Chapter 3

Equilibrium Participation in Public Goods Allocations

The previous chapter addressed certain concerns in mechanism design related to the assumption of equilibrium behavior and the robustness of predictions in repeated interaction scenarios. The current chapter takes a different tack, focusing instead on the enforceability of the outcomes selected by a particular mechanism. The key issues related to enforceability are the credible options of the social planner and the available options of the players. In economies with private goods, Hurwicz [48] assumes that the mechanism designer must allow the agents a 'no-trade' option, which leads naturally to the individual rationality constraint that agents must prefer the chosen allocation to their initial endowment. With public goods, exercising a no-trade option may allow an agent to consume some level of the public good produced by those who participate. Thus, Green & Laffont [41, p. 121] argue that individual rationality is instead founded on the ethical belief that each agent has a natural right to her endowment and the welfare its consumption would generate.

The current chapter reconsiders the mechanism design problem with public goods when the mechanism designer must allow a no-trade option. The resulting constraint – called *equilibrium participation* – requires the mechanism to select an outcome such that every agent prefers to contribute their requested transfer payment rather than withhold it. If an agent withholds her transfer payment, then the level of the public good is reduced to that which can be feasibly produced with the remaining transfers.

In order to induce all agents to choose participation over non-participation, a mechanism can satisfy equilibrium participation by making those agents with the strongest free-riding incentive responsible for the largest share of the production inputs. This is demonstrated in example 3.5 of Section 3.2.4. However, if several agents have strong free-riding incentives, they cannot all be made responsible for the lion's share of production. This problem is exacerbated in larger economies. This is the intuition behind the two main results of this chapter: (1) there are many finite economies in which only the endowment satisfies equilibrium participation, and (2) as *any* classical public goods economy is replicated, the set of outcomes satisfying equilibrium participation converges to the endowment.

The negative results of this chapter imply that coercion is absolutely necessary for mechanisms to successfully implement desirable outcomes. If an agent opts out of the mechanism outcome, some punishment system must be in place so that the dissenting agent cannot free ride on the production of others. This can be obtained explicitly through fines and sanctions, or implicitly by threatening to produce nothing if any agent defects. If explicit coercion is unavailable and implicit threats incredible, then mechanism design cannot avoid the standard free-rider problem.

The next section reviews the relevant literature. The notation and key definition of the chapter are provided in Section 3.2. General properties of the set of allocations satisfying equilibrium participation are explored in Section 3.3, followed by an analysis of the constraint in classical, quasi-concave economies with convex technology in Section 3.4. The main result on convergence to the endowment in large economies is proven in Section 3.5. Concluding comments and open questions are discussed in Section 3.6.

3.1 Relation to Previous Literature

Several authors have tried with limited success to define a notion of the core that is appropriate in a public goods economy. Such definitions must make assumptions about the behavior of non-dissenting coalitions when some coalition blocks an allocation. In the original definition by Foley [36], only the dissenting coalition may produce the public good; non-dissenters withdraw their contributions to production. This maximizes the threat to dissenters and many allocations remain in the core.[1] [82] assumes that non-dissenting agents select levels of production that are 'rational' for themselves (under various meanings) and finds that the subsequent definition of the core may be empty.

Champsaur, Roberts & Rosenthal [15] define the φ-core as the allocations that remain unblocked when blocking coalitions are given the power to tax the remaining agents an amount up to φ, which depends on the *proposed* blocking allocation. If φ

[1] [74] shows that Foley's core does not converge to the set of Lindahl equilibria in large economies.

were a function of the *original* allocation, then this notion of blocking (for single-agent coalitions) could encompass the definition of equilibrium participation. Though the results for both definitions are similarly negative, they are logically independent.

Saijo [89] analyzes the mechanism design problem when the utility of autarkic production is used as a welfare lower bound instead of the utility of the endowment. His notion of *autarkic individual rationality* requires each agent's final utility level to be weakly greater than that which the agent could achieve in isolation with his endowment and access to the production technology. Whereas Ledyard & Roberts [62] demonstrate that the standard notion of individual rationality is incompatible with incentive compatibility among the class of Pareto optimal mechanisms, Saijo [89] shows that autarkic individual rationality is incompatible with incentive compatibility for *all* mechanisms, optimal or not.

Other authors have proposed various models of the outside options of agents in a mechanism design setting. The most general of these is Jackson & Palfrey [53], where an unspecified function maps from any given outcome to another (possibly identical) outcome. The necessary and sufficient conditions of Maskin [67] are then extended in a simple way to accommodate this 'reversion function.' This approach unifies several existing attempts to model renegotiation and participation in the outcomes of mechanisms in private goods settings, such as Ma, Moore & Turnbull [64], Maskin & Moore [68], and Jackson & Palfrey [52]. It also encompasses pubic goods models with an exogenous status quo outcome or mechanism, as in Perez-Nievas [81].

The issue of enforceability has been addressed in the literature on Bayesian mechanism design (where agents have a non-degenerate common knowledge prior belief over the set of possible preference profiles) through the means of an external 'budget breaker' who receives a large transfer from the agents when undesirable performance is observed. This concept was introduced by Holmstrom [47] as a way for managers to incentivize teams of agents. Eswaran & Kotwal [31] argue that such schemes create a strong incentive for the budget breaker to bribe a single agent to deviate. For example, consider a situation where a central planner uses Walker's [100] mechanism to determine the level of some public good. Assume that if some agent submits a transfer smaller than that required by the mechanism outcome, then the planner can credibly commit to giving all received transfers to some disinterested third party, rather than putting those funds into production or refunding them to the agents. If this third party receives some benefit from these transfers, then she has an incentive to bribe one agent in the economy to withhold his transfer. If agents in the economy expect that this budget breaker will offer such a bribe, then the mechanism outcome cannot be supported as an equilibrium because the agents will rationally expect that some agent will be bribed. In the current chapter, it is assumed that the use of such budget breakers is not admissable, either because no dissinterested agent can be found or because the incentive to bribe is sufficiently large so as to make this an ineffective enforcement device.[2]

Finally, it is worth noting that concepts such as dominant strategy incentive compatibility and ex-post equilibrium do not encompass the definition of equilibrium

[2]Alternatively, it could be assumed that the planner has a strong preference for efficiency, so that the use of a budget breaker is simply not credible off the equilibrium path.

participation. Although these concepts do require that the mechanism outcome be preferred by each individual to all other outcomes in the range of the mechanism, there is no guarantee that the allocation obtaining after an agent opts out is in the mechanism's range. Indeed, most 'standard' public goods mechanisms (such as the Groves-Ledyard, Walker, and cVCG mechanisms presented in Section 2.4) do not include the opt-out points in their range. Therefore, the fact that an allocation is selected as part of an equilibrium decision does not preclude the possibility that agents will later prefer to free-ride on the contributions of others.

3.2 Notation & Definitions

This chapter uses the following notational conventions:[3]

$\mathbb{R}$	The real line: $(-\infty, \infty)$
$\mathbb{R}_+$	The non-negative real line: $[0, \infty)$
$\mathbb{R}^n$, $\mathbb{R}^n_+$	The n-fold Cartesian products of $\mathbb{R}$ and $\mathbb{R}_+$, respectively

3.2.1 Environments

Consider the following environment with one private good and one public good:

[3] If $\mathbf{x}$ and $\mathbf{x}'$ are in $\mathbb{R}^n$, then $\mathbf{x} \geq \mathbf{x}' \Leftrightarrow x_i \geq x'_i$ for all i, $\mathbf{x} > \mathbf{x}' \Leftrightarrow \mathbf{x} \geq \mathbf{x}'$ and $x_i > x'_i$ for some i, and $\mathbf{x} \gg \mathbf{x}' \Leftrightarrow x_i > x'_i$ for all i.

$I \geq 2$	The number of individuals
$\mathcal{I} = \{1, \ldots, I\}$	The set of individuals, indexed by i
$\mathbf{x} \in \mathbb{R}_+^I$	An allocation of the private good; $\mathbf{x} = (x_1, \ldots, x_I)$
$y \in \mathbb{R}_+$	A level of the public good
$\mathbf{z} = (\mathbf{x}; y) \in \mathbb{R}_+^{I+1}$	An allocation
$\mathcal{Z} \subset \mathbb{R}_+^{I+1}$	The set of all possible allocations
$\omega \in \mathcal{Z}$	The initial endowment: $\omega_i > 0$ for $i \in \mathcal{I}$, $\omega_{I+1} = 0$
$\mathbf{t} = \omega - \mathbf{x}$	The transfers paid by the agents. $T = \sum_i t_i$, $T_{-i} = \sum_{j \neq i} t_j$
$\succeq_i$	The complete, transitive preference relation of i on $\mathcal{Z} \times \mathcal{Z}$
$\succ_i$	The strict preference relation of i
$u_i : \mathcal{Z} \to \mathbb{R}$	Utility representation of $\succeq_i$
$\mathcal{Y} \subseteq \mathbb{R}^2$	The set of production possibilities: $\mathcal{Y} \cap \mathbb{R}_+^2 = \{(0, 0)\}$
	$\varphi \in \mathcal{Y}$ & $\varphi' \leq \varphi \Rightarrow \varphi' \in \mathcal{Y}$ (comprehensive), $\mathcal{Y}$ closed
$F : \mathbb{R}_+ \to \mathbb{R}_+$	The production function: $F(T) = \sup \{y : (-T, y) \in \mathcal{Y}\}$
$c : F(\mathbb{R}_+) \to \mathbb{R}_+$	The cost function: $c(y) = \inf \{T \geq 0 : (-T, y) \in \mathcal{Y}\}$
$\mathbf{e} = \left(\{\succeq_i\}_{i \in \mathcal{I}}, \mathcal{Y}, \omega\right)$	An economy with I agents
$\mathcal{E}_I$	The set of all economies with I agents

Given an economy $\mathbf{e}$, let $\mathcal{Z}(\mathbf{e}) \subseteq \mathcal{Z}$ be the set of *feasible* allocations of the form $\mathbf{z} = \omega + (-\mathbf{t}; y)$, where

$$y \geq 0$$

and $\mathbf{t} \in \mathbb{R}^I$ satisfies

$$\mathbf{t} \leqq \omega,$$

$$T \geq 0,$$

and

$$(-T; y) \in \mathcal{Y}.$$

A feasible allocation $(\mathbf{x}; y)$ is *balanced* if $y = F(T)$.

The following assumptions are used at various points in the chapter:

A1 (Monotonicity) If $(x_i', y') \geq (x_i, y)$, then $(\mathbf{x}'; y') \succeq_i (\mathbf{x}; y)$.

A2 (Convexity) If $\mathbf{z}' \succeq_i \mathbf{z}$, then $\alpha\mathbf{z}' + (1-\alpha)\mathbf{z} \succeq_i \mathbf{z}$ for all $\alpha \in (0,1)$.

A3 (Continuity) For every $\mathbf{z} \in \mathcal{Z}(\mathbf{e})$, $\{z' \in \mathcal{Z}(\mathbf{e}) : \mathbf{z}' \succeq_i \mathbf{z}\}$ and $\{\mathbf{z}' \in \mathcal{Z}(\mathbf{e}) : \mathbf{z}' \preceq_i \mathbf{z}\}$ are closed.

A4 (Increasing marginal cost) $\mathcal{Y}$ is convex.

A5 (Differentiable utility) Preferences $\succeq_i$ can be represented by a differentiable utility function u_i.

A6 (Differentiable cost) The function F is differentiable.

Denote the set of 'classical' economies satisfying A1 through A4 by $\mathcal{E}_I^C$. Let $\mathcal{E}_I^D$ denote the set of differentiable economies satisfying A1 through A6. Note that under A4 and A6, $c'(y) = 1/F'(T)$.

3.2.2 Mechanisms

The following defines a mechanism and its possible outcomes:

$\mathcal{S}_i$	The set of strategies of i: $\mathcal{S} = \prod_{\mathcal{I}} \mathcal{S}_i$
$\tau : \mathcal{S} \to \mathbb{R}^I$	Transfer function
$\eta : \mathcal{S} \to \mathbb{R}_+$	Outcome function
$\Gamma = (\mathcal{S}, \eta, \tau)$	A mechanism
$\mu_\Gamma (\mathbf{e})$	Equilibrium correspondence mapping Γ and $\mathbf{e}$ into subsets of $\mathcal{S}$
$\mathcal{O}^\mu_\Gamma (\mathbf{e})$	$= \{(\mathbf{x}; y) \in \mathcal{Z} : [\exists \mathbf{s} \in \mu_\Gamma (\mathbf{e})] \; \mathbf{x} = \omega - \tau (\mathbf{s}) \; \& \; y = \eta (\mathbf{s})\}$
$\bar{\mathcal{O}}^\mu_\Gamma (\mathbf{e})$	$= \{(\mathbf{x}; y) \in \mathcal{O}^\mu_\Gamma (\mathbf{e}) : y = F (\sum_i (\omega_i - x_i))\}$

The sets $\mathcal{O}^\mu_\Gamma (\mathbf{e})$ and $\bar{\mathcal{O}}^\mu_\Gamma (\mathbf{e})$ represent the set of outcomes and balanced outcomes, respectively, of an economy $\mathbf{e}$.

Definition 3.1 Γ *is decisive under* μ *if, for all* $\mathbf{e} \in \mathcal{E}_I$, $\mathcal{O}^\mu_\Gamma (\mathbf{e}) \neq \emptyset$.

Definition 3.2 Γ *is feasible under* μ *if it is decisive under* μ *and, for all* $\mathbf{e} \in \mathcal{E}_I$, $\mathcal{O}^\mu_\Gamma (\mathbf{e}) \subseteq \mathcal{Z} (\mathbf{e})$.

Definition 3.3 Γ *is balanced under* μ *if it is feasible under* μ *and, for all* $\mathbf{e} \in \mathcal{E}_I$, $\mathcal{O}^\mu_\Gamma (\mathbf{e}) = \bar{\mathcal{O}}^\mu_\Gamma (\mathbf{e})$.

The set of *Pareto optimal* allocations for e is given by

$$\mathcal{PO} (\mathbf{e}) = \{\mathbf{z} \in \mathcal{Z} (\mathbf{e}) : [\nexists \mathbf{z}' \in \mathcal{Z} (\mathbf{e})] \, \mathbf{z}' \succ \mathbf{z}\} .$$

Definition 3.4 Γ *is efficient under* μ *if it is decisive under* μ *and, for all* $\mathbf{e} \in \mathcal{E}_I$, $\mathcal{O}^\mu_\Gamma (\mathbf{e}) \subseteq \mathcal{PO} (\mathbf{e})$.

If preferences are strictly monotonic, efficient mechanisms must be balanced.

3.2.3 Implementation

In general, if $\mathcal{G}$ is a *social choice correspondence (SCC)* mapping each economy $\mathbf{e}$ to a subset of the feasible allocations $\mathcal{Z}(\mathbf{e})$, then Γ *implements* $\mathcal{G}$ *under* μ if $\mathcal{O}_\Gamma^\mu(\mathbf{e}) \subseteq \mathcal{G}(\mathbf{e})$ for every $\mathbf{e}$ and Γ *fully implements* $\mathcal{G}$ *under* μ if $\mathcal{O}_\Gamma^\mu(\mathbf{e}) = \mathcal{G}(\mathbf{e})$ for every $\mathbf{e}$. For example, if $\mathcal{IR}_i(\mathbf{e}) = \{(\mathbf{x}; y) \in \mathcal{Z}(\mathbf{e}) : (\mathbf{x}; y) \succeq_i (\omega; 0)\}$, then $\mathcal{IR}(\mathbf{e}) = \bigcap_{\mathcal{I}} \mathcal{IR}_i(\mathbf{e})$ is the SCC that selects all points in the economy that are weakly preferred to the endowment by all individuals. If Γ implements $\mathcal{IR}(\mathbf{e})$ under μ, then all agents are made weakly better off by participating in Γ and playing a strategy in $\mu_\Gamma(\mathbf{e})$.

Hurwicz [48] and Ledyard & Roberts [62] have shown that no mechanism implements $\mathcal{PO}(\mathbf{e}) \cap \mathcal{IR}(\mathbf{e})$ in dominant strategies for private or public goods economies, respectively. Hurwicz [49] shows that if a mechanism implements $\mathcal{PO}(\mathbf{e}) \cap \mathcal{IR}(\mathbf{e})$ in Nash equilibrium, then $\mathcal{O}_\Gamma^\mu(\mathbf{e})$ is the set of Walrasian (or Lindahl) allocations.

3.2.4 The Participation Decision

Consider a situation in which agents in economy e participate in a mechanism Γ that is balanced and efficient under μ and receive the outcome $(\omega - \tau; \eta) \in \mathcal{O}_\Gamma^\mu(\mathbf{e})$. If each agent i has the freedom to either contribute τ_i or exercise a 'no-trade' option by withholding τ_i, then the mechanism outcome induces an I-player, two-strategy game. Assume that the final public goods level is the maximum feasible, given the contributions received. If all agents prefer to contribute τ_i over exercising their no-trade option, then full participation is a Nash equilibrium of the induced participation game and the allocation $(\omega - \tau; \eta)$ will be fully realized.

Clearly, there may exist a conflict between the goal of the social planner and the opt-out incentives of the agents. This is clearly seen by the following example:

Example 3.5 *Let* $\mathcal{I} = \{1, 2\}$. *Define*

$$u_1(x_1, y) = x_1 + 21y - 2y^2$$

and

$$u_2(x_2, y) = x_2 + 77y - 9y^2.$$

Fix $\omega_i = 50$ *for each* i *and let* $F(T) = T/10$.

In this example, $\mathcal{PO}(\mathbf{e}) = \{(\mathbf{x}; y) : y = 4 \ \& \ t_1 + t_2 = 40\}$. At the optima, the marginal rate of substitution is 5 for both agents, so the consumers' Lindahl prices are equal. Suppose an efficient mechanism under μ selects the Lindahl solution $\tau = (20, 20)$ and $\eta = 4$. The induced participation game is given in panel (a) of Figure 3.1. Clearly, agent 1 has an incentive to withhold her requested transfer, resulting in a suboptimal outcome of $y = 2$ in equilibrium.

Now consider another efficient mechanism under μ that selects $\eta = 4$ and $\tau = (30, 10)$. In the induced participation game, shown in panel (b) of Figure 3.1, it is an equilibrium for both agents to participate. Agent 1 no longer has an incentive to opt out because her contribution is responsible for a larger share of the production.

Although this redistribution of production 'responsibility' is an effective trick to

$t_1 \setminus t_2$	20	0
20	$82, 194$	$64, 168$
0	$84, 148$	$50, 50$

(a)

$t_1 \setminus t_2$	10	0
30	$72, 204$	$65, 200$
0	$69, 108$	$50, 50$

(b)

Figure 3.1: The induced participation game for Example 3.5 from (a) the equal-price Lindahl allocation, and (b) an unequal-price optimal allocation.

offset free-riding incentives, feasibility constraints limit how many agents can have their tax burden sufficiently increased. Furthermore, some agents may prefer to always defect, regardless how much of the burden they must bear. These difficulties are key to the negative results of the chapter.

Consider the more general case of two players and a constant marginal cost. If an allocation z is proposed such that $t_i > 0$ for each i and $F(T) > 0$, then the allocation that obtains when agent 1 opts out is given by

$$\mathbf{z}^{(-1)} = \left((\omega_1, x_2); y^{(-1)}\right),$$

where

$$y^{(-1)} = F(t_2).$$

The opt-out point $\mathbf{z}^{(-2)}$ is similarly defined. Panel (a) of Figure 3.2 provides a graphical example of these points in the Kolm triangle diagram (Kolm [58]; see Thomson [95] for a detailed exposition.) For the proposal z to satisfy equilibrium participation, both agents must prefer z to their 'opt-out' points $\mathbf{z}^{(-i)}$, as in the figure.

In the case where $t_1 < 0$ while $t_2 > 0$, then $y^{(-2)} = 0$ since negative quantities

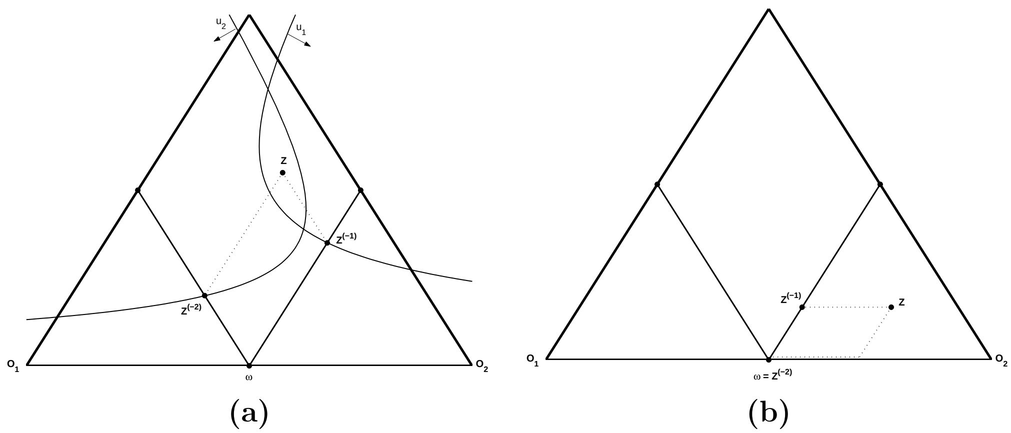

Figure 3.2: (a) The point $z \succeq_i z^{(-i)}$ for each $i \in 1, 2$, so it satisfies equilibrium participation. (b) The points $z^{(-i)}$ when t_1 is negative.

of the public good are not admissible. However, $y^{(-1)} = y$ since agent 1 is not asked to contribute any private good. In this case, it is assumed that the negative transfer rejected by agent 1 is either redistributed among the other agents or destroyed, rather than affecting the level of the public good.[4] Under A1, agent 1 will always prefer participation when $t_1 < 0$ and agent 2 will prefer participation only if $(\mathbf{x}; y) \in \mathcal{IR}_2(\mathbf{e})$. The case of a negative transfer is demonstrated graphically in panel (b) of Figure 3.2.

Generalizing the concepts of the two-player example provides the key definition of this chapter.

Definition 3.6 *For any $I = 1, 2, \ldots$ and any economy $\mathbf{e} \in \mathcal{E}_I$, a feasible allocation $(\mathbf{x}; y) \in \mathcal{Z}(\mathbf{e})$ such that $\mathbf{x} = \omega - t$ satisfies* equilibrium participation for agent i *(EP_i) if and only if*

$$(\mathbf{x}; y) \succeq_i \left(\mathbf{x}^{(-i)}; y^{(-i)}\right),$$

[4]Whether the transfer is redistributed or destroyed will not affect the i's participation decision since $\succeq_i$ depends only on x_i and y.

where

$$x_i^{(-i)} = \omega_i,$$

$$y^{(-i)} = \begin{cases} F\left(T_{-i}\right) & \textit{if} \quad t_i \geq 0,\ T_{-i} \geq 0,\ \textit{and}\ y \geq F\left(T_{-i}\right) \\ 0 & \textit{if} \quad T_{-i} < 0 \\ y & \textit{otherwise} \end{cases}, \tag{3.1}$$

and

$$\left(\mathbf{x}^{(-i)}; y^{(-i)}\right) \in \mathcal{Z}\left(\mathbf{e}\right).$$

The allocation $(\mathbf{x}; y) \in \mathcal{Z}(\mathbf{e})$ *satisfies* equilibrium participation (EP) *if and only if it satisfies* EP_i *for all* $i \in \mathcal{I}$*.*

There are four possible cases in this definition. When $t_i \geq 0$, $T_{-i} \geq 0$, and $y \geq F(T_{-i})$, removing agent i's transfer necessarily reduces production, but not to zero. If $T_{-i} < 0$, then $t_i > 0$ and removing i's transfer results in $y^{(-i)} = 0$. If $t_i < 0$ or $y < F(T_{-i})$, then y can be produced in the absence of i's transfer, so $y^{(-i)} = y$.

For any economy $\mathbf{e} \in \mathcal{E}_I$, let

$$\mathcal{EP}_i\left(\mathbf{e}\right) = \left\{\mathbf{z} \in \mathcal{Z}\left(\mathbf{e}\right) : \mathbf{z} \text{ satisfies } \mathrm{EP}_i\right\},$$

and define

$$\mathcal{EP}\left(\mathbf{e}\right) = \bigcap_{i \in \mathcal{I}} \mathcal{EP}_i\left(\mathbf{e}\right).$$

Referring back to the example of Figure 3.2, $\mathbf{z} \in \mathcal{EP}(\mathbf{e})$ in panel (a), but in panel

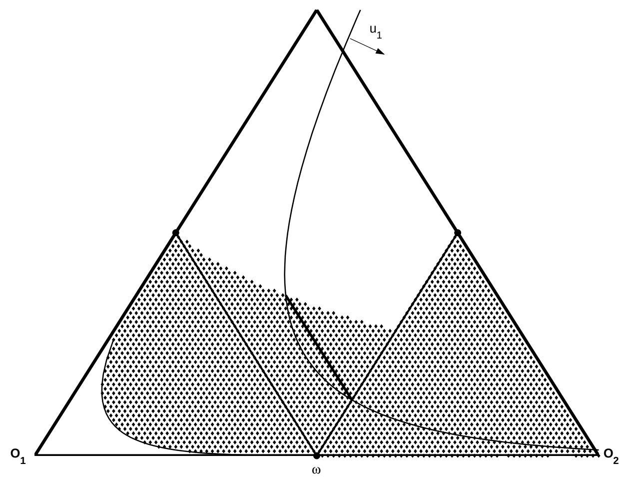

Figure 3.3: The set of balanced allocations satisfying equilibrium participation for agent 1.

(b), $\mathbf{z} \notin \mathcal{EP}_1(\mathbf{e})$, so $\mathbf{z} \notin \mathcal{EP}(\mathbf{e})$.

3.3 Properties of $\mathcal{EP}(\mathbf{e})$

The shaded region of Figure 3.3 demonstrates a typical equilibrium participation set for agent 1 in a two-agent classical economy. Note that $\mathcal{EP}(\mathbf{e})$ is closed and has a continuous boundary, but need not be convex. Clearly, $\mathcal{EP}(\mathbf{e})$ is non-empty for every $\mathbf{e} \in \mathcal{E}_I$ and every I since $(\omega; 0) \in \mathcal{EP}(\mathbf{e})$.

As an alternative to equilibrium participation, consider an environment in which agents can freely choose $t_i \in [0, \omega_i]$, resulting in $y = F(T)$. The set of Nash equilibrium allocations is given by

$$\mathcal{NE}(\mathbf{e}) = \left\{ \begin{array}{c} (\mathbf{x}^*; y^*) \in \mathcal{Z}(\mathbf{e}) : \mathbf{x}^* \leq \omega \text{ and} \\ [\forall i \in \mathcal{I}] \quad [\forall t_i' \geq 0] \quad (x_i^*, y^*) \succeq_i \left(\omega_i - t_i', F\left(T_{-i}^* + t_i'\right)\right) \end{array} \right\}.$$

The notion of equilibrium participation is now shown to be more stringent than the standard notion of individual rationality, but less restrictive than the Nash equilibrium requirement.

Proposition 3.7 *Under monotone increasing preferences (A3), all allocations satisfying equilibrium participation also satisfy individual rationality ($\mathcal{EP}(\mathbf{e}) \subseteq \mathcal{IR}(\mathbf{e})$.)*

Proof. Consider a point $(\mathbf{x}; y)$ such that $(x_i, y) \succeq_i \left(\omega_i, y^{(-i)}\right)$ for all $i \in \mathcal{I}$. Note that $y^{(-i)} \geq 0$ for each i, so A3 implies that $\left(\omega_i, y^{(-i)}\right) \succeq_i (\omega_i, 0)$. By transitivity, $(x_i, y) \succeq_i (\omega_i, 0)$ for every i, proving the result. ■

Proposition 3.8 *All Nash equilibria of the voluntary contributions game satisfy equilibrium participation ($\mathcal{NE}(\mathbf{e}) \subseteq \mathcal{EP}(\mathbf{e})$.)*

Proof. From any Nash equilibrium point, the 'opt-out' allocation for agent i in the participation game is simply $\left(\omega_i, F\left(T_{-i}^*\right)\right)$. Since the definition requires that $(x_i^*, y^*) \succeq_i \left(\omega_i, F\left(T_{-i}^*\right)\right)$ for all i by considering $t_i' = 0$, the point $(\mathbf{x}^*; y^*)$ must satisfy equilibrium participation. ■

In mechanism design with public goods, the most common goal is to implement $\mathcal{PO}(\mathbf{e})$. There exist several mechanisms whose Nash equilibria are guaranteed to be Pareto optimal when utility is transferable. However, if the outcomes of these mechanism fail to satisfy equilibrium participation, then their desirable properties are of little use in environments where agents cannot be coerced to submit their transfers. The following class of examples shows the potential difficulty of finding points in $\mathcal{PO}(\mathbf{e}) \cap \mathcal{EP}(\mathbf{e})$.

Example 3.9 *Let $I \geq 2$. Define $u_i(x_i, y) = v_i(y) + x_i$, where each $v_i(y)$ is continuous and differentiable. Assume $F(T) = T/\kappa$, and let $v_i'(y) < \kappa$ for all $i \in \mathcal{I}$ and $y \geq 0$. Assume that there is a unique $y^o > 0$ such that $\sum_i v_i'(y) > \kappa$ for $y < y^o$ and $\sum_i v_i'(y) < \kappa$ for $y > y^o$. Finally, assume that $\sum_{j \neq i} \omega_j < \kappa y^o$ for each $i \in \mathcal{I}$.*[5]

In this example, no agent is willing to unilaterally fund any amount of the public good at any level and therefore refuses to contribute in any participation game. To see this, pick any allocation $(\mathbf{x}; y) \neq (\omega; 0)$, so $\mathbf{t} \neq \mathbf{0}$. If all agents participate in this allocation, then each agent i receives

$$u_i(x_i, y) = v_i(y) + \omega_i - t_i.$$

If i withholds her transfer, she receives

$$u_i\left(x_i^{(-i)}, y^{(-i)}\right) = v_i\left(y^{(-i)}\right) + \omega_i.$$

There must be some agent i with $t_i > 0$. If $y = 0$ or $T_{-i} \leq 0$, then $y^{(-i)} = 0$ and EP_i is

[5] *One such example is $\kappa = 1$ and*

$$v_i(y) = \begin{cases} \frac{3}{2I} y & \text{if} \quad y \leq 1 \\ \frac{1}{2I} y + \frac{1}{I} & \text{if} \quad y \geq 1 \end{cases}$$

for each i. Here, $y^o = 1$. The point of non-differentiability in v_i is of no consequence.

not satisfied. If $y > 0$ and $T_{-i} > 0$, but $y \leq F(T_{-i})$ then $y^{(-i)} = y$ and EP_i again fails. Therefore, consider the case where $y > 0$, $T_{-i} > 0$, and $y > F(T_{-i})$, so $y^{(-i)} = F(T_{-i})$. By withholding, agent i saves $t_i = \kappa\left(y - y^{(-i)}\right)$ in transfer payments. Her loss in value due to the reduction in public goods production is $v_i(y) - v_i\left(y^{(-i)}\right) = \int_{y^{(-i)}}^{y} v_i'(s)\, ds$, which is less than $\kappa\left(y - y^{(-i)}\right)$ since $v_i'(y) < \kappa$ for all y. Therefore, she will prefer to withhold her transfer regardless of t_i and the allocation will not satisfy equilibrium participation for agent i. In this economy, no allocation can satisfy EP_i for every i, so $\mathcal{EP}(\mathbf{e})$ is simply the endowment. This class of examples proves the following proposition:

Proposition 3.10 *For every $I \geq 2$, there exists economies $\mathbf{e}$ in $\mathcal{E}_I^C$ such that no allocation except the endowment satisfies equilibrium participation ($\mathcal{EP}(\mathbf{e}) = \{\omega\}$).*

The following shows that the notion of voluntary participation implicit in the definition of EP may preclude any optimal allocation from obtaining.

Proposition 3.11 *For every $I \geq 2$, there exists economies $\mathbf{e}$ in $\mathcal{E}_I^C$ in which no allocation $z \in \mathcal{Z}(\mathbf{e})$ can be selected such that the equilibrium of the resulting participation game is Pareto optimal.*

The proof of this result is simple. Any Pareto optimal allocation in the above class of examples must choose $y^o > 0$, from which any agent will defect. Furthermore, optimal allocations cannot obtain *after* an agent defects; if any one agent is consuming $x_i = \omega_i$, then $\sum_{j \neq i} \omega_j < \kappa y^o$ guarantees that y^o cannot be feasibly produced by the remaining agents.

Note that example 3.9 does not represent a knife-edge case. A wide range of economies fits its assumptions and a number of similar examples can be constructed. The key factor is marginal utilities must be smaller than marginal costs at all levels of y.

Since Proposition 3.11 indicates that EP is inconsistent with Pareto optimality, it is natural to ask whether there can exist *any* non-trivial mechanisms that satisfy this constraint.[6] In other words, is there a mechanism and a μ that implements $\mathcal{EP}(\mathbf{e})$ in μ? The results of Gibbard [40], Satterthwaite [91], K. Roberts [85] and Zhou [102] indicate that dominant strategy implementation of $\mathcal{EP}(\mathbf{e})$ is futile, even in classical economies. More positive results may be obtained when μ is weakened to the Nash equilibrium concept; it is simple to show that $\mathcal{EP}(\mathbf{e})$ satisfies Maskin's definition of monotonicity (see Maskin [67], giving the following result[7]):

Proposition 3.12 *The set of allocations satisfying equilibrium participation ($\mathcal{EP}(\mathbf{e})$) can be non-trivially implemented in Nash equilibrium when $I \geq 3$.*

The proof of this proposition for full implementation relies on Maskin's mechanism which is not a particularly 'natural' game form. Proposition 3.8 shows that $\mathcal{EP}(\mathbf{e})$ can be implemented by the voluntary contribution mechanism since $\mathcal{NE}(\mathbf{e}) \subseteq \mathcal{EP}(\mathbf{e})$. However, this mechanism does not fully implement $\mathcal{EP}(\mathbf{e})$. Note that in economies like those of Example 3.5, $\mathcal{EP}(\mathbf{e}) = \{\omega\}$, making implementation of $\mathcal{EP}(\mathbf{e})$ trivial.

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[6] A non-trivial mechanism is defined as one that selects something other than the initial endowment in at least one environment.

[7] The other sufficient condition, 'no-veto power,' is trivially satisfied in economic environments such as this one.

3.4 Quasi-Concave Economies

3.4.1 Necessary and Sufficient Conditions

The additional structure gained by adding assumptions A1 through A6 allows for the derivation of separate necessary and sufficient conditions for an allocation to satisfy equilibrium participation. Although these conditions are not tight, they require only 'local' information about the gradients of utilities and derivative of the production function.

Proposition 3.13 *For any economy in $\mathcal{E}_I^D$, if equilibrium participation is satisfied at a point $(\mathbf{x}; y) = (\omega + \mathbf{t}; y)$, then*

$$\frac{\partial u_i\left(\omega_i; F\left(T_{-i}\right)\right)/\partial y}{\partial u_i\left(\omega_i; F\left(T_{-i}\right)\right)/\partial x_i} \geq c'\left(y^{(-i)}\right) \tag{3.2}$$

for all $i \in \mathcal{I}$ such that $t_i, T_{-i} \geq 0$ and $y \geq F\left(T_{-i}\right)$.

A similar condition is now shown to be sufficient for a point to satisfy equilibrium participation. Whereas the necessary condition compares the marginal rate of substitution to marginal costs at the drop-out point, the sufficient condition compares these quantities at the proposed allocation:

Proposition 3.14 *For any economy in $\mathcal{E}_I^D$, if a point $(\mathbf{x}; y) = (\omega + \mathbf{t}; y)$ satisfies*

$$\frac{\partial u_i\left(\mathbf{x}; y\right)/\partial y}{\partial u_i\left(\mathbf{x}; y\right)/\partial x_i} \geq c'\left(y\right) \tag{3.3}$$

for all i such that t_i,$T_{-i} \geq 0$ and $y \geq F(T_{-i})$ and

$$u_j(\mathbf{x}; y) \geq u_j(\omega; 0) \tag{3.4}$$

for all j such that $T_{-j} < 0$, then equilibrium participation is satisfied at $(\mathbf{x}; y)$.

Unlike the necessary condition, equation (3.4) implies that information about the utilities of some agents at both the suggested allocation and the endowment is needed. This may be undesirable from the standpoint of mechanism design since additional information is necessary to determine that the condition is met.[8] The following condition shows how equation (3.4) could be replaced by a stronger version of equation (3.3) to give a single condition sufficient for all agents that uses only information about preferences and costs at the selected allocation.

Proposition 3.15 *For any economy in $\mathcal{E}_I^D$, if a point $(\mathbf{x}; y) = (\omega + \mathbf{t}; y)$ satisfies*

$$\frac{\partial u_i(\mathbf{x}; y) / \partial y}{\partial u_i(\mathbf{x}; y) / \partial x_i} \geq \frac{t_i}{F(T)} \tag{3.5}$$

for all i, then equilibrium participation is satisfied at $(\mathbf{x}; y)$.

Figure 3.4 demonstrates the interpretation of these conditions. The quantity $(\partial u_i / \partial y) / (\partial u_i / \partial x_i)$ is the slope of the gradient of u_i, while c' is the slope of the normal to the production possibilities frontier. In the figure, F is reflected around the y-axis and horizontally shifted so that its graph represents the production pos-

[8]Of course, there could exist mechanisms whose outcomes satisfy Equilibrium Participation without satisfying this sufficient condition.

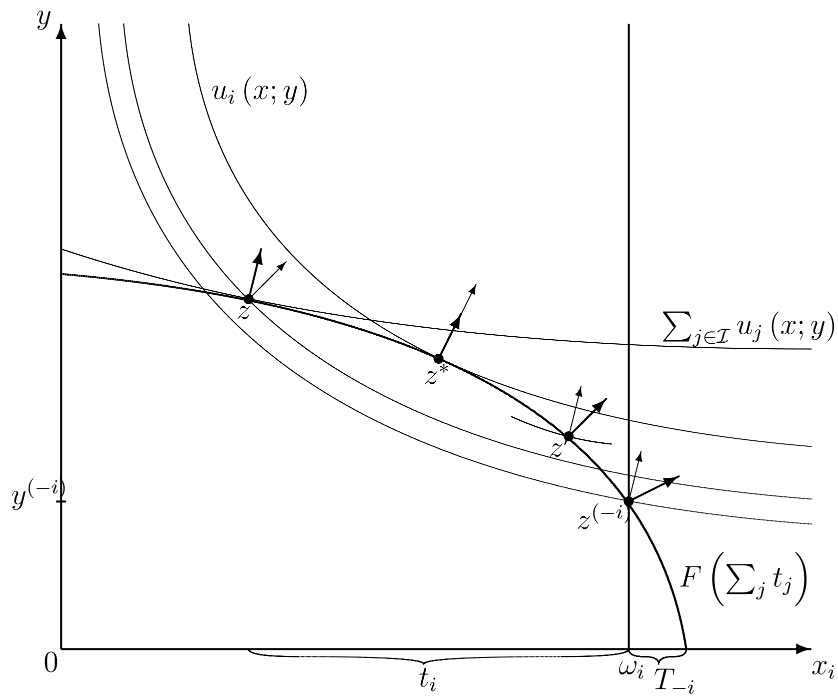

Figure 3.4: An example with quasi-concave utilities and convex production sets. z is Pareto optimal, z^* is i's most-preferred feasible allocation, and $z^{(-i)}$ is i's drop-out point. $z^{(-i)}$ satisfies the sufficient condition for EP. z^* and z' satisfy the sufficient condition.

sibilities set for agent i, given the endowments. If agent i withholds t_i, then the allocation $z^{(-i)}$ results. In this case, i will prefer the Pareto optimal point z to $z^{(-i)}$.

The necessary condition for equilibrium participation is satisfied in the figure since the gradient of utility has a steeper slope than the normal to F at $z^{(-i)}$. The sufficient condition is satisfied at z' since the gradient of utility is steeper than the normal to F at z', but this condition fails at the optimal point, z. In fact, the sufficient condition is satisfied for any point along F between $z^{(-i)}$ and z^*, but nowhere left of z^*. This is intuitive; z' is closer to z^* (i's most preferred point) than $z^{(-i)}$, so i will not opt out of z'.

The Samuelson [90] condition for an interior optimum requires z to be to the left of z^*, where the sufficient condition fails. Thus, equilibrium participation requires that $z^{(-i)}$ be sufficiently to the right of z^*, causing t_i to be large. As in the opening example, large transfers are needed to incentivize participation, but feasibility may constrain how large the transfer can be or how many agents can have these inflated transfers. Clearly, this constraint will be more restrictive in larger economies, as will be demonstrated in Section 3.5.

3.4.2 Quasi-Linear Preferences

The transferable utility environment is especially important in mechanism design as the absence of wealth effects is useful in guaranteeing the ability to satisfy incentive compatibility constraints through transfer payments. It also allows a more precise quantification of the minimal transfer needed to satisfy equilibrium participation.

Assume agents have utility functions $u_i(x_i, y) = v_i(y) + x_i$, where $v_i' > 0$ and $v'' \leq 0$, and let the production function be strictly increasing and concave, so $c(y)$ is strictly increasing and convex. Let y_i^* be the unique solution to $c'(y) = v_i'(y)$. Equilibrium participation at a public good level of $\hat{y}$ requires that

$$t_i \leq \int_{y^{(-i)}}^{\hat{y}} v_i'(y)\, dy.$$

It must be that if t_i is non-negative, then

$$\int_{y^{(-i)}}^{\hat{y}} c'(y)\, dy \leq t_i,$$

with equality if the allocation is non-wasteful. In order for $\hat{y}$ to satisfy equilibrium participation for agent i when $\hat{y} > y_i^*$, it must be the case that

$$\int_{y^{(-i)}}^{y_i^*} \left(v_i'(y) - c'(y)\right) dy \geq \int_{y_i^*}^{\hat{y}} \left(c'(y) - v_i'(y)\right) dy, \tag{3.6}$$

both of which are non-negative quantities.

For an optimal allocation y^o, equation (3.6) provides an exact requirement on how 'far' $y^{(-i)}$ must be from y_i^* to guarantee equilibrium participation. This is demonstrated in Figure 3.5, in which $y^{(-i)}$ is the largest value satisfying (3.6) for the optimal point y^o. The necessary and sufficient conditions from equations (3.2) and (3.3) are also intuitive in this figure; if $y^{(-i)} > y_i^*$, then the necessary condition fails because marginal costs are everywhere larger than the marginal benefit between $y^{(-i)}$ and y^o, and the sufficient condition is satisfied for any $y \in [y^{(-i)}, y_i^*)$ since marginal costs are less than the marginal benefit at every public good level between y and $y^{(-i)}$.

3.5 Equilibrium Participation in Large Economies

The analysis of finite economies indicates that the large transfers needed to guarantee equilibrium participation for optimal allocations conflict with the feasibility constraints, particularly for larger economies. There is a fundamental difficulty in

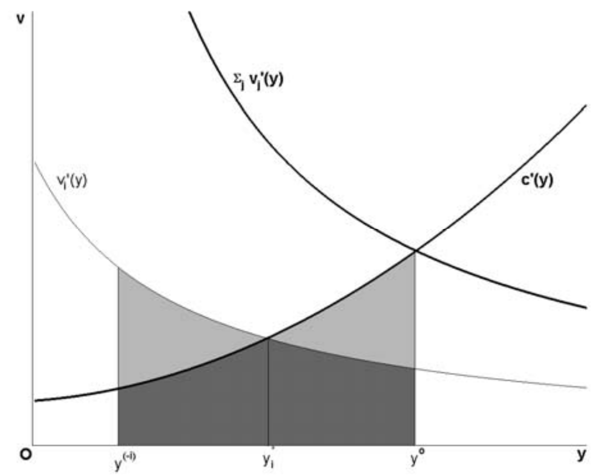

Figure 3.5: The Pareto optimal point y^o exactly satisfies equilibrium participation; if $y^{(-i)}$ were any larger, EP would fail.

the notion of a replica public goods economy. If each replicated agent is given the same endowment, then the total available production input grows without bound. Unless preferences bound the level of production, agents in large economies can find themselves consuming an infinite ratio of public to private goods.

Muench [74], Milleron [73], and Conley [25] discuss the difficulty of replicating public goods economies and offer various possible methods.[9] Milleron [73] provides an intriguing notion of replication; by splitting a fixed endowment among the replicates and adjusting preferences so that agents' concerns for the private good are relative to the size of their endowment, the fundamental difficulties of replication are mitigated. In essence, as the economy is replicated and agents are given a smaller share of the endowment, their preferences adjust proportionally to become more sensitive to the

[9]These authors are examining the convergence of the core of the economy to the Lindahl equilibrium. See Foley [36] for the appropriate definitions.

private goods holding. Thus, a very small shift in the absolute holdings of the private good is more significant to an agent with a small endowment in a big economy than to an agent with a big endowment in a small economy.

Formally, consider a base economy $\mathbf{e} \in \mathcal{E}_I$ with I unique agents such that $\mathbf{e} = \left(\{\succeq_i\}_{i \in \mathcal{I}}, \mathcal{Y}, \omega\right)$. A replica economy $\mathbf{e}^R$ is defined by replicating R times each $i \in \mathcal{I}$. Each replicate of consumer type i, denoted by the pair (i, r) for $r = 1, \ldots, R$, is endowed with ω_i / R units of the private good and a preference relation $\succeq_{i,r}$ such that $(x_{i,r}, y) \succeq_{i,r} \left(x'_{i,r}, y'\right)$, if and only if $(Rx_i, y) \succeq_i (Rx'_i, y')$ because of the scaling of endowments. This assumption on preferences of replicates mimics the approach of Milleron [73] and guarantees that private good consumption trade-offs are significant, even as the magnitude of those trade-offs becomes arbitrarily small. Finally, the production technology of $\mathbf{e}^R$ is assumed to be identical to that of $\mathbf{e}$.

This intuition that equilibrium participation becomes oppressively restrictive as an economy is replicated is confirmed by the following theorem:

Theorem 3.16 *For any economy satisfying A1, A3, and A4 (continuous, monotone preferences and increasing, continuous production technology,) the set of allocations satisfying equilibrium participation converges to the initial endowment as the economy is infinitely replicated.*

The proof of this theorem, available in the chapter appendix (Section 3.7,) demonstrates how the shrinking endowment restricts the amount any agent can be asked to pay in the limit. This, in turn, limits the agent's effect on production. Since agents in large economies care about small changes in their private goods consumption, but not in the level of the public good, agents eventually prefer to opt-out as their individual effect on production vanishes.

This result is sensitive to the definitions of a replica economy. Consider instead a more standard notion of replication in which $\omega_{i,r} = \omega_i$ for each type i and replicate r, and assume $(x_{i,r}, y) \succeq_{i,r} \left(x'_{i,r}, y'\right)$ if and only if $(x_i, y) \succeq_i (x'_i, y')$. To see that the theorem no longer holds, construct a simple base economy $\mathbf{e} \in \mathcal{E}_I$ with an agent i for whom $(0, F(\omega_i)) \succ_i (\omega_i, 0)$. Here, the allocation $(\mathbf{x}, y)$ where $x_i = 0$, $x_j = \omega_j$ for all $j \neq i$, and $y = F(\omega_i)$ satisfies equilibrium participation. This economy can be replicated arbitrarily often, but the sequence of allocations $\left(\mathbf{x}^R, y^R\right)$ such that $x^R_{i,1} = 0$, $x^R_{j,r} = \omega_j$ for all $(j, r) \neq (i, 1)$, $y^R = F(\omega_i)$ satisfies equilibrium participation for every R, but does not converge to the endowment.[10]

Note that this result holds in economies where the set of Pareto optimal allocations remains far from the endowment as the economy grows, so that notion of approximate efficiency is of no benefit. For large economies, it is necessary that the committee or government has the power of coercion in order to overcome the free-rider problem.

3.6 Conclusion

If a mechanism is to implement a desired social choice correspondence with public goods when agents have available a no-trade alternative, it must select an allocation impervious to agents withdrawing their transfers. The incompatibility between equi-

[10]If the limit economy is represented by a measure space of consumers, however, this example fails because the contributions of a single individual are of measure zero and will not affect production of the public good.

librium participation and Pareto optimality is established through simple quasilinear examples, indicating that optimality is unobtainable under the standard assumptions used in mechanism design. In many economies, only the initial endowment is insusceptible to agents withdrawing. Even in those economies for which non-trivial allocations satisfy equilibrium participation, the set of equilibrium participation allocations eventually shrinks to the endowment as the economy is replicated.

The above analysis leaves open important questions about participation in public goods allocations. Perhaps it is possible to characterize those economies for which optimality is not inconsistent with equilibrium participation. If this class of such economies is reasonable to assume as the set of possible economies, then the negative results may be avoided with small numbers of agents. Similarly, there may exist a wide range of economies for which Pareto optimality may be well approximated under equilibrium participation. If such 'approximately desirable' outcomes could be identified, perhaps there exists a more natural mechanism that can implement these outcomes in Nash equilibrium. Given that the equilibrium participation constraint can be thought of as a restriction on the size of transfers, it is conceivable that a total transfer maximizing solution to this system of restrictions may be identified and used to maximize the total size of the public good in a given economy.

Finally, empirical observation demonstrates that non-trivial quantities of public goods are regularly provided in large economies. Governments and other voluntarily established methods of coercion exist as enforcement devices to guarantee that welfare improving allocations are attained. The next chapter provides a repeated-game justification for endogenous enforcement, even when interactions are anonymous and individual reputations cannot be tracked. Such a process is a naturally occurring phenomenon within the larger private ownership/competitive mechanism framework, rather than a formally defined allocation mechanism. A larger model of how allocation mechanisms evolve in time has yet to be developed.

3.7 Appendix

Proof of Proposition 3.13. Pick any agent i such that $t_i, T_{-i} \geq 0$ and $y \geq F(T_{-i})$. Equilibrium participation implies that

$$u_i(\omega_i - t_i, F(T_{-i} + t_i)) \geq u_i(\omega_i, F(T_{-i})).$$

By quasi-concavity of u_i,

$$\nabla u_i(\omega_i, F(T_{-i})) \cdot (-t_i, F(T_{-i} + t_i) - F(T_{-i})) \geq 0,$$

or

$$\frac{F(T_{-i} + t_i) - F(T_{-i})}{t_i} \geq \frac{\partial u_i(\omega_i; F(T_{-i})) / \partial x_i}{\partial u_i(\omega_i; F(T_{-i})) / \partial y}.$$

Thus, by concavity of F,

$$\frac{\partial u_i(\omega_i; F(T_{-i})) / \partial x_i}{\partial u_i(\omega_i; F(T_{-i})) / \partial y} \leq F'(T_{-i}).$$

Inverting this inequality gives the necessary condition. ■

Proof of Proposition 3.14. By monotonicity, equilibrium participation is trivially satisfied for all j such that $t_j < 0$ or $y < F(T_{-j})$. Equation (3.4) guarantees equilibrium participation when $T_{-j} < 0$.

Now consider some $i \in \mathcal{I}$ such that $t_i, T_{-i} \geq 0$ and $y \geq F(T_{-i})$, but for whom equilibrium participation fails. For this agent,

$$u_i(\omega_i, F(T_{-i})) > u_i(\omega_i - t_i, F(T_{-i} + t_i)), \tag{3.7}$$

so that

$$\nabla u_i(\mathbf{x}; y) \cdot (t_i, F(T_{-i}) - F(T_{-i} + t_i)) > 0.$$

This is equivalent to

$$\frac{\partial u_i(\mathbf{x}; y) / \partial x_i}{\partial u_i(\mathbf{x}; y) / \partial y} > \frac{F(T_{-i} + t_i) - F(T_{-i})}{t_i}, \tag{3.8}$$

so applying the concavity of F at $T_{-i} + t_i$ and inverting the resulting relationship gives

$$\frac{\partial u_i(\mathbf{x}; y) / \partial y}{\partial u_i(\mathbf{x}; y) / \partial x_i} < \frac{1}{F'(T_{-i} + t_i)}.$$

Equation (3.3) implies that (3.7) cannot hold, so by the contrapositive of this argument, $(\mathbf{x}; y)$ must satisfy EP_i. ■

Proof of Proposition 3.15. For agents with $T_{-i} < 0$, $y^{(-i)} = 0$, but $F(T_{-i}) < 0$. By replacing $F(T_{-i})$ with zero in the proof of Proposition 3.14, the argument is identical through equation (3.8). At this point, the subsequent relationship with $F'(T)$ cannot be derived from $F(T)/t_i$ when $T_{-i} < 0$, so inverting (3.8) gives the alternative sufficient condition

$$\frac{\partial u_i(\mathbf{x}; y)/\partial y}{\partial u_i(\mathbf{x}; y)/\partial x_i} \geq \frac{1}{F(T)/t_i} \tag{3.9}$$

for all i such that $T_{-i} < 0$. Since this is a stronger condition than (3.3), it is also sufficient for *every* agent. ■

Proof of Theorem 3.16. By way of contradiction, assume that there exists some economy e and some sequence $\left\{\left(\mathbf{x}^R; \hat{y}^R\right)\right\}_{R=1}^{\infty}$ in $\mathcal{EP}\left(e^R\right)$ for each R such that $\left|\hat{y}^R\right|$ fails to converge to zero.

For each (i,r), let $t_{i,r}^R = \omega_{i,r}^R - x_{i,r}^R$. For any $\left(\mathbf{x}^R; \hat{y}^R\right) \in \mathcal{EP}\left(e^R\right)$, if $\hat{y}^R < F\left(\sum_{i,r} t_{i,r}^R\right)$, then by monotonicity, $\left(\mathbf{x}^R; y^R\right) \in \mathcal{EP}\left(e^R\right)$, where $y^R = F\left(\sum_{i,r} t_{i,r}^R\right)$. In other words, if a wasteful allocation $(\mathbf{x}; \hat{y})$ satisfies equilibrium participation, so does the transfer-equivalent non-wasteful allocation $(\mathbf{x}; y)$. Thus, the sequence $\left\{\left(\mathbf{x}^R; y^R\right)\right\}_{R=1}^{\infty}$ satisfies equilibrium participation for each R and $\left\{\left|y^R\right|\right\}_{R=1}^{\infty}$ also fails to converge to zero. This implies that there exists a subsequence $\left\{\left(\mathbf{x}^{R(k)}; y^{R(k)}\right)\right\}_{k=1}^{\infty}$ such that $\left|y^{R(k)}\right| > \varepsilon$ for some $\varepsilon > 0$ all $k \in \mathbb{N} = \{1, 2, \ldots\}$. Letting $c(y)$ represent the minimal cost of producing y (which is the inverse of F,) non-convergence guarantees that $c\left(y^{R(k)}\right) \geq c(\varepsilon) > 0$ for each k since c is an increasing function and $\mathcal{Y} \cap \mathbb{R}_+^2 = \{0\}$.

For any k, if $R(k) > \max_{i \in \mathcal{I}} \left(\omega_i / c(\varepsilon)\right)$, then no one agent (i,r) can unilaterally fund $y^{R(k)}$ using $t_{i,r}^{R(k)}$ since

$$\begin{aligned} t_{i,r}^{R(k)} &\leq \max_{i \in \mathcal{I}} \omega_i / R(k) \\ &< c(\varepsilon) \\ &\leq c\left(y^{R(k)}\right). \end{aligned}$$

Letting

$$k^* = \max\left\{k \in \mathbb{N} : R(k) \leq \max_{i \in \mathcal{I}} \left(\omega_i / c(\varepsilon)\right)\right\},$$

there exists at least one sequence of agents $\{(i_k, r_k)\}_{k=1}^{\infty}$ such that

$$t_{i_k, r_k}^{R(k)} \geq c\left(y^{R(k)}\right) / \left(R(k) I\right)$$

for all k, and $T_{-(i_k, r_k)} > 0$ for all $k > k^*$. In other words, there exists a sequence of agents such that at each k, the identified agent is paying a transfer which is more than the average transfer of $c\left(y^{R(k)}\right) / \left(R(k) I\right) > 0$, and the sum of the others' transfers is eventually positive as individual (i_k, r_k)'s budget constraint becomes restrictive. For example, $\{(i_k, r_k)\}_{k=1}^{\infty}$ might identify the agent (i, r) in each k for whom $t_{i,k}^{R(k)}$ is maximal among all agents (this particular sequence may not have a well-defined limit, but any selection of agents paying an above average proportion of the cost is sufficient.)

Since each $\left(\mathbf{x}^{R(k)}; y^{R(k)}\right)$ satisfies equilibrium participation for all (i, r), it must be the case that

$$\left(\omega_{i,r} - t^{R(k)}_{i_k,r_k}, y^{R(k)}\right) \succeq_{i_k,r_k} \left(\omega_{i,r}, \left(y^{R(k)}\right)^{-(i_k,r_k)}\right),$$

or equivalently,

$$\left(\omega_i - R\left(k\right) t^{R(k)}_{i_k,r_k}, y^{R(k)}\right) \succeq_{i_k} \left(\omega_i, \left(y^{R(k)}\right)^{-(i_k,r_k)}\right).$$

Note that for $k > k^*$,

$$\left(y^{R(k)}\right)^{-(i_k,r_k)} = F\left(\sum_{j,s} t^{R(k)}_{j,s} - t^{R(k)}_{i_k,r_k}\right).$$

By continuity of the production function, $\left(y^{R(k)}\right)^{-(i_k,r_k)}$ becomes arbitrarily close to $y^{R(k)}$ as k grows. However, since $t^{R(k)}_{i_k,r_k} > c\left(\varepsilon\right) / \left(R\left(k\right) I\right)$, then $R\left(k\right) t^{R(k)}_{i_k,r_k}$ is bounded below by $c\left(\varepsilon\right) / I > 0$ at all k. By monotonicity of preferences, it must be the case that

$$\left(\omega_i - \frac{c\left(\varepsilon\right)}{I}, y^{R(k)}\right) \succeq_i \left(\omega_i - R\left(k\right) t^{R(k)}_{i_k,r_k}, y^{R(k)}\right) \succeq_i \left(\omega_i, \left(y^{R(k)}\right)^{-(i_k,r_k)}\right).$$

By continuity of preferences, convergence of $\left(y^{R(k)}\right)^{-(i_k,r_k)}$ to $y^{R(k)}$ implies that for large enough k,

$$\left(\omega_i - \frac{c\left(\varepsilon\right)}{I}, y^{R(k)}\right) \succeq_{i_k} \left(\omega_i, y^{R(k)}\right).$$

However, this violates monotonicity. Since there cannot be an infinite subsequence of allocations with $\left|y^{R(k)}\right| > \varepsilon$ for any $\varepsilon > 0$, it must be the case that $y^R \to 0$ as

$R \to \infty$. Feasibility then requires that $\left\| \mathbf{x}^R - \omega^R \right\|_\infty \to 0$, completing the proof. ■