A WAVE FRONT APPROXIMATION METHOD AND ITS APPLICATION TO ELASTIC STRESS WAVES

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ABSTRACT

This paper presents a new direct method of obtaining wave front approximations for problems involving hyperbolic differential equations. In the problem of a semi-infinite, end-loaded elastic strip (the problem used to illustrate the method), asymptotic solutions are obtained for wave fronts prior to multiple edge interactions. For the special end loading of a step velocity, the results agree with prior results obtained by more complex methods of approximation. Extension of the method to multiple interactions and to other problems of stress wave propagation is briefly discussed.

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I. INTRODUCTION

In this paper we will present a new method for obtaining an approximate solution at wave fronts for wave propagation problems. Although the method has possible uses for many propagation problems which involve hyperbolic partial differential equations, we will illustrate the method only through the analysis of a stress wave propagation problem in the linear plane strain theory of elasticity. By assuming constant material properties, this class of problems resolves into the study of two scalar wave equations.

A common procedure for the analysis of any stress wave propagation problem in linear elasticity is as follows:

1. Perform a Laplace transform (on time) on the basic differential equations and the boundary conditions.
2. Solve the problem exactly in the Laplace transform domain. This is usually done by means of another suitable transform on a space variable.
3. Use the inversion theorem for the Laplace transform to obtain an integral representation for the solution. If wave front approximations are required, they may be obtained by asymptotic analysis of this integral representation.

In this paper we apply the new method to the problem of the end-loaded semi-infinite strip. We will seek information at, and immediately behind, the various wave fronts. For this problem, Rosenfeld and Miklowitz[1, 2] have determined the exact solution by the method described above. They used a large Laplace transform parameter approximation in step 3 to get approximate wave front information valid at any arbitrary point reasonably close to the applied external loading.

The purpose of this paper is to show that, if only wave front approximations are required, it is possible to eliminate step 2 by performing an asymptotic analysis directly on the boundary value problem for the Laplace transform in step 1. Roughly speaking, this analysis is based on the fact that wave front behavior of the solution is determined by the behavior of the Laplace transform of the solution for large values of the Laplace transform parameter. The techniques involved are those commonly used in problems of boundary layer type. In this way we obtain (in part V) some of the wave front approximations of Rosenfeld and Miklowitz, and thus check the present method. Also in the discussion section (part V) the extension of the method is made to other related problems which can also be handled by double transform techniques; that is, those problems involving "mixed"* boundary conditions. The importance of the method, though, lies in its relevance to as yet unsolved transient problems. For these problems it is often step 2 in the procedure described above which presents major difficulties. It is therefore possible that a method which circumvents step 2 would be helpful.

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* "Mixed" boundary conditions are when one condition is on stress; the other, on displacement.

An example of the latter category is the semi-infinite strip with parallel edges free and with prescribed normal and shear stresses at the remaining edge. This problem is briefly discussed.

II. PROBLEM FORMULATION

A. Basic Equations of Dynamic Plane Strain

Consider the dynamic problem of small-strain linear elasticity theory in a medium that is homogeneous and isotropic. The displacement field, $\vec{u}$, is assumed to have small gradients and to be caused by external forces which are independent of $\vec{u}$. We shall also eliminate body moments and forces from consideration. The field equation of motion is thus

(1-1) $$\nabla \cdot \overset{\Rightarrow}{\tau} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$$

where $\nabla \equiv$ vector operator "del"

$\nabla\cdot \equiv$ divergence operator

$\overset{\Rightarrow}{\tau} \equiv$ stress tensor (which is symmetric due to the absence of body moments)

$\rho \equiv$ density, a constant

$t \equiv$ time

For the linearly elastic material, Hooke's Law is just

(1-2) $$\overset{\Rightarrow}{\tau} = \lambda(\nabla \cdot \vec{u})\overset{\Rightarrow}{I} + \mu(\nabla \vec{u} + \vec{u}\nabla)$$

where $\nabla\vec{u}$, $\vec{u}\nabla$ are the dyadic and its conjugate dyadic, respectively.

$\overset{\Rightarrow}{I}$ is the unit tensor.

$\lambda, \mu \equiv$ Lamé material constants

Combining (1-1) and (1-2), we get the Navier equation

(1-3) $$(\lambda + \mu)\nabla(\nabla \cdot \vec{u}) + \mu \nabla^2 \vec{u} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$$

Alternatively, the above equation may be altered by using the identity

$$\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u})$$

equation (1-3) becomes the following equivalent form

(1-3a) $$(\lambda + 2\mu)\nabla(\nabla \cdot \vec{u}) - \mu \nabla \times (\nabla \times \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$$

If we let the displacement field $\vec{u}$ have rectangular cartesian components u, v, w, and consider the case of plane strain where

(1-4) $$w = 0, \quad u = u(x, y, t), \quad v = v(x, y, t)$$

then the previous equations can be simplified to the following two-dimensional problem (using rectangular cartesian system of coordinates x, y, and letting σ, τ denote real stresses).

(1-5) Equations of Motion

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}$$

(1-6) Hooke's Law

$$\sigma_x = (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y}$$

$$\sigma_y = (\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda \frac{\partial u}{\partial x}$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

(1-7) Navier Equations

$$(\lambda + \mu)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho \frac{\partial^2 u}{\partial t^2}$$

$$(\lambda + \mu)\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y}\right) + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = \rho \frac{\partial^2 v}{\partial t^2}$$

B. Use of Wave Potentials

By a theorem due to Helmholtz, any vector field can be expressed as the sum of an irrotational vector field plus a vector field whose divergence is zero. The irrotational part can be expressed as the gradient of a scalar potential; the divergenceless part as the curl of some vector field. Furthermore, this representation is essentially unique. For the displacement field, $\vec{u}$, we can define the scalar field, ϕ, and vector field, $\vec{\Psi}$, by the equation

(2-1) $$\vec{u} \equiv \nabla\phi + \nabla\times\vec{\Psi} \qquad \text{where } \nabla\cdot\vec{\Psi} = 0$$

Taking the divergence of equation (1-3a) and using (2-1) and vector identities, we are led to the following expression

(2-2) $$\nabla^2\left[\frac{\rho}{\lambda+2\mu}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi\right] = 0$$

Taking the curl of equation (1-3a) and using (2-1) and vector identities leads to the second equation condition

(2-3) $$\nabla^2\left[\frac{\rho}{\mu}\frac{\partial^2\vec{\Psi}}{\partial t^2} - \nabla^2\vec{\Psi}\right] = 0$$

Sternberg[3,4] has shown that, assuming zero initial conditions, the unique solution to equations (2-2), (2-3) are the following equations.

(2-4) $$\frac{1}{c_d^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi$$

(2-5) $$\frac{1}{c_s^2}\frac{\partial^2\vec{\Psi}}{\partial t^2} = \nabla^2\vec{\Psi}$$

where

(2-6) $$c_d^2 \equiv \frac{\lambda+2\mu}{\rho}, \quad c_s^2 \equiv \frac{\mu}{\rho}$$

The symbols c_d and c_s are the dilatational (compressional) and shear (equivoluminal) phase velocities, respectively. Accordingly, ϕ is usually called the dilatational displacement potential; $\vec{\Psi}$, the shear displacement potential.

For plane strain problems,

$$\vec{\Psi} \equiv \vec{e}_z \, \Psi(x,y,t) \qquad (\vec{e}_z = \text{unit vector in the } z \text{ direction})$$
$$\phi \equiv \phi(x,y,t) \tag{2-7}$$

The displacement field can now be expressed in terms of ϕ, Ψ.

$$u(x,y,t) = \frac{\partial \phi}{\partial x} - \frac{\partial \Psi}{\partial y}$$
$$v(x,y,t) = \frac{\partial \phi}{\partial y} + \frac{\partial \Psi}{\partial x} \tag{2-8}$$

For stresses, we obtain

$$\sigma_x(x,y,t) = \lambda \nabla^2 \phi + 2\mu\left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial x \partial y}\right)$$
$$\sigma_y(x,y,t) = \lambda \nabla^2 \phi + 2\mu\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial x \partial y}\right) \tag{2-9}$$
$$\tau_{xy}(x,y,t) = \mu\left(2\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2}\right)$$

Thus for any given problem we will seek solutions for $\phi(x,y,t)$ and $\Psi(x,y,t)$ from which the desired physical quantities can be calculated using (2-8) or (2-9).

C. Impact Problem for the Semi-Infinite Strip

Within the linear elasticity theory of section A, we will select the semi-infinite strip as a suitable problem for applying the new approximation method for the wave front approximation. We will consider the strip as bounded by $0 \leq X \leq \infty$ and $0 \leq Y \leq a$,

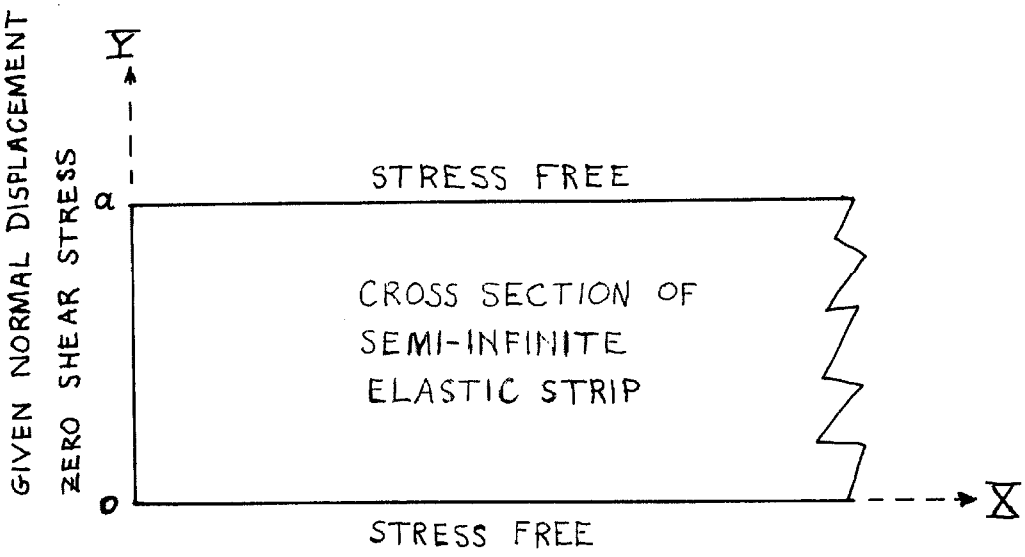


FIG. 1a ACTUAL PROBLEM

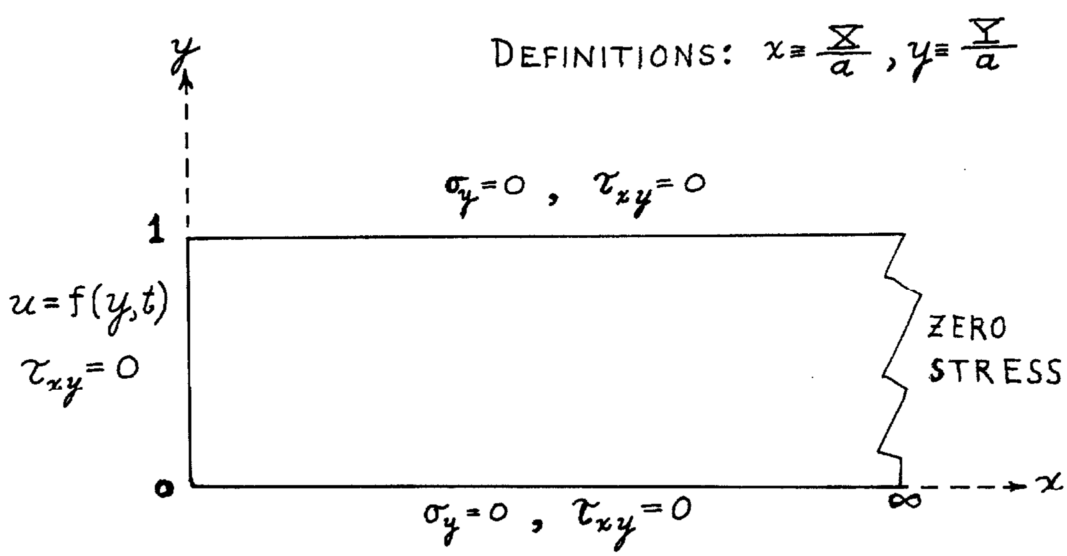


FIG. 1b NORMALIZED PROBLEM

where a is the uniform thickness (Figure 1a). After being free of stress until $t = 0$, a known normal displacement is applied along the edge, and the normal stress at the edge becomes non-zero. The other boundary conditions arise from all the remaining surface stresses being zero for all time.

For convenience, we non-dimensionalize all lengths by dividing by the thickness a. Figure 1b now shows the problem in this new context. Using new variables and constants defined as

$$
\begin{aligned}
x &\equiv \frac{X}{a} & \qquad u(x,y,t) &\equiv \frac{1}{a}\, u(X,Y,t) \\
y &\equiv \frac{Y}{a} & v(x,y,t) &\equiv \frac{1}{a}\, v(X,Y,t) \\
C &\equiv c_d/a & \sigma_x &\equiv a^2 \sigma_X \\
\beta &\equiv c_s/c_d = \frac{\mu}{\lambda+2\mu} & \sigma_y &\equiv a^2 \sigma_Y \\
& & \tau_{xy} &\equiv a^2 \tau_{XY}
\end{aligned}
\tag{3-1}
$$

we can now complete the mathematical formulation for the problem. The field equations of section B become (for $0 \leq x \leq \infty$, $0 \leq y \leq 1$, $t > 0$)

$$
\begin{aligned}
\phi_{xx} + \phi_{yy} - \frac{1}{c^2}\, \phi_{tt} &= 0 \\
\psi_{xx} + \psi_{yy} - \frac{1}{\beta^2 c^2}\, \psi_{tt} &= 0
\end{aligned}
\tag{3-2}
$$

where the subscripts x, y, t denote partial derivatives. The boundary conditions at $y=0$ and $y=1$ (for $0 \leq x \leq \infty$, $t > 0$) are

$$
y = 0, 1 : \begin{cases} 2\phi_{xy} + \psi_{xx} - \psi_{yy} = 0 \\ \phi_{xx} + \phi_{yy} + \dfrac{2\beta^2}{1-2\beta^2}\left(\phi_{yy} + \psi_{xy}\right) = 0 \end{cases}
\tag{3-3}
$$

Boundary conditions at $x=0$, (for $0 \leq y \leq 1$, $t > 0$) are

$$(3\text{-}4) \qquad x=0: \begin{cases} 2\phi_{xy} + \psi_{xx} - \psi_{yy} = 0 \\ \phi_x - \psi_y = f(y,t) \end{cases}$$

where $f(y,t)$ is known. The state of zero stress prior to $t=0$ determines the initial conditions as

$$(3\text{-}5) \qquad \phi = \phi_t = \psi = \psi_t = 0$$

It should be noted that the initial conditions insure that the strip is initially at rest and also eliminate any arbitrariness (like rigid-body motions) in the solutions for displacements u and v. Any pair of functions ϕ, ψ satisfying the Cauchy-Riemann equations would not cause any displacements u or v, but they would satisfy the field equations. If these harmonic solutions were independent of time, they could also satisfy the boundary conditions, but they cannot satisfy the initial conditions (3-5). Thus these unwanted solutions will be automatically discarded.

D. The Laplace-Transformed Problem

For the linear equations of section C, we can conveniently utilize the method of Laplace transforms in order to eliminate the time dependence and to automatically incorporate the initial conditions (3-5). Define the Laplace transform of a time-dependent function $Z(t)$ as

$$(4\text{-}1) \qquad \mathcal{L}\{Z(t)\} = \overline{\overline{Z}}(s) = \int_0^\infty e^{-st} Z(t)\, dt$$

The corresponding inversion formula is

(4-2) $$\mathcal{L}^{-1}\left\{\bar{\bar{Z}}(s)\right\} = Z(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\bar{\bar{Z}}(s)\,ds$$

where γ is a positive, real constant greater than the real part of any singularity of $\bar{\bar{Z}}(s)$ in the complex s-plane.

Applying the definitions (4-1) to the problem of section C gives the following mathematical problem for $\bar{\bar{\phi}}(x,y,s)$, $\bar{\bar{\Psi}}(x,y,s)$:

Field Equations:

(4-3) $$\begin{aligned} \bar{\bar{\phi}}_{xx} + \bar{\bar{\phi}}_{yy} - \frac{1}{\epsilon^2}\bar{\bar{\phi}} &= 0 \\ \bar{\bar{\Psi}}_{xx} + \bar{\bar{\Psi}}_{yy} - \frac{1}{\beta^2\epsilon^2}\bar{\bar{\Psi}} &= 0 \end{aligned}$$

where

$$\epsilon \equiv c/s$$

is a dimensionless parameter which will be greatly utilized in subsequent sections. We regard s as real and positive until Part IV of this paper.

Boundary Conditions for $0 \le x < \infty$, $y=0$, $y=1$:

(4-4) $$\begin{aligned} 2\bar{\bar{\phi}}_{xy} + \bar{\bar{\Psi}}_{xx} - \bar{\bar{\Psi}}_{yy} &= 0 \\ \frac{1}{\epsilon^2}\bar{\bar{\phi}} + \frac{2\beta^2}{1-2\beta^2}\left(\bar{\bar{\phi}}_{yy} + \bar{\bar{\Psi}}_{xy}\right) &= 0 \end{aligned}$$

Note the use of (4-3) on the second equation of (3-3).

Boundary Conditions at $x=0$, $0 \le y \le 1$:

(4-5) $$\begin{aligned} 2\bar{\bar{\phi}}_{xy} + \bar{\bar{\Psi}}_{xx} - \bar{\bar{\Psi}}_{yy} &= 0 \\ \bar{\bar{\phi}}_{x} - \bar{\bar{\Psi}}_{y} &= \bar{\bar{f}}(y,s) \end{aligned}$$

Boundary Conditions at $x=\infty$, $0 \le y \le 1$:

(4-6) $$\bar{\bar{\phi}} = \bar{\bar{\Psi}} = 0$$

III. DIRECT ASYMPTOTIC SOLUTION FOR LARGE s

A. Introduction of Approximations

In the preceding sections, the boundary value problem satisfied by the Laplace transforms of the dilatation and shear potentials $\bar{\bar{\Phi}}(x, y, s)$, $\bar{\bar{\Psi}}(x, y, s)$ has been formulated explicitly for the wave propagation problem of figure 1b. If we now followed the usual procedure of analysis, we would try to find some appropriate method by which the exact solutions for $\bar{\bar{\Phi}}$, $\bar{\bar{\Psi}}$ could be obtained; and then Φ and Ψ would be found by the inversion integral of (II-4-2). Such a program has already been carried out in detail for the specific problem under consideration[1, 2]. Even though the final integral representations for Φ and Ψ that evolve from the usual method are extremely complicated, all required information about Φ, Ψ and all pertinent physical quantities can, at least in principle, be evaluated. For example, the relevant information about the wave front behavior can be derived, in most cases, by appropriate asymptotic operations on the exact integral representation of the solution using the approximation for $\bar{\bar{\Phi}}$ and $\bar{\bar{\Psi}}$ for large s.

Our objective is to find only the wave front approximation. We do it by obtaining the asymptotic expansion of $\bar{\bar{\Phi}}$, $\bar{\bar{\Psi}}$ for large s directly from the boundary value problem without first obtaining the exact solutions. To do this, we can use an already-defined parameter

$$\epsilon = \frac{c}{s} \left(= \frac{c_d}{as} \right)$$

Since large s corresponds to ϵ being very small, we can use the

ϵ as a small parameter for an asymptotic series. In addition, we can utilize the fact that the wave front behavior is, approximately, involved only with external disturbances occurring near $t = 0$, and we assume that the given displacement $f(y,t)$ at $x = 0$ has an asymptotic series in t like

$$f(y,t) = c^2 t F(y) + c^3 t^2 G(y) + c^4 t^3 H(y) + \cdots \tag{1-1}$$

as t approaches zero.* For the Laplace transform of (1-1), we get

$$\bar{\bar{f}}(y,s) = \epsilon^2 F(y) + \epsilon^3 G(y) + \epsilon^4 H(y) + \mathcal{O}(\epsilon^5) \tag{1-2}$$

In terms of $\bar{\bar{f}}(y,s)$, ϵ, $\bar{\bar{\Phi}}$ and $\bar{\bar{\Psi}}$, the problem as defined by equations (II-4-3) to (II-4-5) become

Field Equations:

$0 \le x < \infty$
$0 \le y \le 1$

$$\begin{aligned} \epsilon^2 (\bar{\bar{\Phi}}_{xx} + \bar{\bar{\Phi}}_{yy}) - \bar{\bar{\Phi}} &= 0 \\ \beta^2 \epsilon^2 (\bar{\bar{\Psi}}_{xx} + \bar{\bar{\Psi}}_{yy}) - \bar{\bar{\Psi}} &= 0 \end{aligned} \tag{1-3}$$

Boundary Conditions at $x = 0$, $0 \le y \le 1$:

$$\begin{aligned} 2\bar{\bar{\Phi}}_{xy} + \bar{\bar{\Psi}}_{xx} - \bar{\bar{\Psi}}_{yy} &= 0 \\ \bar{\bar{\Phi}}_x - \bar{\bar{\Psi}}_y &= \epsilon^2 F(y) + \epsilon^3 G(y) + \epsilon^4 H(y) + \cdots \end{aligned} \tag{1-4}$$

Boundary Conditions at $y = 0, 1$; $0 \le x < \infty$:

$$\begin{aligned} 2\bar{\bar{\Phi}}_{xy} + \bar{\bar{\Psi}}_{xx} - \bar{\bar{\Psi}}_{yy} &= 0 \\ \bar{\bar{\Phi}} + \frac{2\beta^2\epsilon^2}{1-2\beta^2}\left(\bar{\bar{\Phi}}_{yy} + \bar{\bar{\Psi}}_{xy}\right) &= 0 \end{aligned} \tag{1-5}$$

---

* Mathematically, we could retain step and delta function behavior in $f(y,t)$, but the physical interpretation of such additional terms make it hard to justify their inclusion in (1-1).

With the problem now formulated for the Laplace transform domain, we seek the asymptotic approximation for ϵ small. Inspection of the previous equations shows that ϵ^2 multiplies the highest order derivatives in the differential equations. Thus the problem appears to be one of "boundary layer" type. Our analysis is based on that assumption. The validity of this assumption will be verified if the derived results compare to previous work[1, 2] and if the results exhibit proper decay away from the boundary layers.[5, 6]

B. Solution Away from $x = 0$

Applying our asymptotic method in a formal way, we can see that the first or crudest approximation to the solution of the problem for small ϵ would be obtained by simply neglecting the terms with ϵ factors in equations (1-3) to (1-5). This would lead to the approximate solution

$$\bar{\bar{\phi}} = \bar{\bar{\psi}} = 0 \tag{2-1}$$

This solution will satisfy all the field equations and all boundary conditions except for the second of equations (1-4).

Let us use a more mathematical procedure by first replacing the $\bar{\bar{\phi}}$ and $\bar{\bar{\psi}}$ by a formal power series in ϵ. Here the coefficients are functions of x, y, so that the solution series are

$$\begin{aligned} \bar{\bar{\phi}}(x,y,s) &\equiv \bar{\bar{\phi}}^{(0)}(x,y) + \epsilon\, \bar{\bar{\phi}}^{(1)}(x,y) + \epsilon^2\, \bar{\bar{\phi}}^{(2)}(x,y) + \cdots \\ \bar{\bar{\psi}}(x,y,s) &\equiv \bar{\bar{\psi}}^{(0)}(x,y) + \epsilon\, \bar{\bar{\psi}}^{(1)}(x,y) + \epsilon^2\, \bar{\bar{\psi}}^{(2)}(x,y) + \cdots \end{aligned} \tag{2-2}$$

But substituting this into the field equations (1-3) again leads to the solution $\bar{\Phi} = \bar{\Psi} = 0$ to <u>any</u> order in ϵ. Inspection of (1-4), (1-5) shows that these zero solutions match the boundary conditions only up to order ϵ^2. The approximation cannot possibly be valid near the edge $x = 0$ for terms of order ϵ^3 because of the non-homogenous boundary condition at $x = 0$. Taking the viewpoint that the problem is one of the boundary layer type, we interpret the above trivial solution as the appropriate one away from the $x = 0$ edge. Thus there must exist a boundary layer along $x = 0$ to match the conditions for order $\epsilon^3, \epsilon^4, \ldots, \epsilon^N$ terms with the trivial solution away from the edge.

C. <u>Boundary Layer Solution Near $x = 0$</u>

Along the $x = 0$ edge of the strip, we seek a solution that will satisfy the boundary conditions at $x = 0$ (equations (1-4)) and also match the solution away from $x = 0$; that is, $\bar{\Phi} = \bar{\Psi} = 0$. To investigate the situation near $x = 0$, we introduce an appropriate stretching of the x-scale. This is a standard procedure in problems of this kind[5,6]. We define a new variable as

(3-1) $$\xi \equiv \epsilon^{-N} x$$

and then (by "trial-and-error" if necessary) pick a real constant, N, such that the solutions will exhibit the proper behavior (Note that this N may or may not exist !). For this problem, N = 1 is an obvious selection to try. This choice leads to the following change of variable and operator definitions:

(3-2) $$\xi \equiv \frac{x}{\epsilon}\ , \quad \frac{\partial}{\partial x} = \frac{1}{\epsilon}\frac{\partial}{\partial \xi}\ , \quad \frac{\partial^2}{\partial x^2} = \frac{1}{\epsilon^2}\frac{\partial^2}{\partial \xi^2}$$

The boundary value problem of equations (1-3) to (1-5) is now

$$\left.\begin{aligned} &\overline{\overline{\Phi}}_{\xi\xi} + \epsilon^2 \overline{\overline{\Phi}}_{yy} - \overline{\overline{\Phi}} = 0 \\ &\overline{\overline{\Psi}}_{\xi\xi} + \epsilon^2 \overline{\overline{\Psi}}_{yy} - \frac{1}{\beta^2} \overline{\overline{\Psi}} = 0 \end{aligned}\right\} \begin{aligned} 0 \le \xi < \infty \\ 0 \le y \le 1 \end{aligned} \tag{3-3}$$

At $\xi = 0$, $0 \le y \le 1$, the boundary conditions are

$$\begin{aligned} &2\epsilon \overline{\overline{\Phi}}_{\xi y} + \overline{\overline{\Psi}}_{\xi\xi} - \epsilon^2 \overline{\overline{\Psi}}_{yy} = 0 \\ &\overline{\overline{\Phi}}_{\xi} - \epsilon \overline{\overline{\Psi}}_{y} = \epsilon^3 F(y) + \epsilon^4 G(y) + \epsilon^5 H(y) + \cdots \end{aligned} \tag{3-4}$$

At $\xi \longrightarrow \infty$, $0 \le y \le 1$, the boundary conditions are

$$\overline{\overline{\Phi}} = \overline{\overline{\Psi}} = 0 \tag{3-5}$$

At $y = 0, 1$; $0 \le \xi < \infty$, the boundary conditions are

$$\begin{aligned} &2\epsilon \overline{\overline{\Phi}}_{\xi y} + \overline{\overline{\Psi}}_{\xi\xi} - \epsilon^2 \overline{\overline{\Psi}}_{yy} = 0 \\ &\overline{\overline{\Phi}} + \frac{2\beta^2}{1-2\beta^2}\left(\epsilon \overline{\overline{\Psi}}_{\xi y} + \epsilon^2 \overline{\overline{\Phi}}_{yy}\right) = 0 \end{aligned} \tag{3-6}$$

Thus for N = 1, the variable change (or scale change) of equations (3-2) has the effect of making the x-derivatives and the undifferentiated terms of the field equations have the same apparent order of magnitude. It therefore exhibits the scale of x on which the terms $\epsilon^2 \overline{\overline{\Phi}}_{xx}$ and $\epsilon^2 \overline{\overline{\Psi}}_{xx}$ cannot be neglected as we did in section B.

Since the scale of y has <u>not</u> been transformed in a similar manner, we expect that the present procedure will not produce suitable approximations in those parts of the strip where $\overline{\overline{\Phi}}$, $\overline{\overline{\Psi}}$ undergo rapid fluctuations in y. Regions such as this do occur in the corners and will be handled later (section D).

We now attempt to determine asymptotic expansions in powers of ϵ for $\overline{\overline{\Phi}}$, $\overline{\overline{\Psi}}$ using an expansion similar to (2-2), but now the coefficients are functions of ξ (not x) and y:

$$\bar{\Phi} \equiv \sum_{n=0}^{\infty} \bar{\Phi}^{(n)}(\xi, y)\,\epsilon^n$$

(3-7)

$$\bar{\Psi} \equiv \sum_{n=0}^{\infty} \bar{\Psi}^{(n)}(\xi, y)\,\epsilon^n$$

The above definitions are now inserted into the field equations and boundary conditions. Note that the boundary conditions at $y = 0, 1$ are not written since they will not be needed to determine the solutions in this region.

Field Equations for $0 \le y \le 1$, $0 \le \xi < \infty$:

$$\left(\bar{\Phi}^{(0)}_{\xi\xi} + \epsilon\bar{\Phi}^{(1)}_{\xi\xi} + \cdots\right) - \left(\bar{\Phi}^{(0)} + \epsilon\bar{\Phi}^{(1)} + \cdots\right) + \epsilon^2\left(\bar{\Phi}^{(0)}_{yy} + \epsilon\bar{\Phi}^{(1)}_{yy} + \cdots\right) = 0$$

(3-8)

$$\left(\bar{\Psi}^{(0)}_{\xi\xi} + \epsilon\bar{\Psi}^{(1)}_{\xi\xi} + \cdots\right) - \frac{1}{\beta^2}\left(\bar{\Psi}^{(0)} + \epsilon\bar{\Psi}^{(1)} + \cdots\right) + \epsilon^2\left(\bar{\Psi}^{(0)}_{yy} + \epsilon\bar{\Psi}^{(1)}_{yy} + \cdots\right) = 0$$

Boundary Conditions at $\xi = 0$, $0 \le y \le 1$:

$$2\epsilon\left(\bar{\Phi}^{(0)}_{\xi y} + \epsilon\bar{\Phi}^{(1)}_{\xi y} + \cdots\right) + \left(\bar{\Psi}^{(0)}_{\xi\xi} + \epsilon\bar{\Psi}^{(1)}_{\xi\xi} + \cdots\right) - \epsilon^2\left(\bar{\Psi}^{(0)}_{yy} + \epsilon\bar{\Psi}^{(1)}_{yy} + \cdots\right) = 0$$

(3-9)

$$\left(\bar{\Phi}^{(0)}_{\xi} + \epsilon\bar{\Phi}^{(1)}_{\xi} + \cdots\right) - \epsilon\left(\bar{\Psi}^{(0)}_{y} + \epsilon\bar{\Psi}^{(1)}_{y} + \cdots\right) = \epsilon^3 F(y) + \epsilon^4 G(y) + \epsilon^5 H(y)$$

Boundary Conditions as $\xi \to \infty$:

(3-10) $$\bar{\Phi}^{(i)} = \bar{\Psi}^{(i)} = 0 \qquad \text{for all } i \text{ and } 0 \le y \le 1.$$

The problem now resolves to a recursive system of boundary value problems for each order in ϵ. For ϵ^0 the field equations are:

$$\bar{\Phi}^{(0)}_{\xi\xi} - \bar{\Phi}^{(0)} = 0$$

(3-11)

$$\bar{\Psi}^{(0)}_{\xi\xi} - \frac{1}{\beta^2}\bar{\Psi}^{(0)} = 0$$

The solutions are:

$$\bar{\Phi}^{(0)} = A_0 e^{\xi} + B_0 e^{-\xi}$$

(3-12)

$$\bar{\Psi}^{(0)} = C_0 e^{\xi/\beta} + D_0 e^{-\xi/\beta}$$

To achieve matching with the zero solution as $\xi \longrightarrow \infty$,

(3-13) $$A_0 = C_0 = 0$$

At $\xi = 0$, $\bar{\bar{\Psi}}^{(0)}_{\xi\xi} = 0$ implies $D_0 = 0$. Also, $\bar{\bar{\Phi}}^{(0)} = 0$ implies $B_0 = 0$.

The solution of (3-11) is thus the trivial solution

(3-14) $$\bar{\bar{\Phi}}^{(0)} = 0, \qquad \bar{\bar{\Psi}}^{(0)} = 0$$

The problems for orders ϵ^1 and ϵ^2 are similar, thus their solutions are similar to the above.

(3-15) $$\bar{\bar{\Phi}}^{(1)} = \bar{\bar{\Phi}}^{(2)} = 0, \qquad \bar{\bar{\Psi}}^{(1)} = \bar{\bar{\Psi}}^{(2)} = 0$$

For order ϵ^3, the boundary conditions along $\xi = 0$ change to

(3-16) $$\bar{\bar{\Psi}}^{(3)}_{\xi\xi} = 0$$
$$\bar{\bar{\Phi}}^{(3)}_{\xi} = F(y)$$

The solution to this order is non-zero for $\bar{\bar{\Phi}}^{(3)}$.

(3-17) $$\bar{\bar{\Psi}}^{(3)} = 0, \qquad \bar{\bar{\Phi}}^{(3)} = -F(y)\,e^{-\xi}$$

For order ϵ^4, the non-zero $\bar{\bar{\Phi}}^{(3)}$ adds yet another non-zero term to the $\xi = 0$ boundary equations. The other equations are still similar to (3-11), (3-12), and (3-13). At $\xi = 0$ the conditions are

(3-18) $$2\,\bar{\bar{\Phi}}^{(3)}_{\xi y} + \bar{\bar{\Psi}}^{(4)}_{\xi\xi} = 0$$
$$\bar{\bar{\Phi}}^{(4)}_{\xi} = G(y)$$

Letting primes denote differentiation with respect to the argument, the solutions for this order become

(3-19) $$\bar{\bar{\Phi}}^{(4)} = -G(y)\,e^{-\xi}, \qquad \bar{\bar{\Psi}}^{(4)} = -2\beta^2 F'(y)\,e^{-\xi/\beta}$$

Similar analysis for order ϵ^5 gives

(3-20) $$\bar{\bar{\Phi}}^{(5)} = \left[-H(y) + \tfrac{1}{2}(\xi - 1 + 4\beta^2)F''(y)\right] e^{-\xi}$$
$$\bar{\bar{\Psi}}^{(5)} = -2\beta^2\, G'(y)\, e^{-\xi/\beta}$$

Obviously, with more algebra, higher order approximations could be obtained. The asymptotic solution for the boundary layer along $x = 0$ is

(3-21) $$\bar{\bar{\Phi}} \approx \epsilon^3 \bar{\bar{\Phi}}^{(3)} + \epsilon^4 \bar{\bar{\Phi}}^{(4)} + \epsilon^5 \bar{\bar{\Phi}}^{(5)}, \qquad \bar{\bar{\Psi}} \approx \epsilon^4 \bar{\bar{\Psi}}^{(4)} + \epsilon^5 \bar{\bar{\Psi}}^{(5)}$$

where the coefficients are given in equations (3-17), (3-19), (3-20). Although three terms are calculated here, the subsequent sections will usually use fewer terms of (3-21) for the sake of clarity.

Before proceeding further, it is appropriate to indicate the mathematical sense in which (3-21) is expected to asymptotically represent the solution of the boundary value problem for the boundary layer along $x = 0$.

Let $\delta > 0$ be an arbitrary positive number. We expect that

(3-22) $$\bar{\bar{\Phi}}(x,y,\epsilon) = \epsilon^3 \bar{\bar{\Phi}}^{(3)}\left(\tfrac{x}{\epsilon}, y\right) + \epsilon^4 \bar{\bar{\Phi}}^{(4)}\left(\tfrac{x}{\epsilon}, y\right) + \epsilon^5 \bar{\bar{\Phi}}^{(5)}\left(\tfrac{x}{\epsilon}, y\right) + O(\epsilon^6)$$

where the error term, $O(\epsilon^6)$ is uniform on the set of points (x, y) in the portion of the strip such that

(3-23) $$x \geq 0, \quad 0 \leq y \leq 1, \quad x^2 + y^2 \geq \delta^2, \quad x^2 + (y-1)^2 \geq \delta^2$$

This corresponds to a closed strip $0 \leq x < \infty$, $0 \leq y \leq 1$ with the corners at $x = 0$, $y = 0$; $x = 0$, $y = 1$ removed by quarter disks of radius δ. A rigorous proof of a similar proposition for a related problem has been given in (7). In this paper, our concern is strictly with the construction of the formal expansions, and no effort is made to rigorously establish our expectations on the error term for this problem.

The validity of solutions (3-21) does not extend entirely to the corners $x = 0$, $y = 0$, and $x = 0$, $y = 1$ since (3-21) was derived without reference to the boundary conditions along the surfaces $y = 0$ and $y = 1$. Thus the approximation (3-21) is not valid when, for fixed ξ, y approaches zero or one. To rectify this error, we must perform another boundary layer type analysis where both x and y coordinates undergo a change in scale. Examples of complicated problems which have used this corner layer technique are given in references (7, 8). This is the subject of the next section.

D. Corner Layer Near $x = 0$, $y = 0$

1. Formulation of the Problem:

Let us now construct the corner layer near $x = 0$, $y = 0$ by a boundary layer procedure.[7,8] As before, we make a scale change in x of $\xi \equiv x/\epsilon$. In addition, we change the y scale by a new variable η which gives rise to the following identities:

$$\eta \equiv y/\epsilon\,, \quad \frac{\partial}{\partial y} = \frac{1}{\epsilon}\frac{\partial}{\partial \eta}\,, \quad \frac{\partial^2}{\partial y^2} = \frac{1}{\epsilon^2}\frac{\partial^2}{\partial \eta^2} \tag{4-1}$$

We will now modify our previous solution, equation (3-21), by adding terms $\bar{\bar{\theta}}$, $\bar{\bar{\chi}}$, which will be significant only in the corner $x = 0$, $y = 0$. The solutions $\bar{\bar{\Phi}}$, $\bar{\bar{\Psi}}$ are then

$$\begin{aligned} \bar{\bar{\Phi}}(x,y;\epsilon) &\approx \epsilon^3\bar{\bar{\Phi}}^{(3)}\left(\tfrac{x}{\epsilon},y\right) + \epsilon^4\bar{\bar{\Phi}}^{(4)}\left(\tfrac{x}{\epsilon},y\right) + \epsilon^5\bar{\bar{\Phi}}^{(5)}\left(\tfrac{x}{\epsilon},y\right) + \bar{\bar{\theta}}\left(\tfrac{x}{\epsilon},\tfrac{y}{\epsilon};\epsilon\right) \\ \bar{\bar{\Psi}}(x,y;\epsilon) &\approx \epsilon^4\bar{\bar{\Psi}}^{(4)}\left(\tfrac{x}{\epsilon},y\right) + \epsilon^5\bar{\bar{\Psi}}\left(\tfrac{x}{\epsilon},y\right) + \bar{\bar{\chi}}\left(\tfrac{x}{\epsilon},\tfrac{y}{\epsilon};\epsilon\right) \end{aligned} \tag{4-2}$$

By substituting the above definitions into the field equations and boundary conditions (see section A), we can formulate the problem for variables ξ, η as the following:

Field Equations:

$$\begin{aligned} &\epsilon^2\left[\frac{1}{\epsilon^2}\bar{\bar{\theta}}_{\xi\xi} + \epsilon^3\cdot\frac{1}{\epsilon^2}\bar{\bar{\Phi}}^{(3)}_{\xi\xi} + \cdots\right] + \epsilon^2\left[\frac{1}{\epsilon^2}\bar{\bar{\theta}}_{\eta\eta} + \epsilon^3\bar{\bar{\Phi}}^{(3)}_{yy} + \cdots\right] - \left[\bar{\bar{\theta}} + \epsilon^3\bar{\bar{\Phi}}^{(3)} + \cdots\right] = 0 \\ &\beta^2\epsilon^2\left[\frac{1}{\epsilon^2}\bar{\bar{\chi}}_{\xi\xi} + \epsilon^4\cdot\frac{1}{\epsilon^2}\bar{\bar{\Psi}}^{(4)}_{\xi\xi} + \cdots\right] + \beta^2\epsilon^2\left[\frac{1}{\epsilon^2}\bar{\bar{\chi}}_{\xi\xi} + \epsilon^4\bar{\bar{\Psi}}^{(4)}_{yy} + \cdots\right] - \left[\bar{\bar{\chi}} + \epsilon^4\Psi^{(4)} + \cdots\right] = 0 \end{aligned} \tag{4-3}$$

Boundary Conditions at $\xi = 0$, $\eta > 0$:

$$\begin{aligned} &2\left[\frac{1}{\epsilon^2}\bar{\bar{\theta}}_{\xi\eta} + \epsilon^3\cdot\frac{1}{\epsilon}\bar{\bar{\Phi}}^{(3)}_{\xi y} + \cdots\right] + \left[\frac{1}{\epsilon^2}\bar{\bar{\chi}}_{\xi\xi} + \epsilon^4\cdot\frac{1}{\epsilon^2}\bar{\bar{\Psi}}^{(4)}_{\xi\xi} + \cdots\right] + \\ &\qquad + (-1)\left[\frac{1}{\epsilon^2}\bar{\bar{\chi}}_{\eta\eta} + \epsilon^4\bar{\bar{\Psi}}^{(4)}_{yy} + \cdots\right] = 0 \\ &\left[\frac{1}{\epsilon}\bar{\bar{\theta}}_{\xi} + \epsilon^3\cdot\frac{1}{\epsilon}\bar{\bar{\Phi}}^{(3)}_{\xi} + \cdots\right] - \left[\frac{1}{\epsilon}\bar{\bar{\chi}}_{\eta} + \epsilon^4\bar{\bar{\Psi}}^{(4)}_{y} + \cdots\right] = \epsilon^3F(y) + \epsilon^4G(y) + \cdots \end{aligned} \tag{4-4}$$

Boundary Conditions at $\eta = 0$, $\xi > 0$:

(4-5)
$$2\left[\frac{1}{\epsilon^2}\bar{\bar{\theta}}_{\xi\eta} + \epsilon^3\cdot\frac{1}{\epsilon}\bar{\bar{\phi}}^{(3)}_{\xi y} + \cdots\right] - \left[\frac{1}{\epsilon^2}\bar{\bar{\chi}}_{\xi\xi} + \epsilon^4\cdot\frac{1}{\epsilon}\bar{\bar{\psi}}^{(4)}_{\xi\xi} + \cdots + \frac{1}{\epsilon^2}\bar{\bar{\chi}}_{\eta\eta} + \epsilon^4\bar{\bar{\psi}}^{(4)}_{yy} + \cdots\right] = 0$$
$$\left[\bar{\bar{\theta}} + \epsilon^3\bar{\bar{\phi}}^{(3)} + \cdots\right] + \left(\frac{2\beta^2}{1-2\beta^2}\right)\epsilon^2\left[\frac{1}{\epsilon^2}\bar{\bar{\theta}}_{\eta\eta} + \epsilon^3\bar{\bar{\phi}}^{(3)}_{yy} + \cdots + \frac{1}{\epsilon^2}\bar{\bar{\chi}}_{\xi\eta} + \epsilon^4\cdot\frac{1}{\epsilon}\bar{\bar{\psi}}^{(4)}_{\xi y} + \cdots\right] = 0$$

In the field equations, it should be noted that all ξ derivatives of $\bar{\bar{\phi}}^{(i)}$ and $\bar{\bar{\psi}}^{(i)}$ cancel with the last $\bar{\bar{\phi}}$ contributions since this was the problem of the x = 0 boundary layer region. Similarly, the terms of the x = 0 boundary layer problem cancel for the equations at $\xi = 0$; thus, the terms exhibiting the given displacement at $\xi = 0$ are cancelled in the second equation of (4-4).

Now we again expand the dependent variables into a power series in ϵ.

(4-6)
$$\bar{\bar{\theta}}(\xi,\eta;\epsilon) \equiv \sum_{i=0}^{\infty}\epsilon^i\,\bar{\bar{\theta}}^{(i)}(\xi,\eta)$$
$$\bar{\bar{\chi}}(\xi,\eta;\epsilon) \equiv \sum_{i=0}^{\infty}\epsilon^i\,\bar{\bar{\chi}}^{(i)}(\xi,\eta)$$

Substituting (4-6) into the equations (4-3) to (4-5), we can again obtain a series of problems for each order in ϵ by the limit process of fixing ξ, η and letting ϵ approach zero. If we take only the first five problems in ϵ for our asymptotic approximation, we get the following boundary value problems:

Field Equations:

(4-7) $\eta > 0$: $\xi > 0$:
$$\bar{\bar{\theta}}^{(i)}_{\xi\xi} + \bar{\bar{\theta}}^{(i)}_{\eta\eta} - \bar{\bar{\theta}}^{(i)} = 0$$
$$\beta^2\left(\bar{\bar{\chi}}^{(i)}_{\xi\xi} + \bar{\bar{\chi}}^{(i)}_{\eta\eta}\right) - \bar{\bar{\chi}}^{(i)} = 0$$
for $i = 0, 1, 2, 3, 4$. (See footnote)

Boundary Conditions at $\xi = 0$:

(4-8)
$$2\,\bar{\bar{\theta}}^{(i)}_{\xi\eta} + \bar{\bar{\chi}}^{(i)}_{\xi\xi} + \bar{\bar{\chi}}^{(i)}_{\eta\eta} = 0$$
$$\bar{\bar{\theta}}^{(i)}_{\xi} - \bar{\bar{\chi}}^{(i)}_{\eta} = 0$$
for $i = 0, 1, 2, \ldots$.

Note that additional terms involving $\bar{\bar{\phi}}^{(i)}(\xi,y)$, $\bar{\bar{\psi}}^{(i)}(\xi,y)$ would appear for $i > 4$.

It is now convenient to change the above boundary conditions at $\xi = 0$. First we take $\left(2\frac{\partial}{\partial\eta}\right)$ of the last equation and substitute the result into the first equation.

$$(4\text{-}9) \qquad 2\bar{\bar{\chi}}^{(i)}_{\eta\eta} + \bar{\bar{\chi}}^{(i)}_{\xi\xi} - \bar{\bar{\chi}}^{(i)}_{\eta\eta} = \bar{\bar{\chi}}^{(i)}_{\xi\xi} + \bar{\bar{\chi}}^{(i)}_{\eta\eta} = 0 \qquad \text{for } i \geq 0,\ \xi = 0.$$

But for $i \leq 4$, the field equations (4-7) make the above condition become

$$(4\text{-}10) \qquad \bar{\bar{\chi}}^{(i)}\Big|_{\xi=0} = 0 \qquad \text{for } i = 0,1,2,3,4.$$

If $\bar{\bar{\chi}}^{(i)} = 0$ along $\xi = 0$, then any η derivative is also zero. The one derivative which will be utilized is

$$(4\text{-}11) \qquad \bar{\bar{\chi}}^{(i)}_{\eta}\Big|_{\xi=0} = 0$$

Using (4-11) in the second equations of (4-8), the boundary conditions at $\xi = 0$ are equivalent to

$$(4\text{-}12) \qquad \xi = 0: \qquad \begin{aligned} \bar{\bar{\chi}}^{(i)} &= 0 \\ \bar{\bar{\theta}}^{(i)}_{\xi} &= 0 \end{aligned} \qquad \text{for } i = 0,1,2,3,4. \text{ and } \eta > 0.$$

Boundary Conditions at $\eta = 0$:

$$(4\text{-}13) \qquad \begin{aligned} &2\bar{\bar{\theta}}^{(i)}_{\xi\eta} + \bar{\bar{\chi}}^{(i)}_{\xi\xi} - \bar{\bar{\chi}}^{(i)}_{\eta\eta} = \delta_{4i}\left[2F'(0)\left(e^{-\xi/\beta} - e^{-\xi}\right)\right] \\ &\bar{\bar{\theta}}^{(i)} + \frac{2\beta^2}{1-2\beta^2}\left(\bar{\bar{\theta}}^{(i)}_{\eta\eta} + \bar{\bar{\chi}}^{(i)}_{\xi\eta}\right) = \delta_{3i}\left[F(0)e^{-\xi}\right] + \delta_{4i}\left[G(0)e^{-\xi}\right] \end{aligned} \qquad \text{for } i = 0,1,2,3,4.$$

In the above equation the Kronecker delta is defined as

$$(4\text{-}14) \qquad \delta_{ji} \equiv \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Boundary Conditions at $\sqrt{\xi^2 + \eta^2} \longrightarrow \infty$:

$$(4\text{-}15) \qquad \bar{\bar{\theta}}^{(i)}\Big|_{\infty} = \bar{\bar{\chi}}^{(i)}\Big|_{\infty} = 0 \qquad \text{for } i \geq 0.$$

Each problem in ϵ can now be analyzed. It should be noted that we are really doing a problem involving the full equations of part I; however, the region has been simplified from a semi-infinite strip to a quadrant. The simplification of the problem geometry is one of the key advantages of using boundary layer techniques.

2. Introduction of Fourier Transforms:

Since each problem for a given order in ϵ still involves a partial differential equation with two independent variables, an additional transformation will be useful. The particular boundary conditions at $\xi = 0$ and $\xi = \infty$ in this corner layer are compatible with the "mixed" boundary conditions that can be handled by Fourier Sine and Fourier Cosine transforms on ξ (see Table 1, p. 56).

Define the Fourier Sine and Fourier Cosine transform pairs as follows:

$$\tilde{\bar{\bar{\chi}}}(\omega,\eta) \equiv \int_0^\infty \bar{\bar{\chi}}(\xi,\eta)\sin(\omega\xi)\,d\xi \equiv \mathcal{F}_s\{\bar{\bar{\chi}}\}$$

(4-16)

$$\bar{\bar{\chi}}(\xi,\eta) = \frac{2}{\pi}\int_0^\infty \tilde{\bar{\bar{\chi}}}(\omega,\eta)\sin(\omega\xi)\,d\omega \equiv \mathcal{F}_s^{-1}\{\tilde{\bar{\bar{\chi}}}\}$$

$$\tilde{\bar{\bar{\theta}}}(\omega,\eta) \equiv \int_0^\infty \bar{\bar{\theta}}(\xi,\eta)\cos(\omega\xi)\,d\xi \equiv \mathcal{F}_c\{\bar{\bar{\theta}}\}$$

(4-17)

$$\bar{\bar{\theta}}(\xi,\eta) = \frac{2}{\pi}\int_0^\infty \tilde{\bar{\bar{\theta}}}(\omega,\eta)\cos(\omega\xi)\,d\omega \equiv \mathcal{F}_c^{-1}\{\tilde{\bar{\bar{\theta}}}\}$$

From the above definitions the transforms of various partial derivatives are obtained (using "integration-by-parts").

$$\mathcal{F}_c\{\bar{\bar{\chi}}_{\xi\eta}\} = \omega\,\tilde{\bar{\bar{\chi}}}_\eta + \bar{\bar{\chi}}_\eta\Big|_{\xi=0}, \quad \mathcal{F}_c\{\bar{\bar{\chi}}_{\xi\xi}\} = -\omega^2\tilde{\bar{\bar{\chi}}} + \omega\,\bar{\bar{\chi}}\Big|_{\xi=0} \tag{4-18}$$

$$\mathcal{F}_s\{\bar{\bar{\theta}}_{\xi\eta}\} = -\omega\,\tilde{\bar{\bar{\theta}}}_\eta, \quad \mathcal{F}_s\{\bar{\bar{\theta}}_{\xi\xi}\} = -\omega^2\tilde{\bar{\bar{\theta}}} - \bar{\bar{\theta}}_\xi\Big|_{\xi=0}$$

Note that all the terms involving $\bar{\bar{\chi}}$ and $\bar{\bar{\theta}}$ at $\xi = 0$ are zero because of equations (4-10), (4-11) and (4-12). Taking the Fourier Sine transform of the first of equations (4-7) and of the second equation of (4-13), and taking the Fourier Cosine transform of the other two equations of (4-7) and (4-13), the following transformed problem is obtained.

Field Equations for $\eta \geq 0$:

$$\begin{aligned} &\tilde{\bar{\bar{\theta}}}^{(i)}_{\eta\eta} - \tilde{\bar{\bar{\theta}}}^{(i)}(1+\omega^2) = 0 \\ &\tilde{\bar{\bar{\chi}}}^{(i)}_{\eta\eta} - \tilde{\bar{\bar{\chi}}}^{(i)}\left(1/\beta^2 + \omega^2\right) = 0 \end{aligned} \qquad \text{for } i = 0, 1, 2, 3, 4. \tag{4-19}$$

Boundary Conditions at $\eta = 0$ (for $i = 0, 1, 2, 3, 4.$)

$$\begin{aligned} &2\omega\,\tilde{\bar{\bar{\theta}}}^{(i)}_\eta + \omega^2\,\tilde{\bar{\bar{\chi}}}^{(i)} + \tilde{\bar{\bar{\chi}}}^{(i)}_{\eta\eta} = \delta_{4i}\left[-2F'(0)\left(\frac{\omega}{1+\omega^2} - \frac{\omega}{\beta^{-2}+\omega^2}\right)\right] \\ &\tilde{\bar{\bar{\theta}}}^{(i)} + \frac{2\beta^2}{1-2\beta^2}\left[\tilde{\bar{\bar{\theta}}}^{(i)}_{\eta\eta} + \omega\,\tilde{\bar{\bar{\chi}}}^{(i)}_\eta\right] = \delta_{3i}\left[F(0)\frac{1}{1+\omega^2}\right] + \delta_{4i}\left[G(0)\frac{1}{1+\omega^2}\right] \end{aligned} \tag{4-20}$$

Boundary Conditions as $\eta \longrightarrow \infty$:

$$\tilde{\bar{\bar{\theta}}}^{(i)} = \tilde{\bar{\bar{\chi}}}^{(i)} = 0 \tag{4-21}$$

The solutions of (4-19) are

$$\begin{aligned} &\tilde{\bar{\bar{\theta}}}^{(i)} = A_i(\omega)\,e^{-\sqrt{1+\omega^2}\,\eta} + C_i(\omega)\,e^{+\sqrt{1+\omega^2}\,\eta} \\ &\tilde{\bar{\bar{\chi}}}^{(i)} = B_i(\omega)\,e^{-\sqrt{\beta^{-2}+\omega^2}\,\eta} + D_i(\omega)\,e^{+\sqrt{\beta^{-2}+\omega^2}\,\eta} \end{aligned} \tag{4-22}$$

Using (4-21), we get $C_i(\omega) = D_i(\omega) = 0$. This makes the solutions of the form

$$\tilde{\tilde{\theta}}^{(i)} = A_i(\omega)\, e^{-\eta\sqrt{1+\omega^2}} \qquad \text{for } i=0,1,2,3,4. \tag{4-23}$$
$$\tilde{\tilde{\chi}}^{(i)} = B_i(\omega)\, e^{-\eta\sqrt{\beta^{-2}+\omega^2}}$$

The final two constants will be determined by using (4-22) in (4-20).

3. <u>Solutions for Order ϵ^0, ϵ^1, ϵ^2:</u>

For these problems, the right-hand side of (4-20) has no non-zero terms. Hence the equations for determining $A_i(\omega)$, $B_i(\omega)$ are homogeneous. Since the determinant of the coefficients (later denoted as $R(\omega)$) is non-zero for most ω values, the only possible solution is the trivial solution

$$A_i(\omega) = B_i(\omega) = 0 \qquad \text{for } i = 0,1,2.$$

The inversion equations (4-16), (4-17) give

$$\bar{\bar{\theta}}^{(0)} = \bar{\bar{\theta}}^{(1)} = \bar{\bar{\theta}}^{(2)} = 0, \quad \bar{\bar{\chi}}^{(0)} = \bar{\bar{\chi}}^{(1)} = \bar{\bar{\chi}}^{(2)} = 0 \tag{4-24}$$

4. <u>Solution for Order ϵ^3:</u>

The second equation of (4-20) has a non-zero contribution, thus the two equations for $A_3(\omega)$, $B_3(\omega)$ become

$$A_3(\omega)\left[-2\omega\sqrt{1+\omega^2}\right] + B_3(\omega)\left[1/\beta^2 + 2\omega^2\right] = 0 \tag{4-25}$$
$$A_3(\omega)\left[1+2\beta^2\omega^2\right] + B_3(\omega)\left[-2\omega\beta^2\sqrt{\beta^{-2}+\omega^2}\right] = F(0)\left[\frac{1-2\beta^2}{1+\omega^2}\right]$$

Solving and substituting into (4-23) gives

$$\tilde{\tilde{\theta}}^{(3)} = \left[\frac{(1-2\beta^2)\,F(0)}{R(\omega)}\right] e^{-\eta\sqrt{1+\omega^2}} \tag{4-26}$$
$$\tilde{\tilde{\chi}}^{(3)} = \left[\frac{(1-2\beta^2)\,F(0)}{R(\omega)}\right] e^{-\eta\sqrt{\beta^{-2}+\omega^2}}$$

where

(4-27) $$R(\omega) \equiv \left[1+2\beta^2\omega^2\right]^2 - 4\beta^4\omega^2\sqrt{1+\omega^2}\sqrt{1/\beta^2+\omega^2}$$

The determinant of the coefficients, $R(\omega)$, is physically related to the Rayleigh surface wave phenomena.

Finally, we use the inversion formulae (4-16), (4-17) and the fact that $\cos(x) = \mathcal{R}e\{e^{ix}\}$, $\sin(x) = \mathcal{I}m\{e^{ix}\}$.

(4-28) $$\bar{\theta}^{(3)}\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) = \frac{2}{\pi}(1-2\beta^2)F(0)\,\mathcal{R}e\left\{\int_0^\infty \frac{1+2\beta^2\omega^2}{1+\omega^2}\frac{1}{R(\omega)}\cdot\right.$$
$$\left.\cdot\exp\left[-\frac{1}{\epsilon}\left(y\sqrt{1+\omega^2}-ix\omega\right)\right]d\omega\right\}$$

(4-29) $$\bar{\chi}^{(3)}\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) = \frac{2}{\pi}(1-2\beta^2)F(0)\,\mathcal{I}m\left\{\int_0^\infty \frac{2\beta^2\omega^2}{\sqrt{1+\omega^2}}\frac{1}{R(\omega)}\cdot\right.$$
$$\left.\cdot\exp\left[\frac{-1}{\epsilon}\left(y\sqrt{\beta^{-2}+\omega^2}-ix\omega\right)\right]d\omega\right\}$$

5. <u>Solution for Order ϵ^4:</u>

The equations for $A_4(\omega)$, $B_4(\omega)$ now become (using (4-19), (4-22))

$$A_4(\omega)\left[-2\omega\sqrt{1+\omega^2}\right] + B_4(\omega)\left[\beta^{-2}+2\omega^2\right] = 2F'(0)\left[\frac{\omega}{1+\omega^2} - \frac{\omega}{\beta^{-2}+\omega^2}\right]$$

(4-30)

$$A_4(\omega)\left[1+2\beta^2\omega^2\right] + B_4(\omega)\left[-2\omega\beta^2\sqrt{\beta^{-2}+\omega^2}\right] = G(0)\left[\frac{1-2\beta^2}{1+\omega^2}\right]$$

Substituting the solutions of (4-30) into (4-23), and then inverting, we get

(4-31) $$\bar{\theta}^{(4)}\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) = \frac{2}{\pi}\mathcal{R}e\left\{\int_0^\infty \left[E_1(\omega)+E_2(\omega)\right]\frac{1}{R(\omega)}\cdot\right.$$
$$\left.\cdot\exp\left[-\frac{1}{\epsilon}\left(y\sqrt{1+\omega^2}-ix\omega\right)\right]d\omega\right\}$$

(4-32) $$\bar{\chi}^{(4)}\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) = \frac{2}{\pi}\mathcal{I}m\left\{\int_0^\infty \left[E_3(\omega)+E_4(\omega)\right]\frac{1}{R(\omega)}\cdot\right.$$
$$\left.\cdot\exp\left[-\frac{1}{\epsilon}\left(y\sqrt{\beta^{-2}+\omega^2}-ix\omega\right)\right]d\omega\right\}$$

where

$$(4\text{-}33)\qquad \begin{aligned} E_1(\omega) &= 4\beta^2(1-\beta^2)\,F'(0)\,\frac{\omega^2}{(1+\omega^2)\sqrt{\beta^{-2}+\omega^2}} \\ E_2(\omega) &= (1-2\beta^2)\;G(0)\;\frac{1+2\beta^2\omega^2}{1+\omega^2} \\ E_3(\omega) &= 2\,(1-\beta^2)\,F'(0)\;\frac{\omega\,(1+2\beta^2\omega^2)}{(1+\omega^2)\,(\beta^{-2}+\omega^2)} \\ E_4(\omega) &= 2\beta^2\,G(0)\,(1-2\beta^2)\,\frac{\omega}{\sqrt{1+\omega^2}} \end{aligned}$$

E. Corner Layer Near $x = 0,\ y = 1$:

Inspection of the results of the $x = 0,\ y = 0$ corner reveals that the solution in the Laplace transformed domain depends on the geometry and constants. The edge loading $f(y,t)$ enters only as constants (like $F(0)$, $F'(0)$, $G(0)$, etc.). Since the geometry of the problem is symmetric about the line $y = \frac{1}{2}$, the wave front locations (from the corner waves) are also symmetric about the same line. The differences in the corner solution at $x = 0,\ y = 1$ from the solution at $x = 0,\ y = 0$ involves a simple change of coordinates and use of different constants. That is, the corner waves from the two corners will be symmetrically located about the line $y = \frac{1}{2}$, but the amplitudes will vary due to different constants (like $F(0)$, $F'(0)$, $G(0)$, etc.). We add terms $\bar{\bar{\Theta}}(\xi,\eta;\epsilon)$, $\bar{\bar{\mathcal{X}}}(\xi,\eta;\epsilon)$ to $\bar{\bar{\Phi}}$ and $\bar{\bar{\Psi}}$. These new terms correspond to the $\bar{\Theta}(\xi,\eta;\epsilon)$, $\bar{\mathcal{X}}(\xi,\eta;\epsilon)$ of section D. Then we expand the new terms in an asymptotic series in ϵ.

To change the results in section D so that they apply to $x = 0$, $y = 1$, we first make the coordinate transformation

(5-1) $$\underline{x} \equiv x \qquad \underline{y} \equiv 1-y \qquad \text{for } 0 \leq y \leq 1, \quad x > 0.$$

where $\underline{x}$ and $\underline{y}$ would be coordinates of a new system centered at $x = 0$, $y = 1$. Now the solution at $x = 0$, $y = 1$ can be written down in terms of $\underline{x}$, $\underline{y}$ by using (5-1), the results of section D, and the following constants:

	Constant in Section D	Replacement for $x = 0, y = 1$
	$F(0)$	$F(1)$
(5-2)	$G(0)$	$G(1)$
	$F'(0)$	$-F'(1)$

Note that the minus sign on $F'(1)$ occurs because $F'(y) = -F'(\underline{y})$.

For ϵ^0, ϵ^1, ϵ^2 the solutions are (using an extra horizontal line to distinguish the new solutions):

(5-3)

For ϵ^3, (4-28) and (4-29) become

(5-4)
$$\bar{\bar{\theta}}^{(3)}\left(\frac{\underline{x}}{\epsilon}, \frac{\underline{y}}{\epsilon}\right) = \frac{2}{\pi}(1-2\beta^2)F(1)\,\mathcal{R}e\left\{\int_0^\infty \frac{1+2\beta^2\omega^2}{1+\omega^2}\,\frac{1}{R(\omega)}\cdot \exp\left[\frac{-1}{\epsilon}(1-\underline{y})\sqrt{1+\omega^2}+\frac{i\underline{x}\omega}{\epsilon}\right]d\omega\right\}$$

$$\bar{\bar{\chi}}^{(3)}\left(\frac{\underline{x}}{\epsilon}, \frac{\underline{y}}{\epsilon}\right) = \frac{2}{\pi}(1-2\beta^2)F(1)\,\mathcal{I}m\left\{\int_0^\infty \frac{2\beta^2\omega^2}{\sqrt{1+\omega^2}}\,\frac{1}{R(\omega)}\cdot \exp\left[\frac{-1}{\epsilon}(1-\underline{y})\sqrt{\beta^{-2}+\omega^2}+\frac{i\underline{x}\omega}{\epsilon}\right]d\omega\right\}$$

For ϵ^4, (4-31) and (4-32) are changed to

(5-5)
$$\bar{\bar{\theta}}^{(4)}\left(\frac{\underline{x}}{\epsilon}, \frac{\underline{y}}{\epsilon}\right) = \frac{2}{\pi}\,\mathcal{R}e\left\{\int_0^\infty\left[\bar{E}_1(\omega)+\bar{E}_2(\omega)\right]\frac{1}{R(\omega)}\exp\left[\frac{-1}{\epsilon}(1-\underline{y})\sqrt{1+\omega^2}+\frac{i\underline{x}\omega}{\epsilon}\right]d\omega\right\}$$

$$\bar{\bar{\chi}}^{(4)}\left(\frac{\underline{x}}{\epsilon}, \frac{\underline{y}}{\epsilon}\right) = \frac{2}{\pi}\,\mathcal{I}m\left\{\int_0^\infty\left[\bar{E}_3(\omega)+\bar{E}_4(\omega)\right]\frac{1}{R(\omega)}\exp\left[\frac{-1}{\epsilon}(1-\underline{y})\sqrt{\beta^{-2}+\omega^2}+\frac{i\underline{x}\omega}{\epsilon}\right]d\omega\right\}$$

where

$$\bar{E}_1(\omega) = -4\beta^2(1-\beta^2)\,F'(1)\,\frac{\omega^2}{(1+\omega^2)\sqrt{\beta^{-2}+\omega^2}}$$

$$\bar{E}_2(\omega) = (1-2\beta^2)\,G(1)\,\frac{1+2\beta^2}{1+\omega^2}$$

(5-6)
$$\bar{E}_3(\omega) = -2(1-\beta^2)\,F'(1)\,\frac{\omega(1+2\omega^2\beta^2)}{(1+\omega^2)(\beta^{-2}+\omega^2)}$$

$$\bar{E}_4(\omega) = 2\beta^2(1-2\beta^2)\,G(1)\,\frac{\omega}{\sqrt{1+\omega^2}}$$

IV. INVERSION FOR WAVE FRONT APPROXIMATIONS

A. Introduction of the Cagniard Method

All of the solutions in the last part are for the Laplace transformed potentials, $\bar{\bar{\Phi}}$ and $\bar{\bar{\Psi}}$. Thus it remains to invert back to the time domain. This can easily be done for some of the terms by using equation (II-4-2), but this approach leads to double integrals for the terms arising from the corner layers.

If we note that $\epsilon = c/s$, then each corner layer solution in the Laplace transform domain has the form of (constant)$\cdot \frac{1}{s^N} \cdot \bar{\bar{Z}}$ where

$$(1\text{-}1) \qquad \bar{\bar{Z}}(x,y;s) = \int_0^\infty \frac{M(\omega)}{R(\omega)} e^{-sg(\omega)} d\omega$$

Now the inversion of the above equation for $Z(x, y, t)$ would be simple if the right-hand side of (1-1) were of the form

$$(1\text{-}2) \qquad \int_0^\infty e^{-s\tau} G(\tau) \frac{\partial\omega}{\partial\tau} d\tau$$

where τ is real and positive. The inversion of the Laplace transform is direct and equals

$$(1\text{-}3) \qquad Z(x,y;t) = G(t) \frac{\partial\omega(t)}{\partial t}$$

To change the integral in (1-1) to the form (1-2), the path of integration in the complex ω-plane must be deformed from the real axis to the particular path which will yield a real and positive τ at all points. The only restriction on performing this deformation is that the path not be deformed through any poles or branch cuts of the integrand, $\mathcal{M}(\omega)\mathcal{R}^{-1}(\omega)$. This ingenious method for inverting the Laplace transform is due to Cagniard[9]. As applied here, the details of the calculation are similar to those in (1, 2).

Now that we can theoretically revert back to the time domain, we must ascertain that the ϕ and Ψ will truly approximate the behavior at, and immediately behind, the wave fronts. With this goal in mind, we consider the Tauberian theorems for Laplace transforms which enable one to obtain an asymptotic series expansion in t from a knowledge of the behavior of the Laplace transform when the parameter s is large[10]. The series expansion is valid for small t only; and, in our problem, this is exactly when we desire to know the effect of the edge disturbance. The nature of wave propagation in elastic media should then enable us to give the correct wave front information throughout the entire body because the wave front consists of phenomena relating only to the geometry of the strip and the nature of the original disturbance for small t.

The relation between the asymptotic approximations for large s and wave front behavior can also be seen from a more mathematical point of view. We illustrate by considering the dilatational potential $\Phi(x,y;t)$. Retaining only first terms in the asymptotic expansions for $\bar{\bar{\Phi}}(x,y;s)$ as $s \longrightarrow \infty$, our approximations are

$$(1\text{-}4) \quad \bar{\bar{\Phi}}(x,y;s) \approx \frac{-F(y)}{s^3} e^{-\frac{sx}{c}} + \frac{1}{s^3}\bar{\Theta}^{(3)}\left(\frac{sx}{c}, \frac{sy}{c}\right) + \frac{1}{s^3}\bar{\bar{\Theta}}^{(3)}\left(\frac{sx}{c}, \frac{sy}{c}\right)$$

We expect (but do not prove) that, in fact, the relation

$$(1\text{-}5) \quad \bar{\bar{\Phi}}(x,y;s) = \left[\frac{-1}{s^3}F(y) + O(s^{-4})\right] e^{-sx/c} + $$
$$+ \frac{1}{s^3}\bar{\bar{\Theta}}^{(3)}(sx/c, sy/c) + O\left[s^{-4} e^{-\frac{s}{c}\sqrt{x^2+y^2}}\right] +$$
$$+ \frac{1}{s^3}\bar{\bar{\Theta}}^{(3)}(sx/c, sy/c) + O\left[s^{-4} e^{-\frac{s}{c}\sqrt{x^2+(1-y)^2}}\right]$$

holds as $s \longrightarrow \infty$. Moreover, we expect that such a relation holds for complex s uniformly in x, y for $x \geqslant 0$, $0 \leqslant y \leqslant 1$, and uniformly in s for $-\infty < \mathcal{I}m\, s < \infty$ as $\mathcal{R}e\, s \longrightarrow \infty$. From the above relation, it follows from the inversion formula for the Laplace transform that

$$(1\text{-}6) \quad \phi(x,y;t) = \frac{-1}{2}F(y)(t-\tfrac{x}{c})^2 \mathcal{H}(t-\tfrac{x}{c}) + O\left[(t-\tfrac{x}{c})^3 \mathcal{H}(t-\tfrac{x}{c})\right] +$$
$$+ \frac{1}{2}\int_0^t (t-\tau)^2 \Theta^{(3)}(x,y;\tau)\,d\tau + O\left[(t-\mathbb{R})^3 \mathcal{H}(t-\mathbb{R})\right] +$$
$$+ \frac{1}{2}\int_0^t (t-\tau)^2 \Theta^{(3)}(x,y;\tau)\,d\tau + O\left[(t-\mathbb{R})^3 \mathcal{H}(t-\mathbb{R})\right]$$

where

$$\mathbb{R} \equiv \frac{1}{c}\sqrt{x^2 + y^2}$$

This gives an approximation which is valid for a short time interval after the wave front arrives at a point x, y.

A similar argument would apply to the potential Ψ and to stresses and displacements computed from ϕ and Ψ. Appropriate refinements of the above statements could be obtained by retaining second and higher order terms in the asymptotic expansions for $\bar{\bar{\Phi}}$ or $\bar{\bar{\Psi}}$.

B. Inversion of $\bar{\bar{\theta}}^{(i)}, \bar{\bar{\chi}}^{(i)}$:

We now apply the Cagniard method to the corner layer terms. Consider $\bar{\bar{\theta}}^{(3)}$, $\bar{\bar{\theta}}^{(4)}$ integrals of equations (III-5-4) and (III-5-6). Since they both have the same functions in the exponential, we define

$$-s\tau \equiv \frac{-1}{\epsilon}\left(y\sqrt{1+\omega^2} - i x \omega\right) \tag{2-1}$$

or

$$\tau \equiv Y\sqrt{1+\omega^2} - iX\omega \tag{2-2}$$

where both X and Y are positive real numbers defined as

$$X \equiv x/c \,, \quad Y \equiv y/c \tag{2-3}$$

The function $[1+\omega^2]^{1/2}$ must now be precisely defined for complex ω. By letting $\omega \equiv \alpha + i\delta$, for α and δ both real, the definitions of variables shown on figure 2a become

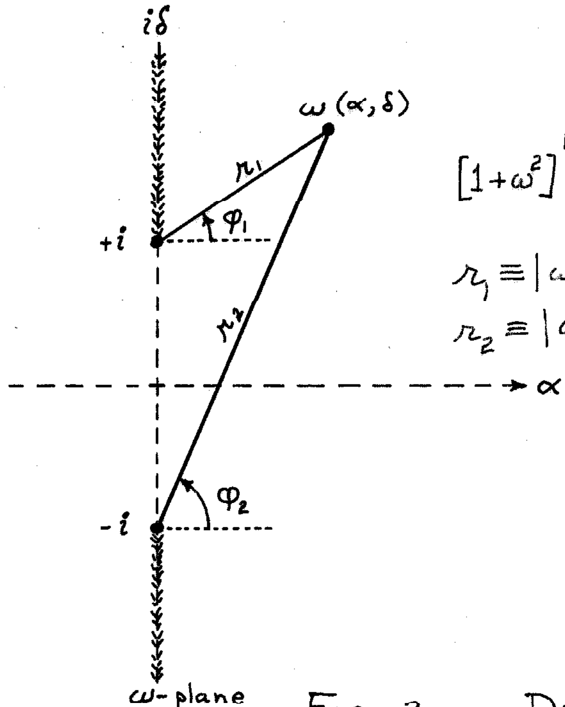
$$r_1 \equiv |\omega - i| = \sqrt{\alpha^2 + (\delta-1)^2} \,, \quad r_2 \equiv |\omega + i| = \sqrt{\alpha^2 + (\delta+1)^2} \tag{2-4}$$

$$\varphi_1 \equiv \arg(\omega - i), \; \frac{-3\pi}{2} < \varphi_1 < \frac{\pi}{2} \,; \quad \varphi_2 \equiv \arg(\omega + i), \; \frac{-\pi}{2} < \varphi_2 < \frac{3\pi}{2} \tag{2-5}$$

Note that the range of angles makes the radicals real and positive for points on the real ω axis. Using (2-4) and (2-5), equation (2-2) can be expressed as follows:

$$\begin{aligned} \tau &\equiv Y\left[1+\omega^2\right]^{1/2} - iX\omega \\ &\equiv \left\{Y\sqrt{r_1 r_2}\cos\left[\frac{\varphi_1+\varphi_2}{2}\right] + X\delta\right\} + i\left\{Y\sqrt{r_1 r_2}\sin\left[\frac{\varphi_1+\varphi_2}{2}\right] - X\alpha\right\} \\ &\equiv \left\{\frac{1}{\sqrt{2}}Y\sqrt{r_1 r_2 + \alpha^2 - \delta^2 + 1} + X\delta\right\} + \\ &\qquad + i\left\{\frac{1}{\sqrt{2}}Y\,\mathrm{sgn}(\varphi_1+\varphi_2)\sqrt{r_1 r_2 - \alpha^2 + \delta^2 - 1} - X\alpha\right\} \end{aligned} \tag{2-6}$$

where $\mathrm{sgn}(\varphi_1+\varphi_2)$ signifies the algebraic sign of $(\varphi_1+\varphi_2)$.

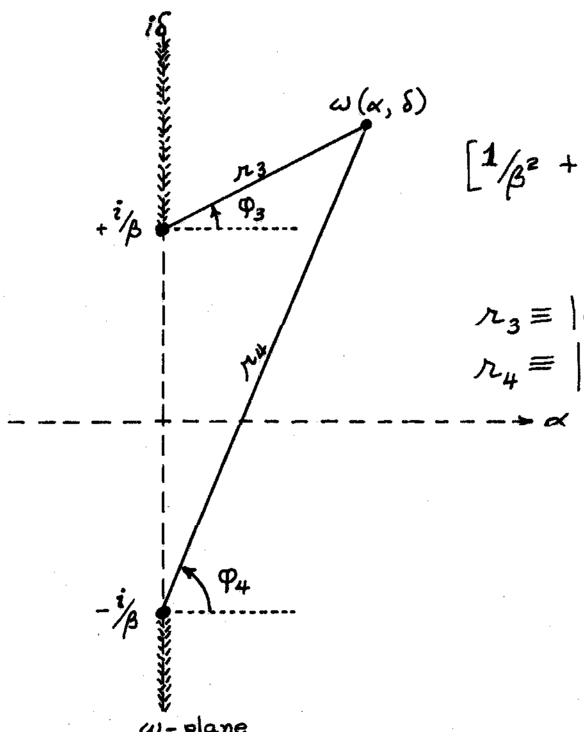


$$[1+\omega^2]^{1/2} \equiv \sqrt{r_1 r_2}\; e^{\frac{i}{2}(\varphi_1+\varphi_2)}$$

$$r_1 \equiv |\omega - i| , \qquad -3\pi/2 < \varphi_1 < \pi/2$$

$$r_2 \equiv |\omega + i| , \qquad -\pi/2 < \varphi_2 < 3\pi/2$$

FIG. 2a DEFINITION OF $[1+\omega^2]^{1/2}$



$$[1/\beta^2 + \omega^2]^{1/2} \equiv \sqrt{r_3 r_4}\; e^{\frac{i}{2}(\varphi_3+\varphi_4)}$$

$$r_3 \equiv |\omega - 1/\beta| , \qquad -3\pi/2 < \varphi_3 < \pi/2$$

$$r_4 \equiv |\omega + 1/\beta| , \qquad -\pi/2 < \varphi_4 < 3\pi/2$$

FIG. 2b DEFINITION OF $[1/\beta^2 + \omega^2]^{1/2}$

For the $\bar{\bar{\mathcal{X}}}^{(3)}$ and $\bar{\bar{\mathcal{X}}}^{(4)}$ integrals, the procedure is similar and uses figure 2b. Corresponding to τ, a new variable $\hat{\tau}$ is

$$\text{(2-7)} \quad \hat{\tau} \equiv Y\left[\beta^{-2}+\omega^{2}\right]^{1/2} - iX\omega$$

$$\equiv \left\{Y\sqrt{r_3 r_4}\cos\left[\frac{\Phi_3+\Phi_4}{2}\right]+X\delta\right\} + i\left\{Y\sqrt{r_3 r_4}\sin\left[\frac{\Phi_3+\Phi_4}{2}\right]-X\alpha\right\}$$

$$\equiv \left\{\frac{1}{\sqrt{2}}Y\sqrt{r_3 r_4+\alpha^2-\delta^2+\beta^{-2}}+X\delta\right\} +$$

$$+ i\left\{\frac{1}{\sqrt{2}}Y\,\mathrm{sgn}[\Phi_3+\Phi_4]\sqrt{r_3 r_4-\alpha^2+\delta^2-\beta^{-2}}-X\alpha\right\}$$

where the new radii and angles are defined by

$$\text{(2-8)} \quad r_3 \equiv |\omega - 1/\beta| = \sqrt{\alpha^2+(\delta-1/\beta)^2}$$

$$r_4 \equiv |\omega + 1/\beta| = \sqrt{\alpha^2+(\delta+1/\beta)^2}$$

$$\text{(2-9)} \quad \Phi_3 \equiv \arg(\omega-1/\beta), \quad -\frac{3\pi}{2} < \Phi_3 < \frac{\pi}{2}$$

$$\Phi_4 \equiv \arg(\omega+1/\beta), \quad -\frac{\pi}{2} < \Phi_4 < \frac{3\pi}{2}$$

Equations (2-6) and (2-7) are used for determining the appropriate Cagniard paths in the ω-plane as shown in figures 3a, 3b. The Γ_1 paths of figure 3 were found by:

(a) finding the locus of points for which $\mathrm{Im}\,\tau = 0$

(b) finding those points on this locus for which $\mathrm{Re}\,\tau \geq \tau\big|_{\omega=0}$

In this problem, step (b) includes the entire hyperbolic curve in the upper half plane, but only the portion of the curve in the first quadrant is used for Γ_1 because of the locus of Γ_0 and the requirement of exponential decay as $|\omega| \to \infty$.

It remains to check that no singularities or branch points of the integrands are inside the $\Gamma_0 - \Gamma_1 - \Gamma_\infty$ paths of figure 3. Consider first $R^{-1}(\omega)$, which appears in all the integrands. The singularities

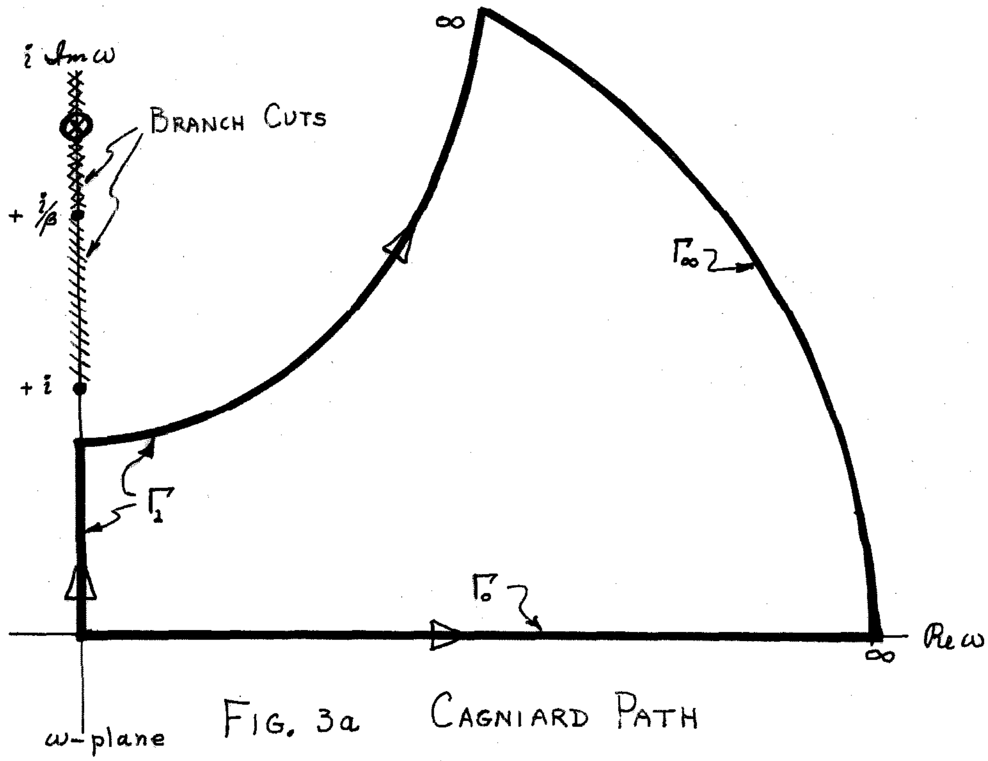


Fig. 3a Cagniard Path

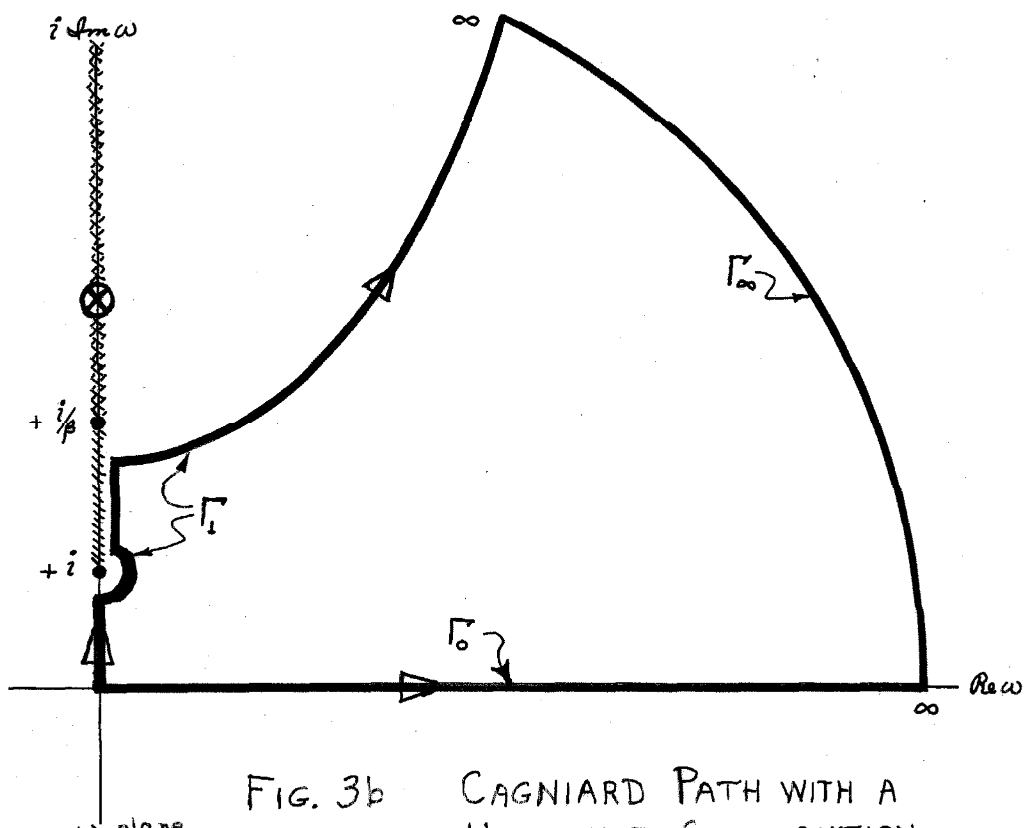


Fig. 3b Cagniard Path with a Headwave Contribution

of the function are as shown in figure 4. Note that the zero of $R(\omega)$ satisfies $W_o > \beta^{-1}$; therefore $\omega = \pm W_o i$ lies outside the branch cuts of the function. Almost all materials of interest have properties such that W_o is located as shown in the figure.

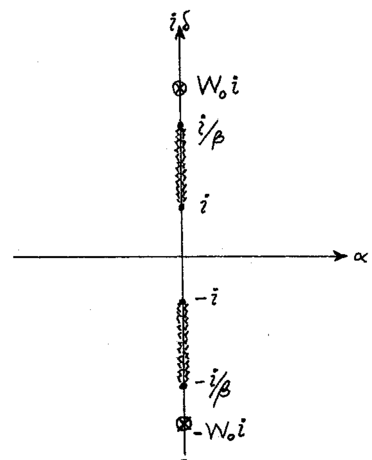


Fig. 4 Singularities of $R(\omega)$

The choice of the branch cuts for the radicals (see figure 2) places all singularities of the integrands along the imaginary axis at, and outside of, the points $\omega = \pm i$. For equations (5-4) and (5-6), the integrands have singularities as shown in figure 3a; for equations (5-5) and (5-7), as shown in figure 3b. The deformation of Γ_o to the path $\Gamma_1 + \Gamma_\infty$ is valid.

The contribution of the Γ_∞ paths in figure 3 are negligible because τ and $\hat{\tau}$ have positive real parts along this curve. This can be easily seen from equations (2-6) and (2-7) since X, Y, $\sqrt{\omega^2 + \text{const.}}$ and ζ are all positive on the Γ_∞ paths. Thus $\mathcal{R}e\,\tau$ and $\mathcal{R}e\,\hat{\tau}$ are positive; and, as all r_i go to infinity, the real parts of τ, $\hat{\tau}$ also go to infinity. The result is that $e^{-\tau s}$ goes to zero along Γ_∞.

With Γ_∞ neglected, Cauchy's theorem gives

$$\int_{\Gamma_o} = \int_{\Gamma_1} \tag{2-10}$$

In figure 3b, the point where the path Γ_1 leaves the imaginary axis in the ω-plane can be at a point above $\omega = i$. Thus for $\left(X/\beta\mathbb{R}\right) > 1$, the path Γ_1 must go around the branch cut associated with $[\omega^2 + 1]^{1/2}$. This segment between $\omega = i$ and $\omega = \frac{iX}{\beta\mathbb{R}}$ will yield the "head wave" contribution.

All factors in the integrands can now be expressed in terms of τ . For the $\bar{\bar{\theta}}^{(3)}, \bar{\bar{\theta}}^{(4)}$ integrals (being careful to note that $\mathbb{R} \equiv \sqrt{X^2 + Y^2}$; it is not $R(\omega)$)

(2-11)
$$\mathbb{R}^2\omega = \begin{cases} i\left[X\tau - Y\sqrt{\mathbb{R}^2 - \tau^2}\right] & \text{for } Y \le \tau \le \mathbb{R} \\ i\tau X + Y\sqrt{\tau^2 - \mathbb{R}^2} & \text{for } \tau > \mathbb{R} \end{cases}$$

(2-12)
$$\mathbb{R}^2\frac{\partial\omega}{\partial\tau} = \begin{cases} i\left[X + Y\tau\left(\mathbb{R}^2 - \tau^2\right)^{-1/2}\right] & \text{for } Y \le \tau \le \mathbb{R} \\ iX + Y\tau\left(\tau^2 - \mathbb{R}^2\right)^{-1/2} & \text{for } \tau > \mathbb{R} \end{cases}$$

(2-13)
$$\mathbb{R}^4\omega^2 = \begin{cases} -\tau^2X^2 - Y^2(\mathbb{R}^2 - \tau^2) + 2\tau XY\sqrt{\mathbb{R}^2 - \tau^2} & \text{for } Y \le \tau \le \mathbb{R} \\ -\tau^2X^2 + Y^2(\tau^2 - \mathbb{R}^2) + i2\tau XY\sqrt{\tau^2 - \mathbb{R}^2} & \text{for } \tau > \mathbb{R} \end{cases}$$

(2-14)
$$\mathbb{R}^8\omega^4 = \begin{cases} \tau^4X^4 + Y^4(\mathbb{R}^2 - \tau^2)^2 + 6\tau^2Y^2X^2(\mathbb{R}^2 - \tau^2) + \\ \quad + (-4)\tau XY\sqrt{\mathbb{R}^2 - \tau^2}\left[\tau^2X^2 + Y^2(\mathbb{R}^2 - \tau^2)\right] & \text{for } Y \le \tau \le \mathbb{R} \\ \tau^4X^4 + Y^4(\tau^2 - \mathbb{R}^2)^2 - 6\tau^2Y^2X^2(\tau^2 - \mathbb{R}^2) + \\ \quad + i4\tau XY\sqrt{\tau^2 - \mathbb{R}^2}\left[-\tau^2X^2 + Y^2(\tau^2 - \mathbb{R}^2)\right] & \text{for } \tau > \mathbb{R} \end{cases}$$

For the $\bar{\bar{\chi}}^{(3)}$ and $\bar{\bar{\chi}}^{(4)}$ integrals

(2-15)
$$\mathbb{R}^2\omega = \begin{cases} i\left[X\hat{\tau} - Y\sqrt{\mathbb{R}^2\beta^{-2} - \hat{\tau}^2}\right] & \text{for } Y/\beta \le \hat{\tau} \le \mathbb{R}/\beta \\ iX\hat{\tau} + Y\sqrt{\hat{\tau}^2 - \mathbb{R}^2\beta^{-2}} & \text{for } \hat{\tau} > \mathbb{R}/\beta \end{cases}$$

$$(2\text{-}16)\quad \mathbb{R}^2 \frac{\partial \omega}{\partial \tau} = \begin{cases} i\left[X\hat{\tau} - Y\left(\mathbb{R}^2\beta^{-2} - \hat{\tau}^2\right)^{-1/2}\right. & \text{for } \frac{Y}{\beta} \le \hat{\tau} \le \frac{\mathbb{R}}{\beta} \\ iX + Y\hat{\tau}\left(\hat{\tau}^2 - \mathbb{R}^2\beta^{-2}\right)^{-1/2} & \text{for } \hat{\tau} > \mathbb{R}/\beta \end{cases}$$

$$(2\text{-}17)\quad \mathbb{R}^4\omega^2 = \begin{cases} -\hat{\tau}^2X^2 - Y^2\mathbb{R}^2\beta^{-2} + Y^2\hat{\tau}^2 + 2\hat{\tau}XY\sqrt{\mathbb{R}^2\beta^{-2} - \hat{\tau}^2} & \text{for } \frac{Y}{\beta} \le \hat{\tau} \le \frac{\mathbb{R}}{\beta} \\ -\hat{\tau}^2X^2 - Y^2\mathbb{R}^2\beta^{-2} + Y^2\hat{\tau}^2 + i2\hat{\tau}XY\sqrt{\hat{\tau}^2 - \mathbb{R}^2\beta^{-2}} & \text{for } \hat{\tau} > \frac{\mathbb{R}}{\beta} \end{cases}$$

$$(2\text{-}18)\quad \mathbb{R}^8\omega^4 = \begin{cases} \hat{\tau}X^4 + Y^4\left(\mathbb{R}^2\beta^{-2} - \hat{\tau}^2\right)^2 + 6\hat{\tau}^2X^2Y^2\left(\mathbb{R}^2\beta^{-2} - \hat{\tau}^2\right) + \\ +(-4)\hat{\tau}XY\sqrt{\mathbb{R}^2\beta^{-2} - \hat{\tau}^2}\left[\hat{\tau}^2X^2 + Y^2\left(\mathbb{R}^2\beta^{-2} - \hat{\tau}^2\right)\right] & \text{for } \frac{Y}{\beta} \le \hat{\tau} \le \frac{\mathbb{R}}{\beta} \\ \hat{\tau}^4X^4 + Y^4\left(\hat{\tau}^2 - \mathbb{R}^2\beta^{-2}\right)^2 - 6\hat{\tau}^2Y^2X^2\left(\hat{\tau}^2 - \mathbb{R}^2\beta^{-2}\right) + \\ +i4\hat{\tau}XY\sqrt{\hat{\tau}^2 - \mathbb{R}^2\beta^{-2}}\left[-\hat{\tau}^2X^2 + Y^2\left(\hat{\tau}^2 - \mathbb{R}^2\beta^{-2}\right)\right] & \text{for } \hat{\tau} > \frac{\mathbb{R}}{\beta} \end{cases}$$

For the region along the branch cut, corresponding to the head wave, the following definitions are used

$$(2\text{-}19)\quad \left.\begin{aligned} \left[\omega^2+1\right]^{1/2} &= \sqrt{r_1 r_2}\, e^{i\pi/2} = i\sqrt{r_1 r_2} \\ \left[\omega^2+1\right]^{-1/2} &= \frac{1}{\sqrt{r_1 r_2}}\, e^{-i\pi/2} = -i\frac{1}{\sqrt{r_1 r_2}} \end{aligned}\right\} \text{for } \left(X + Y\sqrt{\beta^{-2}-1}\right) \le \hat{\tau} \le \mathbb{R}/\beta$$

The $\bar{\bar{\theta}}^{(i)}$ and $\bar{\bar{\chi}}^{(i)}$ equations can now be inverted back to the time domain in the manner of section A. However, this does not give terms of ϕ, ψ because the ϵ^3 and ϵ^4 factors multiplying $\bar{\bar{\phi}}^{(i)}$ and $\bar{\bar{\chi}}^{(i)}$ have not been considered.

There are two related approaches to the inversion of forms like $\bar{\bar{\theta}}^{(i)} s^{-i}$, $\bar{\bar{\chi}}^{(i)} s^{-i}$ where $\bar{\bar{\theta}}^{(i)}$, $\bar{\bar{\chi}}^{(i)}$ are known along with the inversion identity

$$(2\text{-}20)\quad \mathcal{L}^{-1}\left\{\epsilon^N\right\} = c^N \mathcal{L}^{-1}\left\{\frac{1}{s^N}\right\} = c^N t^{N-1}$$

One can either express the result in multiple integrals (as in (1, 2)), or one can use the Convolution Theorem.

$$\mathcal{L}^{-1}\left\{ \bar{\bar{F}}(s)\, \bar{\bar{G}}(s) \right\} = \int_0^t F(\tau) G(t-\tau)\, d\tau \tag{2-21}$$

Using (2-21), the corner layer terms of ϕ and ψ are as shown below

$$\mathcal{L}^{-1}\left\{\epsilon^3 \bar{\bar{\phi}}^{(3)}\right\} = \begin{cases} 0 & \text{for } ct<\sqrt{x^2+y^2} \\ \frac{2}{\pi} c^3 (1-2\beta^2) F(0) \int_{\mathbb{R}}^t \mathcal{R}e\left\{ \frac{1+2\beta^2\omega^2(p)}{1+\omega^2(p)} \frac{1}{R(\omega)} \frac{\partial\omega}{\partial\tau}(p) \right\} \cdot \\ \quad \cdot [t-p]^2 dp & \text{for } t > \mathbb{R} \end{cases} \tag{2-22}$$

$$\mathcal{L}^{-1}\left\{\epsilon^4 \bar{\bar{\phi}}^{(4)}\right\} = \begin{cases} 0 & \text{for } t < \mathbb{R} \\ \frac{8}{\pi} c^4 \beta^4 (1-\beta^2) F'(0) \int_{\mathbb{R}}^t \mathcal{R}e\left\{ \frac{\omega^2}{R(\omega)} \frac{\partial\omega/\partial\tau}{1+\omega^2} \left[\beta^{-2}+\omega^2\right]^{-1/2} \right\} \cdot \\ \quad \cdot [t-p]^3 dp + \frac{2}{\pi} c^4 (1-2\beta^2)\, G(0) \cdot \\ \quad \cdot \int_{\mathbb{R}}^t \mathcal{R}e\left\{ \frac{1+2\beta^2\omega^2}{R(\omega)} \frac{\partial\omega/\partial\tau}{1+\omega^2} \right\} [t-p]^3 dp & \text{for } t > \mathbb{R} \end{cases} \tag{2-23}$$

where the functions of p in (2-22) and (2-23) are equations (2-11) to (2-14) with τ replaced by p.

In the next two equations (the results for χ), equations (2-15) to (2-18) are used with $\hat{\tau}$ replaced by p, and $t_H \equiv \frac{1}{c}\left(x + y\sqrt{\beta^{-2}-1}\right)$.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\epsilon^3 \bar{\bar{\chi}}^{(3)}\right\} &= 0 && \text{for } t \le t_H \le \tfrac{1}{c\beta}\sqrt{x^2+y^2} \\ &= \tfrac{4}{\pi} c^3 \beta^2 (1-2\beta^2) F(0) \int_{t_H}^t \mathcal{A}_{3x}(p,t,x,y)\, dp && \text{for } t_H < t < \tfrac{\mathbb{R}}{\beta} \\ &= \tfrac{4}{\pi} c^3 \beta^3 (1-2\beta^2) F(0) \left[\int_{t_H}^{\mathbb{R}/\beta} \mathcal{A}_{3x}\, dp + \int_{\mathbb{R}/\beta}^t \mathcal{A}_{3x}\, dp \right] && \text{for } t > \mathbb{R}/\beta \end{aligned} \tag{2-24}$$

where

$$\mathcal{A}_{3x}(p,t,x,y) \equiv \mathcal{I}m\left\{ \omega R^{-1}(\omega) \frac{\partial\omega}{\partial\tau} \left[1+\omega^2\right]^{-1/2} \right\} [t-p]^2$$

(2-25) $$\mathcal{L}^{-1}\left\{\epsilon^4 \bar{\bar{\chi}}^{(4)}\right\} = 0 \qquad \text{for} \quad t < t_H < \frac{\mathbb{R}}{\beta}$$

$$= \frac{4}{\pi} c^4 \beta^2 (1-2\beta^2)\, G(0) \int_{t_H}^{t} \mathcal{A}_{3\chi}\, [t-p]\, dp +$$

$$+ \frac{4}{\pi} c^4 (1-\beta^2)\, F'(0) \int_{t_H}^{t} \mathcal{A}_{4\chi}\, dp \qquad \text{for} \quad t_H < t < \frac{\mathbb{R}}{\beta}$$

$$= \frac{4}{\pi} c^4 \beta^2 (1-2\beta^2) G(0) \left[\int_{t_H}^{\mathbb{R}/\beta} \mathcal{A}_{3\chi} [t-p]\, dp + \int_{\mathbb{R}/\beta}^{t} \mathcal{A}_{3\chi}\, [t-p]\, dp \right] +$$

$$+ \frac{4}{\pi} c^4 (1-\beta^2) F'(0) \left[\int_{t_H}^{\mathbb{R}/\beta} \mathcal{A}_{4\chi}\, dp + \int_{\mathbb{R}/\beta}^{t} \mathcal{A}_{4\chi}\, dp \right] \qquad \text{for } t > \frac{\mathbb{R}}{\beta}$$

where

$$\mathcal{A}_{4\chi}(p,t,x,y) \equiv \mathrm{Im}\left\{ \frac{1+2\beta^2\omega^2}{R(\omega)}\, \frac{\partial\omega/\partial\tau}{1+\omega^2}\, \frac{\omega}{\beta^{-2}+\omega^2} \right\} [t-p]^3$$

Note that, for $t > \mathbb{R}/\beta$, the integrals in (2-24) and (2-25) must be split up at $\mathbb{R}/\beta$ because $[1+\omega^2]^{1/2}$ is defined as in (2-19).

C. Solutions in the Time Domain

If we combine the results for the time domain up to order $(t-\frac{x}{c})^2$ we get

(3-1) $$\phi(x,y;t) \approx -c^3 F(y)\, \mathcal{H}(t-\tfrac{x}{c})\, (t-\tfrac{x}{c})^2 + \frac{2}{\pi} c^3 (1-2\beta^2) F(0)\, \cdot$$

$$\cdot\, \mathcal{H}\left(t - \tfrac{1}{c}\sqrt{x^2+y^2}\right) \int_{\frac{1}{c}\sqrt{x^2+y^2}}^{t} \mathrm{Re}\left\{ \frac{1+2\beta^2\omega^2}{1+\omega^2}\, \frac{1}{R(\omega)}\, \frac{\partial\omega}{\partial\tau} \right\} [t-p]^2\, dp +$$

$$+ \frac{2}{\pi} c^3 (1-2\beta^2) F(1)\, \mathcal{H}\left(t - \tfrac{1}{c}\sqrt{x^2+(1-y)^2}\right) \cdot$$

$$\cdot \int_{\frac{1}{c}\sqrt{x^2+(1-y)^2}}^{t} \mathrm{Re}\left\{ \frac{1+2\beta^2\omega^2}{1+\omega^2}\, \frac{1}{R(\omega)}\, \frac{\partial\omega}{\partial\tau} \right\} \Bigg|_{y=\underline{y}} [t-p]^2\, dp$$

$$(3\text{-}2) \quad \Psi(x,y;t) \approx \left[\mathcal{H}(t-t_H)-\mathcal{H}\left(t-\tfrac{1}{\beta c}\sqrt{x^2+y^2}\right)\right]\tfrac{4}{\pi}c^3\beta^2(1-2\beta^2)F(0)\cdot$$

$$\cdot\int_{t_H}^{t}\mathcal{A}_{3x}(p,t,x,y)\,dp+\mathcal{H}\left(t-\tfrac{1}{\beta c}\sqrt{x^2+y^2}\right)\tfrac{4}{\pi}c^3\beta^2(1-2\beta^2)F(0)\cdot$$

$$\cdot\left[\int_{t_H}^{\frac{1}{\beta c}\sqrt{x^2+y^2}}\mathcal{A}_{3x}(p,t,x,y)\,dp+\int_{\frac{1}{\beta c}\sqrt{x^2+y^2}}^{t}\mathcal{A}_{3x}(p,t,x,y)\,dp\right]+$$

$$+\left[\mathcal{H}(t-\hat{t}_H)-\mathcal{H}\left(t-\tfrac{1}{\beta c}\sqrt{x^2+(1-y)^2}\right)\right]\tfrac{4}{\pi}c^3\beta^2(1-2\beta^2)F(1)\cdot$$

$$\cdot\int_{\hat{t}_H}^{t}\mathcal{A}_{3x}(p,t,x,1-y)\,dp+\mathcal{H}\left(t-\tfrac{1}{\beta c}\sqrt{x^2+(1-y)^2}\right)\tfrac{4}{\pi}c^3\beta^2(1-2\beta^2)F(1)\cdot$$

$$\cdot\left[\int_{\hat{t}_H}^{\frac{1}{\beta c}\sqrt{x^2+(1-y)^2}}\mathcal{A}_{3x}(p,t,x,1-y)\,dp+\int_{\frac{1}{\beta c}\sqrt{x^2+(1-y)^2}}^{t}\mathcal{A}_{3x}(p,t,x,1-y)\,dp\right]$$

Stresses and displacements can be approximated to the same order in $\left(t-\frac{x}{c}\right)$ by using the above equations in (II-2-8) and (II-2-9). The arrival times (defined by the step functions, $\mathcal{H}$, in (3-1) and (3-2)) do not change for stress and displacement since all derivatives are taken with respect to space variables.

V. DISCUSSION

A. Description of Wave Fronts

We can now relate the asymptotic expansions of part IV to the physical problem of figure 1a. The task of locating the various wave fronts is indeed easy because of the step functions that appear in the equations (IV-3-1) and (IV-3-2). If we could take a picture of one corner of the strip at some fixed time after $t = 0$, the wave fronts shown on figure 5 would be seen.

Let us first consider the two plane waves which are illustrated by dashed lines on figure 5. The approximate solutions for each of these fronts came from the analysis of the boundary layer near $x = 0$ in the Laplace transform domain. From that analysis (see part III), the series expansion in $\bar{\bar{\Phi}}^{(i)}(\xi, y)$ results in the asymptotic expansion for the dilatational plane wave which propagates along the strip with speed c_d. Correspondingly, the series expansion in $\bar{\bar{\Psi}}^{(i)}(\xi, y)$ eventually makes up the approximation for the plane shear wave with propagation speed c_s. For each expansion arising out of the boundary layer in the transformed domain, the exponential decay in the ξ direction not only gives the proper boundary layer behavior, but also leads to the step functions in the time domain.

In the problem being considered, the magnitude of the plane shear wave is smaller in terms of $\left(t - \frac{x}{c_s}\right)$ than the magnitude of the plane dilatational wave. This is due to the fact that the primary effect of the input at $x = 0$ is dilatational. In the conjugate

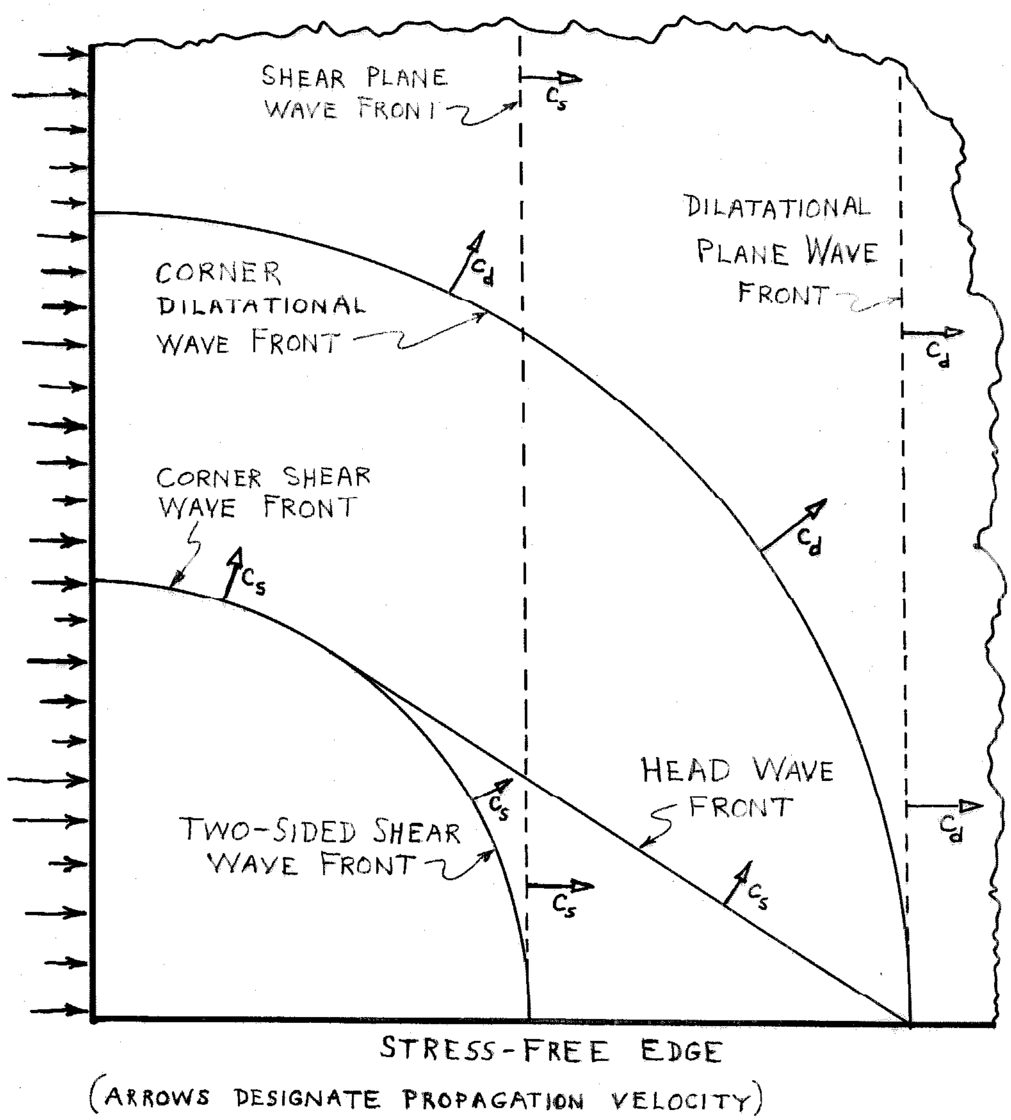


FIG. 5 BASIC WAVE FRONTS NEAR THE CORNER OF THE END-LOADED STRIP.

problem* in which the end loading would involve a prescribed τ_{xy} and $v = 0$, the relative orders of magnitude of the plane waves would be reversed.

Another interesting point is that all of the wave fronts, except for the two plane waves, arise from the corners, as pointed out in (1, 2). Thus any end loading with similar displacements near the corners will have the same asymptotic approximation for the wave fronts arising from the corner layer analysis.

The above characteristic of the method also leads to difficulties when large local displacements of $f(y, t)$ occur away from the corners. With our method, these peaks can only influence the two plane wave fronts shown as dashed lines in figure 5; and the method of this paper keeps the magnitude of the expansion terms (for these two wave fronts) depending only on $f(y, t)$ and its derivatives. However, in the limiting example of a very localized end displacement, the wave front amplitude would diffuse spatially outward from the application point in a more radial manner. Thus our method is definitely restrictive on the allowable smoothness of the displacement function $f(y, t)$ away from the corners. And, of course, derivatives of $f(y, t)$ must exist up to the order of approximation one is calculating in the boundary and corner layers.

The validity of the approximate solution we obtained in part IV also does not hold if we go far enough from the edge (or, equivalently, consider times too long after $t = 0$). This can be seen

* This problem can also be handled be our method. See section D.

from the order ϵ^5 solution for $\overline{\overline{\Phi}}^{(2)}(\xi, y)$ (equation (III-3-20)) which involves a linear function in ξ . Thus if ξ is order $\frac{1}{\epsilon}$ (i.e., if $x = \mathcal{O}(1)$), then the ξ term in $\overline{\overline{\Phi}}^{(5)}(\xi, y)$ should be considered in order ϵ^4 . One can then say that the three-term asymptotic solution for the plane dilatational wave (equation (II-3-21)) is certainly not uniformly valid for all $x \geqslant 0$.

The wave fronts shown in solid lines in figure 5 arise from the corner, and they are calculated through the analysis of the corner layer. For the dilatational corner wave we note that, after $t = \frac{1/\epsilon}{c}$, there are regions of the strip where waves from both corners overlap. The approximation for a wave front in this region requires the superposition of effects from both corners. Point B on figure 6 would represent such a region. For later times, there is also a region (point D in figure 6) where one of the corner shear waves would superimpose on a dilatational wave from the opposite corner.

This superposition process continues only up to the time that the wave fronts from one corner have interacted with the opposite corner, and then have reached the points like F, G of figure 6. At this time, our solution approximation is no longer valid since multiple reflections are involved. These are discussed further in section C.

After the corner dilatational wave, the remaining corner wave fronts involve only Ψ. This fact makes the nomenclature for the wave fronts in figure 5 seem more appropriate since all shear effects arise only from Ψ.

Mathematically, the head wave is the contribution from the segment of the Cagniard path along the imaginary ω axis along the branch cut from $\omega = \hat{i}$ (see figure 3b, p. 37). Physically this wave is just the shear effect of the plane dilatational wave interacting with the stress-free boundary.

The final shear front has two parts due to the head wave interaction over part of the circular arc. Mathematically, this comes from the Cagniard path away from the imaginary axis in figure 3b. Physically it is the effect of the corner load at $t = 0$.

B. Comparison to Previous Methods

Rosenfeld and Miklowitz[1, 2] analyzed the strip impact problem where the end is pushed with a constant velocity V_0 from time zero. In our analysis, this means that the given end displacement is

(2-1) $$f(y,t) = u(0,y;t) = V_0\, t\, \mathcal{H}(t)$$

where V_0 is the constant velocity of impact. When $f(y,t)$ is approximated by a series in t for small t, there is only one term; thus for equation (III-1-1) we would obtain

(2-2) $$F(y) = \frac{V_0}{c^2}\,, \quad G = H = \cdots = 0$$

This problem has no plane shear wave since the boundary layer analysis at $x = 0$ will involve only $\bar{\bar{\phi}}^{(3)}(\xi,y)$. The inversion of $\bar{\bar{\phi}}^{(3)}(\xi,y)$ and the corner layer terms $\bar{\bar{\theta}}^{(i)}$, $\bar{\bar{\bar{\theta}}}^{(i)}$, $\bar{\bar{\chi}}^{(i)}$, $\bar{\bar{\bar{\chi}}}^{(i)}$ leads to exact agreement with the so-called "exact integral solution" of Rosenfeld and Miklowitz when specialized to the case of no multiple reflections.*

C. Multiple Reflections

As mentioned in section A, our solution approximation was valid (at some point) only until a corner wave front had interacted with the opposite corner and then returned to the selected point. Thus the multiple reflections introduce additional wave fronts which have not been considered up to this point in the paper.

Rosenfeld and Miklowitz treat many more of these additional wave fronts mathematically, and, using their methods, they obtained approximate behavior at these fronts. The method of this paper can be applied in a step-by-step fashion to calculate an asymptotic expansion valid at, and immediately behind, these additional wave fronts. Every additional wave front arises out of wave interactions in the corners, thus the additional fronts will arise solely from the corner layer approximation terms.

Consider the first of the additional waves. This front arises when the wave front positions are as shown in figure 7. In this figure, note that only waves from the corner $x = 0$, $y = 0$ are shown since the situation for the other corner is similar. The boundary conditions at $y = 1$, $x = 0$ are the same as before, but the wave potentials are modified from those considered in parts III and IV.

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* Specifically, see equations (111) to (113) of reference (1) with the m and n subscripts set to zero.

POINTS IN FIG. 6	SOLUTION (NON-ZERO WAVEFRONT TERMS)
A	$\Theta^{(i)}, \phi^{(i)}$
B	$\theta^{(i)}, \Theta^{(i)}$
C	$\theta^{(i)}, \chi^{(i)}$ (headwave)
D	$\theta^{(i)}, \psi^{(i)}, \mathfrak{X}^{(i)}$
E	$\chi^{(i)}, \mathfrak{X}^{(i)}$
F, G	MULTIPLE REFLECTION ANALYSIS NEEDED

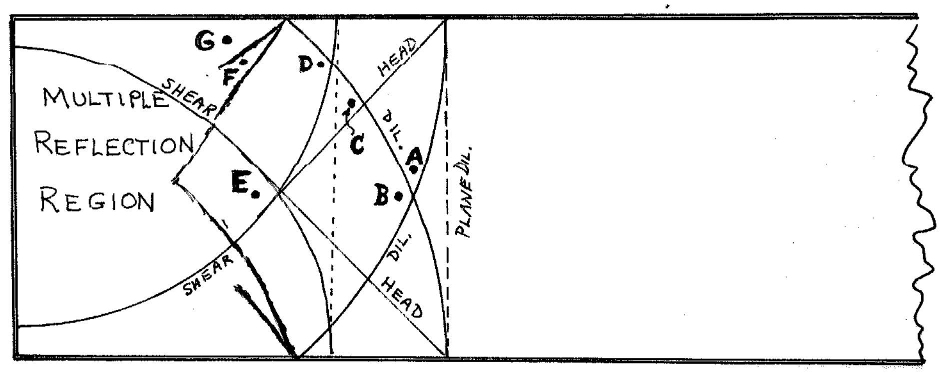

FIG. 6 WAVE FRONTS AT FIXED TIME

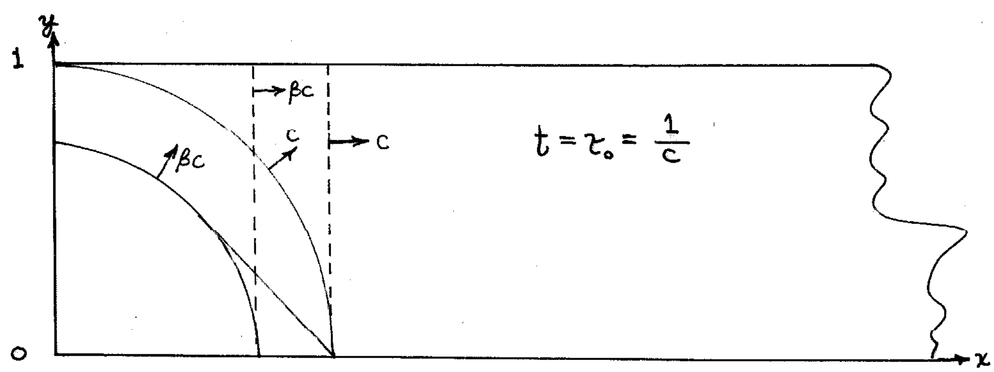

FIG. 7 START OF MULTIPLY-REFLECTED WAVE FRONT

In the corner layer at $y = 1$, $x = 0$, and at $t = \tau_o$, let the $\bar{\bar{\theta}}^{(i)}(\xi, y)$ of (III-5-2) be composed of two parts. The first is the potential of a new wavefront; and the second part is due to the $\bar{\bar{\theta}}^{(i)}(\xi, y)$ expansion for $\tau > \sqrt{x^2 + y^2}$ in equations (III-4-28) and (III-4-31). Since there is, at that time, no shear potential effect from the corner $x = 0$, $y = 0$, the only shear at $x = 0$, $y = 1$ is due to the plane wave. This enables one to define the new corresponding shear potential.

$$(3\text{-}1) \qquad {}_1\bar{\bar{\theta}}^{(i)} \equiv \bar{\bar{\theta}}^{(i)} + \left.\bar{\bar{\Theta}}^{(i)}\right|_{\substack{t = \tau_o + \\ \sqrt{x^2+y^2} = 1+}}$$

$$(3\text{-}2) \qquad {}_1\bar{\bar{\chi}}^{(i)} \equiv \bar{\bar{\chi}}^{(i)} + \left.\bar{\bar{\Upsilon}}^{(i)}\right|_{\substack{t = \tau_o + \\ \sqrt{x^2+y^2} = 1+}}$$

Since the last terms of the above definitions are known, the effect of these terms is to add additional constants to each order ϵ^N problem. Analytically, this does not mean more than increasingly difficult bookkeeping.

The above process is now repeated each time a new wave front reaches an opposite corner. Due to the geometrical symmetry of the problem and the wave front locations, the new wave front expansion in one corner easily converts to the new expansion in the other corner by using the same process as in section E of part III.

There is another approach which can be followed if one is interested in only the first of the multiple reflections. For this approach one considers the effect of the opposite corner solution when solving the corner layer problem in the Laplace transformed domain. Even though the effect on the previous corner layer expansions will be very small due to the $\left[e^{-f(\omega)\eta}\right]$ factor, this exponential only introduces a time delay when the solutions are finally brought back to the time domain.

D. Other Problems

There are several other stress wave propagation problems which clearly can be handled by the method of this paper. Consider the elastic quadrant with one edge stress-free and the other edge having "mixed" boundary conditions. If we just delete the effects from the upper corner in the semi-infinite strip problem of parts II, III, and IV, we obtain an asymptotic expansion for the wave fronts of the quadrant. In this case, there are no additional wave fronts from multiple reflections.

Figure 8a shows an additional semi-infinite strip problem that can be easily handled because the boundary conditions are "mixed" at x = 0. The complete boundary conditions are

(4-1) $\quad y = 0, 1: \qquad \sigma_y = \tau_{xy} = 0$

(4-2) $\quad x = 0: \qquad \sigma_x = F(y,t), \quad v = 0$

where F(y, t) is a given function.*

For the line load on an infinite strip (figure 8b) and the point load on an infinite plate (figure 8d), the use of our method leads to an

* Another physically less interesting set of end conditions (also of the "mixed" type) which can be handled by the method are

at x = 0: $\quad u = 0, \quad \tau_{xy} = F(y,t)$

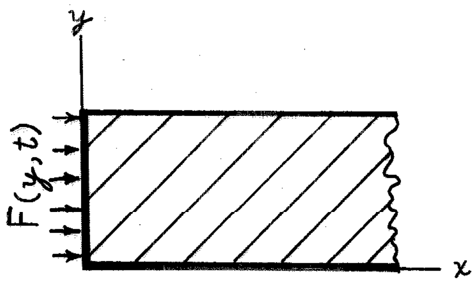

FIG. 8a STEP PRESSURE
(NO z VARIATION)

$\underline{x=0}$: $v = 0$
$\sigma_x = \mathcal{H}(t)[-F(y,t)]$

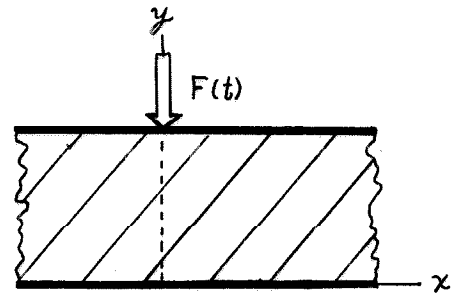

FIG. 8b LINE FORCE ON
INFINITE STRIP
(NO z VARIATION)

$\underline{x=0}$: $u = 0$
$\tau_{xy} = 0$

$\underline{y=1}$: $\sigma_y = \mathcal{H}(t)\,\delta(x)[-F(t)]$
$\tau_{xy} = 0$

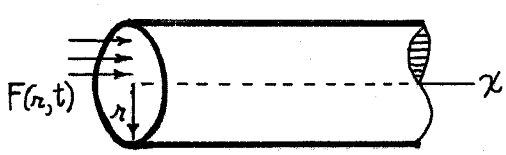

FIG. 8c END-LOADED
BAR PROBLEMS
(AXIAL SYMMETRY)

(i) KNOWN AXIALLY-SYMMETRIC VELOCITY

$\underline{x=0}$: $\tau_{xr} = 0$
$u = \mathcal{H}(t)\,t\,F(r,t)$

(ii) KNOWN AXIALLY-SYMMETRIC PRESSURE

$\underline{x=0}$: $v = 0$
$\sigma_x = \mathcal{H}(t)[-F(r,t)]$

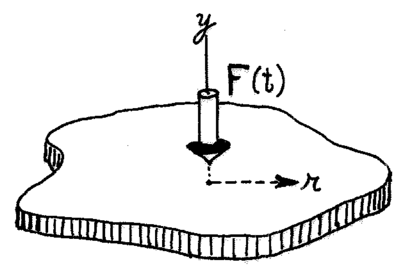

FIG. 8d NORMAL POINT
FORCE
(RADIAL SYMMETRY)

$\underline{y=0}$: $\sigma_y = 0$
$\tau_{ry} = 0 = \tau_{r\theta}$

$\underline{y=1}$: $\tau_{ry} = \tau_{r\theta} = 0$
$\sigma_y = \mathcal{H}(t)\,\frac{\delta(r)}{r}[-F(t)]$

analysis similar to the semi-infinite, end loaded strip. The resulting expansions are expected to give a valid asymptotic behavior for the wave fronts.

With the problems of figures 8c, 8d, we are using cylindrical symmetry and "mixed" boundary conditions so that the method of analysis is similar after the field equations are modified to the new geometry.* The transforms of Table 1, p. 56 are again used.

Reference (1) gives further discussion of the problems in figure 8, and the approximate results are also enumerated. The similarity in these problems is most noticeable in the form of the kernel, $G(\omega)$, in the inversion equation for the solution in the Laplace transform domain (see part IV). The Cagniard paths are also the same ones displayed in figure 3, p. 37.

Since the key to the use of our method is solving a boundary layer and corner layer problem, the method is expected to apply to strips having an end geometry as in our problem, but being curved away from the edge where the external load is applied. Such a curved geometry will, of course, require modified field equations and boundary conditions on the stress free surfaces.

For end loads on the semi-infinite strip corresponding to "non-mixed" end conditions (e.g., σ_x and τ_{xy} prescribed at $x = 0$; or, u, v prescribed at $x = 0$), the present procedure could be successfully applied if the relevant corner (or quadrant) problem with "non-mixed" boundary conditions could be solved. A solution

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* Note that Hankel transforms should be used for 8d.

to such a "non-mixed" problem for a quadrant has not yet been reported in the literature.

Finally, it should be again noted that the method of this paper is not restricted to only stress wave problems in two space variables. The method should prove useful for obtaining wave front asymptotic solutions to many other problems involving hyperbolic partial differential equations.

TABLE 1

Transform Methods for Mixed Boundary Conditions in Stress Wave Propagation Problems

	Method 1	Method 2
Laplace & cosine transformed equations	(1), (4)	(2), (3), (5)
Laplace & sine transformed equations	(2), (3), (5)	(1), (4)
Laplace & cosine transformed variables	u, τ_{xy}, Ψ	$v, \sigma_y, \sigma_x, \phi$
Laplace & sine transformed variables	$v, \sigma_y, \sigma_x, \phi$	u, τ_{xy}, Ψ
Known Boundary Conditions at x = 0	$\sigma_x, v, \phi, \Psi_x$	$u, \tau_{xy}, \Psi, \phi_x$

For x, y coordinates, use the following equations for (1) through (5) (see p. 57 for equations in two other coordinate geometries).

FIELD EQUATIONS

(1) $$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2}$$

(2) $$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = \rho \frac{\partial^2 v}{\partial t^2}$$

STRESS-STRAIN

(3) $$\sigma_x = \lambda \frac{\partial v}{\partial y} + (\lambda + 2\mu) \frac{\partial u}{\partial x}$$

(4) $$\sigma_y = \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu) \frac{\partial v}{\partial y}$$

(5) $$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

Further equations for Table 1 are given below in order to use the transform method on the problems of figures 8c and 8d.

AXIAL SYMMETRY (FIG. 8c)

(1′) $\Phi_{xx} + \frac{1}{r}(r\Phi_r)_r = \frac{1}{c^2}\Phi_{tt}$

(2′) $\Psi_{xx} + \left(\frac{1}{r}(r\Psi)_r\right)_r = \frac{1}{\beta^2 c^2}\Psi_{tt}$

(3′) $\sigma_x = \frac{\lambda}{r}(r\Phi_r)_r - \frac{2\mu}{r}(r\Psi_x)_r + (\lambda+2\mu)\Phi_{xx}$

(4′) $\sigma_r = \lambda\Phi_{xx} + 2\mu\Psi_{rx} + \frac{\lambda}{r}(r\Phi_r)_r + 2\mu\Phi_{rr}$

(5′) $\tau_{xy} = \mu\left\{2\Phi_{rx} + \Psi_{xx} - \left[\frac{1}{r}(r\Psi)_r\right]_r\right\}$

RADIAL SYMMETRY (FIG. 8d)

(1″) $\frac{1}{r}(r\Phi_r)_r + \Phi_{yy} = \frac{1}{c^2}\Phi_{tt}$

(2″) $\left[\frac{1}{r}(r\Psi)_r\right]_r + \Psi_{yy} = \frac{1}{\beta^2 c^2}\Psi_{tt}$

(3″) $\sigma_r = \lambda\Phi_{yy} - 2\mu\Psi_{yr} + \frac{\lambda}{r}\frac{\partial}{\partial r}(r\Phi_r) + 2\mu\Phi_{rr}$

(4″) $\sigma_y = \frac{\lambda}{r}(r\Phi_r)_r + \frac{2\mu}{r}(r\Psi_y)_r + (\lambda+2\mu)\Phi_{yy}$

(5″) $\tau_{yr} = \mu\left\{2\Phi_{yr} + \left[\frac{1}{r}(r\Psi)_r\right]_r - \Psi_{yy}\right\}$

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