho_{\Pi,\lambda}$ is the representation on the Π_f^K-isotypic component of the etale intersection cohomology (for the middle perversity):

$$IH^3_{\text{et}}(\bar{S}_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \bar{\mathbb{Q}}_\ell).$$

Here K is paramodular at primes dividing N, Klingen at q, and hyperspecial outside Nq. Moreover, $\bar{S}_K$ denotes the Satake compactification of the Siegel threefold S_K. The latter has good reduction at $\ell \nmid Nq$, so $\rho_{\Pi,\lambda}$ is crystalline, with Hodge-Tate weights contained in $\{0, 1, 2, 3\}$. See page 41 in [SU]. Now, X and $Y^\vee$ are both torsion subquotients of $\rho_{\Pi,\lambda}(2)$. If $\ell - 1$ is bigger than the Hodge-Tate weights, that is if $\ell > 5$, then X and $Y^\vee$ are Fontaine-Laffaille by Theorem 3.1.3.3 in [BM]. □

From the theory of Fontaine-Laffaille [FL], reviewed by Breuil and Messing in [BM], it follows that the above extensions are reductions of lattices in certain crystalline representations. See Theorem 3.1.3.2 and 3.1.3.3 in [BM]. Consequently, their cohomology classes at ℓ restrict to something in the finite part.

Now consider a prime $p \neq \ell$. Clearly, X and Y are then both unramified at $p \nmid Nq$. We are thus left with the two cases $p|N$ and $p = q$. In the first case we need to show exactness of

$$0 \to \bar{\rho}_{f,\lambda}(1)^{I_p} \to Y^{I_p} \to \mathbf{1} \to 0,$$

where I_p is the inertia group at p, and similarly for X. This requires our minimality assumption that $\bar{\rho}_{f,\lambda}$ is ramified at all primes $p|N$. Moreover, we need to appeal to Conjecture 1.

Lemma 18. *Conjecture 1 implies that X and Y both have good reduction at all $p|N$.*

Proof. Let us first consider X. We need to show it is unramified at $p|N$. Since $\bar{\rho}_{f,\lambda}$ is the unique irreducible quotient of $\Lambda_{\mathbb{F}}$, the quotient of $\Lambda_{\mathbb{F}}(2)$ by X is $\bar{\rho}_{f,\lambda}(2)$. Therefore, we have inequalities

$$3 - \dim X^{I_p} \leq \dim V^{I_p} - \dim X^{I_p} \leq \dim \Lambda_{\mathbb{F}}^{I_p} - \dim X^{I_p} \leq \dim \bar{\rho}_{f,\lambda}^{I_p} \leq 1.$$

The first inequality follows from Conjecture 1, and the last is our minimality assumption. It follows that X is unramified. Next, let us consider Y. Here the dual of the quotient of $\Lambda_{\mathbb{F}}$ by $\bar{\omega}_\ell^{-1}$ equals $Y(2)$. By the same arguments as before, we then have the following string of inequalities:

$$2 \leq \dim V^{I_p} - 1 \leq \dim \Lambda_{\mathbb{F}}^{I_p} - 1 \leq \dim Y^{I_p} \leq \dim \bar{\rho}_{f,\lambda}^{I_p} + 1 \leq 2.$$

We conclude that all these inequalities are in fact equalities, so $\dim Y^{I_p} = 2$. $\square$

To get a contradiction, it now suffices to show that X or Y is unramified at q. For this, we invoke a result of Genestier and Tilouine [GT] on the order of the monodromy operator.

Lemma 19. *X or Y is unramified at q.*

Proof. In this proof, let N be the monodromy operator on V at q. From Appendix B, we see that Π_q has a unique line fixed by the Klingen parahoric since it is of type IIIa. Then part (1) of Theorem 2.2.5 on page 12 in [GT] tells us that $N^2 = 0$. The operator preserves Λ and the composition series of $\Lambda_{\mathbb{F}}$. Suppose X and Y are both ramified. Then, in some basis, N has the form

$$N \sim \begin{pmatrix} 0 & e & a & b \\ & 0 & c & d \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

with e and (c,d) non-zero. However, this cannot happen since $N^2 = 0$. Contradiction. $\square$

This contradicts Lemmas 15 and 16, and therefore $\bar{\omega}_\ell^{-2}$ does embed into $\Lambda_{\mathbb{F}}$.

6.6 Proof of Theorem E

Embed $\bar{\omega}_\ell^{-2}$ as a submodule of $\Lambda_{\mathbb{F}}$, and extend it to a composition series. This gives an extension

$$0 \to \bar{\rho}_{f,\lambda}(2) \to Z \to \mathbf{1} \to 0,$$

which is non-split since $\bar{\rho}_{f,\lambda}$ is the unique irreducible quotient of $\Lambda_{\mathbb{F}}$. Up to a twist, $Z^\vee$ is the quotient of $\Lambda_{\mathbb{F}}$ by $\bar{\omega}_\ell^{-2}$. The exact same arguments as in the previous section then shows that Z has good reduction away from q (assuming Conjecture 1). It remains to deal with the prime q.

Lemma 20. *Z is unramified at q.*

Proof. The extension Z determines a cohomology class in $H^1(\mathbb{Q}_q, \bar{\rho}_{f,\lambda}(2))$. Let c be a cocycle representing this class. Since $\bar{\rho}_{f,\lambda}(2)$ is unramified at q, the cocycle restricts to a homomorpism from the inertia group I_q to the space of $\bar{\rho}_{f,\lambda}(2)$. As $q \neq \ell$ it obviously factors through the tame quotient. Indeed it factors through the homomorphism $t_\ell : I_q \twoheadrightarrow \mathbb{Z}_\ell$. Recall, see [Ta] page 21, that

$$t_\ell(\mathrm{Frob}_q^{-1} \cdot \sigma \cdot \mathrm{Frob}_q) = q \cdot t_\ell(\sigma)$$

for $\sigma \in I_q$. Clearly the left-hand side is independent of the choice of a Frobenius Frob_q in the Galois group of $\mathbb{Q}_q$. We then immediately deduce an analogous relation satisfied by c. Now we invoke the cocycle relation satisfied by c. Using it twice we find that

$$c(\mathrm{Frob}_q^{-1} \cdot \sigma \cdot \mathrm{Frob}_q) = c(\mathrm{Frob}_q^{-1}) + \mathrm{Frob}_q^{-1} \cdot c(\sigma \cdot \mathrm{Frob}_q) = \mathrm{Frob}_q^{-1} \cdot c(\sigma)$$

for $\sigma \in I_q$, since $\bar{\rho}_{f,\lambda}$ is unramified at q, and

$$c(\mathrm{Frob}_q^{-1}) = -\mathrm{Frob}_q^{-1} \cdot c(\mathrm{Frob}_q).$$

The action of Frob_q^{-1} on the vector $c(\sigma)$ is given by the Tate twist $\bar{\rho}_{f,\lambda}(2)$. That is,

$$\mathrm{Frob}_q^{-1} \cdot c(\sigma) = \bar{\rho}_{f,\lambda}(\mathrm{Frob}_q^{-1}) \cdot q^2 c(\sigma).$$

Consequently, we end up with the following identity:

$$\bar{\rho}_{f,\lambda}(\mathrm{Frob}_q) \cdot c(\sigma) = q \cdot c(\sigma).$$

Therefore, if $c(\sigma) \neq 0$ for some $\sigma \in I_q$, we see that $c(\sigma)$ is an eigenvector for $\bar{\rho}_{f,\lambda}(\mathrm{Frob}_q)$ with eigenvalue q. However, the eigenvalues of $\bar{\rho}_{f,\lambda}(\mathrm{Frob}_q)$ are $\{1, q^3\}$ by assumption. □

This finishes the proof. □

Appendix A. Iwahori-Spherical Representations of GSp(4)

The following is essentially Table 1 in the paper [Sch]. We include it here for the convenience of the reader. We are grateful to Ralf Schmidt for his permission to do so. Throughout we use the notation of [ST]. Let B be the Borel subgroup of upper triangular matrices in GSp(4). Let P_α and P_β be the maximal parabolic subgroups containing B. Their matrix realizations are given in section 2.3. If χ_1, χ_2, and σ are characters of GL(1), we denote by $\chi_1 \times \chi_2 \rtimes \sigma$ the representation of GSp(4) obtained by normalized induction from the following character of B:

$$\begin{pmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{pmatrix} \mapsto \chi_1(a)\chi_2(b)\sigma(c).$$

Similarly, if τ is a representation of GL(2), we let $\tau \rtimes \sigma$ be the representation induced from

$$\begin{pmatrix} g & \\ & c\,{}^{\tau}g^{-1} \end{pmatrix} \mapsto \sigma(c)\tau(g).$$

Moreover, $\sigma \rtimes \tau$ denotes the representation induced from the Klingen parabolic,

$$\begin{pmatrix} c & & \\ & g & \\ & & c^{-1}\det g \end{pmatrix} \mapsto \sigma(c)\tau(g).$$

In the table below, $\nu = |\cdot|$ is the normalized absolute value, χ_0 is the unique non-trivial unramified quadratic character, St is the Steinberg representation, $\mathbf{1}$ is the trivial representation, and $L((-))$ denotes the unique irreducible quotient when it exists.

		constituent of	representation	tempered	L^2	generic
I		$\chi_1 \times \chi_2 \rtimes \sigma$	$\chi_1 \times \chi_2 \rtimes \sigma$	$\lvert\chi_i\rvert = \lvert\sigma\rvert = 1$		•
II	a	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$,	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\lvert\chi\rvert = \lvert\sigma\rvert = 1$		•
	b	$\chi^2 \notin \{\nu^{\pm 1}, \nu^{\pm 3}\}$	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$			
III	a	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$,	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GL}(2)}$	$\lvert\chi\rvert = \lvert\sigma\rvert = 1$		•
	b	$\chi \notin \{\mathbf{1}, \nu^{\pm 2}\}$	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GL}(2)}$			
IV	a	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	•	•	•
	b		$L((\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GL}(2)}))$			
	c		$L((\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$			
	d		$\sigma \mathbf{1}_{\mathrm{GSp}(4)}$			
V	a	$\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$,	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$	•	•	•
	b	$\xi_0^2 = \mathbf{1}$, $\xi_0 \neq \mathbf{1}$	$L((\nu^{1/2}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$			
	c		$L((\nu^{1/2}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$			
	d		$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$			
VI	a	$\nu \times \mathbf{1} \rtimes \nu^{-1/2}\sigma$	$\tau(S, \nu^{-1/2}\sigma)$	•		•
	b		$\tau(T, \nu^{-1/2}\sigma)$	•		
	c		$L((\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$			
	d		$L((\nu, \mathbf{1} \rtimes \nu^{-1/2}\sigma))$			

Table A: Iwahori-spherical representations of GSp(4)

Appendix B. Parahoric Fixed Spaces

The following is essentially Table 3 in [Sch]. Here K is hyperspecial, $\tilde{K}$ is paramodular, I is Iwahori, and J_α and J_β denote the Klingen- and Siegel-parahoric subgroups, respectively. For example, J_α is the inverse image of P_α over the residue field under the natural reduction map.

		representation	remarks	K	$\tilde{K}$	J_α	J_β	I
I		$\chi_1 \times \chi_2 \rtimes \sigma$		1	2	4	4	8
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$		0	1	2	1	4
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$		1	1	2	3	4
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GL}(2)}$		0	0	1	2	4
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GL}(2)}$		1	2	3	2	4
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$		0	0	0	0	1
	b	$L((\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GL}(2)}))$	not unitary	0	0	1	2	3
	c	$L((\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$	not unitary	0	1	2	1	3
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(4)}$	irrelevant	1	1	1	1	1
V	a	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$		0	0	1	0	2
	b	$L((\nu^{1/2}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$		0	1	1	1	2
	c	$L((\nu^{1/2}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$		0	1	1	1	2
	d	$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$		1	0	1	2	2
VI	a	$\tau(S, \nu^{-1/2}\sigma)$		0	0	1	1	3
	b	$\tau(T, \nu^{-1/2}\sigma)$		0	0	0	1	1
	c	$L((\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$		0	1	1	0	1
	d	$L((\nu, \mathbf{1} \rtimes \nu^{-1/2}\sigma))$		1	1	2	2	3

Table B: Dimensions of the parahoric fixed spaces

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