String/Gauge Duality and Penrose Limit

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B.S. (Seoul National University, Korea) 1996

In Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

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submitted year 2003

(Defended May 12, 2003)

Abstract

Berenstein, Maldacena, and Nastase have recently discovered a particular limit of AdS/CFT correspondence where string theory in a plane wave background is dual to a sector of $\mathcal{N} = 4$ SYM in a double scaling limit. It is based on the observation that a plane wave background can be obtained by taking Penrose limit of Anti de Sitter background. The corresponding gauge theory limit is identified via AdS/CFT dictionary. This proposal is especially exciting because string worldsheet theory in a plane wave background is exactly solvable, thereby opening a possibility that one can go beyond supergravity approximation. In the absence of string interactions, the duality made a remarkable prediction for anomalous dimension of gauge theory operators from exact free string spectrum, which was soon verified.

In this thesis, we attempt to extend the duality to the interacting theory level. We propose that the correct holographic recipe is to identify the full string field theory Hamiltonian with the dilatation operator of gauge theory. In practice, we must find an identification map between string theory and gauge theory Hilbert spaces and evaluate matrix elements of the two operators accordingly. The requirement that the inner product should be preserved determines a unique identification map assuming that it is hermitian. We show that transition amplitudes of string field theory agree with matrix elements of dilatation operator under this preferred identification for states with two different impurities. We later extend it to states with arbitrary impurities. In doing so, we find a diagrammatic correspondence between string field theory and gauge theory Feynman diagrams thereby providing direct handles on the duality. Our proposal is $universal$ in the sense that it is applicable to any interaction type such as the open-closed interaction, and to all orders in g_2 and λ'. Hopefully, this thesis will be a key step towards proving the novel duality and a beginning of an exciting journey to the stringy regime of string/gauge duality.

Acknowledgment

I am grateful to all the people who helped me in one way or another throughout my education. My advisor, Professor Hirosi Ooguri, has patiently guided me with his creative insights, and ceaselessly encouraged me. Professor John Schwarz has kindly introduced string theory to me and supported me constantly.

I would also like to thank other teachers who have gotten me interested in physics and explained many things to me. These mentors include Prof. Choonkyu Lee, Prof. Soo-Jong Rey, Prof. Yongmin Cho, Prof. Mark Wise, Prof. John Preskill, Prof. Kip Thorn, and Prof. Edward Witten.

I would also like to thank the people in the Caltech theory group for providing a friendly atmosphere. I would especially like to acknowledge Carol Silberstein for all her help and consideration. I have been fortunate enough to collaborate with many outstanding colleagues. Calin Ciocarlie, Jaume Gomis, Peter Lee, Sanefumi Moriyama, and Jonathan Tannenhauser all played an important role. I especially thank Jaume for guiding me academically and non-academically and for showing me his great optimism and enthusiasm towards physics. I appreciate many debates about physics and life with Peter Lee and wish him much success in the future. I have also benefited from many interesting discussions with Andi Brandhuber, Vadim Borokhov, Iouri Chepelev, Kirk Kaminsky, Prof. Anton Kapustin, Christopher Lee, Wilfred Lee, Tristan McLoughlin, Yi Li, Yuji Okawa, Takuya Okuda, Arkadas Ozakin, Jae-Suk Park, Michael Schulz, Ian Swanson, and Xinkai Wu.

Sangmin Lee has originally encouraged me to study string theory and taught me many things from a long distance. Many friends that I met at Caltech, especially Hern hyung, have shared many things with me while I stayed here. My parents and two sisters have been a huge moral support to me. They have convinced me that what I am doing is one of the most valuable things in the world. Without them, I would not be where I am today.

Finally, Soojin Son, my fiance, has supported me in all possible ways. Being always by my side, she has shown me great patience and understanding. She has also helped me a lot to improve my English, and taught me what a complete person is and what a happy life is. I would like to express my deep gratitude and love to her.

This research was supported in part by Caltech Institute Fellowship and DOE grant DE-FG03-92-ER40701.

Contents

Chapter 1

Introduction

String theory was originally born as a theory of hadrons. Strings were a dynamical realization of Faraday flux lines between quarks with a length of about 10^{-13} cm. The theory successfully explained many properties of hadrons including linear Regge trajectories. However, the theory had the unusual feature that it inevitably contained a spin-2 massless particle, which came from the closed string spectrum. It was a mystery at the time because the only consistent way to couple such particles is to treat them as "gravitons," and definitely we were not dealing with gravity. In the early 1970s, it was discovered that hadrons and mesons are made of quarks which are described by Quantum Chromodynamics (QCD), and string theory soon lost its original motivation of describing the strong interaction. Shortly after, string theory was reborn in a more ambitious fashion, as a unified theory of all interactions in nature, particularly, a quantum theory of gravity [1, 2, 3]. Previous attempts to quantize gravity based on point particles were not successful due to divergent short-distance behavior. String theory naturally spreads out the interactions over spacetime in a consistent manner, thereby cutting off the divergences. Presently, string theory is the only known consistent quantum theory of gravity. It is hoped that string theory can address all the important problems in quantum gravity such as Hawking radiation and spacetime singularities. Much progress has been made in this direction, but our understanding is yet far from complete.

Perturbative study of QCD, a non-Abelian gauge theory (also known as Yang-

Mills theory) with $SU(3)$ gauge group, has deepened our understanding of the strong interaction since its discovery. QCD is however strongly coupled at low-energy, and therefore perturbative analysis no longer applies. A full understanding of the theory in this regime is not available to date even though partial success has been achieved in supersymmetric analogues. In particular, quark confinement remains as a challenging problem in theoretical physics. In 1974, 't Hooft conjectured that in the strong coupling regime, the gauge theory should have a dual string theory description [4]. Let us summarize his argument. Assume that we have a gauge theory with $U(N)$ gauge group and that all fields in this theory transform in the adjoint representation of $U(N)$. An adjoint field Φ^a, where a is an index in the adjoint representation, can be written as a matrix acting on the fundamental representation, i.e., $\Phi_i^j = \sum_a \Phi^a [T^a]_i^j$ where T^a's are the generators of $U(N)$ in the fundamental representation and i, j run from 1 to N.[1] Hence, we can think of Φ as carrying two indices, one in the fundamental and the other in the anti-fundamental representation. The propagator for Φ is

$$\langle \Phi_i^j \Phi_k^l \rangle \propto \delta_i^l \delta_k^j. \tag{1.0.1}$$

On drawing Feynman diagrams in this theory, it is natural to represent the propagator for Φ as a double line (or a "fat" line) with each line denoting the fundamental or anti-fundamental index (See figure 1.1). Then any Feynman diagram looks like a web of double lines. We further assume that all cubic interaction vertices are proportional to the Yang-Mills coupling g_{YM}, and all quartic vertices to g_{YM}^2.[2] We are interested in N, g_{YM} dependence of Feynman diagrams. Let us consider a diagram with $V = V_3 + V_4$ vertices, E edges, and F faces. From the interaction vertices, we get $g_{YM}^{V_3+2V_4}$. Each face is bounded by a index loop, which gives a factor of N from summing over colors running around the loop. By counting the number of edges in two independent ways,

[1] More precisely, sub-index and super-index denote fundamental and anti-fundamental representations, respectively.

[2] This is the case for $N = 4$ Super Yang-Mills theory, which will be our main interest in this thesis.

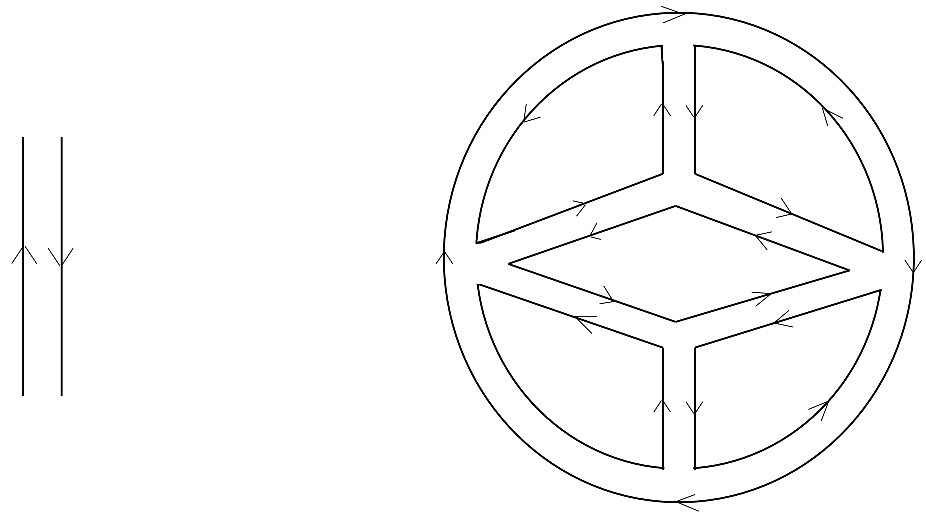

Figure 1.1: The double line representation of adjoint fields and a typical vacuum diagram with genus 0.

we can get a relation $3V_3 + 4V_4 = 2E$. Altogether, this Feynman diagram gives

$$N^F g_{YM}^{V_3+2V_4} = N^F g_{YM}^{2E-2V} = N^{V-E+F}(g_{YM}^2 N)^{E-V} = N^\chi \lambda^{E-V}, \tag{1.0.2}$$

where χ is the Euler number of the underlying two-dimensional surface triangulated by this Feynman diagram and $\lambda = g_{YM}^2 N$ is the so-called 't Hooft coupling. The surface is oriented since the double lines have an orientation and can be compactified by adding one point at infinity. Then the Euler number is completely determined by the genus (the number of handles) of the surface, $\chi = 2 - 2g$. Hence, the ordinary perturbative expansion of the gauge theory can be rearranged into a double expansion in terms of genus and quantum loops:

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{n=0}^{\infty} C_{g,n} \lambda^n. \tag{1.0.3}$$

This sum over the genus or topology of two-dimensional surface is strongly reminiscent of the perturbative string expansion if $1/N$ is identified with the string coupling constant. This double expansion is valid only if N is large and λ is small. When λ becomes large, the string worldsheet emerges from a "fat" Feynman diagram of gauge theory, and the dual string theory takes over. The string theory is still perturbative as long as we keep N large. Therefore, this string/gauge duality is more manifest in the large N limit. From this heuristic argument, we can learn that gauge theory is linked to string theory, and it is not surprising that string theory was originally designed for describing the strong interaction. However, little had been known about whether this argument holds beyond perturbation theory and exactly what the dual string theory is until the recent discovery of the AdS/CFT correspondence.

The AdS/CFT correspondence is a particular realization of 't Hooft's idea [5, 6, 7] (For an extensive review on the subject, see [8]). It is based on consideration of a stack of large number N of parallel D3-branes in type IIB string theory. Dp-branes are special non-perturbative objects in string theory with mass of order $1/g_s$ [3] which are sources of closed strings, in particular, carrying Ramond-Ramond (RR) charges. They are also defined as p-dimensional surfaces where open strings can end on [9]. These two alternative viewpoints provide two equivalent descriptions of the large N D3-brane system. On the one hand, they deform the closed string background, i.e., curve the spacetime and generate RR-flux. We can study the system by considering closed string theory on the resulting background which is a higher dimensional analogue of extremal black holes. On the other hand, we can consider open strings on the D3-branes and their interaction with closed strings in the bulk flat spacetime. Now let us take the low-energy limit with respect to an observer far from the D-brane system. From the first point of view, we have low-energy supergravity modes propagating in the asymptotically flat region of the spacetime and arbitrary modes in the near horizon region. In the limit, these two two types of modes decouple from each other. From the second point of view, we have a four-dimensional $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills (SYM) theory from open strings on the D-branes and supergravity in the bulk flat spacetime. Again, in the limit, they decouple. After cancelling out the common bulk supergravity in the flat spacetime, we come to conjecture that the four-dimensional $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills theory is dual to type IIB string theory on the near horizon geometry which is $AdS_5 \times S^5$ with N units of self-dual RR-flux through S^5. This AdS/CFT correspondence has been the focus of much research since

[3] g_s is the string coupling constant in string perturbation theory.

its discovery. One crucial relation between string theory and gauge theory parameters in this duality is

$$\frac{R^4}{\alpha'^2} = \lambda = g_{YM}^2 N, \tag{1.0.4}$$

where R is the common radius of AdS_5 and S^5, and $\sqrt{\alpha'}$ is the fundamental string length scale. In practice, we take the supergravity approximation of string theory because string theory on $AdS_5 \times S^5$ is hard to quantize due to the notorious difficulty of dealing with RR-flux. This approximation is valid only if the curvature of spacetime is small compared with the string scale, i.e.,

$$\frac{R^4}{\alpha'^2} = \lambda \gg 1, \tag{1.0.5}$$

whereas perturbative SYM computation applies only if the 't Hooft coupling is small, i.e.,

$$\frac{R^4}{\alpha'^2} = \lambda \ll 1. \tag{1.0.6}$$

Therefore, the regimes where we can analyze each side of the duality are mutually exclusive. This fact has been a source of excitement, as well as frustration in the past years. We can make interesting predictions for strongly coupled gauge theory from classical geometry using the duality, for example, but we are not able to prove them. Nevertheless, AdS/CFT has provided us many conceptual advances. It is an explicit realization of the *holographic principle* of quantum gravity [10, 11], which states that a quantum theory of gravity in a region can be described by a theory at the boundary of the region with at most one degree of freedom per Planck area. This is deduced from the fact that the entropy of a black hole is proportional to the area of its horizon and from the subsequent argument of Bekenstein [12] that the entropy of a system surrounded by a surface should be bounded by the area of the surface. Otherwise, we can think of a process of formation of a black hole having the surface as its horizon, and then the entropy would decrease during the process. AdS/CFT could also give us a non-perturbative definition of quantum gravity or string theory at least on a particular background since we know how to define SYM non-perturbatively.

The aforementioned difficulty of studying strings propagating in an RR background has been a challenge for string theorists. There have been many attempts to overcome it, leading finally to a breakthrough. The idea is that we can take a limit of the original $AdS_5 \times S^5$ background to simplify the situation. The limit actually taken is the so-called Penrose limit [13, 14, 15, 16]. The Penrose limit is, roughly speaking, to zoom in on the vicinity of a null geodesic in the original spacetime. It results in a background that an imaginary observer travelling along the null geodesic would observe. In general, one obtains a plane wave (or pp-wave) geometry in this limit. The limit has the nice property that the number of symmetries including supersymmetry of the background never decreases in the limit. In this way, we can obtain a maximally supersymmetric plane wave background of type IIB supergravity or string theory from the $AdS_5 \times S^5$ background. This background is much simpler but still has an interesting light-like RR-flux inherited from the original background.

Another important input came from [17], which showed that free string theory in the plane wave background can be exactly solvable[4] despite the presence of RR-flux. In the seminal paper [18], Berenstein, Maldacena, and Nastase (BMN) identified the corresponding limit in the dual gauge theory via AdS/CFT correspondence. The limit of gauge theory turns out to be a double scaling limit where N, J, λ are taken to infinity while λ/J^2 and J^2/N are kept fixed. Here J is the charge for $U(1)$ inside $SO(6)$ R-symmetry which corresponds to the direction along which the null geodesic winds around S^5. In this limit, the gauge theory truncates to a subset of operators with large R-charge J but finite $\Delta - J$. One would think that 't Hooft's expansion of gauge theory (1.0.3) might not make sense because λ goes to infinity, and that, even if so, only planar diagrams (diagrams with $g = 0$) might contribute since $1/N$ goes to zero. In fact, both expectations are too naive and one finds a well defined double expansion in terms of g_2 and λ' [19, 20],

$$g_2 = \frac{J^2}{N} \qquad\qquad \lambda' = \frac{\lambda}{J^2}, \tag{1.0.7}$$

Now g_2 counts the genus of diagrams and λ' measures quantum loops instead of $1/N$ and λ,

$$\sum_{g=0}^{\infty} g_2^{2-2g} \sum_{n=0}^{\infty} C_{g,n} \lambda'^n, \tag{1.0.8}$$

[4]One should use Green-Schwarz strings in the light-cone gauge.

The reason why we have a sensible expansion in strong 't Hooft coupling limit is that the class of operators we are looking at in this limit nearly satisfies the BPS condition $\Delta = J$, thereby leaving the nonrenormalization theorem partially intact. Therefore, there are quantum corrections, but they are actually controllable in the limit. All the non-planar diagrams do contribute in the $N \to \infty$ limit because a growing number of Feynman diagrams at a given genus can compete with the suppressing factor of N^{2-2g}.

Hence, we have an exciting possibility that we can check AdS/CFT beyond supergravity approximation and also that both sides of the duality may be calculable in the same regime. A crucial step in making the correspondence precise is to identify the charges of string states in spacetime with the charges carried by gauge theory operators. The identification is given by

$$\frac{1}{\mu} H = \Delta - J, \tag{1.0.9}$$

where H is the light-cone Hamiltonian corresponding to a light-cone time of the plane wave background, and μ is a mass scale introduced while taking the Penrose limit of $AdS_5 \times S^5$. The details will be explained later. The first check of the duality is to compare the free string spectrum with anomalous dimensions of operators in the planar limit via (1.0.9). Remarkable agreement is shown perturbatively in [18, 21], and to all orders in λ' using superconformal invariance in [22].

The next question we should address is how to go beyond free string theory. Once we turn on the string coupling, strings start joining and splitting and the free Hamiltonian receives corrections of all orders in the string coupling,

$$H = H_2 + g_2 H_3 + \cdots . \tag{1.0.10}$$

Splitting/joining transition amplitudes via the interaction Hamiltonian should be captured by the dual gauge theory. On the string side, investigation of interacting string theory in plane wave backgrounds was initiated by [23] and soon followed by many authors using light-cone string field theory [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35] or using string bit model [36, 37, 38, 39]. The main interest in this direction is construction of the cubic Hamitonian and evaluation of its matrix elements between single- and two-string states. On the other hand, many computations have been done in the BMN limit of gauge theory [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56]. However, it is not clear how to match both sides of the duality at the interaction level.

In the original AdS/CFT, we have a general holographic description [6, 7], where gauge theory operators inserted on the boundary of AdS_5 give boundary conditions for the corresponding fields in AdS_5. In this way, we can extract, in principle, bulk interactions from boundary gauge theory correlators. Notice that, however, the boundary of AdS_5 is pushed to infinity and eventually lost in the Penrose limit because we are focusing on a trajectory lying deep inside AdS_5. Therefore, we do not have a holographic map of a plane wave background directly inherited from that of AdS/CFT. In fact, it is shown that the conformal boundary of a plane wave geometry is a one-dimensional null line [57]. Therefore, one expects the holographic dual theory to be a quantum mechanical system, but such a system has not yet been found[5].

In this thesis, we attempt to answer this important problem: "*How can we describe string interactions in a plane wave from gauge theory?*" Our guiding principle is simply to take the relation (1.0.9) as the holographic map for *all* orders in the string coupling. This philosophy was first put forward by [40]. In order to use the relation (1.0.9), we need to sandwich the operators on both sides of the equation by states in string and gauge theory. Therefore, we first have to find an explicit map between the Hilbert space of string states and the Hilbert space of states in the gauge theory,

$$|s_A\rangle \to |\tilde{O}_A\rangle, \tag{1.0.11}$$

where $|s_A\rangle$ is a string state and $|\tilde{O}_A\rangle$ the corresponding gauge theory state (or equivalently operators via state/operator correspondence of conformal field theory). Moreover, in order for the comparison to be meaningful, such a map must preserve inner

[5]It is plausible that the double scaling limit of the subset of operators that we consider here may effectively give rise to a quantum mechanical system. See [58, 59] for development along this line.

products of the two Hilbert spaces,

$$\langle s_A | s_B \rangle = \langle \tilde{O}_A | \tilde{O}_B \rangle. \tag{1.0.12}$$

In string theory, there exists a canonical diagonal inner product, namely, states with different number of strings are orthogonal. In gauge theory, a natural inner product is given by mixing of two operators, but operators with different number of traces tend to mix to all orders in g_2. Therefore, we need to diagonalize the gauge theory inner product order by order. After finding this identification, we can test our proposal by checking

$$\frac{1}{\mu} \langle s_A | H | s_B \rangle = \langle \tilde{O}_A | (\Delta - J) | \tilde{O}_B \rangle. \tag{1.0.13}$$

We emphasize that this remarkably simple recipe is *universal*. First, it applies to any *in*- and *out*- states no matter what they are, two-string states, three-string states, or even open string states if any. Computations on both sides are well defined, and we need not find a separate recipe for each interaction type. Furthermore, it can be used for *all* orders in g_2 and λ' without modification. We know how to compute corrections in g_2 and λ' to both sides of (1.0.9) and we just have to compare order by order. Throughout this thesis, however, we restrict ourselves to first order in λ' and leave higher loop analysis for future study. Also, we are mostly interested in calculations to first order in g_2, the so-called cubic Hamiltonian, but we will perform some order g_2^2 contact term analysis.

The rest of the thesis is organized as follows. In the subsequent sections, we explain in detail the Penrose limit of AdS/CFT. In Chapter 2 which is based on [60], we formulate our proposal more precisely and apply it to the simplest class of string and gauge theory states. This will be generalized to arbitrary states in Chapter 3 based on [61] and we complete a proof of agreement of string and gauge theories via our proposal to first order in g_2 and λ'. We also show some agreement to order g_2^2 and λ'. In so doing, we find a correspondence between gauge theory Feynman diagrams and string field theory Feynman diagrams[6]. This tells us that the agreement is not a numerical coincidence, but a physically meaningful duality. In the Penrose limit of

[6] We will define string field theory Feynman diagrams in Chapter 3.

AdS/CFT, we will see explicitly each elementary gauge theory field forming string bits and string worldsheets emerging from gauge theory Feynman diagrams. We conclude the thesis in Chapter 4. Many details of analysis are presented in the appendix.

1.1 AdS/CFT correspondence and Penrose limit

Let us explain AdS/CFT in more detail. In the global coordinate system where the duality is the most conveniently formulated, the metric of $AdS_5 \times S^5$ is given as

$$ds^2 = R^2 \left[-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho \, d\Omega_3^2\right] + R^2 d\Omega_5^2, \tag{1.1.14}$$

where R, the common radius of AdS_5 and S^5, is given by $R^4 = g_{YM}^2 N \alpha'^2$, and $d\Omega_n^2$ is the round metric of an n-sphere. The background is also equipped with a self-dual RR-flux,

$$\int_{S^5} F_5 = N. \tag{1.1.15}$$

The Yang-Mills coupling and the string coupling are related by

$$g_{YM}^2 = 4\pi g_s. \tag{1.1.16}$$

The SYM lives on $\mathbf{R}_t \times S^3$ which is the conformal boundary of AdS_5 located at $\rho = \infty$. Here $\mathbf{R}_t$ is the direction along the global time t. The four-dimensional $\mathcal{N} = 4$ SYM is known to have a conformal symmetry group including ordinary Lorentz symmetry. In a conformal field theory, there are no asymptotic states or S-matrix, and natural objects are operators and correlation functions among them. That said, it is convenient to radially quantize the SYM, meaning that we map $\mathbf{R}_t$ of $\mathbf{R}_t \times S^3$ to the radial direction of $\mathbf{R}^4$ as we do in quantizing the worldsheet theory of strings. Then states on S^3 in the SYM on $\mathbf{R}_t \times S^3$ are mapped to local operators in the SYM on $\mathbf{R}^4$.

The duality implies matching of the global symmetries of the two theories. The string theory has as a global symmetry the isometries of $AdS_5 \times S^5$ which is $SO(4,2) \times SO(6)$. The SYM has the four-dimensional conformal symmetry group $SO(4,2)$ and also the extended $\mathcal{N} = 4$ supersymmetry which includes $SO(6)$ R-symmetry. Using this identification, we can match symmetry generators on both sides. In particular, the Hamiltonian H_t of the string theory associated with the global time t is identified with the scaling operator Δ of the SYM on R^4. Also the rotation symmetry generator along a great circle in S^5 corresponds to the generator of a $U(1)$ subgroup of the $SO(6)$ R-symmetry group.

On taking the Penrose limit of $AdS_5 \times S^5$, it is convenient to express S^5 part of the metric as follows :

$$ds^2 = R^2\left[-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + d\psi^2\cos^2\varphi + d\varphi^2 + \sin^2\varphi\, d\Omega_3^{\prime 2}\right], \tag{1.1.17}$$

where $\psi \sim \psi + 2\pi$ and $\varphi \sim \varphi + 2\pi$. Translation in ψ amount to rotation along a great circle of S^5. We introduce light-cone coordinates

$$x^+ = \frac{t}{\mu}, \quad x^- = \mu R^2(t-\psi), \quad r_1 = R\rho, \quad r_2 = R\theta, \tag{1.1.18}$$

where μ is a free mass scale that can be introduced, and we take the Penrose limit, i.e., $R \to \infty$ with g_s, α' fixed. Then the metric and the RR flux reduce to the form of a plane wave background :

$$ds^2 = 2dx^+dx^- - \mu^2(\vec{r}_1^{\,2} + \vec{r}_2^{\,2})dx^{+2} + d\vec{r}_1^{\,2} + d\vec{r}_2^{\,2}, \tag{1.1.19}$$

$$F_{+1234} = F_{+5678} = \mu, \tag{1.1.20}$$

where $\vec{r}_1, \vec{r}_2 \in \mathbf{R}^4$ and we denote $\vec{r}_1 = (z^1, ..., z^4)$, $\vec{r}_1 = (z^5, ..., z^8)$. Now let us consider the corresponding limit of the dual gauge theory. As explained above, we identify

$$i\partial_t \leftrightarrow \Delta, \qquad -i\partial_\psi \leftrightarrow J, \tag{1.1.21}$$

where Δ is the generator of dilatation and J is a $U(1)$ generator in $SO(6)_R$. Hence, one finds that

$$\begin{aligned} \frac{1}{\mu}H &= \frac{i}{\mu}\partial_{x^+} = i(\partial_t + \partial_\psi) = \Delta - J, \\ \mu R^2 P^+ &= i\mu R^2\partial_{x^-} = -i\partial_\psi = J, \end{aligned} \tag{1.1.22}$$

where H is the generator of x^+ translations, and P^+ is the generator of x^- translations. In the Penrose limit, we keep the light-cone energy and momentum finite.

Therefore, the gauge theory is restricted to operators with large R-charge $J \sim \sqrt{N}$ and finite $\Delta - J$ in the limit $N \to \infty$ with g_{YM} fixed. The true dimensionless parameters of string theory in the plane wave background are $\mu p^+ \alpha'$ and g_s while those of gauge theory are λ' and g_2 as explained before. They are related as follows

$$\lambda' = \frac{1}{(\mu p^+ \alpha')^2}, \qquad g_2 = 4\pi g_s (\mu p^+ \alpha')^2. \tag{1.1.23}$$

In the plane wave background, the Green-Schwarz string action reduces to the following form in the light-cone gauge

$$S = \frac{1}{2\pi\alpha'} \int d\tau \int_0^{\pi\alpha' p^+} d\sigma \left[\frac{1}{2}\dot{z}^2 - \frac{1}{2}z'^2 - \frac{1}{2}\mu^2 z^2 + i\left(\frac{1}{2}S_1 \partial_+ S_1 + \frac{1}{2} S_2 \partial_- S_2 - \mu S_1 \Gamma^{1234} S_2 \right) \right], \tag{1.1.24}$$

where S_i are positive chirality $SO(8)$ spinors. Upon quantization, the Green-Schwarz string gives rise to towers of massive bosonic and fermionic harmonic oscillators, a_n^i and b_n^a $(i, a = 1, \cdots, 8)$ with frequency

$$\omega_n = \sqrt{\mu^2 + \frac{n^2}{(p^+\alpha')^2}}. \tag{1.1.25}$$

The light-cone Hamiltonian is given as

$$H = \sum_{n=0}^{\infty} N_n \omega_n = \mu \sum_{n=0}^{\infty} N_n \sqrt{1 + \frac{n^2}{(\mu p^+ \alpha')^2}}, \tag{1.1.26}$$

where N_n denotes the total occupation number of that mode for both bosonic and fermionic oscillators with frequency ω_n. Notice that we have vanishing zero-point energy due to cancellation between bosons and fermions. Using the identification (1.1.22) and (1.1.23), we can rewrite (1.1.26) in variables better suited for the dual gauge theory,

$$\Delta - J = \sum_{n=0}^{\infty} N_n \sqrt{1 + \lambda' n^2}. \tag{1.1.27}$$

Now our first task is to find a class of gauge theory operators with anomalous dimensions predicted by (1.1.27).

In the next section, we review identification of string states and gauge theory operators in the absence of string interactions, which is done in the original paper [18] by BMN. Those operators are named in the literature as "BMN operators".

1.2 Free strings from BMN operators

Let us briefly review the four-dimensional $\mathcal{N}=4$ SYM. In terms of $\mathcal{N}=1$ superfields, this theory contains one vector superfield V and three chiral superfields Z,Φ,Ψ in the adjoint representation whose lowest components are complex scalar fields named with the same symbols Z,Φ,Ψ. The potential of those complex scalars come from two origins, F-terms and D-terms :

$$\begin{aligned} V_F &= 4g_{YM}^2\left(\left|[Z,\Phi]\right|^2+\left|[Z,\Psi]\right|^2+\left|[\Phi,\Psi]\right|^2\right), \\ V_D &= g_{YM}^2\left([Z,\bar{Z}]+[\Phi,\bar{\Phi}]+[\Psi,\bar{\Psi}]\right)^2. \end{aligned} \tag{1.2.28}$$

We can decompose each complex scalar into two real scalars as

$$\Phi=\frac{\phi_1+i\phi_2}{\sqrt{2}}, \qquad \Psi=\frac{\phi_3+i\phi_4}{\sqrt{2}}, \qquad Z=\frac{\phi_5+i\phi_6}{\sqrt{2}}. \tag{1.2.29}$$

These six scalars actually represent the six transverse directions of the original D3-branes. Without loss of generality, let us set J to be the generator of rotations on the ϕ_5-ϕ_6 plane. Then, Z and $\bar{Z}$ have $U(1)_J$ charge $+1$ and -1, respectively, while the other complex scalars are neutral.

We consider first the gauge theory operator dual to the light-cone ground state $|0,p^+\rangle$. The dual operator must be a chiral primary operator with $\Delta-J=0$. The unique solution to this requirement is $\mathrm{Tr}Z^J$. With a proper normalization factor in the planar limit, we identify

$$|0,p^+\rangle \leftrightarrow \mathcal{O}^J \equiv \frac{1}{\sqrt{JN^J}}\mathrm{Tr}Z^J. \tag{1.2.30}$$

This operator has classical dimension $\Delta=J$ and it does not receive quantum corrections. Hence $\Delta-J=0$ remains true in the full quantum theory as predicted by the duality. Next, consider zero-mode excitations of the ground state, i.e., $a_0^{i\dagger}|0,p^+\rangle$.

The ground state breaks 8 out of 16 bosonic symmetries of the background, and these states are Goldstone bosons corresponding to the broken symmetries. Using the AdS/CFT dictionary, we can find corresponding symmetries in the gauge theory, and the dual operators are obtained by the action of those symmetry generators on $\mathcal{O}^J$. This process is explained in Appendix A and we summarize the result here. The dual operators are obtained by inserting D_iZ or ϕ_i $(i = 1, 2, 3, 4)$ inside the string of J Z's and subsequently symmetrizing over all possible positions, i.e.,

$$a_0^{i\dagger}|0, p^+\rangle \leftrightarrow O_i^J \equiv \frac{1}{\sqrt{JN^J}} \sum_{l=0}^{J-1} \mathrm{Tr}\left(Z^l \frac{\phi_i}{\sqrt{J}} Z^{J-l}\right) = \frac{1}{\sqrt{N^J}} \mathrm{Tr}\left(\phi_i Z^J\right), \tag{1.2.31}$$

similarly for D_iZ insertions. This operator is also a protected operator and its dimension is given as the classical value $J + 1$. It agrees with the fact that the state $a_0^{i\dagger}|0, p^+\rangle$ has $H/\mu = 1$. It is convenient to think of each insertion as carrying a normalization factor $1/\sqrt{J}$ since each insertion generates a sum over J positions. In the last equality of (1.2.31), we use the cyclic property of trace. Here, we are beginning to see the intuitive picture that a Z is representing an unexcited string bit and a ϕ an excitation or impurity on the string. The cyclic property of trace amounts to translational invariance along the string (i.e., σ-direction).

Now, let us think about non-zero modes $a_n^{i\dagger}|0, p^+\rangle$. Since n denotes worldsheet momentum, we assign a worldsheet momentum n to the corresponding impurity meaning that the sum over position l is now weighted by a phase $\exp(2\pi inl/J)$,

$$a_n^{i\dagger}|0, p^+\rangle \leftrightarrow O_{(i,n)}^J \equiv \frac{1}{\sqrt{JN^J}} \sum_{l=0}^{J-1} e^{2\pi inl/J} \mathrm{Tr}\left(Z^l \frac{\phi_i}{\sqrt{J}} Z^{J-l}\right). \tag{1.2.32}$$

In fact, the state $a_n^{i\dagger}|0, p^+\rangle$ is not a physical state since we should impose the level-matching condition $\sum_n nN_n = 0$ coming from the translational invariance along the string. Consistently, the dual operator is identically zero due to the cyclic property of trace,

$$\sum_{l=0}^{J-1} e^{2\pi inl/J} \mathrm{Tr}\left(Z^l \frac{\phi_i}{\sqrt{J}} Z^{J-l}\right) = \mathrm{Tr}\left(\phi_i Z^J\right) \sum_{l=0}^{J-1} e^{2\pi inl/J} = 0. \tag{1.2.33}$$

This is a nice consistency check for the duality. Therefore, we need at least two insertions to talk about non-zero modes. Anyway, this illustrates the general rule of constructing dual operators; for each string oscillator, we insert a corresponding impurity and sum over all possible positions along the string of Z's with a proper phase.

Following the rule, we construct the dual operator of the on-shell string state $a^{i\dagger}_{-n}a^{j\dagger}_{n}|0,p^+\rangle$ as

$$a^{i\dagger}_{-n}a^{j\dagger}_{n}|0,p^+\rangle \leftrightarrow O^J_{ij,n} \equiv \frac{1}{\sqrt{JN^J}}\sum_{l_i=0}^{J-1}\sum_{l_j=0}^{J-1} e^{2\pi in(l_j-l_i)/J}\mathrm{Tr}\left(Z\cdots Z\frac{\phi_i}{\sqrt{J}}Z\cdots Z\frac{\phi_j}{\sqrt{J}}Z\cdots Z\right)$$
$$= \frac{1}{\sqrt{JN^J}}\sum_{l=0}^{J} e^{2\pi inl/J}\mathrm{Tr}\left(\phi_i Z^l \phi_j Z^{J-l}\right), \qquad (1.2.34)$$

where l_i and l_j are the positions of ϕ_i and ϕ_j, respectively. In the last equality, we use the cyclic property again. In general, we can either sum over the positions of all impurities reserving the use of the cyclic property, or we can fix the position of one impurity and sum over the positions of the remaining impurities. We call the two representations of BMN operators as "off-shell"[7] and "on-shell" representaions respectively, which are elaborated in Appendix B.

This identification should be justified by computing all two-point functions among those two-impurity operators and by showing that they diagonalize Δ in the planar limit, because the corresponding string states are eigenstates of the light-cone Hamiltonian in the absence of string interactions. Such an analysis was first performed in [18] by BMN. It is crucial that those operators are nearly BPS and so that we can use a partial nonrenormalization theorem. D-term interaction, gauge boson exchange, and self-energy diagrams cancel among themselves, and we have only to take F-term interaction diagrams into account. The form of F-term interactions and the planar limit allow only interactions between neighboring impurities and thereby imposing locality on the string worldsheet. BMN have obtained the first-order correction to the classical dimension and shown that the above identification is indeed correct.

[7] Here, shell means the level-matching shell.

The second-order analysis is done in [21], and the full square root of the frequency is recovered from the gauge theory in [22] using a clever superconformal symmetry argument.

In the next chapter, we start to explore the string/gauge theory duality at the interaction level.

Chapter 2

Gauge theory description of string interaction in a plane wave background: two impurities

2.1 Introduction

In this chapter, we present our proposal of how gauge theory describes string interactions in a plane wave background, and apply it to the simplest class of string and gauge theory states with two scalar[1] impurities.

The first concrete proposal to this problem was made in [19], which relates these string theory transition amplitudes with three-point functions of BMN operators in the gauge theory. Their proposal is therefore the first attempt to construct a holographic map at the interacting level between string theory in a plane wave and gauge theory. This interesting proposal[2] was put to an explicit test by Spradlin and Volovich in [26], where some Hamiltonian matrix elements were computed using the string field theory vertex constructed previously in [23]. Exact agreement with the proposal given in [19] was reported. Unfortunately both the field theory and string theory computations suffer from errors which, when taken into account, invalidate the proposal. On

[1]By this, we mean ϕ_i $(i = 1, 2, 3, 4)$.

[2]Various aspects of this proposal were considered in [24, 41, 43, 25, 44, 27, 45, 46, 28, 31, 62, 50].

the field theory side, operator mixing is more important than initially contemplated in [20, 19]. Three-point functions of single trace operators with order λ' interactions are not conformally invariant, but the correct form is restored after operator mixing is incorporated, as shown in [47, 48]. On the string field theory side, the prefactor acting on the delta functional overlap of three strings that enters the Hamiltonian H_3 [23] has a minus sign error, which was first reported by Pankiewicz [30]. Once matrix elements are recomputed with the corrected Hamiltonian H_3, which we calculate in Section 2.2, agreement with field theory is lost. Therefore, what is the correct holographic map between string theory in the plane wave [15] and $\mathcal{N} = 4$ SYM at the interacting level?

The most straightforward way to proceed, which was first advocated in a paper by Gross, Mikhailov and Roiban [21], is to take the identification (1.0.9) between the string field theory Hamiltonian H and the generator of scale transformations $\Delta - J$ in $\mathcal{N} = 4$ SYM as the holographic map for all g_2. This holographic map therefore identifies Hamiltonian matrix elements in string field theory with those of the dilatation operator in gauge theory. In order to test this identification one must find an explicit map between the Hilbert space of string states and the Hilbert space of states in the gauge theory. Moreover, in order for the comparison to be meaningful one must compute the matrix elements of these operators in a basis in which the Hilbert space inner product in gauge theory is the same as that in string field theory. The obvious inner product in the Hilbert space of string theory is the familiar inner product where, for example, the one-string states are orthogonal to two-string states.

In gauge theory, the Hilbert space inner product is induced by the matrix of two-point functions of BMN operators[3]

$$|x|^{2\Delta_0}\langle O_A \bar{O}_B\rangle = G_{AB} + \Gamma_{AB}\ln(x^2\Lambda^2)^{-1}, \tag{2.1.1}$$

where G_{AB} is the Hilbert space inner product and Γ_{AB} is the matrix of anomalous dimensions. Unlike with the usual Hilbert space inner product in string field theory,

[3]In this formula the various rows and columns describe single trace, double trace, etc components. A more complete characterization of this matrix is given in Section 2.3.

which remains diagonal to all orders in g_2, perturbative corrections in gauge theory induce operator mixing at each order in g_2 in perturbation theory and the Hilbert space inner product is no longer diagonal. Direct comparison with string field theory calculations requires correcting for operator mixing systematically, order by order in the g_2, expansion by making G_{AB} orthonormal via a change of basis.

In order to relate string theory to gauge theory calculations via (1.0.9) we must calculate the matrix elements of the dilatation operator. It is straightforward to show that the matrix elements of the dilatation operator between states created by BMN operators are given by the matrix of anomalous dimensions[4]

$$\langle O_A|(\Delta - J)|O_B\rangle = (\Delta^0 - J)G_{AB} + \Gamma_{AB} = nG_{AB} + \Gamma_{AB}, \tag{2.1.2}$$

where n is the number of impurities. Comparison of matrix elements of H and Δ requires first making the gauge theory inner product orthonormal order by order in perturbation theory. We can accomplish this by finding a new basis of operators $\tilde{O}_A = U_{AB}O_B$ such that they are orthonormal

$$UGU^\dagger = 1. \tag{2.1.3}$$

When $g_2 = 0$, the correct identification between string states and gauge theory operators was given by BMN. Namely, an n-string state is described by an n-trace operator. Once g_2 corrections are taken into account this identification has to be modified. Therefore, the precise mapping between string field theory states $|s_A\rangle$ and gauge theory states $|\tilde{O}_A\rangle$ when $g_2 \neq 0$ is given by[5]

$$|s_A\rangle \to |\tilde{O}_A\rangle = U_{AB}|O_B\rangle, \qquad \langle s_A|s_B\rangle = \langle\tilde{O}_A|\tilde{O}_B\rangle = \delta_{AB}. \tag{2.1.4}$$

[4]Since BMN operators are BPS or nearly BPS, the contribution from the action of Δ coming from the bare dimension of the operator is almost cancelled by the contribution from the R-charge and thereby giving n in (2.1.2).

[5]In light-cone string field theory the canonical normalization of states is the usual delta function normalization $\langle s'_A|s'_B\rangle = p^+_A\delta(p^+_A - p^+_B) = J_A\delta_{J_A,J_B}$, so that $|s'_A\rangle = \sqrt{J_A}|s_A\rangle$. Therefore, when comparing string field theory results with gauge theory results we will have to take into account this normalization factor, since gauge theory states have unit norm [57, 19].

In Section 2.3 we give an expression for the change of basis to order g_2^2. Once the right basis is found, the holographic map reads

$$\frac{1}{\mu}\langle s_A|H|s_B\rangle = \langle\tilde{O}_A|(\Delta - J)|\tilde{O}_B\rangle = n\delta_{AB} + (U\Gamma U^\dagger)_{AB}. \tag{2.1.5}$$

An important subtlety in the holographic map (2.1.5) is that the orthonormalization procedure of the gauge theory inner product is not unique, namely, the transformation matrix U that makes G_{AB} orthonormal is not unique[6]. We uniquely fix the form U to order g_2 by demanding that the dilatation operator matrix elements agree with the matrix elements of the corrected string field theory Hamiltonian that we calculate in Section 2.2. We then evaluate U to order g_2^2 which via (2.1.5) makes a non-trivial prediction for string field theory Hamiltonian matrix elements to order g_2^2 which have not yet been evaluated. Remarkably, this purely gauge theory result we present is reproduced by the order g_2^2 string field theory contact term calculation in [35].[7]

The rest of the chapter is organized as follows. In Section 2.2 we revisit the string field theory Hamiltonian H_3 and point out that there is an incorrect relative minus sign in [26][8]. We recompute Hamiltonian matrix elements with the corrected sign, which we will use in Section 2.3. In Section 2.3 we fix the form of the change of basis in gauge theory to order g_2 by demanding that (2.1.5) holds when the string field theory matrix elements in Section 2.2 are used. We also evaluate the dilatation operator matrix elements to order g_2^2 by making a particular choice of basis to order g_2^2 which allows us to make a calculable prediction by using (2.1.5) about the string field theory Hamiltonian matrix elements to that order. We note that the final answer agrees with the recent string field theory result in [35]. A discussion is added in Section 2.4.

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[6]This ambiguity was first pointed out in [21].

[7]It also agrees with the order g_2^2 string bit Hamiltonian [36, 38] matrix elements computed in [39].

[8]This sign has been corrected in a recent revision.

2.2 SFT computation revisited

Before going into our gauge theory computation, let us perform the correct string field theory calculation that we will compare it to. This also corrects some errors in the previous literature. In string field theory, the Hilbert space is a direct product of ℓ-string states,

$$\mathcal{H} = \oplus_\ell \mathcal{H}_\ell. \tag{2.2.6}$$

All these states are orthogonal with respect to each other and they have an orthonormal inner product. The full Hamiltonian H, representing infinitesimal evolution along x^+, can be expanded in g_2 :

$$H = H_2 + g_2 H_3 + g_2^2 H_2' \cdots . \tag{2.2.7}$$

In the plane wave background the freely propagating part H_2 is simply the energy of an infinite collection of harmonic oscillators α_n^i. The three-string interaction part H_3 is the leading interaction coupling an n-string state to an $(n \pm 1)$-string state, and H_2' is a contact term. Following the flat space results in [67, 66] the plane wave vertex H_3 has been studied in [23, 26, 30, 32, 33]. The properly normalized cubic interaction term in the case of purely bosonic excitations along $\mathbf{R}^4$ in the exponential (BMN) basis of oscillators is given by[9]

$$\frac{1}{\mu}|H_3\rangle = -\frac{y(1-y)}{2} P|V\rangle, \tag{2.2.8}$$

where $p^+_{(r)}$ is the length of string r and P is the prefactor

$$P = \sum_{r=1}^{3} \sum_{n=-\infty}^{\infty} \frac{\omega_{n(r)}}{\mu p^+_{(r)} \alpha'} \alpha^{i\dagger}_{n(r)} \alpha^i_{-n(r)}, \tag{2.2.9}$$

[9]We take without loss of generality $\alpha' p^+_{(3)} = -1, \alpha' p^+_{(1)} = y$ and $\alpha' p^+_{(2)} = 1 - y$, where $0 < y < 1$. Therefore, $\lambda' = 1/\mu^2$. The large μ normalization was fixed in [39, 60] by comparison with a field theory amplitude.

with $\omega_{n(r)} = \sqrt{(\mu p^+_{(r)}\alpha')^2 + n^2}$ and[10]

$$|V\rangle = \exp\Big(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\alpha^{i\dagger}_{m(r)}\tilde{N}^{(rs)}_{mn}\alpha^{i\dagger}_{n(s)}\Big)|\text{vac}\rangle. \tag{2.2.10}$$

$|V\rangle$ represents the delta functional overlap of three strings in the oscillator basis, and the prefactor P is necessary to appropriately realize the supersymmetry algebra.

The string states, which are dual to a certain class of two impurity BMN operators when $g_2 = 0$, are given by

$$|n\rangle = \alpha^{1\dagger}_n\alpha^{2\dagger}_{-n}|0,1\rangle \tag{2.2.11}$$

$$|n,y\rangle\rangle = \alpha^{1\dagger}_n\alpha^{2\dagger}_{-n}|0,y\rangle\otimes|0,1-y\rangle \tag{2.2.12}$$

$$|y\rangle\rangle = \alpha^{1\dagger}_0|0,y\rangle\otimes\alpha^{2\dagger}_0|0,1-y\rangle, \tag{2.2.13}$$

where the first state represents a single-string and the other two represent two-string states. Here $|0,y\rangle$ is the one-string vacuum state carrying a fraction $0<y<1$ of the total longitudinal momentum p^+ of the multi-string state. The Hamiltonian matrix elements of these states are given by

$$\begin{aligned}
\langle n|H_3|m,y\rangle\rangle &\sim (F^+_{(1)|m|}F^+_{(3)|n|} + F^-_{(1)|m|}F^-_{(3)|n|})(\bar{N}^{(13)}_{|m|,|n|} - \bar{N}^{(13)}_{-|m|,-|n|}), \quad \text{if } mn>0\\
\langle n|H_3|m,y\rangle\rangle &\sim (F^+_{(1)|m|}F^+_{(3)|n|} - F^-_{(1)|m|}F^-_{(3)|n|})(\bar{N}^{(13)}_{|m|,|n|} + \bar{N}^{(13)}_{-|m|,-|n|}), \quad \text{if } mn<0\\
\langle n|H_3|y\rangle\rangle &\sim F^+_{(3)|n|}(F^+_{(1)0}\bar{N}^{(23)}_{0,|n|} + F^+_{(2)0}\bar{N}^{(13)}_{0,|n|}), \quad \text{for } \forall n\neq 0
\end{aligned} \tag{2.2.14}$$

where $F_{m(r)}$ comes from the prefactor and the Neumann matrices $\bar{N}^{(rs)}_{mn}$ come from the delta functional overlap. The negative modes are related to the positive modes by (here $m,n>0$)

$$\bar{N}^{(rs)}_{-m,-n} = -(U_{(r)}\bar{N}^{(rs)}U_{(s)})_{m,n}, \tag{2.2.15}$$

$$F^-_{(r)m} = i(U_{(r)}F^+_{(r)})_m, \tag{2.2.16}$$

with $(U_{(r)})_{m,n} = \delta_{m,n}\Big(\sqrt{m^2+(\mu p^+_{(r)}\alpha')^2} - \mu p^+_{(r)}\alpha'\Big)/m$. Note that in (2.2.16) we have an extra factor of i compared to the original literature [26][11], which was first pointed

[10] Here we omit the overall p^+ conservation factor, $|p^+_{(3)}|\delta(p^+_{(1)}+p^+_{(2)}+p^+_{(3)})$.

[11] More explicitly, formulas (3.15) and (3.21) in the original version of [26] should have an extra factor of i. This invalidates the evaluation of (4.4) [26].

out in [30]. Using (2.2.15) and (2.2.16), we can show that both (2.2.14) and (2.2.14) reduce to the same expression.

In order to compare these answers with perturbative gauge theory we must analyze the $\mu \to \infty$ limit of (2.2.14), (2.2.14) and (2.2.14). The $\mu \to \infty$ behavior of both the prefactor and the Neumann matrices was evaluated in [26]. In the $\mu \to \infty$ limit, (2.2.14), (2.2.14) and (2.2.14) yield[12]

$$\frac{1}{\mu}\langle n|H_3|m,y\rangle\rangle = \frac{\lambda'}{2\pi^2}(1-y)\sin^2(n\pi y), \tag{2.2.17}$$

$$\frac{1}{\mu}\langle n|H_3|y\rangle\rangle = -\frac{\lambda'}{2\pi^2}\sqrt{y(1-y)}\sin^2(n\pi y). \tag{2.2.18}$$

These corrected results invalidate the agreement previously found in the literature[13]. In the next section we will use these results to test the holographic proposal in (2.1.5) and will show that agreement is found for a particular choice of basis.

2.3 Orthonormalization and comparison with SFT

In this section, we make a change of operator basis such that the gauge theory inner product is orthonormal order by order in g_2. We then compute the matrix elements of the operator $\Delta - J$ in this basis. An important subtlety in this procedure is that the basis change that makes the Hilbert space inner product G orthonormal is not unique. The leading term at $g_2 = 0$ is uniquely fixed by the original correspondence explained in [18] between single/double-string states and single/double-trace operators. When $g_2 \neq 0$ the field theory operators start mixing and the matrix with which we make the inner product orthonormal is not unique, due to the familiar ambiguity when diagonalizing a matrix. We propose to fix this ambiguity in the change of basis by taking seriously the proposal in (2.1.5) and demanding that exact agreement with

[12] The precise overall numerical factor of the cubic string field theory Hamiltonian is not known. It is fixed by comparing with the gauge theory calculation in the next section.

[13] Also due to this extra i, the result proven in [31] that the prefactor reduces to energy difference between incoming and outgoing states should be modified. Instead it reduces to the energy difference in cos modes minus that in sin modes that appear in the worldsheet Fourier decomposition.

the corrected string field theory results computed in the previous section is obtained. Exact agreement is obtained for a unique choice of basis. This particular choice is the unique choice for which the transformation matrix is hermitian to order g_2. We then proceed to calculate to next-to-leading order, i.e., to order g_2^2. Here there is also an ambiguity in the orthonormalization procedure. If we make the strong assumption that the transformation matrix is still hermitian to this order, we can uniquely determine the transformation matrix to order g_2^2.[14] Given these assumptions we can calculate explicitly the order g_2^2 matrix element between the orthonormal basis states which reduce when $g_2 = 0$ to single trace operators. This result gives a prediction using the proposal (2.1.5) for light-cone string field theory matrix elements to order g_2^2 involving single-string states, which has been confirmed in [35].[15]

The particular set of gauge theory operators that we are interested in are the following single trace and double trace operators[16]

$$\mathcal{O}^J = \frac{1}{\sqrt{JN^J}}\mathrm{Tr}Z^J, \tag{2.3.19}$$

$$\mathcal{O}^J_{(i)} = \frac{1}{\sqrt{N^{J+1}}}\mathrm{Tr}\left(\phi_i Z^J\right), \tag{2.3.20}$$

$$\mathcal{O}^J_n = \frac{1}{\sqrt{JN^{J+2}}}\sum_{l=0}^{J} e^{2\pi i l n/J}\mathrm{Tr}\left(\phi_1 Z^l \phi_2 Z^{J-l}\right), \tag{2.3.21}$$

$$\mathcal{T}^{J,y}_p = :\mathcal{O}^{y\cdot J}_p \mathcal{O}^{(1-y)\cdot J}:, \tag{2.3.22}$$

$$\mathcal{T}^{J,y} = :\mathcal{O}^{y\cdot J}_{(1)} \mathcal{O}^{(1-y)\cdot J}_{(2)}:. \tag{2.3.23}$$

We need to make the inner product G appearing in the matrix of two-point functions in (2.1.1) orthonormal and eventually compute the matrix elements of Γ in the orthonormal basis (2.1.5). Both matrices G_{AB} and Γ_{AB} have a systematic expansion in

[14] This choice is compatible with the proposal in [39].

[15] the matrix elements we predict from the gauge theory computation also match the order g_2^2 matrix elements computed in [39] using the string bit formalism [36, 38]. It would be very desirable to understand more precisely the relation between light-cone string field theory and the string bit formalism.

[16] Here we are using the notation in [47].

powers of g_2

$$G = \mathbf{1} + g_2 G^{(1)} + g_2^2 G^{(2)} + \mathcal{O}(g_2^3), \tag{2.3.24}$$

$$\Gamma = \Gamma^{(0)} + g_2 \Gamma^{(1)} + g_2^2 \Gamma^{(2)} + \mathcal{O}(g_2^3). \tag{2.3.25}$$

In the following we split these matrices into 3×3 blocks representing matrix elements involving $\mathcal{O}_n^J$, $\mathcal{T}_p^{J,y}$ and $\mathcal{T}^{J,y}$, respectively. The indices $(n, m, \cdots)$ denote the worldsheet momentum of the single trace BMN operators like, for example, $\mathcal{O}_n^J$. The double indices $(py, qz, \cdots)$ represent for example the worldsheet momentum and light-cone momentum fraction of the double trace operators $\mathcal{T}_p^{J,y}$, while $(y, z, \cdots)$ represent the fraction of momentum carried by the operator $\mathcal{T}^{J,y}$. These matrices have been computed in [47, 48]. They are given by :

$$G = \mathbf{1} + g_2 \begin{pmatrix} 0 & C_{n,qz} & C_{n,z} \\ C_{py,m} & 0 & 0 \\ C_{y,m} & 0 & 0 \end{pmatrix} + g_2^2 \begin{pmatrix} M^1_{n,m} & 0 & 0 \\ 0 & \langle ? \rangle & \langle ? \rangle \\ 0 & \langle ? \rangle & \langle ? \rangle \end{pmatrix}, \tag{2.3.26}$$

$$\begin{aligned} \frac{\Gamma}{\lambda'} = & \begin{pmatrix} n^2 \delta_{n,m} & 0 & 0 \\ 0 & \frac{p^2}{y^2}\delta_{p,q}\delta_{y,z} & 0 \\ 0 & 0 & 0 \end{pmatrix} + g_2 \begin{pmatrix} 0 & \Gamma^{(1)}_{n,qz} & \Gamma^{(1)}_{n,z} \\ \Gamma^{(1)}_{py,m} & 0 & 0 \\ \Gamma^{(1)}_{y,m} & 0 & 0 \end{pmatrix} \\ + \ & g_2^2 \begin{pmatrix} nmM^1_{n,m} + \frac{1}{8\pi^2}\mathcal{D}^1_{n,m} & 0 & 0 \\ 0 & \langle ? \rangle & \langle ? \rangle \\ 0 & \langle ? \rangle & \langle ? \rangle \end{pmatrix}. \end{aligned} \tag{2.3.27}$$

The explicit form of the matrix elements are summarized in Appendix C and we denote by $\langle ? \rangle$ matrix elements that have not yet been computed. Luckily, we will not need them for our computations. Finding them is, however, an important enterprise since via the holographic map (2.1.5) they predict yet unknown matrix elements in string field theory, like the order g_2^2 Hamiltonian matrix element of a two-string state[17].

[17] One should also compute, however, the mixing between double and triple trace operators to get this result.

Let us now apply a linear transformation U to make the inner product orthonormal. We require that

$$UGU^{\dagger} = \mathbf{1} \tag{2.3.28}$$

and solve this equation order by order. We can expand U in a power series in g_2 and express it as

$$U = \mathbf{1} + g_2 U^{(1)} + g_2^2 U^{(2)} + \mathcal{O}(g_2^3). \tag{2.3.29}$$

As explained in the beginning of this section, we restrict our attention to hermitian matrices. This is motivated in part by the fact that the hermitian choice uniquely leads to exact agreement with string field theory via (2.1.5) to order g_2 as we will see below. Clearly, having a better understanding of why this choice works is very desirable. Therefore, by assuming that U is a hermitian matrix, we need to solve (2.3.28). Solving this equation order by order we get

$$U^{(1)} = -\frac{1}{2} G^{(1)}, \tag{2.3.30}$$

$$U^{(2)} = -\frac{1}{2} G^{(2)} + \frac{3}{8} (G^{(1)})^2. \tag{2.3.31}$$

In the new orthonormal basis, Γ is transformed to

$$\tilde{\Gamma} = U \Gamma U^{\dagger}. \tag{2.3.32}$$

We can determine Γ order by order in g_2 by expanding

$$\tilde{\Gamma} = \tilde{\Gamma}^{(0)} + g_2 \tilde{\Gamma}^{(1)} + g_2^2 \tilde{\Gamma}^{(2)} + \mathcal{O}(g_2^3). \tag{2.3.33}$$

The matrix of anomalous dimensions in the new basis is therefore

$$\tilde{\Gamma}^{(0)} = \Gamma^{(0)}, \tag{2.3.34}$$

$$\tilde{\Gamma}^{(1)} = \Gamma^{(1)} - \frac{1}{2}\{G^{(1)}, \Gamma^{(0)}\}, \tag{2.3.35}$$

$$\tilde{\Gamma}^{(2)} = \Gamma^{(2)} - \frac{1}{2}\{G^{(2)}, \Gamma^{(0)}\} - \frac{1}{2}\{G^{(1)}, \Gamma^{(1)}\} + \frac{3}{8}\{(G^{(1)})^2, \Gamma^{(0)}\} + \frac{1}{4} G^{(1)} \Gamma^{(0)} G^{(1)}. \tag{2.3.36}$$

Using (2.3.26) and (2.3.27) we can evaluate (2.3.35) to be

$$\tilde{\Gamma}^{(1)} = \begin{pmatrix} 0 & \tilde{\Gamma}^{(1)}_{n,qz} & \tilde{\Gamma}^{(1)}_{n,z} \\ \tilde{\Gamma}^{(1)}_{py,m} & 0 & 0 \\ \tilde{\Gamma}^{(1)}_{y,m} & 0 & 0 \end{pmatrix}, \tag{2.3.37}$$

where

$$\tilde{\Gamma}^{(1)}_{n,py} = \tilde{\Gamma}^{(1)}_{py,n} = \lambda' \frac{\sqrt{1-y}}{\sqrt{Jy}} \frac{\sin^2(\pi n y)}{2\pi^2}, \tag{2.3.38}$$

$$\tilde{\Gamma}^{(1)}_{n,y} = \tilde{\Gamma}^{(1)}_{y,n} = -\lambda' \frac{1}{\sqrt{J}} \frac{\sin^2(\pi n y)}{2\pi^2}. \tag{2.3.39}$$

We note that after using the proposed holographic map (2.1.5) that the gauge theory results (2.3.38), (2.3.39) match with the string field theory results (2.2.17), (2.2.18)[18].

We can now make a prediction about the order g_2^2 matrix elements in string field theory by using (2.1.5). In order to do that we must calculate the matrix of anomalous dimensions in the new basis to order g_2^2. By using (2.3.26), (2.3.27) we can perform the sums in (2.3.36) to get[19]

$$\tilde{\Gamma}^{(2)} = \begin{pmatrix} \tilde{\Gamma}^{(2)}_{n,m} & 0 & 0 \\ 0 & \langle ? \rangle & \langle ? \rangle \\ 0 & \langle ? \rangle & \langle ? \rangle \end{pmatrix}, \tag{2.3.40}$$

where[20]

$$\tilde{\Gamma}^{(2)}_{n,m} = \begin{cases} \frac{\lambda'}{32\pi^4}\left(\frac{3}{nm} + \frac{1}{(n-m)^2}\right) & \text{if } n \neq m, -m \\ \frac{\lambda'}{16\pi^2}\left(\frac{1}{3} + \frac{5}{2\pi^2 n^2}\right) & \text{if } n = m \\ -\frac{15\lambda'}{128\pi^4 n^2} & \text{if } n = -m \end{cases}, \tag{2.3.41}$$

[18] As explained in footnote 5, in order to compare the string field theory answer with the gauge theory answer, one must divide the string result by $\sqrt{Jy(1-y)}$ so that both string field theory states and gauge theory states have unit norm.

[19] The formulas we need to compute the required sums are summarized in Appendix D.

[20] Numerically, this expression is the same as the one in [19] for the non-nearest neighbor genus 1 single-trace two-point function.

and $\langle ? \rangle$ are quantities that we cannot determine since the full matrix of two-point functions has not been computed to order g_2^2. We can nevertheless make the following prediction

$$\frac{1}{\mu}\langle n|H|m\rangle\Big|_{g_2^2} = \tilde{\Gamma}^{(2)}_{n,m}. \tag{2.3.42}$$

The corresponding string field theory computation of order g_2^2 contact term was recently done in [35], and it precisely agrees with our prediction (2.3.42).

2.4 Discussion

In this chapter we have studied the gauge theory realization of string field theory Hamiltonian matrix elements. The answer, which was already anticipated in [21], is that these matrix elements correspond to matrix elements of the dilatation operator. Using the corrected string field theory results in Section 2.2 we find a preferred basis of states which yields agreement between gauge theory and string theory calculations. Moreover, we make a prediction using a gauge theory computation for the Hamiltonian matrix elements of single-string states to order g_2^2, which precisely agrees with a recent string field theory computation of order g_2^2 contact term in [35] [21]. An outcome of the corrected string field theory calculation in Section 2.2 is that the proposal of [19] no longer holds. This paper gives evidence that the correct correspondence is between matrix elements of the string Hamiltonian and the dilatation operator in the gauge theory. This proposal suggests that the only observables that can be holographically computed in the plane wave string theory are gauge theory two-point functions [36, 21]. Nevertheless, operator mixing between multi-trace operators contains information about higher point functions in gauge theory. In string field theory, the Hamiltonian matrix elements compute the matrix of anomalous dimensions in the orthonormal basis via (2.1.5), which can be read from the gauge theory two-point functions.

The mapping between the string field theory and gauge theory Hilbert spaces is non-unique. By comparing the calculation of the dilatation operator matrix elements

[21] It also agrees with the recent string bit [36, 38] Hamiltonian calculation presented in [39].

with the corrected string field theory Hamiltonian matrix elements in Section 2.2 we fixed the ambiguity, which picks a particular basis of states in the gauge theory to be identified with string states. The unique transformation matrix U is *hermitian.* Notice that just assuming U is hermitian is enough to find the unique gauge theory basis. However, it would be very desirable to have a first principle explanation of why this choice is the correct one. In [39], the authors motivate this choice by proving that the string bit Hamiltonian simplifies in this basis, that is, it truncates at finite order in g_2. It would be desirable to understand the uniqueness of the basis choice more directly. In this choice of basis the gauge theory computation we present in Section 2.3 exactly agrees with the calculation [39] performed at order g_2^2 using the string bit [36, 38] Hamiltonian.

A fascinating open problem is to understand more precisely the relation between light-cone string field theory and the string bit formalism. The advantage of the string bit formalism is, as shown in [39], that the Hamiltonian truncates at order g_2^2. This however seems to raise a puzzle. The Hamiltonian matrix elements truncate at order g_2^2, which via the map (2.1.5) predicts that matrix elements of the dilatation operator in some basis truncate at order g_2^2 even though the Hilbert space inner product G_{AB} and the matrix of anomalous dimensions Γ_{AB} have corrections to all orders in g_2. It would be very interesting to study this prediction in detail.

The light-cone string field theory and the string bit model have some complementary features. In the string bit formalism it is easier to compute the g_2^2 corrections, while the same problem is notoriously difficult in light-cone string field theory. On the other hand, in light-cone string field theory we can systematically evaluate $1/\mu$ corrections, which are hard to obtain in the string bit model. Computing these corrections is crucial in extending the duality beyond leading order in λ' in the gauge theory. In particular, the results in (2.2.14) make non-trivial predictions about λ'-corrections to the matrix of anomalous dimensions via (2.1.5). The complete string field theory formulas to all orders in $1/\mu$ is recently found in [33] using the factorization theorems [29, 30]. It would be very desirable to compute these corrections directly in gauge theory.

Chapter 3

Generalization to abitrary impurities

3.1 Introduction

In the previous chapter and also in [39, 60, 21], a basis of operators in $\mathcal{N} = 4$ SYM was found such that the $\mathcal{O}(g_2)$ matrix elements of the string Hamiltonian were reproduced using (1.0.9) from gauge theory computations. The analysis in the previous chapter and in [39, 60, 21] was restricted to string states with two different scalar impurities along an $\mathbf{R}^4$ plane in the transverse $\mathbf{R}^8$ directions of the plane wave.

In this chapter we compute the $\mathcal{O}(g_2)$ and $\mathcal{O}(g_2^2)$ Hamiltonian matrix elements for string states with two *identical* scalar impurities along $\mathbf{R}^4$ and reproduce them from gauge theory computations. We find that the matrix elements of the dilatation operator in the basis described in Chapter 2 exactly reproduce the string theory answer. When considering string states with identical impurities we find that there are new classes of Feynman diagrams that contribute to the string theory and gauge theory computations. In this work we find a direct connection between the Feynman diagrams that appear in the string calculation and the Feynman diagrams that contribute to the gauge theory matrix elements. Roughly, the action of the prefactor in string field theory is captured by the interaction vertex in gauge theory while the Neumann matrices are captured by the sum over all free contractions in gauge theory. This correspondence could be an important step in deriving the duality. We then compute the $\mathcal{O}(g_2)$ Hamiltonian matrix elements for string states with an *arbitrary* number of impurities along $\mathbf{R}^4$ and exactly reproduce them using gauge theory using (1.0.9) and the basis of states in Chapter 2, after identifying gauge theory Feynman diagrams with corresponding diagrams in string theory. These results give strong supporting evidence of the holographic map (1.0.9) and of the basis of gauge theory states proposed in Chapter 2 as a dual description of string states.

The rest of the chapter is organized as follows. In Section 3.2 we consider the string states and gauge theory operators with two identical scalar impurities. We perform computations up to $\mathcal{O}(g_2^2)$ of the string Hamiltonian matrix elements, emphasizing the extra diagrams that contribute beyond those that appear when considering string states with two different scalar impurities. Using the basis change proposed in Chapter 2, we exactly reproduce the string theory results from a gauge theory analysis. In Section 3.3 we show equivalence between string theory and gauge theory computations for arbitrary string states by identifying string theory Feynman diagrams with gauge theory Feynman diagrams. Appendix J, which is outside the main focus of the paper, contains the $\mathcal{O}(g_2)$ calculation of a two-impurity p-string state transition into a $p+1$-string state. We find precise agreement with the gauge theory calculation in [52] once we change to the basis in Chapter 2.

3.2 Correspondence in two impurity singlet sector

In this section, we study string states and BMN operators with two real scalar impurities along the same direction in $\mathbf{R}^4$. Since $SO(4)$ is a symmetry, we can decompose two scalar impurity states into $\mathbf{4}\otimes\mathbf{4}=\mathbf{1}\oplus\mathbf{6}\oplus\mathbf{9}$ irreducible representations of $SO(4)$, with two repeated impurities belonging to the singlet. We will consider states with two impurities in one direction $i\in\{1,2,3,4\}$ instead of looking at the singlet state and later on extend the analysis to arbitrary number of impurities.

The single-string states we will consider are given by (no sum over i):

$$|ii,n\rangle \;=\; \alpha_n^{i\dagger}\alpha_{-n}^{i\dagger}|\mathrm{vac}\rangle,$$

$$|ii,0\rangle = \frac{1}{\sqrt{2}}\alpha_0^{i\dagger}\alpha_0^{i\dagger}|\text{vac}\rangle. \tag{3.2.1}$$

As shown by [64, 47], the corresponding gauge theory operators when $g_2 = 0$ are given respectively by

$$\begin{aligned}
\mathcal{O}_{ii,n}^J &= \frac{1}{\sqrt{JN^{J+2}}}\left(\sum_{l=0}^{J} e^{2\pi i l n/J}\mathrm{Tr}\left(\phi_i Z^l \phi_i Z^{J-l}\right) - \mathrm{Tr}\left(\bar{Z}Z^{J+1}\right)\right), \\
\mathcal{O}_{ii,0}^J &= \frac{1}{\sqrt{2JN^{J+2}}}\left(\sum_{l=0}^{J} \mathrm{Tr}\left(\phi_i Z^l \phi_i Z^{J-l}\right) - \mathrm{Tr}\left(\bar{Z}Z^{J+1}\right)\right),
\end{aligned} \tag{3.2.2}$$

without summing over i. The extra contribution involving $\bar{Z}$ is crucial [64, 47] for the existence of the BMN limit, where $N, J \to \infty$, with $g,\ g_2 = J^2/N$ and $\lambda' = g^2N/J^2$ fixed and as we will see leads to interesting new effects.

The interaction term H_3 couples single-string states to two-string states. These are given by

$$\begin{aligned}
|ii,m,y\rangle\rangle &= \alpha_m^{i\dagger}\alpha_{-m}^{i\dagger}|\text{vac},y\rangle \otimes |\text{vac},1-y\rangle, \\
|ii,0,y\rangle\rangle &= \frac{1}{\sqrt{2}}\alpha_0^{i\dagger}\alpha_0^{i\dagger}|\text{vac},y\rangle \otimes |\text{vac},1-y\rangle, \\
|ii,y\rangle\rangle &= \alpha_0^{i\dagger}|\text{vac},y\rangle \otimes \alpha_0^{i\dagger}|\text{vac},1-y\rangle,
\end{aligned} \tag{3.2.3}$$

where $0 < y < 1$ is the fraction of the total momentum carried by the first string in the two-string state. These states are represented when $g_2 = 0$ by the following gauge theory operators

$$\begin{aligned}
\mathcal{T}_{ii,m}^{J,y} &= :\mathcal{O}_{ii,m}^{y\cdot J}\cdot\mathcal{O}^{(1-y)\cdot J}:, \\
\mathcal{T}_{ii}^{J,y} &= :\mathcal{O}_{i}^{y\cdot J}\cdot\mathcal{O}_{i}^{(1-y)\cdot J}:,
\end{aligned} \tag{3.2.4}$$

where $y = J_1/J$ and $1-y = J_2/J$ and

$$\begin{aligned}
\mathcal{O}^J &= \frac{1}{\sqrt{JN^J}}\mathrm{Tr}\left(Z^J\right), \\
\mathcal{O}_i^J &= \frac{1}{\sqrt{N^{J+1}}}\mathrm{Tr}\left(\phi_i Z^J\right).
\end{aligned} \tag{3.2.5}$$

We now proceed to describe string interactions among these states using string field theory and reproduce the results from a gauge theory analysis.

3.2.1 SFT computations

• *The $\mathcal{O}(g_2)$ Computation*

We now compute the matrix elements between single-string and two-string states. It is convenient to introduce Feynman rules to evaluate these amplitudes, specially in later sections when we consider arbitrary impurities. They are given by

$$(r,m) \text{ ——— } (s,n) \quad \Longleftrightarrow \quad \tilde{N}^{(rs)}_{m,n},$$

$$(r,m) \text{ —×— } (s,n) \quad \Longleftrightarrow \quad \left[\frac{\omega_{m(r)}}{\mu p^+_{(r)}\alpha'} + \frac{\omega_{n(s)}}{\mu p^+_{(s)}\alpha'}\right]\tilde{N}^{(rs)}_{m,-n}, \quad (3.2.6)$$

where $r, s \in \{1,2,3\}$ label the string and m, n label the worldsheet momentum of the oscillator. Then, the Neumann matrix $\tilde{N}^{(rs)}_{m,n}$ introduced in (2.2.10) is the propagator between oscillators $\alpha_{m(r)}$ and $\alpha_{n(s)}$. We can eliminate the prefactor P in (2.2.8) by sequentially commuting it through the external states oscillators, which has the effect of reversing the sign of the worldsheet momentum of the oscillator which P is acting on. After elimination of the prefactor, we are left with contractions between external states oscillators. The $\times$ symbol in the vertex (3.2.6) signifies the total effect of commuting the prefactor P in (2.2.8) through both oscillators and their contraction.

Using these Feynman rules and the following symmetry relations satisfied by the Neumann matrices

$$\tilde{N}^{(rs)}_{m,n} = \tilde{N}^{(sr)}_{n,m}, \qquad \tilde{N}^{(rs)}_{m,n} = \tilde{N}^{(rs)}_{-m,-n}, \tag{3.2.7}$$

we can now evaluate any Hamiltonian matrix element using combinatorics of Feynman diagrams. In the case of two identical impurities, the amplitudes are given by

$$\begin{aligned} \frac{1}{\mu}\langle ii,n|H_3|jj,m,y\rangle\rangle &= -\frac{y(1-y)}{2}\Big[\delta_{ij}4\tilde{N}^{(13)}_{m,n}\tilde{N}^{(13)}_{m,-n}\Big(\frac{\omega_{m(1)}}{\mu y} - \frac{\omega_{n(3)}}{\mu}\Big) \\ &\qquad +2\tilde{N}^{(33)}_{n,-n}\tilde{N}^{(11)}_{m,m}\frac{\omega_{m(1)}}{\mu y} - 2\tilde{N}^{(33)}_{n,n}\tilde{N}^{(11)}_{m,-m}\frac{\omega_{n(3)}}{\mu}\Big], \end{aligned}$$

$$\frac{1}{\mu}\langle ii,n|H_3|jj,y\rangle\rangle = -\frac{y(1-y)}{2}\Big[\delta_{ij}4\tilde{N}^{(13)}_{0,n}\tilde{N}^{(23)}_{0,n}\Big(1 - \frac{\omega_{n(3)}}{\mu}\Big)$$

$$+2\tilde{N}^{(33)}_{n,-n}\tilde{N}^{(12)}_{0,0} - 2\tilde{N}^{(33)}_{n,n}\tilde{N}^{(12)}_{0,0}\frac{\omega_{n(3)}}{\mu}\Big]. \qquad (3.2.8)$$

We note that there are Feynman diagrams in which the identical impurities in a given string are connected via Neumann matrices involving only that string. Such contributions are absent when considering strings with different impurities due to the $SO(8)$ invariance of the Neumann matrices. We can evaluate the expression in the large[1] μ limit, which corresponds to the perturbative gauge theory regime. Even though $\tilde{N}^{(11)}$, $\tilde{N}^{(12)}$ and $\tilde{N}^{(33)}$ are suppressed by $1/\mu$ as compared to $\tilde{N}^{(13)}$, the self-contraction contributions are of the same order as the contractions between different strings due to cancellations in the contribution of contractions between different strings. The large μ expressions are given by

$$\begin{aligned}
\frac{1}{\mu}\langle ii,n|H_3|jj,m,y\rangle\rangle &= \delta_{ij}\left(\tilde{\Gamma}^{(1)}_{n,my} + \tilde{\Gamma}^{(1)}_{-n,my}\right) - \frac{1}{2}\Gamma^{(1)}_{n,0y}, \\
\frac{1}{\mu}\langle ii,n|H_3|jj,y\rangle\rangle &= \delta_{ij}\left(\tilde{\Gamma}^{(1)}_{n,y} + \tilde{\Gamma}^{(1)}_{-n,y}\right) - \frac{1}{2}\Gamma^{(1)}_{n,y}, \qquad (3.2.9)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Gamma}^{(1)}_{n,my} &= \lambda'\frac{\sqrt{1-y}}{\sqrt{Jy}}\frac{\sin^2(\pi n y)}{2\pi^2}, \\
\tilde{\Gamma}^{(1)}_{n,y} &= -\lambda'\frac{1}{\sqrt{J}}\frac{\sin^2(\pi n y)}{2\pi^2}. \qquad (3.2.10)
\end{aligned}$$

$\Gamma^{(1)}_{n,0y}$ and $\Gamma^{(1)}_{n,y}$ are defined in Appendix C and as we shall see have a direct gauge theory origin. The splitting of the first term in (3.2.9) into two identical contributions is convenient when comparing with the gauge theory analysis in the next subsection.

The first contribution in (3.2.9) is twice as large as compared to the answer one gets when considering string states with two different impurities [39, 60] . The reason is that there are twice as many ways of contracting impurities among different strings. This is reproduced in the gauge theory computation because the scalar impurities have two ways of contracting when they are both the same. The last terms in (3.2.9) are due to self-contractions and only appear when two impurities are repeated. In the

[1] We summarize the large μ expansion of the Neumann matrices in Appendix E.

gauge theory computation in the next subsection these extra contractions are due to the extra diagrams that one gets when considering the operators (3.2.2)(3.2.4). The new contractions in string field theory correspond to gauge theory diagrams involving $\bar{Z}$ and diagrams coupling all four scalar impurities. In Section 3.3 the connection between gauge theory diagrams and string field theory diagrams will be made explicit.

• *The* $\mathcal{O}(g_2^2)$ *Computation*

We now consider the $\mathcal{O}(g_2^2)$ matrix elements between single string states, that is, the contact term contribution. We will also reproduce this result from gauge theory considerations.

The single-string contact term in the plane wave geometry has been recently analyzed in [35]. It is constructed from the plane wave dynamical supersymmetry generators via $H_2' = \{Q_3, \bar{Q}_3\}$, where Q_3 is the leading g_2 correction to the free supercharge. In [35] it was shown that by considering the contact term contribution for two different impurity string states the gauge theory results in the orthonormal basis of [39, 60, 21] could be reproduced if one truncated the intermediate states to the two impurity sector. We will perform a similar calculation for string states with two identical impurities using the same truncation and reproduce these results from gauge theory in the next subsection. Understanding more precisely why the truncation works is an important open problem.

The intermediate two impurity states that contribute are given by

$$\begin{aligned}
|j, m, y, 1\rangle\rangle &= \alpha_m^{j\dagger} \frac{1}{\sqrt{2}} (b_m^{d\dagger} - ie(m) b_{-m}^{d\dagger}) |\text{vac}, y\rangle \otimes |\text{vac}, 1-y\rangle, \\
|j, 0, y, 1\rangle\rangle &= \alpha_0^{j\dagger} b_0^{d\dagger} |\text{vac}, y\rangle \otimes |\text{vac}, 1-y\rangle, \\
|j, 0, y, 1\rangle\rangle' &= \alpha_0^{j\dagger} |\text{vac}, y\rangle \otimes b_0^{d\dagger} |\text{vac}, 1-y\rangle,
\end{aligned} \tag{3.2.11}$$

and $|j, m, y, 2\rangle\rangle(')$ defined by changing the string on which the operators act. The b oscillators are the fermionic oscillators. Using the expression in [32] for the supersymmetry charge Q we can calculate its matrix elements in the large μ limit[2] (see

[2] The zero mode contribution vanishes in the large μ regime.

Appendix F for details)

$$
\begin{aligned}
Q_{n,m(s)} &= \langle ii,n|Q_{\dot{a}}|j,m,y,s\rangle\rangle \\
&\simeq \sqrt{1+\mu\alpha k}\,\delta_{ij}u^{i}_{abc\dot{a}}\delta^{abcd}_{1234}\frac{Y_{m(s)}}{\sqrt{2}}\left[\tilde{F}^{-}_{(3)-n}\tilde{N}^{(s3)}_{m,n}+\tilde{F}^{-}_{(3)n}\tilde{N}^{(s3)}_{m,-n}\right],
\end{aligned}
\tag{3.2.12}
$$

for $s = 1, 2$. Therefore the $\mathcal{O}(g_2^2)$ Hamiltonian matrix element in the case of two impurities in the same direction is given by

$$
\langle ii,n|H_2'|jj,m\rangle = \delta_{ij}\int_0^1 \frac{dy}{y(1-y)}\sum_{s=1}^{2}\sum_{l=-\infty}^{\infty} Q_{n,l(s)}Q^{*}_{m,l(s)}. \tag{3.2.13}
$$

Performing the relevant sums and integral one arrives at the final result(see Appendix G):

$$
\langle ii,n|H_2'|jj,m\rangle = \delta_{ij}\frac{1}{16\pi^2}(B_{n,m}+B_{n,-m}). \tag{3.2.14}
$$

The result in (3.2.14) has an extra term as compared to the calculation for two different impurities, which is identical to the first one except for the sign of the worldsheet momentum.

In this subsection we have calculated the Hamiltonian matrix elements using string field theory up to $\mathcal{O}(g_2^2)$. We now turn to the gauge theory analysis.

3.2.2 Gauge theory computations

The BMN operators with two identical scalar impurities (3.2.2)(3.2.4) are insensitive to the sign of the worldsheet momentum since $\mathcal{O}^J_{ii,n} = \mathcal{O}^J_{ii,-n}$ and $\mathcal{T}^{J,y}_{ii,m} = \mathcal{T}^{J,y}_{ii,-m}$, so we will consider without loss of generality $n, m \geq 0$. Moreover, the BPS double trace operator $\mathcal{T}^{J,y}_{ii}$ is invariant under $y \to 1-y$, so we can restrict to $0 < y \leq 1/2$.

As explained in Section 2.1, in order to compute string interactions from gauge theory we must compute the matrix of two-point functions of BMN operators $\mathcal{O}^J_{ii,n}$, $\mathcal{T}^{J,y}_{ii,p}$ and $\mathcal{T}^{J,y}_{ii}$. The relevant inner product metric and matrix of anomalous dimensions can be extracted from [47]. They are given by[3]

$$G = \mathbf{1} \;+\; g_2\,\delta_{ij}\begin{pmatrix} 0 & C_{n,qz}+C_{-n,qz} & 2C_{n,z} \\ C_{py,m}+C_{py,-m} & 0 & 0 \\ 2C_{y,m} & 0 & 0 \end{pmatrix}$$

$$+\; g_2^2\,\delta_{ij}\begin{pmatrix} M^1_{n,m}+M^1_{n,-m} & 0 & 0 \\ 0 & \langle ? \rangle & \langle ? \rangle \\ 0 & \langle ? \rangle & \langle ? \rangle \end{pmatrix}, \qquad (3.2.15)$$

and

$$\Gamma = \delta_{ij}\begin{pmatrix} \lambda' n^2 \delta_{nm} & 0 & 0 \\ 0 & \lambda' \frac{p^2}{y^2}\delta_{p,q}\delta_{y,z} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+g_2\begin{pmatrix} 0 & \delta_{ij}(\Gamma^{(1)}_{n,qz}+\Gamma^{(1)}_{-n,qz})-\frac{1}{2}\Gamma^{(1)}_{n,0z} & 2\delta_{ij}\Gamma^{(1)}_{n,z}-\frac{1}{2}\Gamma^{(1)}_{n,z} \\ \delta_{ij}(\Gamma^{(1)}_{py,m}+\Gamma^{(1)}_{py,-m})-\frac{1}{2}\Gamma^{(1)}_{0y,m} & 0 & 0 \\ 2\delta_{ij}\Gamma^{(1)}_{y,m}-\frac{1}{2}\Gamma^{(1)}_{y,m} & 0 & 0 \end{pmatrix}$$

$$+g_2^2\begin{pmatrix} \delta_{ij}(\Gamma^{(2)}_{n,m}+\Gamma^{(2)}_{n,-m})-\frac{1}{16\pi^2}\mathcal{D}^1_{n,m} & 0 & 0 \\ 0 & \langle ? \rangle & \langle ? \rangle \\ 0 & \langle ? \rangle & \langle ? \rangle \end{pmatrix}, \qquad (3.2.16)$$

where $\langle ? \rangle$ denotes matrix elements that have not yet been computed. We note that whenever the worldsheet momentum index in (3.2.15), (3.2.16) vanishes, that we must divide the matrix element by $\sqrt{2}$. Likewise, when both operators have vanishing momentum, we must divide that matrix element by 2. These extra factors arise from our normalization of the operators in (3.2.2), (3.2.4) which differ from those in [47]. In this way we get an orthonormal inner product for $n, m \geq 0$.

The inner product metric can be computed in the free theory while the matrix of anomalous dimensions comes with a power of λ' from evaluating one loop graphs. In

[3]We have summarized in Appendix C the explicit expressions for the matrix elements.

the free theory the $\bar{Z}$ portion of the gauge theory operators (3.2.2), (3.2.4) does not couple to the terms in (3.2.2), (3.2.4) without the $\bar{Z}$. Moreover, the diagrams involving only $\bar{Z}$ are suppressed by a power of $1/J$ with respect to the leading contribution, which only involves the part of the operator with the two impurities(terms without $\bar{Z}$). Therefore, in the computation of the mixing matrix the extra term in the operators (3.2.2), (3.2.4) does not contribute in the BMN limit, so that at any order in g_2 the inner product metric can be calculated neglecting the $\bar{Z}$ term. It then follows that there are twice as many contributions in the inner product of (3.2.2), (3.2.4) as compared to the case of two different impurities. This is easy to understand since there are now twice as many ways of contracting the impurities and they come with the opposite sign of the worldsheet momentum. An analogous phenomenon occurs when extending the analysis to arbitrary number of impurities.

The matrix of anomalous dimensions also has twice as many contributions of the type appearing for different impurities. These gauge theory Feynman diagrams can be identified in the string field theory calculation with contractions involving impurities living in different strings. However, there is an extra term arising from vertices involving $\bar{Z}$ in (3.2.2)(3.2.4) and the coupling of all scalar impurities[4] (see Fig. 1).

These extra Feynman diagrams can be identified in the string field theory calculation with contractions of impurities living on the same string as can be inferred by looking at (3.2.9).

We can now test the holographic correspondence (2.1.5). Using the formula for the matrix of anomalous dimensions in the orthonormal basis in terms of G and Γ we find:

$$\begin{aligned}
\tilde{\Gamma}^{(1)}_{ii;n,jj;my} &= \delta_{ij}\left(\tilde{\Gamma}^{(1)}_{n,my} + \tilde{\Gamma}^{(1)}_{-n,my}\right) - \frac{1}{2}\Gamma^{(1)}_{n,0y}, \\
\tilde{\Gamma}^{(1)}_{ii;n,jj;y} &= \delta_{ij}\left(\tilde{\Gamma}^{(1)}_{n,y} + \tilde{\Gamma}^{(1)}_{-n,y}\right) - \frac{1}{2}\Gamma^{(1)}_{n,y},
\end{aligned} \tag{3.2.17}$$

where terms with $\tilde{\Gamma}$ on the right hand side come from the usual Feynman diagrams

[4]The quartic scalar coupling denotes the effective interaction after taking into account self-energy and gluon exchange diagrams [47].

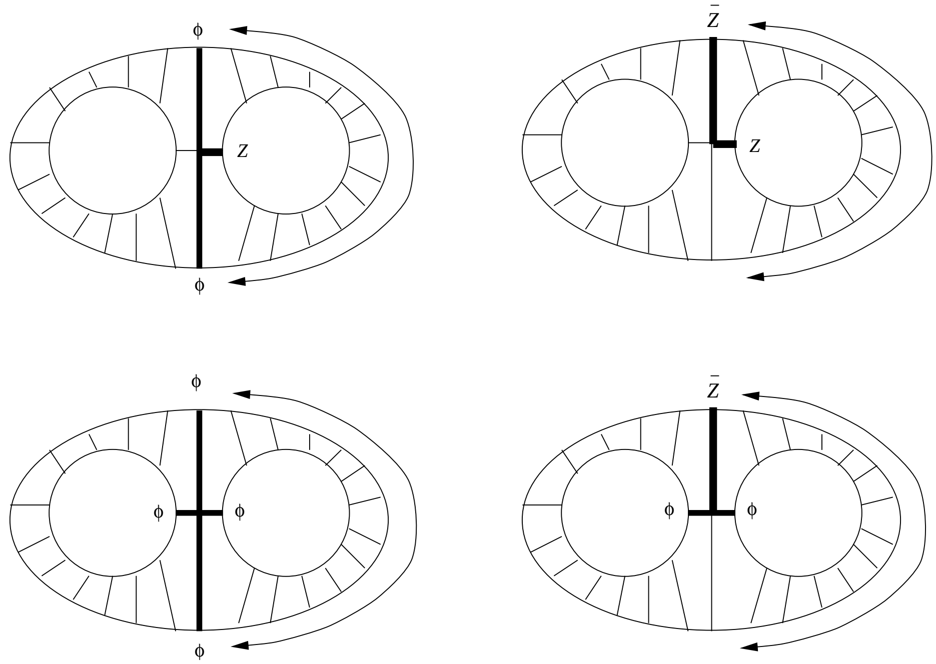

Figure 3.1: The new diagrams. The thick lines represent the impurities or $\bar{Z}$ while the thin lines denote Z. The first line is for diagrams involving $\mathcal{T}^{J,y}_{ii,m}$ while the second is for diagrams with $\mathcal{T}^{J,y}_{ii}$.

present also for two different impurities and the last term comes from new diagrams only present when two impurities are the same. By comparing with the string field theory calculation (3.2.9) we find precise agreement.

We now proceed to computing the matrix elements of the mostly single trace operators[5] to order g_2^2. Using (2.3.36) we find after some computation[6]

$$\tilde{\Gamma}^{(2)}_{ii;n,jj;m} = \delta_{ij}\left(\tilde{\Gamma}^{(2)}_{n,m} + \tilde{\Gamma}^{(2)}_{n,-m}\right) + \delta\tilde{\Gamma}^{(2)}_{ii;n,jj;m}, \tag{3.2.18}$$

where

$$\tilde{\Gamma}^{(2)}_{n,m} = \frac{1}{16\pi^2} B_{n,m}, \tag{3.2.19}$$

is the result obtained for different impurity operators [60] and $\delta\tilde{\Gamma}^{(2)}_{ii;n,jj;m}$ are the new

[5]In order to compute the matrix elements of the mostly double trace operators to this order, we would need to know the expressions for ⟨?⟩.

[6]For the detailed computation, see Appendix H.

contributions only arising for identical impurity operators. They are given by[7]

$$\delta\tilde{\Gamma}^{(2)}_{ii;n,jj;m} = \delta\Gamma^{(2)}_{ii;n,jj;m} - \frac{1}{2}\{G^{(1)}, \delta\Gamma^{(1)}\}_{ii;n,jj;m}, \tag{3.2.20}$$

since as explained above only the matrix of anomalous dimensions receives genuine new contributions while the inner product contributions have the same form as in the case of different impurities. From (3.2.16) we read

$$\begin{aligned} \delta\Gamma^{(2)}_{ii;n,jj;m} &= -\frac{1}{16\pi^2}\mathcal{D}^1_{n,m}, \\ \delta\Gamma^{(1)}_{ii;n,jj;my} = \delta\Gamma^{(1)}_{ii;my,jj;n} &= -\frac{1}{2}\Gamma^{(1)}_{n,0y}, \\ \delta\Gamma^{(1)}_{ii;n,jj;y} = \delta\Gamma^{(1)}_{ii;y,jj;n} &= -\frac{1}{2}\Gamma^{(1)}_{n,y}. \end{aligned} \tag{3.2.21}$$

After some computation one finds (see Appendix H for details)

$$\{G^{(1)}, \delta\Gamma^{(1)}\}_{ii;n,jj;m} = -\frac{1}{8\pi^2}\mathcal{D}^1_{n,m}, \tag{3.2.22}$$

giving us the simple result:

$$\delta\tilde{\Gamma}^{(2)}_{ii;n,jj;m} = 0. \tag{3.2.23}$$

Hence, the final expression is

$$\tilde{\Gamma}^{(2)}_{ii;n,jj;m} = \delta_{ij}\frac{1}{16\pi^2}(B_{n,m} + B_{n,-m}), \tag{3.2.24}$$

which exactly matches the $\mathcal{O}(g_2^2)$ contact term contribution in the string field theory calculation (3.2.14).

We now turn to the analysis of arbitrary string states.

3.3 Generalization to arbitrary impurities

Thus far we have analyzed the correspondence for string states with two impurities. In this section we construct a proof that shows the equivalence between the string theory and gauge theory computations for an arbitrary number of impurities. The

[7] We use the notation δA for the new contributions to A due to having identical impurities.

idea is to find a direct link between the Feynman diagrams of string theory and the Feynman diagrams of gauge theory, so that the equality between string theory and gauge theory for arbitrary states follows diagram by diagram. We first outline the strategy of the proof and then give the explicit details of the string theory and gauge theory computation.

Let's first consider which diagrams in string theory contribute to leading order in the $1/\mu$ expansion, which is of $\mathcal{O}(1/\mu^2)$. These diagrams will have corresponding contributions in the one loop – which is of $\mathcal{O}(\lambda')$ – gauge theory computation. We consider matrix elements between single-string states and two-string states with n impurities each, that is impurity preserving[8] processes. The impurities can be distributed at will among the four directions in $\mathbf{R}^4$.

As explained in Section 3, in order to compute $\mathcal{O}(g_2)$ Hamiltonian matrix elements, we must commute the prefactor (2.2.9) of the cubic vertex (2.2.8) through all the impurities. This gives us a sum of $2n$ terms with $2n$ oscillators each in which the sign of the worldsheet momentum of one of the oscillators is reserved. Each term now can be calculated using the Feynman rules (3.2.6). Each diagram is multiplied by the frequency of the oscillator whose worldsheet momentum is reversed when commuting through the prefactor. Now, given the $SO(8)$ invariance of the string field theory vertex (2.2.10), the oscillators in different directions in $\mathbf{R}^4$ completely decouple, so we can concentrate on the case in which all the impurities are in one direction. The final answer for arbitrary string states is just the product of the contribution along each of the $\mathbf{R}^4$ directions.

We can now classify Feynman diagrams in terms of the number of self-contractions (propagators) in the single-string state, that is the number of $\tilde{N}^{(33)}$'s. It is clear that to $\mathcal{O}(1/\mu^2)$ there can be *at most* one self-contraction. Since we are looking at impurity preserving processes, a self-contraction $\tilde{N}^{(33)}$ always is accompanied by a self-contraction in the two-string state of the type $\tilde{N}^{(rs)}$, where r, s is either 1 or 2. Since $\tilde{N}^{(33)}$ and $\tilde{N}^{(rs)}$ are of $\mathcal{O}(1/\mu)$, we can have at most one self-contraction to leading order in the $1/\mu$ expansion. This simple observation greatly diminishes the

[8] Impurity non-preserving processes are inherently non-perturbative [19] .

Feynman diagrams that need to be considered. We now study the two possibilities.

Let us consider first the case in which there are no self-contractions. In this case all impurities in the single-string are contracted with impurities of the two-string state, so the result is the product of Neumann matrices of the type $\tilde{N}^{(r3)}$, $(r = 1, 2)$, where $\tilde{N}^{(r3)} \simeq \mathcal{O}(1)$. In any of the $2n$ terms one gets after commuting the prefactor through the oscillators there is precisely one oscillator with reversed worldsheet momentum. This oscillator can now contract with any oscillator in the single-string state or two-string state depending on whether the reversed oscillator belongs to the two-string or single-string state. For each such contraction there is a corresponding one in which the sign of the worldsheet momentum of the two oscillators involved in the contraction is reversed[9]. The combination of these two contractions we represent by the vertex $(r, m) \xrightarrow{\times} (3, l)$ in (3.2.6), where $\times$ signifies the action of the prefactor on the oscillators $\alpha_{m(r)}$ and $\alpha_{l(3)}$. These two terms combine to yield an expression of $\mathcal{O}(1/\mu^2)$ due to the leading cancellation of the energy difference $\left(\frac{\omega_{m(r)}}{\mu p^+_{(r)}} - \frac{\omega_{l(3)}}{\mu}\right)$ of these two oscillators in the large μ limit. Therefore, this class of diagrams yields an expression given by the product of n Neumann matrices of the type $\tilde{N}^{(r3)}$ for $r = 1$ or 2 times the energy difference between one oscillator in the single-string state and one oscillator in the two-string state.

We now consider the case with one self-contraction on the single string state. As mentioned above, and due to the impurity conservation condition, this self-contraction is always accompanied by a self-contraction on the two-string state. Therefore we have a contribution of the form $\tilde{N}^{(33)} \cdot \tilde{N}^{(rs)}$, where r, s is 1 or 2, which is already of order $\mathcal{O}(1/\mu^2)$. There are now two possibilities to be considered. Either any of the oscillators involved in the self-contraction have their worldsheet momentum reversed due to action of the prefactor or they don't. If they do not, then there is a contraction connecting the single-string state with the two-string state involving the oscillator with the worldsheet momentum reversed. Just as in the previous case of no self-contractions, such diagram always comes accompanied with another one in which the sign of the worldsheet momentum is reversed on both oscillators involved in the

[9]This appears from the term one gets after commuting the prefactor through the other oscillator.

contraction, yielding the vertex (r,m) ——×—— $(3,l)$ for $r=1$ or 2. Therefore, in this case, the diagram is proportional to $\left(\frac{\omega_{m(r)}}{\mu p^{+}_{(r)}} - \frac{\omega_{l(3)}}{\mu}\right) \cdot \tilde{N}^{(33)} \cdot \tilde{N}^{(rs)} \simeq \mathcal{O}(1/\mu^4)$, so it does not contribute to the leading order result. Therefore, in the case of one self-contraction the only possibility left is the case in which the self-contractions involve one oscillator which has the worldsheet momentum reversed due to the prefactor, so that only diagrams with the vertex $(3,m)$ ——×—— $(3,l)$ or (r,m) ——×—— (s,l) for $r,s=1$ or 2 contribute to the leading order result.

From now on, let us focus on a particular Feynman diagram and show agreement between the string field theory and gauge theory computation. The string states with n impurities that we need to consider are given by[10]

$$
\begin{aligned}
|(d_i,n_i)\rangle &= i^n \delta_{\sum_i n_i,0} \prod_i {\alpha^{d_i}_{n_i}}^{\dagger} |\text{vac}\rangle, \\
|(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle &= i^n \delta_{\sum_{j\in\mathcal{I}_1} p_j,0}\, \delta_{\sum_{k\in\mathcal{I}_2} p_k,0} \prod_{j\in\mathcal{I}_1} {\alpha^{e_j}_{p_j}}^{\dagger} |\text{vac},y\rangle \otimes \prod_{k\in\mathcal{I}_2} {\alpha^{e_k}_{p_k}}^{\dagger} |\text{vac},1-y\rangle,
\end{aligned}
\tag{3.3.25}
$$

where the δ-functions impose the familiar level matching condition. The corresponding level-matched gauge theory operators are given by[11]

$$
\begin{aligned}
\mathcal{O}^J_{(d_i,n_i)} &= \frac{1}{\sqrt{JN^{J+n}}} \sum_{0\le l_1,\cdots,l_n\le J} \text{Tr}\left(Z\ldots Z\frac{\phi_{d_1}}{\sqrt{J}}Z\ldots Z\frac{\phi_{d_2}}{\sqrt{J}}Z\ldots Z\frac{\phi_{d_n}}{\sqrt{J}}Z\ldots Z\right)\prod_{i=1}^{n} t_i^{l_i} \\
&\quad +\text{terms involving } \bar{Z} \qquad\qquad \text{with} \quad \sum_{i=1}^{n} n_i = 0, \\
\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2} &= \,:\mathcal{O}^{y\cdot J}_{(e_j,p_j)_{j\in\mathcal{I}_1}} \cdot \mathcal{O}^{(1-y)\cdot J}_{(e_k,p_k)_{k\in\mathcal{I}_2}}: \qquad \text{with} \quad \sum_{j\in\mathcal{I}_1} p_j = \sum_{k\in\mathcal{I}_2} p_k = 0,
\end{aligned}
\tag{3.3.26}
$$

The labels $d_i, e_i \in \{1,2,3,4\}$ denote the direction along $\mathbf{R}^4$ of a string oscillator and the corresponding gauge theory impurity, and $n_i, p_i \in \mathbf{Z}$ are their worldsheet momenta where $t_i = \exp(2\pi i n_i/J)$, and $s_j = \exp(2\pi i p_j/J_1)$ for $j \in \mathcal{I}_1$ and $s_k = \exp(2\pi i p_k/J_2)$

[10]The arbitrary phase of the state is determined by comparison with gauge theory.

[11]Here are using a simplified notation for the operators, their precise description is given in Appendix A and B.

for $k \in \mathcal{I}_2$. Also we explicitly assign a factor of $1/\sqrt{J}$ to each impurity which aids in keeping track of factors of J during the computation. $\mathcal{I}_1$ and $\mathcal{I}_2$ is a partition of the index set $\{1, \cdots, n\}$, which describes a particular way of distributing the n impurities among string/trace 1 and 2, respectively.

Let us explain the gauge theory computation of the two-point function of single-trace and double-trace BMN operators defined above and exhibit analogies with the string theory computation. At one loop order, that is to $\mathcal{O}(\lambda')$, we can have at most a quartic interaction[12] vertex, coupling four fields, with two of them contracted with the *in*-operator and the other two with the *out*-operator. There are three kinds of interaction vertices depending on how far the two fields in the same operator are separated:

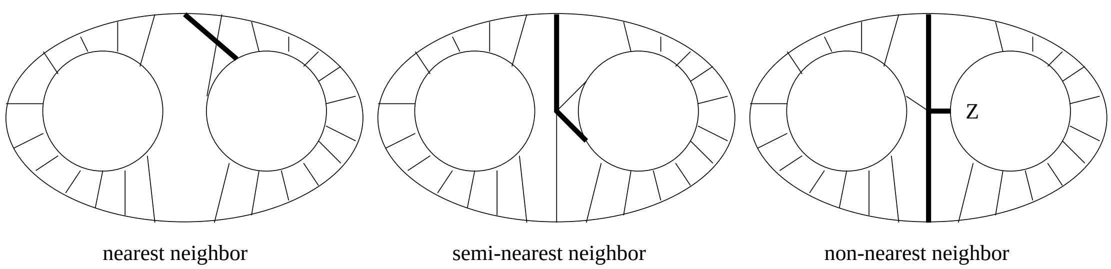

Figure 3.2: The three classes of interaction vertices.

• The nearest neighbor interaction[13] vertex, where two fields on each operator coupled by the interaction sit next to each other, involves one impurity in the *in*-operator, and the same impurity and Z in the *out*-operator. This interaction can occur at $\mathcal{O}(J)$ sites along the smaller trace operator and we have to sum over the position of the interaction in the trace.

• The semi-nearest neighbor interaction vertex has two fields on one side sitting next to each other but the two fields on the other side are separated by $\mathcal{O}(J)$ sites. The vertex can be inserted only at a particular place along the trace and so we do not sum over the position of the vertex.

[12]As shown in [65, 19, 47] the other possible interactions cancel among themselves due to supersymmetry.

[13]This terminology was first introduced in [19].

• The non-nearest neighbor interaction vertex has the two fields on each side of the interaction point separated by $\mathcal{O}(J)$ sites. In this vertex, two impurities or $\bar{Z}$ are involved in the two operators and this is possible only when we have two identical impurities in each operator. This interaction can also occur at a specific location in the trace, so we do not sum over the position of the vertex.

The contribution of each interaction vertex is given as

$$\begin{aligned} I^{\text{nearest}}_{n_i,p_i}(l_i) = \quad & \frac{1}{\sqrt{JJ_1}}\frac{g^2N}{8\pi^2}(1-t_i)(1-\bar{s}_i)(t_i\bar{s}_i)^{l_i} \qquad \text{for } i\in\mathcal{I}_1, \\ & \text{or} \\ & \frac{1}{\sqrt{JJ_2}}\frac{g^2N}{8\pi^2}(1-t_i)(1-\bar{s}_i)t_i^{J_1}(t_i\bar{s}_i)^{l_i} \qquad \text{for } i\in\mathcal{I}_2, \end{aligned} \tag{3.3.27}$$

$$\begin{aligned} I^{\text{semi-nearest}}_{n_i,p_i} = \quad & -\frac{1}{\sqrt{JJ_1}}\frac{g^2N}{8\pi^2}[(1-t_i)+(1-\bar{s}_i)](1-t_i^{J_1}) \qquad \text{for } i\in\mathcal{I}_1, \\ & \text{or} \\ & \frac{1}{\sqrt{JJ_2}}\frac{g^2N}{8\pi^2}[(1-t_i)+(1-\bar{s}_i)](1-t_i^{-J_2}) \qquad \text{for } i\in\mathcal{I}_2, \end{aligned} \tag{3.3.28}$$

$$\begin{aligned} I^{\text{non-nearest}}_{n_i,n_j,p_i,p_j} = \quad & -\frac{1}{\sqrt{JJJ_1J_1}}\frac{g^2N}{8\pi^2}(1-t_i^{J_1})(1-t_j^{J_1}) \qquad \text{for } i,j\in\mathcal{I}_1, \\ & \text{or} \\ & -\frac{1}{\sqrt{JJJ_2J_2}}\frac{g^2N}{8\pi^2}(1-t_i^{J_1})(1-t_j^{J_1}) \qquad \text{for } i,j\in\mathcal{I}_2, \\ & \text{or} \\ & \frac{1}{\sqrt{JJJ_1J_2}}\frac{g^2N}{8\pi^2}(1-t_i^{J_1})(1-t_j^{J_1}) \qquad \text{for } i\in\mathcal{I}_1, j\in\mathcal{I}_2, \end{aligned} \tag{3.3.29}$$

where l_i in $I^{\text{nearest}}_{n_i,p_i}(l_i)$ denotes the position of the nearest neighbor interaction vertex to be summed over. Here each factor of $1/\sqrt{J}$ or $1/\sqrt{J_r}$ $(r=1,2)$ comes from each impurity participating in the interaction. The rest of impurities in the *in*-operator are freely contracted with the remaining impurities in the *out*-operator and each free contraction contributes

$$\frac{1}{\sqrt{JJ_1}}(t_i\bar{s}_i)^{l_i} \text{ for } i\in\mathcal{I}_1 \qquad \text{or} \qquad \frac{1}{\sqrt{JJ_2}}t_i^{J_1}(t_i\bar{s}_i)^{l_1} \text{ for } i\in\mathcal{I}_2. \tag{3.3.30}$$

Now we have to multiply all the different contributions, coming from the interaction vertex and the free contractions and sum over all possible positions of the impurities.

However, the whole summation is simply factorized in the large J limit into sums over each contribution since each contribution is independent of the positions of the rest of impurities:

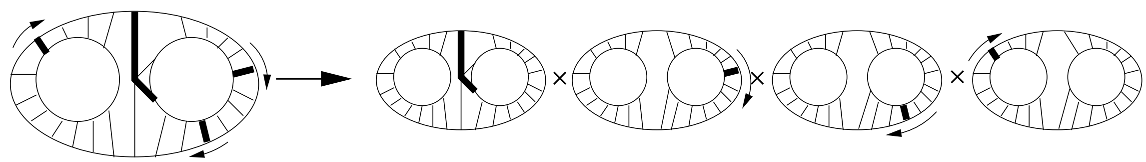

Figure 3.3: The factorization property of gauge theory amplitudes.

The computation of each contribution then essentially reduces to the one or two-impurity cases. This factorization property, which as we have seen earlier has an analog in the string field theory computation, will turn out to be useful in comparing the gauge theory and string theory expressions for Feynman diagrams. As we will see, the effect of the prefactor interaction (r,m) ——×—— (s,l) in string field theory is essentially captured by the interaction vertex in gauge theory while the sum over free contractions in gauge theory capture the Neumann matrices.

Now, let us start to compute the string field theory amplitudes and compare them with the gauge theory results. As discussed earlier, there are two cases to be considered.

1) Case 1: Diagrams without self-contraction

First, let us consider a particular way of contracting the oscillators without self-contractions. In this case, without loss of generality, we can assume that $d_i = e_i$ and take the d_i-th oscillator to contract with e_i-th oscillator for all $i \in \{1, \cdots, n\}$. More specifically, the j-th oscillator in string 3 contracts with the j-th oscillator in string 1 for $j \in \mathcal{I}_1$ and the k-th oscillator in string 3 contracts with the k-th oscillator in string 2 for $k \in \mathcal{I}_2$. On the string field theory side, using the Feynman rules in (3.2.6) we can compute the matrix elements between these states as in the previous section:

$$\frac{1}{\mu}\langle (d_i, n_i)|H_3|(d_i, p_i); \mathcal{I}_1, \mathcal{I}_2; y\rangle\rangle$$

$$= \quad -(-1)^n \frac{y(1-y)}{2} \left\{ \sum_{l\in\mathcal{I}_1} \left[\left(\frac{\omega_{p_l(1)}}{\mu y} - \frac{\omega_{n_l(3)}}{\mu} \right) \tilde{N}^{(13)}_{p_l,-n_l} \prod_{j\in\mathcal{I}_1-\{l\}} \tilde{N}^{(13)}_{p_j,n_j} \prod_{k\in\mathcal{I}_2} \tilde{N}^{(23)}_{p_k,n_k} \right] \right.$$

$$\left. + \sum_{l\in\mathcal{I}_2} \left[\left(\frac{\omega_{p_l(2)}}{\mu(1-y)} - \frac{\omega_{n_l(3)}}{\mu} \right) \tilde{N}^{(23)}_{p_l,-n_l} \prod_{j\in\mathcal{I}_1} \tilde{N}^{(13)}_{p_j,n_j} \prod_{k\in\mathcal{I}_2-\{l\}} \tilde{N}^{(23)}_{p_k,n_k} \right] \right\}. \quad (3.3.31)$$

Now let us explain how to match each term above with specific Feynman diagrams in gauge theory.

- $l \in \mathcal{I}_1$

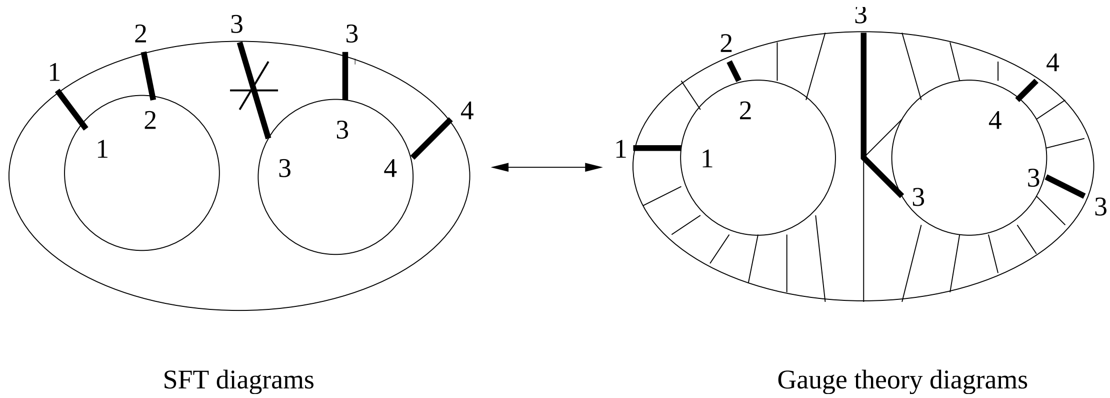

Figure 3.4: Diagrams without self-contractions, $l \in \mathcal{I}_1$. The numbers represent the direction of the SFT oscillators and the corresponding gauge theory impurities.

For each $l \in \mathcal{I}_1$, the particular term

$$(-1)^n \frac{y(1-y)}{2} \left(\frac{\omega_{n_l(3)}}{\mu} - \frac{\omega_{p_l(1)}}{\mu y} \right) \tilde{N}^{(13)}_{p_l,-n_l} \prod_{j\in\mathcal{I}_1-\{l\}} \tilde{N}^{(13)}_{p_j,n_j} \prod_{k\in\mathcal{I}_2} \tilde{N}^{(23)}_{p_k,n_k}, \quad (3.3.32)$$

arises when the l-th oscillator in string 1 and string 3 go through the prefactor and contract while the rest of the oscillators get contracted among themselves. The pair of l-th oscillators produce

$$\frac{1}{2} \left(\frac{\omega_{n_l(3)}}{\mu} - \frac{\omega_{p_l(1)}}{\mu y} \right) \tilde{N}^{(13)}_{p_l,-n_l} \simeq \frac{1}{4\mu^2} \left(n_l - \frac{p_l}{y} \right)^2 \tilde{N}^{(13)}_{p_l,n_l}, \quad (3.3.33)$$

where we have used the large μ relation

$$\tilde{N}^{(r3)}_{p,-n} \simeq \frac{n-\frac{p}{y}}{n+\frac{p}{y}} \tilde{N}^{(r3)}_{p,n} \qquad r = 1 \text{ or } 2. \quad (3.3.34)$$

The other pairs of oscillators bring down one Neumann coefficient $\tilde{N}^{(13)}_{p_j,n_j}$ or $\tilde{N}^{(23)}_{p_k,n_k}$. Therefore the contribution to the Hamiltonian matrix element due to this diagram is[14]

$$\begin{aligned}
&\frac{1}{\mu}\langle (d_i,n_i)|H_3|(d_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle\Big|_l \\
&\simeq (-1)^n\frac{1}{4\mu^2}\sqrt{\frac{y(1-y)}{J}}\left(n_l-\frac{p_l}{y}\right)^2\tilde{N}^{(13)}_{p_l,n_l}\prod_{j\in\mathcal{I}_1-\{l\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}. \quad (3.3.35)
\end{aligned}$$

We claim that this particular term corresponds to the interaction Feynman diagrams where two ϕ_l's are involved in the interaction vertex and the rest of the impurities are freely contracted. The contributions come from two classes of diagrams. The nearest neighbor diagrams give

$$\begin{aligned}
\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l^{\mathrm{nearest}} &= \sqrt{\frac{y(1-y)}{J}}\times\left[\frac{g^2N}{8\pi^2}(1-t_l)(1-\bar{s}_l)\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_l\bar{s}_l)^a\right] \\
&\times\prod_{j\in\mathcal{I}_1-\{l\}}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2}\frac{1}{\sqrt{JJ_2}}\sum_{a=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^a,
\end{aligned}$$

(3.3.36)

whereas the semi-nearest neighbor diagrams contribute

$$\begin{aligned}
\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l^{\mathrm{semi-nearest}} &= \sqrt{\frac{y(1-y)}{J}}\times\left[-\frac{1}{\sqrt{JJ_1}}\frac{g^2N}{8\pi^2}[(1-t_l)+(1-\bar{s}_l)](1-t_l^{J_1})\right] \\
&\times\prod_{j\in\mathcal{I}_1-\{l\}}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2}\frac{1}{\sqrt{JJ_2}}\sum_{a=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^a,
\end{aligned}$$

(3.3.37)

where the phases are defined as $t_i=\exp(2\pi i n_i/J)$, and $s_j=\exp(2\pi i p_j/J_1)$ for $j\in\mathcal{I}_1$ and $s_k=\exp(2\pi i p_k/J_2)$ for $k\in\mathcal{I}_2$. The subscript l means that only Feynman

[14] After going to the unit norm basis.

diagrams with ϕ_l's involved in the interaction vertex are included. The first factor in (3.3.36) and (3.3.48) comes from the interaction vertices involving ϕ_l and the rest of the expression comes from free contraction of the other impurities. We can compute each factor and express it in terms of purely string field theory quantities and show that the interaction essentially captures the energy difference factor in the string theory computation while the free contractions yield the Neumann matrices. For $j \in \mathcal{I}_1$ and $k \in \mathcal{I}_2$, the free contraction contribution is

$$\begin{aligned}
\frac{1}{\sqrt{JJ_1}} \sum_{a=0}^{J_1-1} (t_j \bar{s}_j)^a &\simeq (-1)^{n_j+p_j+1} e^{i\pi n_j y} \tilde{N}^{(13)}_{p_j,n_j}, \\
\frac{1}{\sqrt{JJ_2}} \sum_{a=0}^{J_2-1} t_k^{J_1} (t_k \bar{s}_k)^a &\simeq (-1)^{n_k+1} e^{i\pi n_k y} \tilde{N}^{(23)}_{p_k,n_k},
\end{aligned} \tag{3.3.38}$$

while the interaction vertex contribution is

$$\begin{aligned}
\frac{1}{\sqrt{JJ_1}} \frac{g^2 N}{8\pi^2} (1-t_l)(1-\bar{s}_l) \sum_{a=0}^{J_1-1} (t_l \bar{s}_l)^a &\simeq (-1)^{n_l+p_l+1} e^{i\pi n_l y} \times \frac{\lambda'}{2} \left(\frac{n_l p_l}{y} \right) \tilde{N}^{(13)}_{p_l,n_l}, \\
-\frac{1}{\sqrt{JJ_1}} \frac{g^2 N}{8\pi^2} [(1-t_l)+(1-\bar{s}_l)](1-t_l^{J_1}) &\simeq (-1)^{n_l+p_l+1} e^{i\pi n_l y} \times \frac{\lambda'}{2} \left(n_l - \frac{p_l}{y} \right)^2 \tilde{N}^{(13)}_{p_l,n_l}.
\end{aligned} \tag{3.3.39}$$

Altogether, we obtain

$$\begin{aligned}
&\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l \\
&\simeq (-1)^n \frac{\lambda'}{2} \sqrt{\frac{y(1-y)}{J}} \left[\left(n_l - \frac{p_l}{y} \right)^2 + n_l \frac{p_l}{y} \right] \tilde{N}^{(13)}_{p_l,n_l} \prod_{j\in\mathcal{I}_1-\{l\}} \tilde{N}^{(13)}_{p_j,n_j} \prod_{k\in\mathcal{I}_2} \tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned} \tag{3.3.40}$$

Notice that all the phase factors except $(-1)^n$ disappear upon imposing the level-matching conditions. In order to compare with the string theory result, we must evaluate these expressions in the string field theory basis (2.3.36). In order to compute

$$\tilde{\Gamma}^{(1)}|_l = \Gamma^{(1)}|_l - \frac{1}{2}\{G^{(1)}, \Gamma^{(0)}|_l\} \tag{3.3.41}$$

we also need to compute $G^{(1)}$ and $\Gamma^{(0)}|_l$. They are given by[15]

$$
\begin{aligned}
G^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y} &= \sqrt{\frac{y(1-y)}{J}} \prod_{j\in\mathcal{I}_1} \frac{1}{\sqrt{JJ_1}} \sum_{a=0}^{J_1-1} (t_j\bar{s}_j)^a \prod_{k\in\mathcal{I}_2} \frac{1}{\sqrt{JJ_2}} \sum_{a=0}^{J_2-1} t_k^{J_1}(t_k\bar{s}_k)^a \\
&\simeq (-1)^n \sqrt{\frac{y(1-y)}{J}} \prod_{j\in\mathcal{I}_1} \tilde{N}^{(13)}_{p_j,n_j} \prod_{k\in\mathcal{I}_2} \tilde{N}^{(23)}_{p_k,n_k}, \\
\Gamma^{(0)}_{\{n_i\},\{m_i\}}|_l &= \frac{\lambda'}{2} n_l^2 \prod_i \delta_{n_i,m_i}, \\
\Gamma^{(0)}_{\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y,\{q_i;\mathcal{I}_1,\mathcal{I}_2\}z}|_l &= \frac{\lambda'}{2}\left(\frac{p_l}{y}\right)^2 \delta_{y,z} \prod_i \delta_{p_i,q_i}.
\end{aligned} \tag{3.3.42}
$$

Hence,

$$
\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l \simeq (-1)^n \frac{\lambda'}{4} \sqrt{\frac{y(1-y)}{J}} \left(n_l - \frac{p_l}{y}\right)^2 \tilde{N}^{(13)}_{p_l,n_l} \prod_{j\in\mathcal{I}_1-\{l\}} \tilde{N}^{(13)}_{p_j,n_j} \prod_{k\in\mathcal{I}_2} \tilde{N}^{(23)}_{p_k,n_k}. \tag{3.3.43}
$$

which precisely reproduces the string field theory result (3.3.35).

- $l \in \mathcal{I}_2$

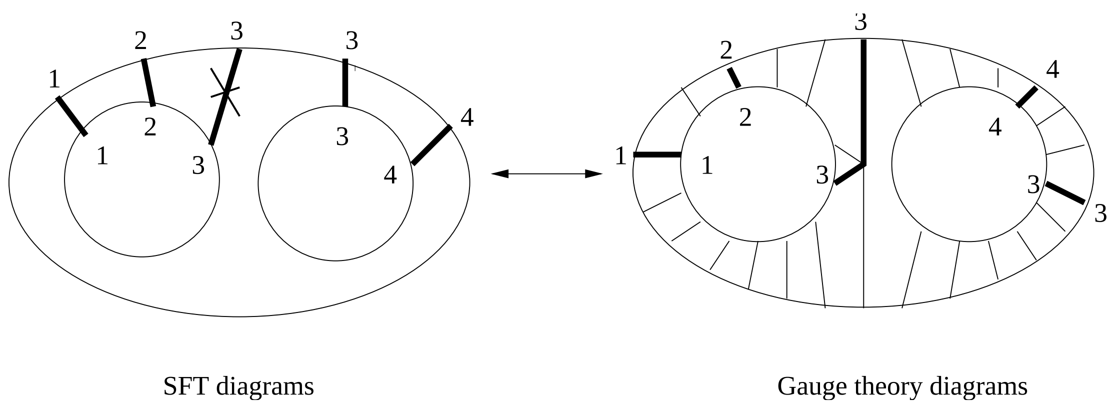

Figure 3.5: Diagrams without self-contractions, $l \in \mathcal{I}_2$.

Now we consider the other type of contraction in the string field theory computation, where the prefactor acts on the l-th oscillator in string 2 and string 3. The

[15] Here we use the large μ relation (3.3.38) to rewrite $G^{(1)}$ in terms of string field theory quantities.

expression for this diagram is

$$(-1)^n \frac{y(1-y)}{2}\left(\frac{\omega_{n_l(3)}}{\mu}-\frac{\omega_{p_l(2)}}{\mu(1-y)}\right)\tilde{N}^{(23)}_{p_l,-n_l}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{l\}}\tilde{N}^{(23)}_{p_k,n_k}. \qquad (3.3.44)$$

As before, it is convenient to express the contribution from the prefactor as

$$\frac{1}{2}\left(\frac{\omega_{n_l(3)}}{\mu}-\frac{\omega_{p_l(2)}}{\mu(1-y)}\right)\tilde{N}^{(23)}_{p_l,-n_l}\simeq\frac{1}{4\mu^2}\left(n_l-\frac{p_l}{1-y}\right)^2\tilde{N}^{(23)}_{p_l,n_l}, \qquad (3.3.45)$$

where we have used the large μ relation (3.3.34). Therefore, the contribution of this diagram to the Hamiltonian matrix element of unit normalized states is

$$\begin{aligned}&\frac{1}{\mu}\langle(d_i,n_i)|H_3|(d_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle\Big|_l\\ \simeq\ &(-1)^n\frac{1}{4\mu^2}\sqrt{\frac{y(1-y)}{J}}\left(n_l-\frac{p_l}{1-y}\right)^2\tilde{N}^{(23)}_{p_l,n_l}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{l\}}\tilde{N}^{(23)}_{p_k,n_k}.\end{aligned} \qquad (3.3.46)$$

The corresponding gauge theory diagrams are again classified into two classes. The nearest neighbor diagrams yields

$$\begin{aligned}\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\text{nearest}}_l &= \sqrt{\frac{y(1-y)}{J}}\times\left[\frac{g^2N}{8\pi^2}(1-t_l)(1-\bar{s}_l)\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_l^{J_1}(t_l\bar{s}_l)^b\right]\\ &\quad\times\prod_{j\in\mathcal{I}_1}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2-\{l\}}\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^b,\end{aligned} \qquad (3.3.47)$$

whereas the semi-nearest neighbor diagrams contribute

$$\begin{aligned}\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\text{semi-nearest}}_l &= \sqrt{\frac{y(1-y)}{J}}\times\left[\frac{1}{\sqrt{JJ_2}}\frac{g^2N}{8\pi^2}[(1-t_l)+(1-\bar{s}_l)](1-t_l^{J_1})\right]\\ &\quad\times\prod_{j\in\mathcal{I}_1}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2-\{l\}}\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^b.\end{aligned}$$

(3.3.48)

We can also express the various contributions in terms of string field theory quantities. The interaction vertex contribution is given by

$$\begin{aligned}
\frac{g^2N}{8\pi^2}(1-t_l)(1-\bar{s}_l)\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1} t_l^{J_1}(t_l\bar{s}_l)^b &\simeq (-1)^{n_l+1}e^{i\pi n_l y}\times\frac{\lambda'}{2}\left(\frac{n_l p_l}{1-y}\right)\tilde{N}^{(23)}_{p_l,n_l}, \\
\frac{1}{\sqrt{JJ_2}}\frac{g^2N}{8\pi^2}[(1-t_l)+(1-\bar{s}_l)](1-t_l^{J_1}) &\simeq (-1)^{n_l+1}e^{i\pi n_l y}\times\frac{\lambda'}{2}\left(n_l-\frac{p_l}{1-y}\right)^2\tilde{N}^{(23)}_{p_l,n_l},
\end{aligned} \tag{3.3.49}$$

whereas the free contraction (3.3.38) yields the product of Neumann matrix after imposing the level matching constraint.

In order to compute the matrix of anomalous dimensions in the string field theory basis we need also $G^{(1)}$ and $\Gamma^{(0)}\Big|_l$. It is easy to show that these quantities are the same as in (3.3.42) except for the last formula which can be correctly obtained by replacing $y\to 1-y$. Therefore, using (3.3.41), we obtain

$$\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l \simeq (-1)^n\frac{\lambda'}{4}\sqrt{\frac{y(1-y)}{J}}\left(n_l-\frac{p_l}{1-y}\right)^2\tilde{N}^{(23)}_{p_l,n_l}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{l\}}\tilde{N}^{(23)}_{p_k,n_k} \tag{3.3.50}$$

and again we find agreement with the string theory result (3.3.46).

2) Terms with self-contractions

As explained in the beginning of this section, to leading order in the $1/\mu$ expansion we can have at most one self-contraction in string 3 and the prefactor has to go through any of the oscillators involved in the self-contraction.

Without loss of generality, we can assume that $d_1=d_2$, $e_1=e_2$, $d_i=e_i$ for $i\in\{3,\cdots,n\}$ and we will consider contractions between d_1-d_2, e_1-e_2, and d_i-e_i for $i\in\{3,\cdots,n\}$. There are three cases depending on how the 1st and the 2nd impurities are distributed on the two-string state and the double-trace operator: $1,2\in\mathcal{I}_1$, $1,2\in\mathcal{I}_2$ and $1\in\mathcal{I}_1$, $2\in\mathcal{I}_2$.

- $1,2\in\mathcal{I}_1$

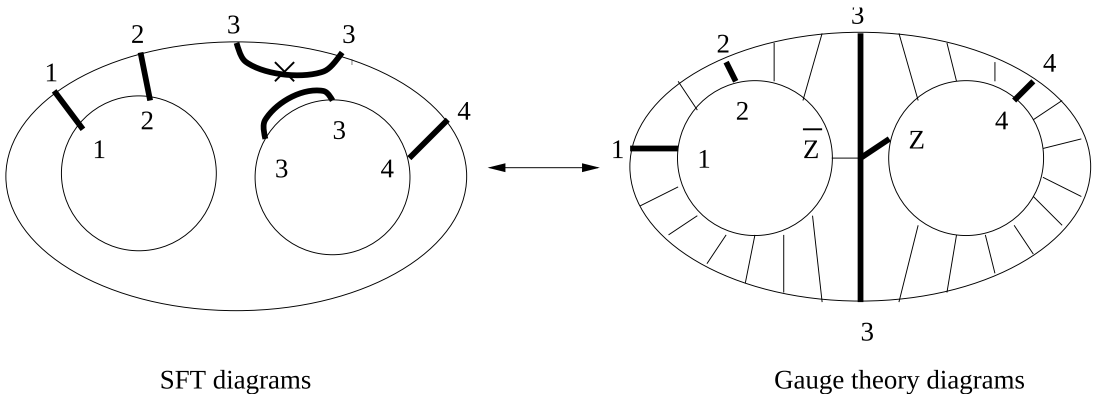

Figure 3.6: Diagrams with self-contractions, $1, 2 \in \mathcal{I}_1$.

The string theory computation of this particular Feynman diagram is

$$\begin{aligned}
&\frac{1}{\mu}\langle (d_i, n_i)|H_3|(e_i, p_i); \mathcal{I}_1, \mathcal{I}_2; y\rangle\rangle \\
= &-(-1)^n \frac{y(1-y)}{2}\left[\left(\frac{\omega_{p_1(1)}+\omega_{p_2(1)}}{\mu y}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,-p_2} - \left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(11)}_{p_1,p_2}\right] \\
&\times \prod_{j\in\mathcal{I}_1-\{1,2\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}. \qquad (3.3.51)
\end{aligned}$$

The first line in (3.3.51) is due to the self-contractions while the rest is due to the contraction between oscillators in the single string state with the two-string state.

The self-contraction contribution is to the leading order in $1/\mu$:

$$\begin{aligned}
&-\frac{1}{2}\left[\left(\frac{\omega_{p_1(1)}+\omega_{p_2(1)}}{\mu y}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,-p_2} - \left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(11)}_{p_1,p_2}\right] \\
\simeq &-2\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,p_2}. \qquad (3.3.52)
\end{aligned}$$

Therefore, the matrix element of unit normalized states is given by

$$\begin{aligned}
&\frac{1}{\mu}\langle (d_i, n_i)|H_3|(e_i, p_i); \mathcal{I}_1, \mathcal{I}_2; y\rangle\rangle\Big|_{1-2;1-2} \\
\simeq &-2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1-\{1,2\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}. \qquad (3.3.53)
\end{aligned}$$

We now show that the corresponding gauge theory diagrams are those with an interaction vertex involving ϕ_{d_1}, ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1}, ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$. In this case, only non-nearest interaction diagrams contribute and the result is

$$\begin{aligned}\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\mathrm{non-nearest}}_{1-2;1-2} &= \sqrt{\frac{y(1-y)}{J}} \times \left[-\frac{1}{\sqrt{JJJ_1J_1}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1})\right] \\ &\quad \times \prod_{j\in\mathcal{I}_1-\{1,2\}} \frac{1}{\sqrt{JJ_1}} \sum_{a=0}^{J_1-1} (t_j\bar{s}_j)^a \prod_{k\in\mathcal{I}_2} \frac{1}{\sqrt{JJ_2}} \sum_{b=0}^{J_2-1} t_k^{J_1}(t_k\bar{s}_k)^b.\end{aligned} \tag{3.3.54}$$

The interaction contribution reduces in the BMN limit to

$$\begin{aligned}-\frac{1}{\sqrt{JJJ_1J_1}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1}) &= e^{\pi i(n_1+n_2)y}\lambda'\frac{\sin(\pi n_1 y)\sin(\pi n_2 y)}{2\pi^2 y} \\ &\simeq -2(-1)^{n_1+n_2+p_1+p_2}e^{\pi i(n_1+n_2)y}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,p_2},\end{aligned} \tag{3.3.55}$$

while the rest can be rewritten in string field theory language using (3.3.38). Again the various phase factors disappear after imposing the level matching condition on each trace.

In order to compare with string field theory we must go to the string field theory basis. However, the particular class of Feynman diagrams we are considering, which are those with an interaction vertex involving ϕ_{d_1}, ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1}, ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$ do not contribute to $\Gamma^{(0)}\Big|_l$. Therefore, in this case (3.3.41) yields

$$\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\mathrm{non-nearest}}_{1-2;1-2} \simeq -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1-\{1,2\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}, \tag{3.3.56}$$

which agrees with the SFT result (3.3.51).

- $1, 2 \in \mathcal{I}_2$

The string theory computation of this particular Feynman diagram is similar to

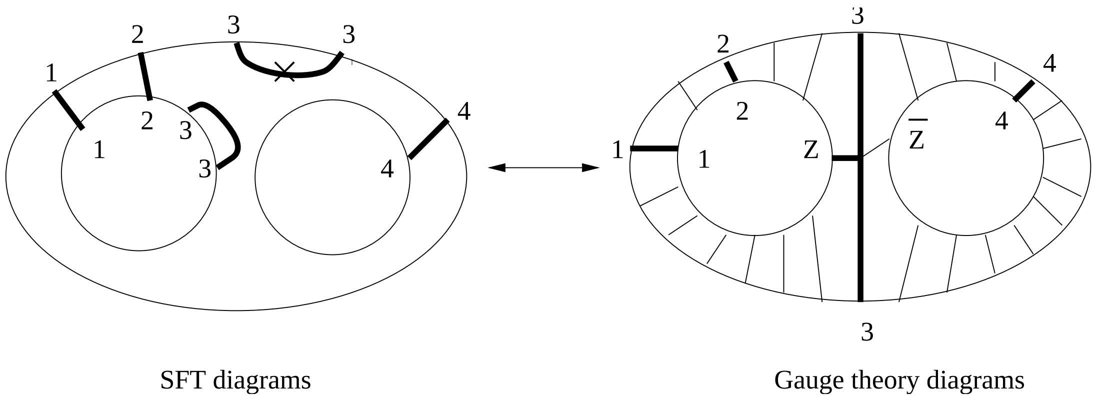

Figure 3.7: Diagrams with self-contractions, $1, 2 \in \mathcal{I}_2$.

the previous one:

$$
\begin{aligned}
&\frac{1}{\mu}\langle(d_i, n_i)|H_3|(e_i, p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle \\
= \quad & -(-1)^n\frac{y(1-y)}{2}\left[\left(\frac{\omega_{p_1(2)}+\omega_{p_2(2)}}{\mu(1-y)}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,-p_2} - \left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(22)}_{p_1,p_2}\right] \\
& \times\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{1,2\}}\tilde{N}^{(23)}_{p_k,n_k}. \qquad (3.3.57)
\end{aligned}
$$

The self-contraction contribution is to the leading order in $1/\mu$:

$$
\begin{aligned}
& -\frac{1}{2}\left[\left(\frac{\omega_{p_1(2)}+\omega_{p_2(2)}}{\mu(1-y)}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,-p_2} - \left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(22)}_{p_1,p_2}\right] \\
\simeq \quad & -2\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,p_2}. \qquad (3.3.58)
\end{aligned}
$$

Therefore, the matrix element of unit normalized states is given by

$$
\begin{aligned}
&\frac{1}{\mu}\langle(d_i, n_i)|H_3|(e_i, p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle\Big|_{1-2;1-2} \\
\simeq \quad & -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{1,2\}}\tilde{N}^{(23)}_{p_k,n_k}. \qquad (3.3.59)
\end{aligned}
$$

We now show that the corresponding gauge theory diagrams are those with an interaction vertex involving ϕ_{d_1}, ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1}, ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$. In this case, only non-nearest interaction diagrams contribute and the result is

$$\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\text{non-nearest}}_{1-2;1-2} = \sqrt{\frac{y(1-y)}{J}} \times \left[-\frac{1}{\sqrt{JJJ_2J_2}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1})\right] \times \prod_{j\in\mathcal{I}_1}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a \prod_{k\in\mathcal{I}_2-\{1,2\}}\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^b. \tag{3.3.60}$$

The interaction contribution reduces in the BMN limit to

$$-\frac{1}{\sqrt{JJJ_2J_2}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1}) = e^{\pi i(n_1+n_2)y}\lambda'\frac{\sin(\pi n_1 y)\sin(\pi n_2 y)}{2\pi^2(1-y)} \simeq -2(-1)^{n_1+n_2}e^{\pi i(n_1+n_2)y}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,p_2}, \tag{3.3.61}$$

while the rest can be rewritten in string field theory language using (3.3.38). Again the various phase factors disappear after imposing the level matching condition on each trace.

In order to compare with string field theory we must go to the string field theory basis. However, the particular class of Feynman diagrams we are considering, which are those with an interaction vertex involving ϕ_{d_1}, ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1}, ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$, do not contribute to $\Gamma^{(0)}\Big|_l$. Therefore, in this case (3.3.41) yields

$$\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\text{non-nearest}}_{1-2;1-2} \simeq -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{1,2\}}\tilde{N}^{(23)}_{p_k,n_k}, \tag{3.3.62}$$

which agrees with the SFT result (3.3.57).

- $1\in\mathcal{I}_1$, $2\in\mathcal{I}_2$

The string field theory computation of this particular contraction term is

$$\frac{1}{\mu}\langle(d_i,n_i)|H_3|(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle$$

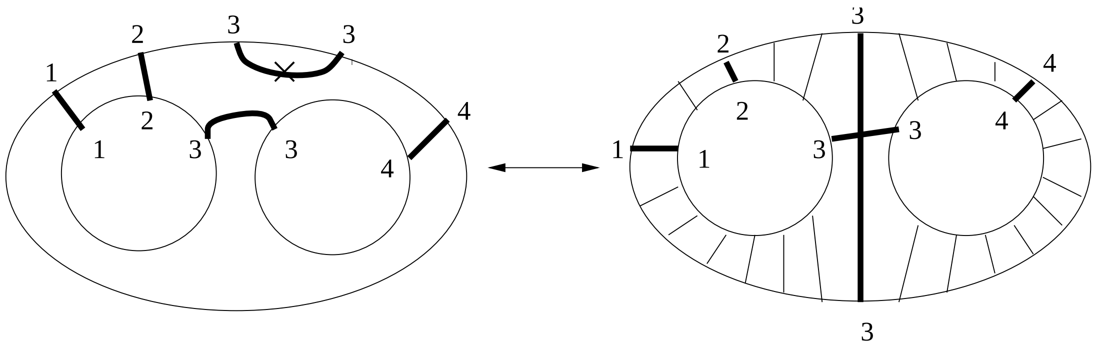

Figure 3.8: Diagrams with self-contractions, $1 \in \mathcal{I}_1$, $2 \in \mathcal{I}_2$.

$$= \quad -(-1)^n \frac{y(1-y)}{2}\left[\left(\frac{\omega_{p_1(1)}}{\mu y}+\frac{\omega_{p_2(2)}}{\mu(1-y)}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,-p_2}-\left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(12)}_{p_1,p_2}\right]$$

$$\times \prod_{j\in\mathcal{I}_1-\{1\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{2\}}\tilde{N}^{(23)}_{p_k,n_k}. \tag{3.3.63}$$

The first factor which is the result of the self-contraction between the single and two-string state, is to the leading order in $1/\mu$:

$$-\frac{1}{2}\left[\left(\frac{\omega_{p_1(1)}}{\mu y}+\frac{\omega_{p_2(2)}}{\mu(1-y)}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,-p_2}-\left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(12)}_{p_1,p_2}\right]$$

$$\simeq \quad -2\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,p_2}. \tag{3.3.64}$$

Therefore, the contribution of this Feynman diagram to the matrix element of unit normalized states is given by

$$\frac{1}{\mu}\langle (d_i,n_i)|H_3|(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle\Big|_{1-2;1-2}$$

$$\simeq \quad -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1-\{1\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{2\}}\tilde{N}^{(23)}_{p_k,n_k}. \tag{3.3.65}$$

Now let us compute the corresponding gauge theory diagrams with an interaction vertex involving ϕ_{d_1},ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1},ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$. The result is

$$\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\text{semi-nearest}}_{1-2;1-2} \quad = \quad \sqrt{\frac{y(1-y)}{J}}\times\left[\frac{1}{\sqrt{JJJ_1J_2}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1})\right]$$

$$\times \prod_{j\in\mathcal{I}_1-\{1\}} \frac{1}{\sqrt{JJ_1}} \sum_{a=0}^{J_1-1} (t_j\bar{s}_j)^a \prod_{k\in\mathcal{I}_2-\{2\}} \frac{1}{\sqrt{JJ_2}} \sum_{b=0}^{J_2-1} t_k^{J_1}(t_k\bar{s}_k)^b. \tag{3.3.66}$$

The interaction part of the diagram reduces to

$$\begin{aligned} \frac{1}{\sqrt{JJJ_1J_2}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1}) &= -e^{\pi i(n_1+n_2)y}\lambda'\frac{\sin(\pi n_1 y)\sin(\pi n_2 y)}{2\pi^2\sqrt{y(1-y)}} \\ &\simeq -2(-1)^{n_1+n_2+p_1}e^{\pi i(n_1+n_2)y}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,p_2}, \end{aligned} \tag{3.3.67}$$

while the rest of the diagram, the free contraction contribution, can be computed using (3.3.38) making the phase disappear after imposing the level matching condition on each trace.

Just as in the previous case, the Feynman diagrams we are considering do not contribute to $\Gamma^{(0)}\Big|_l$ so that their contribution to the matrix of anomalous dimensions in the string field theory basis is given by

$$\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\mathrm{semi-nearest}}_{1-2;1-2} \simeq -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1-\{1\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{2\}}\tilde{N}^{(23)}_{p_k,n_k}, \tag{3.3.68}$$

which agrees with the string theory result (3.3.65).

Chapter 4

Conclusion

In this thesis, we have attempted to extend the duality between string theory in a plane wave background and a sector of $\mathcal{N}=4$ SYM in a double scaling limit, which is derived from the Penrose limit of AdS/CFT, to the interacting theory level. In Chapter 2, we have proposed that the correct holographic map is simply (1.0.9). In practice, in order to test our proposal, we need to evaluate (1.0.9) with a particular basis of states. In the string field theory, there exists a canonical basis in the string Fock space where states with different numbers of strings are orthogonal to all orders in g_2. In the gauge theory, a natural and computationally convenient basis is BMN basis, the set of operators with a definite number of traces, i.e., single-trace operators, double-trace operators, and so on. The gauge theory inner product is given by the mixing matrix of two-point function, and BMN operators mix to all orders in g_2. By orthonormalizing the gauge theory inner product, we find another basis, "string basis," where the holographic map (1.0.9) can be directly tested. In doing so, we needed an assumption that the transformation matrix U from BMN basis to string basis is hermitian in order to fix an ambiguity in orthonormalization. It is necessary to understand the hermiticity of U from a first principle. We have shown that this proposal works in the two-impurity sector at $\mathcal{O}(g_2)$ and produces a non-trivial prediction for $\mathcal{O}(g_2^2)$ contact term, which is verified in [35].

Using the holographic map and the basis of gauge theory states proposed in Chapter 2, we have extended our argument to states with arbitrary impurities in Chapter

3. We have exactly reproduced all string amplitudes from gauge theory considerations. The calculations have been carried up to $\mathcal{O}(g_2^2)$ for the case of two identical impurities and to $\mathcal{O}(g_2)$ for arbitrary impurities. The precise agreement found here gives strong support to the validity of the holographic map (1.0.9) and the basis of gauge theory states in Chapter 2. The $\mathcal{O}(g_2^2)$ computation has been performed in the string field theory by truncating by hand the allowed intermediate states [35]. With this truncation, we get precise agreement with the gauge theory calculation. It is desirable to understand whether the truncation is necessary.

While considering arbitrary string states, we have found that there is a direct correspondence between the Feynman diagrams of gauge theory and the string field theory Feynman diagrams that contribute to a given amplitude. This diagrammatic correspondence is specially powerful when we consider general string states, in which new classes of Feynman diagrams appear as compared to the case with two different impurities. In particular, we have shown which interaction vertex in gauge theory corresponds to which string field theory vertex arising from the action of the prefactor. Likewise, the various Neumann matrices in string theory have been derived from purely field theoretic considerations as arising from various free contractions in gauge theory. The diagrammatic equivalence between gauge theory one loop diagrams and string theory diagrams shows explicitly the picture that each gauge field plays the role of a string bit and a string of gauge fields realizes a physical string. Also, it may be useful in extending the correspondence to higher orders in λ'. In fact, the string side computation is already done to all orders in λ' or equivalently in $1/\mu p^+ \alpha'$ in [33]. It would be interesting to reproduce it from gauge theory. An argument analogous to [22] might be helpful to do so without order by order consideration.

Our proposal seems to suggest that only two-point functions of the gauge theory are relevant in this duality. This is because, from the original AdS/CFT viewpoint, we are probing the vicinity of a null geodesic which lies deep inside AdS_5. Therefore, the only string worldsheet configurations in AdS_5, which survive in the Penrose limit, are those that are stretched along the null geodesic without probing near the AdS_5 boundary. Relevant observables are quantum mechanical transition amplitudes between *in* and *out* states. Hence, it is natural to consider two-point functions of the corresponding *in* and *out* operators in the gauge theory. Nevertheless, higher-point functions can be deformed to two-point functions in a pinching limit, which actually takes place in the Penrose limit. Consequently, the information about higher-point functions is encoded in operator mixing between operators with a different number of trace.

Eventually, it would also be very desirable to represent the degrees of freedom of the BMN sector of $\mathcal{N} = 4$ SYM as a complete theory, without any truncation. It is shown that the conformal boundary of the plane wave geometry is a one-dimensional null line[57]. Together with the fact that only two-point functions are relevant, holography strongly suggests that there should be an effective quantum mechanical model which describes the BMN sector of $\mathcal{N} = 4$ SYM and at the same time captures all the physics of string theory in the plane wave geometry[1].

The universality of our proposal allows one to apply it to other dualities. For example, we can introduce D-branes in a plane wave background [68, 69], and study open-closed string transition or splitting/joining process of open strings. The dual gauge theories have been studied in [70, 71, 69, 72]. Using the same holographic map and following the same procedure to find "string basis" in dual gauge theories, the string amplitudes can be reproduced from the gauge theories. This would be a severe test and a strong evidence for the proposal. A work along this line is in progress [73].

Throughout this thesis, we consider only the regime of small λ' and small g_2, in which both the string theory and the gauge theory admit perturbations. There are other interesting regimes. If $\lambda' \gg 1$ and $g_2 \ll 1$, the perturbation in the gauge theory breaks down. However, on the string theory side, we know the exact answer in λ' due to [33]. Therefore, this will give us non-trivial predictions for the gauge theory in the strong λ' region. Another important regime is when $\lambda' \ll 1$, $g_2 \gg 1$ with $g_2^2\lambda' \gg 1$. This is the regime that the size of giant gravitons [74] is bigger than that of strings, and giant gravitons become essential [18]. In gauge theory, nonplanar quantum loop corrections as well as free nonplanar diagrams are important, and both G and Γ need

[1]See [58, 59] for recent attempts along this line

to be exactly treated or resumed. The exact treatment of G may be achieved along the line of [75]. In this case, we may have to find "giant graviton basis" rather than string basis which is presumably better suited for resummation. However, how to resume gauge theory diagrams in this regime is not yet clear. One may speculate that a physical three-dimensional volume of a giant graviton can be reconstructed from proper gauge theory fields in a similar way as a string emerges from elementary gauge fields.

So far we have limited our discussion to scalar impurities which corresponds to one $SO(4)$ directions among $SO(4) \times SO(4)$ transverse directions of the plane wave background. The anomalous dimensions of BMN operators with vector impurities $D_i Z$ have been calculated in [46, 55] to give agreement with string theory predictions. It would be desirable to apply our analysis to vector impurities also.

Finally, finding black hole solutions in a plane wave or asymptotic plane wave solutions would be intriguing[76, 77]. Then, the dual gauge theory description would be a thermalization of the BMN sector. Hence, it is important to see if the truncation to the BMN sector in the double scaling limit also arises under thermalization.

The duality between a pp-wave string theory and a certain limit of a gauge theory discussed in this thesis is the first example of string/gauge duality in which we can explore a "stringy" regime explicitly. This is just beginning of a new avenue for studying string theory and gauge theory. Understanding obtained here may be useful to understand holography in a flat spacetime or in a more general situation. Eventually, we would like to address interesting problems in quantum gravity such as black holes and Hawking radiations in this context. Hopefully, the development in this thesis will shed new light on these subjects.

Appendix A

BMN operators in complex scalar notation

In the main text, we have used real scalar field notation to define BMN operators with arbitrary combination of impurities. In this case, we have four kinds of scalar impurities $\phi_1, \phi_2, \phi_3, \phi_4$ which can be inserted, and a subtlety arises when two identical impurities collide. In this appendix, we study the same problem in the complex scalar field formulation. In this formulation, there are also four kinds of impurities $\Phi, \bar{\Phi}, \Psi, \bar{\Psi}$. First, we want to see if BMN operators with anti-holomorphic insertions are well defined in the BMN limit. For example, let us consider Φ and $\bar{\Psi}$ insertions:

$$\mathcal{O}^J_{\Phi\bar{\Psi},n} = \frac{1}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} e^{2\pi i l n/J} \mathrm{Tr}\left(\Phi Z^l \bar{\Psi} Z^{J-l}\right). \qquad \text{(A.0.1)}$$

From the original Lagrangian of $\mathcal{N}=4$ SYM theory, it is easy to see that there is a symmetry which maps ϕ_4 to $-\phi_4$, thereby transforming Ψ to $\bar{\Psi}$ without changing Z and Φ. From the ten-dimensional $\mathcal{N}=1$ SYM viewpoint, it is just the reflection along one of the internal directions. In terms of an $\mathcal{N}=1$ superfield formulation of $\mathcal{N}=4$ SYM, it is equivalent to treating $Z, \Phi, \bar{\Psi}$ as chiral superfields instead of Z, Φ, Ψ. (The original D-term potential and F-term potential should regroup to give the same form of D-term and F-term potential in terms of $Z, \Phi, \bar{\Psi}$.) Therefore, the Feynman diagram computation is identical to that for BMN operators with Φ and Ψ

insertions, as it should be because in terms of the real scalar representation the four impurities are equivalent as far as same impurities do not collide. Hence, we conclude that the four complex impurities are equivalent in the dilute gas approximation.

Now let us think about the subtlety arising when two impurities collide. In the real scalar representation, only when two same impurities collide we had to add an extra term with $\bar{Z}$ insertion. In the complex scalar representation, this extra term is necessary only when Φ and $\bar{\Phi}$ collide or Ψ and $\bar{\Psi}$ collide. This can be understood from the action of R-symmetry generators on the BMN operators. (See also [64].) Let us denote the R-symmetry generator of the rotation on ϕ_i-ϕ_j plane by $R_{ij} = -R_{ji}$. More precisely,

$$R_{ij} \cdot \phi_j = \phi_i, \qquad R_{ij} \cdot \phi_i = -\phi_j. \tag{A.0.2}$$

Then define

$$R_{\Phi Z} = \frac{1}{2}\left(R_{15} + R_{26} + iR_{25} - iR_{16}\right), \qquad R_{\bar{\Phi} Z} = \frac{1}{2}\left(R_{15} - R_{26} - iR_{25} - iR_{16}\right). \tag{A.0.3}$$

Their actions are given as

$$\begin{aligned} &R_{\Phi Z} \cdot Z = \Phi, \qquad R_{\Phi Z} \cdot \bar{\Phi} = -\bar{Z}, \qquad R_{\Phi Z} \cdot \bar{Z} = R_{\Phi Z} \cdot \Phi = 0, \\ &R_{\bar{\Phi} Z} \cdot Z = \bar{\Phi}, \qquad R_{\bar{\Phi} Z} \cdot \Phi = -\bar{Z}, \qquad R_{\bar{\Phi} Z} \cdot \bar{Z} = R_{\bar{\Phi} Z} \cdot \bar{\Phi} = 0, \end{aligned} \tag{A.0.4}$$

and likewise for $R_{\Psi Z}$ and $R_{\bar{\Psi} Z}$. BPS BMN operators can be obtained by acting these generators successively on the vacuum operator $\mathrm{Tr}(Z^J)$. For example, if we want to insert Φ and Ψ, we act with $R_{\Phi Z}$ and $R_{\Psi Z}$ on $\mathrm{Tr}(Z^{J+2})$,

$$R_{\Psi Z} \cdot \Big(R_{\Phi Z} \cdot \mathrm{Tr}(Z^{J+2})\Big) = R_{\Psi Z} \cdot \left(\sum_{l=0}^{J+1} \mathrm{Tr}(Z^l \Phi Z^{J+1-l})\right) = (J+2) \sum_{l=0}^{J} \mathrm{Tr}(\Phi Z^l \Psi Z^{J-l}). \tag{A.0.5}$$

Since $R_{\Psi Z} \cdot \Phi = 0$, we don't have any extra term arising when $R_{\Psi Z}$ acts on Φ. It is also the case when we insert two Φ's because $R_{\Phi Z} \cdot \Phi = 0$. Now let us consider Φ and $\bar{\Phi}$ insertions.

$$R_{\bar{\Phi} Z} \cdot \Big(R_{\Phi Z} \cdot \mathrm{Tr}(Z^{J+2})\Big) = R_{\bar{\Phi} Z} \cdot \left(\sum_{l=0}^{J+1} \mathrm{Tr}(Z^l \Phi Z^{J+1-l})\right)$$

$$= (J+2)\left(\sum_{l=0}^{J} \mathrm{Tr}(\Phi Z^l \bar{\Phi} Z^{J-l}) - \mathrm{Tr}(\bar{Z} Z^{J+1})\right). \tag{A.0.6}$$

The $\bar{Z}$ term arises when $R_{\bar{\Phi}Z}$ acts on Φ, in other words, when Φ and $\bar{\Phi}$ "collide". We conclude that only when holomorphic and antiholomorphic insertions of the same kind collide, we need to add an extra $\bar{Z}$ term. From this consideration, we can also learn that no extra term is necessary when $\bar{Z}$ collides with the four impurities because all the four generators annihilate $\bar{Z}$. For example, when $R_{\Phi Z}$ acts on $\mathrm{Tr}(\bar{Z} Z^{J+1})$,

$$R_{\Phi Z} \cdot \mathrm{Tr}(\bar{Z} Z^{J+1}) = \sum_{l=0}^{J} \mathrm{Tr}(\bar{Z} Z^l \Phi Z^{J-l}). \tag{A.0.7}$$

This implies that we don't have to worry about collision of more than two impurities. In general, we have only to take care of holomorphic and antiholomorphic impurities of the same kind pairwise.

Appendix B

Off-shell representation of BMN operators

In this appendix, we carefully define "on-shell" and "off-shell" representations of BMN operators which are introduced in the main text. (See also [19].) Here by shell we mean the level matching condition shell, which states that the sum of all worldsheet momentum vanishes. In the on-shell representation, we fix the position of one scalar impurity and sum over positions of the rest of the impurities. To explain more explicitly, let us consider the case of three impurities. In this case, we have

$$\mathcal{O}^J_{\rm on} = \sum_{0\le l_2,l_3\le J} {\rm Tr}\,(\phi_{d_1} Z\cdots Z\phi_{d_2} Z\cdots Z\phi_{d_3} Z\cdots Z)\, s_2^{l_2} s_3^{l_3}, \tag{B.0.1}$$

where $d_i \in \{1,2,3,4\}$ is the direction of the i-th impurity, l_i is the number[1] of Z in front of ϕ_{d_i} and $s_i = e^{2\pi i n_i/J}$ is the phase assigned to ϕ_{d_i}. This definition gets ambiguous when two impurities sit next to each other. Therefore, we need a rigorous definition:

$$\mathcal{O}^J_{\rm on} = \mathcal{O}^J_{\rm on,c} + \mathcal{O}^J_{\rm on,a}, \tag{B.0.2}$$

with

$$\mathcal{O}^J_{\rm on,c} \;=\; \sum_{0\le a_2\le a_3\le J} {\rm Tr}(\phi_{d_1} Z^{a_2}\phi_{d_2} Z^{a_3-a_2}\phi_{d_3} Z^{J-a_3}) s_2^{a_2} s_3^{a_3},$$

[1]In [64], l_i is argued to include the number of other impurities in front of it, but the difference is only subleading in $1/J$ and inconsequential throughout this paper.

$$\mathcal{O}^J_{\text{on,a}} = \sum_{0\le a_3\le a_2\le J} \text{Tr}(\phi_{d_1} Z^{a_3}\phi_{d_3} Z^{a_2-a_3}\phi_{d_2} Z^{J-a_2}) s_2^{a_2} s_3^{a_3}. \tag{B.0.3}$$

To normalize this operator canonically, let us compute its free two-point function:

$$\langle \bar{\mathcal{O}}^J_{\text{on}} \mathcal{O}^J_{\text{on}}\rangle = \langle \bar{\mathcal{O}}^J_{\text{on,c}} \mathcal{O}^J_{\text{on,c}}\rangle + \langle \bar{\mathcal{O}}^J_{\text{on,a}} \mathcal{O}^J_{\text{on,a}}\rangle = (J+1)(J+2)N^{J+3}. \tag{B.0.4}$$

Here we have counted the number of pairs (a_2, a_3) such that $0 \le a_2 \le a_3 \le J$, which is $\binom{J+2}{2}$. Hence, in the BMN limit, the correct normalization is

$$\mathcal{O}^J_{\text{BMN}} = \frac{1}{J\sqrt{N^{J+3}}} \mathcal{O}^J_{\text{on}}. \tag{B.0.5}$$

Now let us move on to the off-shell representation. In the off-shell representation, we do not fix the position of any scalar impurity and treat them on equal footing by summing over all possible positions of all impurities. For our present case of 3 impurities, we define

$$\mathcal{O}^J_{\text{off}} = \sum_{0\le l_1,l_2,l_3\le J} \text{Tr}\left(Z\cdots Z\phi_{d_1} Z\cdots Z\phi_{d_2} Z\cdots Z\phi_{d_3} Z\cdots Z\right) s_1^{l_1} s_2^{l_2} s_3^{l_3}, \tag{B.0.6}$$

where l_i is defined in the same way as above. Again, a rigorous definition is given by

$$\mathcal{O}^J_{\text{off}} = \mathcal{O}^J_{\text{off}}(1,2,3) + \mathcal{O}^J_{\text{off}}(2,3,1) + \mathcal{O}^J_{\text{off}}(3,1,2) + \mathcal{O}^J_{\text{off}}(1,3,2) + \mathcal{O}^J_{\text{off}}(3,2,1) + \mathcal{O}^J_{\text{off}}(2,1,3), \tag{B.0.7}$$

with

$$\begin{aligned}
\mathcal{O}^J_{\text{off}}(1,2,3) &= \sum_{0\le a_1\le a_2\le a_3\le J} \text{Tr}\left(Z^{a_1}\phi_{d_1} Z^{a_2-a_1}\phi_{d_2} Z^{a_3-a_2}\phi_{d_3} Z^{J-a_3}\right) s_1^{a_1} s_2^{a_2} s_3^{a_3}, \\
\mathcal{O}^J_{\text{off}}(2,3,1) &= \sum_{0\le a_2\le a_3\le a_1\le J} \text{Tr}\left(Z^{a_2}\phi_{d_2} Z^{a_3-a_2}\phi_{d_3} Z^{a_1-a_3}\phi_{d_1} Z^{J-a_1}\right) s_1^{a_1} s_2^{a_2} s_3^{a_3}, \\
\vdots \quad & \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad ,
\end{aligned} \tag{B.0.8}$$

where the other operators are defined likewise. Now the claim is that $\mathcal{O}^J_{\text{off}}$ is non-vanishing if and only if $n_1 + n_2 + n_3 = 0$:

$$\mathcal{O}^J_{\text{off}} \neq 0 \Longleftrightarrow n_1 + n_2 + n_3 = 0. \tag{B.0.9}$$

Note that this condition is exactly the level-matching condition in string field theory. Furthermore, if this condition holds, we have

$$\begin{aligned} \mathcal{O}^J_{\text{off}}(1,2,3)+\mathcal{O}^J_{\text{off}}(2,3,1)+\mathcal{O}^J_{\text{off}}(3,1,2) &= J\mathcal{O}^J_{\text{on,c}}, \\ \mathcal{O}^J_{\text{off}}(1,3,2)+\mathcal{O}^J_{\text{off}}(3,2,1)+\mathcal{O}^J_{\text{off}}(2,1,3) &= J\mathcal{O}^J_{\text{on,a}}, \end{aligned} \tag{B.0.10}$$

and the off-shell representation (B.0.7) is reduced to the on-shell one (B.0.2):

$$\mathcal{O}^J_{\text{off}} = J\mathcal{O}^J_{\text{on}}. \tag{B.0.11}$$

This explains the terminology of "on-shell/off-shell" representation. Consequently, the correct normalization of the off-shell operator is

$$\mathcal{O}^J_{\text{BMN}} = \frac{1}{\sqrt{J}\sqrt{J^3}\sqrt{N^{J+3}}}\mathcal{O}^J_{\text{off}}. \tag{B.0.12}$$

This argument can be immediately generalized to n impurities assuming all of them are different. The on-shell operator is

$$\mathcal{O}^J_{\text{on}} = \sum_{0\le l_2,\cdots,l_n\le J} \text{Tr}\left(\phi_{d_1}Z\cdots Z\phi_{d_2}Z\cdots\cdots Z\phi_{d_n}Z\cdots Z\right)\prod_{i=2}^{n} s_i^{l_i}, \tag{B.0.13}$$

with l_i being the number of Z's in front of ϕ_i as before. Or more rigorously the definition of it is given as the sum of $(n-1)!$ operators corresponding to permutations after fixing the position of one impurity:

$$\mathcal{O}^J_{\text{on}} = \sum_{\sigma\in\text{Perm}\{2,\cdots,n\}} \mathcal{O}^J_{\text{on},\sigma}, \tag{B.0.14}$$

with

$$\mathcal{O}^J_{\text{on},\sigma} = \sum_{0\le a_{\sigma(2)}\le\cdots\le a_{\sigma(n)}\le J} \text{Tr}\left(\phi_{d_1}Z^{a_{\sigma(2)}}\phi_{d_{\sigma(2)}}Z^{a_{\sigma(3)}-a_{\sigma(2)}}\phi_{d_{\sigma(3)}}\cdots\phi_{d_{\sigma(n)}}Z^{J-a_{\sigma(n)}}\right)\prod_{i=2}^{n} s_i^{a_i}. \tag{B.0.15}$$

Each $\mathcal{O}^{J,n}_{on,\sigma}$ is composed of $\binom{J+n-1}{n-1} \simeq \frac{J^{n-1}}{(n-1)!}$ terms, where the combinatoric number comes from the number of $(n-1)$-tuple $(a_2, a_3, \cdots, a_n)$ satisfying $0 \le a_{\sigma(2)} \le \cdots \le a_{\sigma(n)} \le J$. Hence, the normalization is

$$\mathcal{O}^J_{\text{BMN}} = \frac{1}{\sqrt{J^{n-1}}\sqrt{N^{J+n}}}\mathcal{O}^J_{\text{on}}. \tag{B.0.16}$$

Similarly, the off-shell operator with n impurities is

$$\mathcal{O}^J_{\rm off} = \sum_{0\le l_1,\cdots,l_n\le J} \mathrm{Tr}\,(Z\cdots Z\phi_{d_1}Z\cdots Z\phi_{d_2}Z\cdots\cdots Z\phi_{d_n}Z\cdots Z)\prod_{i=1}^{n} s_i^{l_i}, \tag{B.0.17}$$

with a rigorous definition given by a sum over $n!$ terms. As in 3-impurity case, we have

$$\mathcal{O}^J_{\rm off} = J\mathcal{O}^J_{\rm on}, \tag{B.0.18}$$

if and only if the level-matching (on-shell) condition holds for the off-shell operator. Hence the normalization for the off-shell operator is

$$\mathcal{O}^J_{\rm BMN} = \frac{1}{\sqrt{J}\sqrt{J^n}\sqrt{N^{J+n}}}\mathcal{O}^J_{\rm off}. \tag{B.0.19}$$

Here we can think of each impurity as carrying a normalization factor $1/\sqrt{J}$, since we sum over J possible positions for each impurity. The leftover factor $1/\sqrt{J}$ is the original normalization of the vacuum operator and it originates in the cyclic property of Tr.

So far, we have defined on-shell and off-shell operators assuming that all impurities are distinct. However, we have to deal with same impurities eventually since there are only 4 directions. When two impurities, say ϕ_{d_1} and ϕ_{d_2}, are the same, we have to insert $-\bar{Z}$ when they collide as discussed in Appendix A. Then the correct definition is in the off-shell representation,

$$\begin{aligned}\mathcal{O}^J_{\rm off} = &\sum_{0\le l_1,l_2,l_3,\cdots,l_n\le J} \mathrm{Tr}\,(Z\cdots Z\phi_{d_1}Z\cdots Z\phi_{d_2}Z\cdots Z\phi_{d_3}Z\cdots\cdots Z\phi_{d_n}Z\cdots Z)\prod_{i=1}^{n} s_i^{l_i} \\ &- \sum_{0\le l_{(1,2)},l_3,\cdots,l_n\le J+1} \mathrm{Tr}\left(Z\cdots Z\bar{Z}Z\cdots Z\phi_{d_3}Z\cdots\cdots Z\phi_{d_n}Z\cdots Z\right)(s_1s_2)^{l_{(1,2)}}\prod_{i=3}^{n} s_i^{l_i},\end{aligned} \tag{B.0.20}$$

where $l_{(1,2)}$ is the number of Z's in front of $\bar{Z}$ arising when ϕ_{d_1} and ϕ_{d_2} collide. Now we have to do this modification whenever we have a pair (i,j) such that $d_i = d_j$. However, as argued in Appendix A, we do not have to worry about collision of more than two impurities. The normalization is not changed since the number of $\bar{Z}$ terms is subleading in $1/J$ compared with the original terms because $\bar{Z}$ terms arise only when two impurities collide.

As an example, let us consider a BMN operator with 4 same impurities, i.e., $d_1 = d_2 = d_3 = d_4$. In this case the BMN operator should be modified by $-\bar{Z}$ as

$$\begin{aligned}
\mathcal{O}^J_{\text{off}} = & \sum_{0 \leq l_1, l_2, l_3, l_4 \leq J} \mathrm{Tr}\,(Z \cdots Z \phi_{d_1} Z \cdots Z \phi_{d_2} Z \cdots Z \phi_{d_3} Z \cdots Z \phi_{d_4} Z \cdots Z) \prod_{i=1}^{4} s_i^{l_i} \\
& - \sum_{0 \leq l_{(1,2)}, l_3, l_4 \leq J+1} \mathrm{Tr}\left(Z \cdots Z \bar{Z} Z \cdots Z \phi_{d_3} Z \cdots Z \phi_{d_4} Z \cdots Z\right) (s_1 s_2)^{l_{(1,2)}} s_3^{l_3} s_4^{l_4} \\
& - \sum_{0 \leq l_{(1,3)}, l_2, l_4 \leq J+1} \mathrm{Tr}\left(Z \cdots Z \bar{Z} Z \cdots Z \phi_{d_2} Z \cdots Z \phi_{d_4} Z \cdots Z\right) (s_1 s_3)^{l_{(1,3)}} s_2^{l_2} s_4^{l_4} \\
& \qquad \vdots \\
& + \sum_{0 \leq l_{(1,2)}, l_{(3,4)} \leq J+2} \mathrm{Tr}\left(Z \cdots Z \bar{Z} Z \cdots Z \bar{Z} Z \cdots Z\right) (s_1 s_2)^{l_{(1,2)}} (s_3 s_4)^{l_{(3,4)}} \\
& \qquad \vdots \qquad .
\end{aligned} \tag{B.0.21}$$

Appendix C

Matrix elements

The definition of various matrices appearing in G and Γ on the gauge theory calculation are given as follows.

2-impurity matrix elements $(|m| \neq |n|, m \neq 0, n \neq 0, p \in Z, 0 < y < 1)$

- $$C_{n,py} = C_{py,n} = \frac{y^{3/2}\sqrt{1-y}}{\sqrt{J}\pi^2}\frac{\sin^2(\pi n y)}{(p-ny)^2}$$

  $$C_{n,y} = C_{y,n} = -\frac{1}{\sqrt{J}\pi^2}\frac{\sin^2(\pi n y)}{n^2}$$

- $$M^1_{n,n} = \frac{1}{60} - \frac{1}{24\pi^2 n^2} + \frac{7}{16\pi^4 n^4}$$

  $$M^1_{n,-n} = \frac{1}{48\pi^2 n^2} + \frac{35}{128\pi^4 n^4}$$

  $$M^1_{n,m} = \frac{1}{12\pi^2(n-m)^2} - \frac{1}{8\pi^4(n-m)^4} + \frac{1}{4\pi^4 n^2 m^2} + \frac{1}{8\pi^4 nm(n-m)^2}$$

- $$\Gamma^{(1)}_{n,py} = \Gamma^{(1)}_{py,n} = \lambda'\left(\frac{p^2}{y^2} - \frac{pn}{y} + n^2\right)C_{n,py}$$

  $$\Gamma^{(1)}_{n,y} = \Gamma^{(1)}_{y,n} = \lambda' n^2 C_{n,y}$$
  $$\Gamma^{(2)}_{n,m} = \lambda' nm M^1_{n,m} + \frac{1}{8\pi^2}\mathcal{D}^1_{n,m}$$

- $$\mathcal{D}^1_{n,n} = \mathcal{D}^1_{n,-n} = \lambda'\left(\frac{2}{3} + \frac{5}{\pi^2 n^2}\right)$$

$$\mathcal{D}^1_{n,m} = \lambda'\left(\frac{2}{3} + \frac{2}{\pi^2 n^2} + \frac{2}{\pi^2 m^2}\right)$$

- $$B_{n,n} = \frac{1}{3} + \frac{5}{2\pi^2 n^2}$$

$$B_{n,-n} = -\frac{15}{8\pi^2 n^2}$$

$$B_{n,m} = \frac{3}{2\pi^2 mn} + \frac{1}{2\pi^2(m-n)^2} \tag{C.0.1}$$

***n*-impurity matrix elements**

- $$G^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y} = G^{(1)}_{\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y,\{n_i\}}$$

$$= (-1)^{n+\sum_{k\in\mathcal{I}_2} n_k} \frac{\sqrt{y^{n_1+1}}\sqrt{(1-y)^{n_2+1}}}{\sqrt{J}} \prod_{j\in\mathcal{I}_1} \frac{\sin(\pi n_j y)}{\pi(p_j - n_j y)} \prod_{k\in\mathcal{I}_2} \frac{\sin(\pi n_k(1-y))}{\pi(p_k - n_k(1-y))},$$

- $$\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y} = \Gamma^{(1)}_{\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y,\{n_i\}}$$

$$= \frac{\lambda'}{2}\left[\sum_{j\in\mathcal{I}_1}\left(\left(n_j - \frac{p_j}{y}\right)^2 + n_j\frac{p_j}{y}\right) + \sum_{k\in\mathcal{I}_2}\left(\left(n_k - \frac{p_k}{1-y}\right)^2 + n_k\frac{p_k}{1-y}\right)\right]$$

$$\times G^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}. \tag{C.0.2}$$

where $n_1 = |\mathcal{I}_1|, \;\; n_2 = |\mathcal{I}_2|$.

Appendix D

Useful summation formulas

When one multiplies two matrices in (2.3.36), the following formulae are useful[1]:

- $$\sum_{p,y} C_{n,py}C_{py,m} = \frac{1}{J\pi^4} J\int_0^1 dy y^3(1-y)\sin^2(\pi n y)\sin^2(\pi m y)\sum_{p=-\infty}^{\infty}\frac{1}{(p-nr)^2(p-mr)^2}$$

$$= \begin{cases} \frac{1}{6\pi^2(n-m)^2} + \frac{1}{4\pi^4 n^2 m^2} + \frac{1}{\pi^4 nm(n-m)^2} - \frac{1}{4\pi^4(n-m)^4} & \text{if } n \neq m \\ \frac{1}{30} - \frac{1}{12\pi^2 n^2} + \frac{1}{2\pi^4 n^4} & \text{if } n = m \end{cases}$$

- $$\sum_{y} C_{n,y}C_{y,m} = \frac{1}{J\pi^4} J\int_0^1 dy \frac{\sin^2(\pi n y)}{n^2}\frac{\sin^2(\pi m y)}{m^2}$$

$$= \begin{cases} \frac{1}{4\pi^4 n^2 m^2} & \text{if } n \neq m, -m \\ \frac{3}{8\pi^4 n^4} & \text{if } n = m, -m \end{cases}$$

- $$\sum_{p,y} \frac{p}{y} C_{n,py}C_{py,m} = \frac{1}{J\pi^4} J\int_0^1 dy y^2(1-y)\sin^2(\pi n y)\sin^2(\pi m y)\sum_{p=-\infty}^{\infty}\frac{p}{(p-nr)^2(p-mr)^2}$$

$$= \begin{cases} (n+m)\left\{\frac{1}{12\pi^2(n-m)^2} + \frac{1}{4\pi^4 n^2 m^2} + \frac{1}{8\pi^4 nm(n-m)^2} - \frac{1}{8\pi^4(n-m)^4}\right\} & \text{if } n \neq m \\ \frac{n}{30} - \frac{1}{12\pi^2 n} + \frac{7}{8\pi^4 n^3} & \text{if } n = m \end{cases}$$

- $$\sum_{p,y} \frac{p^2}{y^2} C_{n,py}C_{py,m} = \frac{1}{J\pi^4} J\int_0^1 dy y(1-y)\sin^2(\pi n y)\sin^2(\pi m y)\sum_{p=-\infty}^{\infty}\frac{p^2}{(p-nr)^2(p-mr)^2}$$

[1] Similar identities can also be found in the Appendix of [48].

$$= \begin{cases} \frac{n^2+m^2}{12\pi^2(n-m)^2} + \frac{n^6+m^6-2nm(n^4+m^4)+n^3m^3}{4\pi^4 n^2 m^2 (n-m)^4} & \text{if } n \neq m \\ \frac{n^2}{30} + \frac{3}{2\pi^4 n^2} & \text{if } n = m \end{cases} .$$

Appendix E

Asymptotic behavior of Neumann matrices

In this appendix we present the asymptotic large μ behavior of all Neumann matrices in the exponential basis. These can be obtained from $(m, n \neq 0)$

$$\tilde{N}^{(rs)}_{m,n} = \frac{1}{2}(\bar{N}^{(rs)}_{|m|,|n|} - e(mn)\bar{N}^{(rs)}_{-|m|,-|n|}), \qquad \tilde{N}^{(rs)}_{m,0} = \frac{1}{\sqrt{2}}\bar{N}^{(rs)}_{|m|,0}, \qquad \tilde{N}^{(rs)}_{0,0} = \bar{N}^{(rs)}_{0,0}, \tag{E.0.1}$$

where $e(m) = \mathrm{sign}(m)$ and the asymptotic behavior of Neumann matrices in the cos/sin basis in [33]:

$$\tilde{N}^{(11)}_{m,n} \simeq \frac{(-1)^{m+n}}{4\pi\mu y}$$

$$\tilde{N}^{(12)}_{m,n} \simeq \frac{(-1)^{m+1}}{4\pi\mu\sqrt{y(1-y)}}$$

$$\tilde{N}^{(22)}_{m,n} \simeq \frac{1}{4\pi\mu(1-y)}$$

$$\tilde{N}^{(13)}_{m,n} \simeq \frac{(-1)^{m+n+1}\sin n\pi y}{\pi\sqrt{y}(n - m/y)}$$

$$\tilde{N}^{(23)}_{m,n} \simeq \frac{(-1)^{n}\sin n\pi y}{\pi\sqrt{1-y}(n - m/(1-y))}$$

$$\tilde{N}^{(33)}_{m,n} \simeq \frac{(-1)^{m+n+1}\sin m\pi y \sin n\pi y}{\pi\mu}. \tag{E.0.2}$$

For the computation of the contact term, we also need $\tilde{F}^{\pm}$ in the exponential basis $(n \neq 0)$

$$\tilde{F}^{\pm}_{n(r)} = \frac{1}{\sqrt{2}}F^{\pm}_{|n|(r)}, \qquad \tilde{F}^{\pm}_{0(r)} = F^{\pm}_{0(r)}. \tag{E.0.3}$$

and the scalar quantity k and fermionic Neumann matrices $\bar{Y}$. Using again the results in [33], we have

$$\begin{aligned}
\tilde{F}^{+}_{(1)n} &\simeq (-1)^{n+1}\sqrt{\mu y}(1-y) \\
\tilde{F}^{+}_{(2)n} &\simeq \sqrt{\mu(1-y)}y \\
\tilde{F}^{+}_{(3)n} &\simeq \frac{(-1)^{n+1}ny(1-y)\sin\pi ny}{\sqrt{\mu}},
\end{aligned} \tag{E.0.4}$$

$$\begin{aligned}
\tilde{F}^{-}_{(1)n} &\simeq \frac{(-1)^{n+1}n(1-y)}{2\sqrt{\mu y}} \\
\tilde{F}^{-}_{(2)n} &\simeq \frac{ny}{2\sqrt{\mu(1-y)}} \\
\tilde{F}^{-}_{(3)n} &\simeq 2\sqrt{\mu}y(1-y)(-1)^{n+1}\sin\pi ny.
\end{aligned} \tag{E.0.5}$$

$$1-\mu y(1-y)k \simeq \frac{1}{4\pi\mu y(1-y)}, \tag{E.0.6}$$

$$\bar{Y}_0 \simeq \frac{1}{\sqrt{4\pi\mu y(1-y)}}, \qquad \bar{Y}_{n(1)} \simeq \sqrt{\frac{1-y}{4\pi\mu}}(-1)^{n+1}, \qquad \bar{Y}_{n(2)} \simeq \sqrt{\frac{y}{4\pi\mu}}. \tag{E.0.7}$$

Appendix F

Calculation of supersymmetry charge matrix elements

In this appendix we shall show explicitly how to reduce the supersymmetry vertex in [32] to our simple formula (3.2.12) when we assume the external state to have two bosonic impurities and the intermediate state to have one bosonic and one fermionic impurity. See also [35].

The Hamiltonian and supersymmetry charge vertices in [32] are given by

$$
\begin{aligned}
|H_3\rangle &= c\Big((1+\mu\alpha k)(K_+^i - K_-^i)(K_+^j + K_-^j) - \mu\alpha\delta^{ij}\Big)v_{ij}(Y)E_aE_bE_{b0}|0\rangle, \\
|Q_{3\dot a}\rangle &= c(1+\mu\alpha k)^{1/2}(K_+^i - K_-^i)s^i_{\dot a}(Y)E_aE_bE_{b0}|0\rangle, \\
|\bar Q_{3\dot a}\rangle &= c(1+\mu\alpha k)^{1/2}(K_+^i + K_-^i)\tilde s^i_{\dot a}(Y)E_aE_bE_{b0}|0\rangle. \qquad \text{(F.0.1)}
\end{aligned}
$$

Various constituents of the prefactor, $K^i_\pm$, v^{ij} $s^i_{\dot a} = -i\sqrt{2}(\eta s^i_{1\dot a} + \bar\eta s^i_{2\dot a})$ and $\tilde s^i_{\dot a} = i\sqrt{2}(\bar\eta s^i_{1\dot a} + \eta s^i_{2\dot a})$ are given as

$$
\begin{aligned}
K_+^i &= \sum_{r=1}^{3}\sum_{m=-\infty}^{\infty}\tilde F^+_{m(r)}\alpha^{i\dagger}_{m(r)}, \\
K_-^i &= \sum_{r=1}^{3}\sum_{m=-\infty}^{\infty}\tilde F^-_{m(r)}\alpha^{i\dagger}_{m(r)}, \qquad \text{(F.0.2)}
\end{aligned}
$$

$$
v^{ij} = \delta^{ij} + \frac{1}{4!\alpha^2}t^{ij}_{abcd}Y^aY^bY^cY^d + \frac{1}{8!\alpha^4}\delta^{ij}\epsilon_{abcdefgh}Y^a\cdots Y^h
$$

$$+\frac{1}{2!\alpha}\gamma^{ij}_{ab}Y^aY^b+\frac{1}{2!6!\alpha^3}\gamma^{ij}_{ab}\epsilon^{ab}{}_{cdefgh}Y^c\cdots Y^h,$$

$$\frac{1}{\sqrt{2}}s^i_{1\dot{a}} = \gamma^i_{a\dot{a}}Y^a+\frac{1}{3!5!\alpha^2}u^i_{abc\dot{a}}\epsilon^{abc}{}_{defgh}Y^d\cdots Y^h,$$

$$\frac{1}{\sqrt{2}}s^i_{2\dot{a}} = -\frac{1}{3!\alpha}u^i_{abc\dot{a}}Y^aY^bY^c+\frac{1}{7!\alpha^3}\gamma^i_{a\dot{a}}\epsilon^a{}_{bcdefgh}Y^b\cdots Y^h, \tag{F.0.3}$$

where Y^a reads

$$Y^a=\sqrt{2}Y_0(\alpha_{(1)}\lambda^a_{(2)}-\alpha_{(2)}\lambda^a_{(1)})+\sum_{r=1}^{3}\sum_{m=1}^{\infty}Y_{m(r)}b^{a\dagger}_{m(r)}, \tag{F.0.4}$$

with

$$\lambda^a_{(r)}=\sqrt{\frac{\alpha_{(r)}}{2}}\begin{pmatrix} b^{a\dagger}_{0(r)} \\ b^a_{0(r)}\end{pmatrix} \quad (r=1,2), \qquad \lambda^a_{(3)}=\frac{1}{\sqrt{2}}\begin{pmatrix} b^a_{0(3)} \\ b^{a\dagger}_{0(3)}\end{pmatrix}, \tag{F.0.5}$$

and

$$Y_0=\bar{Y}_0\begin{pmatrix}1 & \\ & 0\end{pmatrix}+\frac{1}{\bar{Y}_0}\begin{pmatrix}0 & \\ & 1\end{pmatrix}, \tag{F.0.6}$$

$$Y_{n(1)}=\bar{Y}_{n(1)}\begin{pmatrix}1 & \\ & 0\end{pmatrix}, \qquad Y_{n(2)}=\bar{Y}_{n(2)}\begin{pmatrix}1 & \\ & 0\end{pmatrix}. \tag{F.0.7}$$

Note that in the matrix representation of (F.0.5) (F.0.6) (F.0.7), the upper(left) entries denote the components with spinor indices $a=1,\cdots,4$, while the lower(right) ones denote $a=5,\cdots,8$. Also E_a, E_b and E_{b0} come from the overlapping condition of bosonic modes, fermionic non-zero modes and fermionic zero modes, respectively,

$$E_a=\exp\Big(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\alpha^{i\dagger}_{(r)m}\tilde{N}^{(rs)}_{mn}\alpha^{i\dagger}_{(s)n}\Big), \tag{F.0.8}$$

$$E_{b0}=\frac{1}{2^4}\prod_{a=1}^{4}(\sqrt{\alpha_{(1)}}b^{a\dagger}_{(1)}+\sqrt{\alpha_{(2)}}b^{a\dagger}_{(2)}+b^a_{(3)})\prod_{b=5}^{8}(\sqrt{\alpha_{(1)}}b^b_{(1)}+\sqrt{\alpha_{(2)}}b^b_{(2)}+b^{b\dagger}_{(3)}), \tag{F.0.9}$$

and the explicit expression of E_b is not necessary in our analysis. Finally the "ground" state $|0\rangle$ is related to the "vacuum" state with the lowest energy by

$$|0\rangle=\prod_{a=5}^{8}b^{a\dagger}_{(1)}\prod_{b=5}^{8}b^{b\dagger}_{(2)}\prod_{c=1}^{4}b^{c\dagger}_{(3)}|\mathrm{vac}\rangle. \tag{F.0.10}$$

Now we would like to calculate the supersymmetry charge matrix elements

$$Q_{n,m(s)} = \langle \text{vac}|\alpha^i_{n(3)}\alpha^i_{-n(3)}\alpha^k_{m(s)}\frac{1}{\sqrt{2}}(b^d_{m(s)} - ie(m)b^d_{-m(s)})|Q_{\dot{a}}\rangle, \tag{F.0.11}$$

where we assume the external states to be two bosonic impurity states and the intermediate states to be states with one bosonic and one fermionic impurity. The supersymmetry charge matrix elements under this assumption will be greatly simplified. We find the $(Y)^1$ and $(Y)^7$ terms in (F.0.3) vanish and the $(Y)^3$ and $(Y)^5$ terms reduce to (3.2.12).

The typical matrix element of the supersymmetry charge (F.0.11) is

$$\langle \text{vac}|\alpha^i_{n(3)}\alpha^i_{-n(3)}\alpha^k_{m(s)}\frac{1}{\sqrt{2}}(b^d_{m(s)} - ie(m)b^d_{-m(s)})K^{\pm}Y^{\ell}E_aE_bE_{b0}|0\rangle, \tag{F.0.12}$$

where ℓ denotes the number of fermions in the expression . First of all, let us concentrate on the zero mode $b_{0(3)}$ operators. Since we only have $b_{0(3)}$ in E_{b0} and $|0\rangle$, all the $b_{0(3)}$ operators should cancel out to obtain a non-vanishing contribution. The only possibility is that $b^a_{0(3)}$ $(a = 1, \cdots, 4)$ in E_{b0} cancels those in $|0\rangle$ and we never use $b^b_{0(3)}$ $(b = 5, \cdots, 8)$ in E_{b0}. Using this fact, our typical matrix elements become

$$\langle \text{vac}|\alpha^i_{n(3)}\alpha^i_{-n(3)}\alpha^k_{m(s)}\frac{1}{\sqrt{2}}(b^d_{m(s)} - ie(m)b^d_{-m(s)})K^{\pm}Y^{\ell}E_aE_{b0}\prod_{a=5}^{8} b^{a\dagger}_{0(1,2)}|\text{vac}\rangle, \tag{F.0.13}$$

with $b^a_{0(1,2)}$ meaning $b^a_{0(1)}$ or $b^a_{0(2)}$. Next, let us concentrate on the zero mode $b_{0(1,2)}$ operators. In case of $\ell = 1$, we will not have enough annihilation operators to cancel all the leftover zero modes in $|0\rangle$. In case $\ell = 7$, we use four of $b(Y)^7$ to cancel the zero modes. But the rest must all be the creation operators and now we have too many of them. If $\ell = 3$, exactly four operators in $b(Y)^3$ are used to cancel the leftover in $|0\rangle$. If $\ell = 5$, four in $b(Y)^5$ are used to cancel. Since Y does not have both the creation operators and annihilation ones for the same operator, two of the Y's cannot cancel each other. Therefore we have to choose four operators in Y to cancel the creation operators in $|0\rangle$ and let the remaining Y's be cancelled by the b of the intermediate state.

For the $(Y)^3$ term, only the zero modes contribute:

$$\sim \bar{Y}_0(\tilde{F}^{\pm}_{(1)0}\tilde{N}^{(33)}_{n,-n} + \tilde{F}^{\pm}_{(3)n}\tilde{N}^{(13)}_{0,-n} + \tilde{F}^{\pm}_{(3)-n}\tilde{N}^{(13)}_{0,n}). \tag{F.0.14}$$

For the $(Y)^5$ term, besides the zero modes contribution, the non-zero modes also contribute as:

$$\sim \frac{\bar{Y}_{m(1)}}{\sqrt{2}}(\tilde{F}^{\pm}_{(1)m}\tilde{N}^{(33)}_{n,-n} + \tilde{F}^{\pm}_{(3)n}\tilde{N}^{(13)}_{m,-n} + \tilde{F}^{\pm}_{(3)-n}\tilde{N}^{(13)}_{m,n}). \tag{F.0.15}$$

Using the large μ behavior of various Neumann coefficients in Appendix A, we find that $\tilde{F}^{-}_{(3)}\tilde{N}^{(13)}$ gives the leading contribution. Besides, from the symmetry of the Neumann coefficients, we have

$$\tilde{F}^{-}_{(3)n}\tilde{N}^{(13)}_{0,-n} + \tilde{F}^{-}_{(3)-n}\tilde{N}^{(13)}_{0,n} \simeq 0. \tag{F.0.16}$$

Therefore the only relevant matrix element of the supersymmetry charge comes from $m \neq 0$.

For the analysis of the normalization of the contact term, let us be careful about the overall factor. Since

$$(\alpha_{(1)}\sqrt{\alpha_{(2)}}b^a_{0(2)} - \alpha_{(2)}\sqrt{\alpha_{(1)}}b^a_{0(1)})(\sqrt{\alpha_{(1)}}b^a_{0(1)} + \sqrt{\alpha_{(2)}}b^a_{0(2)})b^{a\dagger}_{0(1)}b^{a\dagger}_{0(2)}|\mathrm{vac}\rangle = -\sqrt{\alpha_{(1)}\alpha_{(2)}}|\mathrm{vac}\rangle, \tag{F.0.17}$$

for $a = 5, \cdots, 8$, from the cancellation of the fermionic zero modes we have an extra factor of

$$c_0 = \left(\frac{\sqrt{\alpha_{(1)}\alpha_{(2)}}}{2\bar{Y}_0}\right)^4. \tag{F.0.18}$$

Taking it into account, we find that the only non-trivial contribution is

$$Q_{n,m(s)} = i\eta\frac{2cc_0}{\alpha^2}\sqrt{1+\mu\alpha k}\,u^i_{abc\dot{a}}\delta^{abcd}_{1234}\frac{\bar{Y}_{m(1)}}{\sqrt{2}}(\tilde{F}^{-}_{(3)-n}\tilde{N}^{(s3)}_{m,n} + \tilde{F}^{-}_{(3)n}\tilde{N}^{(s3)}_{m,-n}). \tag{F.0.19}$$

To fix the overall normalization, let us compare the supersymmetry charge matrix element with the Hamiltonian. If we restrict the external states to be purely bosonic ones, we also have the same fermionic zero mode factor c_0 in the Hamiltonian matrix element:

$$\begin{aligned} |H_3\rangle &= \frac{cc_0}{\alpha^2}(1+\mu\alpha k)(K^i_+ - K^i_-)(K^j_+ + K^j_-)t^{ij}_{5678}E_a|\mathrm{vac}\rangle \\ &= \frac{2cc_0}{\alpha^2}\left(-\frac{y(1-y)}{2}\right)\sum_{r=1}^{3}\sum_{n=-\infty}^{\infty}\frac{\omega_{n(r)}}{\alpha_{(r)}}\alpha^{i\dagger}_{n(r)}\alpha^i_{-n(r)}E_a|\mathrm{vac}\rangle. \end{aligned} \tag{F.0.20}$$

In the final step, we have used the formula derived in [31]. Comparing our final expression with (2.2.8) whose normalization factor was determined in Chapter 2 and [39] by comparing the string field theory result with a gauge theory computation, we find that

$$\frac{2cc_0}{\alpha^2} = 1. \tag{F.0.21}$$

Appendix G

Formulas for calculating the contact term

The necessary summation and integration we need to calculate the contact term are the following ones:

$$\begin{aligned} \sum_{l=-\infty}^{\infty} \tilde{N}_{l,n}^{(13)} \tilde{N}_{l,m}^{(13)} &= \frac{(-1)^{m+n} \sin(n-m)\pi y}{\pi(n-m)}, \\ \sum_{l=-\infty}^{\infty} \tilde{N}_{l,n}^{(23)} \tilde{N}_{l,m}^{(23)} &= \frac{\sin(n-m)\pi(1-y)}{\pi(n-m)}. \end{aligned} \tag{G.0.1}$$

Also,

$$\begin{aligned} &\int_0^1 dy (-1)^{m+n} \frac{\sin \pi m y \sin \pi n y}{\pi^2} \Big\{ (-1)^{l+n} \Big(\frac{\sin \pi(m-n) y}{\pi(m-n)} - \frac{\sin \pi(m+n) y}{\pi(m+n)} \Big) (1-y) \\ &\qquad\qquad + \Big(\frac{\sin \pi(m-n)(1-y)}{\pi(m-n)} - \frac{\sin \pi(m+n)(1-y)}{\pi(m+n)} \Big) y \Big\} \\ &\quad = \frac{1}{4\pi^4(m-n)^2} + \frac{1}{4\pi^4(m+n)^2}, \\ &\int_0^1 dy \frac{\sin^2 \pi n y}{\pi^2} \Big\{ \Big(y - \frac{\sin 2\pi n y}{2\pi n} \Big)(1-y) + \Big((1-y) - \frac{\sin 2\pi n(1-y)}{2\pi n} \Big) y \Big\} \\ &\quad = \frac{1}{2\pi^2} \Big(\frac{1}{3} + \frac{5}{8\pi^2 n^2} \Big). \end{aligned} \tag{G.0.2}$$

Appendix H

$\tilde{\Gamma}^{(2)}$ computation in singlet sector

In this appendix, we explain the details of the computation of $\tilde{\Gamma}^{(2)}$ matrix elements for the operators with two impurities in the same direction, as discussed in Section 3. The following identity will be useful throughout the computation:

$$C_{n,py} = C_{-n,-py}. \tag{H.0.1}$$

As in (2.3.36), $\tilde{\Gamma}^{(2)}$ is given by

$$\tilde{\Gamma}^{(2)} = \Gamma^{(2)} - \frac{1}{2}\{G^{(2)}, \Gamma^{(0)}\} - \frac{1}{2}\{G^{(1)}, \Gamma^{(1)}\} + \frac{3}{8}\{G^{(1)2}, \Gamma^{(0)}\} + \frac{1}{4}G^{(1)}\Gamma^{(0)}G^{(1)}. \tag{H.0.2}$$

Here we shall compute all the terms and show that $\tilde{\Gamma}^{(2)}$ reduces to (3.2.24).

Our strategy is to split each matrix element in (3.2.16) into a part proportional to δ_{ij} and a part coming from extra diagrams. More precisely, we have

$$\begin{aligned}
\Gamma^{(1)}_{iin,jjqz} &= \delta_{ij}(\Gamma^{(1)}_{n,qz} + \Gamma^{(1)}_{-n,qz}) + \delta\Gamma^{(1)}_{n,qz}, \\
\Gamma^{(1)}_{iin,jjz} &= \delta_{ij}(\Gamma^{(1)}_{n,z} + \Gamma^{(1)}_{-n,z}) + \delta\Gamma^{(1)}_{n,z}, \\
\Gamma^{(2)}_{iin,jjm} &= \delta_{ij}(\Gamma^{(2)}_{n,m} + \Gamma^{(2)}_{n,-m}) + \delta\Gamma^{(2)}_{n,m},
\end{aligned} \tag{H.0.3}$$

with

$$\begin{aligned}
\delta\Gamma^{(1)}_{n,qz} &= -\frac{1}{2}\Gamma^{(1)}_{n,0z}, \\
\delta\Gamma^{(1)}_{n,z} &= -\frac{1}{2}\Gamma^{(1)}_{n,z},
\end{aligned}$$

$$\delta\Gamma^{(2)}_{n,m} = -\frac{1}{16\pi^2}\mathcal{D}^1_{n,m}. \tag{H.0.4}$$

As a preliminary computation let us consider $(G^{(1)})^2$:

$$\begin{aligned}
(G^{(1)})^2 &= J\int_0^1 dy\left(\sum_{p=1}^{\infty}(C_{n,py}+C_{n,-py})(C_{py,m}+C_{-py,m})+2C_{n,0y}C_{0y,m}\right) \\
&\quad +J\int_0^{1/2} dy 2C_{n,y}2C_{y,m} \\
&= J\int_0^1 dy\sum_{p=-\infty}^{\infty}\Big(C_{n,py}C_{py,m}+C_{n,py}C_{py,-m}\Big)+2J\int_0^1 dyC_{n,y}C_{y,m} \\
&= \frac{1}{2}(M^1_{n,m}+M^1_{n,-m}).
\end{aligned} \tag{H.0.5}$$

Here we have to be careful about the extra normalization factor $1/\sqrt{2}$ for zero modes as explained around (3.2.16). Note that originally in the first line we sum only over positive integers the product of two terms. One of the terms is the product of two contributions with opposite worldsheet momentum. But with the help of (H.0.1), we can rewrite these cross terms into the summation of two terms over all the integers, with still one of them carrying the reversed *external* worldsheet momentum as in the second equation in (H.0.5). Since one of two terms is identical to the one arising for operators with two impurities in different directions, we can perform the summation and integration easily and add the other term by reversing the external worldsheet momentum. This kind of mechanism happens everywhere, also in the computation of $\tilde{\Gamma}^{(2)}$. Therefore, the naive expectation of $\tilde{\Gamma}^{(2)}$ is obtained by adding a term with the external worldsheet momentum reversed:

$$\tilde{\Gamma}^{(2)}_{iin,jjm} = \delta_{ij}(\tilde{\Gamma}^{(2)}_{n,m}+\tilde{\Gamma}^{(2)}_{n,-m}) = \delta_{ij}\frac{1}{16\pi^2}(B_{n,m}+B_{n,-m}). \tag{H.0.6}$$

The only point we have to be careful with is whether (H.0.4) will give a non-trivial contribution.

Let us postpone the effect of (H.0.4) and concentrate on the dominant contribution to see whether the results have an additional contribution of reversing the worldsheet momentum, as compared to the case of operators with two impurities in different directions. Now it is quite trivial to calculate terms involving $\Gamma^{(0)}$ in (H.0.2) such as $\{(G^{(1)})^2,\Gamma^{(0)}\}$, $G^{(1)}\Gamma^{(0)}G^{(1)}$ and $\{G^{(2)},\Gamma^{(0)}\}$. They are given by

$$\begin{aligned}
\{(G^{(1)})^2,\Gamma^{(0)}\} &= \frac{n^2+m^2}{2}(M^1_{n,m}+M^1_{n,-m}),\\
G^{(1)}\Gamma^{(0)}G^{(1)} &= J\int_0^1 dy\sum_{p=0}^{\infty}(C_{n,py}+C_{n,-py})\frac{p^2}{y^2}(C_{py,m}+C_{-py,m})\\
&= J\int_0^1 dy\sum_{p=-\infty}^{\infty}(C_{n,py}\frac{p^2}{y^2}C_{py,m}+C_{n,py}\frac{p^2}{y^2}C_{py,-m}),\\
\{G^{(2)},\Gamma^{(0)}\} &= (n^2+m^2)(M^1_{n,m}+M^1_{n,-m}).
\end{aligned} \tag{H.0.7}$$

Let us turn to the term involving $\Gamma^{(1)}$ in (H.0.2), but ignoring the effect of (H.0.4). It is given by

$$\begin{aligned}
&\{G^{(1)},(\Gamma^{(1)}-\delta\Gamma^{(1)})\}\\
=\;& J\int_0^1 dy\sum_{p=1}^{\infty}(C_{n,py}+C_{n,-py})(\Gamma^1_{py,m}+\Gamma^1_{-py,m})+\frac{1}{2}(C_{n,0y}+C_{n,0y})(\Gamma^1_{0y,m}+\Gamma^1_{0y,m})\\
&+J\int_0^1 dy\sum_{p=1}^{\infty}(\Gamma^1_{n,py}+\Gamma^1_{n,-py})(C_{py,m}+C_{-py,m})+\frac{1}{2}(\Gamma^1_{n,0y}+\Gamma^1_{n,0y})(C_{0y,m}+C_{0y,m})\\
&+J\int_0^{1/2} dy(4C_{n,y}\Gamma^1_{y,m}+4\Gamma^1_{n,y}C_{y,m})\\
=\;& J\int_0^1 dy\Big\{\sum_{p=-\infty}^{\infty}(C_{n,py}\Gamma^1_{py,m}+\Gamma^1_{n,py}C_{py,m})+(C_{n,y}\Gamma^1_{y,m}+\Gamma^1_{n,y}C_{y,m})\Big\}\\
&+J\int_0^1 dy\Big\{\sum_{p=-\infty}^{\infty}(C_{n,py}\Gamma^1_{py,-m}+\Gamma^1_{n,py}C_{py,-m})+(C_{n,y}\Gamma^1_{y,-m}+\Gamma^1_{n,y}C_{y,-m})\Big\}.
\end{aligned} \tag{H.0.8}$$

Also if we ignore the effect of (H.0.4), $\Gamma^{(2)}$ also has the same additional contribution, as seen in (H.0.3). As promised, all the results come paired with (n,m) and $(n,-m)$, where the first group of terms adds up to give the same result as for the case of two different impurities.

Now let us consider the contribution of $\delta\Gamma$'s to $\tilde{\Gamma}^{(2)}$

$$\delta\tilde{\Gamma}^{(2)}=\delta\Gamma^{(2)}-\frac{1}{2}\{G^{(1)},\delta\Gamma^{(1)}\}, \tag{H.0.9}$$

that we have not taken into account so far. We can compute the second term as before:

$$\begin{aligned}
&\{G^{(1)}, \delta\Gamma^{(1)}\}_{iin,jjm} \\
= \quad & J\int_0^1 dy \left\{ \sum_{p=1}^{\infty} (C_{n,py} + C_{n,-py}) \Big(-\frac{1}{2}\Gamma^{(1)}_{0y,m} \Big) + C_{n,0y} \Big(-\frac{1}{2}\Gamma^{(1)}_{0y,m} \Big) \right\} \\
& +J\int_0^1 dy \left\{ \sum_{p=1}^{\infty} \Big(-\frac{1}{2}\Gamma^{(1)}_{n,0y} \Big) (C_{py,m} + C_{-py,m}) + \Big(-\frac{1}{2}\Gamma^{(1)}_{n,0y} \Big) C_{0y,m} \right\} \\
& +J\int_0^{\frac{1}{2}} dy (2C_{n,y}) \Big(-\frac{1}{2}\Gamma^{(1)}_{y,m} \Big) + \Big(-\frac{1}{2}\Gamma^{(1)}_{n,y} \Big) (2C_{y,n}) \\
= \quad & -\frac{J}{2}\int_0^1 dy \sum_{p=-\infty}^{\infty} \Big(C_{n,py}\Gamma^1_{0y,m} + \Gamma^1_{n,0y}C_{py,m} \Big) - \frac{n^2+m^2}{2} J\int_0^1 dy C_{n,y}C_{y,m}. \qquad \text{(H.0.10)}
\end{aligned}$$

Using the formula,

$$J\int_0^1 dy \sum_{p=-\infty}^{\infty} C_{n,py}\Gamma^1_{0y,m} = \frac{1}{12\pi^2}\Big(1 + \frac{3}{\pi^2 m^2} \Big), \qquad \text{(H.0.11)}$$

and the summation formula in the appendix of [60], we obtain

$$\{G^{(1)}, \delta\Gamma^{(1)}\}_{iin,jjm} = -\frac{1}{8\pi^2}\mathcal{D}^1_{n,m}, \qquad \text{(H.0.12)}$$

which precisely cancels $\delta\Gamma^{(2)}$:

$$\delta\tilde{\Gamma}^{(2)} = 0. \qquad \text{(H.0.13)}$$

Therefore we find that (H.0.6) is exact. In Section 3.2 and Appendix F, we saw that this result is correctly reproduced from the contact term calculation in string field theory.

Appendix I

Anomalous dimension of the singlet operators

In this appendix we shall calculate the anomalous dimension of the operator with two impurities in the same direction, using the perturbation theory. This calculation has essentially been done in [47] by diagonalizing the matrix of two-point functions in the BMN basis. Here, we perform the calculation using the string field theory basis and it serves as a consistency check of the evaluation of $\tilde{\Gamma}^{(2)}$ in the previous appendix.

In perturbation theory the eigenvalue at $O(g_2^2)$ is given by

$$\Delta^{(2)} = J\int_0^1 dy \sum_{p=0}^{\infty}\sum_{j=1}^{4} \frac{(\tilde{\Gamma}^{(1)}_{ii;n,jj;py})^2}{n^2 - p^2/y^2} + J\int_0^{1/2} dy \sum_{j=1}^{4} \frac{(\tilde{\Gamma}^{(1)}_{ii;n,jj;y})^2}{n^2} + \tilde{\Gamma}^{(2)}_{ii;n,ii;n}. \tag{I.0.1}$$

Using the following relations

$$\begin{aligned} \tilde{\Gamma}^{(1)}_{n,py} &= \frac{1}{2}\Gamma^{(1)}_{n,0y}, \\ \tilde{\Gamma}^{(1)}_{n,y} &= \frac{1}{2}\Gamma^{(1)}_{n,y}, \end{aligned} \tag{I.0.2}$$

(I.0.1) can be rewritten as

$$\Delta^{(2)} = \frac{J}{2}\int_0^1 dy \sum_{p=-\infty}^{\infty} \frac{(\Gamma^{(1)}_{n,0y})^2}{n^2 - p^2/y^2} + \frac{J}{2}\int_0^1 dy \frac{(\Gamma^{(1)}_{n,y})^2}{n^2} + \tilde{\Gamma}^{(2)}_{ii;n,ii;n}. \tag{I.0.3}$$

Now let us proceed to evaluate each term. The first term is

$$\frac{1}{2\pi^4}\int_0^1 dy\frac{1-y}{y}\sin^4(\pi ny)\sum_{p=-\infty}^{\infty}\frac{1}{n^2-p^2/y^2}=\frac{3}{64\pi^4 n^2}, \tag{I.0.4}$$

and the second term is simply reduced to the integration of $(C_{n,y})^2$, whose result can be found in [60]:

$$\frac{n^2}{2}J\int_0^1 dyC_{n,y}^2=\frac{3}{16\pi^4 n^2}. \tag{I.0.5}$$

Consequently, the anomalous dimension of the singlet operator is

$$\Delta^{(2)}=\frac{3}{64\pi^4 n^2}+\frac{3}{16\pi^4 n^2}+\frac{1}{16\pi^2}(B_{n,n}+B_{n,-n})=\frac{1}{16\pi^2}\left(\frac{1}{3}+\frac{35}{8\pi^2 n^2}\right), \tag{I.0.6}$$

which as explained in [47, 48] is the same as the operators transforming in the **6** and **9** representations of $SO(4)$.

Appendix J

The $\mathcal{O}(g_2)$ coupling a p-th string state to a $p+1$-th string state

In a recent paper Gursoy [52] has analyzed the two-point function of multi-trace BMN operators with two different impurities. The class of operators that the author considers are

$$\mathcal{T}^{J,y_1,y_2,\ldots,y_p}_{ij,n} =: \mathcal{O}^{y_1\cdot J}_n \cdot \mathcal{O}^{y_2\cdot J} \cdots \mathcal{O}^{y_p\cdot J} : \delta_{y_1+\ldots+y_p,1}. \qquad \text{(J.0.1)}$$

The $\mathcal{O}(g_2)$ result for the mixing of the p-th trace with the $p+1$ trace BMN operator is given by [52]

$$\begin{aligned} G^{p,p+1(1)}_{ny_1\ldots y_p;mz_1\ldots z_p} &= y_1^{3/2} C_{n,mz_1/y_1} \sum_P \delta_{y_2,z_{P(2)}} \ldots \delta_{y_p,z_{P(p)}} \delta_{y_1,z_{P(p)}} \delta_{y_1,z_1+z_{P(i+1)}} + \\ & \frac{1}{J}\delta_{n,m}\delta_{y_1,z_1} \sum_{P,P'} \delta_{y_{P(2)},z_{P'(2)}} \ldots \delta_{y_{P(p-1)},z_{P'(p-1)}} \delta_{y_{P(p)},z_{P'(p)}+z_{P'(p+1)}} \end{aligned} \qquad \text{(J.0.2)}$$

The contribution in the first line is due to contractions in which the two impurities in the p-trace operator contract with the two impurities and any vacuum operator in the $p+1$-trace operator. Therefore, the quantity $C_{n,mz_1/y_1}$ is the mixing between a single trace and double trace two-impurity BMN operator. The contribution in the second line comes from Wick contractions where the operators with the impurities just connect among themselves and the vacuum operators in the p-th trace BMN operator contract with the vacuum operator in the $p+1$-th trace BMN operator.

The contribution to the matrix of anomalous dimensions is given by [52]

$$\Gamma^{p,p+1(1)}_{ny_1\ldots y_p;mz_1\ldots z_p} = \left(\frac{n^2}{y_1^2}+\frac{m^2}{z_1^2}-\frac{nm}{y_1 z_1}\right) G^{p,p+1(1)}_{ny_1\ldots y_p;mz_1\ldots z_p}. \tag{J.0.3}$$

Using the holographic proposal we can calculate these matrix elements in the orthonormal basis

$$\begin{aligned}\tilde{\Gamma}^{p,p+1(1)}_{ny_1\ldots y_p;mz_1\ldots z_p} &= \Gamma^{p,p+1(1)}_{ny_1\ldots y_p;mz_1\ldots z_p} - \frac{1}{2}\left\{G^{p,p+1(1)}_{ny_1\ldots y_p;mz_1\ldots z_p}, \Gamma^{p,p+1(0)}_{ny_1\ldots y_p;mz_1\ldots z_p}\right\} \\ &= \frac{1}{2}\left(\frac{n}{y_1}-\frac{m}{z_1}\right)^2 G^{p,p+1(1)}_{ny_1\ldots y_p;mz_1\ldots z_p}.\end{aligned} \tag{J.0.4}$$

We note that, in the orthonormal basis, the second term in (J.0.2) does not contribute due to the δ function constraints and the prefactor in (J.0.4). Therefore, the final answer can be written as

$$\tilde{\Gamma}^{p,p+1(1)}_{ny_1\ldots y_p;mz_1\ldots z_p} = \frac{1}{\sqrt{y_1}}\tilde{\Gamma}_{n,mz_1/y_1} \times \sum_P \delta_{y_2,z_{P(2)}}\ldots\delta_{y_p,z_{P(p)}}\delta_{y_1,z_{P(p)}}\delta_{y_1,z_1+z_{P(i+1)}}. \tag{J.0.5}$$

We now perform the relevant string field theory calculation. The string states dual to the BMN operators (J.0.1 are given by

$$|n,y_1,y_2,\ldots,y_p\rangle = \alpha^i_n\alpha^j_{-n}|\mathrm{vac},y_1\rangle\otimes|\mathrm{vac},y_2\rangle\otimes\ldots\otimes|\mathrm{vac},y_p\rangle\ \delta_{y_1+\ldots+y_p,1}. \tag{J.0.6}$$

It follows from the expression for the cubic Hamiltonian vertex (2.2.8) (2.2.9) that any contraction coupling only vacua is zero. The only possible non-zero contributions are those in which the string carrying the two impurities contracts with the string carrying two-impurities and a vacuum string state. Therefore, the matrix elements of the unitly normalized states are

$$\begin{aligned}&\frac{1}{\mu}\langle n,y_1\ldots y_p|H_3|m,z_1\ldots z_p\rangle \\ &= \frac{1}{\mu}\langle n,y_1|H_3|m,z_1\rangle \times \sum_P \delta_{y_2,z_{P(2)}}\ldots\delta_{y_p,z_{P(p)}}\delta_{y_1,z_{P(p)}}\delta_{y_1,z_1+z_{P(i+1)}} \\ &= \frac{1}{\sqrt{y_1}}\tilde{\Gamma}^{(1)}_{n,mz_1/y_1} \times \sum_P \delta_{y_2,z_{P(2)}}\ldots\delta_{y_p,z_{P(p)}}\delta_{y_1,z_{P(p)}}\delta_{y_1,z_1+z_{P(i+1)}},\end{aligned} \tag{J.0.7}$$

which match the gauge theory computation.

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