" Some Contributions to the Theory

of the

Stiffened Suspension Bridge".

Thesis

by

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SUMMARY

PART I OF THIS THESIS CONTAINS A REVIEW OF THE BASIC THEORY OF THE CABLE AND STIFFENING GIRDER COMBINATION ACCORDING TO THE METHOD OF DR. RODE WHICH TAKES INTO ACCOUNT THE INEVITABLE LONGITUDINAL DISPLACEMENTS OF THE SUSPENDER CONNECTIONS, WHEN THE CABLE IS DEFLECTED FROM ITS NORMAL CONFIGURATION.

NUMERICAL CALCULATIONS HAVE BEEN CARRIED OUT TO ESTABLISH THE ORDER OF MAGNITUDE OF APPROXIMATIONS IN THE ORDINARY DEFLECTION THEORY AS DEVELOPED IN THIS COUNTRY BY F. E. TURNEARE FROM THE WORK OF J. MELAN. BESIDES THE ESSENTIAL OMISSION WHICH IS EVIDENT IN THE COMPARISON OF THE FUNDAMENTAL DIFFERENTIAL EQUATION OF RODE'S THEORY WITH THAT OF THE ORDINARY DEFLECTION THEORY, THE MOST SIGNIFICANT ERROR IS THE NEGLECT OF THE EFFECT OF INCLINATION OF THE SUSPENDERS ON CABLE DEFLECTION. IT WOULD APPEAR THAT AN ERROR OF THREE PERCENT CAN ARISE FROM THIS SOURCE WHICH ALTHOUGH UNIMPORTANT FROM THE STRUCTURAL ENGINEER'S POINT OF VIEW, IS IMPORTANT AS A LIMIT ON ACCURACY REQUIRED OF ANY THEORY WHICH NEGLECTS IT.

RODE'S DIFFERENTIAL EQUATION IS NOT INTEGRABLE AND AN APPROXIMATION HAS BEEN DEVELOPED WHICH IS TRACTABLE BY THE METHODS OF PART II AND YET DOES NOT SACRIFICE ENTIRELY THE IMPROVED REPRESENTATION OF STRUCTURAL ACTION GIVEN BY RODE'S THEORY.

PART II UTILISES TRIGONOMETRIC SERIES FOR THE DEVELOPMENT OF A SIMPLE METHOD OF DETERMINATION OF THE DEFLECTIONS AND GIRDER BENDING MOMENTS IN A SPAN. AS APPLIED TO THE ORDINARY DEFLECTION THEORY THERE RESULT FORMULAE OF EXTREME SIMPLICITY WHICH BRING INTO SHARP PERSPECTIVE THE RELATED FUNCTIONS OF CABLE AND STIFFENING GIRDER. FROM THESE FORMULAE A PICTORIAL REPRESENTATION OF BRIDGE STIFFNESS IN TERMS OF CERTAIN BASIC PARAMETERS HAS BEEN DEVELOPED.

THE THEORY AND METHOD OF COMPUTATION HAVE THEN BEEN EXTENDED TO TAKE INTO ACCOUNT THE EFFECT OF A PRESTRESS INTRODUCED BY ARBITRARY ADJUSTMENT OF SUSPENDER LENGTHS WHICH MAY BE NECESSARY IN THE REHABILITATION OF OLD STRUCTURES OR REDUCTION OF PEAK BENDING STRESSES IN NEW DESIGNS.

THE APPROXIMATION TO RODE'S THEORY DEVELOPED IN PART I IS THEN SOLVED IN TERMS OF THE METHODS DEVELOPED FOR THE SIMPLE THEORY AND FORMULAE RESULT WHICH ARE BUT SLIGHTLY MORE COMPLICATED THAN THE ELEGANT FORMULAE OF THE PREVIOUS WORK. THE PRACTICABILITY OF APPLICATION HAS BEEN TESTED BY EXAMPLES.

FINALLY, IN ORDER TO DISPEL CERTAIN ERRONEOUS CONCEPTIONS OF THE EFFICACY OF THE STIFFENING GIRDER IN CONTROLLING THE BENDING MOMENTS, (AND CONSEQUENTLY, THE DEFLECTIONS) IN A SPAN, THE METHOD HAS BEEN EXTENDED TO PERMIT THE INVESTIGATION OF THE INFLUENCE OF VARIATION IN FLEXURAL RIGIDITY OF THE GIRDER. THIS LEADS TO FORMULAE, NOT OVERLY LENGTHY, WHICH SHOW IN PROPER RELATION THE INFLUENCE OF THE VARIOUS HARMONIC COMPONENTS OF THE FLEXURAL RIGIDITY OF THE GIRDER.

THE METHODS OF THIS THESIS ARE SIMPLE IN CONCEPTION AND APPLICATION WHILE STILL RETAINING A PROPER PHYSICAL BASIS, AND ARE CAPABLE OF EXTENSION BEYOND THE BOUNDS OF CURRENT METHODS WITHOUT LOSS OF ALGEBRAICAL AND ARITHMETICAL TRACTABILITY.

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PART 1.

A Critical Review of the Basic Theory of Cable and Stiffening Girder Systems.

1.01. The Assumptions of the "Deflection Theory".

The usual assumptions introduced in the analysis of two-hinged- truss suspension bridges are as follows:

1. The initial cable curve is a parabola.
2. Initial dead load, w, is carried by the cable with no bending stresses in the truss.
3. The stiffening truss has constant moment of inertia(E.I.), although this is not essential to the application of the "Deflection Theory".
4. There is a continuous sheet of vertical hangers connecting the cable to the stiffening girder.
5. The hangers are vertical in all deflected configurations of the structure i.e. inclination of hangers may be neglected under live loads.
6. Axial elastic deformations of towers and hangers are neglected. Apropos of this L.Moissieff (reference 1,p.1207) says: "Deflection Theory assumes that the effect of elongation of the suspenders is negligible, and that its effect on the cable pull is insignificant. This assumption has been justified by computations in several instances as well as by observations on models.

   The effect of suspender elongation cannot however be neglected locally. Near the towers, where the suspenders are longest the effect of lengthening under live load cannot be neglected.

   Moments and shears will increase up to 25% towards the ends".

1.02. For a critical evaluation of the assumptions involved in the theory of cable and stiffening girder systems, it is convenient to begin with a study of the geometrical relations between the positions and dimensions of a given element of the cable in the initial and deflected configurations.

In Fig.1.01 an element of cable length defined by cd subtends a length ab when undeflected. After deflection the points c and d move to new positions c' and d' and the length c'd' is in general not equal to cd.

The vertical deflection of c is denoted by η and the lateral shift of the same point is ξ . The suspenders which connect cd to ef in the initial configuration, are attached to c'd' in the deflected position since c' and d' are the new positions of c and d. The deflection of the girder at e is denoted by ν and is equal to the sum of η and changes due to the suspenders. Structural details decide whether g shall be vertically below e and are discussed later. The slope of the undeflected cable, ψ , changes to $\psi + \delta\psi$ after deflection.
Lower order quantities are defined in Fig. 1.01.

The relation between ν and η may be written

$$\nu = \eta + h.\omega t + h.\text{versin}\ \frac{\xi}{h} + \frac{hq''}{A_h E_h} \qquad (1)$$

where ω = temperature coefficient.

t = temperature change contributing to the change in configuration.

h = length of suspender.

A_h = area of cross section of suspender.

E_h = Young's Modulus of suspender.

q'' = increase of suspender force due to added live load.

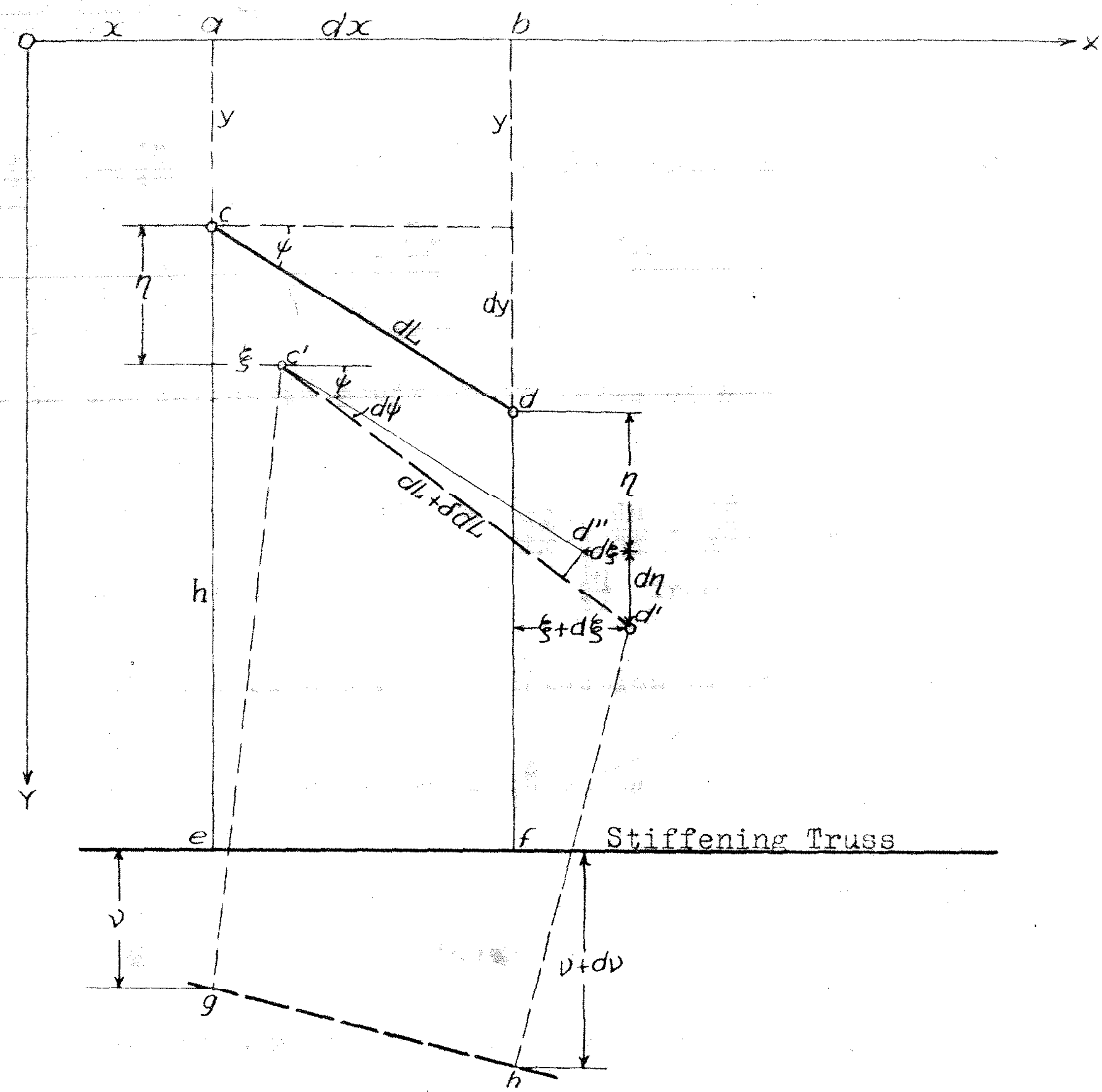


Fig. 1.01

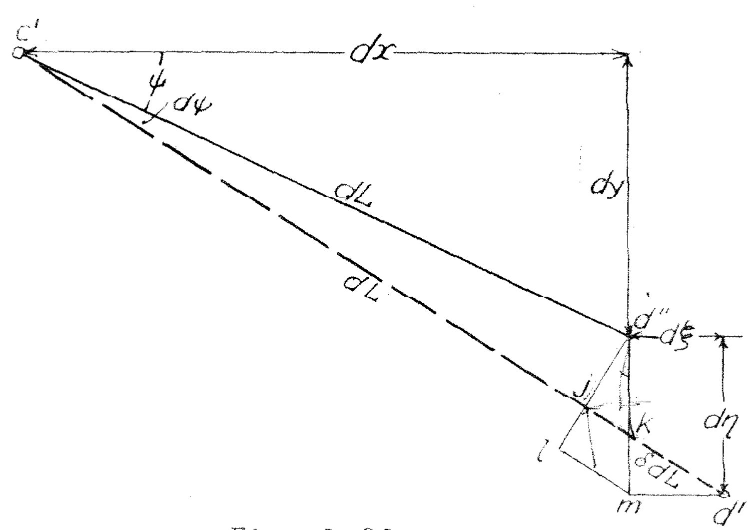


Fig. 1.02

From (2) and (7) $\frac{d\xi}{dx}$ can be expressed in terms of $\frac{d\eta}{dx}$ and vice versa. Thus

$$\frac{d\xi}{dx} = \frac{-\frac{d\eta}{dx}\tan\psi + \frac{d\eta}{dx}\frac{H_w}{EA}\tan\psi\sec\psi + \frac{H}{EA}\sec^3(\psi+\delta\psi) + \omega t\sec^2\psi}{1 + \frac{H_w}{EA}\sec\psi\tan^2\psi} \qquad (8).$$

It will be of interest to express the slope of the element of the cable in the deflected position in terms of η .

In Fig.1.02,

$$\tan(\psi + \delta\psi) = \frac{dy + d''k}{dx} = \frac{dy}{dx} + \frac{d\eta}{dx} - \frac{d\xi}{dx}\tan\psi \qquad (9).$$

and $\frac{d\xi}{dx}$ can be expressed in terms of $\frac{d\eta}{dx}$ from (8).

From (9) the rate of change of deflection of the cable is

$$\tan(\psi + \delta\psi) - \tan\psi = \frac{d\eta}{dx} - \frac{d\xi}{dx}\tan\psi \qquad (10).$$

1.03. The Differential Equation of the Stiffened Suspension Bridge.

In what follows, the deflections of the stiffening girder are of prime importance, and the differential equation will be expressed in terms of ν , the deflection of the girder at x.

By differentiation of (1) with respect to x and transposing terms

$$\frac{d\eta}{dx} = \frac{d\nu}{dx} - \omega t\frac{dh}{dx} - \frac{d}{dx}\left(h \operatorname{vers}\frac{\xi}{h}\right) - \frac{d}{dx}\left(\frac{hq''}{A_h E_h}\right) \qquad (11).$$

Using (11) and (8) in (10) the rate of change of cable deflection can be expressed in terms of the rate of change of girder deflection.

In Fig.1.01 c'd" is drawn parallel and equal to cd and the section including cd"d' is redrawn to larger scale in Fig.1.02. d"j is drawn perpendicular to c'd' and since $d\psi$ is small the angle d" jk is practically a right angle and c' j is equal to dL.

$$\delta dL = jd' = d''m \sin\psi + d'm \cos\psi$$

$$\delta dL = d\eta \sin\psi + d\xi \cos\psi \qquad (2).$$

Furthermore, if during the change in configuration, a change in stress, $d\sigma$, and a change in temperature, t, occur then

$$\delta dL = \left(\frac{d\sigma}{E} + wt\right) dL \qquad (3).$$

In the initial condition the stress of the cable, σ , is given by

$$\sigma = \frac{H_w}{A} \operatorname{Sec}\psi$$

In the final condition the stress has become

$$\sigma + \delta\sigma = \frac{H_w + H}{A} \sec(\psi + \delta\psi)$$

$$\delta\sigma = \frac{H_w + H}{A} \sec(\psi + \delta\psi) - \frac{H_w}{A} \sec\psi \qquad (4).$$

From Fig.1.02,

$$\sec(\psi + \delta\psi) = \sec\psi + \frac{d\eta - d\xi \tan\psi}{dx} \sin\psi$$

$$= \sec\psi + \left(\frac{d\eta}{dx} - \frac{d\xi}{dx} \tan\psi\right)\sin\psi \qquad (5).$$

Substituting (5) in (4) we obtain

$$\delta\sigma = \frac{H_w}{A} \left(\frac{d\eta}{dx} - \frac{d\xi}{dx} \tan\psi\right)\sin\psi + \frac{H}{A} \sec(\psi + \delta\psi) \qquad (6).$$

and substituting (6) in (3) there results

$$\delta dL = \frac{dL}{EA} \left[H_w \left(\frac{d\eta}{dx} - \frac{d\xi}{dx} \tan\psi\right) \sin\psi + H\sec(\psi + \delta\psi) \right] + \omega t.dL \qquad (7).$$

The load per unit length supported by the element of cable which originally subtended the length dx, is equal to the product of the horizontal tension H_w+H, and the rate of change of slope of the cable curve as given by (9), and is given by

$$-(H_w + H) \frac{d}{dx} \left(\frac{dy}{dx} + \frac{d\eta}{dx} - \frac{d\xi}{dx} \tan \psi \right) = p' \qquad (12).$$

The loads per unit length supported by the girder on the length dx is given by the simple theory of bending as

$$\frac{d^2}{dx^2} \left(EI \frac{d^2\nu}{dx^2} \right) = p'' \qquad (13).$$

Therefore, the total load per unit length supported by the combination is given by

$$p' + p'' = \frac{d^2}{dx^2} \left(EI \frac{d^2\nu}{dx^2} \right) - (H_w + H)\frac{d}{dx} \left(\frac{dy}{dx} + \frac{d\eta}{dx} - \frac{d\xi}{dx} \tan\psi\right) \quad (14).$$

Equation (14) combined with (11) and (8) and (1), constitute the equation of the stiffened suspension bridge.

Equation (14) contains the coefficient (H_w + H) which must be determined from the condition of continuity of the cable and its supports. For the present it will be assumed that the cable is supported at two points whose distance apart is fixed for all loading conditions.

Then, the integration of equation (8) is to be executed with the boundary condition that ξ vanish at the supports and the result will provide a means of determining H.

1.04. Before proceeding further it is desirable that an investigation be made of the order of the various terms occuring in the basic equations. A critical review will also be made of the implicit or explicit assumptions involved in the analysis and this will lead to a considerable simplification of the results, which is desirable in the light of the overall accuracy which can be achieved in Bridge Design.

In order to make the problem amenable to analytic formulation the discontinuous cable and suspender system has been replaced by a continuous distribution of suspenders, which in effect replaces the discontinuous cable curve by a continuous curve. However, in actual fact the cable contains from 10 to 30% of the structural weight, and the number of hangers is of the order of 70 to 100 in a main span. In practice, then, the departure of the continuous curve from the actual curve is of very low order, provided the dead load distribution contains no significant discontinuities, and the suspender lengths are correctly determined. From this cause alone the error in determinations of girder bending moments is H.e, where e is the error in the assumed cable curve ordinate.

This is to be compared with the computed cable moment due to live load, $H_w\eta + H(y + \eta)$. The ratio is

$$r = \frac{He}{H_w\eta + H(y + \eta)} = \frac{e}{\frac{H_w}{H} + (y + \eta)}$$

For a practical case we can take $\frac{H_w}{H} = 4$, $\eta = 6'$, $y + \eta = 206'$ and $r = \frac{e}{230}$. e will never be greater than a few inches in good design and construction, so the error in computed moments will be much less than 1%. Inability to achieve proper girder construction to zero stress under dead load may increase the effective error, beyond the order indicated. Such a condition necessitates adjustment of suspender lengths to achieve the optimum conditions.

Of at least equal significance, is the neglect of the bending stiffness of the cable. For the main span of the Golden Gate Bridge at San Francisco, the moment of inertia of one cable is approximately

$$\frac{\pi \times 3^4 \times 12^2}{64} = 572 \text{ in}^2 \text{ ft}^2.$$

The moment of inertia of one stiffening girder is 43,150 in^{2}ft^2

$$\frac{572}{43,150} = 0.0132.$$

While admitting the approximate nature of these numbers, there is still an apparent effect of the order of 1%.

In the same connection it should be noted that areas and weights and the value of E will not in general be realised to a correctness of 1%. These are only a few of the sources of practical error in the achievement of a design, and are introduced here to provide a criterion with which to judge the importance of the various terms which occur in the analysis.

The establishment of some such criterion is necessary to avoid overloading a solution with inconsequential terms sometimes in direct contradiction to actual realisation. For example, when a main span is deflected anti-symmetrically the suspenders all suffer an inclination from the vertical (as shown in Fig. 1.03) all in the same direction. As a result the horizontal thrust departs from the constancy assumed in the derivation by a maximum amount

$$\sum_{o}^{l} F_h \,.\mathrm{Sin}\, \frac{\xi}{h}$$

where F_h is the suspender force and $\sum_{0}^{l}$ indicates summation over the whole span. Fig. 1.03 illustrates the resultant horizontal force system.

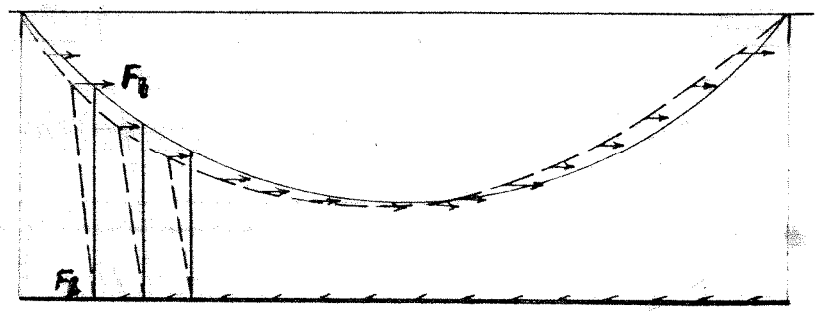


Fig. 1.03

Although the total change of the horizontal force produced may be comparatively small, the couples produced may be significant.

Furthermore, there is a strong tendency to shift the total deck load to the left in Fig. 1.03. Since expansion joints are provided at both ends of long spans, this does occur but is conveniently ignored in conventional analyses.

R.J.Atkinson and R.V.Southwell (4) have approached this aspect of the problem from another direction by formulating the fact that the girder load at (x,y) is transmitted to the cable at a point $x+\xi$, $y+\eta$, while neglecting the horizontal components of suspender forces, and prohibiting longitudinal displacement of the girder. These authors are completely in error in another respect (see Appendix I) which removes their derivation of a " corrected " differential equation from further consideration.

In the presentation given above, the longitudinal shift of the girder is immaterial, provided horizontal force components are neglected. Because of the methods of supporting the ends of the girder, the support moves with the suspended structure, and the origin of coordinate is always at this support. Expressions involving the cable displacement are all referred to the initial configuration in such a manner that the lateral shift of the cable relative to the girder does not affect the result.

Parcel and Maney, in the book " Statically Indeterminate Stresses " p.396, ascribe to this load shift the discrepancy between the usual Deflection Theory equation

$$P = EI \frac{d^4\eta}{dx^4} - H \frac{d^2y}{dx^2} - (H + H_w) \frac{d^2\eta}{dx^2}$$

and Rode's Deflection Theory equation,

$$P = EI \frac{d^4\eta}{dx^4} - H \frac{d^2y}{dx^2} - (H + H_w) \frac{d}{dx} \left(\frac{d\eta}{dx} \mathrm{Sec}^2\psi \right)$$

whereas in point of fact, both theories neglect all the consequences of such a shift.

The discrepancy lies in the somewhat naive assumption that at a given vertical section defined by x, the deflection of the cable is identical with the deflection of the girder, although a little consideration will show that a pure vertical motion of a point on the cable can occur only under very exceptional circumstances.

Put in another way, the ordinary Deflection Theory assumes that the cable bands can slide along the cable in order to satisfy the conditions that the suspenders remain vertical. Such a slip would prove disastrous in practice.

The longitudinal shift of the suspended load and the effect of inclined suspenders are amenable to treatment in a discussion directed towards development of a practicable analytical treatment of the problem, only if all the load is assumed to be at the girder level and the horizontal forces due to inclined suspenders are neglected. A longitudinal shift of the girder means all the suspended load is moved towards a support, and the support points of girder and cable are no longer in the same vertical line. If the origin of coordinates is regarded as lying at the girder support the analysis of section 1.02 applies.

The computation of the longitudinal shift of the cable bands, ξ , given in Table III, Column (4), shows that for that case at least the assumption that all the load moves longitudinally with the girder is closely satisfied in fact.

The ends of the girder are usually supported by some form of rocker link much shorter and with greater cross section than the theoretical end suspenders, which are approximately equal to the tower height.

The substitution of a short rocker link for a full suspender invalidates F.E.Turneare's formulae (6) for effect of cable stretch on girder bending moments.

Numerical calculations carried out by the method of Dana and Rapp have established the importance of the effect of end supports (see Section 1.01, assumption 6) and suitable means for evaluation of suspender elongation are discussed im Part 2. The methods there developed can be applied to deflection effects arising from suspender inclination, if necessary, but the computation of this inclination presupposes a knowledge of the lateral shift of the girder which can only be carried out numerically.

In the above analysis leading to equation (14) which is based on the work of Dr.Rode(3), the essential sources of difference in deflection of cable and girder have been included in order to emphasise the fundamental nature of the difference. Some of the contributions are of insignificant order and their inclusion in (14) is no criterion of the order of accuracy of that equation which still involves the assumptions already discussed.

Engineers are accustomed to base judgment of the desirable accuracy of structural calculations on a standard of 1%-2% and there seems no reason to raise the standard for suspension bridge calculations. A false impression has arisen from the necessity of using a large number of figures in the calculations of the exponentional functions of the Deflection Theory in order to achieve a final result correct to three significant figures.

Investigations of lower order terms are useful in establishing beyond doubt the order of the quantities involved, but need play no part in the final working method of design. A few such investigations will be undertaken herein for that purpose.

1.05. Where numerical data is needed the proportions of the main span of the Golden Gate Bridge at San Francisco will be used.

The Golden Gate Bridge ------ Significant Dimensions and Weights.

Length of Main Span		4,200 ft.
Sag of Main Span		470 ft.
Length of Side Spans		1,125 ft.
Weight of Main Span per lineal ft.	Deck	10,300 lbs.
	Cables, Suspenders etc.	6,670 lbs.
	Stiffening Trusses	3,330 lbs.
	Bracing	600 lbs.
	Miscellaneous	400 lbs.
	Total	21,300 lbs.
Dead load for side spans per linear ft.		21,500 lbs.
Live load capacity per linear ft.		4,000 lbs.
Maximum downward deflection of Main Span		10.8 ft.
Longitudinal Tower deflections	Shoreward	22 ins.
	Channelward	18 ins.
Diameter of Cables over wrapping		$36\frac{3}{8}$ ins.
Length of one cable		7,650 ft.
Number of wires in each cable		27,572
Size of wire, diameter		0.196 ins.
Weight of cable, suspenders etc.		24,500 tons.

Moment of Inertia of one main stiffening girder	43,150 $in^2 ft^2$.
Normal Temperature	70°F.
Maximum Temperature	110°F.
Minimum Temperature	30°F.
Unit stress in cable wire	82,000 p.s.i.

1.06. Some numerical evaluations of the various terms of equation (8) will be made for hypothetical loading conditions which are fairly representative of maximum conditions on the bridge.

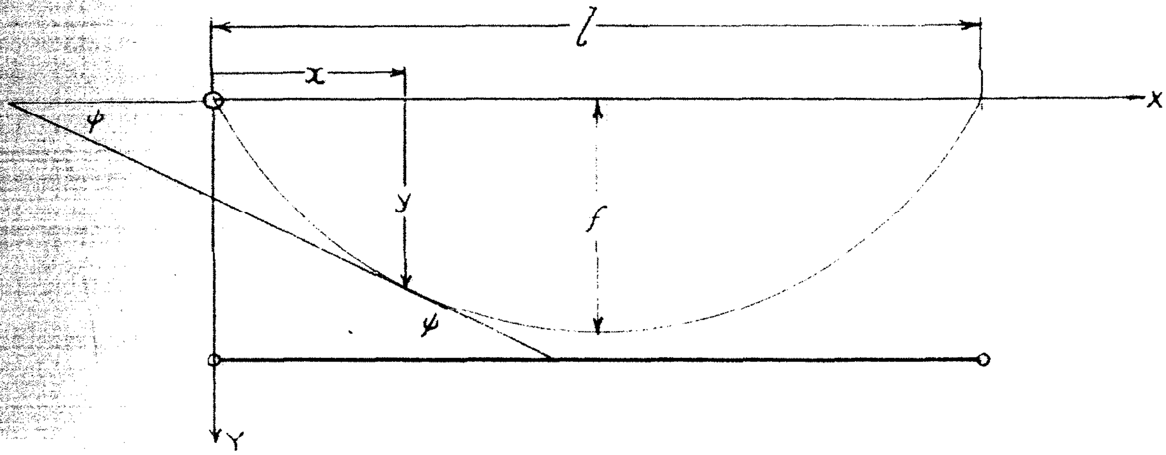


Fig. 1.04

The equation of the cable curve regarded as a parabola with origin at the top of the left hand tower is

$$y = \frac{4f}{l^2}(lx - x^2) = 4f(z - z^2) \qquad (15)$$

where f is the centre sag of the cable, l is the span length, and $z = \frac{x}{l}$.

For dead load alone on the cable at mean temperature

$$H_w = \frac{wl^2}{8f} \qquad (16)$$

wherein w is the dead load per lineal foot.

From (15)

$$\frac{dy}{dx} = \frac{dy}{dz} = \tan\psi = \frac{4f}{l}(1-2Z) \qquad (17).$$

To investigate the effect of $\delta\psi$ in the function $\sec(\psi+\delta\psi)$ an anti-symmetrical expression for η is used, given by

$$\eta = a_2 \sin\frac{2\pi x}{l} = a_2 \sin 2\pi Z \qquad (18)$$

where a_2 is chosen to represent the maximum deflection of the structure in the region of the quarter-point of the span. Then

$$\frac{d\eta}{dx} = \frac{d\eta}{l\,dZ} = \frac{2\pi a_2}{l} \cos 2\pi Z \qquad (19)$$

For the chosen example, the Golden Gate Bridge

$$H_w = \frac{21,300 \times 4200 \times 4200}{2 \times 8 \times 470} = 5 \times 10^7 \text{ lbs. per cable.}$$

$$\frac{H_w}{EA} = \frac{5 \times 10^7}{29 \times 10^6 \times 27,572 \times 0.0312} = 2 \times 10^{-3} \text{ unit strain.}$$

$$\tan\psi = \frac{4 \times 470}{4200}(1-2Z) = 0.4477(1-2Z).$$

a_2 has been chosen as 10 feet making

$$\eta = 10 \sin 2\pi Z \text{ where } \eta \text{ will be in feet}$$

$$\frac{d\eta}{dx} = \frac{2\pi \times 10}{4200} \cos 2\pi Z = 0.01496 \cos 2\pi Z.$$

From Table I it is evident that for the purposes of design it is satisfactory to take

$$1 - \frac{H_w}{EA}\sec\psi = 0.998 \approx 1.000$$

$$1 + \frac{H_w}{EA}\sec\psi\tan^2\psi = 1.000.$$

Then equation (8) can be simplified to

$$\frac{d\xi}{dx} = -0.998\frac{d\eta}{dx}\tan\psi + \frac{H}{EA}\sec^3(\psi+\delta\psi) + \omega t \sec^2\psi \qquad (20).$$

In this equation ωt takes account of temperature changes, and will be shown later to take care also of changes in span in later developments. For the present purpose ωt is taken as zero, and under

TABLE I.

$\frac{H_W}{EA} = 2 \times 10^{-3}$ $\tan\psi = 0.4477 \; (1-2z)$

1.	2.	3.	4.	5.	6.	7.	8.
Z	$\tan\psi$	$\sec\psi$	$\frac{H_W}{EA}\sec\psi$	$1-\frac{H_W}{EA}\sec\psi$	$\frac{H_W}{EA}\sec\psi\tan^2\psi$	$1+\frac{H_W}{EA}\sec\psi\tan^2\psi$	ψ
0	0.4477	1.0955	2.19×10^{-3}	0.9978	4.39×10^{-4}	1.00044	24°07'
0.1	0.3581	1.062	2.12×10^{-3}	0.9979	2.72×10^{-4}	1.00027	19°42'
0.2	0.2686	1.036	2.07×10^{-3}	0.9979	1.49×10^{-4}	1.00015	15°02'
0.3	0.1790	1.016	2.03×10^{-3}	0.9980	0.65×10^{-4}	1.00006	10°09'
0.4	0.0895	1.004	2.01×10^{-3}	0.9980	0.16×10^{-4}	1.00002	5°07'
0.5	0000	1.000	2.00×10^{-3}	0.9980	00×10^{-4}	1.00000	0°00'
0.6	-0.0895	1.004	2.01×10^{-3}	0.9980	0.16×10^{-4}	1.00002	5°07'
0.7	-0.1790	1.016	2.03×10^{-3}	0.9980	0.65×10^{-4}	1.00006	10°09'
0.8	-0.2686	1.036	2.07×10^{-3}	0.9979	1.49×10^{-4}	1.00015	15°02'
0.9	-0.3581	1.062	2.12×10^{-3}	0.9979	2.72×10^{-4}	1.00027	19°42'
1.0	-0.4477	1.0955	2.19×10^{-3}	0.9978	4.39×10^{-4}	1.00044	24°07'

this limitation equation (20) becomes

$$\frac{d\xi}{dx} = -0.998\ \frac{d\eta}{dx}\ \tan\psi\ +\ \frac{H}{EA}\ \sec^3\ (\psi + \delta\psi). \qquad (21).$$

Substituting $\frac{dy}{dx}$ for $\tan\psi$, we have

$$\frac{d\xi}{dx} = -0.998\ .\ \frac{d\eta}{dx}\ .\ \frac{dy}{dx} + \frac{H}{EA}\ \mathrm{Sec}^3\ (\psi + \delta\psi).$$

Integrating from 0 to x

$$\xi\ =\ +\ 0.998\int_0^x \eta\ .\ \frac{d^2y}{dx^2}\ dx\ -\ 0.998\left[\eta\ .\ \frac{dy}{dx}\right]_0^x + \frac{H}{EA}\ \ \mathrm{Sec}^3(\psi + \delta\psi)dx \quad (22)$$

From the condition of fixity of position of the cable supports

$$\xi\ = \eta =\ \ 0 \text{ at } x = 0 \text{ and } x = l\ .$$

and an equation is obtained from determining H, thus:

$$\frac{H}{EA}\int_0^l \mathrm{Sec}^3\ (\psi + \delta\psi)dx = -\ 0.998\int_0^x \eta\ .\ \frac{d^2y}{dx^2}\ .\ dx \qquad (23).$$

Substituting $\frac{d^2y}{dx^2} = -\frac{8f}{l^2}$ from equation (15), and η from equation(18) and carrying out the integration, the right hand side of (23) vanishes. On the left hand side $\sec(\psi + \delta\psi)$ is always positive, which can be seen from Table I, so in order that the left hand side vanish it is necessary that

$$H = 0.$$

The explanation of this very useful and unexpected result lies in the cable load distribution necessary to produce such an assumes curve of deflection, and is discussed further in Part II. The same conclusion applies to all even harmonics. The assumption of equation (18) differs from practical quarter point deflection curves only by the omission of the odd harmonic terms which produce the changes in horizontal force. This can be verified numerically by comparison with the computed deflection curves of reference (7). Since the second harmonic predominates in the quarter point deflection curves, this investigation is not limited in value by the apparently impractical conclusion H = 0.

The restricted equation (21) can now be written be for all even harmonics

$$\frac{d\xi}{dx} = -\ 0.998\ \frac{d\eta}{dx}\ .\ \tan\psi = -\ 0.998\ \frac{d\eta}{dx}\ .\ \frac{dy}{dx}. \qquad (24).$$

Substituting (24) in (10)

$$\tan\ (\psi + \delta\psi)\ -\ \tan\psi = \frac{d\eta}{dx} + 0.998\ \frac{d\eta}{dx}\ \tan^2\psi$$

$$= \frac{d}{dx}(\ 1 + 0.998\ \tan^2\psi\ )$$

$$\tan\ (\psi + \delta\psi) = \tan\psi + \frac{d\eta}{dx}\ (\ 1 + 0.998\ \tan^2\psi\ ) \approx \tan\psi + \frac{d\eta}{dx}\mathrm{Sec}^2\,\psi \qquad (25)$$

From equations (17), (19) and (25) values of Sec $(\psi+\delta\psi)$ have been calculated for ten points in the span and a_2 = 10' and tabulated in Table II.

From Table II it is evident that the substitution of secψ for Sec $(\psi+\delta\psi)$ may produce errors up to 3% in the value of $\sec^3(\psi+\delta\psi)$. However the errors due to this substitution vanish for all practical purposes when that quantity is integrated over the span length as in equation (23) for the determination of H; this is evident from the agreement of the totals of the columns (9) and (10) in Table II. This conclusion can be generalised for all even harmonic deflection curves.

In the case of odd harmonic deflection curve shapes the errors due to the substitution of ψ for $(\psi+\delta\psi)$ will also compensate, although not completely, in the integration of equation(23) except in the case of the first harmonic.

TABLE II.

$\tan\psi = 0.44762\ (1-2Z) \qquad \frac{d\eta}{dx} = \frac{d\eta}{l\,dz} = 0.01496\ \mathrm{Cos}\,2\pi Z.$

1. Z	2. $\tan\psi$	3. $\mathrm{Sec}\,\psi$	4. $0.998\tan^2\psi+1.$	5. $\frac{d\eta}{dx}$	6. $\frac{d\eta}{dx}(1+0.998\tan^2\psi).$
0	0.44762	1.0956	1.2000	0.01496	0.017952
0.1	0.35810	1.0622	1.1280	0.01210	0.013967
0.2	0.26857	1.0354	1.0720	0.00462	0.004952
0.3	0.17904	1.0159	1.0320	-0.00462	-0.004768
0.4	0.08952	1.0040	1.0080	-0.01210	-0.01220
0.5	0000	1.0000	1.0000	-0.01496	-0.01496
0.6	-0.08952	1.0040	1.0080	-0.01210	-0.01220
0.7	-0.17904	1.0159	1.0320	-0.00462	-0.004768
0.8	-0.26857	1.0354	1.0720	0.00462	0.004952
0.9	-0.35810	1.0622	1.1280	0.01210	0.013967
1.0	-0.44762	1.0956	1.2000	0.01496	0.017952

Z	7. $\tan(\psi+\delta\psi)$	8. $\mathrm{Sec}(\psi+\delta\psi)$	9. $\mathrm{Sec}^3\psi$	10. $\mathrm{Sec}^3(\psi+\delta\psi).$
0	0.46557	1.1031	1.315	1.342
0.1	0.37207	1.0669	1.198	1.214
0.2	0.27352	1.0367	1.110	1.114
0.3	0.17427	1.0151	1.048	1.046
0.4	0.07732	1.0030	1.012	1.009
0.5	-0.01496	1.0001	1.000	1.000
0.6	-0.10172	1.0051	1.012	1.015
0.7	-0.18381	1.0167	1.048	1.051
0.8	-0.26382	1.0341	1.110	1.106
0.9	-0.34413	1.0575	1.198	1.183
1.0	-0.42967	1.0884	1.315	1.289
			12.366	12.369

In the case of the first harmonic the error is cumulative but of lower maximum value due to the smaller amplitude and the longer wave length. For practical cases the error in determining H will be in error by less than 1% due to the neglect of $\delta\psi$. We can then, substitute ψ for $(\psi + \delta\psi)$ in equation (20) without introducing errors of significant magnitude, in the values of or H, subject of course, to later discussion of the influence of the term $\omega t \operatorname{Sec}^2\psi$.

Equation (22) can be used to evaluate the longitudinal displacement of the cable bands at any point in the span. It has been shown above that for the particular example being discussed (22) can be written,

$$\xi = -0.998 \left\{ \frac{8f}{l^2} \int_0^x \eta \cdot dx + \left[\eta \cdot \frac{dy}{dx} \right]_0^x \right\}$$

$$= -0.998 \left\{ \frac{40f}{l} (1-2Z) \operatorname{Sin} 2\pi Z + \frac{80f}{l} \int_0^Z \operatorname{Sin} 2\pi Z \, dZ \right\}$$

Finally,

$$\xi = -4.465 \left\{ (1-2Z) \operatorname{Sin} 2\pi Z + \frac{1}{\pi} (1 - \operatorname{Cos} 2\pi Z) \right\} \text{ ft.} \qquad (26).$$

Table III has been computed for nineteen points in the span of the Golden Gate Bridge. It contains the values of ξ from equation(26), suspender lengths, and $\operatorname{Sin} \frac{\xi}{h}$. It would be desirable to carry Table III through for each cable band (83 points) but in the interest of brevity 19 points only were calculated.

In order to determine the longitudinal movement of the girder, due to the inclination of the suspenders it is necessary to know the forces in the suspenders, F_h. These have been determined by differentiating twice the equation to the deflected cable curve, and subtracting the weight of the cable thus:

TABLE III

$\xi = -4.465\left\{(1-2Z)\ \sin 2\pi Z + \frac{1}{\pi}\ (1-\cos 2\pi Z)\right\}$ $F_H = 2.24\times10^6(1+0.105\ \sin 2\pi Z) - 0.701\times10^6$ $F_L \approx F_H \cdot \sin\frac{\xi'}{H} \approx \frac{\xi'}{H}$ $H_W \cdot \eta = 5\times10^8\ \sin 2\pi Z$

1 Z	2 $\sin 2\pi Z$	3 $\cos 2\pi Z$	4 ξ	5 ξ/H	6 ξ/H	7 F_H	8 $F_{L(s=0)}$	9 $F_{L(s=3)}$	10 $F_{L(s=3.1)}$	11 $\xi' = \xi + 3.04$	12 ξ'/H H. VERS.	13 $F_{L(s=3.04)}$	14 $\sum F_L.H$	15 $H_W.\eta$
0	0.000	1.000	000	485	000									
0.05	0.3090	0.9511	-1.31	396	-0.00331	1.613×10^6	-0.53×10^4	0.689×10^4	0.737×10^4	1.73	0.004	0.705×10^4	2.79×10^6	154×10^6
0.1	0.5878	0.8090	-2.37	316	-0.0075	1.677×10^6	-1.26×10^4	0.334×10^4	0.405×10^4	0.67	0.000	0.356×10^4	3.91×10^6	294×10^6
0.15	0.8090	0.5878	-3.11	245.2	-0.0127	1.732×10^6	-2.20×10^4	-0.073×10^4	-0.002×10^4	-0.07	0.000	-0.050×10^4	3.79×10^6	404×10^6
0.2	0.9511	0.3090	-3.53	184.2	-0.0192	1.764×10^6	-3.39×10^4	-0.507×10^4	-0.411×10^4	-0.49	0.000	-0.469×10^4	2.93×10^6	475×10^6
0.25	1.000	0000	-3.65	132.5	-0.0275	1.776×10^6	-4.88×10^4	-0.871×10^4	-0.737×10^4	-0.61	0.001	-0.818×10^6	1.85×10^6	500×10^6
0.3	0.9511	-0.3090	-3.56	90.2	-0.0394	1.764×10^6	-6.95×10^4	-1.096×10^4	-0.900×10^4	-0.52	0.002	-1.018×10^6	0.93×10^6	475×10^6
0.35	0.8090	-0.5878	-3.34	57.3	-0.0583	1.732×10^6	-10.01×10^4	-1.028×10^4	-0.706×10^4	-0.30	0.000	-0.908×10^6	0.41×10^6	404×10^6
0.4	0.5878	-0.8090	-3.09	33.8	-0.0914	1.677×10^6	-15.34×10^4	-0.447×10^4	0.049×10^4	-0.05	0.000	-0.25×10^6	0.33×10^6	294×10^6
0.45	0.3090	-0.9511	-2.91	19.7	-0.1476	1.613×10^6	-23.82×10^4	0.737×10^4	0.081×10^4	0.13	0.000	1.06×10^4	0.54×10^6	154×10^6
0.5	0.000	-1.000	-2.84	15	-0.1893	1.539×10^6	-29.13×10^4	1.640×10^4	2.660×10^4	0.20	0.001	2.054×10^4	0.85×10^6	000×10^6
0.55	-0.3090	-0.9511	-2.91	19.7	-0.1476	1.464×10^6	-21.60×10^4	0.669×10^4	1.412×10^4	0.13	0.000	0.967×10^4	1.04×10^6	-154×10^6
0.6	-0.5878	-0.8090	-3.09	33.8	-0.0914	1.399×10^6	-12.79×10^4	-0.372×10^4	0.042×10^4	-0.05	0.000	-0.207×10^4	0.97×10^6	-294×10^6
0.65	-0.8090	-0.5878	-3.34	57.3	-0.0583	1.349×10^6	-7.87×10^4	-0.799×10^4	-0.564×10^4	-0.30	0.000	-0.707×10^4	0.56×10^6	-404×10^6
0.7	-0.9511	-0.3090	-3.56	90.2	-0.0394	1.316×10^6	-5.19×10^4	-0.817×10^4	-0.671×10^4	-0.52	0.002	-0.759×10^4	-0.12×10^6	-475×10^6
0.75	-1.000	0.000	-3.65	132.5	-0.0275	1.303×10^6	-3.59×10^4	-0.640×10^4	-0.541×10^4	-0.61	0.001	-0.601×10^4	-0.92×10^6	-500×10^6
0.8	-0.9511	0.3090	-3.53	184.2	-0.0192	1.316×10^6	-2.53×10^4	-0.373×10^4	-0.307×10^4	-0.49	0.000	-0.350×10^4	-1.56×10^6	-475×10^6
0.85	-0.8090	0.5878	-3.11	245.2	-0.0127	1.349×10^6	-1.71×10^4	-0.060×10^4	-0.005×10^4	0.07	0.000	0.038×10^4	-1.47×10^6	-404×10^6
0.9	-0.5878	0.8090	-2.37	316	-0.0075	1.399×10^6	-1.05×10^4	0.279×10^4	0.323×10^4	0.67	0.000	0.296×10^4	-0.54×10^6	-294×10^6
0.95	-0.3090	0.9511	-1.31	396	-0.0033	1.464×10^6	-0.48×10^4	0.625×10^4	0.662×10^4	1.73	0.004	0.640×10^4	2.04×10^6	-154×10^6
1.0	0.0000	1.0000	0.00	485	0.000	Totals.	-154.3×10^4	-2.115×10^4	2.964×10^4			-0.20×10^4		

$$M = H_w(y + \eta) = -\frac{4f}{l^2}(lx - x^2) - a_2 \sin\frac{2\pi x}{l}$$

$$w = -H_w(y + \eta) = H_w\left[\frac{8f}{l^2} + \frac{4\pi^2}{l^2} a_2 \sin\frac{2\pi x}{l}\right]$$

where w is the cable load per foot run.

Substituting numerical values

$$w = 5 \times 10^7 \left[\frac{8.470}{4,200^2} + \frac{4.\pi^2.10}{4,200^2} \operatorname{Sin} 2\pi Z\right]$$

$$= 10,650 \left[1 + 0.105 \operatorname{Sin} 2\pi Z\right] \text{ lb/ft.} \qquad (27).$$

For an assumed hanger spacing of 210 ft., which is a consequence of the choice of nineteen points in the 4,200 ft. span, the hanger force is

$$F_h = 210 \times 10,650 \left[1 + 0.105 \sin 2\pi Z\right] - 210 \times \frac{6,670}{2}$$

$$= 2,240,000 \left[1 + 0.105 \sin 2\pi Z\right] - 701,000 \text{ lb.}$$

From this equation F_h has been evaluated as shown in Table III.

It is now possible to calculate the horizontal forces, F_l, at the cable bands, which are given by

$$F = F_h . \sin\frac{\xi}{h} \approx F_h . \frac{\xi}{h}$$

Column(8) shows that the longitudinal forces are quite appreciable and their magnitude leaves no doubt that the deck will move longitudinally. The amount of longitudinal displacement is determined by the condition that

$$\sum F_l = 0.$$

To determine this shift the following artifice has been adopted: The nett longitudinal force, $\sum F_l$, has been determined for ξ as tabulated, and for $\xi' = \xi + S$ for two values of S, which are chosen to embrace the condition

$$\sum F_l = 0.$$

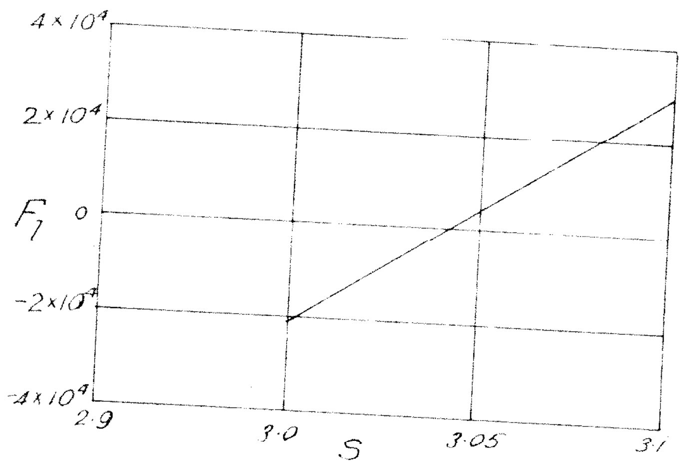


Fig. 1.05

Then, by means of the plot of Fig. 1.05 the value of S is interpolated which will result in the equilibrium

$$\sum F_l = 0.$$

With the known shift it is possible to compute the inclination of the hangers, and the effect of this inclination on deflection, as included in equation (1). This is shown in Table III, Column(12), and it is evident that it is negligible in comparison with the maximum deflection of 10', being 0.04% at most for this representative example.

It is evident from Table III, Column (10), that the short hangers exert very large forces for very small shifts of the cable bands, which is to be expected. This enables us to state in general, that the effect of obliquity of the hangers on the deflection of the girder will always be small, since the short hangers will tend to remain vertical, because of their predominant control overthe longitudinal shift of the girder.

Odd harmonic deflection curves will produce much smaller longitudinal displacements of the cable bands and, from symmetry, no longitudinal shift. The magnitude of the longitudinal forces, however, requires further investigation.

Table III, Column (8), shows how large forces can occur at the suspender connections if the girder is prevented from moving longitudinally. This occurs when the deflection of the cables is caused by twist of the bridge about a longitudinal axis, and the total longitudinal forces due to the suspenders are equal and opposite on the two cables. If the deck is strong enough to withstand these forces without sidewise buckling in the deck plane or shearing distortion of the deck, it is evident that torsional distortion of the bridge is strongly opposed. That this consideration becomes important in long span bridges has been tragically demonstrated by the failure of the Tacoma Bridge. The weak point for this condition lies at the centre cable band which contributes the largest control force.

It is now possible to compute the magnitude of the error in the cable live load bending moment due to the longitudinal suspender forces, F_h. This has been done in Table III, Column(14), and it may be compared with the live load bending moment carried by the cable, $H_w \eta$, in Column (15).

The moment of the longitudinal forces is given by

$$\sum_{o}^{x} F_{l} . h$$

and positive values indicate that the neglect of these moments underestimates the deflection of the cable. The maximum percentage of the corresponding cable live load moment is 2.4% occurring at $0.1 l$. If the cable were totally unstiffened this would represent an error in deflection of practically 2.4% at that point. Because of the girder the actual percentage error in deflection will be less depending on the effectiveness of the girder.

It is important to note that bending moments in the girder in the critical locations, are increased in absolute value by the lateral forces, although it is somewhat unsafe to generalise this statement for higher harmonic deflection curves. However, it appears quite possible to have 2% error in the maximum girder bending moment due to this cause.

1.07. In practice, the points of support of the cables move longitudinally due to a variety of causes. Thus it is not proper to assume that the span length remains unchanged under actual conditions. The amount of this change of span can reach 1/1000 of the span length.

The deviation of the fundamental equation is not invalidated by this variation except insofar as the neglect of longitudinal cable band forces will affect the result. However, this does not remove the necessity for consideration of the effects of the cable deflection resulting from such a change in cable span.

In the consideration of temperature change it was assumed that the cable length changed and all vertical dimensions of the structure changed in the same proportion, but the span of the cable remained unchanged. For such small changes we could equally correctly assume that all the structural parts remained unchanged while the span changed a small amount due to temperature.

When temperature changes are considered in this light a rise of temperature above normal corresponds to a decrease in span length. It is at once evident that a change of the distance between cable saddles due to any other cause can also be regarded as resulting from a temperature change.

The term ωt in the present analysis must now be taken ~~longitudinally~~ to include also the span change due to longitudinal shift of the tower tops. A decrease of span length dl, when divided by the span and the coefficient of expansion ω, will give a number which may be regarded as an equivalent rise of temperature, t, thus: $t = \frac{dl}{\omega \cdot l}$.

1.08. It appears that the acceptance of assumptions(4) and (5) of section 1.01 will not produce serious errors in the evaluation of deflections of the structure under live load. If, furthermore, we accept (6) with the proviso that the effect of suspender elongation can be calculated as a correction if necessary, then equation (1) can be written

$$\vartheta = \eta + h.wt \qquad (28)$$

and equation (11) can be written

$$\frac{d\eta}{dx} = \frac{d\vartheta}{dx} - \omega t\,\frac{dh}{dx} \qquad (29).$$

If we eliminate the terms that have been shown to be trivial in equation (8) then we obtain

$$\frac{d\xi}{dx} = -\frac{d\eta}{dx}\tan\psi + \frac{H}{EA}\,\text{Sec}^3\psi \quad wt\ \text{Sec}^2 \qquad (30).$$

Differentiating (28) and transposing we obtain

$$\frac{d\eta}{dx} = \frac{d\vartheta}{dx} - \omega t\,\frac{dh}{dx} \qquad (31).$$

But $\frac{dh}{dx} = -\tan\psi$

which, when substituted in (31) gives

$$\frac{d\eta}{dx} = \frac{d\nu}{dx} + \omega t\,\tan\psi \qquad (32).$$

Eliminating $\frac{d\eta}{dx}$ from (30) by using (32) there results

$$\frac{d\xi}{dx} = -\frac{d\nu}{dx}.\tan\psi - \omega t\,\tan^2\psi + wt\ \text{Sec}^2\psi + \frac{H}{EA}\,\text{Sec}^3\psi$$

$$= -\frac{d\vartheta}{dx}\tan\psi + \omega t + \frac{H}{EA}\,\text{Sec}^3\psi \qquad (33)$$

$$\frac{d\eta}{dx} - \frac{d\xi}{dx}\tan\psi = \frac{d\vartheta}{dx}\tan^2\psi + \frac{d\vartheta}{dx} - \frac{H}{EA}\,\text{Sec}^3\psi\,\tan\psi$$

$$= \left[\frac{d\vartheta}{dx} - \frac{H}{EA}\,\text{Sec}\,\psi\,\tan\psi\right]\text{Sec}^2\psi$$

and substituting this into equation (14) we obtain the fundamental equation of the stiffened suspension bridge

$$p' + p'' = w_l + w_d = \frac{d^2}{dx^2}\left(EI\,\frac{d^2\nu}{dx^2}\right) - (H_w + H)\frac{d}{dx}\left\{\frac{dy}{dx} + \left[\frac{d\nu}{dx} - \frac{H}{EA}\,\text{Sec}\,\psi\tan\psi\right]\text{Sec}^2\psi\right\} \qquad (34).$$

Where w_l and w_d represent live and dead load per unit length respectively.

The equations (33) and (34) are free of insignificant terms and must be integrated before the stresses in a suspension bridge can be satisfactorily evaluated.

1.09. Main Suspension Span. Horizontal Tension due to Live Load.

In the case of a main span the cable saddles are at the same level and the ordinates of the cable curve are measured from the horizontal. Then

$$\tan\psi = \frac{dy}{dx}$$

and equation (33) becomes

$$\frac{d\xi}{dx} = -\frac{d\nu}{dx}\cdot\frac{dy}{dx} + \omega t + \frac{H}{EA}\,\text{Sec}^3\psi \qquad (35).$$

This may be integrated as follows:

$$\int_0^l \frac{d\xi}{dx}\cdot dx = \int_0^l d\xi = \xi = -\nu\,\frac{dy}{dx}\Bigg|_0^l + \int_0^l \nu\,\frac{d^2y}{dx^2}\,dx + \omega t\,l + \frac{H}{EA}\int_0^l \text{Sec}^3\psi\,dx$$

The first term on the right hand side is zero because ν is zero at the two supports. In the unstrained condition the cable curve is approximately a parabola. Since the terms in H are already small the error introduced by using the geometry of the parabola in the evaluation of the integrals will be unimportant.

We write then

$$\frac{d^2y}{dx^2} = -\frac{8f}{l^2}$$

and integrate again

$$\xi = -\frac{8f}{l^2}\int_0^l \vartheta\, dx + \omega t l + \frac{H}{EA}\int_0^l \mathrm{Sec}^3\psi.$$

$$= -\frac{8f}{l^2}\int_0^l \vartheta dx + wtl + \frac{H}{EA} L_c \qquad (36)$$

where $L_c = \int_0^l \mathrm{Sec}^3\psi\, dx = \int_0^l (\frac{dL}{dx})^3 dx = l\left\{\left(\frac{5}{8} + \frac{4f^2}{l^2}\right)\left(1 + \frac{16f^2}{l^2}\right)^{\frac{1}{2}} + \frac{3\,l}{32f}\log_e\left[\frac{4f}{l} + (1 + \frac{16f^2}{l^2})\right]\right\}$

Then H can be determined from the equation $\xi = 0$ whence

$$\frac{H}{EA} = \left\{\frac{8f}{l^2}\int_0^l \vartheta dx - wtl\right\}\frac{1}{L_c} \qquad (37)$$

wherein a rise of temperature, actual or calculated from a span change is considered positive.

1.10. Equation (34) m[illegible] [illegible]e integrated once directly

$$\int (w_1 + w_d)\, dx = \frac{d}{\cdot}\left(EI\, \frac{d^2 \nu}{dx^2}\right) - (H_w + H)\left\{ \frac{dy}{dx} + \frac{d\nu}{dx}\mathrm{Sec}^2\psi - \frac{H}{EA}\mathrm{Sec}^3\psi \tan\psi \right\} + c_1$$

Integrating again we obtain the bending moment, M:

$$M = EI\, \frac{d^2\nu}{dx^2} - (H_w + H)y - (H_w + H)\int_o^x \frac{d\nu}{dx}\, \mathrm{Sec}^2\psi\, dx + \frac{H}{EA}\int_o^x \mathrm{Sec}^3\psi \tan\psi\, dx + c_1 x \qquad (38)$$

The ordinary deflection theory writes $\sec^2\psi = 1$ and both of the integrals in (38) can then be evaluated. This approximation introduces large errors and cannot be accepted in a refined theory. A better approximation is obtained as follows:

$$\int_o^x \frac{d\nu}{dx}\, \mathrm{Sec}^2\psi\, dx = \nu\, \mathrm{Sec}^2\psi - 2\int_o^x \nu\, \mathrm{Sec}^2\psi \tan\psi \frac{d\psi}{dx}\, dx$$

and $\frac{d\psi}{dx} = -\frac{8f}{l^2}\,\frac{1}{\mathrm{Sec}^2\psi}$ therefore

$$\int_o^x \frac{d\nu}{dx}\, \mathrm{Sec}^2\psi = \nu\, \mathrm{Sec}^2\psi + \frac{16f}{l^2}\int_o^x \nu\, \tan\psi\, dx.$$

The second term on the right hand side will be neglected on the basis of a later discussion.

The second integral on the right hand side of equation (38) may also be evaluated

$$\int_o^x \mathrm{Sec}^3\psi \tan\psi \frac{dx}{d\psi}\, d\psi = -\frac{l^2}{8f}\int_o^x \mathrm{Sec}^4\psi\, \frac{d(\mathrm{Sec}\,\psi)}{d\psi}\,.\, d\psi$$

$$= \frac{l^2}{40f}\left(\mathrm{Sec}^5\psi_o - \mathrm{Sec}^5\psi \right)$$

The constant c_1 is evaluated from the condition that $M = 0$ at $x = 0$ for the condition of no rotational restraint and no deflection at the ends. This gives $c_1 = 0$. Substituting these modifications in equation (38) we obtain

$$M = M_L + M_d = EI \frac{d^2\nu}{dx^2} - (H_w + H)\ (y + \nu \operatorname{Sec}^2\psi) + \frac{H l^2}{40EAf}(\operatorname{Sec}^5\psi_0 - \operatorname{Sec}^5\psi) \tag{39}$$

The external bending moment is due to live load and dead load and the two constituents have been shown explicitly by writing $M = M_L + M_d$.

1.11. In order to obtain a numerical idea of the effect of the term $\operatorname{Sec}^2\psi$ in equation (39) calculations have been made of the cable deflection of various points in a span, for the assumed girder deflection used in section 1.06.

The deflection theory assumes the cable deflection, η_D, is the same as the girder deflection, ν, at every section if suspender elongation is neglected.

Rode's theory is free of this assumption and the deflection of the cable can be expressed with satisfactory accuracy by

$$\eta_R = \nu - \xi \tan(\psi + \delta\psi).$$

The modification of Rode's Theory included in equation (39) involves the assumption

$$\eta_M = \nu \operatorname{Sec}^2\psi = \nu(1 + \tan^2\psi).$$

Table IV contains the tabulated values from these equations for the deflection curve

$$\nu = 10 \operatorname{Sin} 2\pi Z.$$

Table IV shows the improved approximation obtained by the assumptions of equation (39). It should also be noted that the stretch of the suspenders will improve the approximation, and it is stated here for the record that the effect of suspender elongation should not be included in design calculations based on the Deflection Theory, or the improved equation (39), as this elongation tends to improve the approximation slightly.

The discrepancies between model test and calculated values observed by Beggs, Davis and Davis (2), are not surprising in view of the disagreement between the values in columns (6) and (7) in Table IV.

TABLE IV.

z	ν	ξ	$\tan(\psi+\delta\psi)$	$\sec^2\psi$	η_D	η_R	η_M
1	2	3	4	5	6	7	8
0 0	0	0			0	0	0
0.1	5.878	-2.37	0.3721	1.128	5.878	6.76	6.63
0.2	9.511	-3.53	0.2735	1.072	9.511	10.48	10.31
0.3	9.511	-3.56	0.1743	1.031	9.511	10.13	9.81
0.4	5.878	-3.09	0.0773	1.008	5.878	6.12	5.93
0.5	0	-2.84	-0.015	1.00	0	-0.04	0
0.6	-5.878	-3.09	-0.1017	1.008	-5.878	-6.19	-5.93
0.7	-9.511	-3.56	-0.1838	1.031	-9.511	-10.16	-9.81
0.8	-9.511	-3.53	-0.2636	1.072	-9.511	-10.44	-10.31
0.9	-5.878	-2.37	-0.3441	1.128	-5.878	-6.69	-6.63
1.0	0	0			0	0	0

PART II.

The Determination of Deflections and Bending Moments in Cable and Stiffening Girder Systems.

2.01. Approximate Solution of Structural Problems by Use of a Finite Series of Chosen Functions.

Structural problems involving deflections, bending moments, and loads arise in the daily practice of the structural engineer. In many cases the solution by the "exact" approach leads to cumbersome or intractable equations, which, considering the great uncertainties in loading, and the somewhat idealised structural details adopted to render the problem amenable to mathematical treatment, are not justifiable for general use.

In many such cases solutions in terms of a convergent infinite series are possible and desirable, insofar as only a few terms of the series, consistent with the approach of the structure to ideal behaviour, are retained. Furthermore, an intelligent procedure of design is made possible by the consideration of major terms only in the early stages and introducing terms of lesser magnitude in the later stages.

In the application of this method the wanted function Y is expressed by a series of functions Y_1, Y_2, Y_3.......which severally satisfy the boundary conditions of the problem.

Then $Y = a_1Y_1 + a_2Y_2 + a_3Y_3 + \ldots\ldots\ldots\ldots$

where Y_1, Y_2, Y_3 ...... are the chosen functions, and a_1, a_2, a_3 ..... are numerical coefficients whose values must be determined so that $\sum_n a_nY_n$ is a good fit for Y.

The criterion of fit is a matter of choice in practical cases. One can by arbitrary guess and trial, find values of a_n which will result in a satisfactory fit of $\sum_n a_n Y_n$ to the plotted values of Y. This method can lead to good results in simple cases but it lacks the theoretical background which is necessary to guarantee a result of sufficient accuracy with a finite amount of labour.

Alternatively, the coefficients of the finite series, $\sum_n a_n Y_n$ can be chosen so that at n special points the equation

$$Y = \sum_n a_n Y_n$$

is satisfied exactly. This procedure leads to a system of n linear simultaneous equations for the coefficients. This is a definite procedure and a result can be obtained in a reasonable time. It has the disadvantage that there is no guide to the best points to select to obtain a good fit with a few terms, so in general, it is necessary to retain a larger number than is desirable from the standpoint of office use.

The third method is to select each coefficient to obtain the best "least square" fit to the residual remaining from the previous term. This procedure gives a criterion for the determination of each coefficient in turn. The latter condition is automatically satisfied by trigonometrical series which are specially significant to the structural engineer.

2.02. The problems of structural mechanics involve the bending moments and deflections in beams and columns. The fundamental equation of bending

$$\frac{d^2}{dx^2}\left(EI \frac{d^2y}{dx^2}\right) = - \frac{d^2M}{dx^2} = p,$$

can be satisfied, under certain very commonly occuring boundary conditions, by a Fourier Series of the form

$$y = f(x) = \sum_{n=1}^{\infty} a_n \operatorname{Sin} \frac{n\pi x}{l} + \sum_{n=0}^{\infty} b_n \operatorname{Cos} \frac{n\pi x}{l}$$

y may have a finite number of discontinuities, and except in the neighbourhood of such discontinuities, the series converges to f(x).

In those problems which require evaluation of deflections as an intermediate or final step in design this series has particular interest because of the recurrence of sines and cosines after every two differentiations.

$$\frac{d}{dx}\left(\operatorname{Sin}\frac{n\pi x}{l}\right) = \frac{n\pi}{l}\operatorname{Cos}\frac{n\pi x}{l}, \quad \frac{d^2}{dx^2}\left(\operatorname{Sin}\frac{n\pi x}{l}\right) = -\frac{n^2\pi^2}{l^2}\operatorname{Sin}\frac{n\pi x}{l}$$

This allows many results to be obtained by the technique of equating coefficients.

It can be shown (9) that the finite series obtained by cutting off all terms beyond a certain order n, is the best approximation possible by a trigometric series of the same order. The coefficients remain unchanged if more terms are taken in a new approximation. Furthermore, the coefficient of each term results in the best "least square" fit of the term to the remainder of the function and each additional term of the series results in an improved "least square" fit of the finite series to the function Y.

This series, then, has desirable characteristics from the point of view of the structural analyst. It lends itself well to successive improvement in accuracy of calculation as design proceeds. Tables of Sines and Cosines make calculations and plotting of values a very simple matter. In many cases, the influence of design changes on stresses can be detected by their effect on coefficients of terms of certain orders and the result of such changes can be rapidly evaluated.

The evaluation of the coefficients a_n, b_n and b_o follow certain rules which depend on the orthogonal properties of these trigonometrical functions. They are quoted here without derivation

$$a_n = \frac{1}{l}\int_0^{2l} f\left(\frac{\pi x}{l}\right) \operatorname{Sin} \frac{n\pi x}{l}\, dx$$

$$b_o = \frac{1}{2l}\int_0^{2l} f\left(\frac{\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \operatorname{Cos} \frac{n\pi x}{l}\, dx$$

wherein $f(x)$ is the quantity which is to be expressed as a Fourier Series over a certain interval and $2l$ is the length of one cycle of this function.

In cases where the beam of length, l, is freely supported at the two ends as in Fig. 2.01 an expansion in Sine terms alone satisfies the boundary conditions, and in effect, we are assuming a cycle over the distance $2l$ as shown.

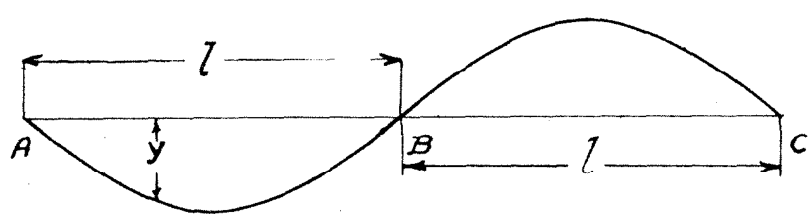


Fig. 2.01

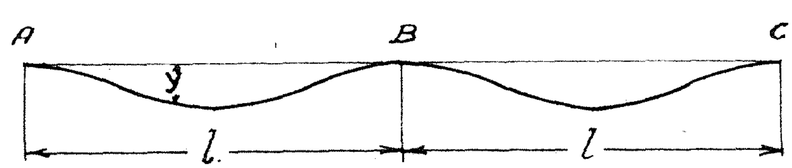


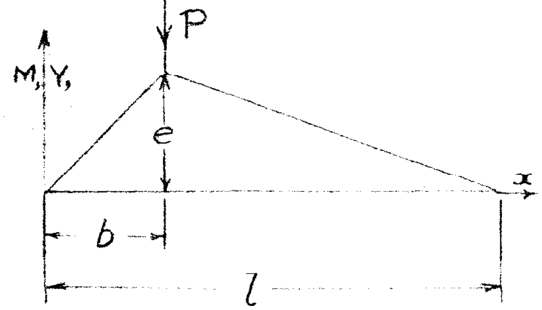
Fig 2.02.

Where the beam is fixed with zero slope and deflection at the ends the cosine series alone satisfies the boundary conditions over a span length, l. As shown in Fig.2.02 the function Y is assumed to repeat beyond the end of the span, B.

Boundary conditions other than these two types are not easily treated by trigonometric series.

2.03. The work which follows requires that the bending moment due to the external loads, be expressed in a trigonometric series. The few necessary types of expansion encountered in usual structural analysis are collected here for convenient reference.

(1)



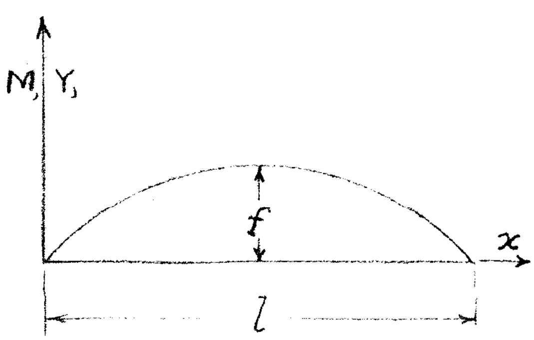
The shape of the triangle is given by

$$y = \frac{2el^2}{\pi^2 b(l-b)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi b}{l} \sin\frac{n\pi x}{l}$$

A triangular bending moment diagram due to a load P is defined by

$$M = \frac{2Pl}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi b}{l} \sin\frac{n\pi x}{l}$$

(2)



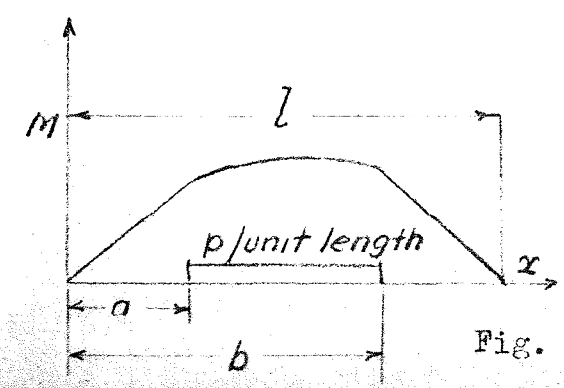
The parabola $y = \frac{4f}{l^2}(lx - x^2)$ is also defined by

$$y = \frac{32f}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{l} \quad \text{for n odd only.}$$

The bending moment due to unit load p per unit length is given by

$$M = -\frac{4pl^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{l} \quad \text{for n odd.}$$

(3)



The bending moment due to a uniform load p per unit length extending from x = a to x = b on a span l, is given by

$$M = -\frac{2pl^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\left(\cos\frac{n\pi b}{l} - \cos\frac{n\pi a}{l}\right) \sin\frac{n\pi x}{l}}{n^3}$$

Fig. 2.03

From (3) all the formulae for uniform load p per unit length over part of the span, can be derived. A great simplification is achieved here since the single equation replaces three separate equations in the usual method of calculation.

2.04. For the theory which follows we need to know the coefficients of the expansion

$$\operatorname{Sin} \frac{i\pi x}{l} \operatorname{Sin} \frac{k\pi x}{l} = \sum_{m=1}^{\infty} a_m \operatorname{Sin} \frac{m\pi x}{l} = \sum_{m=1}^{\infty} a_m \operatorname{Sin} m\pi Z$$

where i may have any odd ppsitive integral value and k may have any positive integral values.

For the purposes of calculation it will be sufficient to tabulate values of i, k, and m up to 7.

The coefficients a_m, are determined by

$$a_m = \frac{2}{l}\int_0^l \operatorname{Sin} \frac{m\pi x}{l} \cdot \operatorname{Sin} \frac{i\pi x}{l} \cdot \operatorname{Sin} \frac{k\pi x}{l}\, dx.$$

$$= -\frac{1}{2\pi}\left[\frac{\operatorname{Cos}(i-k+m)\frac{\pi x}{l}}{(i-k+n)} + \frac{\operatorname{Cos}(k-i+n)\frac{\pi x}{l}}{(k-i+n)} + \frac{\operatorname{Cos}(i+k-n)\frac{\pi x}{l}}{(i+k-n)} - \frac{\operatorname{Cos}(i+k+n)\frac{\pi x}{l}}{(i+k+n)}\right]_0^l$$

The coefficients have been calculated and tabulated in Table V.

TABLE V.

i =1.

Values of k.

m	1.	2.	3.	4.	5.	6.	7
1.	0.849	0	-0.1696	0	-0.02425	0	-0.00808
2.	0	0.679	0	-0.1939	0	-0.03231	0
3.	-0.1696	0	0.655	0	-0.2021	0	-0.0360
4.	0	-0.1939	0	0.682	0	-0.2056	0
5.	-0.02425	0	-0.2021	0	0.6435	0	-0.2075
6.	0	-0.03234	0	-0.2056	0	0.641	0
7.	-0.00808	0	-0.0360	0	-0.2075	0	0.640
8.	0	-0.01177	0	-0.0380	0	-0.2088	0

i = 3.

Values of k.

m	1.	2.	3.	4.	5.	6.	7.
1.	-0.1696	0	0.655	0	-0.2021	0	-0.03602
2.	0	0.485	0	0.453	0	-0.238	0
3.	0.655	0	0.283	0	0.4165	0	-0.2518
4.	0	0.453	0	0.2470	0	0.4025	0
5.	-0.2021	0	0.4156	0	0.2331	0	0.3962
6.	0	-0.2382	0	0.4025	0	0.2263	0
7.	-0.0360	0	-0.2516	0	0.3962	0	0.2224
8.	0	-0.0497	0	-0.2585	0	0.392	0

i =5.

Values of k.

m	1.	2.	3.	4.	5.	6.	7.
1.	-0.02425	0	-0.2021	0	0.6435	0	-0.2075
2.	0	-0.2263	0	0.441	0	0.435	0
3.	-0.2021	0	0.4165	0	0.2331	0	0.3962
4.	0	0.441	0	0.2088	0	0.1939	0
5.	0.6435	0	0.2331	0	0.1696	0	0.1782
6.	0	0.435	0	0.1939	0	0.1542	0
7.	-0.2075	0	0.3962	0	0.1782	0	0.1459
8.	0	-0.2466	0	0.3805	0	0.1701	0

i =7.

Values of k.

m.	1.	2.	3.	4.	5.	6.	7.
1.	-0.00808	0	-0.0360	0	-0.2075	0	0.640
2.	0	-0.0441	0	-0.243	0	-0.432	0
3.	-0.0360	0	-0.2518	0	0.3962	0	0.2224
4.	0	-0.243	0	0.3878	0	0.1864	0
5.	-0.2075	0	0.3962	0	0.1783	0	0.1459
6.	0	-0.432	0	0.1864	0	0.1379	0
7.	0.640	0	0.2224	0	0.1459	0	0.01212
8.	0	0.4295	0	0.1820	0	0.1291	0

2.05. It is necessary to expand the function

$$Z(1 - Z)\text{Sin } i\pi Z$$

as a Sine Series, and the coefficients of the expansion are given by

$$a_m = 2\int_0^1 Z(1 - Z)\text{ Sin } i\pi Z \text{ Sin } m\pi Z \, dZ.$$

After integration the coefficients, a_m, are determined by the expressions

$$a_m = -\frac{1}{\pi^2}\left[\frac{1+\text{Cos}(i - m)\pi}{(i - m)^2} - \frac{1 + \text{Cos}(i + m)\pi}{(i + m)^2}\right] \text{if } m^2 \neq i^2$$

and

$$a_m = \frac{1}{6} + \frac{1}{2(i\pi)^2} \quad \text{if } m = i.$$

It will be sufficient for the purposes of this thesis to tabulate a_n for values of i and m up to 5.

TABLE VI.

Values of i.

m.	1.	2.	3.	4.	5.
1.	0.2173	0	-0.038	0	-0.007
2.	0	0.1793	0	-0.0455	0
3.	-0.0380	0	0.1722	0	-0.0475
4.	0	-0.0455	0	0.1698	0
5.	-0.007	0	-0.0475	0	0.1686

2.06. It will also be necessary to represent the function

$$\text{Sec}^5\psi_0 - \text{Sec}^5\psi$$

in terms of a Sine Series. The influence of the term containing the function on the equation of the suspension bridge is quite small, and for all practical purposes it is sufficient to compute this term for a parabolic cable curve.

Then

$$y = \frac{4f}{l^2}(lx - x^2) = 4f(Z - Z^2)$$

$$\frac{dy}{dx} = \frac{dy}{l\,dz} = \tan\psi = \frac{4f}{l}(1 - 2Z).$$

The analytic determination of the harmonic coefficients has not proved practicable so a numerical method has been used. For three values of $\frac{f}{l}$ the value of the function, $\mathrm{Sec}^5\psi_0 - \mathrm{Sec}^5\psi$ has been computed for twenty four points in main span and the sine series coefficients determined by standard harmonic analysis.

These coefficients are plotted against f/l in Fig.2.04. Two coefficients are given. No great accuracy is warranted in these factors as the term which involves them is relatively unimportant.

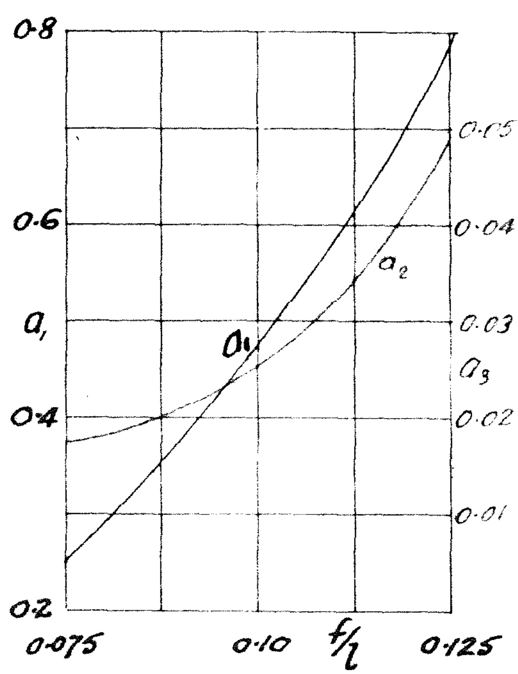


Fig. 2.04

2.06. Deflection of a Simple Beam.

Consider the simple beam under any loading which produces a moment diagram defined by M.

We may express M as follows

$$M = \sum_{n=1}^{\infty} a_n \, \mathrm{Sin} \, \frac{n\pi x}{l} \qquad 2.01.$$

Furthermore, we may write the deflection

$$y = \sum_{n=1}^{\infty} b_n \mathrm{Sin} \, \frac{n\pi x}{l} \qquad 2.02.$$

From the relation

$$-EI \frac{d^2y}{dx^2} = M \qquad 2.03.$$

we obtain, using 2.01 and 2.02

$$\frac{\pi^2}{l^2} EI \sum_{n=1}^{\infty} n^2 b_n \, \mathrm{Sin} \, \frac{n\pi x}{l} = \sum_{n=1}^{\infty} a_n \, \mathrm{Sin} \, \frac{n\pi x}{l} \qquad 2.04.$$

Equating coefficients of like terms in 2.04 we derive

$$b_n = \frac{l^2}{EI\pi^2 n^2} \cdot a_n \qquad 2.05.$$

Now, a_n is easily found for all the usual loadings, from the expansions of Fig.2.03.

$\frac{\pi^2 EI}{l^2}$ is a property of the beam and n is an integer. Hence, the derivation of the deflection of a beam is extremely simple.

This approach avoids entirely the introduction of the "virtual work" concept and the consequent integration of virtual work as developed by Timoshenko in reference (10).

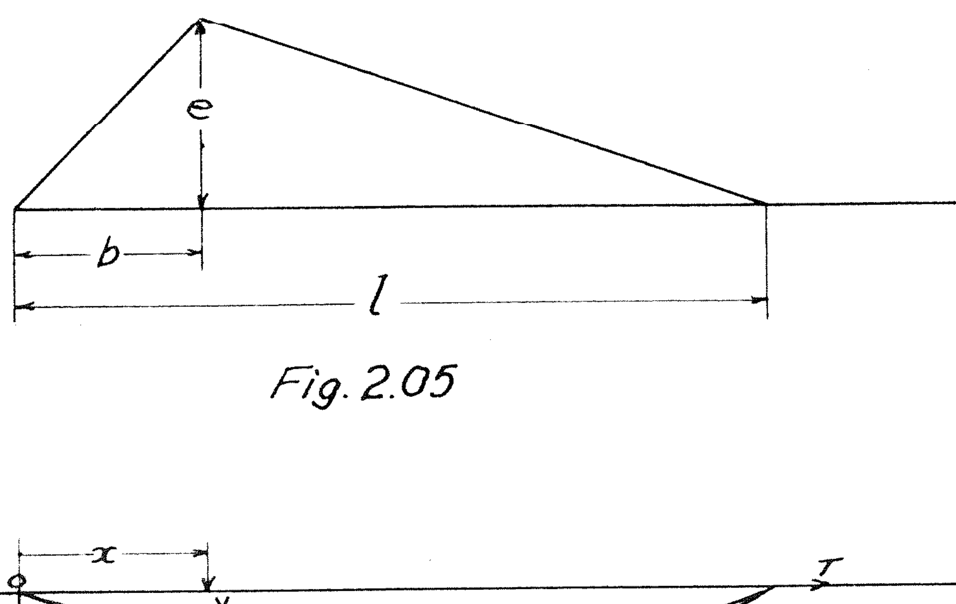


Fig. 2.05

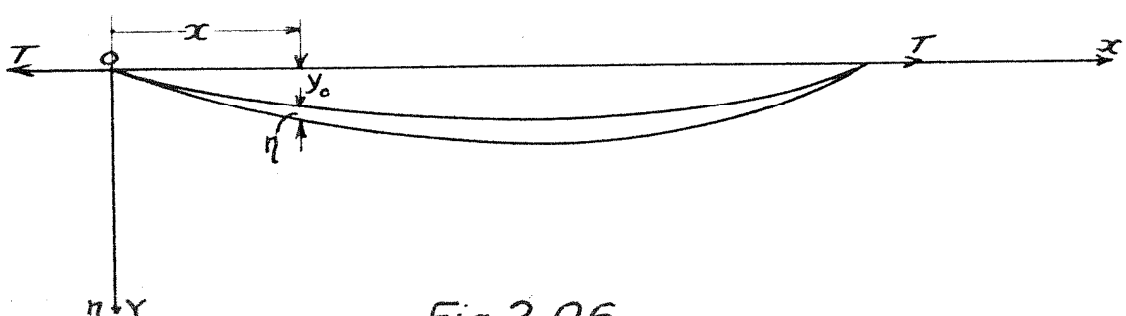


Fig 2.06

The expansion for the figure shown in Fig.2.05 is, from Fig.2.03,

$$y = \frac{2e\,l^2}{\pi^2 b(l-b)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi b}{l} \sin\frac{n\pi x}{l}$$

For a load P at x = b $\qquad M_b = \frac{(l-b)b}{l} . P = e$

$\therefore$

$$M = \frac{2l}{\pi^2} . \; P \sum \frac{1}{n^2} \sin\frac{n\pi b}{l} \sin\frac{n\pi x}{l}$$

$$b_n = \frac{l^2}{EI\pi^2 n^2} \quad \frac{2lP}{\pi^2} \quad \frac{1}{n^2} \sin\frac{n\pi b}{l}$$

$$= \frac{2Pl^3}{\pi^4 EIn^4} \sin\frac{n\pi b}{l}$$

which result is identical with that obtained by Timoshenko (10).

The elegance of the trigonometric series method is even more evident here than in Timoshenko's derivation.

2.07. Deflections or Stresses in Beams or Plates.

1. Edges simply supported.

The deflection of the plate if initially straight can be represented by the series

$$\eta = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

where the origin of x and y is as shown in Fig.2.06.

If the plate is initially bent to the curve $y_o = \sum_{n=1}^{\infty} e_n \text{Sin} \frac{n\pi x}{l}$ the total deviation from the x axis under load is

$$y = y_o + \eta = \sum_{n=1}^{\infty} e_n \text{ Sin } \frac{n\pi x}{l} + \sum_{n=1}^{\infty} a_n \text{ Sin } \frac{n\pi x}{l} \qquad 2.06.$$

The external moment, M_s, due to the load can be similarly expressed as

$$M_s = \sum_{n=1}^{\infty} b_n \text{ Sin } \frac{n\pi x}{l} \qquad 2.07.$$

If the ends are not free to approach each other without restraint a tension, T, exists at the supports.

Equating moments at a section distant x from the origin we have $M_s - T(y_o + \eta) = -EI \frac{d^2\eta}{dx^2}$
or substituting from 2.06 and 2.07

$$\sum_{n=1}^{\infty} b_n \text{ Sin } \frac{n\pi x}{l} - T(y_o + \eta) = EI \frac{\pi^2}{l^2} \sum_{n=1}^{\infty} n^2 a_n \text{Sin } \frac{n\pi x}{l}$$

And equating coefficients we obtain

$$b_n - T(e_n + a_n) = \frac{\pi^2 EI}{l^2} n^2 a_n$$

whence

$$a_n = \frac{b_n - Te_n}{\frac{\pi^2 EI}{l^2} n^2 + T} \qquad 2.08.$$

T is determined from the condition of restraint of the longitudinal dimensions of the member.

2. For the plate with clamped edges we must introduce the restriction that the external bending moment due to the load is symmetrical about the centre line of the plate.

It is then possible to express the deflection due to load $\eta = \sum_{n=0}^{\infty} a_n \text{ Cos } \frac{n\pi x}{l}$ and the symmetry of the problem excludes at once the odd values of n.

$$\frac{d\eta}{dx} = -\frac{\pi}{l}\sum_{0}^{\infty} n\, a_n \operatorname{Sin}\frac{n\pi x}{l}$$

$$\frac{d^2\eta}{dx^2} = -\frac{\pi^2}{l^2}\sum_{0}^{\infty} n^2\, a_n \operatorname{Cos}\frac{n\pi x}{l}$$

Further we know that, in order that $\eta = 0$ at $x = 0$ and $x = l$

$$\sum_{0}^{\infty} a_n = 0 \tag{2.09.}$$

The static bending moment diagram can be expressed in a series of the form

$$M_s = \sum_{n=0}^{\infty} b_n \operatorname{Cos}\frac{n\pi x}{l}$$

and the clamped edge conditions prescribe a fixing moment M_f.

If the supports are restrained longitudinally a tension T is induced in the member.

Equating moments at section distant x from the left hand support

$$M_s - M_f - T.\eta = -\ EI\frac{d^2\eta}{dx^2}$$

Substituting for η and M_s we obtain

$$\sum_{n=0}^{\infty} b_n \operatorname{Cos}\frac{n\pi x}{l} - M_f - T\sum_{0}^{\infty} a_n \operatorname{Cos}\frac{n\pi x}{l} = \frac{\pi^2 EI}{l^2}\sum_{0}^{\infty} n^2 a_n \operatorname{Cos}\frac{n\pi x}{l} \tag{2.10}$$

Equating coefficients we have

$$b_o - M_f - T\, a_o = 0 \tag{2.11}$$

$$b_n - T\, a_n = \frac{\pi^2 EI}{l^2} n^2\, a_n$$

or

$$a_n = \frac{b_n}{\frac{\pi^2 EI}{l^2}n^2 + T} \tag{2.12.}$$

3. The change in length of the member due to the curvature is given by (10)

$$\Delta l = \frac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx = \frac{\pi^2}{2l^2}\int_0^l \left(\sum_0^\infty n\, a_n \sin\frac{n\pi x}{l}\right)^2 dx$$

$$= \frac{\pi^2}{4l}\sum_0^\infty n^2 a_n^2.$$

If the supports do not approach each other the stretch is

$$\Delta l = \frac{T.l}{EA}$$

$$\therefore \quad T = \frac{\pi^2 EA}{4l^2}\sum_0^\infty n^2 a_n^2 \qquad 2.13.$$

For thin plates EI is taken as zero to obtain a first value for T. In the case of thick plates T is taken as zero. This first value is used to obtain a new set of coefficients. The procedure converges.

2.08. The method of approximate solution exemplified by the content of sections 2.06 and 2.07 can be applied to a wide variety of problems in structural mechanics. It is a variation of the trigonometric series method developed by S. Timoshenko and differs in the method of determination of the coefficients of the series representing the deflection curve.

All plate and beam bending problems of Chapter I, reference(11) can be treated effectively by this method, and all the beam-column formulae which have been treated by trigonometric series using the energy method can be simply derived by this alternative procedure.

At first glance there seems to be no advantage to be gained by the variation. However, in the usual method, the determination of the coefficients requires the evaluation of integrals representing the internal strain energy and, except in a few cases, this is not easily done.

Furthermore, the technique of equating coefficients permits us to introduce arbitrary differences in the initial conditions, e.g. the initial bent shape of the beam of section 2.07, by simply varying a coefficient, and the most advantageous variations are at once evident from the form of the expansion of the applied bending moment.

This advantage is much more in evidence when two members participate in supporting a load as in the flitched beam, deck and rim in an arch, and, of more immediate interest, the cable and girder of the suspension bridge.

If for some reason, e.g. hanger elongation in the suspension bridge, the second supporting member deflects more or less than the first by a determinable amount it is very simply taken into account by writing in the appropriate harmonic coefficients in the static equilibrium equation as typified by equation 2.10.

It may be desirable to investigate the possibilities of applying a prestress in the stiffening girder of a suspension bridge in order to relieve a member which is overloaded. This can be done fairly readily by adjustment of the suspender lengths. The critical load conditions are set up in the form of a static moment equation with all terms expanded in harmonic series. Then it will be evident that a certain harmonic has greater effect at the point considered than others and harmonic prestress of the same order but of opposite sign is most effective in reducing the critical conditions.

2.10. The Deflection Theory ordinarily takes no account of the difference in the deflection curves of the cable and stiffening girder due to the longitudinal shift of the cable bands. In the present section it is proposed to present another approach to this simple theory for the main span, of a suspension bridge. The development of the equations for a side span follows the same pattern closely and is not presented in this work.

We accept the assumptions (2) to (6) of section 1.01 which involve, by assumption (5), the condition that the supports of the cable are vertically above the corresponding girder pin joints. The fundamental equation of the Deflection Theory can then be written down briefly for a point distant x from the origin(Fig.2.07).

$$M_t = M - (H + H_w)\ (y + \eta_c) \qquad 2.14$$

where M_t = bending moment supported by the girder.

$M = M_L + M_d$ = bending moment due to all loads in the span.

y = initial cable ordinate under dead load above .

η_c = cable deflection due to added live load.

H_w = Horizontal tension due to dead load above.

H = change in horizontal tension due to change in conditions of load and deflection.

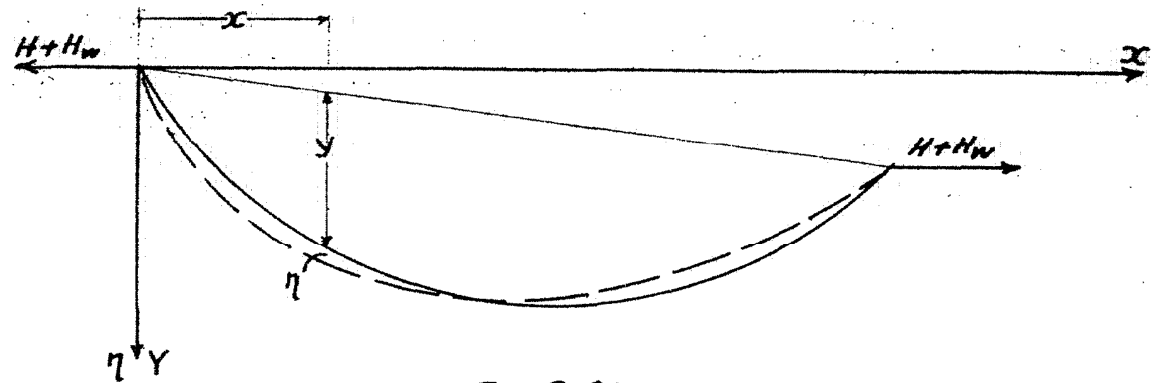


Fig. 2.07

From assumption (2), section 1.01, the dead load bending moment, M_d, was carried initially by the cable alone, whence

$$M_d = H_w . y$$

and equation 2.14 can be reduced to the familiar equation

$$M_t = M_L - H.y - (H + H_w)\eta_c \qquad 2.15.$$

From the theory of bending with the usual convention of signs for bending moments in the girder

$$M_t = - EI \frac{d^2\eta_t}{dx^2} \qquad 2.16.$$

where η_t is the truss deflection corresponding to, but not necessarily equal to η_c. By substituting from 2.16 in 2.15 and transposing there results the fundamental equation of this section*

$$EI \frac{d^2\eta_t}{dx^2} - Hy - (H + H_w)\eta_c = - M_L \qquad 2.17.$$

No restrictions have been placed on the form of the cable curve as defined by y in equation 2.17. This is determined by the dead load distribution in the span. Furthermore, the form of equation 2.17 is such that it is possible to introduce arbitrary differences between the corresponding values of η_c and η_t to permit investigations of prestress and other sources of difference between cable and girder deflection.

In the study of prestress, the effect of an arbitrary change in suspender lengths is included by writing

$$\eta_t - \eta_c = f(x) \qquad 2.18.$$

In the study of errors in theory due to the longitudinal shift of the cable bands we begin by accepting the approximation $\eta_t \approx \eta_c$ from the first order theory. By proper manipulation it is possible to calculate from the deflection η_t the true deflection of the cable. The values of $\eta_t - \eta_c$ can then be determined and introduced as a prestress to determine closer values.

Regardless of the dead load distribution it will be possible to express the cable ordinate y as a sine series to any degree of accuracy required for computation. Some idea of the number of terms necessary for the usual accuracy of structural calculations can be obtained from a comparison of the ordinates of a parabola as derived from the exact expression

$$\frac{y}{f} = 4(Z - Z^2)$$

and from the expansion

$$\frac{y}{f} = \frac{32}{\pi^3} \sum_{n = 1,3,5,7,}^{\infty} \frac{1}{n^3} \cdot \text{Sin } n\pi z$$

$$= 1.031 \text{ Sin}\,\pi Z + 0.0382 \text{ Sin}\,3\pi Z + 0.0082 \text{ Sin}\,5\pi Z + \ldots$$

The convergence is evidently very rapid.

The dead load configuration of the cable is written then as

$$\frac{y}{f} = \sum_{n=1}^{\infty} C_n \text{ Sin } \frac{n\pi x}{l} = \sum_{n=1}^{\infty} C_n \text{ Sin } n\pi Z \qquad 2.19.$$

where the coefficients ~~of~~ C_n are determined by the usual methods for Fourier series. For symmetrical spans only odd terms appear in equation 2.19.

In the same manner the curve of bending moment due to live loads can be expressed as a sine series, and for all the usual loads encountered in design, Fig.2.03 gives the coefficients in terms of the loads and dimensions of the span. For the purposes of a general discussion the live load bending moment will be written

$$\frac{M_L}{H_w f} = \sum_{n=1}^{\infty} a_n \operatorname{Sin} \frac{n\pi x}{l} = \sum_{n=1}^{\infty} a_n \operatorname{Sin} n\pi Z \qquad 2.20.$$

A convenient and simple expression for f(x) in equation 2.18 is written as follows

$$\frac{\eta_t - \eta_c}{f} = \sum_{n=1}^{\infty} \frac{d_n}{f} \operatorname{Sin} \frac{n\pi x}{l} = \sum_{n=1}^{\infty} \frac{d_n}{f} \operatorname{Sin} n\pi Z \qquad 2.21.$$

and usually this equation will be limited to a single term and always to odd terms for symmetrical spans. then

$$\frac{\eta_c}{f} = \frac{\eta_t}{f} - \sum_{n=1}^{\infty} d_n \operatorname{Sin} n\pi Z \qquad 2.22.$$

Substituting from equations 2.19, 2.20, and 2.22 in equation 2.17, the following expression for η_t is obtained. It should be noted that EI is assumed constant.

$$\frac{d^2}{dZ^2}\left(\frac{\eta_t}{f}\right) - (1+\beta)\pi^2\alpha\frac{\eta_t}{f} = \pi^2\alpha\left[\beta \sum_{n=1}^{\infty} C_n \operatorname{Sin} n\pi Z - \sum_{n=1}^{\infty} a_n \operatorname{Sin} n\pi Z - (1+\beta)\sum_{n=1}^{\infty} d_n \operatorname{Sin} n\pi Z\right] \qquad 2.23$$

where

$$\alpha = \frac{H_w l^2}{\pi^2 EI} \quad \text{and} \quad \beta = \frac{H}{H_w}$$

All the quantities on the right-hand side, except β, are known from the loads and dimensions of the structure. β must be determined from the equation of continuity of the cable with the foundations and will be discussed in the next section.

If an expression for η_t is assumed of the form

$$\frac{\eta_t}{f} = \sum_{n=1}^{\infty} D_n \operatorname{Sin} \frac{n\pi x}{l} = \sum_{n=1}^{\infty} D_n \operatorname{Sin} n\pi Z \qquad 2.24$$

the boundary conditions for the girder and cable are satisfied identically. Then

$$\frac{d^2}{dZ}\left(\frac{\eta_t}{f}\right) = -\pi^2 \sum_{n=1}^{\infty} n^2 D_n \operatorname{Sin} n\pi Z \qquad 2.25.$$

Substituting from 2.25 and 2.24 in equation 2.23 and equating the coefficients of terms of the same order, the coefficients D_n in 2.25 can be evaluated.

$$D_n = \frac{\left[a_n - \beta C_n + (1+\beta)\, d_n\right]}{1 + \beta + \frac{n^2}{\alpha}} \qquad 2.26.$$

β is the only unknown in equation 2.26 and it occurs in both the numerator and denominator. Evidently the coefficients for the serie for deflection, equation 2.24, are not linear functions of β. As the value of the $\frac{n^2}{\alpha}$ in the denominator becomes large, the influence of β on the denominator decreases. This occurs if α is small which defines a structure with a relatively stiff girder.

For α very small equation 2.26 shows deflections are linear functions of the applied bending moment. Since bending moments are determinable by the method of superposition and β multiplies the cable curve coefficients in the numerator, which is effectively a constant for given design dimensions, the only bar to the use of the method of superposition in the determination of deflections is the β in the denominator. When α is very small the term $\frac{n^2}{\alpha}$ will become large and β may be neglected in the denominator.

A criterion is now available for determining the applicability of the "Elastic Theory". First order harmonics are not usually very important so we will consider the second harmonic which makes $n^2 = 4$. Then $\alpha = 2$ makes the denominator $3 + \beta$, and β may be 0.3. The neglect of β in the denominator will mean an error of 10% in the second order harmonic of the deflection curve. In the first harmonic there will be an error of about 16% and in harmonics above the third a negligible amount. Table IX shows that for modern bridges, especially those for highway traffic alone, the value of α is much larger than 2, and in some cases so large that the term in α can be neglected. It will not be satisfactory in such cases to use the "Elastic Theory" and the method of Influence Lines, except for qualitative discussions. Where the ratio of live load, w_l, to dead load, w_d, becomes small with α remaining large, β may always be small and the problem again approaches linearity.

Leaving prestress out of account the equation for coefficients of deflection of the girder will be given by

$$D_n = \frac{a_n - \beta C_n}{1 + \beta + \frac{n^2}{\alpha}} \qquad 2.26(a).$$

2.11. For the evaluation of β it may be assumed that the initial cable curve is sufficiently close to parabolic to justify the use of equation (37) of Part I where ν, corresponds to η_c of equation 2.22. From that equation

$$\beta = \frac{H}{H_w} = \frac{EA\,l}{H_w L_c}\left[\frac{8f^2}{l^2}\int_0^1 \frac{\eta_c}{f}\,dZ - \omega t\right] \qquad 2.27,$$

and applying equations 2.22 and 2.24

$$\beta = \gamma\left[\frac{8f^2}{l^2}\int_0^1 (D_n - d_n)\ \mathrm{Sin}\ n\pi Z\ dZ - \omega t\right] \quad \text{where } \gamma = \frac{EA\,l}{H_w L_c}$$

which after integration yields

$$\beta = \gamma\left[\frac{8f^2}{\pi l^2}\sum_{n=1}^{\infty}\frac{(D_n - d_n)}{n}(1 - \mathrm{Cos}\ n\pi) - \omega t\right] \qquad 2.28.$$

If prestress is left out, the equation 2.28 becomes

$$\beta = \gamma\left[\frac{8f^2}{\pi l^2}\sum_{n=1}^{\infty}\frac{D_n}{n}(1 - \mathrm{Cos}\ n\pi) - \omega t\right] \qquad 2.28(a).$$

It should be noted that $(1 - \mathrm{Cos}\ n\pi)$ vanishes for even values of n so only odd terms occur in the evaluation of β. This is a useful simplification.

A rise of temperature or a decrease in span are considered positive in the application of equation 2.28.

2.12. Equations 2.28 and 2.26 together enable us to determine the curve of deflection of the stiffening girder for any specified load and dimensions. Evidently for a particular case β must be determined by trial, and it is not difficult to guess a sufficiently good value of β to determine first values of D_n for use in evaluation of β from equation 2.28. This new value of β

is then used to obtain new values of D_n for a better approximation, and so on.

However, the value of β depends also on the tower top deflection due to side span interaction, and it must be chosen so that for a given loading condition the value of β in the main span differs from the value of β in the sidespan by the small proportion of the dead load horizontal tension necessary to deflect the tower. When the cables are employed they also affect the value of β .

In order to satisfy these various conditions it is necessary in practice to determine β for various changes in span length for main spans and sidespans, and plot β against span change. Under these circumstances the evaluation of β no longer presents any difficulty if we proceed in reverse order thus: For various values of β, D_n is evaluated from equation 2.26 and substituted into equation 2.28, which is then solved for ωt. It has been stated in section 1.07 that ωt can be regarded as the result of temperature change, span change, or both together.

The valuesof ωt thus obtained are plotted against β . In the case where the structural action is such that $\omega t = 0$ the correct value of β to satisfy 2.26 and 2.28 is given by the intersection of the plot with the β-axis.

For the use of such plotted curves when side span interaction and tower stiffness exists, the reader is referred to R.A.Tudor's discussion of the "Modified Dana Method" in Reference 2. It will be evident that the form of equations 2.26 and 2.28 is very convenient for use in this manner.

2.13. An example is presented here to illustrate the use of the results of sections 2.10 and 2.11. At the same time, the example has been taken from reference (8) so that a comparison can be made with the well-known trigonometric series' method of Priester and Timoshenko.

Manhattan Bridge.

$\alpha = 1.746$ $\quad H_w = 10.48 \times 10^6$ lb. $\quad w_l = 4084$ lb/ft.

$\frac{f}{l} = 0.1004$ $\quad l = 1.447$ ft. $\quad \beta = 0.10$

$\gamma = 704$ $\quad \omega = 6.6 \times 10^{-6}$ per^o F.

The live load, w_l, is applied over the first quarter of the span, then, in Fig.2.03, (3), $\frac{a}{l} = 0$ and $\frac{b}{l} = \frac{1}{4}$. Therefore

$$\frac{M_L}{H_w f} = 0.3614 \sum_{n=1}^{\infty} \frac{1 - \cos\frac{n\pi}{4}}{n^3} \sin n\pi Z$$

whence $a_n = 0.3614 \dfrac{\left(1 - \cos\frac{n\pi}{4}\right)}{n^3}$.

The cable shape is the expansion of a parabola and

$$C_n = \frac{1.032}{n^3} \text{ for n odd only.}$$

No prestress is involved so D_n is determined from equation 2.26(a) for the assumed value of β . The values of D_n thus obtained are used in equation 2.28(a) to find the value of ωt corresponding to the assumed value of β . Table VIII shows the computations for one value of β .

TABLE VIII

$\beta = 0.1$

n	C_n	α_n	βC_n	D_n	$\frac{D_n}{n}(1-\cos n)$
1	1.0323	0.1058	0.10323	0.001536	0.003072
2	0	0.0452	0	0.01332	0.0000
3	0.0382	0.02283	0.00382	0.003039	0.002026
4	0	0.01129	0	0.001101	0.00000
5	0.00825	0.00493	0.00082	0.000266	0.000106
6	0	0.00167	0	0.0000769	0.00000
7	0.0030	0.00031	0.00030	0.0000003	0.00000
					0.005,204

Computed t = -1.29° F

$\beta = 0.101$

n	C_n	α_n	βC_n	D_n	$\frac{D_n}{n}(1-\cos n)$
1	1.0323	0.10580	0.10426	0.00092	0.001,840
2					
3	0.0382	0.0228	0.00386	0.003026	0.002017
4					
5	0.00825	0.00493	0.00083	0.000266	0.000,106
					0.003963

Computed t = -6.3° F

By linear interplation, $\beta = 0.0997$ for t = 0

$\beta = 0.0997$

1	1.0323	0.1058		0.001732
2	0	0.0452		0.01332
3	0.0382	0.02283		0.003039
4	0	0.01129		0.001101
5	0.00825	0.00493		0.000266

The results agree with the " H - approximate " values tabulated by Priester (reference (8)) and lie a small percentage below his " H- exact " values. However, the developments of Part I of this thesis show that the equation from which β is derived in this work is highly accurate and it takes into account the actual structural behaviour of the cable.

It would seem that the " H - approximate " values of reference (8) are more near the truth than the " H - exact" values. On the basis of the present work the writer feels that the method of this text is most reliable for general use, although for particular cases it can be shown that the methods of references (6) and (8), and section 2.11, will agree within 3%. It is not possible to generalise this statement without extensive computations.

For stiff bridges, into which class the example falls, three terms prove sufficient, but for more flexible bridges a fourth and perhaps a fifth term may be used. The calculations are very simple and involve no difficulties when used with Tudor's procedure(reference 2). Table IX gives the basic design parameters for a few modern bridge spans.

2.14. For modern suspension bridges α can be large and equation 2.26 shows that when α is large the term which it influences loses importance. This leads to the deduction that any attempts to reduce girder stress by local increases of the moment of inertia

TABLE IX

NAME	W_D	W_l	W_l/W_D	l	F	I	E-TRUSS	E-CABLE	H_W	A	F/l	α	ϕ	γ
TRIBOROUGH	9,580	2,000	0.209	1,358	133.7	15,000	29×10^6	28×10^6	16.45×10^6	276.6	0.0985	7.07	0.0206	
SAN FRANCISCOBAY	9,200	3,000	0.326	2,294	228.7	69,500	29×10^6	28×10^6	25.81×10^6	500	0.0997	6.81	0.0325	
MANHATTAN	5,820	4,000	0.687	1,447	145.3	43,900	29×10^6	29×10^6	10.48×10^6	275	0.1004	1.746	0.0690	704
MAUMEE RIVER	6,500	2,000	0.308	777.6	96.0	2,600	29×10^6	27×10^6	5.12×10^6	106.6	0.1234	4.16	0.0380	
MT. HOPE	2,650	750	0.283	1,188	118.8	4,259	29×10^6	29×10^6	3.94×10^6	73.9	0.100	4.55	0.0283	
MOLINE, ILL.	2,185	1,120	0.512	735	73.0	1,835	29×10^6	24×10^6	2.03×10^6	44.9	0.0994	2.08	0.0509	
GOLDEN GATE	10,650	2,000	0.188	4,200	470.0	43,150	29×10^6	29×10^6	50.00×10^6	860.0	0.1118	71.4	0.0210	430
TACOMA	2,850	500	0.175	2,800	232.0	1,018	29×10^6	28×10^6	14.00×10^6	191.5	0.0828	377.0	0.0145	
BRONX-WHITESTONE	5,500	1,500	0.273	2,300	200.0	2,930	29×10^6	28×10^6	18.20×10^6	297.0	0.0870	114.7	0.0237	
WASHINGTON	15,790	2,000	0.1266	3,500	319.2	168	29×10^6	28×10^6	75.80×10^6	1,598.0	0.0912	119,000.0	0.0115	
DEER ISLAND	1,200	350	0.292	1,080	108.0	424	29×10^6	24×10^6	1.89×10^6	25.5	0.100	18.14	0.0292	
THOUSAND IS.	1,800	500	0.278	800	80.0	252	29×10^6	24×10^6	1.80×10^6	35.5	0.100	15.93	0.0278	

is of no great avail. This point cannot be over - emphasised for too often in the reports of constructed bridges one notes the statement that the girder chords are proportioned to take the bending moment at the section. On the contrary, if attempt is made to reduce stress by increasing the chord area the result will be weight increase and practically no change in chord stress which is completely undesirable. If, instead, the girder depth is increased, then the result will be an increase of chord stress which degrades the design.

In spans with large values of α, then, the girder depth defines the maximum chord stress and any attempt to proportion the chords to the bending moment is futile. The Golden Gate, Tacoma, andBronx - Whitestone Bridges are in this class.

Spans with smaller values of α, say from 1 to 50, which includes a large class in Table IX, can be designed with variable chord sections to some advantage and means of evaluating the influence of variable α will be developed in later sections. Better results in this direction can be attained with the lower values of α , because as α increases the effect of a given increase of moment of inertia rapidly decreases.

A given amount of chord material should, for maximum effectiveness, be arranged with the greatest possible depth of truss consistent with the stress limitation of the material. Since, for construction reasons, the chords are generally parallel, the moment at one section determines the chord depth, and the stress at all lower order maxima will be less than the allowable.

One can with advantage introduce a prestress which will reduce the girder bending at the first order maximum and increase it at others, and as a result, the chord depth can be increased. It is to permit the calculation of the influence of prestress that the additional term, d_n, has been included in equation 2.26 and 2.28.

The procedure is as follows: All maximum moments are calculated assuming no prestress, and experience shows that there will be positive and negative maxima at the center and near the quarter points. Positive and negative maxima at a given point will not in general be equal, and it will be found that suitably chosen first order and third order deflection curves will help to equalise the positive and negative values at each point, and the values at the quarter point and the centre. Even order prestressing deflections will not appear because of symmetry, and higher orders than the third are unwarranted.

Thus the values of d_1 and d_3 are determined and resubstituted in equations 2.26 and 2.28 to check the work. The effect of the prestress terms on β will be small but in any case any discrepancy is readily adjusted by carrying through again the calculations of β for the critical loadings.

The prestress is introduced during or after completion of construction by adjustment of suspender lengths and involves no difficulties. Some guidance in estimating the desirable amount of prestress can be had from the curves of reference (7) but in general the proper maxima must be calculated for the particular case because of sidespan and temperature effects.

2.15. The emphasis in all the preceding sections has been placed on deflections because in suspension bridges permissible deflections (or permissible flexibility) controls the design. The girder proportions are determined by the moment of inertia necessary to limit deflections to the chosen values, and the allowable stress which depends on the depth. The dependent variable is then the area of the chord cross-section and reduction in that area can be achieved solely by increase of allowable stress which permits a greater depth for a given value of I. A proper recognition of this philosophy is necessary for intelligent design of such bridges.

Equation 2.26 enables us to develop a pictorial representation of the influence of the several design variables on the flexibility of the structure. Usually a span length and traffic capacity are the first known quantities, and the representation developed below will enable us to determine the influence of the design parameters f/l , $\frac{w_L}{w_d}$, and α on flexibility for any definite live load, W_L, and span l .

The type of loading assumed is such that only the second order harmonic deflection curve appears. This is an entirely fictitious loading that gives β = 0 from equation 2.28(a). Since we seek only the influence ~~the influence~~ of parameters on flexibility the fictitious character of the loading assumed is of no great importance. We may write then for the quarter point

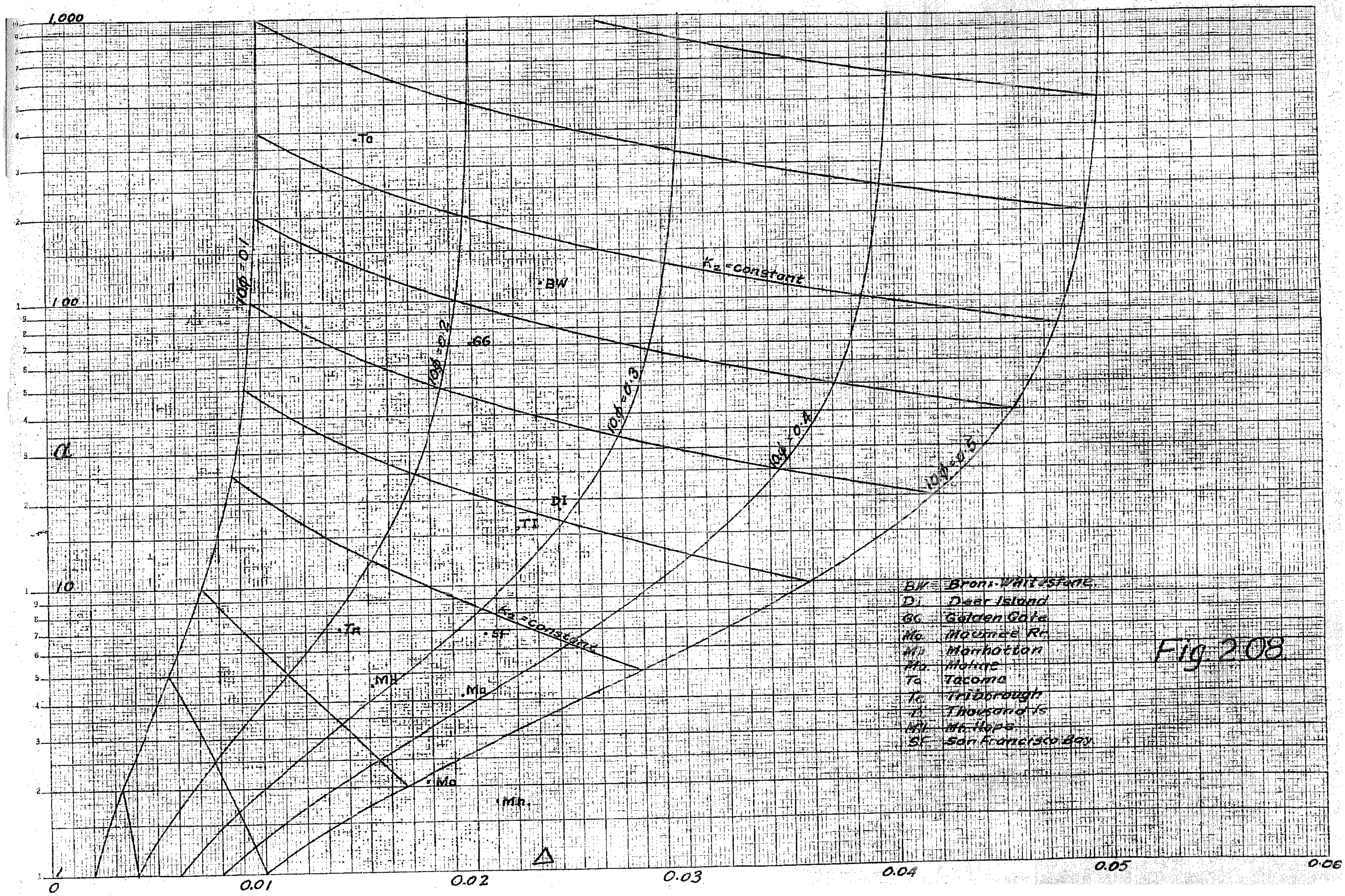

1,000
100
10
1
α
0
0.01
0.02
0.03
0.04
0.05
0.06
Δ
10φ = 0.1
10φ = 0.2
10φ = 0.3
10φ = 0.4
10φ = 0.5
K₂ = constant
K₂ = constant
Ta
BW
GG
DI
TI
Tr
SF
MH
Ma
Mo
Mn
BW Bronx-Whitestone
DI Deer Island
GG Golden Gate
Mo Maumee Rr.
Mn Manhattan
Ma Moline
Ta Tacoma
Tr Triborough
TI Thousand Is.
MH Mt. Hope
SF San Francisco Bay

Fig. 2.08

$$\frac{\eta_c}{f} = D_2 = \frac{a_2}{1 + \frac{4}{\alpha}} = \frac{M_L}{H_w f} \; \frac{1}{1 + \frac{4}{\alpha}}$$

$$\frac{\eta_c}{l} = \frac{M_L}{H_w l} \quad \frac{1}{1 + \frac{4}{\alpha}} = K_1 \cdot \frac{w_l f}{w_d l} \cdot \frac{1}{1 + \frac{4}{\alpha}}$$

$$\Delta = \frac{\eta_c}{K_1 l} = \frac{w_l}{w_d} \cdot \frac{f}{l} \cdot \frac{1}{1 + \frac{4}{\alpha}} \qquad 2.29.$$

where K_1 is a constant whose numerical value is of no consequence in the present discussion.

$$\alpha = \frac{H_w l^2}{\pi^2 EI} = \frac{w_d l^4}{8f\pi^2 EI} = \frac{w_d}{w_l} \cdot \frac{l}{f} \cdot K_2 = \frac{K_2}{\phi} \qquad 2.30.$$

where $K_2 = \dfrac{w_l l^3}{8\pi^2 EI}$

From equation 2.29 Δ is evaluated and platted against α for various values of ϕ in Fig. 2.08. The K_2 = constant lines in Fig. 2.08 are determined from equation 2.30 and connect points related by fixed values of w_l, l, and EI because K_2 is then constant, and for any two values of ϕ, equation 2.30 gives

$$\frac{\phi_1}{\phi_2} = \frac{\alpha_2}{\alpha_1} \qquad 2.31.$$

The effect on flexibility of a proposed change in the parameter $\frac{w_l}{w_d} \cdot \frac{f}{l} = \phi$ due to a change in dead load or sag is estimated by locating the point on Fig.2.08 corresponding to the existing design, then following the K_2 - lines to the new proposed value of ϕ and reading off the value of Δ.

The influence of an increase in stiffening by increasing EI without change in dead load, can be found by following the full lines to the point determined by the new value of α, and reading off the corresponding value of Δ. Mixed changes are treated in two steps by taking the changes separately and consecutively.

2.16. The theory of section 2.10 leads to an expression for the curve of deflection of the cable, in terms of an infinite sine series whose coefficients are known. These coefficients do not decrease in value very rapidly, compared with those obtained by the method of variation of strain energy. It will be found that usually two more terms will be needed to obtain satisfactory approximation but this is a small disadvantage compared with the gain in flexibility of this alternative technique.

Mathematically speaking it is possible to improve the convergence of the coefficients in the manner of Karman in reference (14), which is equivalent in our case, to separating equation 2.26 into two parts; one part is the deflection of the unstiffened cable under load, and the other is a correction to be subtracted due to the action of the stiffening girder. Theoretically, the first part is simply a matter of statical equilibrium and easily evaluated algebraically, and the second part will be small and rapidly convergent if the girder is relatively flexible, which is usually the case in modern bridges.

However as a matter of computation, which, after all, is the final destiny of structural theory, it is much easier to evaluate up to ten terms of a sine series and add the results, than it is to evaluate the algebraic expressions for the static deflection curve of a cable and two or three correction terms. The first is simply a matter of reading off tabulated values of the sine and multiplying by a coefficient and the coefficients are determined without any special difficulty. For all usual load distributions only four or five terms of the series are necessary for satisfactory accuracy, so the writer has not included here any mathematical rearrangements of the basic equation. From the standpoint of office computation the form given is ideal for calculation and checking, and has the further advantage that a single series represents the deflection curve for the whole span.

Very stiff bridges, which are characterised by low values of α, produce a satisfactory rate of convergence with few terms, because then the term in α, strongly influences the denominator of equation 2.26. Flexible bridges such as the Tacoma or Bronx-Whitestone are scarcely influenced by the term in α. A few constructed bridges are marked up in Fig. 2.08 according to their parameters ϕ and α.

2.17. Although deflections have become the foremost design considerations in suspension bridges and are dependent on the parameters α and ϕ, it is still necessary to determine the maximum bending moments in the girder in order to determine the maximum permissible girder depth(and consequently the cross - sections) consistent with the chosen values of moment of inertia and maximum chord stress. We repeat that the designer must not mistakenly attempt to design for the maximum bending moment of all sections as for a simple girder.

Rewriting equation 2.16

$$\frac{M_T}{EI} = - \frac{d^2\eta_t}{dx^2}$$

and introducing dimensionless variables

$$\frac{M_T}{EIf} = - \frac{d^2}{dZ^2} \left(\frac{\eta_t}{f} \right) \qquad 2.32$$

and using equation 2.25 the expression for bending moment in the girder is

$$\frac{M_T l^2}{EIf} = \sum_{n=1}^{\infty} \pi^2 m_n \operatorname{Sin} \frac{n\pi x}{l} = \sum_{n=1}^{\infty} \pi^2 m_n \operatorname{Sin} n\pi Z = \sum_{n=1}^{\infty} n^2\pi^2 D_n \operatorname{Sin} n\pi Z \qquad 2.33.$$

from which we obtain, by equating coefficients

$$m_n = n^2 D_n \qquad 2.34.$$

which enables us to determine the truss bending moment curve at once from

$$\frac{M_T l^2}{\pi^2 EIf} = \sum_{n=1}^{\infty} m_n \operatorname{Sin} n\pi Z \qquad 2.35.$$

The remarks on convergence of section 2.16 apply also to equation 2.34 of the present section. However it is found that five terms usually are sufficient and calculations for additional terms are simple as is evident from equations 2.34 qnd 2.26. After determination of the desirable prestress only the term of the order of the prestress curve will be much affected.

2.18. The writer has made an attempt to use the fundamental formulae of section 2.10 of this thesis as foundation for the preliminary design method of Wessman and Hardesty in Reference(7), and is forced to the conclusion that the method advocated by those authors is safe only insofar, as it has been checked against computations by the Deflection Theory for the determination of the empirical coefficients. If we accept the parameters α and ϕ as being descriptive of a suspension bridge as shown in Fig. 2.08, then the bridges upon which Wessman and Hardesty have based their method of computation cover a very small range; their examples are the first six bridges listed in Table IX and show values of α between 1.7 and 7.1 with ϕ ranging from 0.2 to 0.7. The range of ϕ is large but the range of α defines only a very small region low down in Fig. 2.08, and it cannot be expected that an empirical approach can lead to equations which will represent the action of the stiffening girder over a very large range without recourse to changes in the empirical coefficients.

If only one harmonic proved important in the determination of critical deflections then ~~is~~ some proper physical basis for the Preliminary Design Method can be established. In general, there are two important harmonics and the applicability of the method must be proved by calculations based on the usual theories which are more nearly representative of the facts.

Each suspension bridge is a problem in itself and exact calculations for the two major design points treated in Reference (7), namely centre and quarter points, are as easily made by the present theory with better physical basis. Furthermore, Moissieff has stated, and adduced data for actual designs as evidence, that the quarter point is not always a point of maximum bending but the maximum lies appreciably nearer the support. Under the latter circumstances the empirical approach of Wessman and Hardesty fails because of its lack of consideration of the proper physical relation between the basic parameters of the suspension bridge.

2.19. In the present analysis it is possible to remove the limitation of assumption (1) of section 1.01 from all except low order terms because of the orthogonal properties of the sine series.

Repeating the integration of equation (35) of section(1.09 there results again

$$\xi = \int_0^l \nu \frac{d^2y}{dx^2}\, dx + \omega\, t\, l + \frac{H}{EA} \int_0^l \mathrm{Sec}^3\phi\, dx \qquad 2.36.$$

The present section concerns itself with the integration of the first integral on the right- hand side. The expression for ν we may write in general

$$\nu = \eta_t - f\sum_{n=1}^{\infty} d_n \text{Sin}\,\frac{n\pi x}{l} = f \sum_{n=1}^{\infty} (D_n - d_n) \text{ Sin}\,\frac{n\pi x}{l}$$

and from equation 2.19

$$y = f \sum_{n=1}^{\infty} C_n \text{ Sin}\,\frac{n\pi x}{l}$$

whence

$$\frac{d^2y}{dx^2} = -\frac{\pi^2 f^2}{l^2} \sum_{n=1}^{\infty} n^2 C_n \text{ Sin}\,\frac{n\pi x}{l}$$

and the integral becomes

$$-\frac{\pi^2 f^2}{l^2} \int_0^l \sum_{n=1}^{\infty} n^2 C_n \text{ Sin}\,\frac{n\pi x}{l} \sum_{n=1}^{\infty} (D_n - d_n) \text{ Sin}\frac{n\pi x}{l}\, dx$$

$$= -\frac{\pi^2 f^2 l}{l^2} \sum_{n=1}^{\infty} n^2 C_n (D_n - d_n) \qquad 2.37.$$

Substituting 2.37 for the first integral in 2.36 and writing $\xi = 0$ for the support conditions the more general equation for the horizontal tension is

$$\frac{HL_c}{EA} = \frac{\pi^2 l}{2} . \frac{f^2}{l^2} \sum_{n=1}^{\infty} C_n (D_n - d_n) - \omega t l$$

For symmetrical cable curves only odd harmonics occur in the summation. Then

$$\beta = \frac{EA\,l}{H_w L_c} \left[\frac{\pi^2 f^2}{2\,l^2} \sum_{n=1}^{\infty} n^2 C_n (D_n - d_n) - \omega t \right]$$

$$= \gamma \left[\frac{\pi^2 f^2}{2\,l^2} \sum_{n=1}^{\infty} n^2 C_n (D_n - d_n) - \omega t \right] \qquad 2.39.$$

which is a little more general than equation 2.28 and contains the assumption of a parabolic cable curve in L_c only. For a parabolic cable curve equations 2.28 and 2.39 give identical numerical results and, for the usual curves, they are interchangeable.

2.20. We recall now equation (39) of section 1.10.

$$M = M_L + M_d = EI \frac{d^2 \nu}{dx^2} - (H_w + H)\left[(y + \nu) \mathrm{Sec}^{\ 2}\ \right) - \frac{H l^2}{40EAf}$$

$$(\mathrm{Sec}^5 \psi_0 - \mathrm{Sec}^5 \psi).$$

If we accept assumption (2) of section 1.01 which states in effect that the curve of the cable defined by y is determined to match the dead load bending moment curve, then

$$M_d = H_w . y$$

and the equation above reduces to

$$M_L = EI \frac{d^2 \nu}{dx^2} - H\left[y + \frac{l^2 (H_w + H)}{40\ EAf}(\mathrm{Sec}^5 \psi_0 - \mathrm{Sec}^5 \psi)\right] - (H_w + H) \nu\, \mathrm{Sec}^2 \psi$$

If we use the expansion of section 2.06 for $\mathrm{Sec}^5 \psi_0 - \mathrm{Sec}^5 \psi$ then that term can be included with y and offers no further difficulty. In actual cases it results in a correction oof much less than 0.1% in the coefficients of the expansion for y and will henceforth be ignored as trivial. There remains

$$- M_L = EI \frac{d^2 \nu}{dx^2} - Hy - (H_w + H) \nu\, \mathrm{Sec}^2 \psi$$

where ν is the deflection of the lower ends of the suspenders and the sign convention for bending moment has been changed to agree with section 2.10.

$$\eta_t = \nu + f \sum_{n=1}^{\infty} d_n \ \mathrm{Sin} \frac{n \pi x}{l} \qquad 2.40$$

where η_t and d_n are defined in section 2.10, and if we wish to include prestress the equation becomes

$$- M_L = EI \frac{d^2 \eta_t}{dx^2} - H_y - (H_w + H) \nu \, Sec^2 \qquad 2.41.$$

Introducing dimensionless quantities

$$- \frac{M_L}{H_w f} = \frac{EI}{H_w l^2} \frac{d^2}{dZ^2} \left(\frac{\eta_t}{f} \right) - \beta . \frac{y}{f} - (1 + \beta) \frac{\nu}{f} Sec^2 \psi \qquad 2.42.$$

Using the geometry of the parabola

$$Sec^2 \psi = 1 + tan^2 \psi = 1 + \frac{16f^2}{l^2} (1 - 2Z)^2$$

$$= 1 + \frac{16f^2}{l^2} - \frac{64f^2}{l^2} Z(1 - Z)$$

and substituting into equation 2.42

$$- \frac{M_L}{H_w f} = \frac{EI}{H_w l^2} \frac{d^2}{dZ^2} \left(\frac{\eta_t}{f} \right) - \beta \frac{y}{f} - (1 + \beta)\left(1 + \frac{16f^2}{l^2}\right) \frac{\nu}{f} +$$

$$(1 + \beta) \frac{64f^2}{l^2} Z(1 - Z) \frac{\nu}{f} \qquad 2.43.$$

Writing $\alpha = \dfrac{H_w l^2}{\pi^2 EI}$, $A = 1 + \dfrac{16f^2}{l^2}$, and $B = \dfrac{64f^2}{l^2}$

we obtain

$$\frac{- M_L}{H_w f} = \frac{1}{\pi^2 \alpha} \frac{d^2}{dZ^2} \left(\frac{\eta_t}{f} \right) - \beta \frac{y}{f} - (1 + \beta) A \frac{\nu}{f} + (1 + \beta) B \; Z(1 - Z) \frac{\nu}{f}$$

and substituting from equation 2.19, 2.20, 2.22 and 2.24 and reducing we obtain

$$\sum_{n=1}^{\infty} \left[a_n - \beta C_n + A(1 + \beta) d_n \right] Sin\, n\pi Z - B(1 + \beta) Z(1 - Z) \sum_{n=1}^{\infty} d_n \, Sin\, n\pi Z \qquad 2.44$$

$$= \sum_{n=1}^{\infty} \left[\frac{n^2}{\alpha} + A(1+\beta) \right] D_n \operatorname{Sin} n\pi Z - B(1+\beta) Z(1-Z) \sum_{n=1}^{\infty} D_n \operatorname{Sin} n\pi Z \qquad 2.44.$$

Equation 2.44 is general and can be used to discover the effect of the correction term in $Z(1-Z)$ and the influence of variable moment of inertia of the girder. It is not yet in a form which enables the coefficients, D_n, to be determined. If we expand the product

$$Z(1-Z) \sum_{n=1}^{\infty} d_n \operatorname{Sin} n\pi Z$$

by means of Table VI for first and third order deflection curves we obtain

$$B(1+\beta)Z(1-Z)\left[d_1 \operatorname{Sin}\pi Z + d_3 \operatorname{Sin} 3\pi Z \right] = B(1+\beta)\left\{ d_1 \left[0.2173 \operatorname{Sin}\pi Z - 0.038 \operatorname{Sin} 3\pi Z - 0.007 \operatorname{Sin} 5\pi Z \right] - d_3 \left[0.038 \operatorname{Sin}\pi Z - 0.1722 \operatorname{Sin} 3\pi Z + 0.0475 \operatorname{Sin} 5\pi Z \right] \right\} \qquad 2.45$$

The left hand side of equation 2.44 is now fully expanded in trigonometric series and makes possible the inclusion of prestress in the more accurate deflection theory equation 2.41. For what follows there is no necessity to include the prestress explicitly if it is understood that its inclusion requires only that the proper harmonic coefficients from equation 2.45 be included in the numerator of the expression for D_n as typified by equation 2.26.

The numerator for n =1 is quoted here as a guide

$$N_1 = a_1 - \beta C_1 + A(1+\beta)d_1 - B(1+\beta)\left\{0.2173d_1 - 0.038d_3\right\}$$

Equation 2.44 is simplified to

$$\sum_{n=1}^{\infty} \left[a_n - \beta C_n\right] \text{Sin } n\pi Z = \sum_{n=1}^{\infty} \frac{n^2}{\alpha} + A(1+\beta) \quad D_n \text{ Sin } n\pi Z$$

$$- B(1+\beta) \quad Z(1-Z) \sum_{n=1}^{\infty} D_n \text{Sin } n\pi Z \qquad 2.46.$$

and the second term on the right hand side is expanded using Table VI as before

$$Z(1-Z) \sum_{n=1}^{\infty} D_n \text{ Sin } n\pi Z = D_1\left[0.2173 \text{ Sin}\pi Z - 0.038 \text{ Sin } 3\pi Z - 0.007 \text{ Sin } 5\pi Z\right] + D_2\left[0.1793 \text{ Sin } 2\pi Z - 0.0455 \text{ Sin } 4\pi Z\right] - D_3\left[0.038 \text{ Sin}\pi Z - 0.1722 \text{ Sin } 3\pi Z + 0.0475 \text{ Sin } 5\pi Z\right] - D_4\left[0.0455 \text{ Sin } 2\pi Z - 0.1698 \text{ Sin } 4\pi Z\right] - D_5\left[0.007 \text{ Sin}\pi Z + 0.0475 \text{ Sin } 3\pi Z - 0.1686 \text{ Sin } 5\pi Z\right] + \ldots.. \qquad 2.47.$$

If α is constant, which signifies a girder of constant flexural rigidity, then it is possible to write down the coefficients, D_n, from 2.46 and 2.47. For example, equating coefficients of terms of the first order and solving for D_1 we obtain

$$D_1 = \frac{a_1 - \beta C_1 - B(1+\beta)(0.038D_3 + 0.007D_5 + \ldots)}{(A - 0.2173\ B)(1+\beta) + \frac{1}{\alpha}} \qquad 2.48(a)$$

Similarly

$$D_2 = \frac{a_2 - \beta C_2 - B(1+\beta)(0.0455D_4 + \ldots.)}{(A - 0.1793\ B)\ (1+\beta) + \frac{4}{\alpha}} \qquad 2.48(b)$$

and

$$D_3 = \frac{a_3 - \beta C_3 - B(1+\beta)(0.038\, D_1 + 0.0475\, D_5 + \dots)}{(A - 0.1722B)(1+\beta) + \frac{9}{\alpha}} \qquad 2.48(c)$$

and so on.

It will be noted that the denominator is increased in value over that given by the ordinary Deflection Theory. This is evident if one compares the denominator of the second harmonic from equation 2.26(a) with equation 2.48(b). From 2.26(a)

$$\begin{aligned} 1 + \beta + \frac{4}{\alpha} &= A(1+\beta) - \frac{B}{4}(1+\beta) + \frac{4}{\alpha} \\ &= (A - 0.25B)(1+\beta) + \frac{4}{\alpha} < (A - 0.1793B) \\ &\qquad (1+\beta) + \frac{4}{\alpha} \quad \text{from } 2.48(b). \end{aligned}$$

Deflections will be decreased but generalisation as to the amount of decrease is not possible because of the influence of other harmonic coefficients in the numerator.

It would seem that the appearance of other unknown coefficients in the numerator renders the equations 2.48 useless for computation, but this is not the case. The coefficients in the numerator appear as part of a small correction term and need to be known only approximately. They can be obtained by using the equation 2.26(a) which would in any case be computed as a first step in design calculations. Then equations 2.48 are as tractable as equation 2.26(a) and can be used to find β in the manner discussed in section 2.12.

The general appearance of the numerator suggests that the correction has the same nature as a prestress term. We may readily include the prestress term discussed earlier in this section; for example, the first harmonic coefficient will then be

$$D_1 = \frac{a_1 - \beta C_1 + B(1+\beta)\left[\left(\frac{A}{B} - 0.2173\right)d_1 - 0.038(D_3 - d_3) - 0.007(D_5 - d_5) + , ..\right.}{(A - 0.2173\ B(1+\beta) + \frac{1}{\alpha}}$$

To illustrate the use of equation 2.48 the example used for Table VIII will be used. This example is not altogether satisfactory because the value of $\alpha = 1.746$ is low according to modern standards and obscures the issue somewhat. The equations for the coefficients are derived from equation 2.48 as follows:

From Table VIII

$D_1 = 0.001536$ $D_3 = .003039$ $D_5 = 0.000266$

$A = 1 + 16 \times 0.1004^2$ $B = 64 \times 0.1004^2$

$= 1.161$ $= 0.425$

$\alpha = 1.746$

Using these values equation 2.48 gives

$$D_1 = \frac{a_1 - \beta(C_1 + 0.00005) - 0.00005}{1.642 + 1.069\beta}$$

$$D_3 = \frac{a_3 - \beta(C_3 + 0.00003) - 0.00003}{6.24 + 1.088\beta}$$

$$D_5 = \frac{a_5 - \beta(C_5 + 0.000066) - 0.000066}{15.41 + 1.089\beta}$$

To determine the influence of the correction terms Table X has been computed. It is evident that the uncorrected Deflection

TABLE X

$\beta = 0.1$

n	C_n	A_n	D_n	$\frac{2D_n}{n}$
1	1.0323	0.1058	0.001441	.002882
2	0.0	0.0452		
3	0.0382	0.02283	0.002992	.001995
4	0.0	0.01129		
5	0.00825	0.00493	0.000260	.000104
				.004981

Computed $t = -2.1^{\circ}$ F

$\beta = 0.0996$

1	1.0323	0.1058	0.001628	.003256
2	0.0	0.0452		
3	0.0382	0.02283	0.002994	.001996
4	0.0	0.01129		
5	0.00825	0.00493	0.000260	.000104
				.005356

Computed $t = -0.6^{\circ}$ F

By linear interpolation $\beta = 0.0994$ for $t = 0$

$\beta = 0.0994$

1	0.00172
2	0.01295
3	0.002994
4	0.001066
5	0.000260

Theory overestimates the deflection of the cable, and gives too high values for β. This effect will become more apparent in bridges with larger values of α. For a stiff bridge such as the Manhattan Bridge used in Table X the value of α is sufficiently small to reduce appreciably the effect of the correction terms, because of the large contribution of the α-term to the value of the denominators in equation 2.48. In a bridge such as the Golden Gate, where $\alpha = 70$, the effect on β can reach 10%. This accounts very satisfactorily fot the deficiency of 6% in the measured values of β from a model of the San Francisco Bay Bridge which is reported in Reference(2).

The difference in deflections of the Deflection Theory and the corrected Deflection Theory is shown in Table XI which collects and compares the harmonic coefficients from Tables VIII and X.

TABLE XI

n	D_n Deflection Theory	D_n Corrected Theory
1	0.00173	0.00172
2	0.01332	0.01295
3	0.00304	0.00299
4	0.00110	0.00107
5	0.000260	0.000266

The second order term shows the largest discrepancy amounting to 3%. It α were four times larger the error would be more than 6%.

2.21. There remains to be investigated the influence of variable moment of inertia on the deflection coefficients. We eliminate from equation 2.24, the terms involving prestress and the correction term discussed in section 2.20. There results

$$\sum_{n=1}^{\infty}\left[a_n - \beta C_n\right]\text{Sin } n\pi Z = \sum_{n=1}^{\infty}\frac{n^2}{\alpha}D_n\text{Sin } n\pi Z + (H\beta)\sum_{n=1}^{\infty}D_n\text{Sin } n\pi Z \qquad 2.49$$

where α is now considered variable. We may write

$$\frac{1}{\alpha} = \frac{\pi^2 E}{H_w l^2}(I_o + I_1\text{Sin}\,\pi Z + I_3\text{Sin } 3\pi Z + \ldots)$$ wherein

symmetry excludes the even harmonics. Then, we write

$$\frac{n^2}{\alpha} = \frac{n^2}{\alpha_o} + n^2\left(\frac{1}{\alpha_1}\text{Sin}\,\pi Z + \frac{1}{\alpha_3}\text{ Sin } 3\pi Z + \ldots\right) \qquad 2.50$$

where $\frac{1}{\alpha_n} = \frac{\pi^2 E}{H_w l^2} \cdot I_n$

I_o is the moment of inertia at the ends of the girder, and I_n is an additional amount. This subdivision of the flexural rigidity is convenient in that it permits us to discover the influence of a particular change superimposed on the uniform girder. For convenience of algebraic presentation, only the first harmonic will be retained in the following treatment.

Now

$$\sum_{n=1}^{\infty}\frac{n^2}{\alpha}D_n\text{Sin } n\pi Z = \sum_{n=1}^{\infty}n^2 D_n\text{Sin } n\pi Z\left(\frac{1}{\alpha_o} + \frac{1}{\alpha_1}\text{Sin}\,\pi Z\right)$$

$$= \sum_{n=1}^{\infty}\frac{n^2}{\alpha_o}D_n\text{Sin } n\pi Z + \sum_{n=1}^{\infty}\frac{n^2}{\alpha_1}D_n\text{ Sin } n\pi Z\text{ Sin}\,\pi Z$$

Using Table V

$$\sum_{n=1}^{\infty}\frac{n^2}{\alpha_1}D_n\text{ Sin}\,\pi Z\text{ Sin } n\pi Z = \frac{D_1}{\alpha_1}(0.849\text{ Sin}\,\pi Z - 0.1696\text{Sin}3\pi Z$$

$$- 0.02435 \text{ Sin } 5\pi Z) + \frac{D_2}{\alpha_1}(2.72 \text{ Sin } 2\pi Z - 0.776 \text{ Sin} 4\pi Z -)$$

$$- \frac{D_3}{\alpha_1}(1.525 \text{ Sin} \pi Z - 5.89 \text{ Sin } 3\pi Z + 1.82 \text{ Sin } 5\pi Z +)$$

$$- \frac{D_4}{\alpha_1}(3.1 \text{ Sin } 2\pi Z - 10.9 \text{ Sin } 4\pi Z +)$$

$$- \frac{D_5}{\alpha_1}(0.606 \text{ Sin} \pi Z + 5.06 \text{ Sin } 3\pi Z - 16.1 \text{ Sin } 5\pi Z +)$$

We may now write down the coefficients of D_n using the foregoing expansion and equation 2.49

$$D_1 = \frac{a_1 - \beta C_1 + \frac{1}{\alpha_1}(1.525 D_3 + 0.606 D_5 +)}{1 + \beta + \frac{1}{\alpha_o} + \frac{0.849}{\alpha_1}} \qquad 2.51(a)$$

$$D_2 = \frac{a_2 - \beta C_2 + \frac{1}{\alpha_1}(3.1 D_4 + \quad)}{1 + \beta + \frac{4}{\alpha_o} + \frac{2.72}{\alpha_1}} \qquad 2.51(b)$$

$$D_3 = \frac{a_3 - \beta C_3 + \frac{1}{\alpha_1}(0.1696 D_1 + 5.06 D_5 +)}{1 + \beta + \frac{9}{\alpha_o} + \frac{5.89}{\alpha_1}} \qquad 2.51(c).$$

and so on.

With the aid of Table V it is a simple matter to carry through a similar analysis for any other sinusoidal variation of the moment of inertia.

Equation 2.51 show one important point, namely, that a relatively small variation in the moment of inertia, which means essentially a large value of α_n, can have very little effect on the lower order deflection curves.

The method of calculation requires first values of D_n to be known and these are found from the deflection theory equation 2.26(a) taking into account only I_o. If α_1 is positive these values will be too large. Using these values in equation 2.51 permits the determination of more accurate values of β and D_n.

It is very easy to check the structural value of a proposed variation of I in a design which has already been calculated for a truss of constant moment of inertia I_o.

Taking into account now the first and third harmonic variation of I we have

$$\sum_{n=1}^{\infty} \frac{n^2}{\alpha} D_n \text{Sin } n\pi Z = \sum_{n=1}^{\infty} \frac{n^2}{\alpha_0} D_n + \sum_{n=1}^{\infty} n^2 D_n \text{Sin } n\pi Z \left(\frac{1}{\alpha_1} \text{Sin } \pi Z + \frac{1}{\alpha_3} \text{Sin } 3\pi Z\right).$$

Using Table V, $\sum_{n=1}^{\infty} \frac{n^2}{\alpha_1} D_n \text{ Sin} \pi Z \text{ Sin } n\pi Z =$

$$\frac{D_1}{\alpha_1}(0.849 \text{Sin } \pi Z - 0.1696 \text{Sin } 3\pi Z - 0.02425 \text{ Sin } 5\pi Z +)$$

$$+ \frac{D_2}{\alpha_1}(2.72 \text{ Sin } 2\pi Z - 0.776 \text{ Sin } 4\pi Z +)$$

$$- \frac{D_3}{\alpha_1}(1.525 \text{ Sin } \pi Z - 5.89 \text{ Sin } 3\pi Z + 1.82 \text{ Sin } 5\pi Z +)$$

$$- \frac{D_4}{\alpha_1}(3.1 \text{ Sin } 2\pi Z - 10.9 \text{ Sin } 4\pi Z +)$$

$$- \frac{D_5}{\alpha_1}(0.606 \text{ Sin } \pi Z + 5.06 \text{ Sin } 3\pi Z - 16.1 \text{ Sin } 5\pi Z +)$$

and $\sum_{n=1}^{\infty} \frac{n^2}{\alpha_3} D_n \text{ Sin } 3\pi Z \text{ Sin } n\pi Z =$

$$- \frac{D_1}{\alpha_3}(0.1696 \text{ Sin } \pi Z - 0.655 \text{ Sin } 3\pi Z + 0.202 \text{ Sin } 5\pi Z +)$$

$$+ \frac{D_2}{\alpha_3}(1.94 \text{ Sin } 2\pi Z + 1.81 \text{ Sin } 4\pi Z - 0.953 \text{ Sin } 6\pi Z +)$$

$$+ \frac{D_3}{\alpha_3}(5.9 \text{ Sin } \pi Z + 2.55 \text{ Sin } 3\pi Z + 3.74 \text{ Sin } 5\pi Z +)$$

$$+ \frac{D_4}{\alpha_3}(7.25 \text{ Sin } 2\pi Z + 3.95 \text{ Sin } 4\pi Z +)$$

$$- \frac{D_5}{\alpha_3} (5.06 \sin \pi Z - 10.4 \sin 3\pi Z - 5.83 \sin 5\pi Z +)$$

Collecting the coefficients of the terms of each order and solving for each value of D_n in terms of the others we obtain as before.

$$D_1 = \frac{a_1 - \beta C_1 + \frac{1}{\alpha_1}(1.525 D_3 + 0.606 D_5 +) - \frac{1}{\alpha_3} (5.9 D_3 - 5.06 D_5 +)}{1 + \beta + \frac{1}{\alpha_0} + \frac{0.849}{\alpha_1} - \frac{0.1696}{\alpha_3}} \qquad 2.52(a)$$

$$D_2 = \frac{a_2 - \beta C_2 + \frac{1}{\alpha_1}(3.1 D_4 +) - \frac{1}{\alpha_3}(7.25 D_4 +)}{1 + \beta + \frac{1}{\alpha_0} + \frac{2.72}{\alpha_1} + \frac{1.94}{\alpha_3}} \qquad 2.52(b)$$

$$D_3 = \frac{a_3 - \beta C_3 + \frac{1}{\alpha_1}(0.1696 D_1 + 5.06 D_5 +) - \frac{1}{\alpha_3}(0.655 D_1 + 10.4 D_5 +)}{1 + \beta + \frac{1}{\alpha_0} + \frac{5.89}{\alpha_1} + \frac{2.55}{\alpha_3}} \qquad 2.52(c)$$

and so on.

It will be seen that equations 2.52 differ ~~equations~~ from equations 2.51 only by the addition of the terms involving α_3. This may be generalised for any number of harmonic variations of I. It will be noticed that the equations 2.51 and 2.52 are linear in the coefficients, D_n. If we use Tudor's method of reference(2) to solve for β it will be necessary to solve simultaneously equations for D_n. When only one harmonic variation of flexural rigidity is present this is very simple because there are only three unknowns in any equation if we retain five harmonic coefficients. For investigation of the influence of variation of I when α_n is small this procedure is the best and proves very convenient because even and odd harmonics can be solved for independently and only even harmonics are concerned when β is being determined.