Embeddings of One-Factorizations of Hypergraphs and Decompositions of Partitions

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Abstract

We look at one-factorizations of complete k-uniform hypergraphs, and investigate the problem of determining when, for $U \subset V$, one can embed a one-factorization of the complete k-uniform hypergraph on U in a one-factorization of the complete k-uniform hypergraph on V. We give a brief history of the problem, and find our own independent results for specific values of k and $v = |V|$, in the process making explicit a theorem implicitly used by Häggkvist and Hellgren in their solution to the problem in general. We provide our own independent proof of this theorem, and subsequently use it to extend our results to certain nonuniform hypergraphs. This, in particular, allows us to find alternate proofs about two results involving the extension of symmetric Latin squares, originally shown by Cruse and by Hoffman. We then explain the connection between the hypergraph-embedding problem and a problem involving the decomposition of partitions of an integer N into subpartitions of an integer n, where n divides N. This in turn leads to a problem involving the cone generated by a subset V of $\mathbb{R}^n$, the properties of which we investigate thoroughly.

Contents

Chapter 1

Embeddings of One-Factorizations of Hypergraphs

1.1 Preliminary Definitions

Given a finite set V, let $\binom{V}{k}$ denote the complete k-uniform hypergraph with vertex set V. We will refer to V as the set of vertices, and the k-element subsets of V as the k-edges. Then, by a *one-factor* of $\binom{V}{k}$, we mean a collection of k-edges that are disjoint and partition the set of vertices. A *one-factorization* of $\binom{V}{k}$ will be a collection of one-factors of $\binom{V}{k}$ such that each k-edge occurs in precisely one such one-factor. When dealing with finite sets labeled U and V, unless stated otherwise, we will let $u = |U|$ and $v = |V|$.

For a given natural number n and partition π of n (given by $\pi_1 + \pi_2 + \cdots + \pi_N = n$) we define $v_\pi \in \mathbb{Z}^n$ by $v_\pi = (a_1, a_2, \ldots, a_n)$, where a_i is the number of parts of π of size i. Then, for $k, m \in \mathbb{N}$ with $1 \leq k, m \leq n$ we define the following:

$$V_n = \{v_\pi \ : \pi \text{ is a partition of } n\}$$

$$V_{n,k,m} = \{v_\pi : \pi \text{ is a partition of } n \text{ into at most } m \text{ parts of size at most } k\}.$$

Also, we define $[n] = \{1, 2, \ldots, n\}$.

1.2 One-Factorizations of $\binom{\mathbf{V}}{\mathbf{k}}$

Clearly, a necessary condition for the existence of a one-factorization of $\binom{V}{k}$ is that k divides v; the following theorem of Baranyai (one proof is given, for example, in [1]) shows that this condition is, in fact, sufficient as well:

Theorem 1.2.1 *If k divides v, there exists a one-factorization of $\binom{V}{k}$.*

Note that each one-factor of $\binom{V}{k}$ must contain $\frac{v}{k}$ k-edges, and as there are a total of $\binom{v}{k}$ k-edges in $\binom{V}{k}$, it follows that any given one-factorization of $\binom{V}{k}$ must contain exactly $\frac{k}{v}\binom{v}{k} = \binom{v-1}{k-1}$ one-factors. We will denote such a one-factorization by $(V, k, \mathcal{P})$ where $\mathcal{P} = \{\mathcal{P}_i : i = 1, 2, \dots, \binom{v-1}{k-1}\}$ is the corresponding set of one-factors. Also, given one-factorizations $(U, k, \mathcal{Q})$ and $(V, k, \mathcal{P})$, we say that $(U, k, \mathcal{Q})$ is a subsystem of $(V, k, \mathcal{P})$ if $U \subseteq V$ and if there is a labeling of $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{Q}_i \subseteq \mathcal{P}_i$ for each i with $1 \leq i \leq \binom{u-1}{k-1}$. If $(U, k, \mathcal{Q})$ is a subsystem of $(V, k, \mathcal{P})$, we will also say that we can *embed* $(U, k, \mathcal{Q})$ in $(V, k, \mathcal{P})$. The following result is shown, in particular, in [2]: we present our own proof for completeness.

Theorem 1.2.2 *If $(U, k, \mathcal{Q})$ is a subsystem of $(V, k, \mathcal{P})$ and $U \neq V$, then $v \geq 2u$.*

Proof. Suppose that $(U, k, \mathcal{Q})$ is a subsystem of $(V, k, \mathcal{P})$ with $U \neq V$. Consider the labeling of $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{Q}_i \subseteq \mathcal{P}_i$ for each i with $1 \leq i \leq \binom{u-1}{k-1}$ and for each such i let $R_i = \mathcal{P}_i \setminus \mathcal{Q}_i$. Let $W = V \setminus U$. Then, it follows that each R_i is a one-factor of $\binom{W}{k}$ and thus $|R_i| = \frac{|W|}{k}$. Further, since each of the $\mathcal{P}_i$ are pairwise disjoint, it follows that each of the R_i are pairwise disjoint. Thus

$$\binom{|W|}{k} \geq \sum_{i=1}^{\binom{u-1}{k-1}} |R_i| = \sum_{i=1}^{\binom{u-1}{k-1}} \frac{|W|}{k} = \binom{u-1}{k-1}\frac{|W|}{k}.$$

This implies that

$$\binom{u-1}{k-1} \leq \frac{k}{|W|}\binom{|W|}{k} = \binom{|W|-1}{k-1},$$

which implies that $u - 1 \leq |W| - 1$ and thus $u \leq |W| = v - u$. But this implies that $2u \leq v$. $\square$

This leads to the following question: given a one-factorization $(U, k, \mathcal{Q})$ and $v \geq 2u$ such that k divides v, can we always find a V with a one-factorization $(V, k, \mathcal{P})$ that has $(U, k, \mathcal{Q})$ as a subsystem? First, note that it is easy to see that the answer is independent of the one-factorization of U that we choose; simply permute the k-sets of U in the first $\binom{u-1}{k-1}$ elements of $\mathcal{P}$ accordingly. It follows that the answer depends only on the cardinalities of the sets, i.e., u and v. So if there exists a U, V, $\mathcal{Q}$ and $\mathcal{P}$ with the one-factorization $(U, k, \mathcal{Q})$ embeddable in the one-factorization $(V, k, \mathcal{P})$, we will simply say that we can embed U in V and that (u, v) is an embeddable pair of order k. To avoid trivialities, we require that $u < v$ for (u, v) to be considered an embeddable pair. Let the set of all embeddable pairs of order k be denoted by $\mathcal{EP}_k$. Thus, the question can be reworded: is it true that

$$\mathcal{EP}_k = \{(u, v) : v \geq 2u, \text{ and } k \mid u, v\}?$$

In fact, the answer is yes, and the following theorem is proven in [3]:

Theorem 1.2.3 *For all $k \in \mathbb{N}$, (u, v) is in $\mathcal{EP}_k$ if and only if $v \geq 2u$ and k divides u and v.*

We provide our own independent proofs that for any u, v with $k \mid u, v$ and either $v \geq ku$ or $v = 2u$, we have $(u, v) \in \mathcal{EP}_k$. We will also provide independent proofs of theorem 1.2.3 for small k (in particular, for $2 \leq k \leq 5$). First, we need one more definition.

By an *m-partition* of V, we mean a multiset $\mathcal{A}$ of m pairwise disjoint subsets of V, some of which may be empty, whose union is V. Note that this definition implies that the only set that may be repeated in $\mathcal{A}$ is the empty set. Note that if $(V, k, \mathcal{P})$ is a one-factorization, and $U \subseteq V$, if we let $\mathcal{Q}$ be the restriction of $\mathcal{P}$ to U, i.e., let $\mathcal{Q} = \{\mathcal{Q}_i : i = 1, 2, \ldots, \binom{v-1}{k-1}\}$, where $\mathcal{Q}_i = \{P \cap U : P \in \mathcal{P}_i\}$, it follows that each $\mathcal{Q}_i$ is an m-partition of U, where $m = \frac{v}{k}$. Further, as each j element subset (which we will hereby refer to as a j-subset) of U is the restriction of exactly $\binom{v-u}{k-j}$ k-subsets of V to U, it follows that each j-subset of U occurs exactly $\binom{v-u}{k-j}$ times in the $\mathcal{Q}_i$, for $0 \leq j \leq u$ (with the usual convention that $\binom{n}{k} = 0$ if $k < 0$). The following theorem, which again can be found in [1], in the proof of their theorem 38.1 (which is equivalent to our theorem 1.2.1), shows the converse of this:

Theorem 1.2.4 *Given $U \subseteq V$ and a set $\mathcal{A} = \{\mathcal{A}_i : i = 1, 2, \ldots, \binom{v-1}{k-1}\}$, where each $\mathcal{A}_i$ is a $\frac{v}{k}$-partition of U, such that, for $0 \leq j \leq u$, every j-subset of U occurs precisely $\binom{v-u}{k-j}$ times in the $\mathcal{A}_i$, then $\mathcal{A}$ can be extended to a one-factorization $(V, k, \mathcal{P})$ of V (meaning that $\mathcal{A}$ is the restriction of $\mathcal{P}$ to U).*

Thus, to determine whether $(u, v) \in \mathcal{EP}_k$, it remains to find a system $\mathcal{A} = \{\mathcal{A}_i : i = 1, 2, \ldots, \binom{v-1}{k-1}\}$ of $\frac{v}{k}$-partitions of U such that each j-subset of U occurs exactly $\binom{v-u}{k-j}$ times in the $\mathcal{A}_i$ and such that, for $1 \leq i \leq \binom{u-1}{k-1}$ we have $\mathcal{A}_i$ equal to $\mathcal{Q}_i$ (for some $\mathcal{Q}$ such that $(U, k, \mathcal{Q}$ is a one-factorization) unioned with $\frac{v-u}{k}$ copies of the empty set, so that the $\mathcal{Q}_i$ become $\frac{v}{k}$-partitions. We now prove the following result, which will be useful in applying this technique:

Theorem 1.2.5 *Given a set $\mathcal{A} = \{\mathcal{A}_i : i = 1, 2, \ldots N\}$, where each $\mathcal{A}_i$ is a $\frac{v}{k}$-partition of U, for some natural numbers N and v, with $v \geq u$, such that for $1 \leq j \leq u$ each j-subset of U occurs exactly $\binom{v-u}{k-j}$ times in the $\mathcal{A}_i$, it follows that $N = \binom{v-1}{k-1}$ and the empty set occurs exactly $\binom{v-u}{k}$ times in the $\mathcal{A}_i$.*

Proof. We will be using the following combinatorial identity (and some trivial variations of it):

$$\sum_{i=0}^{k} \binom{n-j}{k-i}\binom{j}{i} = \binom{n}{k}.$$

To see that this is true, we count the number of k-subsets of an n-set by first breaking that n-set into two parts; one of size j and the other of size $n - j$. Then, we choose i of the elements in the k-subset to be in the chosen j-subset, and the other $k - i$ to be among the remaining $n - j$ elements of the n-set. There are clearly $\binom{n-j}{k-i}\binom{j}{i}$ ways to do this, and thus as we sum over all i from $i = 0$ to $i = k$ we count each k-subset of the n-set exactly once, implying that the left-hand side is in fact equal to the right-hand side, and thus the identity is true.

So, returning to the notation specified in the statement of theorem 1.2.5, let $x_{i,j}$ be the number of times an j-subset of U occurs in $\mathcal{A}_i$ and let $m = \frac{v}{k}$. Note also that the conditions given above imply that for $j > k$, no j-subset occurs in any of the $\mathcal{A}_i$. Then, it follows that, since each A_i is an m-partition of U, that for $1 \leq i \leq N$, we have

$$\sum_{j=1}^{k} jx_{i,j} = \sum_{i=0}^{k} jx_{i,j} = u.$$

Further, by the property given above, and the fact that there are $\binom{u}{j}$ j-subsets of U, we have, for $1 \leq j \leq u$

$$\sum_{i=1}^{N} x_{i,j} = \binom{v-u}{k-j}\binom{u}{j}.$$

Thus, putting these together, we have

$$\begin{aligned} Nu &= \sum_{i=1}^{N} u = \sum_{i=1}^{N}\sum_{j=1}^{k} jx_{i,j} = \sum_{j=1}^{k} j \sum_{i=1}^{N} x_{i,j} = \sum_{j=1}^{k} j \binom{v-u}{k-j}\binom{u}{j} \\ &= \sum_{j=1}^{k} \binom{v-u}{k-j} j \binom{u}{j} = \sum_{j=1}^{k} \binom{v-u}{k-j} u \binom{u-1}{j-1} \\ &= u \sum_{j=1}^{k} \binom{v-u}{k-j}\binom{u-1}{j-1} = u \binom{v-1}{k-1}, \end{aligned}$$

and thus $N = \binom{v-1}{k-1}$. As for the empty set, note that, as each $\mathcal{A}_i$ is an m partition of U, it follows that, for $1 \leq i \leq N$,

$$x_{i,0} + \sum_{j=1}^{k} x_{i,j} = m,$$

so we must have

$$\sum_{i=1}^{N} x_{i,0} + \sum_{i=1}^{N}\sum_{j=1}^{k} x_{i,j} = \sum_{i=1}^{N} m,$$

which implies

$$\sum_{i=1}^{N} x_{i,0} = \sum_{i=1}^{N} m - \sum_{i=1}^{N}\sum_{j=1}^{k} x_{i,j} = Nm - \sum_{j=1}^{k}\sum_{i=1}^{N} x_{i,j} = Nm - \sum_{j=1}^{k}\binom{v-u}{k-j}\binom{u}{j}$$
$$= \binom{v-1}{k-1}\frac{v}{k} - \sum_{j=1}^{k}\binom{v-u}{k-j}\binom{u}{j} = \binom{v}{k} - \sum_{j=1}^{k}\binom{v-u}{k-j}\binom{u}{j}$$
$$= \binom{v-u}{k}\binom{u}{0} = \binom{v-u}{k}.$$

But as $\sum_{i=1}^{N} x_{i,0}$ is precisely the number of times the empty set occurs in the $\mathcal{A}_i$, this proves our result. □

Thus, to check whether a collection of m-partitions satisfy the conditions of theorem 1.2.4, it suffices to only consider the case $1 \leq j \leq k$ and to show that no j-subset of U occurs at all for $j > k$.

1.3 The Cases $\mathbf{v \geq ku}$ and $\mathbf{v = 2u}$

We first give our proof for the case $v \geq ku$:

Theorem 1.3.1 *If $k \mid u, v,$ and $v \geq ku$ then $(u, v) \in \mathcal{EP}_k$.*

Proof. Let $U = [u]$, $V = [v]$, with $v \geq ku$. For $1 \leq i \leq k$, let j_i be the smallest nonnegative integer such that $j_i + u$ is divisible by i, and let $U_i = U$ if $j_i = 0$ and $U_i = U \cup \{x_1, \ldots, x_{j_i}\}$ where the x_j are points distinct from the elements of U, so that $|U_i|$ is divisible by i. Then, let $\mathcal{T}^{(i)}$ be such that $(U_i, i, \mathcal{T}^{(i)})$ is a one-factorization of $\binom{U_i}{i}$ (we know such a $\mathcal{T}^{(i)}$ exists by theorem 1.2.1). Note that, since $k \mid u$, we must have $U_k = U$. Let $\mathcal{Q} = \mathcal{T}^{(k)}$. Then, let $m = \frac{v}{k}$ and let $\mathcal{A}^{(i)}$ be a collection of m-partitions defined as follows: $\mathcal{A}^{(i)} = \{\mathcal{A}_j^{(i)} : j = 1, 2, \ldots, \binom{|U_i|-1}{i-1}\}$ where

$$\mathcal{A}_j^{(i)} = \{T \cap U : T \in \mathcal{T}_j^{(i)}\} \cup R_i,$$

where R_i is $m - |\mathcal{T}_j^{(i)}|$ copies of the empty set (this is to ensure $\mathcal{A}_j^{(i)}$ is an m-partition). Note that for $1 \leq i \leq k$, we always have $m - |\mathcal{T}_j^{(i)}| \geq 0$ because of our condition requiring $v \geq ku$. Thus, it follows that, for $1 \leq s \leq i$, as each s-subset of U is contained in $\binom{j_i}{i-s}$ i-subsets of U_i, it follows that each s-subset of U occurs $\binom{j_i}{i-s}$ times in the $\mathcal{A}_j^{(i)}$ (and of course each s-subset of U with $s > i$ does not occur at all in the $\mathcal{A}_j^{(i)}$ by construction).

Now, we define a sequence of integers α_i for $1 \leq i \leq k$ inductively. First, let $\alpha_k = 1$. Then, assuming we have defined $\alpha_{s+1}, \alpha_{s+2}, \ldots, \alpha_k$ for some s with $1 \leq s < k$, we define α_s by

$$\alpha_s = \binom{v-u}{k-s} - \sum_{i=s+1}^{k} \alpha_i \binom{j_i}{i-s}.$$

We now claim that each of the α_s is in fact nonnegative. We prove this by induction. Note that we already have $\alpha_k > 0$, so assume that, for some s with $1 \leq s < k$, each of $\alpha_{s+1}, \alpha_{s+2}, \ldots, \alpha_k$ is nonnegative. Further, note that this implies that $\alpha_i \leq \binom{v-u}{k-i}$ for $s < i \leq k$, since in our above construction of α_i all terms in the sum will be nonnegative. Also, note that, by definition, $j_i \leq i-1$. So, note that

$$\begin{aligned}\alpha_s &= \binom{v-u}{k-s} - \sum_{i=s+1}^{k} \alpha_i \binom{j_i}{i-s} = \binom{v-u}{k-s} - \alpha_{s+1} j_{s+1} - \sum_{i=s+2}^{k} \alpha_i \binom{j_i}{i-s} \\ &\geq \binom{v-u}{k-s} - s\binom{v-u}{k-s-1} - \sum_{i=s+2}^{k} \alpha_i \frac{j_i - i + s + 1}{i-s} \binom{j_i}{i-s-1}.\end{aligned}$$

But note that $j_i - i + s + 1 \leq i - 1 - i + s + 1 = s$. Also, for $i \geq s+2$, $i - s \geq 2$ and thus for such i we have $\frac{j_i - i + s + 1}{i-s} \leq \frac{s}{2}$. Thus

$$\alpha_s \geq \binom{v-u}{k-s} - s\binom{v-u}{k-s-1} - \frac{s}{2} \sum_{i=s+2}^{k} \alpha_i \binom{j_i}{i-s-1}.$$

But note that by our induction assumption $a_{s+1} \geq 0$ and thus, using its definition

$$\sum_{i=s+2}^{k} \alpha_i \binom{j_i}{i-s-1} \leq \binom{v-u}{k-s-1},$$

which, when applied to our above inequality, yields

$$\alpha_s \geq \binom{v-u}{k-s} - s\binom{v-u}{k-s-1} - \frac{s}{2}\binom{v-u}{k-s-1} = \binom{v-u}{k-s} - \frac{3s}{2}\binom{v-u}{k-s-1}$$
$$\geq \frac{v-u-k+s+1}{k-s}\binom{v-u}{k-s-1} - \frac{3s}{2}\binom{v-u}{k-s-1}$$
$$\geq \binom{v-u}{k-s-1}(\frac{v-u-k+s+1}{k-s} - \frac{3s}{2}).$$

But note that as per our assumptions above, we have $v \geq ku$. Also, theorem 1.3.1 is trivially true in the case $u = k$, so we can also assume that $u \geq 2k$. Using these facts, we have

$$\frac{v-u-k+s+1}{k-s} = \frac{v-u+1}{k-s} - 1 \geq \frac{v-u}{k-s} - 1 \geq \frac{2k(k-1)}{k-s} - 1$$
$$\geq \frac{2k(k-s)}{k-s} - 1 = 2k - 1 = \frac{3k}{2} + \frac{k}{2} - 1 \geq \frac{3k-1}{2} \geq \frac{3s}{2}.$$

Thus, it follows that $\alpha_s \geq 0$.

So, each α_i is a nonnegative integer. Now, to form our collection of m-partitions, we simply take the union of α_i copies of $\mathcal{A}^{(i)}$, for $1 \leq i \leq k$. Call this collection $\mathcal{A}$. Note that this will include taking one copy of $\mathcal{A}^{(k)}$ (since $\alpha_k = 1$), which is just the one-factorization $\mathcal{Q}$ with the appropriate number of empty sets added to make it into a collection of m-partitions, so if we can extend $\mathcal{A}$ to be a one-factorization $(V, k, \mathcal{P})$ we will be done, as such a $(V, k, \mathcal{P})$ will have $(U, k, \mathcal{Q})$ as a subsystem of it. Note that by theorem 1.2.4 and theorem 1.2.5, it follows that to show this we must show that each s-subset of U, for $1 \leq s \leq u$, occurs $\binom{v-u}{k-s}$ times in the m-partitions contained in $\mathcal{A}$. Note that for $k < s \leq u$ this is trivial, as we constructed the m-partitions such that no such s-subset of U is contained in any of them. Also, each k-subset can only be contained in the m-partitions contained in $\mathcal{A}^{(k)}$, and as they are contained once in it, it follows that they occur once in $\mathcal{A}$, and as $1 = \binom{v-u}{k-k}$, this is the required value. So it remains to consider $1 \leq s \leq k-1$. Note that each s-subset of U can only be contained in one of the $\mathcal{A}^{(i)}$ with $i \geq s$ and are contained exactly $\binom{j_i}{i-s}$ times in it, by above. So thus, as we have α_i copies of $\mathcal{A}^{(i)}$ in $\mathcal{A}$, each s-subset occurs exactly $\sum_{i=s}^{k} \alpha_i \binom{j_i}{i-s}$ times in $\mathcal{A}$. But note that

$$\sum_{i=s}^{k} \alpha_i \binom{j_i}{i-s} = \alpha_s \binom{j_i}{s-s} + \sum_{i=s+1}^{k} \alpha_i \binom{j_i}{i-s} = \alpha_s + \sum_{i=s+1}^{k} \alpha_i \binom{j_i}{i-s}$$
$$= \binom{v-u}{k-s} - \sum_{i=s+1}^{k} \alpha_i \binom{j_i}{i-s} + \sum_{i=s+1}^{k} \alpha_i \binom{j_i}{i-s} = \binom{v-u}{k-s},$$

and thus each s-subset occurs the proper amount of times, showing that $\mathcal{A}$ can be extended to being a one-factorization of V, which, by definition, has $(U, k, \mathcal{Q})$ as a subsystem, implying that $(u, v) \in \mathcal{EP}_k$. □

Now, before considering $v = 2u$ in general, consider the special case when $k!$ (and not just k) divides u:

Theorem 1.3.2 *If $k! \mid u$ and $v = 2u$, then $(u, v) \in \mathcal{EP}_k$.*

Proof. Let $U = [u], V = [v]$. For each integer r with $\frac{u}{2} \leq r < k$, as r divides u, let $\mathcal{T}^{(r)}$ be such that $(U, r, \mathcal{T}^{(r)})$ is a one-factorization of $\binom{U}{r}$. Then, for each s with $1 \leq s \leq \binom{u-1}{r-1}$, using $\mathcal{T}_s^{(r)}$, we will construct a collection $\mathcal{A}_{r,s}$ of m-partitions, where $m = \frac{v}{k}$ as before. Each m-partition constructed in this way will contain precisely $\frac{u}{k}$ r-subsets of U and $\frac{u}{k}$ $(k-r)$-subsets of U. Let $\sigma_1, \sigma_2, \ldots, \sigma_{m_r}$, where $m_r = \frac{u}{r}$, be the r-subsets of U that are contained in $\mathcal{T}_s^{(r)}$. Then, for i with $1 \leq i \leq k$, if we let $t = \frac{u}{rk}$, let

$$B_i = \{\sigma_{(i-1)t+1}, \sigma_{(i-1)t+2}, \ldots, \sigma_{it}\},$$

and let

$$C_i = \bigcup_{\sigma \in B_i} \sigma,$$

so B_i is a collection of t different r-subsets of U and C_i is the union of those r-subsets, making it a collection of $\frac{u}{k}$ elements of U. Then, let X be any $(k-r)$-subset of $[k]$ and let $X_1, \ldots, X_n$ (for some

n with $1 \leq n \leq k-r$) be any partition of X, where here we mean partition in the usual sense, i.e., each element of X is in exactly one of the X_i and each X_i is nonempty. Then, let $x_i = |X_i|$. Then, for any $j \in X_i$, let $\mathcal{S}^{(j)}$ be such that $(C_j, x_i, \mathcal{S}^{(j)})$ is a one-factorization (such an $\mathcal{S}^{(j)}$ exists because of the fact that $|C_j| = \frac{u}{k}$ and as $k!$ divides u and we have that $x_i \leq k-r < k$ it follows that x_i divides $|C_j|$). Then, fix a given one factor S_j in each of the $\mathcal{S}^{(j)}$ and let

$$T_i = \bigcup_{j \in X_i} S_j.$$

Each T_i will thus be a collection of x_i-subsets of U, and by construction $|T_i| = \frac{u}{k}$. Then, consider a fixed labeling of the elements of each T_i as

$$T_i = \{y_1^{(i)}, y_2^{(i)}, \ldots, y_{\frac{u}{k}}^{(i)}\}.$$

Finally, for that labeling, for each integer j with $1 \leq j \leq \frac{u}{k}$ let

$$Y_j = \bigcup_{i=1}^{n} y_j^{(i)}.$$

Note that this implies that

$$|Y_j| = \sum_{i=1}^{n} |y_j^{(i)}| = \sum_{i=1}^{n} x_i = |X| = k-r.$$

Finally, let W be the following m-partition of U:

$$W = \{Y_1, Y_2, \ldots, Y_{\frac{u}{k}}\} \cup \{B_i : i \in [k] \setminus X\}.$$

One can easily check that such a W is in fact an m-partition of U. Also, it is clear that W consists $\frac{u}{k}$ sets of size r and $\frac{u}{k}$ sets of size $k-r$. Further, we take $a_{\{x_1,\ldots,x_n\}}$ copies of W, where

$$a_{\{x_1,\ldots,x_n\}} = \frac{u(x_1-1)!\ldots(x_n-1)!(r-1)!}{(k-n)!}.$$

Note that this is an integer, as u is divisible by $k!$ and thus is divisible by $(k-n)!$ Then, we repeat this process for every possible ordering of each of the T_i, and consider the collection of $(a_{\{x_1,\ldots,x_n\}})(x_1!)\ldots(x_n!)$ different m-partitions. Note that by our construction, each $(k-r)$-subset of U that, for each i with $1 \le i \le n$ contains precisely x_i elements in one of the C_j with $j \in X_i$ will occur in exactly one of the above W and thus will occur $a_{\{x_1,\ldots,x_n\}}$ times in this collection.

Now, we repeat the above construction for every possible $(k-r)$-set X in $[k]$ and every possible partition of such an X and let the collection of all of these m-partitions be $\mathcal{A}_{r,s}$. Now, consider Y, any arbitrary $(k-r)$-subset of U. Let $z_i = |Y \cap C_i|$ and let $D = \{i : z_i > 0\}$, say $D = \{i_1, \ldots, i_n\}$ for some n with $1 \le n \le k-r$. Then, by our above construction, Y will occur $a_{\{z_{i_1},\ldots,z_{i_n}\}}$ times in $\mathcal{A}_{r,s}$ for each possible $(k-r)$-subset X of $[k]$ and partition of X into n sets $X_1, \ldots X_n$ with $|X_j| = z_{i_j}$ and $i_j \in X_j$. Note that such an $X_1, \ldots, X_n$ is equivalent to choosing $z_{i_j} - 1$ elements out of the $k-n$ element set $[k] \setminus D$ to put in the set X_j with each i_j (and also choosing r elements not to included in X). Thus, there are

$$\binom{k-n}{z_{i_1}-1,\ldots,z_{i_n}-1,r}$$

ways to pick the $X_1, \ldots, X_n$. This implies that Y will occur precisely

$$a_{\{z_{i_1},\ldots,z_{i_n}\}}\binom{k-n}{z_{i_1}-1,\ldots,z_{i_n}-1,r} =$$
$$\frac{u(z_{i_1}-1)!\ldots(z_{i_n}-1)!(r-1)!}{(k-n)!} * \frac{(k-n)!}{(z_{i_1}-1)!\ldots(z_{i_n}-1)!r!} = \frac{u}{r}$$

times in $\mathcal{A}_{r,s}$. Further, if we let

$$\mathcal{A}_r = \bigcup_{s=1}^{\binom{u-1}{r-1}} \mathcal{A}_{r,s},$$

it follows that each $(k-r)$-subset of U occurs exactly

$$\frac{u}{r}\binom{u-1}{r-1} = \binom{u}{r}$$

times in $\mathcal{A}_r$. As each m-partition contained in $\mathcal{A}_r$ contains exactly $\frac{u}{k}$ $(k-r)$-subsets of U, and there are $\binom{u}{k-r}$ such subsets, it follows that there must have been precisely

$$\frac{\binom{u}{r}\binom{u}{k-r}}{\frac{u}{k}}$$

m-partitions in $\mathcal{A}_r$. Further, it is clear by our above construction that every r-subset of U must occur the same number of times in $\mathcal{A}_r$ and thus, as there were $\frac{u}{k}$ in each m-partition, and $\binom{u}{r}$ of them total, it follows that each of them occurred precisely

$$\frac{\binom{u}{r}\binom{u}{k-r}}{\frac{u}{k}} * \frac{\frac{u}{k}}{\binom{u}{r}} = \binom{u}{k-r}$$

times in $\mathcal{A}_r$. Thus, for both $i = r$ and $i = k - r$, each i-subset of U occurs exactly $\binom{u}{k-i}$ times in $\mathcal{A}_r$ (and no sets of any other size occur in $\mathcal{A}_r$ by construction).

Thus, if we let R be $\frac{u}{k}$ copies of the empty set, and let

$$\mathcal{A}_k = \{Q \cup R : Q \in \mathcal{Q}\},$$

and let

$$\mathcal{A} = \bigcup_{r=\lfloor \frac{u}{2} \rfloor}^{k} \mathcal{A}_r,$$

it follows that each i-subset of U, for $1 \leq i \leq k$, occurs exactly $\binom{u}{k-i}$ times in $\mathcal{A}$ and thus by our above theorems, we can embed U in V so $(u, v) \in \mathcal{EP}_k$. $\square$

To prove the result when $k!$ does not divide u requires us to first prove a generalization of theorem 1.2.1. First, we give the following definition:

Definition Given an m-partition $\mathcal{A}$ of U, and nonnegative integers $t_0, t_1, \ldots, t_u$ such that $\sum_{i=0}^{u} t_i = m$ and $\sum_{i=0}^{u} i t_i = u$, we say $\mathcal{A}$ is of type $(t_0, t_1, \ldots, t_u)$ if and only if $\mathcal{A}$ contains precisely t_i elements of size i for $0 \leq i \leq u$.

Note the original problem of finding a $(V, k, \mathcal{P})$ that contained $(U, k, \mathcal{Q})$ as a subsystem could be reduced to finding a collection of m-partitions of U such that each j-subset occurred a fixed number of times. In our theorem below, we further reduce that problem to a problem of considering a certain partition π of $u(\binom{v-1}{k-1} - \binom{u-1}{k-1})$ into parts of size at most $k-1$ and finding a decomposition of π into partitions of u each of which has at most m parts. We note that this theorem is in fact an implicit consequence of theorem 1 in [3], though the authors never explicitly describe it as such. Our proof was found independently of theirs. Further, the result in [3] is incredibly general, whereas ours is specifically formulated to fit this investigation. As such, the following proof is simpler then the proof of their theorem 1, and uses terminology that is consistent with our above work.

Theorem 1.3.3 *If there exist nonnegative integers a_i for $1 \leq i \leq u$, natural number N, and nonnegative integers $t_{i,j}$ for $1 \leq i \leq u$ and $1 \leq j \leq N$ such that, for each $i \in [u]$ we have*

$$a_i \binom{u}{i} = \sum_{j=1}^{N} t_{i,j},$$

and, for each $j \in [N]$, we have

$$\sum_{i=1}^{u} t_{i,j} \leq m,$$

$$\sum_{i=1}^{u} i t_{i,j} = u,$$

then there exists m-partitions $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N$, of $U = [u]$ where, for all $j \in [n]$, $\mathcal{A}_j$ is of type $(t_{0,j}, t_{1,j}, \ldots, t_{u,j})$, where $t_{0,j} = m - \sum_{i=1}^{u} t_{i,j}$. Further, it will be the case that each $S \subseteq U$ with $S \neq \emptyset$ occurs exactly $a_{|S|}$ times in the $\mathcal{A}_j$.

Proof. This proof in fact is a generalization of the proof of theorem 38.1 in [1]. First, we let $a_0 = \sum_{j=1}^{N} t_{0,j}$, since then it will be trivially true that the equation $a_i\binom{u}{i} = \sum_{j=1}^{N} t_{i,j}$ holds for $i = 0$. Also, we have

$$\sum_{i=0}^{u} t_{i,j} = t_0 + \sum_{i=1}^{u} t_{i,j} = \left(m - \sum_{i=1}^{u} t_{i,j}\right) + \sum_{i=1}^{u} t_{i,j} = m,$$

$$\sum_{i=0}^{u} it_{i,j} = 0 + \sum_{i=1}^{u} it_{i,j} = u.$$

Now, we will inductively show, for any integer w with $0 \leq w \leq u$, that we can find m-partitions $\mathcal{A}_1^w, \mathcal{A}_2^w, \ldots, \mathcal{A}_N^w$ of the set $[w]$ (when $w = 0$ this is just the empty set) and functions $c_1^w, c_2^w, \ldots, c_N^w$ with

$$c_j^w : \mathcal{A}_j^w \to \{0, 1, \ldots u\},$$

such that for each $S \subseteq [u]$ and for each i with $0 \leq i \leq u$, there are exactly $a_i \binom{u-w}{i-|S|}$ values of $j \in \{1, 2, \ldots, N\}$ such that $S \in \mathcal{A}_j^w$ and $c_j^w(S) = i$ (for $S = \emptyset$ we count each j with multiplicity the number of times S appears in $\mathcal{A}_j$ and has c_j^w map it to i). Also, for each $j \in [N]$, there will be exactly $t_{i,j}$ elements of $\mathcal{A}_j^w$ that map to i under c_j^w for each $i \in \{0, 1, \ldots, u\}$.

For $w = 0$, for each $j \in [N]$, let $\mathcal{A}_j^0$ contain m copies of the empty set, and arbitrarily choose $t_{i,j}$ of them to have the function c_i^0 take the value i (this can be done since $\sum_{i=0}^{u} t_{i,j} = m$). Then, the empty set will be sent to the value i a total of $\sum_{j=1}^{N} t_{i,j}$ times (counting multiplicities), but, as

$$\sum_{j=1}^{N} t_{i,j} = a_i \binom{u}{i} = a_i \binom{u-w}{i-0},$$

this shows our conditions are satisfied.

So assume for some w with $0 \leq w < u$ we have $\mathcal{A}_1^w, \mathcal{A}_2^w, \ldots, \mathcal{A}_N^w$ and $c_1^w, c_2^w, \ldots, c_N^w$ with our above conditions being satisfied. Then, we form a transportation network as follows. The vertex set will be

$$\{\sigma\} \cup \{\tau\} \cup A \cup \bigcup_{i=0}^{u} B_i,$$

where

$$A = \{\mathcal{A}_j^w : 1 \leq j \leq N\},$$

$$B_i = \{(S,i) : S \subseteq \{1,2,\ldots,w\}\}.$$

The source will be σ and there will be a directed edge from σ to each $\mathcal{A}_j^w$ with capacity 1. There will be a directed edge from $\mathcal{A}_j^w$ to (S,i) with capacity 1 if and only if $S \in \mathcal{A}_j^w$ and $c_j^w(S) = i$ (if $S = \emptyset$, there will be p such edges, where p is the number of copies of $\emptyset$ contained in A_j^w). The sink will be τ and there will be a directed edge from each pair (S,i) to τ with capacity $a_i\binom{u-w-1}{i-|S|-1}$.

We define a flow f on this network as follows:

$$f(\sigma, \mathcal{A}_j^w) = 1 \text{ for } 1 \le j \le N,$$

$$f(\mathcal{A}_j^w, (S,i)) = \frac{i-|S|}{u-w} \text{ for each } S \in \mathcal{A}_j^w \text{ with } c_j^w(S) = i,$$

$$f((S,i),\tau) = a_i\binom{u-w-1}{i-|S|-1} \text{ for each } S \subseteq \{1,2,\ldots,w\} \text{ and } 0 \le i \le u.$$

To see that this is a flow, it suffices to check on each $\mathcal{A}_j^w$ and each (S,i). So note that the value of f into $\mathcal{A}_j^w$ is 1 and the value out of $\mathcal{A}_j^w$ is

$$\sum_{S\in\mathcal{A}_j^w} \frac{c_j^w(S)-|S|}{u-w} = \frac{1}{u-w}\left(\sum_{S\in\mathcal{A}_j^w} c_j^w(S) - \sum_{S\in\mathcal{A}_j^w} |S|\right) = \frac{1}{u-w}\left(\left(\sum_{i=0}^{u} it_{i,j}\right) - w\right)$$

$$\sum_{S\in\mathcal{A}_j^w} \frac{c_j^w(S)-|S|}{u-w} = \frac{1}{u-w}(u-w) = 1.$$

The value of f out of (S,i) is $a_i\binom{u-w-1}{i-|S|-1}$ and, since S takes the value i on precisely $a_i\binom{u-w}{i-|S|}$ of the c_j^w, the value into (S,i) is

$$\frac{i-|S|}{u-w}a_i\binom{u-w}{i-|S|} = a_i\frac{(i-|S|)(u-w)!}{(u-w)(i-|S|)!(u-w-i+|S|)!} = a_i\binom{u-w-1}{i-|S|-1}.$$

Thus f is a flow, and furthermore, it is maximal, since each edge out of σ and each edge into τ is saturated (and thus each such edge is saturated in any maximal flow). Thus, by theorem 7.2 in [1], it follows there is an integer-valued maximal flow f'. Since each edge leaving σ is saturated, f' assigns value 1 to exactly one of the edges leaving each $\mathcal{A}_j^w$, which distinguishes one of the elements $S \in \mathcal{A}_j^w$

(corresponding to the unique (S,i) with $f(\mathcal{A}_j^w,(S,i))=1$). Call it S_j and let $S_j' = S_j \cup \{w+1\}$. Then, define $\mathcal{A}_j^{w+1}$ and c_j^{w+1} as follows:

$$\mathcal{A}_j^{w+1} = \mathcal{A}_j^w \setminus \{S_j\} \cup \{S_j'\},$$

$$c_j^{w+1}(S) = c_j^w(S) \ \text{ if } S \neq S_j',$$

$$c_j^{w+1}(S_j') = c_j^w(S_j).$$

Immediately it follows that c_j^{w+1} takes the value i exactly as many times as c_j^w takes the value i, which is $t_{i,j}$. Also, it is clear that each $\mathcal{A}_j^{w+1}$ is an m-partition of $\{1,2,\ldots,w+1\}$. Now, each $S \subseteq \{1,2,\ldots,w+1\}$ that contains $(w+1)$ will have $c_j^{w+1}(S)=i$ whenever $c_j^w(S\setminus\{w+1\})=i$ and the edge entering $(S\setminus\{w+1\},i)$ takes flow value 1. This is precisely the total flow leaving $(S\setminus\{w+1\},i)$, which, since each edge into τ must be saturated, is

$$a_i\binom{u-w-1}{i-|S\setminus\{w+1\}|-1} = a_i\binom{u-(w+1)}{i-|S|},$$

and thus such an S satisfies our conditions. It remains to consider S with $(w+1)\notin S$. Then, S will have $c_j^{w+1}(S)=i$ once for each time an edge entering (S,i) takes flow value 0. This will be the total number of edges into (S,i) minus the flow value leaving (S,i), which is

$$a_i\binom{u-w}{i-|S|} - a_i\binom{u-w-1}{i-|S|-1} = a_i\binom{u-(w+1)}{i-|S|},$$

and thus all S satisfy our condition.

Thus our induction is complete and it follows that the above result holds for $w=u$. Let $\mathcal{A}_j = \mathcal{A}_j^u$. Then, each set S will have $c_j^u(S)=i$ precisely $a_i\binom{u-u}{i-|S|} = a_i\binom{0}{i-|S|}$ times. But $\binom{0}{i-|S|}$ is nonzero only when $i=|S|$, in which case it is 1. So each set $|S|$ will occur precisely $a_{|S|}$ times in the $\mathcal{A}_j$. Further, each $\mathcal{A}_j$ has $t_{i,j}$ elements taking value i under c_j^u, and the only elements that can take value i are i element sets. Thus $\mathcal{A}_j$ contains precisely $t_{i,j}$ elements of size i. But that means that $\mathcal{A}_j$ has type $(t_{0,j}, t_{1,j}, \ldots, t_{u,j})$. □

Note that, in theorem 1.3.3, if we consider the set V instead of U, let $N = \binom{v}{k}$, let $m = \frac{v}{k}$, let $t_{k,j} = m$ for $1 \leq j \leq N$, let $a_k = 1$ and let $t_{i,j} = a_i = 0$ for $1 \leq j \leq N$ and all $i \neq k$, it is easily checked that this gives us precisely theorem 1.2.1. In addition, if we let U be the set W in any one of the intermediate steps in the above proof (with each of the terms interpreted in the same way), and let each c_j^U for $1 \leq j \leq N$ be the constant function that sends each element to k, one can check that we have the exact result of theorem 1.2.4.

However, theorem 1.3.3 does more than just give alternate proofs of our above results; to see this, we first apply it to the case $v = 2u$. We need the following lemma:

Lemma 1.3.4 *For all nonnegative integers u, k, i with $u \neq 0$, $\frac{k}{u}\binom{u}{i}\binom{u}{k-i}$ is an integer.*

Proof. Consider two sets U_1, U_2 with $|U_1| = |U_2| = u$. Then, let A be the number of k element sets containing i elements from U_1, $k - i$ elements from U_2 and consisting of one distinguished element. Clearly, the number of such k-sets is $\binom{u}{i}\binom{u}{k-i}$ and the number of ways to pick a distinguished element from this k-set is k and thus $A = k\binom{u}{i}\binom{u}{k-i}$. However, if we choose the distinguished element first, it can either be chosen from U_1 or U_2. If it is chosen in U_1, there are u ways to choose it, $\binom{u-1}{i-1}$ ways to choose the rest of the elements from U_1 and $\binom{u}{k-i}$ ways to choose the elements from U_2. If the distinguished element is in U_2, there are u ways to choose it, $\binom{u-1}{k-i-1}$ ways to choose the rest of the elements from U_2 and $\binom{u}{i}$ ways to choose the elements from U_1. Thus,

$$A = u\binom{u-1}{i-1}\binom{u}{k-i} + u\binom{u}{i}\binom{u-1}{k-i-1},$$

and it follows that

$$\begin{aligned}\frac{k}{u}\binom{u}{i}\binom{u}{k-i} = \frac{A}{u} &= \frac{1}{u}\left(u\binom{u-1}{i-1}\binom{u}{k-i} + u\binom{u}{i}\binom{u-1}{k-i-1}\right)\\ &= \binom{u-1}{i-1}\binom{u}{k-i} + \binom{u}{i}\binom{u-1}{k-i-1},\end{aligned}$$

which is certainly an integer. □

We are now ready to deal with the case $v = 2u$ in full generality.

Theorem 1.3.5 *If $k|u$ and $v = 2u$, then $(u, v) \in \mathcal{EP}_k$.*

Proof. Let $U = [u], V = [v]$. We use a similar technique to the one used earlier in this section, which involves finding a collection $\mathcal{A}$ of $m = \frac{v}{k}$ partitions of Usuch that each r-element-subset occurs exactly $\binom{v-u}{k-r} = \binom{u}{k-r}$ times, for $1 \leq r \leq k-1$ (as, again, adding an appropriate number of copies of the empty set to each element of $\mathcal{Q}$ will insure every k-subset occurs once). First, note that if $\frac{k}{2}$ is an integer, we can simply take a one-factorization $(U, \frac{k}{2}, \mathcal{T})$, add an appropriate number of empty sets, and take $\binom{u}{\frac{k}{2}}$ copies of it. So it remains to consider $r \neq \frac{k}{2}$. We will in fact take m-partitions that contain r-sets and $(k-r)$-sets, for $1 \leq r < \frac{k}{2}$. So, for such an r, first let $N = \frac{k}{u}\binom{u}{r}\binom{u}{k-r}$, which, by lemma 1.3.4, is an integer (further it is clearly positive as $0 < r < k \leq u$).Then, for all $j \in \{1, 2, \ldots, N\}$ let $t_{0,j}, t_{1,j}, \ldots, t_{u,j}$ be defined by $t_{r,j} = t_{k-r,j} = \frac{u}{k}$ (note that k must divide u for a one-factorization to exist) and $t_{i,j} = 0$ if i is not equal to either r or $k-r$. Then we have

$$\sum_{i=0}^{u} t_{i,j} = \frac{2u}{k} = m,$$

$$\sum_{i=0}^{u} it_{i,j} = (r)(\frac{u}{k}) + (k-r)(\frac{u}{k}) = \frac{(r+k-r)(u)}{k} = \frac{ku}{k} = u.$$

Further, let $a_0, a_1, \ldots, a_u$ be defined by $a_r = \binom{u}{k-r}$ and $a_{k-r} = \binom{u}{r}$, and $a_i = 0$ for i not equal to r or $k-r$. Also, for i not equal to r or $k-r$ we trivially have

$$a_i \binom{u}{i} = \sum_{j=1}^{N} t_{i,j},$$

as $a_i = 0$ and for any $j \in \{1, 2, \ldots, N\}$ we have $t_{i,j} = 0$. Also,

$$a_r \binom{u}{r} = \binom{u}{k-r}\binom{u}{r} = \frac{u}{k} N = \sum_{j=1}^{N} t_{r,j},$$

$$a_{k-r}\binom{u}{k-r} = \binom{u}{r}\binom{u}{k-r} = \frac{u}{k}N = \sum_{j=1}^{N} t_{k-r,j}.$$

Thus, by theorem 1.3.3, it follows that we can find a collection $\mathcal{A}^r$ of m-partitions that have each r-subset of U occur $\binom{u}{k-r}$ times and each $(k-r)$-subset of U will occur $\binom{u}{r}$ times (and every other subset will not occur). So simply by taking the union of the $\mathcal{A}^r$ for $1 \leq r < \frac{k}{2}$ we will have our desired collection of m-partitions, finishing the proof. □

Before proving theorem 1.2.3 for $k \in \{3, 4, 5\}$, we provide an application of theorem 1.3.3 to the problem of completing Latin squares.

1.4 An Application of Theorem 1.3.3 to Symmetric Latin Squares and Nonuniform Hypergraphs

We will use theorem 1.3.3 to give alternate proofs of a theorem of Cruse's (which can be found in [4]) and a theorem of Hoffman's (which can be found in [5]), both of which talk about extending incomplete symmetric Latin squares. First, we point out that in [3], the authors noticed the connection between their theorem 1 (of which our theorem 1.3.3 is a special case) and the results in [4] and [5], but did not explicitly give an alternate proof of either result. Also, it was shown by the authors in [3] that the result in [4] was a corollary of their result. However, we provide separate proofs of each result for completeness and to show the different ways in which theorem 1.3.3 can be applied.

We remind the reader that a (possibly incomplete) Latin square L can be defined as a function $L : R \times C \to S$, where $|R| = |C| = k$ and $|S| = n$ for some $k, n \in \mathbb{N}$ with $k \leq n$, such that $L(x, y) = L(x, y')$ implies $y = y'$ and $L(x, y) = L(x', y)$ implies $x = x'$. A symmetric Latin square is a Latin square where $L(x, y) = L(y, x)$ for all $x \in R, y \in C$. If $k < n$, then we say L is incomplete, otherwise we say L is complete. Unless stated otherwise, we will assume that $R = C = [k]$ and $S = [n]$. We say that the Latin square $L' : R' \times C' \to S$ is an extension of the Latin square $L : R \times C \to S$ if $R \subset R'$, $C \subset C'$ and L' restricted to $R \times C$ is equal to L.

Consider a fixed Latin square $L : [k] \times [k] \to [n]$. For any $j \in [n]$, define $N_L(j) = |L^{-1}(j)|$, define

$d_L(x) = |L^{-1}(j) \cap D|$ where $D = \{(i, i) : i \in [k]\}$ and define $e_L(j) = N_L(j) - d_L(j)$. So $N_L(j)$ counts the number of times j occurs as a symbol in L, $d_L(j)$ counts the number of times j occurs on the diagonal of L, and $e_L(j)$ counts the number of times j occurs off of the diagonal in L (where R, C, and S have the common interpretations as the row, column, and symbol set of L, respectively).

We are now ready to give our alternate proof of Cruse's theorem, using our theorem 1.3.3.

Theorem 1.4.1 *Let $k, n \in \mathbb{N}$ with $k < n$. Then, let $L : [k] \times [k] \to [n]$ be an incomplete symmetric Latin square. L is extendible to a complete symmetric Latin square L' if and only if $N_L(j) \geq 2k - n$ for each $j \in [n]$ and $N_L(j) \equiv n \pmod 2$ for at least k different $j \in [n]$.*

Proof. For $j \in [n]$, we define the set B_j by, for $x \in [k]$, $\{x\} \in B_j$ if and only if $L(x, x) = j$ and, for $x, y \in [k]$ with $x \neq y$, $\{x, y\} \in B_j$ if and only if $L(x, y) = j$. Note that each 1-element subset of $[k]$ corresponds to an occurrence of j on the diagonal of L, so there are $d_L(j)$ 1-element sets in B_j. Since L is symmetric, if $x \neq y$, $\{x, y\} \in B_j$ implies that $L(x, y) = L(y, x) = j$. Thus, each 2-element subset of $[k]$ in B_j corresponds to two different off-diagonal occurrences of j in L and thus $\frac{1}{2}e_L(j)$ is the number of 2-element sets in B_j (and thus $e_L(j)$ must be even). Thus $|B_j| = d_L(j) + \frac{1}{2}e_L(j)$. Further, define C_j in the following way:

$$C_j = \{\{x\} : x \in [k] \setminus \bigcup_{S \in B_j} S\}.$$

Then, note that $B_j \cup C_j$ is a partition of $[k]$ into 1- and 2-element subsets. We can thus define n-partitions $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ of $[k]$ by taking $B_j \cup C_j$ and adding an appropriate number of copies of the empty set to obtain $\mathcal{D}_j$.

Now, suppose that a complete symmetric L' exists that is an extension of L. Then, we let B'_j be defined (similar to B_j above) by, for $x \in [n]$, $\{x\} \in B'_j$ if and only if $L'(x, x) = j$ and, for $x, y \in [n]$ with $x \neq y$, $\{x, y\} \in B'_j$ if and only if $L'(x, y) = j$. Note that, by definition, we must have $B_j \subset B'_j$. Also, by a well-known property of complete Latin squares (see, for example, [1]), each symbol must appear once in each row (and column). This implies that each symbol occurs exactly n times in L and thus $N_{L'}(j) = n$ for all $j \in [n]$ and that that each B'_j must be a partition of $[n]$

into 1- and 2-element sets. Further, for any $j \in [n]$, the number of 1-element sets in B'_j is just $d_{L'}(j)$ and the number of two element sets in B'_j is $\frac{1}{2}e_{L'}(j)$, which implies (as above) that $e_{L'}(j)$ is even. Thus, since $n = N_{L'}(j) = d_{L'}(j) + e_{L'}(j)$, it follows that $d_{L'}(j) \equiv n \pmod{n}$. Now, consider any $j \in [n]$ with $N_L(j) \not\equiv n \pmod{2}$; thus $N_L(j) \not\equiv d_{L'}(j) \pmod{2}$. Since $N_L(j) = d_L(j) + e_L(j)$ we have $N_L(j) \equiv d_L(j) \pmod{2}$ so $d_L(j) \not\equiv d_{L'}(j) \pmod{2}$. Thus, there is at least one $y \in [n] \setminus [k]$ with $L'(y, y) = j$. Since there are only $n - k$ such y, it follows that there can be at most $n - k$ such j, and thus, for at least k choices of j, we have $N(j) \equiv n \pmod{2}$.

Further, consider any $j \in [n]$ and any $x \in [k]$ with $x \in C_j$. There must be some $y \in [n]$ with $L'(x, y) = j$ but since $x \notin \cup_{S \in B_j} S$, we must have $y \in [n] \setminus [k]$. Further, by definition, for each $x \in C_j$, we must have a unique such y, implying that $n - k \geq |C_j|$. But $|C_j|$ just counts the number of rows (or columns) of L that do not contain the symbol j, and thus $|C_j| = k - N_L(j)$. Thus we have $n - k \geq k - N_L(j)$ implying that $N_L(j) \geq 2k - n$.

Conversely, assume that, for our incomplete symmetric Latin square L, we have $N_L(j) \geq 2k - n$ for each $j \in [n]$ and $N_L(j) \equiv n \pmod{2}$ for at least k different $j \in [n]$. Below, we show how to choose a_i, $t_{i,j}$ and c_j^k for $1 \leq j \leq n$ and $0 \leq i \leq n$ to satisfy both the initial conditions and one of the inductive steps of theorem 1.3.3. First, we let $a_i = t_{i,j} = 0$ for $i > 2$, $a_2 = a_1 = 1$, and $a_0 = \binom{n}{2}$. Now, we define functions $c_j, c_j^k : \mathcal{D}_j \to \{0, 1, 2\}$ for all $j \in [n]$.

For any $S \in \mathcal{D}_i$ with $|S| = 2$ we define $c_j(S) = c_j^k(S) = 2$. Then, for any $S \in \mathcal{D}_i$ with $|S| = 1$ and $S \in B_j$, we define $c_j(S) = c_j^k(S) = 1$. Further, for any $S \in \mathcal{D}_i$ with $|S| = 1$ and $S \in C_j$, we define $c_j(S) = c_j^k(S) = 2$. This defines $c_j(S)$ and $c_j^k(S)$ for all nonempty S. We are left with $n - N_L(j) - |C_j|$ copies of the empty set (this is nonnegative since $|C_j| = k - N_L(j)$ by above). If $N_L(j) \equiv n \pmod{2}$, for $\frac{1}{2}(n - N_L(j)) - |C_j|$ copies of the empty set, we define $c_j(\emptyset) = 2$, and for the other $\frac{1}{2}(n - N_L(j))$ copies we define $c_j(\emptyset) = 0$. Now, to see that $\frac{1}{2}(n - N_L(j)) - |C_j|$ is in fact nonnegative, just note that by assumption, we have $N_L(j) \geq 2k - n$, and thus

$$\frac{1}{2}(n - N_L(j)) - |C_j| = \frac{1}{2}(n - N_L(j)) - (k - N_L(j)) = \frac{1}{2}n + \frac{1}{2}N_L(j) - k$$
$$\geq \frac{1}{2}n + \frac{1}{2}(2k - n) - k = 0.$$

If $N_L(j) \not\equiv n \pmod 2$, we define $c_j(\emptyset) = 1$ for one copy of the empty set, $c_j(\emptyset) = 2$ for $\frac{1}{2}(n - N_L(j) - 1) - |C_j|$ copies of the empty set, and $c_j(\emptyset) = 0$ for the other $\frac{1}{2}(n - N_L(j) - 1)$ copies of the empty set. Again, all these numbers are nonnegative: since $\frac{1}{2}(n - N_L(j)) - |C_j| \geq 0$, we have $\frac{1}{2}(n - N_L(j) - 1) - |C_j| \geq -\frac{1}{2}$, and since this quantity must be an integer, it is nonnegative.

Note that in either case, for a fixed $j \in [n]$ we have at least as many copies of the empty set in $\mathcal{D}_j$ satisfying $c_j(\emptyset) = 0$ as those satisfying $c_j(\emptyset) = 2$. Then, let R be the set of all $j \in [n]$ such that $N_L(j) \not\equiv n \pmod 2$. Thus, if $r = |R|$ there are precisely r copies of the empty set with $c_j(\emptyset) = 1$ for some j. If we let p be the number of copies of the empty set such that $c_j(\emptyset) = 2$, we have

$$\begin{aligned}
p &= \sum_{j \in R} (\frac{1}{2}(n - N_L(j) - 1) - |C_j|) + \sum_{j \in [n] \setminus R} (\frac{1}{2}(n - N_L(j)) - |C_j|) \\
&= \sum_{j \in R} (\frac{1}{2}(n - N_L(j)) - |C_j|) + \sum_{j \in [n] \setminus R} (\frac{1}{2}(n - N_L(j)) - |C_j|) - \sum_{j \in R} \frac{1}{2} \\
&= \sum_{j \in [n]} (\frac{1}{2}(n - N_L(j)) - |C_j|) - \frac{r}{2} \\
&= 2\frac{1}{2} \sum_{j \in [n]} n - \frac{1}{2} \sum_{j \in [n]} N_L(j) - \sum_{j \in [n]} |C_j| - \frac{r}{2}.
\end{aligned}$$

Further, note that $\sum_{j \in [n]} |N_L(j)|$ simply counts the number of entries in L, which is k^2. By above, we have $|C_j| = k - N_L(j)$ and thus

$$\sum_{j \in [n]} |C_j| = \sum_{j \in [n]} |C_j|(k - N_L(j)) = kn - \sum_{j \in [n]} N_L(j) = kn - k^2.$$

Thus, it follows that

$$p = \frac{1}{2}n^2 - \frac{1}{2}k^2 - (kn - k^2) - \frac{r}{2} = \frac{1}{2}(n^2 + k^2 - r - 2kn) = \frac{1}{2}((n-k)^2 - r).$$

Further, since p is an integer, we must have $r \equiv (n-k)^2 \pmod 2$, and since $(n-k)^2 \equiv (n-k) \pmod 2$, we have $r \equiv n - k \pmod 2$, implying that $n - k - r$ is even (it must be nonnegative since by assumption $m \leq n - k$). Thus, choose $\frac{1}{2}(n - k - r)$ copies of the empty set with $c_j(\emptyset) = 2$, and for each, choose a unique copy of the empty set also in $\mathcal{D}_j$ with $c_j(\emptyset) = 0$ (such unique copies exist by above). Then, for all such copies of the empty set, let $c_j^k(\emptyset) = 1$. For all other copies of the empty set, simply let $c_j^k(\emptyset) = c_j(\emptyset)$.Note that, for each pair of copies of the empty set with $c_j(\emptyset) \neq c_j^k(\emptyset)$ we have (if $\emptyset_1$ and $\emptyset_2$ denote the two copies)

$$c_j^k(\emptyset_1) + c_j^k(\emptyset_2) = 1 + 1 = 2 = c_j(\emptyset_1) + c_j(\emptyset_2).$$

So, it follows that, for all $j \in [n]$,

$$\sum_{S \in \mathcal{D}_j} c_j^k(S) = \sum_{S \in \mathcal{D}_j} c_j(S).$$

Now, there will be

$$r + 2(\frac{1}{2})(n - k - r) = r + n - k - r = n - k = \binom{n-k}{1}$$

copies of the empty set with $c_j^k(\emptyset) = 1$, and

$$p - \frac{1}{2}(n - k - r) = \frac{1}{2}((n-k)^2 - r) - \frac{1}{2}(n - k - r) = \frac{1}{2}((n-k)^2 - (n-k)) = \binom{n-k}{2}$$

copies of the empty set with $c_j^k(\emptyset) = 2$. Finally, if q is the number of copies of the empty set with $c_j^k(\emptyset) = 0$, q is simply

$$\begin{aligned}
q &= \sum_{j\in R}\frac{1}{2}(n-N_L(j)-1)+\sum_{j\in[n]\setminus R}\frac{1}{2}(n-N_L(j))-\frac{1}{2}(n-k-r)\\
&= \frac{1}{2}\sum_{j\in[n]}(n-N_L(j))-\frac{1}{2}\sum_{j\in R}1-\frac{1}{2}(n-k-r)\\
&= \frac{1}{2}(n^2-k-r-(n-k-r))=\frac{1}{2}(n^2-n)=\binom{n}{2}.
\end{aligned}$$

Now, for each $j \in [n]$, for $i \in \{0,1,2\}$ simply let $t_{i,j}$ be the number of $S \in \mathcal{D}_j$ with $c_j^k(S) = i$ (and then let all other values of $t_{i,j} = 0$). Then, for any $j \in [n]$, we trivially have $\sum_{i=0}^n t_{i,j} = |\mathcal{D}_j| = n$ and

$$\begin{aligned}
\sum_{i=0}^{n} i t_{i,j} &= \sum_{S\in\mathcal{D}_j} c_j^k(S) = \sum_{S\in\mathcal{D}_j} c_j(S)\\
&= \sum_{S\in\mathcal{D}_j,|S|=0} c_j(S) + \sum_{S\in\mathcal{D}_j,|S|=1} c_j(S) + \sum_{S\in\mathcal{D}_j,|S|=2} c_j(S)\\
&= \sum_{S\in\mathcal{D}_j,|S|=0} c_j(S) + d_L(j) + 2(\frac{1}{2}e_L(j)+|C_j|)\\
&= d_L(j)+e_L(j)+2|C_j| + \sum_{S\in\mathcal{D}_j,|S|=0} c_j(S)\\
&= N_L(j)+2|C_j| + \sum_{S\in\mathcal{D}_j,|S|=0} c_j(S).
\end{aligned}$$

Now, note that if $N_L(j) \equiv n \pmod 2$, we have

$$\sum_{S\in\mathcal{D}_j,|S|=0} c_j(S) = 2(\frac{1}{2}(n-N_L(j))-|C_j|) = n-N_L(j)-2|C_j|.$$

Otherwise, we have

$$\sum_{S\in\mathcal{D}_j,|S|=0} c_j(S) = 2(\frac{1}{2}(n-N_L(j)-1)-|C_j|)+1 = n-N_L(j)-2|C_j|.$$

Thus, in either case, we have

$$\sum_{i=0}^{n} it_{i,j} = N_L(j) + 2|C_j| + n - N_L(j) - 2|C_j| = n.$$

Finally, note that for $i \in \{0, 1, 2\}$, $\sum_{j=1}^{n} t_{i,j}$ simply counts the total number of $S \in \mathcal{D}_j$ with $c_j^k(S) = i$ as j ranges from 1 to n. Thus,

$$\begin{aligned}
\sum_{j=1}^{n} t_{0,j} &= q = \binom{n}{2} = a_0\binom{n}{0}, \\
\sum_{j=1}^{n} t_{1,j} &= k + n - k = n = a_1\binom{n}{1}, \\
\sum_{j=1}^{n} t_{2,j} &= \binom{k}{2} + kn - k^2 + \frac{1}{2}((n-k)^2 - (n-k)) \\
&= \frac{1}{2}(k^2 - k + 2kn - 2k^2 + n^2 - 2kn + k^2 - n + k) \\
&= \frac{1}{2}(n^2 - n) = a_2\binom{n}{2}, \\
\sum_{j=1}^{n} t_{i,j} &= 0 = a_i\binom{n}{i},
\end{aligned}$$

for $i > 2$. Thus, we satisfy all of the initial conditions for theorem 1.3.3 (with $U = [n]$ and $m = N = n$). Further, let $w = k$ and let $\mathcal{A}_j^w = \mathcal{D}_j$ for $j \in [n]$. Then, by construction, we have $t_{i,j}$ elements of $\mathcal{D}_j$ that map to i under c_j^k for $i, j \in [n]$. Further, consider any $S \subseteq [k]$. If $|S| > 2$, S does not appear in any $\mathcal{D}_j$. Otherwise, if $|S| = 2$, S is in exactly one of the $\mathcal{D}_j$ and maps to 2 under c_j^k. If $|S| = 1$, there is one j where $S \in B_j$ and thus $c_j^k(S) = 1$, and there are $n - k$ such j with $S \in C_j$ with $c_j^k(S) = 2$. For the empty set, by above, with multiplicity, there are $\binom{n-k}{2}$ copies that map to 2 under some c_j^k, there are $\binom{n-k}{1}$ copies that map to 1, and there are $\binom{n}{2}$ copies that map to 0. Thus, in all cases, for any $S \subseteq [k]$ and $i \in [n]$, there are $a_i\binom{n-k}{i-|S|}$ values of j (with multiplicity) with $c_j^k(S) = i$. So we satisfy the induction hypothesis at this step of the proof of theorem 1.3.3, and we can follow the method of the proof to obtain n-partitions $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ of $[n]$ such that each 1 or 2 element subset of $[n]$ occurs in exactly one of the $\mathcal{A}_j$. We now define the complete Latin square $L' : [n] \times [n] \to [n]$ by letting $L'(x, x) = j$ where $\{x\} \in A_j$ for $x \in [n]$, and by letting $L'(x, y) = L'(y, x) = j$ where

$\{x,y\} \in A_j$ for $x, y \in [n]$ with $x \neq y$. Consider any $j \in [n]$ and any $S \in \mathcal{B}_j$. By above, we have $c_j^k(S) = |S|$. Then, the directed edge from $\mathcal{D}_j$ to (S, i) (in the transportation network defined in the proof of theorem 1.3.3) with capacity 1 must have $i = |S|$. But we know that the edge from $(S, |S|)$ to the sink has capacity 0 by definition. Thus, such an S can never be the distinguished element in $\mathcal{D}_j$, and from our construction, we will have $S \in \mathcal{A}_j^{k+1}$ and $c_j^{k+1}(S) = c_j^k(S) = |S|$. So, by a simple induction, it follows that $S \in \mathcal{A}_j$. Therefore, given $x, y \in [k]$, if $L(x,y) = j$, we have $\{x,y\} \in B_j$, which implies that $\{x,y\} \in \mathcal{A}_j$ and thus $L'(x,y) = j$. And thus we conclude that L' (which by construction is symmetric) is an extension of L. □

We now prove Hoffman's theorem, again using theorem 1.3.3.

Theorem 1.4.2 *Let $k, n \in \mathbb{N}$ with $k < n$. Then, let $L : [k] \times [k] \to [n]$ be an incomplete symmetric Latin square and let $f_{k+1}, f_{k+2}, \ldots, f_n \in [n]$. Further, for $j \in [n]$, let F_j be defined by*

$$F_j = |\{i \in [n] \setminus [k] : f_i = j\}|.$$

Then, L is extendible to a complete symmetric Latin square L' with $L'(i,i) = f_i$ for $i \in [n] \setminus [k]$ if and only if, for each $j \in [n]$, we have $d_L(j) + F_j \equiv n \pmod 2$ and $N_L(j) \geq 2k - n + F_j$.

Proof. First, for each $j \in [n]$, we define B_j, C_j and $\mathcal{D}_j$ as in the proof of theorem 1.4.1. Now, assume that such an L' as described in the statement of theorem 1.4.2 exits. Then, for each $j \in [n]$, define B'_j as in the proof of theorem 1.4.1. Then, note that $d_{L'}(j) = d_L(j) + F_j$, and further $e_{L'}(j) + d_{L'}(j) = N_{L'}(j) = n$, and, since $e_{L'}(j)$ must be even (by above) we have:

$$d_{L'}(j) \equiv n \pmod 2,$$
$$d_L(j) + F_j \equiv n \pmod 2.$$

Also, as in our proof of theorem 1.4.1, for any $j \in [n]$, for each $x \in C_j$, we must have a unique $y \in [n] \setminus [k]$ with $L'(x,y) = j$. Further, by assumption, if we have $f_y = j$, we must also have

$L'(y,y) = j$, which implies that $L'(x,y) \neq j$. Thus, it follows that each such y must also have $f_y \neq j$. So we must have $|C_j| \leq n-k-F_j$. But, as above, we have $|C_j| = k - N_L(j)$, and it follows that $n-k-F_j \leq k - N_L(j)$ implying that $N_L(j) \geq 2k-n+F_j$.

Conversely, assume that for each $j \in [n]$ we have $d_L(j)+F_j \equiv n \pmod 2$ and $N_L(j) \geq 2k-n+F_j$. As before, let $a_i = 0$ for $i > 2$, let $a_2 = a_1 = 1$, and let $a_0 = \binom{n}{2}$. Then, for each $j \in [n]$, let $t_{i,j} = 0$ for $i > 2$, and let

$$\begin{aligned} t_{1,j} &= d_L(j) + F_j, \\ t_{2,j} &= \frac{1}{2}(n - d_L(j) - F_j), \\ t_{0,j} &= t_{2,j}. \end{aligned}$$

Note that $t_{2,j}$ is an integer, since $d_L(j) + F_j \equiv n \pmod 2$, and nonnegative, since $d_L(j) \leq k$ and $F_j \leq n-k$ by definition. Further,

$$\begin{aligned} \sum_{i=0}^{n} t_{i,j} &= d_L(j) + F_j + n - d_L(j) - F_j = n, \\ \sum_{i=0}^{n} i t_{i,j} &= d_L(j) + F_j + 2(\frac{1}{2}(n - d_L(j) - F_j)) = n. \end{aligned}$$

Also, note that $\sum_{j=1}^{n} d_L(j) = k$ and $\sum_{j=1}^{n} F_j = n-k$, which implies that

$$\begin{aligned}
\sum_{j=1}^{n} t_{1,j} &= \sum_{j=1}^{n}(d_L(j) + F_j) = \sum_{j=1}^{n} d_L(j) + \sum_{j=1}^{n} F_j \\
&= k + n - k = n = a_1\binom{n}{1}, \\
\sum_{j=1}^{n} t_{2,j} &= \sum_{j=1}^{n} \frac{1}{2}(n - d_L(j) - F_j) \\
&= \frac{1}{2}\sum_{j=1}^{n} n - \frac{1}{2}\sum_{j=1}^{n} d_L(j) - \frac{1}{2}\sum_{j=1}^{n} F_j \\
&= \frac{n^2 - k - (n-k)}{2} = \frac{n^2 - n}{2} = \binom{n}{2} = a_2\binom{n}{2}, \\
\sum_{j=1}^{n} t_{0,j} &= \sum_{j=1}^{n} t_{2,j} = \binom{n}{2} = a_0\binom{n}{0},
\end{aligned}$$

and, for $i > 2$,

$$\sum_{j=1}^{n} t_{i,j} = 0 = a_i\binom{n}{i}.$$

So if $U = [n]$ and $m = N = n$, we again satisfy the initial conditions for theorem 1.3.3. We now define functions $c_j^k : \mathcal{D}_j \to \{0, 1, 2\}$ for $j \in [n]$. Consider any $S \in \mathcal{D}_j$. If $|S| = 2$, let $c_j^k(S) = 2$. If $|S| = 1$, if $S \in B_j$, let $c_j^k(S) = 1$; if $S \in C_j$, let $c_j^k(S) = 2$. Further, for F_j copies of the empty set, let $c_j^k(\emptyset) = 1$. Then, for $t_{0,j}$ copies of the empty set, let $c_j^k(\emptyset) = 0$. And for all remaining copies of the empty set, let $c_j^k(\emptyset) = 2$. Let E_j be equal to the number of copies of the empty set in $\mathcal{D}_j$. Note that $E_j = n - |B_j| - |C_j|$. And since, by above, $|B_j| = d_L(j) + \frac{1}{2}e_L(j)$ and $|C_j| = n - N_L(j)$, we have:

$$\begin{aligned}
E_j = n - |B_j| - |C_j| &= n - (d_L(j) + \frac{1}{2}e_L(j) + k - N_L(j)) \\
&= n - k + N_L(j) - \frac{1}{2}e_L(j) - d_L(j) \\
&= n - k + d_L(j) + e_L(j) - \frac{1}{2}e_L(j) - d_L(j) \\
&= n - k + \frac{1}{2}e_L(j).
\end{aligned}$$

Now, note that:

$$\begin{aligned}E_j - F_j - t_{0,j} &= n - k + \frac{1}{2}e_L(j) - F_j - \frac{1}{2}(n - d_L(j) - F_j)\\&= \frac{1}{2}(n - F_j + d_L(j) + e_L(j)) - k\\&= \frac{1}{2}(n + N_L(j) - F_j - 2k),\end{aligned}$$

and, since by our assumption we have $N_L(j) \geq 2k - n + F_j$, it follows that $n + N_L(j) - F_j - 2k \geq 0$ and thus $E_j - F_j - t_{0,j} \geq 0$. So our function c_j^k is well defined. By definition, the number of elements of $\mathcal{D}_j$ that map to 0 under c_j^k is $t_{0,j}$. The number of elements of $\mathcal{D}_j$ that map to 1 under c_j^k is $d_L(j) + F_j = t_{1,j}$. Thus, since $\mathcal{D}_j$ contains n elements, the number of elements that map to 2 must be

$$\begin{aligned}n - t_{0,j} - t_{1,j} &= n - \frac{1}{2}(n - d_L(j) - F_j) - d_L(j) - F_j\\&= \frac{1}{2}(n - d_L(j) - F_j) = t_{2,j}.\end{aligned}$$

Now, much like in the proof of theorem 1.4.1, let $w = k$ and let $\mathcal{A}_j^w = \mathcal{D}_j$ for $j \in [n]$. Consider any $S \subseteq [k]$ with $|S| > 0$. Since, for any $i, j \in [n]$ we define $c_j^k(S)$ and a_i precisely the same as in the proof of theorem 1.4.1, it follows that each such S has precisely $a_i\binom{n-k}{i-|S|}$ values of $j \in [n]$ with $S \in D_j$ and $c_j^k(S) = i$. So it remains to consider the empty set. For each $j \in [n]$, there are F_j copies of the empty set with $c_j^k(\emptyset) = 1$, and thus there are a total of

$$\sum_{j=1}^{n} F_j = n - k = \binom{n-k}{1}$$

copies of the empty set mapping to 1 under some c_j^k. Similarly, for $j \in [n]$ there are $t_{0,j}$ copies of the empty set with $c_j^k(\emptyset) = 0$ and thus there are a total of $\sum_{j=1}^{n} t_{0,j} = \binom{n}{2}$ (by above) copies of the empty set mapping to 0 under some c_j^k. Finally, for $j \in [n]$, by above, there are $\frac{1}{2}(n+N_L(j)-F_j-2k)$

copies of the empty set with $c_j^k(\emptyset) = 2$. Thus, if we let q be the total number of copies of the empty set mapping to 2 under some c_j^k, we have

$$\begin{aligned} q &= \sum_{j=1}^{n} \frac{1}{2}(n + N_L(j) - F_j - 2k) \\ &= \frac{1}{2}\sum_{j=1}^{n} n + \frac{1}{2}\sum_{j=1}^{n} N_L(j) - \frac{1}{2}\sum_{j=1}^{n} F_j - \sum_{j=1}^{n} k \\ &= \frac{1}{2}n^2 + \frac{1}{2}k^2 - \frac{1}{2}(n-k) - nk = \frac{1}{2}\left(n^2 - 2nk + k^2 - (n-k)\right) \\ &= \frac{1}{2}\left((n-k)^2 - (n-k)\right) = \binom{n-k}{2}. \end{aligned}$$

Thus, the induction hypothesis found in the proof of theorem 1.3.3 is satisfied. Let $\mathcal{N}$ be the network described in that proof. Recall, the vertices in $\mathcal{N}$ will either be σ (the source), τ (the sink), $\mathcal{D}_j$ for $j \in [n]$ or (S, i) with $S \subseteq [n]$ and $i \in [n]$. Let f be the maximal flow explicitly described in that proof. Then, let $\mathcal{M}$ be the induced subnetwork of $\mathcal{N}$ formed by removing vertices $\mathcal{D}_{f_{k+1}}$ and $(\emptyset, 1)$. We now describe a flow g on $\mathcal{M}$. First, for all directed edges in $\mathcal{M}$ of the form $(\mathcal{D}_j, (S, i))$ for some $i, j \in [n]$, $S \subseteq [k]$, let $g(\mathcal{D}_j, (S, i)) = f(\mathcal{D}_j, (S, i))$. Since all capacities in $\mathcal{M}$ are the same as in $\mathcal{N}$, this is allowed. For notational purposes, let $g(\mathcal{D}_j, (S, i)) = 0$ for any $i, j \in [n]$, $S \subseteq [k]$ where $(\mathcal{D}_j, (S, i))$ is not an edge in $\mathcal{M}$. Now, we simply define g on all remaining edges to insure that it is a flow. For all $j \in [n] \setminus \{f_{k+1}\}$, let

$$g(\sigma, \mathcal{D}_j) = \sum_{S \subseteq [k], i \in [n]} g(\mathcal{D}_j, (S, i)),$$

and for all $S \subseteq [k]$, $i \in [n]$, let

$$g((S, i), \tau) = \sum_{j \in [n]} g(\mathcal{D}_j, (S, i)).$$

It is clear from the way we define g that it in fact is a flow on $\mathcal{M}$. Now, let $v(f)$ and $v(g)$ denote the values of the flows f and g, respectively. Again, for notational purposes, we define $f(\mathcal{D}_j, (S, i)) = 0$

for all $i, j \in [n]$, $S \subseteq [k]$ where $(\mathcal{D}_j, (S, i))$ is not an edge in $\mathcal{N}$. Since for any $j \in [n] \setminus \{f_{k+1}\}$, the only edges of the form $(\mathcal{D}_j, (S, i))$ that are in $\mathcal{N}$ but not in $\mathcal{M}$ are those with $i = 1$ and $S = \emptyset$ with $c_j^k(S) = 1$. By above, there are precisely F_j such copies of the empty set. Thus, we have

$$\begin{aligned} g(\sigma, \mathcal{D}_j) &= \sum_{S \subseteq [k], i \in [n]} g(\mathcal{D}_j, (S, i)) = \sum_{S \subseteq [k], i \in [n]} f(\mathcal{D}_j, (S, i)) - F_j f(\mathcal{D}_j, (\emptyset, 1)) \\ &= f(\sigma, \mathcal{D}_j) - F_j f(\mathcal{D}_j, (\emptyset, 1)) = 1 - F_j f(\mathcal{D}_j, (\emptyset, 1)). \end{aligned}$$

Further, by the construction given in theorem 1.3.3, we have $f(\mathcal{D}_j, (\emptyset, 1)) = \frac{1}{n-k}$, and it follows that $g(\sigma, \mathcal{D}_j) = 1 - \frac{F_j}{n-k}$. So

$$\begin{aligned} v(g) &= \sum_{j \in [n] \setminus \{f_{k+1}\}} g(\sigma, \mathcal{D}_j) = \sum_{j \in [n] \setminus \{f_{k+1}\}} \left(1 - \frac{F_j}{n-k}\right) \\ &= \sum_{j \in [n] \setminus \{f_{k+1}\}} 1 - \sum_{j \in [n] \setminus \{f_{k+1}\}} \frac{F_j}{n-k} = n - 1 - \frac{1}{n-k} \sum_{j \in [n] \setminus \{f_{k+1}\}} F_j. \end{aligned}$$

Further, by definition $\sum_{j \in [n]} F_j = n - k$ by above, so, since $F_{f_{k+1}} \geq 1$ by definition, we have $\sum_{j \in [n] \setminus \{f_{k+1}\}} F_j < n - k$. Thus

$$v(g) = n - 1 - \frac{1}{n-k} \sum_{j \in [n] \setminus \{f_{k+1}\}} F_j > n - 1 - \frac{n-k}{n-k} = n - 2.$$

Thus, since all capacities of edges in $\mathcal{M}$ are integers, there must be some integer-valued maximal flow h on $\mathcal{M}$ with value $v(h) \geq v(g) > n - 2$ so $v(h) \geq n - 1$. And since there are precisely $n - 1$ edges of the form $(\sigma, \mathcal{D}_j)$, each with capacity 1, $v(h) \leq n - 1$ so $v(h) = n - 1$. Then, we define a flow h' on $\mathcal{N}$ by letting $h' = h$ on all edges that are in $\mathcal{M}$, $h'(\sigma, D_{f_{k+1}}) = 1$, $h'(D_{f_{k+1}}, (\emptyset, 1)) = 1$ for one of those such edges, $h'((\emptyset, 1), \tau) = 1$. This is clearly a flow, provided that the capacity of $((\emptyset, 1), \tau)$ is at least 1. But note that this capacity is equal to

$$\binom{n-k-1}{1-|\emptyset|-1} = \binom{n-k-1}{0} = 1,$$

ensuring that h' is a well-defined flow. Further, the value $v(h')$ of this flow is $v(h') = v(h) + 1 = n$. Thus, since all edges leaving σ are saturated, h' is a maximal, integer flow on $\mathcal{N}$. So, as in the proof of theorem 1.3.3, for each $j \in [n]$ there exists a distinguished S_j in $\mathcal{D}_j$. Consider $\{x\}$ for any $x \in [k]$. Since

$$h'((\{x\},1),\tau) = a_1\binom{n-k-1}{1-|\{x\}|-1} = \binom{n-k-1}{-1} = 0,$$

$$h'((\{x\},2),\tau) = a_2\binom{n-k-1}{2-|\{x\}|-1} = \binom{n-k-1}{0}\ 1,$$

there must be precisely one j with $S_j = \{x\}$; call this j_x. Then, we define $L^{k+1} : [k+1] \times [k+1] \to [n]$ as follows:

$$L^{k+1}(x,y) \begin{cases} L(x,y) & \text{if } x \in [k],\ y \in [k], \\ j_x & \text{if } x \in [k],\ y = k+1, \\ j_y & \text{if } x = k+1,\ y \in [k], \\ f_{k+1} & \text{if } x = k+1,\ y = k+1. \end{cases}$$

Note that it immediately follows that $L^{k+1}(x,y) = L^{k+1}(y,x)$ for all $x, y \in [k+1]$. Further, suppose we have $L^{k+1}(x,y) = L^{k+1}(x,y')$ for some $x, y, y' \in [k+1]$. Note that if $x, y, y' \in [k]$, the fact that L is a Latin square implies that $y = y'$. Consider the case when $x = k+1$. Then, if both $y, y' \in [k]$, we have $j_y = j_{y'}$. But then, since each $\mathcal{D}_j$ has only one distinguished subset, we must have $y = y'$. So, consider the case $x = y' = k+1$, $y \in [k]$. Then, we have $j_y = f_{k+1}$. But note that, by above, we have $h'(\mathcal{D}_{f_{k+1}}, (\emptyset, 1)) = 1$, it follows that the empty set is the distinguished set in $\mathcal{D}_{f_{k+1}}$, contradicting the fact that $\{y\}$ is the distinguished set in $\mathcal{D}_{j_y} = \mathcal{D}_{f_{k+1}}$. The remaining case (without loss of generality) is when $x, y \in [k]$ and $y' = k+1$. In this case, we must have $L(x,y) = j_x$. This implies that $\{x,y\} \in B_{j_x}$, and thus $\{x\} \notin C_{j_x}$. Note that, by above, for any

$S \subseteq [k]$ with $|S| = 1$, the only way that S can be the distinguished subset of D_{j_x} is if $c^k_{j_x}(S) = 2$. This can only happen, by definition, if $S \in C_{j_x}$. Thus, $\{x\}$ cannot be the distinguished subset of $\mathcal{D}_{j_x}$, a contradiction. As this covers all cases, we must have $y = y'$. Further, since L^{k+1} is symmetric, it follows that $L^{k+1}(x, y) = L^{k+1}(x', y)$ implies that $x = x'$. Thus L^{k+1} is a symmetric Latin square, and, by construction, it is an extension of L such that $L^{k+1}(k+1, k+1) = f_{k+1}$. Further, consider the numbers $f_{k+2}, f_{k+3}, \ldots, f_n$ and let

$$F'_j = |\{i \in [n] \setminus [k+1] : f_i = j\}|.$$

Then, for any $j \neq f_{k+1}$, we have $d_{L^{k+1}}(j) = d_L(j)$ and $F'_j = F_j$, so

$$d_{L^{k+1}}(j) + F'_j = d_L(j) + F_j \equiv n \pmod 2.$$

And if $j = f_{k+1}$ we have $d_{L^{k+1}}(j) = d_L(j) + 1$, and $F'_j = F_j - 1$, so

$$d_{L^{k+1}}(j) + F'_j = d_L(j) - 1 + F_j + 1 = d_L(j) + F_j \equiv n \pmod 2.$$

Thus, L^{k+1} satisfies the first condition of theorem 1.4.2. Now, suppose there is some $j \neq f_{k+1}$ with $N_{L^{k+1}}(j) < 2(k+1) - n + F'_j$. Then, it follows that

$$N_{L^{k+1}}(j) < 2(k+1) - n + F'_j = 2k - n + F_j + 2 \leq N_L(j) + 2.$$

Thus, we must have $N_{L^{k+1}}(j) \leq N_L(j) + 1$. But, since $L^{k+1}(k+1, k+1) = f_{k+1} \neq j$, if $N_{L^{k+1}}(j) > N_L(j)$ there must be some $x \in [k]$ with $L^{k+1}(x, k+1) = L^{k+1}(k+1, x) = j$ and thus $N_{L^{k+1}}(j) = N_L(j) + 2$. But by above, $N_{L^{k+1}}(j) \leq N_L(j) + 1$, implying that $N_{L^{k+1}}(j) = N_L(j)$. This implies that S_j, the distinguished set in $\mathcal{D}_j$ must be equal to the empty set (if we had $|S_j| = 1$, by above, the symbol j must occur in the $k+1$ row and column, contradicting the fact that $N_{L^{k+1}}(j) = N_L(j)$). Further, the empty set is chosen as the distinguished set in $\mathcal{D}_{f_{k+1}}$ and this copy has $c^k_{f_{k+1}}(\emptyset) = 1$, and, by above, this is the only such copy of the empty set. Thus, not only must the empty set be chosen as the distinguished set in $\mathcal{D}_j$, it must be some copy with $c_j^k(\emptyset) = 2$. By above, there must be

$$E_j - F_j - t_{0,j} = \frac{1}{2}(n + N_L(j) - F_j - 2k)$$

such copies of the empty set, and thus $\frac{1}{2}(n + N_L(j) - F_j - 2k) \geq 1$. But this implies that $2 \leq n + N_L(j) - F_j - 2k$, which in turn implies that

$$N_{L^{k+1}}(j) = N_L(j) \geq 2 + 2k - n + F_j = 2(k+1) - n + F'_j,$$

contradicting our above assumption that $N_{L^{k+1}}(j) < 2(k+1) - n + F'_j$. Thus, no such j exists. It follows that L^{k+1} satisfies the conditions of theorem 1.4.2 as well, and if we let $L^k = L$, we can inductively construct $L^k, L^{k+1}, L^{k+2}, \ldots, L^n$ such that, for $k+1 \leq i \leq n$, L^i is symmetric, L^i is an extension of L^{i-1} and $L^i(i,i) = f_i$. Thus, if we simply let $L' = L^n$, L' is symmetric, $L'(i,i) = f_i$ for $k+1 \leq i \leq n$ and L' is an extension of L, thus completing our proof. □

Note that the collection of n-partitions associated with a complete symmetric Latin square $L : [n] \times [n] \rightarrow [n]$ are such that each 1-subset and 2-subset of $[n]$ occur exactly once. For $N \in \mathbb{N}$ and $j_1, j_2 \ldots, j_N, \alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{N}$, we define the hypergraph $\bigcup_{i=1}^{N} \alpha_i \binom{V}{j_i}$ to be the edge-disjoint union of each $\alpha_i \binom{V}{j_i}$, where $\alpha_i \binom{V}{j_i}$ denotes the hypergraph formed by taking α_i edge-disjoint copies of $\binom{V}{j_i}$. Then, a complete symmetric Latin square is equivalent to a one-factorization of $\binom{[n]}{1} \cup \binom{[n]}{2}$. We now prove a theorem that gives necessary and sufficient conditions for the existence of a one-factorization of $\alpha\binom{V}{r} \cup \beta\binom{V}{s}$ for arbitrary $\alpha, \beta \in \mathbb{N}$ and $r, s \in [v]$ with $r \neq s$. First, let $n = n(r,s)$ be the number of pairs (a,b) where a, b are nonnegative integer solutions to the equation $ar + bs = v$, and let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_b)$ be such that $a_i r + b_i s = v$ for each $i \in [n]$ and $a_1 > a_2 > \cdots > a_n$ (and thus $b_1 < b_2 \cdots < b_n$). Next, we give a preliminary lemma.

Lemma 1.4.3 *Given $v, \alpha, \beta \in \mathbb{N}$ and $r, s \in [v]$ with $n = n(r,s)$ as above, let $d = \gcd(r,s)$. If there exists some $j_0 \in [n]$ with $\frac{v}{d}$ dividing $a_{j_0}\beta\binom{v}{s} - b_{j_0}\alpha\binom{v}{r}$, then for all $j \in [n]$, $\frac{v}{d}$ divides $a_j\beta\binom{v}{s} - b_j\alpha\binom{v}{r}$.*

Proof. Suppose such a j_0 exists, and let $j \in [n]$ with $j \neq j_0$. Then, note that by definition we have $a_{j_0}r + b_{j_0}s = v = a_jr + b_js$, and thus $(a_j - a_{j_0})r = (b_{j_0} - b_j)s$, which implies that $\frac{a_j - a_{j_0}}{s} = \frac{b_{j_0} - b_j}{r}$. Then,

$$\begin{aligned} a_j\beta\binom{v}{s} - b_j\alpha\binom{v}{r} &= (a_j - a_{j_0} + a_{j_0})\beta\binom{v}{s} - (b_j - b_{j_0} + b_{j_0})\alpha\binom{v}{r} \\ &= (a_j - a_{j_0})\beta\binom{v}{s} + (b_{j_0} - b_j)\alpha\binom{v}{r} + a_{j_0}\beta\binom{v}{s} - b_{j_0}\alpha\binom{v}{r}. \end{aligned}$$

Further, since $\frac{v}{d}$ divides $a_{j_0}\beta\binom{v}{s} - b_{j_0}\alpha\binom{v}{r}$ by assumption, it remains to show that $\frac{v}{d}$ divides $(a_j - a_{j_0})\beta\binom{v}{s} + (b_{j_0} - b_j)\alpha\binom{v}{r}$. If we let $A = (a_j - a_{j_0})\beta\binom{v}{s} + (b_{j_0} - b_j)\alpha\binom{v}{r}$, note that

$$\begin{aligned} A &= (a_j - a_{j_0})\beta\frac{v}{s}\binom{v-1}{s-1} + (b_{j_0} - b_j)\alpha\frac{v}{r}\binom{v-1}{r-1} \\ &= v\left(\frac{a_j - a_{j_0}}{s}\beta\binom{v-1}{s-1} + \frac{b_{j_0} - b_j}{r}\alpha\binom{v-1}{r-1}\right) \\ &= v\left(\frac{a_j - a_{j_0}}{s}\beta\binom{v-1}{s-1} + \frac{a_j - a_{j_0}}{s}\alpha\binom{v-1}{r-1}\right) \\ &= \frac{v}{s}\left(a_j - a_{j_0}\right)\left(\beta\binom{v-1}{s-1} + \alpha\binom{v-1}{r-1}\right). \end{aligned}$$

Thus, it follows that

$$A \div \frac{v}{d} = \frac{d}{s}\left(a_j - a_{j_0}\right)\left(\beta\binom{v-1}{s-1} + \alpha\binom{v-1}{r-1}\right).$$

Further, by above, $\frac{r}{d}\left(a_j - a_{j_0}\right) = \frac{s}{d}\left(b_{j_0} - b_j\right)$. But, by definition, $\frac{s}{d}$ and $\frac{r}{d}$ are relatively prime, and thus it follows that $\frac{s}{d}$ divides $(a_j - a_{j_0})$. So $\frac{d}{s}(a_j - a_{j_0})$ is an integer, which implies that $\frac{v}{d}$ divides A, which in turn implies that $\frac{v}{d}$ divides $a_j\beta\binom{v}{s} - b_j\alpha\binom{v}{r}$. □

We are now ready to state our theorem.

Theorem 1.4.4 *Given $v, \alpha, \beta \in \mathbb{N}$ and $r, s \in [v]$ with $n = n(r, s)$ as defined above, if $V = [v]$, there* *exists a one-factorization of* $\alpha\binom{V}{r} \cup \beta\binom{V}{s}$ *if and only if one of the following holds:*

(1) $n = 1$, a_1 *divides* $\alpha\binom{v}{r}$, b_1 *divides* $\beta\binom{v}{s}$, *and* $b_1\alpha\binom{v}{r} = a_1\beta\binom{v}{s}$

(2) $n > 1$, $b_1\alpha\binom{v}{r} \leq a_1\beta\binom{v}{s}$, $a_n\beta\binom{v}{r} \leq b_n\alpha\binom{v}{s}$, *and there exists some* $j \in [n]$ *such that* $\frac{v}{d}$ *divides* $a_j\beta\binom{v}{s} - b_j\alpha\binom{v}{r}$, *where* $d = \gcd(r, s)$.

Proof. First, note that the existence of such a one-factorization is equivalent to a collection of v-partitions $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N$ (for some $N \in \mathbb{N}$) of V such that each r-subset occurs α times, each s-subset occurs β times, and no other nonempty subset occurs at all in the $\mathcal{A}_i$. theorem 1.3.3 shows that this is equivalent to the existence of nonnegative integers $t_{i,j}$ for $i \in [v], j \in [N]$ such that $t_{i,j} = 0$ unless $i \in \{0, r, s\}$, $\alpha\binom{v}{r} = \sum_{j=1}^{N} t_{r,j}$, $\beta\binom{v}{s} = \sum_{j=1}^{N} t_{s,j}$, and

$$\sum_{i=1}^{v} t_{i,j} = \sum_{i=1}^{v} i t_{i,j} = v,$$

for all $j \in [N]$. Note that this last conditions will be true if and only if, for each $j \in [N]$, there exists some $i \in [n]$ such that $t_{r,j} = a_i$, $t_{s,j} = b_i$ and $t_{0,j} = v - a_i - b_i$, where (a_i, b_i) are defined as above. Thus, we can define m_i to be the number of times this occurs for each $i \in [n]$, and the condition from theorem 1.3.3 is equivalent to the existence of nonnegative integers $m_1, m_2, \ldots, m_n$ such that $\alpha\binom{v}{j} = \sum_{i=1}^{n} a_i m_i$ and $\beta\binom{v}{k} = \sum_{i=1}^{n} b_i m_i$.

So assume that such $m_1, m_2, \ldots, m_n$ exist. Then, if $n = 1$, we must have $a_1 m_1 = \alpha\binom{v}{r}$ and $b_1 m_1 = \beta\binom{v}{s}$ and thus $m_1 = \frac{\alpha\binom{v}{r}}{a_1} = \frac{\beta\binom{v}{s}}{b_1}$. Since m_1 must be an integer, this implies that a_1 divides $\alpha\binom{v}{r}$ and b_1 divides $\beta\binom{v}{s}$, and we also have $b_1\alpha\binom{v}{r} = a_1\beta\binom{v}{s}$. This gives condition (1). If $n > 1$, then, since $a_1 > a_2 > \cdots > a_n$, and $b_1 < b_2 < \cdots < b_n$, we have

$$\begin{aligned} a_1\beta\binom{v}{s} - b_1\alpha\binom{v}{r} &= a_1\sum_{i=1}^{n} b_i m_i - b_1\sum_{i=1}^{n} a_i m_i = \sum_{i=1}^{n} a_1 b_i m_i - \sum_{i=1}^{n} a_i b_1 m_i \\ &= \sum_{i=1}^{n}(a_1 b_i - a_i b_1) m_i \geq \sum_{i=1}^{n}(a_1 b_1 - a_1 b_1) m_i = 0, \end{aligned}$$

and thus $a_1\beta\binom{v}{s} \geq b_1\alpha\binom{v}{r}$. Similarly,

$$b_n\alpha\binom{v}{r} - a_n\beta\binom{v}{s} = b_n\sum_{i=1}^{n} a_i m_i - a_n\sum_{i=1}^{n} b_i m_i = \sum_{i=1}^{n} a_i b_n m_i - \sum_{i=1}^{n} a_n b_i m_i$$
$$= \sum_{i=1}^{n}(a_i b_n - a_n b_i)m_i \geq \sum_{i=1}^{n}(a_n b_n - a_n b_n)m_n = 0,$$

and thus $a_n\beta\binom{v}{r} \leq b_n\alpha\binom{v}{s}$. Further, consider any $j \in [n]$. Then, note that

$$a_j\beta\binom{v}{s} - b_j\alpha\binom{v}{r} = a_j\sum_{i=1}^{n} b_i m_i - b_j\sum_{i=1}^{n} a_i m_i = \sum_{i=1}^{n} a_j b_i m_i - \sum_{i=1}^{n} a_i b_j m_i$$
$$= \sum_{i=1}^{n}(a_j b_i - a_i b_j)m_i.$$

It is an elementary consequence of Bezout's identity (see Proposition 5 in [6]) that there exists $k \in \mathbb{Z}$ such that $a_i = a_j - k\frac{s}{d}$ and $b_i = b_j + k\frac{r}{d}$. Thus,

$$a_j b_i - a_i b_j = a_j(b_j + k\frac{r}{d}) - (a_j - k\frac{s}{d})b_j = a_j b_j + \frac{k a_j r}{d} - a_j b_j + \frac{k b_j s}{d}$$
$$= \frac{k a_j r + k b_j s}{d} = \frac{k(a_j r + b_j s)}{d} = \frac{kv}{d},$$

and thus $\frac{v}{d}$ divides $a_j b_i - a_i b_j$ for all $i \in [n]$. This implies, by above, that $\frac{v}{d}$ divides $a_j\beta\binom{v}{s} - b_j\alpha\binom{v}{r}$, which shows that condition (2) holds.

Conversely, assume that either condition (1) or (2) holds. If we have condition (1), we can simply let $m_1 = \frac{\alpha\binom{v}{r}}{a_1} = \frac{\beta\binom{v}{s}}{b_1}$ and it follows that $a_1 m_1 = \alpha\binom{v}{r}$ and $b_1 m_1 = \beta\binom{v}{s}$, implying a one-factorization of $\alpha\binom{V}{r} \cup \beta\binom{V}{s}$ exists. So assume that condition (2) holds. Then, choose $j \in [n-1]$ such that $b_j\alpha\binom{v}{r} \leq a_j\beta\binom{v}{s}$ and $b_{j+1}\alpha\binom{v}{r} \geq a_{j+1}\beta\binom{v}{s}$. Since condition (2) insures us that $b_1\alpha\binom{v}{r} \leq a_1\beta\binom{v}{s}$ and $b_n\alpha\binom{v}{r} \geq a_n\beta\binom{v}{s}$, such a j must exist. Then, let

$$m_j = \frac{d}{v}\left(b_{j+1}\alpha\binom{v}{r} - a_{j+1}\beta\binom{v}{s}\right),$$

$$m_{j+1} = \frac{d}{v}\left(a_j\beta\binom{v}{s} - b_j\alpha\binom{v}{r}\right),$$

and let $m_i = 0$ for all $i \in [n] \setminus \{j, j+1\}$. Condition (2) and lemma 1.4.3 insure that both m_j and m_{j+1} are integers. Now, again, Proposition 5 in [6] states that, for any $i \in [n]$, there exists some $k \in \mathbb{Z}$ such that $a_i = a_j - k\frac{s}{d}$ and $b_i = b_j + k\frac{r}{d}$. Proposition 5 also states that this condition is sufficient, i.e., if we let $a = a_j - k\frac{s}{d}$ and $b = b_j + k\frac{r}{d}$ then $ar + bs = v$. Thus, since the a_i are indexed in decreasing order, a_{j+1} is the next smallest a_i after a_j, and it follows that $a_{j+1} = a_j - \frac{s}{d}$ and thus $b_{j+1} = b_j + \frac{r}{d}$. Thus

$$\begin{aligned}
\sum_{i=1}^{n} a_i m_i &= a_j m_j + a_{j+1} m_{j+1} \\
&= a_j \frac{d}{v}\left(b_{j+1}\alpha\binom{v}{r} - a_{j+1}\beta\binom{v}{s}\right) + a_{j+1}\frac{d}{v}\left(a_j\beta\binom{v}{s} - b_j\alpha\binom{v}{r}\right) \\
&= \frac{d}{v}\left(\alpha\binom{v}{r}(a_j b_{j+1} - a_{j+1}b_j) + \beta\binom{v}{s}(a_j a_{j+1} - a_j a_{j+1})\right) \\
&= \frac{d\alpha}{v}\binom{v}{r}\left(a_j(b_j + \frac{r}{d}) - (a_j - \frac{s}{d})b_j\right) = \frac{d\alpha}{v}\binom{v}{r}\left(a_j b_j - a_j b_j + \frac{a_j r + b_j s}{d}\right) \\
&= \frac{d\alpha}{v}\binom{v}{r}\frac{v}{d} = \alpha\binom{v}{r},
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n} b_i m_i &= b_j m_j + b_{j+1} m_{j+1} \\
&= b_j \frac{d}{v}\left(b_{j+1}\alpha\binom{v}{r} - a_{j+1}\beta\binom{v}{s}\right) + b_{j+1}\frac{d}{v}\left(a_j\beta\binom{v}{s} - b_j\alpha\binom{v}{r}\right) \\
&= \frac{d}{v}\left(\alpha\binom{v}{r}(b_j b_{j+1} - b_j b_{j+1}) + \beta\binom{v}{s}(a_j b_{j+1} - a_{j+1} b_j)\right) \\
&= \frac{d\beta}{v}\binom{v}{s}\left(a_j(b_j + \frac{r}{d}) - (a_j - \frac{s}{d})b_j\right) = \frac{d\beta}{v}\binom{v}{s}\left(a_j b_j - a_j b_j + \frac{a_j r + b_j s}{d}\right) \\
&= \frac{d\beta}{v}\binom{v}{s}\frac{v}{d} = \beta\binom{v}{s}.
\end{aligned}$$

This, by above, shows that a one-factorization of $\alpha\binom{v}{r} \cup \beta\binom{v}{s}$ exists. □

In the next section we show how theorem 1.2.3, for the case $k = 3$, is a direct consequence of theorem 1.4.4. We will then use a modified form of that argument, along with theorem 1.3.3, to prove theorem 1.2.3 for $k = 4$ and $k = 5$.

1.5 Proving Theorem 1.2.3 for k=3 and k=4

We begin by considering the case $k = 3$.

Theorem 1.5.1 *For any $u, v \in \mathbb{N}$ such that 3 divides both u and v, $(u, v) \in \mathcal{EP}_3$ if and only if $v \geq 2u$.*

Proof. Note that theorem 1.3.5 takes care of the case $v = 2u$. So, consider any $u, v \in N$ such that 3 divides both u and v, and such that $v > 2u$; since 3 divides both u and v, we must have $v \geq 2u + 3$. Now, consider the hypergraph $H = \alpha\binom{U}{1} \cup \beta\binom{U}{2}$, where $\alpha = \binom{v-u}{2}$ and $\beta = \binom{v-u}{1}$. Then, defining $n = n(1, 2)$ as in theorem 1.4.4, it is clear that $n = \lfloor \frac{u}{2} \rfloor + 1$, since if we define a_i and b_i as in theorem 1.4.4, for $1 \leq i \leq \lfloor \frac{u}{2} \rfloor + 1$, it is clear that $b_i = i - 1$, $a_i = u - 2i + 2$, and these are the only such solutions to $a + 2b = u$. Note that, since $a_1 = u$ and $b_1 = 0$, we have

$$a_1\beta\binom{u}{2} - b_1\alpha\binom{u}{1} = u\binom{v-u}{1}\binom{u}{2},$$

which is obviously divisible by u. Further,

$$b_1\alpha\binom{u}{1} = 0 \leq a_1\beta\binom{u}{2},$$

so, to satisfy condition (2) of theorem 1.4.4, it remains to check whether $a_n\beta\binom{u}{2} \leq b_n\alpha\binom{u}{1}$ is true or not. Note that if u is even, then $a_n = 0$, making this statement trivially true. So consider the case when u is odd. Then, $a_n = 1$ and $b_n = \frac{u-1}{2}$. So

$$a_n\beta\binom{v}{r} = (v-u)\binom{u}{2} = (v-u)\frac{u(u-1)}{2} = b_n u(v-u)$$
$$\leq b_n u(v-u)\frac{v-u-1}{2} = b_n u\binom{v-u}{2} = b_n\alpha\binom{u}{1}.$$

This shows that condition (2) is satisfied, which implies that there exists a one-factorization of $\alpha\binom{U}{1} \cup \beta\binom{U}{2}$. Further, the proof of theorem 1.4.4 shows that this one-factorization can be taken to only contain two types of one-factors: those that contain a_j 1-subsets b_j 2-subsets of U, and those that contain a_{j+1} 1-subsets and b_{j+1} 2-subsets of U, for some $j \in [n]$. Further, it is implied that the number of the first type of one-factors occurring in our one-factorization is m_j and the number of the second type of one-factors occurring in our one-factorization is m_{j+1}, where, as in that proof,

$$m_j = \frac{1}{u}\left(b_{j+1}\alpha\binom{u}{1} - a_{j+1}\beta\binom{u}{2}\right),$$
$$m_{j+1} = \frac{1}{u}\left(a_j\beta\binom{u}{2} - b_j\alpha\binom{u}{1}\right).$$

Thus, in particular, we must have $b_{j+1}\alpha\binom{u}{1} \geq a_{j+1}\beta\binom{u}{2}$. Further, since $a_j = u-2j+2$, $a_{j+1} = u-2j$, $b_j = j-1$ and $b_{j+1} = j$, as well as $\alpha = \binom{v-u}{2}$ and $\beta = \binom{v-u}{1}$, it follows that:

$$j\binom{v-u}{2}\binom{u}{1} \geq (u-2j)\binom{v-u}{1}\binom{u}{2},$$
$$j\frac{(v-u)(v-u-1)}{2}u \geq (u-2j)(v-u)\frac{u(u-1)}{2},$$
$$j(v-u-1) \geq (u-2j)(u-1),$$
$$j(v-u-1) \geq u^2 - u - j(2u-2),$$
$$j(v+u-3) \geq u^2 - u,$$

and thus, $j \geq \frac{u^2-u}{v+u-3}$. Note that the number of elements in the first type of one-factor is $a_j + b_j = u-2j+2+j-1 = u-j+1$ and the number of elements in the second type of one-factor is

$a_{j+1} + b_{j+1} = u - 2j + j = u - j$. Thus, all one-factors in our one-factorization have cardinality less than or equal to

$$
\begin{aligned}
u - j + 1 \le u - \frac{u^2 - u}{v + u - 3} + 1 &= \frac{vu + u^2 - 3u - u^2 + u + v + u - 3}{v + u - 3} \\
&= \frac{v + vu - u - 3}{v + u - 3}.
\end{aligned}
$$

We claim this last quantity is strictly less than $\frac{v+3}{3}$. To see this, note that

$$
\begin{aligned}
\frac{v+3}{3} - \frac{v + vu - u - 3}{v + u - 3} &= \frac{v^2 + vu - 3v + 3v + 3u - 9}{3(v + u - 3)} - \frac{3v + 3vu - 3u - 9}{3(v + u - 3)} \\
&= \frac{v^2 - 2vu - 3v + 6u}{3(v + u - 3)} = \frac{v(v - 2u - 3) + 6u}{3(v + u - 3)} \\
&\ge \frac{6u}{3(v + u - 3)} > 0,
\end{aligned}
$$

since $v - 2u - 3 \ge 0$ by above. Thus, our claim is proven, and it follows that $u - j + 1 < \frac{v+3}{3}$. Further, since $u - j + 1$ must be an integer, we have $u - j + 1 \le \frac{v}{3}$, and it follows that each one-factor in our one-factorization has cardinality at most $m = \frac{v}{3}$. We can thus add an appropriate number of copies of the empty set to each one-factor so that it becomes an m-partition of U. Thus, we have found a collection of m-partitions of U such that each 1-subset of U occurs $\alpha = \binom{v-u}{2}$ times, and each 2-subset of U occurs $\beta = \binom{v-u}{1}$ times. It follows, by the methods used above, that $(u, v) \in \mathcal{EP}_3$. $\square$

Next, we consider the case when $k = 4$.

Theorem 1.5.2 *For any $u, v \in \mathbb{N}$ such that 4 divides both u and v, $(u, v) \in \mathcal{EP}_4$ if and only if $v \ge 2u$.*

Proof. Note that theorem 1.3.5 deals with the case $v = 2u$, and theorem 1.3.1 deals with the case $v \ge 4u$. So consider any v, u such that 4 divides both u and v and such that $2u < v < 4u$. We will consider three different types of m-partitions of u, where $m = \frac{v}{4}$. One will consist of $\frac{u}{4}$ 3-subsets of

U, $\frac{u}{4}$ 1-subsets of U and $\frac{v-2u}{4}$ copies of the empty set. The second type will consist of $\frac{u}{2}$ 2-subsets of U and $\frac{v-2u}{4}$ copies of the empty set. The last type of m-partition will consist of a 3-subsets of U, b 2-subsets of U, c 1−subsets of U and $\frac{v}{4} - a - b - c$ copies of the empty set, where a, b and c depend on u and v. Obviously, we must have $3a + 2b + c = u$ and $a + b + c \leq \frac{v}{4}$. Then, let

$$m_1 = \frac{4}{u}\left(\binom{v-u}{1}\binom{u}{3} - a\frac{\binom{v-u}{3}\binom{u}{1} - \binom{v-u}{1}\binom{u}{3}}{c-a}\right),$$

$$m_2 = \frac{2}{u}\left(\binom{v-u}{2}\binom{u}{2} - b\frac{\binom{v-u}{3}\binom{u}{1} - \binom{v-u}{1}\binom{u}{3}}{c-a}\right),$$

$$m_3 = \frac{\binom{u}{1}\binom{v-u}{3} - \binom{u}{3}\binom{v-u}{1}}{c-a}.$$

We claim that there exist suitable natural numbers a, b, c such that each m_i is a nonnegative integer and such that $3a + 2b + c = u$ and $a + b + c \leq \frac{v}{4}$. Then, note that

$$\begin{aligned} m_1\frac{u}{4} + m_3 a &= \binom{v-u}{1}\binom{u}{3} - a\frac{\binom{v-u}{3}\binom{u}{1} - \binom{v-u}{1}\binom{u}{3}}{c-a} + a\frac{\binom{v-u}{3}\binom{u}{1} - \binom{v-u}{1}\binom{u}{3}}{c-a} \\ &= \binom{v-u}{1}\binom{u}{3}, \end{aligned}$$

$$\begin{aligned} m_2\frac{u}{2} + m_3 b &= \binom{v-u}{2}\binom{u}{2} - b\frac{\binom{v-u}{3}\binom{u}{1} - \binom{v-u}{1}\binom{u}{3}}{c-a} + b\frac{\binom{u}{1}\binom{v-u}{3} - \binom{u}{3}\binom{v-u}{1}}{c-a} \\ &= \binom{v-u}{2}\binom{u}{2}, \end{aligned}$$

$$
\begin{aligned}
m_1\frac{u}{4}+m_3c &= \binom{v-u}{1}\binom{u}{3} - a\frac{\binom{v-u}{3}\binom{u}{1}-\binom{v-u}{1}\binom{u}{3}}{c-a} + c\frac{\binom{u}{1}\binom{v-u}{3}-\binom{u}{3}\binom{v-u}{1}}{c-a} \\
&= \binom{v-u}{1}\binom{u}{3} + (c-a)\frac{\binom{v-u}{3}\binom{u}{1}-\binom{v-u}{1}\binom{u}{3}}{c-a} \\
&= \binom{v-u}{1}\binom{u}{3} + \binom{v-u}{3}\binom{u}{1} - \binom{v-u}{1}\binom{u}{3} = \binom{v-u}{3}\binom{u}{1}.
\end{aligned}
$$

So, by theorem 1.3.3, we can find a collection of m_1, m_2 and m_3 m-partitions of U of the first, second and third type, respectively, described above, such that each 3-subset of U occurs $\binom{v-u}{1}$ times, each 2-subset of U occurs $\binom{v-u}{2}$ times and each 1-subset of U occurs $\binom{v-u}{3}$ times. Thus $(u, v) \in \mathcal{EP}_4$.

All that is left to do is determine a, b, and c. First, let $S = \binom{v-u}{3}\binom{u}{1} - \binom{v-u}{1}\binom{u}{3}$. Then,

$$
\begin{aligned}
S &= \binom{v-u}{3}\binom{u}{1} - \binom{v-u}{1}\binom{u}{3} \\
&= \frac{1}{6}u(v-u)(v-u-1)(v-u-2)(u) - \frac{1}{6}u(v-u)(u-1)(u-2) \\
&= \frac{u(v-u)}{6}\left(v^2-(2u+3)v+(u+1)(u+2)-(u-1)(u-2)\right) \\
&= \frac{u(v-u)}{6}\left(v^2-(2u+3)v+6u\right) = \frac{u(v-u)(v-2u)(v-3)}{6}.
\end{aligned}
$$

Now, we will consider three different cases.

Case 1: $v \equiv 2u \pmod 3$.

Note that this implies that $\frac{v-2u}{3}$ is a nonnegative integer, and, in fact, it is even, since both u and v are. Let $c = \frac{v-2u}{3} + a$ and $b = \frac{u-3a-c}{2} = \frac{5u-v-12a}{6}$, where a will be a nonnegative integer determined below. Note that by construction, we have $3a+2b+c = u$. Trivially, c is a nonnegative integer and, because $\frac{v-2u}{3}$ is even a and c have the same parity. This implies that $3a+c$ is even, and thus b is an integer. Since b must nonnegative as well, we must have $5u-v-12a \geq 0$, which is equivalent to

$$a \leq \frac{5u - v}{12}.$$

Also,

$$\begin{aligned}
a + b + c = a + \frac{5u - v - 12a}{6} + \frac{v - 2u}{3} + a = \frac{v + u}{6} \\
\leq \frac{v + \frac{v}{2}}{6} = \frac{3v}{12} = \frac{v}{4},
\end{aligned}$$

which is one of our requirements. Further, note that

$$\begin{aligned}
m_1 &= \frac{4}{u}\left(\binom{v-u}{1}\binom{u}{3} - a\frac{S}{c-a}\right) \\
&= \frac{4}{u}\left(\frac{u(v-u)(u-1)(u-2)}{6} - a\frac{u(v-u)(v-2u)(v-3)}{2(v-2u)}\right) \\
&= \frac{2(v-u)}{3}\left((u-1)(u-2) - 3a(v-3)\right),
\end{aligned}$$

$$\begin{aligned}
m_2 &= \frac{2}{u}\left(\binom{v-u}{2}\binom{u}{2} - b\frac{S}{c-a}\right) \\
&= \frac{2}{u}\left(\frac{u(v-u)(u-1)(v-u-1)}{4} - b\frac{u(v-u)(v-2u)(v-3)}{2(v-2u)}\right) \\
&= \frac{(v-u)}{2}\left((u-1)(v-u-1) - 2b(v-3)\right),
\end{aligned}$$

$$m_3 = \frac{S}{c-a} = \frac{u(v-u)(v-3)}{2}.$$

Since $v - u$ is divisible by 2, it follows that m_2 and m_3 are integers, and clearly m_3 is nonnegative. If $v - u$ is not divisible by 3, since $v - 2u$ is, it follows that $u = (v - u) - (v - 2u)$ is not divisible by 3 either, implying either $u \equiv 1 \pmod 3$ or $u \equiv 2 \pmod 3$. So one of $(v - u)$, $(u - 1)$ or $(u - 2)$ must be divisible by 3, which implies that m_1 is an integer. It remains to find sufficient conditions for a such that m_1 and m_2 are nonnegative (and such that b is nonnegative). Note that m_1 is nonnegative if and only if $a \leq \frac{(u-1)(u-2)}{3(v-3)}$. Further, m_2 is nonnegative if and only if

$$\begin{aligned} 0 &\leq (u-1)(v-u-1) - 2\frac{5u-v-12a}{6}(v-3) \\ &= vu - u^2 - u - v + u + 1 - \frac{(v-3)(5u-v)}{3} + 4a(v-3) \\ &= \frac{-3u^2 + 3vu - 3v + 3 - 5vu + v^2 + 15u - 3v}{3} + 4a(v-3) \\ &= \frac{v^2 - 2vu - 3u^2 - 6v + 15u + 3}{3} + 4a(v-3), \end{aligned}$$

which is true if and only if

$$a \geq \frac{-v^2 + 2uv + 3u^2 + 6v - 15u - 3}{12(v-3)}.$$

To insure that both b and m_1 are nonnegative, we simply let a be the minimum of $\frac{5u-v}{12}$ and $\lfloor \frac{(u-1)(u-2)}{3(v-3)} \rfloor$. Note that, since $v \leq 4u$, $\frac{5u-v}{12}$ is nonnegative, and, since v and u are divisible by 4 and

$$5u - v \equiv 3u + 2u - v \equiv 2u - v \equiv 0 \pmod{3},$$

$5u - v$ is divisible by 3 and 4, implying that $\frac{5u-v}{12}$ is an integer. So, if $a = \frac{5u-v}{12}$, a is a nonnegative integer. Further, by above, $b = 0$, implying immediately that m_2 is nonnegative, and we are done. Otherwise, $a = \lfloor \frac{(u-1)(u-2)}{3(v-3)} \rfloor$, which of course is a nonnegative integer. So we need only check that m_2 is nonnegative. If we let

$$Y = \frac{-v^2 + 2vu + 3u^2 + 6v - 15u - 3}{12(v-3)},$$

we have

$$\begin{aligned}
a - Y &\geq \frac{(u-1)(u-2)}{3(v-3)} - 1 - Y \\
&= \frac{(u-1)(u-2) - 3(v-3)}{3(v-3)} - \frac{-v^2 + 2vu + 3u^2 + 6v - 15u - 3}{12(v-3)} \\
&= \frac{v^2 - 2vu + u^2 - 18v + 3u + 47}{12(v-3)} \\
&= \frac{v^2 - 2vu - 12v - 6v + 12u + 72 + u^2 - 9u - 25}{12(v-3)} \\
&= \frac{(v-6)(v-2u-12) + (u-12)(u+3) + 11}{12(v-3)}.
\end{aligned}$$

Note that $v-2u$ must be a multiple of 3 and it must be a multiple of 4, which implies that $v-2u \geq 12$, and, of course, $v \geq 6$. Thus, if $u \geq 12$, m_2 is nonnegative. This just leaves us to consider $u = 8$ (since $u = 4$, and, in general, $u = k$, is trivial) but the only $v \leq 32$ that also has $v - 2u$ a multiple of 12 is $v = 28$. In that case, it is trivial to check that $a = 0$, $b = 2$, $c = 4$, $m_1 = 560$, $m_2 = 330$, and $m_3 = 2000$, all nonnegative integers. This finishes the $u \equiv 2v \pmod 3$ case.

Case 2: $v \not\equiv 2u \pmod 3$ *and* $u \not\equiv 0 \pmod 3$.

Here we let $c = \frac{v-2u}{2} + a$ and $b = \frac{u-3a-c}{2} = \frac{4u-v-4a}{4}$, where a again will be determined below. Since 4 divides u and v, it follows that c is a nonnegative integer with the same parity as a and thus b is also an integer. We will have b nonnegative if and only if $a \leq u - \frac{v}{4}$. It is again clear that by construction $3a + 2b + c = u$ and in this case we have

$$a + b + c = a + \frac{4u - v - 4a}{4} + \frac{v - 2u}{2} + a = \frac{v}{4}.$$

So it remains to check m_1, m_2 and m_3 for our other conditions on a.

$$m_1 = \frac{4}{u}\left(\binom{v-u}{1}\binom{u}{3} - a\frac{S}{c-a}\right)$$
$$= \frac{4}{u}\left(\frac{u(v-u)(u-1)(u-2)}{6} - a\frac{u(v-u)(v-2u)(v-3)}{3(v-2u)}\right)$$
$$= \frac{2(v-u)}{3}\left((u-1)(u-2) - 2a(v-3)\right),$$

$$m_2 = \frac{2}{u}\left(\binom{v-u}{2}\binom{u}{2} - b\frac{S}{c-a}\right)$$
$$= \frac{2}{u}\left(\frac{u(v-u)(u-1)(v-u-1)}{4} - b\frac{u(v-u)(v-2u)(v-3)}{3(v-2u)}\right)$$
$$= \frac{(v-u)}{6}\left(3(u-1)(v-u-1) - 4b(v-3)\right),$$

$$m_3 = \frac{S}{c-a} = \frac{u(v-u)(v-3)}{3}.$$

If $v-u$ is divisible by 3, all three m_i are integers. Otherwise, since $v-2u$ and u are also not divisible by 3, it follows that $v-2u \not\equiv v-u \pmod 3$, and thus $v-2u+v-u \equiv 0 \pmod 3$, which implies that $2v \equiv 0 \pmod 3$ and thus $v \equiv 0 \pmod 3$. This shows that $v-3$ is divisible by 3, which immediately shows that m_2 and m_3 are integers. Further, since $u \not\equiv 0 \pmod 3$, it follows that $(u-1)(u-2)$ is divisible by 3, and this now shows that m_1 is an integer as well. Thus, all we need do is insure that m_1 and m_2 are nonnegative. Note that m_1 is nonnegative if and only if $a \leq \frac{(u-1)(u-2)}{2}$, and m_2 is nonnegative if and only if

$$0 \leq 3(u-1)(v-u-1) - 4b(v-3)$$
$$= 3(u-1)(v-u-1) - (4u-v-4a)(v-3)$$
$$= 3vu - 3u^2 - 3v + 3 - (4u-v)(v-3) + 2a(v-3)$$
$$= v^2 - uv - 3u^2 - 6v + 12u + 3 + 2a(v-3),$$

which is true if and only if

$$a \geq \frac{-v^2 + vu + 3u^2 + 6v - 12u + 3}{2(v-3)}.$$

Similar to the first case, we let a be the minimum of $u - \frac{v}{4}$ and $\lfloor \frac{(u-1)(u-2)}{2(v-3)} \rfloor$. It is clear that either way a is a nonnegative integer. If $a = u - \frac{v}{4}$, again, we have $b = 0$, implying that m_2 is nonnegative and we are done. Otherwise, $a = \lfloor \frac{(u-1)(u-2)}{2(v-3)} \rfloor$. Then, if $Y' = \frac{-v^2+vu+3u^2+6v-12u+3}{2(v-3)}$, we have

$$\begin{aligned} a - Y' &\geq \frac{(u-1)(u-2)}{2(v-3)} - 1 - Y' \\ &= \frac{u^2 - 3u + 2}{2(v-3)} - \frac{-v^2 + vu + 3u^2 + 6v - 12u + 3}{2(v-3)} \\ &= \frac{v^2 - uv - 2u^2 - 6v + 9u - 1}{2(v-3)} \\ &= \frac{v^2 - 2uv - 4v + uv - 2u^2 + 8u - 2v + u - 1}{2(v-3)} \\ &= \frac{v(v - 2u - 4) + u(v - 2u) + 2(4u - v) + (u - 1)}{2(v-u)} \geq 0, \end{aligned}$$

since $v \geq 2u + 4$, $4u \geq v$ (and of course $u \geq 1$). Thus, m_2 is nonnegative.

Case 3: $v \not\equiv 2u \pmod 3$ *and* $u \equiv 0 \pmod 3$.

In this case, we let a, b, and c be defined precisely the same as in the previous case. Note that if m_1 and m_2 are integers, the same analysis holds as in that case, and we are done. Otherwise, let $m_1' = \lfloor m_1 \rfloor$ and let $m_2' = \lfloor m_2 \rfloor$. Then, let $\epsilon_i = m_i - m_i'$ for $i = 1, 2$. Further, let $n_i = 3m_i$ for $i = 1, 2$. Then

$$\begin{aligned} n_1 &= 2(v-u)\left((u-1)(u-2) - 2a(v-3)\right) \\ &= 2(v-u)(u-1)(u-2) - 4a(v-u)(v-3), \end{aligned}$$

$$\begin{aligned} n_2 &= \frac{(v-u)}{2}\left(3(u-1)(v-u-1) - 4b(v-3)\right) \\ &= \frac{3}{2}(v-u)(u-1)(v-u-1) - 2b(v-u)(v-3), \end{aligned}$$

and, since $v-u$ is even, it follows that both n_i are integers. This implies that $\epsilon_i \in \{\frac{1}{3}, \frac{2}{3}\}$. Further, since $u \equiv 0 \pmod 3$ and $v \not\equiv 0 \pmod 3$, which in turn implies that $v^2 \equiv 1 \pmod 3$,

$$n_1 \equiv 2(v-u)(u-1)(u-2) - 4a(v-u)(v-3) \equiv 4v - 4av^2 \equiv v - a \pmod 3,$$

$$n_2 \equiv \frac{3}{2}(v-u)(u-1)(v-u-1) - 2b(v-u)(v-3) \equiv -2bv^2 \equiv b \pmod 3.$$

Note that $b = u - \frac{v}{4} - a$, so $b \equiv -v - a \pmod 3$, and thus $n_1 \not\equiv n_2 \pmod 3$, which implies that either $\epsilon_1 = \frac{1}{3}$ and $\epsilon_2 = \frac{2}{3}$ or $\epsilon_1 = \frac{2}{3}$ and $\epsilon_2 = \frac{1}{3}$. Note that either way,

$$3\epsilon_1 \frac{u}{4} + 2\epsilon_2 \frac{u}{2} + \epsilon_1 \frac{u}{4} = \epsilon_1 u + \epsilon_2 u = u,$$

$$\epsilon_1 \frac{u}{4} + \epsilon_2 \frac{u}{2} + \epsilon_3 \frac{u}{4} = \epsilon_1 \frac{u}{2} + \epsilon_2 \frac{u}{2} = \frac{u}{2} \leq \frac{v}{4},$$

which shows that we can find an m-partition of u with $\epsilon_1 \frac{u}{4}$ parts of size 1 and 3, and $\epsilon_2 \frac{u}{2}$ parts of size 2. Finally, we see that

$$m_1' \frac{u}{4} + m_3 a + \epsilon_1 \frac{u}{4} = (m_1' + \epsilon_1)\frac{u}{4} + m_3 a = m_1 \frac{u}{4} + m_3 a = \binom{v-u}{1}\binom{u}{3},$$

$$m_2' \frac{u}{2} + m_3 b + \epsilon_2 \frac{u}{2} = (m_1' + \epsilon_2) \frac{u}{2} + m_3 b = m_2 \frac{u}{2} + m_3 b = \binom{v-u}{2} \binom{u}{2},$$

$$m_1' \frac{u}{4} + m_3 c + \epsilon_1 \frac{u}{4} = (m_1' + \epsilon_1) \frac{u}{4} + m_3 c = m_1 \frac{u}{4} + m_3 c = \binom{v-u}{3} \binom{u}{1},$$

which, using theorem 1.3.3 again, shows that we can find a suitable collection of m-partitions of u (this time including the single m-partition with $\epsilon_1 \frac{u}{4}$ parts of size 1 and 3 and $\epsilon_2 \frac{u}{2}$ parts of size 2) such that each subset of u occurs the correct number of times. Thus, in all cases, $(u, v) \in \mathcal{EP}_4$. □

1.6 Proving Theorem 1.2.3 for k=5

Before considering the case $k = 5$, we will describe a new method of constructing our collection of m-partitions of U (where $m = \frac{v}{k}$). We will construct each m-partition $\mathcal{A}$ by first partitioning U into $\frac{u}{k}$ parts of size k, and then partitioning each such k-subset into parts of size 2 or more. To do this, we will find positive integers $p_{i,j}$ for $1 \leq i \leq N = \binom{v-1}{k-1} - \binom{u-1}{k-1}$ and $1 \leq j \leq \frac{u}{k}$, and positive integers $q_{i,j,t}$ for $1 \leq i \leq N$, $1 \leq j \leq \frac{u}{k}$, and $1 \leq t \leq p_{i,j}$, satisfying the following properties: for all $(i, j) \in [N] \times [\frac{u}{k}]$ we have

$$p_{i,j} \geq 2 \text{ and } \sum_{t=1}^{p_{i,j}} q_{i,j,t} = k,$$

and we also have

$$\sum_{j=1}^{\frac{u}{k}} p_{i,j} \leq m \text{ for all } i \in [N],$$

$$|\{(i, j, t) : q_{i,j,t} = s\}| = \binom{v-u}{k-s} \binom{u}{s} \text{ for all } s \in [k-1].$$

Then, it will follow from theorem 1.3.3 that there exists a collection of m-partitions $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N$ of U such such that each s-subset of U occurs $\binom{v-u}{k-s}\binom{u}{s}$, for $1 \leq s \leq k-1$. Further, for $i \in [N]$, there will exist pairwise disjoint k-subsets $A_{i,1}, A_{i,2}, \ldots, A_{i,\frac{u}{k}}$ of U and, for $j \in [\frac{u}{k}]$, partitions $\{B_{i,j,1}, B_{i,j,2}, \ldots, B_{i,j,p_{i,j}}\}$ of $A_{i,j}$ into $p_{i,j}$ parts, with $q_{i,j,t} = |B_{i,j,t}|$ for $1 \leq t \leq p_{i,j}$, such that

$$\mathcal{A}_i = \{B_{i,j,p_{i,j}} : (i,j) \in [N] \times [\frac{u}{k}]\} \cup \mathcal{R}_i,$$

where $\mathcal{R}_i$ consists of an appropriate number of copies of the empty set. Below we show how to apply this method to the case $k = 5$; it is our hope that it can be generalized for $k > 5$. We in fact break this up into two theorems.

Theorem 1.6.1 *For any $u, v \in \mathbb{N}$ such that 5 divides both u and v, and such that $v \leq 3u + 5$, $(u, v) \in \mathcal{EP}_5$ if and only if $v \geq 2u$.*

Proof. Suppose we have u, v with 5 dividing both u and v and $v \leq 3u$. As above, we need only consider $v > 2u$. Further, as the case $u = 5$ is trivial, we will assume $u \geq 10$. We will show how to construct $p_{i,j}, q_{i,j,t}$ as above. First, we consider the case $2u + 5 \leq v \leq 3u$. To simplify notation, we let $w = v - u$; thus, it follows that $u + 5 \leq w \leq 2u$ Let n_1, n_2, n_3 and n_4 be defined as follows:

$$n_1 = \binom{w}{1}\binom{u}{4},$$

$$n_2 = \frac{1}{5}\left(2\binom{w}{1}\binom{u}{4} + 4\binom{w}{2}\binom{u}{3} + \binom{w}{3}\binom{u}{2} - 2\binom{w}{4}\binom{u}{1}\right),$$

$$n_3 = \frac{1}{5}\left(-2\binom{w}{1}\binom{u}{4} + \binom{w}{2}\binom{u}{3} - \binom{w}{3}\binom{u}{2} + 2\binom{w}{4}\binom{u}{1}\right),$$

$$n_4 = \frac{1}{5}\left(-\binom{w}{1}\binom{u}{4} - 2\binom{w}{2}\binom{u}{3} + 2\binom{w}{3}\binom{u}{2} + \binom{w}{4}\binom{u}{1}\right).$$

Note that each n_i is an integer, since $\binom{u}{i}$ is divisible by 5 for $1 \leq i \leq 4$. Now, obviously $n_1 \geq 0$. Further, since $w \geq u + 5$,

$$\binom{w}{1}\binom{u}{4} = \frac{wu}{24}(u-1)(u-2)(u-3) \le \frac{wu}{24}(w-1)(w-2)(w-3)$$
$$\le \binom{w}{4}\binom{u}{1},$$

$$\binom{w}{2}\binom{u}{3} = \frac{wu(u-1)(w-1)}{12}(u-2) \le \frac{wu(u-1)(w-1)}{12}(w-2)$$
$$\le \binom{w}{3}\binom{u}{2}.$$

This shows that $n_4 \ge 0$. Also, note that

$$\binom{w}{2}\binom{u}{3} - 2\binom{w}{1}\binom{u}{4} = wu(u-1)\,(u-2)\left(\frac{w-1}{12} - \frac{2(u-3)}{24}\right)$$
$$= \frac{uw(u-1)(u-2)}{12}(w-u+2) \ge 0,$$

$$2\binom{w}{4}\binom{u}{1} - \binom{w}{3}\binom{u}{2} = wu(w-1)(w-2)\left(\frac{2(w-3)}{24} - \frac{u-1}{12}\right)$$
$$= \frac{uw(u-1)(u-2)}{12}(w-u-2)$$
$$= \frac{uw(u-1)(u-2)}{12}\,(w-(u+2)) \ge 0,$$

which shows that $n_3 \ge 0$. Further, note that if we expand n_2 out as a polynomial in u and w, and then simplify it, using, for example, Mathematica, we get

$$n_2 = \frac{uw}{60}\left(-w^3 + (u+5)w^2 + (4u^2 - 15u)w + (u^3 - 10u^2 + 25u - 10)\right).$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = -x^3 + (u+5)x^2 + (4u^2 - 15u)x + (u^3 - 10u^2 + 25u - 10).$$

Then, we have

$$f'(x) = -3x^2 + (2u+10)x + 4u^2 - 15u,$$

$$f''(x) = -6x + (2u+10).$$

Thus the only solution to $f''(x) = 0$ is $x = \frac{u+5}{3}$. Since $\frac{u+5}{3} < \frac{2u}{3} < u$, this implies that f is concave down on the interval $[u, 2u+5]$. Further, we have

$$f(u) = -u^3 + u^3 + 5u^2 + 4u^3 - 15u^2 + u^3 - 10u^2 + 25u - 10$$

$$= 5u^3 - 20u^2 + 25u - 10 = 5(u-1)(u-1)(u-2) > 0,$$

$$f(2u+5) = 5u^3 - 40u^2 - 75u - 10 = 5u^3 - 40u^2 - 75u - 250 + 240$$

$$= 5(u-10)(u^2 + 2u + 5) + 240 > 0,$$

since $u \geq 10$. By an elementary calculus result, it follows that f is positive on the interval $[u, 2u+5]$. So, since $u < w \leq 2u+5$, we must have $f(w) > 0$, which, in turn, implies that $n_2 > 0$.

Now, for $i \in [N], j \in [\frac{u}{5}]$ (where $N = \binom{v-1}{4} - \binom{u-1}{4}$, the total number of m-partitions of U we need to construct) we will define our $p_{i,j}$. For any $(i,j) \in [N] \times [\frac{u}{5}]$, it will either be the case that $p_{i,j} = 2$ or $p_{i,j} = 3$. We will choose $n_1 + n_2$ of the $p_{i,j}$ to be equal to 2 and $n_3 + n_4$ of the $p_{i,j}$ to be equal to 3. To see that this is possible, note that,

$$\sum_{i=1}^{4} n_i = \frac{1}{5}\left(4\binom{w}{1}\binom{u}{4} + 3\binom{w}{2}\binom{u}{3} + 2\binom{w}{3}\binom{u}{2} + \binom{w}{4}\binom{u}{1}\right)$$
$$= \frac{u}{5}\left(\binom{w}{1}\binom{u-1}{3} + \binom{w}{2}\binom{u-1}{2} + \binom{w}{3}\binom{u-1}{1} + \binom{w}{4}\right),$$

and, by applying the combinatorial identity described in the proof of theorem 1.2.5, we have

$$\sum_{i=1}^{4} n_i = \frac{u}{5}\left(\binom{w+u-1}{4} - \binom{u-1}{4}\right) = \frac{u}{5}\left(\binom{v-1}{4} - \binom{u-1}{4}\right) = \frac{Nu}{5}.$$

And this is, of course, the cardinality of $[N]\times[\frac{u}{5}]$, which is precisely the total number of $p_{i,j}$. However, we also need to ensure that, for any $i \in [N]$, we have $\sum_{j=1}^{\frac{u}{5}} p_{i,j} \leq \frac{v}{5}$. If we let m_i be the number of $p_{i,j}$ that are equal to 3, with the rest equal to 2, the required inequality will be true if and only if

$$\frac{v}{5} \geq 3m_i + 2(\frac{u}{5} - m_i) = \frac{2u}{5} + m_i,$$

which is true if and only if $m_i \leq \frac{v-2u}{5}$. So, the number of $p_{i,j}$ that are equal to 3 can be 0 (if all $m_i = 0$). Also, the number of $p_{i,j}$ that are equal to 3 will be maximized if each $m_i = \frac{v-2u}{5} = \frac{w-u}{5}$. Then, it will be the case that $\frac{N(w-u)}{5}$ of the $p_{i,j}$ are equal to 3. Note that $\frac{N(w-u)}{5} = \frac{1}{5}(wN - uN)$, and

$$wN = w\left(\binom{w-u-1}{4} - \binom{u-1}{4}\right)$$
$$= w\binom{u}{4} + w\binom{w-1}{1}\binom{u}{3} + w\binom{w-1}{2}\binom{u}{2} + w\binom{w-1}{3}\binom{u}{1}$$
$$+ w\binom{w-1}{4} - w\binom{u-1}{4}$$
$$= \binom{w}{1}\binom{u}{4} + 2\binom{w}{2}\binom{u}{3} + 3\binom{w}{3}\binom{u}{2} + 4\binom{w}{4}\binom{u}{1}$$
$$+ w\left(\binom{w-1}{4} - \binom{u-1}{4}\right),$$

$$\begin{aligned} uN &= u\left(\binom{w-u-1}{4} - \binom{u-1}{4}\right) \\ &= u\binom{w}{1}\binom{u-1}{3} + u\binom{w}{2}\binom{u-1}{2} + u\binom{w}{3}\binom{u-1}{1} + u\binom{w}{4} \\ &= 4\binom{w}{1}\binom{u}{4} + 3\binom{w}{2}\binom{u}{3} + 2\binom{w}{3}\binom{u}{2} + \binom{w}{4}\binom{u}{1}, \end{aligned}$$

and thus, since $w \geq u$,

$$\begin{aligned} wN - uN &\geq -3\binom{w}{1}\binom{u}{4} - \binom{w}{2}\binom{u}{3} + \binom{w}{3}\binom{u}{2} + 3\binom{w}{4}\binom{u}{1} \\ &= 5(n_3 + n_4). \end{aligned}$$

This shows that $0 \leq n_3 + n_4 \leq \frac{N(w-u)}{5}$, which implies that, since we can change the values of $p_{i,j}$ from 3 to 2 one at a time, we can find values of m_i such that exactly $n_3 + n_4$ of the $p_{i,j}$ are equal to 3 and the other $n_1 + n_2$ must be equal to 2. Consider the $n_1 + n_2$ ordered pairs $(i, j) \in [N] \times [\frac{u}{5}]$ such that $p_{i,j} = 2$. For n_1 of these pairs, let $q_{i,j,1} = 4$ and $q_{i,j,2} = 1$, and for the other n_2 such pairs, let $q_{i,j,1} = 3$ and $q_{i,j,2} = 2$. Then, for n_3 of the pairs (i, j) with $p_{i,j} = 3$, let $q_{i,j,1} = 3$ and $q_{i,j,2} = q_{i,j,3} = 1$. For the other n_4 such pairs, let $q_{i,j,1} = q_{i,j,2} = 2$ and let $q_{i,j,3} = 1$. Then, note first for each $(i, j) \in [N] \times [\frac{u}{5}]$, it is the case that $\sum_{t=1}^{p_{i,j}} q_{i,j,t} = 5$. Further, it is easy to check that

$$|\{(i,j,t) : q_{i,j,t} = 4\}| = n_1 = \binom{v-u}{1}\binom{u}{4},$$

$$|\{(i,j,t) : q_{i,j,t} = 3\}| = n_2 + n_3 = \binom{v-u}{2}\binom{u}{3},$$

$$|\{(i,j,t) : q_{i,j,t} = 2\}| = n_2 + 2n_4 = \binom{v-u}{3}\binom{u}{2},$$

$$|\{(i,j,t) : q_{i,j,t} = 1\}| = n_1 + 2n_3 + n_4 = \binom{v-u}{4}\binom{u}{1}.$$

Thus, by our above discussion, we can apply theorem 1.3.3 to find a collection of m-partitions of U that satisfies theorems 1.2.4 and 1.2.5, showing that $(u,v) \in \mathcal{EP}_5$. $\square$

Theorem 1.6.2 *For any $u, v \in \mathbb{N}$ such that 5 divides both u and v, and such that $v \geq 3u + 10$, $(u,v) \in \mathcal{EP}_5$.*

Proof. Our proof will be similar to the proof of theorem 1.6.1. Again, let $w = v - u$. Further, again by theorem 1.3.1, we need only consider $v < 5u$, so we have $2u + 10 \leq w \leq 4u - 5$. Let $N = \left(\binom{v-1}{4} - \binom{u-1}{4}\right)$ and let

$$n_1 = \binom{w}{1}\binom{u}{4}, \quad n_2 = \binom{w}{2}\binom{u}{3}.$$

Further, let

$$n_3' = \frac{1}{2}\left(-\binom{w}{2}\binom{u}{3} + \binom{w}{3}\binom{u}{2}\right),$$

$$n_4' = \frac{1}{10}\left(-2\binom{w}{1}\binom{u}{4} + \binom{w}{2}\binom{u}{3} - \binom{w}{3}\binom{u}{2} + 2\binom{w}{4}\binom{u}{1}\right),$$

and let $n_3 = \lfloor n_3' \rfloor$, $n_4 = \lfloor n_4' \rfloor$, and $n_5 = n_3' + n_4' - n_3 - n_4$. Note that n_3' is nonnegative because (similar to above) $w > u$. Further, in theorem 1.6.1, it was shown that

$$-2\binom{w}{1}\binom{u}{4} + \binom{w}{2}\binom{u}{3} - \binom{w}{3}\binom{u}{2} + 2\binom{w}{4}\binom{u}{1} \geq 0,$$

and since proof of this never used the fact that $w < 2u$, it follows that the same proof shows that $n_4 \geq 0$. Also, since u is divisible by 5, $2n_4'$ must be an integer, implying that $n_4' - n_4 \in \{0, \frac{1}{2}\}$. Clearly, $n_3' - n_3 \in \{0, \frac{1}{2}\}$ as well. Further, note that

$$2n_4' \equiv 10n_4' \equiv \binom{w}{2}\binom{u}{3} - \binom{w}{3}\binom{u}{2} \equiv 2n_3' \pmod{2},$$

which implies that either $n_3' - n_3 = n_4' - n_4 = 0$, and thus $n_5 = 0$, or $n_3' - n_3 = n_4' - n_4 = \frac{1}{2}$, and thus $n_5 = 1$. Either way n_5 is a nonnegative integer, which shows that each of the n_i are nonnegative integers, for $i \in [5]$.

We now show how to choose $p_{i,j}$ for $(i,j) \in [N] \times [\frac{u}{5}]$ (where $N = \binom{v-1}{4} - \binom{u-1}{4}$ as above) and $q_{i,j,t}$ for $1 \le t \le p_{i,j}$ satisfying the above inequalities. We will let $n_1 + n_2$ of the $p_{i,j}$ be equal to 2, n_3 of the $p_{i,j,}$ be equal to 3, n_5 of the $p_{i,j}$ be equal to 4, and n_4 of the $p_{i,j}$ be equal to 5. First, note that:

$$\begin{aligned}\sum_{i=1}^{5} n_i &= n_1 + n_2 + n_3 + n_4 + n_3' + n_4' - n_3 - n_4 = n_1 + n_2 + n_3' + n_4' \\ &= \frac{1}{10}\left(8\binom{w}{1}\binom{u}{4} + 6\binom{w}{2}\binom{u}{3} + 4\binom{w}{3}\binom{u}{2} + 2\binom{w}{4}\binom{u}{1}\right) \\ &= \frac{u}{5}\left(\binom{w}{1}\binom{u-1}{3} + \binom{w}{2}\binom{u-1}{2} + \binom{w}{3}\binom{u-1}{1} + \binom{w}{4}\right) \\ &= \frac{u}{5}\left(\binom{w+u-1}{4} - \binom{u-1}{4}\right) = \frac{Nu}{5},\end{aligned}$$

implying that it is possible to assign the above values to the $p_{i,j}$. Next, we show how to assign these values in such a way that $\sum_{j=1}^{\frac{u}{5}} p_{i,j} \le \frac{v}{5}$ for all $i \in [N]$. Let $a = \lceil \frac{w}{60} \rceil$ and let $b = \lceil \frac{w}{36} \rceil$. We claim that the following are all true:

$$a + b \le \frac{u}{5},\ 3a + b \le \frac{w-u}{5},$$

$$aN \ge n_4 + n_5,\ bN \ge n_3.$$

First, note that for $u < 155$, we can simply check each of the above inequalities for each pair (u, w) where $2u + 10 \le w\ 4u - 5$ (and of course 5 divides both u and w). This can be done in fractions of a second on Mathematica. So assume that $u \ge 155$. Then, note that

$$\begin{aligned}a + b &\le \frac{w}{60} + 1 + \frac{w}{36} + 1 = \frac{8w + 360}{180} \le \frac{8(4u-5) + 360}{180} \\ &= \frac{32u + 320}{180} \le \frac{36u}{180} = \frac{u}{5},\end{aligned}$$

$$\begin{aligned}3a+b &\le 3(\frac{w}{60}+1)+\frac{w}{36}+1=\frac{14w+720}{180}\le\frac{14w+8u+80}{180}\\&=\frac{36w-22w+8u+80}{180}\le\frac{36w-22(2u+10)+8u+80}{180}\\&\le\frac{36w-36u}{180}=\frac{w-u}{5}.\end{aligned}$$

Also, similar to above,

$$\begin{aligned}wN &= w\left(\binom{w+u-1}{4}-\binom{u-1}{4}\right)\\&= w\binom{w-1}{4}+w\binom{w-1}{3}\binom{u}{1}+w\binom{w-1}{2}\binom{u}{2}+w\binom{w-1}{1}\binom{u}{3}\\&+w\left(\binom{u}{4}-\binom{u-1}{4}\right)\\&=5\binom{w}{5}+4\binom{w}{4}\binom{u}{1}+3\binom{w}{3}\binom{u}{2}+2\binom{w}{2}\binom{u}{3}+\binom{w}{1}\binom{u-1}{3}.\end{aligned}$$

Thus,

$$\begin{aligned}aN-n_4-n_5 &\ge \frac{w}{60}N-\frac{1}{2}\\&-\frac{1}{10}\left(-2\binom{w}{1}\binom{u}{4}+\binom{w}{2}\binom{u}{3}-\binom{w}{3}\binom{u}{2}+2\binom{w}{4}\binom{u}{1}\right)\\&=\frac{1}{12}\binom{w}{5}-\frac{2}{15}\binom{w}{4}\binom{u}{1}+\frac{3}{20}\binom{w}{3}\binom{u}{2}-\frac{1}{15}\binom{w}{2}\binom{u}{3}\\&+\frac{1}{5}\binom{w}{1}\binom{u}{4}+\frac{1}{60}\binom{w}{1}\binom{u-1}{3}-\frac{1}{2}.\end{aligned}$$

We will show this last quantity to be nonnegative. Since $u \ge 155$, we obviously have $\frac{1}{60}\binom{w}{1}\binom{u-1}{3}-\frac{1}{2}\ge 0$, so it remains to show that, if we let

$$f_1(x) = \frac{1}{12}\binom{w}{5} - \frac{2}{15}\binom{w}{4}\binom{x}{1} + \frac{3}{20}\binom{w}{3}\binom{x}{2} - \frac{1}{15}\binom{w}{2}\binom{x}{3} + \frac{1}{5}\binom{w}{1}\binom{x}{4}$$

we have $f_1(u) \geq 0$. We will in fact prove the stronger statement that $f(x) \geq 0$ for integer values of x such that $x \geq \frac{w}{4}$. Let

$$g_1(x) = -\frac{2}{15}\binom{w}{4} + \frac{3}{20}\binom{w}{3}\binom{x}{1} - \frac{1}{15}\binom{w}{2}\binom{x}{2} + \frac{1}{5}\binom{w}{1}\binom{x}{3},$$

$$h_1(x) = \frac{3}{20}\binom{w}{3} - \frac{1}{15}\binom{w}{2}\binom{x}{1} + \frac{1}{5}\binom{w}{1}\binom{x}{2}.$$

First, note that

$$h_1(x) = \frac{w}{60}\left(6x^2 - (2w+4)x + 15w^2 - 45w + 30\right),$$

and if we let D_1 denote the discriminant of $6x^2 - (2w+4)x + 15w^2 + 45w - 30$, we have

$$\begin{aligned} D_1 &= (2w+4)^2 - 4(6)(15w^2 - 45w + 30) = -356w^2 + 1096w - 704 \\ &= -356(w^2 - 4w + 3) - 328w + 364 \\ &= -356(w-3)(w-1) - 328(w-2) - 292 < 0, \end{aligned}$$

since $w > 3$ is of course true. Thus, $h_1(x)$ has no real roots, and, since $h_1(0) = \frac{3}{20}\binom{w}{3} > 0$, it follows that $h_1(x) > 0$ for all $x \in \mathbb{R}$. Next, we show, via induction, that $g_1(x) > 0$ for all integers x with $x \geq \frac{w}{4}$. Let $x_0 = \lceil \frac{w}{4} \rceil$. Then, $\frac{w}{4} \leq x_0 \leq \frac{w}{4} + 1$, so

$$\begin{aligned} g_1(x_0) &\geq -\frac{2}{15}\binom{w}{4} + \frac{3}{20}\binom{w}{3}\binom{\frac{w}{4}}{1} - \frac{1}{15}\binom{w}{2}\binom{\frac{w}{4}+1}{2} + \frac{1}{5}\binom{w}{1}\binom{\frac{w}{4}}{3} \\ &= \frac{w}{5760}\left(w^2 + 32w - 96\right)(w-2) = \frac{w}{5760}(w^2 + 32w - 96)(w-2) \\ &= \frac{w}{5760}\left((w-3)(w+35) + 9\right)(w-2) > 0. \end{aligned}$$

Now, assume $g_1(x) > 0$ for some $x \geq x_0$. Then, note that the recursion

$$\binom{x+1}{t} = \binom{x}{t} + \binom{x}{t-1}$$

implies that $g_1(x+1) = g_1(x) + h_1(x) > 0$, and thus if follows that $g_1(x) > 0$ for all $x \in \mathbb{Z}$ with $x \geq \frac{w}{4}$. Now, we will use a similar induction to show that $f_1(x) > 0$ for all $x \in \mathbb{Z}$ with $x \geq \frac{w}{4}$. We have

$$\begin{aligned}
f_1(x_0) \geq & \frac{1}{12}\binom{w}{5} - \frac{2}{15}\binom{w}{4}\binom{\frac{w}{4}+1}{1} + \frac{3}{20}\binom{w}{3}\binom{\frac{w}{4}}{2} - \frac{1}{15}\binom{w}{2}\binom{\frac{w}{4}+1}{3} \\
& + \frac{1}{5}\binom{w}{1}\binom{\frac{w}{4}}{4} \\
= & \frac{w}{92160}\left(3w^4 - 952w^3 + 5568w^2 - 9920w + 4608\right) \\
= & \frac{w}{92160}\left(3w^3(w-318) + 2w^3 + 5569w(w-2) + 1216w + 4608\right).
\end{aligned}$$

Since $u \geq 155$ and $w \geq 2u + 10$, $w \geq 310$, which implies that $f_1(x_0) > 0$. Assume $f_1(x) > 0$ for some $x \geq x_0$. Using the above recursion again, it is easy to see that $f_1(x+1) = f_1(x) + g_1(x) > 0$, implying that $f_1(x) > 0$ for all $x \in \mathbb{Z}$ with $x \geq x_0$. Thus, in particular, $f_1(u) > 0$, showing that $aN \geq n_4 + n_5$. To see that $b_N \geq n_3$, note that

$$\begin{aligned}
bN - n_3 \geq & \frac{w}{36}N - \frac{1}{2}\left(-\binom{w}{2}\binom{u}{3} + \binom{w}{3}\binom{u}{2}\right) \\
= & \frac{5}{36}\binom{w}{5} + \frac{1}{9}\binom{w}{4}\binom{u}{1} - \frac{5}{12}\binom{w}{3}\binom{u}{2} + \frac{5}{9}\binom{w}{2}\binom{u}{3} \\
& + \frac{1}{36}\binom{w}{1}\binom{u-1}{3}.
\end{aligned}$$

We let

$$f_2(x) = \frac{5}{36}\binom{x}{5} + \frac{1}{9}\binom{x}{4}\binom{u}{1} - \frac{5}{12}\binom{x}{3}\binom{u}{2} + \frac{5}{9}\binom{x}{2}\binom{u}{3} + \frac{1}{36}\binom{x}{1}\binom{u-1}{3},$$

and it will be sufficient to show that $f_2(w) \geq 0$. Let

$$g_2(x) = \frac{1}{36}\binom{x}{4} + \frac{1}{36}\binom{x}{3}\binom{u}{1} - \frac{5}{36}\binom{x}{2}\binom{u}{2} + \frac{5}{18}\binom{x}{1}\binom{u}{3} + \frac{1}{36}\binom{u-1}{3},$$

$$h_2(x) = \frac{1}{36}\binom{x}{3} + \frac{1}{36}\binom{x}{2}\binom{u}{1} - \frac{5}{36}\binom{x}{1}\binom{u}{2} + \frac{5}{18}\binom{u}{3},$$

$$p_2(x) = \frac{1}{36}\binom{x}{2} + \frac{1}{36}\binom{x}{1}\binom{u}{1} - \frac{5}{36}\binom{u}{2}.$$

Then, note that

$$p_2(x) = \frac{1}{72}(x^2 + (2u-1)x - 5u^2 + 5u) = \frac{1}{72}\left((x-2u)(x+4u-1) + 3u^2 + 3u\right),$$

so, in particular, $p_2(x) > 0$ if $x \geq 2u$. Further,

$$\begin{aligned} h_2(2u+9) &= \frac{1}{36}\binom{2u+9}{3} + \frac{1}{36}\binom{2u+9}{2}\binom{u}{1} - \frac{5}{36}\binom{2u+9}{1}\binom{u}{2} + \frac{5}{18}\binom{u}{3} \\ &= \frac{7}{24}u^2 + \frac{251}{72}u + \frac{7}{2} > 0. \end{aligned}$$

And since, similar to above, $h_2(x+1) = h_2(x) + p_2(x)$, if, for $x \geq 2u+9$, we have $h_2(x) > 0$, then $h_2(x+1) > 0$. Thus, $h_2(x) > 0$ for all integer values of $x \geq 2u+9$. Also,

$$\begin{aligned} g_2(2u+9) &= \frac{1}{36}\binom{2u+9}{4} + \frac{1}{36}\binom{2u+9}{3}\binom{u}{1} - \frac{5}{36}\binom{2u+9}{2}\binom{u}{2} \\ &\quad + \frac{5}{18}\binom{2u+9}{1}\binom{u}{3} + \frac{1}{36}\binom{u-1}{3} \\ &= \frac{1}{108}(u^4 - 19u^3 + 98u^2 + 1030u + 375) \\ &= \frac{1}{108}(u^3(u-19) + 98u^2 + 1030u + 375) > 0. \end{aligned}$$

Since $g_2(x+1) = g_2(x) + h_2(x)$, it is again true by induction that $g_2(x) > 0$ for all integer values of $x \geq 2u+9$. Finally, $f_2(w) = \frac{g_2(w-1)}{w} > 0$, since $w - 1 \geq 2u + 10 - 1 = 2u + 9$. This proves that $bN \geq n_3$.

Now, for each $i \in [N]$, let $p_{i,j} = 5$ for a values of $j \in [\frac{u}{5}]$, $p_{i,j} = 3$ for b values of $j \in [\frac{u}{5}]$ and $p_{i,j} = 2$ for $\frac{u}{5} - a - b$ values of $j \in [\frac{u}{5}]$. Note that

$$\begin{aligned}\sum_{j=1}^{\frac{u}{5}} p_{i,j} &= 5a + 3b + 2\left(\frac{u}{5} - a - b\right) = 5a + 3b + \frac{2u}{5} - 2a - 2b = 3a + b + \frac{2u}{5} \\ &\leq \frac{w-u+2u}{5} = \frac{w+u}{5} = \frac{v}{5}.\end{aligned}$$

Then, we simply choose n_5 of the $p_{i,j}$ to switch from 5 to 4, $aN - n_4 - n_5$ of the $p_{i,j}$ to switch from 5 to 2, and $bN - n_3$ of the $p_{i,j}$ to switch from 3 to 2. There will be exactly n_5 of the $p_{i,j,}$ equal to 4, n_4 of the $p_{i,j}$ equal to 5, and n_3 of the $p_{i,j}$ equal to 3. This will imply that $\frac{Nu}{5} - \sum_{i=2} 5n_i$ of the $p_{i,j}$ will be equal to 2, but, by above, this number is equal to $n_1 + n_2$. Note that since we only decreased the value of any particular $p_{i,j}$, it will still be the case, for all $i \in [N]$, that $\sum_{j=1}^{\frac{u}{5}} p_{i,j} \leq \frac{v}{5}$. Now, we finish by choosing n_1 of the $(i,j) \in [N] \times [\frac{u}{5}]$ with $p_{i,j} = 2$ and let $q_{i,j,1} = 4$ and $q_{i,j,2} = 1$. For the other n_2 such (i,j), we let $q_{i,j,1} = 3$ and $q_{i,j,2} = 2$. For the n_3 values of (i,j) with $p_{i,j} = 3$, we let $q_{i,j,1} = q_{i,j,2} = 2$ and $q_{i,j,3} = 1$. For the n_4 values of (i,j) with $p_{i,j} = 5$, let $q_{i,j,t} = 1$ for $1 \leq t \leq 5$. Finally, for the n_5 values of (i,j) with $p_{i,j} = 4$, let $q_{i,j,1} = 2$ and let $q_{i,j,2} = q_{i,j,3} = q_{i,j,4} = 1$. It is the case that, for all $(i,j) \in [N] \times [\frac{u}{5}]$, $\sum_{t=1}^{p_{i,j}} q_{i,j,t} = 5$ and it is easy to check that

$$|\{q_{i,j,t} = 4\}| = n_1 = \binom{v-u}{1}\binom{u}{4},$$

$$|\{q_{i,j,t} = 3\}| = n_2 = \binom{v-u}{2}\binom{u}{3},$$

$$|\{q_{i,j,t} = 2\}| = n_2 + 2n_3 + n_5 = \binom{v-u}{3}\binom{u}{2},$$

$$|\{q_{i,j,t} = 1\}| = n_1 + n_3 + 5n_4 + 3n_5 = \binom{v-u}{4}\binom{u}{1}.$$

Thus, just like the proof above of theorem 1.6.1, it follows that $(u, v) \in \mathcal{EP}_5$. $\square$

Chapter 2

Decompositions of Partitions

2.1 The Connection between Partitions and One-Factorizations

Note that for each case illustrated in chapter 1, our strategy has been essentially the same: we must find a collection of m-partitions of U (where $m = \frac{v}{k}$) such that each i-subset of U occurs $\binom{v-u}{k-i}$ times, for $1 \leq i \leq k-1$. To accomplish this, we simply use theorem 1.3.3, setting $a_i = \binom{v-u}{k-i}$ for $1 \leq i \leq k-1$, $a_0 = \binom{v-u}{k} - \frac{v-u}{k}\binom{u-1}{k-1}$ (since the empty set occurs $\frac{v-u}{k}$ times in each of the $\binom{u-1}{k-1}$ one-factors in our given one-factorization of $\binom{U}{k}$) and $a_i = 0$ for $i \geq k$. Then, it suffices to find a collection $\{t_{i,j}\}$ for $0 \leq i \leq k-1$ and $1 \leq j \leq N$ (where $N = \binom{v-1}{k-1} - \binom{u-1}{k-1}$ as that is precisely the number of m-partitions we need in addition to the $\binom{u-1}{k-1}$ used for our given one-factorization of $\binom{U}{k}$) that satisfy the following sets of equations:

$$(1) \quad \binom{v-u}{k-i}\binom{u}{i} = \sum_{j=1}^{N} t_{i,j} \text{ for } 1 \leq i \leq k-1 \ ,$$

$$(1') \quad \binom{v-u}{k} - \frac{v-u}{k}\binom{u-1}{k-1} = \sum_{j=1}^{N} t_{0,j},$$

$$(2) \quad \sum_{i=0}^{k-1} t_{i,j} = m \text{ for } 1 \leq j \leq N,$$

$$(3) \quad \sum_{i=0}^{k-1} i t_{i,j} = u \text{ for } 1 \leq j \leq N.$$

To see this in a different light, first, assume we have such a collection $\{t_{i,j}\}$. Then, let π^j be the partition of u such that it has $t_{i,j}$ parts of size i (it must be a partition of u because of (3)), with each part of size at most $k-1$; note that (2) implies that it must have less than or equal to m parts. Further, (1) implies that if we take the partition π generated by summing up each of the π^j we obtain a partition of uN such that there are precisely $\binom{v-u}{k-i}\binom{u}{i}$ parts of size i for $1 \le i \le k-1$.

Conversely, assume that the partition π of uN with $\binom{v-u}{k-i}\binom{u}{i}$ parts of size i for $1 \le i \le k-1$ is decomposable into N partitions of U, each having less than or equal to m parts; denote them as π^j for $1 \le j \le N$. Then, for $1 \le i \le k-1$, let $t_{i,j}$ be equal to the number of parts of size i in π^j and let $t_{0,j} = m - \sum_{i=1}^{k-1} t_{i,j}$; it follows immediately that we have equations (2) and (3), and (1). To see that we have (1′), note that

$$\begin{aligned}
\sum_{j=1}^{N} t_{0,j} &= \sum_{j=1}^{N}(m - \sum_{i=1}^{k-1} t_{i,j}) = \sum_{j=1}^{N} m - \sum_{j=1}^{N}\sum_{i=1}^{k-1} t_{i,j} = mN - \sum_{i=1}^{k-1}\sum_{j=1}^{N} t_{i,j} \\
&= \frac{v}{k}(\binom{v-1}{k-1} - \binom{u-1}{k-1}) - \sum_{i=1}^{k-1}\binom{v-u}{k-i}\binom{u}{i} \\
&= \binom{v}{k} - \frac{v}{k}\binom{u-1}{k-1} - \sum_{i=1}^{k-1}\binom{v-u}{k-i}\binom{u}{i} \\
&= \binom{v}{k} - \frac{v}{k}\binom{u-1}{k-1} - (\binom{v}{k} - \binom{u}{k} - \binom{v-u}{k}) \\
&= \binom{v-u}{k} + \binom{u}{k} - \frac{v}{k}\binom{u-1}{k-1} = \binom{v-u}{k} + \frac{u}{k}\binom{u-1}{k-1} - \frac{v}{k}\binom{u-1}{k-1} \\
&= \binom{v-u}{k} - \frac{v-u}{k}\binom{u-1}{k-1},
\end{aligned}$$

which is precisely (1′).

Thus, to determine if $(u, v) \in \mathcal{EP}_k$, it suffices to find a decomposition of the above partition π into N partitions of u, each with at most m parts. This leads to the problem of determining when a given partition can be decomposed into smaller partitions (all of the same number). Note that, using the notation defined in section 1.1, a partition π of nt (where n and t are arbitrary positive integers) can be decomposed into t partitions of n if and only if v_π can be written as a nonnegative integer linear combination of elements of V_n. Further, π can be decomposed into t partitions of n, each having less than or equal to m parts, each of which is of size less than or equal to k if and only if v_π can be written as a nonnegative integer linear combination of elements in $V_{n,k,m}$. We will use the term *integer cone generated by* S to denote the set of all nonnegative integer linear combinations of elements of S, where S is any arbitrary subset of $\mathbb{Z}^n$. Thus, a one-factorization $(V, k, \mathcal{P})$ that contains $(U, k, \mathcal{Q})$ as a subsystem exists if and only if w is in the integer cone generated by $V_{u,k-1,\frac{v}{k}}$, where $w_i = \binom{v-u}{k-i}\binom{u}{i}$ for $1 \leq i \leq k-1$ and $w_i = 0$ for $i > k-1$. In fact, [3] shows that such a w is in always in the integer cone generated by $V_{u,k-1,\frac{v}{k}}$ (when $v \geq 2u$ and k divides v), through a long and complicated construction. It is our hope that instead, a general theorem about membership in the integer cone generated by $V_{u,k-1,m}$ will be found that will apply to both this case and many others. Thus we state the following conjecture:

Conjecture 2.1.1 *There exists an easily describable set $S_{u,v,k}$ that is in the integer cone generated by $V_{u,k-1,\frac{v}{k}}$ such that $w \in S_{u,v,k}$ (for w defined above).*

However, this in general seems to be a difficult question, and thus we, for the time being, focus our efforts on the integer cone generated by V_n. One can ask what happens if we remove the nonnegativity requirement, and instead ask what is the span of V_n over $\mathbb{Z}$.

Theorem 2.1.2 *If we let*

$$S = \{v \in \mathbb{Z}^n : \sum_{i=1}^{n} i v_i = tn \text{ for some } t \in \mathbb{Z}\},$$

then S is the span of V_n over $\mathbb{Z}$.

Proof. First, note that if v is in the span of V_n over $\mathbb{Z}$, then $v = \sum_{j=1}^{M} \alpha_j v^j$ with $\alpha_j \in \mathbb{Z}$ and $v^j \in V_n$ so

$$\sum_{i=1}^{n} i v_i = \sum_{i=1}^{n} i \sum_{j=1}^{M} \alpha_j v_i^j = \sum_{j=1}^{M} \alpha_j \sum_{i=1}^{n} i v_i^j = \sum_{j=1}^{M} \alpha_j n,$$

so if we let $t = \sum_{j=1}^{M} \alpha_j$ (which is clearly in $\mathbb{Z}$)we see that $\sum_{i=1}^{n} i v_i = tn$ and thus $v \in S$.

For the other direction, we first show that if we define S' by

$$S' = \{v \in \mathbb{Z}^n : \sum_{i=1}^{n} iv_i = 0\},$$

then S' is contained in the span of V_n over $\mathbb{Z}$. To see this, for $2 \leq j \leq n$, let $u^j \in \mathbb{Z}^n$ be defined by letting $u^j_j = 1$, $u^j_1 = -j$ and $u^j_i = 0$ for all other i. Then, clearly each u^j is in S' and further, for any $v \in S'$, consider the vector $y = \sum_{j=2}^{n} v_j u^j$, it is clear that, for $2 \leq j \leq n$, $y_j = v_j$ (since the only u^i with nonzero j coordinate is u_j). Further, note that:

$$y_1 = \sum_{j=2}^{n} v_j u^j_1 = \sum_{j=2}^{n} v_j(-j) = -\sum_{j=1}^{n} jv_j + v_1 = 0 + v_1 = v_1,$$

and thus $y = v$, which shows that the u^j form a basis for S' over $\mathbb{Z}$. Then, if we let $z^1 = (n, 0, \ldots, 0)$ and for $2 \leq j \leq n$ let $z^j_j = 1$, $z^j_1 = n - j$, and let all other coordinates of z^j be zero, it is clear that $z^j \in V_n$ for $1 \leq j \leq n$ and further $u^j = z^j - z^1$. This implies that each of the u^j are in the span of V_n over $\mathbb{Z}$ and thus so is S'. So, now consider any $v \in S$. Then, if we consider $w = v - tz^1$, note that:

$$\sum_{i=1}^{n} iw_i = \sum_{i=1}^{n} i(v_i - tz^1_i) = \sum_{i=1}^{n} iv_i - t\sum_{i=1}^{n} iz^1_i = tn - tn = 0,$$

which shows that w is in S' and thus is in the span of V_n over $\mathbb{Z}$, which implies that $v = w + tz^1$ is also in the span of V_n over $\mathbb{Z}$, which completes the proof. □

This implies that if π is any partition of tn into parts of size at most n, that v_π is contained in the span of V_n over $\mathbb{Z}$. Thus, since there clearly are partitions of tn that are not decomposable into t partitions of n, removing the nonnegativity constraint does not seem helpful to the process at hand. One other way of modifying the above conditions is to, instead of removing the nonnegativity constraint, we can look at what happens when we remove the integer constraint, and look at the nonnegative span of V_n over $\mathbb{R}$, which we refer to simply as the *cone generated by* V_n. This condition appears much more restrictive, and thus this approach seems more fruitful. It in fact led us to the following conjecture relating the cone generated by V_n to the integer cone generated by V_n:

Conjecture 2.1.3 *If C is the cone generated by V_n, and C' is the integer-cone generated by V_n, then*

$$C' = \{v \in C : \sum_{i=1}^{n} iv_i \in n\mathbb{N}\} \cap \mathbb{Z}^n.$$

If we let $C'' = \{v \in C : \sum_{i=1}^{n} iv_i \in n\mathbb{N}\} \cap \mathbb{Z}^n$, it is clear that $C' \subseteq C''$, so to prove Conjecture 2.1.3, it would be sufficient that $C'' \subseteq C'$. One tool that could be used to settle conjecture 2.1.3 in general involves the notion of discrete convexity. We give the following definition, which can be found in [7]:

Definition Given a finite subset S of $\mathbb{Z}^n$, we say that S is *pseudoconvex* if $S = \overline{S} \cap \mathbb{Z}^n$, where $\overline{S}$ denotes the convex hull of S (in $\mathbb{R}^n$).

Note that under this definition, V_n is pseudoconvex. Consider any $v \in \overline{V_n} \cap \mathbb{Z}^n$. There exists $N \in \mathbb{N}$ with $v = \sum_{j=1}^{N} \alpha_j v^j$, with each $v^j \in V_n$ and each $\alpha_j \geq 0$. Further, by definition, we have $\sum_{i=1}^{n} v_i^j = n$. Thus,

$$\sum_{i=1}^{n} iv_i = \sum_{i=1}^{n}\sum_{j=1}^{N} i\alpha_j v_i^j = \sum_{j=1}^{N} \alpha_j \sum_{i=1}^{n} iv_i^j = n\sum_{j^1}^{N} \alpha_j = n.$$

Thus, since it follows that each coordinate of v is a nonnegative integer, it must be the case that $v \in V_n$, showing V_n is pseudoconvex.

We now let $[m]S$, for any $m \in \mathbb{N}$, denote the Minkowski sum of m copies of S, i.e.,

$$[m]S = \{v \in \mathbb{Z}^n : v = \sum_{i=1}^{m} v^i, v^i \in S \text{ for } 1 \leq i \leq m\}.$$

Note that this implies that $[m]V_n$ is the set of all v_π, where π is a partition of nm that can be decomposed into m partitions of n, and thus $\bigcup_{m\in\mathbb{N}}[m]V_n$ is the integer cone generated by V_n. So consider any $v \in C''$, and let $M = \sum_{i=1}^{n} iv_i$. By definition, n divides M, so we have $m \in \mathbb{N}$ with $M = nm$. Further, we must also have some $N \in \mathbb{N}$ such that $v = \sum_{j=1}^{N} \alpha_j v^j$, where $\alpha_j \geq 0$ and $v^j \in V_n$ for $1 \leq j \leq N$. Thus, we have

$$M = mn = \sum_{i=1}^{n} iv_i = \sum_{i=1}^{n}\sum_{j=1}^{N} i\alpha_j v_i^j = \sum_{j=1}^{N}\alpha_j \sum_{i=1}^{n} iv_i^j = n\sum_{j=1}^{n}\alpha_j.$$

Then, letting $\beta_j = \frac{\alpha_j}{m}$ for $1 \leq j \leq N$, we have $\sum_{j=1}^{N}\beta_j = 1$. Note that, for $1 \leq j \leq N$, we have $mv^j = \sum_{i=1}^{m} v^j$, and thus $mv_j \in [m]V_n$. Further,

$$v = \sum_{j=1}^{N}\alpha_j v^j = \sum_{j=1}^{N} m\beta_j v^j = \sum_{j=1}^{N}\beta_j(mv^j),$$

showing that v is in the convex hull of $[m]V_n$. So, if we could show that $[m]V_n$ is pseudoconvex, it would follow that $v \in [m]V_n$. Unfortunately, though it is true that convex sets are closed under Minkowski sums, this need not be the case for pseudoconvex sets, as the following simple example from [8] shows:

Example Let

$$A = \{(0,0),(1,1)\},$$

$$B = \{(0,1),(1,0)\}.$$

It is trivial to check that both A and B are pseudoconvex. However, $A+B$ is not, since $(1,1)$ is in $\overline{A+B}$ but not $A+B$.

Note that we have reduced Conjecture 2.1.3 to the following:

Conjecture 2.1.4 *For all $m, n \in \mathbb{N}$, $[m]V_n$ is pseudoconvex.*

Unfortunately, Conjecture 2.1.4 is not true. Define v_i for $i \in \{1,2,3,4\}$ by

$$v_1 = (0,0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,0,0,0),$$

$$v_2 = (0,1,0,0,0,0,0,0,2,0,0,0,0,0,0,0,0,0,0,0),$$

$$v_3 = (0,0,0,0,0,1,2,0,0,0,0,0,0,0,0,0,0,0,0,0),$$

$$v_4 = (0,1,0,0,0,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0).$$

It is immediately checked that each $v_i \in V_{20}$. Then, let $x = v_1 + v_2$, $y = v_3 + v_4$. It follows by definition that $x, y \in [2]V_{20}$. However, if

$$w = \frac{1}{2}x + \frac{1}{2}y = (0, 1, 0, 0, 0, 2, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$

it is not hard to see that $w \notin [2]V_{20}$.

This does lead to the problem of determining when, for $m, n \in \mathbb{N}$, $[m]V_n$ is pseudoconvex, but this seems like a difficult problem in general. Instead, we turn our attention directly to the cone generated by V_n, which is is interesting in its own right, as is seen in the next section.

2.2 The Cone Generated by $\mathbf{V_n}$

For any arbitrary $S \subseteq \mathbb{R}^n$, we define the dimension of S to simply be the dimension of subspace of $\mathbb{R}^n$ spanned by S. Since we are motivated by the case of V_n, which is finite, we will only consider finite such S. Also, note that, by our above result, V_n has dimension n. Returning to arbitrary finite $S \subseteq \mathbb{R}^n$, if we let m be the dimension of S we define a *facet* of S to be a nonzero linear functional f such that $f(v) \geq 0$ for all $v \in S$ and such that f vanishes on some $m-1$ dimensional subset of S. Note that for any v in the cone generated by S, since v is a nonnegative linear combination of elements of S, it follows that $f(v) \geq 0$ for any facet f of S. The converse is also true, which we will show below. First, we remind the reader of the Minkowski-Farkas lemma (a proof can be found, for example, in [9])

Lemma 2.2.1 *(Minkowski-Farkas) For any finite $S \subseteq \mathbb{R}^n, v \in R^n$, either v is in the cone generated by S or there exists a $\lambda \in \mathbb{R}^n$ such that $\lambda \cdot x \geq 0$ for all $x \in S$ but $\lambda \cdot v < 0$.*

This in fact implies a stronger result, which is well known but whose proof we add for completeness

Theorem 2.2.2 *For any finite $S \subseteq \mathbb{R}^n$, if v is not in the cone generated by S, there exists a facet f of S such that $f(v) < 0$.*

Proof. Assume v is not in the cone generated by S. Let λ be such that $\lambda \cdot x \geq 0$ for all $x \in S$ but $\lambda \cdot v < 0$; such an λ exists by lemma 2.2.1. In fact, we may assume that $\lambda \cdot v = -1$; otherwise, we simply take the appropriate scalar multiple of λ. Define the linear functional g on $\mathbb{R}^n$ by $g(x) = \lambda \cdot x$. Let $T \subseteq S$ be defined by

$$T = \{x \in S : g(x) = 0\}.$$

Then, if we let m be the dimension of S, if T has dimension $m-1$ or m, g is a facet of S (if it has dimension m then it vanishes on all of S and in particular it vanishes on some $m-1$ dimensional subset of S) and we are done. Otherwise, let r be the dimension of T; thus $r \leq m-2$. We then will construct a nonzero linear functional g' such that g' is nonnegative on S, g' vanishes on a subset of S of dimension at least $r+1$ and $g'(v) < 0$; then, if the dimension of the subspace of S that g' vanishes on is less than $m-1$, we repeat the process. As the dimension of the intersection of the null space of these functionals with S continues to strictly increase, this process must terminate with us finding a facet f of S with $f(v) < 0$.

Thus, it remains to find such a g'. Let $B = \{z_1, \ldots, z_r\}$ be a basis of T. Note that $g(v) \neq 0$ implies that v is not in the span of B and thus $B \cup \{v\}$ is a linearly independent set of dimension $r+1 \leq m-1$. Thus, since S has dimension m, it follows that there exists at least one $x_1 \in \mathcal{S}$ such that $B \cup \{v, x_1\}$ is also an independent set. Then, define the linear functional h by $h(z_i) = 0$ for $1 \leq i \leq r$, $h(v) = -1$ and $h(x_1) = 0$, and extend h in any arbitrary way to the rest of $\mathbb{R}^n$. Then, note that since $g(x_1) \neq 0$ and $h(x_1) = 0$, h is a nonzero linear functional that vanishes on T but is linearly independent from g. Now, if h happens to be nonnegative on all of S, we can simply let $g' = h$ and are done, as g' will vanish on $T \cup \{x_1\}$, a set of dimension $r+1$. So assume there is at least one $x \in S$ with $h(x) < 0$. Further note that, since h vanishes on T, if $h(x) < 0$, x is not in the span of T and thus $g(x) > 0$. Then, for any x with $h(x) < 0$, define α_x by $\alpha_x = \frac{-h(x)}{g(x)}$ and let α be the maximum over all such α_x (such a maximum exists of course since S is finite). It follows that, since each $\alpha_x > 0$, $\alpha > 0$. Then, let $g' = \alpha g + h$; it follows that g' vanishes on T and

$g'(v) = \alpha g(v) + h(g) = -\alpha - 1 < 0$. Further, for any $x \in S$, by definition $g(x) \geq 0$, so if $h(x) \geq 0$ we have $g'(x) \geq 0$. If $h(x) < 0$, by construction we have $\alpha \geq \alpha_x$, which implies that

$$g'(x) = \alpha g(x) + h(x) \geq \alpha_x g(x) + h(x) = \frac{-h(x)}{g(x)} g(x) + h(x) = 0,$$

and thus g' is nonnegative on all of S. Further, we constructed α such that there is at least one x such that $\alpha_x = \alpha$ (call it x_2) that has

$$g'(x_2) = \alpha g(x_2) + h(x_2) = \alpha_{x_2} g(x_2) + h(x_2) = \frac{-h(x_2)}{g(x_2)} g(x_2) + h(x_2) = 0.$$

Also, $g(x_2) \neq 0$ so x_2 is not in the span of T, which implies that $T \cup \{x_2\}$ is a subset of S of dimension $r + 1$ where g' vanishes. Thus, g' has the required properties, which finishes the proof. □

Thus, to determine the cone of V_n we only need to determine each of the facets of V_n. Also, note that any facet f of V_n can be represented by a unique $d \in \mathbb{R}^n$ such that $f(v) = d \cdot v$. If $d = (d_1, d_2, \ldots, d_n)$, for any $v = (v_1, v_2, \ldots, v_n) \in V_n$, this yields the following inequality

$$d_1 v_1 + d_2 v_2 + \cdots + d_n v_n \geq 0,$$

which is true for all $v \in V_n$ with equality holding for some $n - 1$ dimensional subset of V_n if and only if d corresponds to a facet of V_n. We will henceforth use the term *facet* to refer to either the functional, the associated vector, or the associated inequality; it will be clear by context to which of these we are referring. We have the following simple result:

Theorem 2.2.3 *For any integer i with $2 \leq i \leq n$, $v_i \geq 0$ is a facet of V_n, while $v_1 \geq 0$ is not.*

Proof. Note that $v_i \geq 0$ is trivially true for all $v \in V_n$ so it remains to show that we have $v_i = 0$ true for some $n - 1$ dimensional subset of V_n when $2 \leq i \leq n$ and that no such $n - 1$ dimensional subset exists when $i = 1$. First, consider the case $2 \leq i \leq n$. Then, for $1 \leq j \leq n$ and $j \neq i$, let v^j be defined by $v^j_j = 1$, $v^j_1 = n - j$ and all other coordinates of v^j are zero. Then, it is clear that $v^j \in V_n$. Now, let

$$U = \{v^j : 1 \leq j \leq n, i \neq j\}.$$

It is easy to see that U has dimension $n-1$; also, by definition $v_i^j = 0$. Thus, we have equality on the $n-1$ dimensional subset W, which shows that $v_i \geq 0$ is a facet.

So it remains to consider the case where $i = 1$. Note that the only $v \in V_n$ such that $v_{n-1} \neq 0$ must have $v_{n-1} = 1$ and thus $v_1 = 1$. Thus, any $v \in V_n$ with $v_1 = 0$ must also have $v_{n-1} = 0$. So the set of all $v \in V_n$ that satisfy $v_1 = 0$ is contained in the set

$$W = \{v \in \mathbb{R}^n : v_1 = v_{n-1} = 0\},$$

which has dimension $n-2$. Thus, there does not exist any $n-1$ dimensional subset of V_n where $v_1 = 0$ and it follows that $v_1 \geq 0$ is not a facet of V_n. □

Note that any positive scalar multiple of a facet is a facet. We will refer to any facet corresponding to $\alpha v_i \geq 0$ for $\alpha > 0$ a *trivial* facet. We now prove a theorem about nontrivial facets.

Theorem 2.2.4 *Suppose $d \in \mathbb{R}^n$ is a nontrivial facet of V_n, and consider any i with $2 \leq i \leq n$. Then, the following are true:*

(a) $d_1 > 0$,

(b) There exists some $u \in V_n$ with $u_i > 0$ and $d \cdot u = 0$,

(c) $d_i \leq i d_1$.

Proof.

(a) Consider any facet d of V_n. Note first that the vector $v = (n, 0, 0, \ldots, 0)$ is in V_n and since $0 \leq d \cdot v = d_1 v_1 = n d_1$ we must have $d_1 \geq 0$. Suppose $d_1 = 0$. Then, for $2 \leq j \leq n$ let v^j be defined as above. Note that

$$d \cdot v^j = d_j v_j^j + d_1 v_1^j = d_j * 1 + 0 * (n-j) = d_j.$$

Thus, we must have $d_j \geq 0$ for all such j. Now, suppose there exist $j \neq k$ with d_j and d_k both not equal to zero, and thus positive. Then, for any vector v with $d \cdot v = 0$ we must have

$$0 = d \cdot v = d_1 v_1 + d_2 v_2 + \cdots + d_n v_n \geq d_j v_j + d_k v_k \geq 0,$$

and thus it follows that $d_j v_j + d_k v_k = 0$, which implies that $v_j = v_k = 0$. Thus, the set of all $v \in V_n$ that satisfy $d \cdot v = 0$ must be contained in the set

$$U = \{v \in \mathbb{R}^n : v_i = v_j = 0\},$$

which has dimension $n-2$. This contradicts our assumption that d is a facet, and thus d has at most one nonzero coordinate. But that in turn contradicts our assumption that d is nontrivial. Thus, we must have $d_1 \neq 0$, which implies $d_1 > 0$.

(b) Assume that for any $v \in V_n$ with $d \cdot v = 0$, we must have $v_i = 0$. Then, it follows that the set of all $v \in V_n$ with $d \cdot v = 0$ must be a subset of

$$W = \{v \in \mathbb{R}^n : v_i = 0\}.$$

However, since this set has dimension $n-1$, it follows that any vector $v \in W \cap V_n$ must have $d \cdot v = 0$. So, in particular,

$$0 = d \cdot (n, 0, \ldots, 0) = d_1 n > 0,$$

which is a contradiction. Thus, we must have some $u \in V_n$ with $d \cdot u = 0$ and $u_i > 0$.

(c) Suppose for some i we have $d_i > i d_1$. Then, by part (b), there exists some u with $d \cdot u = 0$ and $u_i > 0$. Consider the vector u' with $u'_i = u_i - 1$, $u'_1 = u_1 + i$, and $u'_j = u_j$ for any j satisfying $j \neq 1$ and $j \neq i$. Note that

$$\sum_{j=1}^{n} ju'_j = u'_1 + iu'_i + \sum_{j=2}^{i-1} u'_j + \sum_{j=i+1}^{n} u'_j = u_1 + i + i(u_i - 1) + \sum_{j=2}^{i-1} u_j + \sum_{j=i+1}^{n} u_j$$
$$= u_1 + iu_i + \sum_{j=2}^{i-1} u_j + \sum_{j=i+1}^{n} u_j = \sum_{j=1}^{n} ju_j = n,$$

which shows that $u' \in V_n$ and thus $d \cdot u' \geq 0$. So

$$0 \leq d \cdot u' = \sum_{j=1}^{n} d_j u'_j = d_1 u'_1 + d_i u'_i + \sum_{j=2}^{i-1} d_j u'_j + \sum_{j=i+1}^{n} d_j u'_j$$
$$\leq d_1(u_1 + i) + d_i(u_i - 1) + \sum_{j=2}^{i-1} d_j u_j + \sum_{j=i+1}^{n} d_j u_j = d_1 i - d_i + \sum_{j=1}^{n} d_j u_j$$
$$\leq d_1 i - d_i + d \cdot u = d_1 i - d_i$$
$$< d_1 i - i d_1 = 0,$$

a contradiction. Thus, $d_i \leq i d_1$. □

Now, for any vector $v \in V_n$ we have, by definition, $\sum_{i=1}^{n} iv_i = n$. So, consider any nontrivial facet d of V_n. Then, for i with $1 \leq i \leq n$, let $c_i = id_1 - d_i$. By above, each $c_i \geq 0$. Also, note that, for $v \in V_n$, we have

$$\sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} (id_1 - d_i) v_i = \sum_{i=1}^{n} id_1 v_i - \sum_{i=1}^{n} d_i v_i = d_1 \sum_{i=1}^{n} iv_i - \sum_{i=1}^{n} d_i v_i$$
$$= d_1 n - d \cdot v \leq d_1 n,$$

and we have equality if and only if we have $d \cdot v = 0$, which occurs on some $n-1$ dimensional subspace of V_n. Also, note that $c_1 = d_1 - d_1 = 0$. Conversely, assume we have some affine inequality of the form

$$\sum_{i=1}^{n} c_i v_i \leq b,$$

for some $b > 0$, which holds on all of V_n and has equality on some $n-1$ dimensional subspace of V_n and additionally has $c_1 = 0$. Then, by letting $d_1 = \frac{b}{n}$ and let $d_i = id_1 - c_i$, it is clear by a similar method to the above that d is a nontrivial facet of V_n. Thus, there is a one-to-one correspondence between nontrivial facets of V_n and affine inequalities of the form

$$\sum_{i=1}^{n} c_i v_i \leq b,$$

for some $b > 0$ with $c_1 = 0$, and which hold on an $n-1$ dimensional subset of V_n. We will call such inequalities *affine-facet inequalities* or AF-inequalities for short. Then, we have the following rather technical-looking theorem:

Theorem 2.2.5 *Suppose $\sum_{i=1}^{n} c_i v_i \leq b$ is an AF-inequality. Assume for some k with $2 \leq k \leq n$ we have $N \in \mathbb{N}$ with $k_1, k_2, \ldots, k_N, a_1, a_2, \ldots, a_N \in \mathbb{N}$ such that $\sum_{j=1}^{N} a_j k_j = k$. Then, it must be true that $\sum_{j=1}^{N} a_j c_{k_j} \leq c_k$.*

Proof. Suppose we have positive integers satisfying the assumptions of the theorem 2.2.5. Let

$$S = \{i \in \mathbb{N} : i \leq n, i \neq k, i \neq k_1, i \neq k_2, \ldots, i \neq k_N\}.$$

Now, by theorem 2.2.4 part (b), there must be some $u \in V_n$ with $\sum_{i=1}^{n} c_i u_i = b$ and $u_k > 0$ (since there must be some such u with equality in the corresponding facet). Define u' by letting $u'_k = u_k - 1$, $u'_{k_j} = u_{k_j} + a_j$ for $1 \leq j \leq N$ and $u'_i = u_i$ for any $i \in S$.

$$\sum_{i=1}^{n} iu'_i = ku'_k + \sum_{j=1}^{N} k_j u'_{k_j} + \sum_{i \in S} iu'_i = k(u_k - 1) + \sum_{j=1}^{N} k_j (u_{k_j} + a_j) + \sum_{i \in S} iu_i$$
$$= -k + \sum_{j=1}^{N} a_j k_j + \sum_{i=1}^{n} iu_i = 0 + \sum_{i=1}^{n} iu_i = n,$$

showing that $u' \in V_n$. Thus, we must have:

$$b \geq \sum_{i=1}^{n} c_i u'_i = c_k u'_k + \sum_{j=1}^{N} c_{k_j} u'_{k_j} + \sum_{i \in S} c_i u'_i = c_k(u_k - 1) + \sum_{j=1}^{N} c_{k_j}(u_{k_j} + a_j) + \sum_{i \in S} c_i u_i$$
$$\geq -c_k + \sum_{j=1}^{N} c_{k_j} a_j + \sum_{i=1}^{n} c_i u_i = -c_k + \sum_{j=1}^{N} c_{k_j} a_j + b.$$

This implies that

$$c_k \geq \sum_{j=1}^{N} c_{k_j} a_j,$$

which is our desired result. □

This has the following corollary:

Corollary 2.2.6 *Suppose* $\sum_{i=1}^{n} c_i v_i \leq b$ *is an AF-inequality. Then:*

(a) If $1 \leq i \leq n-1$, *then* $c_i \leq c_{i+1}$,

(b) If $1 \leq i \leq \frac{n}{2}$, *then* $c_i + c_{n-i} = b$.

Proof.

(a) Let $k = i + 1$, let $k_1 = i$, let $k_2 = 1$, and let $a_1 = a_2 = 1$. Then,

$$a_1 k_1 + a_2 k_2 = i + 1 = k,$$

so by theorem 2.2.5 we have $a_1 c_1 + a_2 c_i \leq c_{i+1}$ and since $c_1 = 0$ we have $c_i \leq c_{i+1}$.

(b) By theorem 2.2.4 part (b), there exists some u with $u_{n-i} \neq 0$ and $\sum_{j=1}^{n} c_j u_j = b$. Then, since $2(n-i) \geq n$, either $u_i = 2$, which implies that n is even and $i = \frac{n}{2}$, or we must have $u_{n-i} = 1$. Now, if we are in the first case, by assumption we have $2c_i = b = c_i + c_{n-i}$ since $i = n - i$ in this case. Otherwise, we have $u_{n-i} = 1$. Let $k = i$ and let $k_1, k_2, \ldots, k_N$ be the indices of the other coordinates of u that are nonzero, and let $a_i = u_{k_i}$. Then, by definition we have

$$n = \sum_{j=1}^{n} ju_j = (n-i)u_{n-i} + \sum_{j=1}^{N} k_j u_{k_j} = n - k + \sum_{j=1}^{N} a_j k_j.$$

This implies that $k = \sum_{j=1}^{N} a_j k_j$, so by theorem 2.2.5 we have

$$c_i = c_k \geq \sum_{j=1}^{N} a_j c_{k_j} = \sum_{j=1}^{N} u_j c_{k_j} = \sum_{j=1}^{n} c_j u_j - c_{n-i} u_{n-i} = b - c_{n-i},$$

which implies that $c_1 + c_{n-i} \geq b$. However, note that the vector v defined by $v_{n-i} = v_i = 1$ (all other coordinates zero) is in V_n, which implies that $c_i + c_{n-i} \leq b$ and thus $c_i + c_{n-i} = b$. □

Note that we can also consider the nontrivial facet d corresponding to any AF-inequality, yielding another corollary:

Corollary 2.2.7 *Suppose $d \in \mathbb{R}^n$ is a nontrivial facet of V_n. Then:*

(a) If $1 \leq i \leq n-1$, then $d_{i+1} \leq d_1 + d_i$,

(b) If $1 \leq i \leq \frac{n}{2}$, then $d_i = -d_{n-i}$.

Proof. Consider the AF-inequality defined by letting $c_i = id_1 - d_i$.

(a) By Corollary 2.2.6 part (a) we have $c_i \leq c_{i+1}$, which implies that

$$id_1 - d_i \leq (i+1)d_1 - d_{i+1},$$

$$d_{i+1} \leq (i+1)d_1 - id_1 + d_i = d_1 + d_i.$$

(b) By Corollary 2.2.6 part (b) we have

$$b = c_i + c_{n-i} = id_1 - d_i + (n-i)d_1 - d_{n-i} = nd_1 - d_i - d_{n-i}.$$

But, since by above we have $b = d_1 n$, this simplifies to $0 = -d_i - d_{n-i}$, which implies that $d_i = -d_{n-i}$.

□

Now, to find every facet of V_n, we simply check every $n-1$ element subset of V_n, determine if it has dimension $n-1$, and, if it does, find the unique (up to scalar multiplication) vector $d \in \mathbb{R}^n$ perpendicular to it, and check whether or not d is a facet. However, if we let $P(n)$ denote the number of partitions of n, there are $\binom{\mathcal{P}(n)}{n-1}$ different such subsets to check. Since $P(n)$ grows exponentially, this becomes quite difficult for relatively small values of n. In the next section, we will show how to narrow our search considerably.

2.3 Narrowing the Search for Facets

We first prove two related lemmas:

Lemma 2.3.1 *If π is a partition of n into $m \geq 4$ parts, there exists partitions σ, τ, and ρ of n, where σ has $m-1$ parts, τ has three parts and ρ has two parts, such that $v_\pi = v_\sigma + v_\tau - v_\rho$.*

Proof. Let π be given by $\pi_1 + \pi_2 + \cdots + \pi_m = n$; without loss of generality, we can assume $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_m$. Note that this implies that $\pi_1 + \pi_2 \leq \frac{n}{2}$. Further, define σ by $\sigma_1 = \pi_1 + \pi_2$ and $\sigma_i = \pi_{i+1}$ for $2 \leq i \leq m-1$; it is clear that by this definition σ is a partition of n into $m-1$ parts. Then, define τ by $\tau_1 = \pi_1$, $\tau_2 = \pi_2$ and $\tau_3 = n - \sigma_1$, and since

$$\pi_1 + \pi_2 + n - \sigma_1 = \pi_1 + \pi_2 + n - \pi_1 - \pi_2 = n,$$

it follows that τ is a partition of n into three parts. Finally, define ρ by $\rho_1 = \sigma_1$, $\rho_2 = n - \sigma_1$, so ρ is a partition of n into two parts. Then, let ϕ^1 be the partition of $2n$ defined by $\phi^1_i = \pi_i$ for $1 \leq i \leq m$ and $\phi^1_{m+j} = \rho_j$ for $1 \leq j \leq 2$, and let ϕ^2 be the partition of $2n$ defined by $\phi^2_i = \sigma_i$ for $1 \leq i \leq m-1$ and $\phi^2_{m+j-1} = \tau_j$ for $1 \leq j \leq 3$ (so ϕ^1 is formed by concatenating π and ρ and ϕ^2 is formed by concatenating σ and τ). Then, we claim that ϕ^1 and ϕ^2 are in fact the same partition. To see this, just note that

$$\phi^1_1 = \pi_1 = \tau_1 = \phi^2_m,\ \phi^1_2 = \pi_2 = \phi^2_{m+1},$$

$$\phi^1_i = \pi_i = \sigma_{i-1} = \phi^2_{i-1} \text{ for } 3 \leq i \leq m,$$

$$\phi^1_{m+1} = \rho_1 = \sigma_1 = \phi^2_1, \text{ and } \phi^1_{m+2} = \rho_2 = n - \sigma_1 = \tau_3 = \phi^2_{m+2}.$$

Thus ϕ^1 and ϕ^2 have precisely the same parts. This implies that $v_{\phi^1} = v_{\phi^2}$, and since $v_{\phi^1} = v_\pi + v_\rho$ and $v_{\phi^2} = v_\sigma + v_\tau$, we have $v_\pi + v_\rho = v_\sigma + v_\tau$, which implies that $v_\pi = v_\sigma + v_\tau - v_\rho$. □

Lemma 2.3.2 *If π is a partition of n into m parts, where $m \geq 4$, there exists $v^1, v^2, \ldots, v^{m-2} \in V_{n,n,3} \setminus V_{n,n,2}$ and $w^1, w^2, \ldots, w^{m-3} \in V_{n,n,2}$ such that*

$$v_\pi = \sum_{i=1}^{m-2} v^i - \sum_{i=1}^{m-3} w^i.$$

Proof. We prove this by induction on m. When $m = 4$, by lemma 2.3.1, we can find σ, τ and ρ, partitions of n into three, three and two parts, respectively, with $v_\pi = v_\sigma + v_\tau - v_\rho$, and since $v_\sigma, v_t \in V_{n,n,3} \setminus V_{n,n,2}$ and $v_\rho \in V_{n,n,2}$, our result is proven by letting $v^1 = v_\sigma$, $v^2 = v_\tau$, and $w^1 = v_\rho$. So now, assume lemma 2.3.2 is true for all partitions of n into m parts, and consider the case when π is a partition of n into $m + 1$ parts. Then, again choose σ, τ, and ρ as in lemma 2.3.1; in this case, σ is a partition of n into m parts, so by our induction hypothesis, we have $v_\sigma = \sum_{i=1}^{m-2} v^i - \sum_{i=1}^{m-3} w^i$ with $v^1, v^2, \ldots, v^{m-2} \in V_{n,n,3} \setminus V_{n,n,2}$ and $w^1, w^2, \ldots, w^{m-3} \in V_{n,n,2}$. Then, simply let $v^{m-1} = v_\tau \in V_{n,n,3} \setminus V_{n,n,2}$ and let $w^{m-2} = v_\rho \in V_{n,n,2}$, and note that:

$$v_\pi = v_\sigma + v_\tau - v_\rho = \sum_{i=1}^{m-2} v^i - \sum_{i=1}^{m-3} w^i + v^{m-1} - w^{m-2} = \sum_{i=1}^{m-1} v^i - \sum_{i=1}^{m-2} w^i,$$

which is our desired result. Thus, lemma 2.3.2 is true for all $m \geq 4$. □

Note that an immediate consequence of lemma 2.3.2 is that $V_{n,n,3}$ spans V_n, and since V_n has dimension n, $V_{n,n,3}$ must have dimension n as well.

Corollary 2.3.3 *$V_{n,n,3}$ has dimension n.* □

We now show the result that narrows our search for facets from V_n to $V_{n,n,3}$.

Theorem 2.3.4 *If f is a nontrivial facet of V_n, it follows that f vanishes on all of $V_{n,n,2}$ and f vanishes on some $n-1$ dimensional subset of $V_{n,n,3}$.*

Proof. Let f be any nontrivial facet of V_n, and let d be the vector associated with it. Now, note that the only elements of $V_{n,n,2}$ are either $u^n = (0, 0, \dots, 0, 1)$ or u^j defined by $u^j_j = u^j_{n-j} = 1$ and the rest of the coordinates of u^j are zero for $1 \le \frac{n-1}{2}$ or, if n is even, $u^{\frac{n}{2}} = 2$ and the rest of the coordinates of $u^{\frac{n}{2}}$ are zero. Since $V_n \setminus \{u^n\}$ is orthogonal to u^n it follows that any $n-1$ dimensional subspace of V_n that does not contain u^n must span $V_n \setminus \{u^n\}$. This implies that any facet that is not zero on u^n must be zero on the rest of V_n. Thus, since theorem 2.2.4 part (a) tells us $d_1 > 0$, it follows that $f(n, 0, 0, \dots, 0) = nd_1 \neq 0$. So f does not vanish on all of $V_n \setminus \{u^n\}$, which implies that $f(u^n) = 0$. Further, for $1 \le j \le \frac{n}{2}$

$$f(u^j) = d \cdot u^j = d_j + d_{n-j} = 0,$$

by Corollary 2.2.7 part (b). Thus, f vanishes on all of $V_{n,n,2}$.

Suppose that f does not vanish on any $n-1$ dimensional subset of $P_{n,n,3}$. Let

$$T = \{v \in V_n : f(v) = 0\},$$

and let $W = T \cap P_{n,n,3}$. Note that W has dimension less than $n-1$, by definition but T has dimension $n-1$. Thus, there is some $v \in T$ that is not in the subspace of $\mathbb{R}^n$ spanned by W. If m is the number of parts in the partition of n that corresponds to v, we have, by lemma 2.3.2, $v^1, v^2, \dots, v^{m-2} \in V_{n,n,3} \setminus V_{n,n,2}$ and $w^1, w^2, \dots, w^{m-3} \in V_{n,n,2}$ such that $v = \sum_{i=1}^{m-2} v^i - \sum_{i=1}^{m-3} w^i$. In particular, by above, each $w^i \in P_{n,n,2}$ implies that $f(w^i) = 0$. Also, note that $v \in T$ means that $f(v) = 0$. So

$$0 = f(v) = \sum_{i=1}^{m-2} f(v^i) - \sum_{i=1}^{m-3} f(w^i) = \sum_{i=1}^{m-2} f(v^i) - \sum_{i=1}^{m-3} 0 = \sum_{i=1}^{m-2} f(v^i).$$

But f is a facet, which implies that $f(v^i) \geq 0$ for $1 \leq i \leq m-2$, so for the sum to vanish, we must have each term equal to zero, so $f(v^i) = 0$ for $1 \leq i \leq m-2$. But then, each $v^i \in W$ and, by above, each $w^i \in W$, implying that v is in fact in the subspace of $\mathbb{R}^n$ spanned by W, which is a contradiction. Thus, no such v exists, which implies that f does vanish on some $n-1$ dimensional subspace of $P_{n,n,3}$. □

Since any nontrivial facet of V_n is completely determined (up to scalar multiplication) by any $n-1$ dimensional subset it vanishes on, it follows that each such facet is completely determined by its action on $V_{n,n,3}$. Thus, instead of checking all $n-1$ element subsets of V_n, we need only check all $n-1$ element subsets of $V_{n,n,3}$. Further, since each nontrivial facet must vanish on $V_{n,n,2}$, we need only check the $n-1$ element subsets that contain all of $V_{n,n,2}$. In other words, we just need to check each of the $n-1-|V_{n,n,2}|$ element subsets of $V_{n,n,3} \setminus P_{n,n,2}$. Note that $|V_{n,n,2}|$ is simply the number of partitions of n into one or two parts, and $|V_{n,n,3} \setminus V_{n,n,2}|$ is the number of partitions of n into exactly three parts. These numbers are well known (for example, see [1]) to be $\lfloor \frac{n}{2} \rfloor + 1$ and $[\frac{1}{12}n^2]$, respectively, where $[x]$ is the integer nearest to x. Thus, as $n-1-|V_{n,n,2}| = \lfloor \frac{n-3}{2} \rfloor$, we have reduced the number of subsets of V_n we need to check to $\binom{[\frac{1}{12}n^2]}{\lfloor \frac{n-3}{2} \rfloor}$. To get some perspective, when $n = 20$, this is the difference between $\binom{627}{19}$, which is roughly 8.78×10^{35} and $\binom{30}{8}$, which is roughly 5.85×10^6. At the end of this paper, in appendix A, we list A_n for $n = 3$ to $n = 16$, where the rows of A_n correspond to each of the distinct (up to scalar multiplication) nontrivial facets of the cone generated by V_n.

Thus, the procedure is clear. To find each nontrivial facet of V_n, we simply choose some $n-1$ element subset S of $V_{n,n,3}$ that contains $V_{n,n,2}$, check that it in fact has dimension $n-1$, then find the unique (up to scalar multiplication) functional f that vanishes on S and is positive on at least one element of V_n. We then check whether or not it is nonnegative on all of V_n; if so, it is a facet. We then repeat this process over all such S. theorem 2.3.4 insures that all such facets can be found this way. However, the following theorem shows that we only need check whether f is nonnegative on $V_{n,n,3}$:

Theorem 2.3.5 *If f is a nonzero linear functional on $\mathbb{R}^n$ such that f vanishes on some $n-1$ dimensional subset of $V_{n,n,3}$ that contains $V_{n,n,2}$, and f is nonnegative on $V_{n,n,3}$, then f is a nontrivial facet of V_n.*

Proof. Let f be any nonzero linear functional on $\mathbb{R}^n$ that vanishes on an $n-1$ dimensional subset of $V_{n,n,3}$ that contains $V_{n,n,2}$ and $f$is nonnegative on $V_{n,n,3}$. Then, suppose that $f(v) < 0$ for some $v \in V_n$ and let

$$S = \{\pi : \pi \text{ is a partition of } n \text{ such that } f(v_\pi) < 0\},$$

and choose a $\pi \in S$ with the minimum number of parts. Note that, by assumption, if we let m be the number of parts of π, $m \geq 4$. So then, by lemma 2.3.1, there exists partitions σ, τ, and ρ of n into $m-1$, three and two parts, respectively, with $v_\pi = v_\sigma + v_\tau - v_\rho$. Further, since σ has $m-1 < m$ parts, $f(\sigma) \geq 0$ by the minimality of m. Also, $v_\tau \in P_{n,n,3}$ and $v_\rho \in P_{n,n,2}$, which implies $f(v_\tau) \geq 0$ and $f(v_\rho) = 0$, respectively. Thus,

$$0 > f(v_\pi) = f(v_\sigma) + f(v_\tau) - f(v_\rho) = f(v_\sigma) + f(v_\tau) \geq 0,$$

which is a contradiction. Thus, f is nonnegative on all of V_n, which implies that f is a facet of V_n. Further, if f is a trivial facet, then there exists some j, such that, if $d \in \mathbb{R}^n$ is the vector associated with f, then $d_j \neq 0$ and $d_i = 0$ for $i \neq j$. But then, consider the vector $u \in V$ such that $u_j = u_{n-j} = 1$ if $j \neq \frac{n}{2}$ and $u_j = 2$ if $j = \frac{n}{2}$. Then, $f(u) = d_j$ or $f(u) = 2d_j$, respectively. But in either case, $f(u) \neq 0$, but $u \in V_{n,n,2}$. This is a contradiction, so f is in fact a nontrivial facet of V_n.

□

We now define the linear transformation $T_n : \mathbb{R}^n \to \mathbb{R}^m$, where $m = \lfloor \frac{n-1}{2} \rfloor$ for the rest of this section, as follows. Let $e^1, e^2, \ldots, e^n$ denote the standard basis of $\mathbb{R}^n$, with $e^j_j = 1$ and $e^j_i = 0$ for $i \neq j$, and, similarly, let $b^1, b^2, \ldots, b^m$ be the standard basis of $\mathbb{R}^m$. Then, let

$$T_n(e^j) = \begin{cases} b^j & \text{if } j < \frac{n}{2}, \\ 0 & \text{if } j = \frac{n}{2}, \\ -b^{n-j} & \text{if } j > \frac{n}{2}, \end{cases}$$

and we extend T_n linearly. We note the following:

Lemma 2.3.6 *$T_n(V_{n,n,2}) = \{0\}$ and T_n restricted to $V_{n,n,3} \setminus V_{n,n,2}$ is one-to-one.*

Proof. If $v \in V_{n,n,2}$ then either there exists some j with $v_j = v_{n-j} = 1$, and all other coordinates of v are zero, so $T_n(v) = b^j - b^{n-(n-j)} = b^j - b^j = 0$, or $v_{\frac{n}{2}} = 2$ and all other coordinates of v are zero, so $T_n(v) = 0$.

Now, suppose that we have $v, w \in V_{n,n,3} \setminus V_{n,n,2}$ with $T_n(v) = T_n(w)$ but $w \neq v$. Then, $T_n(v-w) = 0$, which implies that, if $y = v - w$, $y_j = y_{n-j}$ for $1 \leq j \leq n$. Further, if $v_i = w_i$ for all $i \neq \frac{n}{2}$, it follows that, since both represent partitions of n, $v = w$. Since this is not true, there must be some j with $j \neq \frac{n}{2}$ and $v_j \neq w_j$ so $y_j \neq 0$. But then, $y_{n-j} \neq 0$ as well, so, without loss of generality, $j > \frac{n}{2}$. So either $v_j = 1$ or $w_j = 1$; without loss of generality, $v_j = 1$, which implies that $w_j = 0$ so $y_j = 1$. Thus, $y_{n-j} = 1$, but since $v \notin V_{n,n,2}$, we must have $v_{n-j} = 0$. But this implies that $w_{n-j} = -1$, an impossibility. So no such v, w exist implying that T_n restricted to $V_{n,n,3} \setminus V_{n,n,2}$ is one-to-one. □

Let $Q_n = T_n(V_{n,n,3})$. Now, we show that Q_n has full dimension:

Theorem 2.3.7 *Q_n has dimension m.*

Proof. Define v^j, for $1 \leq j \leq m$ as follows (all coordinates not specifically mentioned are zero)

$$v^1_1 = 2,\ v^1_{n-2} = 1,$$

$$v^j_1 = 1,\ v^j_j = 1,\ v^j_{n-j-1} = 1 \text{ for } 2 \leq j \leq m-1,$$

$$v^m_1 = 1,\ v^m_m = 2 \text{ if } n \text{ is odd},$$

$$v^m_2 = 1,\ v^m_m = 2 \text{ if } n \text{ is even}.$$

It is easy to see that each $v^j \in V_{n,n,3} \setminus V_{n,n,2}$ and thus $T_n(v^j) \in Q_n$. Further, if $w^j = T_n(v^j)$, we have

$$w_1^1 = 2, \; w_2^1 = -1,$$

$$w_1^j = 1, \; w_j^j = 1, \; w_{j+1}^j = -1 \text{ for } 2 \leq j \leq m-1,$$

$$w_1^m = 1, \; w_m^m = 2 \text{ if } n \text{ is odd},$$

$$w_2^m = 1, \; w_m^m = 2 \text{ if } n \text{ is even}.$$

Thus, for $1 \leq j \leq m-1$, we have $w_{j+1}^j = -1$ and $w_i^j = 0$ for $i > j+1$. This implies that $\{w_1, w_2, \ldots, w_{m-1}\}$ is a linearly independent set. Further, note that $\sum_{i=1}^m i v_i^j = 0$ for $1 \leq j \leq m-1$ and $\sum_{i=1}^m i v_i^m \neq 0$, which implies that v^m is not in the subspace of $\mathbb{R}^m$ spanned by $\{w_1, w_2, \ldots, w_{m-1}\}$, showing that $\{w_1, w_2, \ldots, w_{m-1}, w_m\}$ is in fact a linearly independent set. But this implies that the dimension of Q_n is at least m, and since $Q_n \subset \mathbb{R}^m$, the dimension of Q_n is exactly m. □

Now, note that if g is any linear functional on $\mathbb{R}^m$, $f = g \circ T_n$ is a linear functional on $\mathbb{R}^n$, and the fact that T_n vanishes on $V_{n,n,2}$ shows that f must vanish on $V_{n,n,2}$. For the converse, we have the following lemma:

Lemma 2.3.8 *If f is any linear functional on $\mathbb{R}^n$ that vanishes on $V_{n,n,2}$, then there exists a unique linear functional g on $\mathbb{R}^m$ such that $f = g \circ T_n$.*

Proof. Note that if such a g exists, it must be unique. Assume we have $f = g_1 \circ T_n = g_2 \circ T_n$, for linear functionals g_1, g_2 on $\mathbb{R}^m$. Then, it is clear that by definition T_n is onto, so for any $x \in \mathbb{R}^m$ we have some $y \in \mathbb{R}^n$ with $T(y) = x$ and thus $g_1(x) = g_1(T(y)) = f(y) = g_2(T(y)) = g_2(x)$ showing that $g_1 = g_2$.

Conversely, let $e^1, e^2, \ldots, e^n$ and $b^1, b^2, \ldots, b^m$ be the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, as above. Then, let g be defined by $g(b^j) = f(e^j)$ for $1 \leq j \leq m$ and extending linearly. Then, for

$1 \leq j \leq n-1$, let $u^j \in V_{n,n,2}$ be defined as before, with $u^j_j = u^j_{n-j} = 1$ and all other coordinates are zero if $j \neq \frac{n}{2}$ and $u^j_j = 2$ with all other coordinates zero if $j = \frac{n}{2}$. Then, for any $x \in \mathbb{R}^n$, define $w \in R^n$ by:

$$w = \begin{cases} x - \sum_{i=m+1}^{n} x_i u^i & \text{if } n \text{ is odd,} \\ x - \frac{x_{m+1}}{2} u^{m+1} - \sum_{i=m+2}^{n} x_i u^i & \text{if } n \text{ is even.} \end{cases}$$

Note that by construction, $w_i = 0$ for $i > m$. This shows that $T_n(w) = \sum_{i=1}^{m} w_i b^i$. Also, the fact that f vanishes on $V_{n,n,2}$ implies that $f(x) = f(w)$. Further, lemma 2.3.6 shows us that $T_n(x) = T_n(w)$. Putting this all together, we have:

$$\begin{aligned} g \circ T_n(x) = g \circ T_n(w) = g(\sum_{i=1}^{m} w_i b^i) = \sum_{i=1}^{m} w_i g(b^i) = \sum_{i=1}^{m} w_i f(e^i) \\ = f(\sum_{i=1}^{m} w_i e^i) = f(w) = f(x), \end{aligned}$$

and thus $f = g \circ T_n$. □

We now prove a theorem showing a one-to-one correspondence between nontrivial facets of V_n and facets of Q_n.

Theorem 2.3.9 *Let f be a nonzero linear functional on $\mathbb{R}^n$ that vanishes on $V_{n,n,2}$ and let g be the unique linear functional on $\mathbb{R}^m$ such that $f = g \circ T_n$. Then, f is a nontrivial facet of V_n if and only if g is a facet of Q_n.*

Proof. First, note that f is nonnegative on $V_{n,n,3}$ if and only if g is nonnegative on Q_n. To see this, first assume that f is nonnegative on $V_{n,n,3}$. Then, for any $y \in Q_n$ we have $y = T_n(x)$ for some $x \in V_{n,n,3}$ and thus

$$g(y) = g(T_n(x)) = f(x) \geq 0.$$

Conversely, assume g is nonnegative on Q_n. Then, since f vanishes on $V_{n,n,2}$, it remains to consider any $x \in V_{n,n,3} \setminus V_{n,n,2}$. We have $f(x) = g(T(x)) \geq 0$, which proves our result.

Thus, we are ready to prove theorem 2.3.9. Assume f is a facet of V_n. Then, let

$$S = \{x \in V_{n,n,3} : f(x) = 0\}.$$

By theorem 2.3.5, S has dimension $n-1$. Then, since $V_{n,n,3}$ has dimension n, let $v \in V_{n,n,3}$ with v not in the span of S (so, as a consequence, since f is nonzero, $f(v) \neq 0$). Thus, $S' = S \cup \{v\}$ has dimension n. Since S' has dimension n and T_n is onto, $T_n(S')$ spans $\mathbb{R}^m$. For any $x \in S'$, note that either $x \in V_{n,n,2}$ and $T_n(x) = 0$ or $x \in V_{n,n,3} \setminus V_{n,n,2}$, in which case $T_n(x) \in Q_n$. Thus, the nonzero elements of $T_n(S')$ are in Q_n, and since they span $\mathbb{R}^m$, they form a set of dimension m in Q_n. This implies that the removal of one element results in a set of dimension at least $m-1$, and thus $T_n(S) \setminus \{0\}$ is a subset of Q_n of dimension at least $m-1$. For any $y \in T_n(S) \setminus \{0\}$, we have $y = T_n(x)$, where $x \in S$ implies $f(x) = 0$ and thus $g(y) = g(T_n(x)) = f(x) = 0$. And, since $g(T_n(v)) = f(v) \neq 0$, which shows g is a nonzero linear functional on $\mathbb{R}^m$ that vanishes on an $m-1$ dimensional subset of Q_n, and, since f is a facet, f is nonnegative on $V_{n,n,3}$. This shows, by above, that g is nonnegative on Q_n. Thus, g is a facet of Q_n.

Conversely, assume g is a facet of Q_n. Then, let

$$R = \{y \in Q_n : g(y) = 0\},$$

and let $R' = R \cup \{z\}$, where z is some element of Q_n with $g(z) \neq 0$ (such a z exists because g is nonzero and Q_n has dimension m). is Then, note that R has dimension $m-1$ (since g is a facet) and R' thus has dimension m. Further, let

$$A = \{x \in V_{n,n,3} : T_n(x) \in R\},$$

and let v be the unique element of $V_{n,n,3} \setminus V_{n,n,2}$ such that $T_n(v) = z$ (such a v exists by the fact that T_n is onto and by lemma 2.3.6). Then, let $B = A \cup V_{n,n,2} \cup \{v\}$. We claim B has dimension n. If it does not, there must be some $u \in V_{n,n,3}$ that is not in the span of B, and thus $u \notin V_{n,n,2}$. This implies that $T_n(u) \in Q_n$. But note that the elements of R' span $\mathbb{R}^m$ so in particular, $T_n(u)$ is in the span of R'. Thus, if $R = \{y^1, y^2, \ldots, y^N\}$ for some $N \in \mathbb{N}$, we must have:

$$T_n(u) = \sum_{i=1}^{N} \alpha_i y^i + \beta z,$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_N, \beta \in \mathbb{R}$. But then, if we let x^i be the unique element of $V_{n,n,3} \setminus V_{n,n,2}$ such that $T_n(x^i) = y^i$ for $1 \leq i \leq N$, we have:

$$T_n(u) = \sum_{i=1}^{N} \alpha_i T_n(x^i) + \beta T_n(v) = T_n(\sum_{i=1}^{N} \alpha_i (x^i) + \beta(v)).$$

By lemma 2.3.6, this implies $u = \sum_{i=1}^{N} \alpha_i (x^i) + \beta(v)$ and since each $x^i \in A \subset B$ and $v \in B$, we have u is in the span of B, a contradiction. Thus, B has dimension n. This implies that $B \setminus \{v\}$ has dimension at least $n-1$. Further, for $x \in A$, $f(x) = g(T_n(x)) = 0$ by construction, and for $x \in V_{n,n,2}$ $f(x) = 0$ by assumption, so $A \cup V_{n,n,2} = B \setminus \{v\}$ is a set of dimension $n-1$ in $V_{n,n,3}$ containing $V_{n,n,2}$ that f vanishes on. And since $f(v) = g(T_n(v)) = g(z) \neq 0$, f is in fact nonzero. Further, by above, g being a facet implies g is nonnegative on Q_n, which, in turn, implies f is nonnegative on $V_{n,n,3}$. Thus, by theorem 2.3.5, f is a nontrivial facet of V_n. $\square$

So we have again reduced the problem of finding all facets of V_n further, this time to finding facets of Q_n. However, the problem of classifying all facets seems difficult. An inspection of the facets in appendix A yields some patterns, but many facets appear fairly random. We will show below some infinite families of facets, writing them in inequality form. First, we describe Q_n in more detail.

Theorem 2.3.10 *For any $x \in \mathbb{R}^m$, $x \in Q_n$ if and only if one of the following conditions holds:*

(a) $\sum_{i=1}^{m} ix_i = 0$, $\sum_{i=1}^{n} |x_i| = 3$ and there exists some j with $x_j = -1$, $x_i \geq 0$ for $i < j$ and

$x_i = 0$ for $i > j$,

(b) $x = v_\pi$ for some partition π of n into exactly three parts, each of size at most m,

(c) n is even and $v = \pi$ for some partition π of $\frac{n}{2}$ into exactly two parts.

Proof. First, consider any $v \in V_{n,n,3} \setminus V_{n,n,2}$ such that there exists some $j > \frac{n}{2}$ with $v_j \neq 0$. Since v corresponds to a partition of n into three parts, it follows that any other nonzero coordinate of v must have index less than $n - j$. Thus, if $x = T_n(v)$, we have $x_i = 0$ if $i > n - j$, $x_j = -1$ and $x_i \geq 0$ for all other coordinates, so in particular for $i < n - j$. Also, note that we have $\sum_{i=1}^n |x_i| = \sum_{i=1}^n v_i = 3$ (since $v_{n-j} = 0$ and $v_{n-i} = 0$ for $i \leq m$ with $i \neq j$). Thus, x satisfies condition (a). Conversely, for any x satisfying condition (a), if we let v be defined by $v_{n-j} = 1$, and $v_i = x_i$ for $i < j$ (and all other coordinates of v be zero) we have $T_n(v) = x$ and

$$\sum_{i=1}^{n} iv_i = (n-j)v_{n-j} + \sum_{i=1}^{j-1} ivi = n - j + \sum_{i=1}^{j-1} ix_i = n - j + \sum_{i=1}^{m} ix_i - jx_j$$
$$= n - j + j = n,$$

showing $v \in V_n$. Further, $\sum_{i=1}^n v_i = \sum_{i=1}^n |x_i| = 3$ by construction, showing that $v \in V_{n,n,3} \setminus Q_n$, and thus $x \in Q_n$.

Now consider any $v \in V_{n,n,3} \setminus V_{n,n,2}$ with $v_i = 0$ for $i \geq \frac{n}{2}$. Then, we simply have $T_n(v)$ is equal to the first m coordinates of v, and thus $T_n(v) = v_\pi$ (in $\mathbb{R}^m$) where π is the partition corresponding to v; it thus has 3 parts, and all parts less than or equal to m, showing x satisfies condition (b). Conversely, if x satisfies condition (b), by simply defining v by letting $v_i = x_i$ for $i \leq m$ and $v_i = 0$ for $i > m$, we have $T_n(v) = x$ and clearly $v \in V_{n,n,3} \setminus V_{n,n,2}$, showing $x \in Q_n$.

The only other case is when n is even, and we have some $v \in \setminus V_{n,n,2}$ with $v_{\frac{n}{2}} = 1$ (since if $v_{\frac{n}{2}} = 2$, then $v \in V_{n,n,2}$). Thus, $v_i = 0$ for $i > \frac{n}{2}$ and we also have $x = T_n(v)$ satisfying $x_i = v_i$ for $i \leq m$. This shows that

$$\sum_{i=1}^{m} ix_i = \sum_{i=1}^{m} iv_i = \sum_{i=1}^{n} iv_i - \frac{n}{2}v_{\frac{n}{2}} = n - \frac{n}{2} = \frac{n}{2}$$
$$= \sum_{i=1}^{m} y_i = \sum_{i=1}^{n} v_i - v_{\frac{n}{2}} = 3 - 1 = 2.$$

Thus, x satisfies condition (c) and $x \in Q_n$. Conversely, if x satisfies condition (c), we can simply define v by $v_{\frac{n}{2}} = 1$, $v_i = x_i$ for $i \leq m$, and $v_i = 0$ for $i > \frac{n}{2}$. Then, $T_n(v) = x$ by construction, and

$$\sum_{i=1}^{n} iv_i = \sum_{i=1}^{m} iv_i + \frac{n}{2}v_{\frac{n}{2}} = \sum_{i=1}^{m} ix_i = \frac{n}{2} = \frac{n}{2} + \frac{n}{2} = n$$
$$= \sum_{i=1}^{m} v_i + v_{\frac{n}{2}} = \sum_{i=1}^{m} x_i + 1 = 2 + 1 = 3,$$

showing that $v \in V_{n,n,3} \setminus V_{n,n,2}$, which implies that $x \in Q_n$.

Now, if $x \in Q_n$, it is the case that for some $v \in V_{n,n,3} \setminus V_{n,n,2}$ we have $T_n(v)$. Since the above exhausts all possibilities of such v, we must have x satisfying one of the above conditions. And since we showed that any x satisfying the above conditions is in Q_n, we are done. $\square$

We now describe four infinite families of facets of V_n.

Theorem 2.3.11 *The following all are facets of V_n (for $n \geq 3$):*

(a) $\sum_{i=1}^{m} iv_i - \sum_{i=\lfloor \frac{n+2}{2} \rfloor}^{n-1} (n-i)v_i \geq 0$,

(b) $v_1 - v_{n-1} \geq 0$ *if* $n \neq 5, 7$,

(c) $\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} v_{2i-1} - \sum_{i=\lfloor \frac{n+2}{4} \rfloor}^{n-1} v_{2i+1}$ *if n is even,*

(d) $\sum_{i=1}^{m-1} 2iv_i - v_m + v_{m+1} - \sum_{i=m+2}^{n-1} 2iv_i \geq 0$ *if n is odd and $n > 3$.*

Proof. Let f^1, f^2, f^3 and f^4 be the linear functionals on $\mathbb{R}^n$ associated with the inequalities in parts (a), (b), (c) and (d), respectively. It it easy to check that in each of the above inequalities, the coefficient of v_i is equal to the negation of coefficient of v_{n-i} for $1 \leq i \leq n$. Thus, each f^i vanishes on $V_{n,n,2}$. Then, let g^i be the unique linear functional on $\mathbb{R}^m$ such that $f^i = g^i \circ T_n$ for $1 \leq i \leq 4$.

We will then prove that each g_i is a facet of Q_n. By theorem 2.3.9, this will prove that each of the f_i is a facet of V_n.

First, we consider g^1. Note that the proof of theorem 2.3.9 shows that $g^1(b^j) = j$ for $1 \leq j \leq m$ (where b^j is the jth basis element of $\mathbb{R}^m$ described above). Thus, g^1 is nonnegative on any element of Q_n that is nonnegative. Further, for any element x of Q_n satisfying $\sum_{i=1}^{m} ix_i$ will have $g^1(x) = 0$. By theorem 5.17, this shows that g^1 is nonnegative on Q^n. Further, for $1 \leq i \leq m-1$, we define u^i as follows. Let $u_1^1 = 2$ and $u_2^1 = -1$, and let all other coordinates of u^1 be zero. Then, let $u_1^i = 1$, $u_i^i = 1$ and $u_{i+1}^i - 1$, for $2 \leq i \leq m-1$. It is easy to see each $u^i \in Q_n$, since they satisfy condition (a) of theorem 5.17. Further, if $S = \{u^i : 1 \leq i \leq m-1\}$, since each u^i has $u_{i+1}^i \neq 0$ but $u_j^i = 0$ for $j > i+1$, the elements of S are linearly independent. And since

$$g^1(u^i) = \sum_{j=1}^{m} u_j^i g^1(b^j) = \sum_{j=1}^{m} j u_j^i = 0$$

(as can easily be checked), it follows that g^1 vanishes on S, a set of dimension $m-1$, and thus g^1 is a facet of Q_n.

Now, note that g^2 has $g^2(b^1) = 1$ and $g^2(b^j) = 0$ for $2 \leq j \leq m$. This implies that g^2 is nonnegative on Q_n, since by theorem 2.3.10, it is impossible for $x \in Q_n$ to have $x_1 < 0$ (since either x corresponds to some partition, or we would have $\sum_{i=1}^{m} ix_i = x_1 \neq 0$). Further, an inspection of appendix A (and any computer search) shows us theorem 2.3.11 part (b) is true in particular for $n \leq 12$ and thus we need only consider $n > 13$. So, we now define u^i for $1 \leq i \leq m-2$ as follows (all coordinates not specifically mentioned are zero): first, let $u_3^1 = 2$ and $u_6^1 = -1$. Then, let $u_2^2 = 2$ and $u_4^2 = -1$. For $3 \leq j \leq m-2$, let $u_2^j = 1$, $u_j^j = 1$, and $u_{j+2}^j = -1$. Note that by construction, each $u^j \in Q_n$. It is easy to see that $S = \{u^i : 2 \leq i \leq m-2\}$ is an independent set, since for $2 \leq j \leq m-2$ we have $u_{j+2}^j \neq 0$ but $u_i^j = 0$ for $i > j+2$. Further, u^1 is not in the span of S. For if it were, we would have $\alpha_2, \ldots, \alpha_{m-2}$ such that $u^1 = \sum_{i=2}^{m-2} \alpha_i u^i$. But u^3 is the only u^j with $u_3^j \neq 0$, so $\alpha_3 = 2$. Similarly, u^5 is the only u^j other than u^3 with $u_5^j \neq 0$, so since $u_5^3 = -1$ and $u_5^5 = 1$, we must have $a_5 = 2$. Similarly, by induction, for odd $i > 3$, we must have $a^i = 2$ since $u_i^1 = 0$, and

$u_i^{i-2} = -1$ and $u_i^i = 1$. But then, for either $i = m-2$ or $i = m-3$, whichever is odd, we have $a^i = 2$, and since u^i is the only such j with $u_{i+2}^j \neq 0$, we would have $u_{i+2}^1 = a^i u_{i+2}^i = -2$, contradicting the fact that we must have $u_{i+2}^1 = 0$. Thus, u^1 is not in the span of S, implying that $S' = S \cup \{u^1\}$ has dimension $m-2$. Further, let v be defined by $v_m = v_{m-1} = 1$, and, if n is odd, $v_2 = 1$, and if n is even, $v_3 = 1$. Either way, $v \in Q_n$. Further, since $g^1(u^j) = 0$ for $1 \leq j \leq m-2$ and $g^1(v) = n \neq 0$, v is not in the span of S', implying that $S'' = S' \cup \{v\}$ has dimension $m-1$. Further, since each $x \in S''$ has $x_1 = 0$, $g^2(x) = 0$ and thus g^2 vanishes on S'', showing g^2 is a facet of Q_n.

Now, assume that n is even. Again, it can be checked that theorem 2.3.11 part (c) is true for $n \leq 12$ so assume $n \geq 14$. Note that $g^3(b^j) = 1$ if j is odd and $g^3(b^j) = 0$ if j is even, for $1 \leq j \leq m$. Since $g^3(b^j) \geq 0$ in any case, $g^3(x) \geq 0$ for $x \in Q_n$ satisfying conditions (b) or (c) of theorem 2.3.10 (since such x has nonnegative coordinates). Further, for any $x \in Q_n$ satisfying condition (a) of theorem 2.3.10, we either have the j with $x^j = -1$ even, in which case $g^3(x)$ trivially is greater than or equal to zero, or we such a j odd, in which case there must be some $i < j$ with $x_i \geq 1$ and i odd, since we must have $\sum_{i=1}^{j-1} ix_i = 0 - jx_j = j$, an odd number. Then, since all other coordinates of x are nonnegative, we have

$$g^3(x) \geq g^3(b^j) + g^3(b^i) = -1 + 1 = 0,$$

showing g^3 in fact is nonnegative on Q_n. Further, we define, in this case, for $1 \leq j \leq m-2$, u^j by $u_1^j = u_{j+1}^j = 1$ and $u_{j+2}^j = -1$ if j is odd, and $u_2^2 = 2$, $u_4^2 = -1$, and $u_2^j = u_j^j = 1$ and $u_{j+2}^j = -1$ if j is even and $j > 2$. It is easy to see tht each $u^j \in Q_n$. Further, it follows that $S = \{u^j : 1 \leq j \leq m-2\}$ is independent, by reasons similar to those described above. Further, if j is even, $u_i^j = 0$ for odd i, implying that $g^3(u^j) = 0$. If j is odd, then

$$g^3(u^j) = g^3(b^1) + g^3(b^{j+1}) - g^3(b^{j+2}) = 1 + 0 - 1 = 0.$$

Thus, g^3 vanishes on S. Now, if we let v be defined by $v_m = 2$, $v_2 = 1$ if m is even and $v_{m-1} = 2$, $v_4 = 1$ if m is odd. Either way, we have $v \in Q_n$ and $v_i = 0$ if i is odd, so $g^3(v) = 0$. Further, similar to above, $g^1(u^j) = 0$ for $1 \leq j \leq m-2$ and $g^1(v) \neq 0$ imply that $S' = S \cup \{v\}$ is linearly independent. Thus, S' is a subset of Q_n of dimension $m-1$ that g^3 vanishes on, implying that g^3 is a facet of Q_n.

Finally, we assume that n is odd and consider g^4. We first note that $g^4(b^j) = 2j$ if $j < m$ and $g^4(b^m) = -1$. So, for any $v \in Q_n$ such that v satisfies condition (a) of theorem 2.3.10 with $v_m = 0$, we have $g^4(v) = 2g^1(v) = 0$. Further, for any $v \in Q_n$ satisfying condition (a) with $v_m \neq 0$, we must have $v_m = -1$, and $v_i \geq 0$ for $i < m$. Then, we have

$$g^4(v) = \sum_{i=1}^{m} v_i g^4(b^i) = \sum_{i=1}^{m-1} v_i g^4(b^i) + g^4(b^m)v_m \geq 0 + (-1)(-1) = 1,$$

so g^4 is nonnegative on all $v \in Q_n$ satisfying condition (a) of theorem 2.3.10. The only other $v \in Q_n$ are those that correspond to partitions of n. Either $v_m = 0$, in which case $g^4(v) geq 0$ or $v_m = 1$ or $v_m = 2$ (since $3m > n$). If $v_m = 2$, then $v_1 = 1$ and the rest of the coordinates of v are zero, and thus

$$g^4(v) = g^4(b^1) + 2g^4(b^m) = 2 - 2 = 0.$$

Otherwise, $v_m = 1$ and there must be at least one $i < m$ with $v_i > 0$. Then

$$g^4(v) \geq g^4(b^m) + v_i g^4(b^i) = -1 + 2i > 0.$$

This exhausts all cases, implying g^4 is nonnegative on Q_n. Further, in this case, let u^j for $1 \leq j \leq m-2$ be defined by $u_1^1 = 2$, $u_2^1 = -1$, and $u_1^j = u_j^j = 1$, $u_{j+1}^j = -1$ for $2 \leq j \leq m-2$. Then, each u^j has $u_m^j = 0$ and each satisfies condition (a) of theorem 2.3.10, showing each $u^j \in Q_n$ and $g^4(u^j) = 0$. Further, we again have $S = \{u^j : 1 \leq j \leq m-2$ is an independent set. So if we let v have $v_m = 2$ and $v_1 = 1$, our above analysis shows that $v \in Q_n$ and $g^4(v) = 0$. And again, $g^1(v) \neq 0$, $g^1(u^j) = 0$ for $1 \leq j \leq m-2$, and thus we have $S' = S \cup \{v\}$ is an independent set. And again, S' has dimension $m-1$ and g^4 vanishes on S'. Thus g^4 is a face of Q_n. $\square$

Appendix A

The Matrices $\mathbf{A_n}$ for n = 3 to n = 16

$$A_3 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix},\ A_4 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix},\ A_5 = \begin{bmatrix} 2 & -1 & 1 & -2 & 0 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 1 & 2 & 0 & -2 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},\ A_7 = \begin{bmatrix} 4 & 1 & -2 & 2 & -1 & -4 & 0 \\ 3 & -1 & 2 & -2 & 1 & -3 & 0 \\ 2 & 4 & -1 & 1 & -4 & -2 & 0 \\ 1 & 2 & 3 & -3 & -2 & -1 & 0 \end{bmatrix},$$

$$A_8 = \begin{bmatrix} 1 & 2 & 3 & 0 & -3 & -2 & -1 & 0 \\ 1 & 2 & -1 & 0 & 1 & -2 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$A_9 = \begin{bmatrix} 4 & -1 & 3 & -2 & 2 & -3 & 1 & -4 & 0 \\ 2 & 4 & 6 & -1 & 1 & -6 & -4 & -2 & 0 \\ 2 & 4 & 0 & -1 & 1 & 0 & -4 & -2 & 0 \\ 2 & 1 & 0 & 2 & -2 & 0 & -1 & -2 & 0 \\ 2 & 1 & 0 & -1 & 1 & 0 & -1 & -2 & 0 \\ 1 & 2 & 3 & 4 & -4 & -3 & -2 & -1 & 0 \\ 1 & 2 & 0 & 1 & -1 & 0 & -2 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 3 & 6 & -1 & 2 & 0 & -2 & 1 & -6 & -3 & 0 \\ 3 & 1 & -1 & 2 & 0 & -2 & 1 & -1 & -3 & 0 \\ 2 & 4 & 1 & -2 & 0 & 2 & -1 & -4 & -2 & 0 \\ 1 & 2 & 3 & 4 & 0 & -4 & -3 & -2 & -1 & 0 \\ 1 & 2 & 3 & -1 & 0 & 1 & -3 & -2 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} 12 & 2 & 3 & 4 & -6 & 6 & -4 & -3 & -2 & -12 & 0 \\ 8 & 16 & 2 & -1 & -4 & 4 & 1 & -2 & -16 & -8 & 0 \\ 8 & 5 & 2 & 10 & -4 & 4 & -10 & -2 & -5 & -8 & 0 \\ 8 & 5 & 2 & -1 & -4 & 4 & 1 & -2 & -5 & -8 & 0 \\ 7 & 3 & -1 & 6 & 2 & -2 & -6 & 1 & -3 & -7 & 0 \\ 6 & 1 & 7 & 2 & -3 & 3 & -2 & -7 & -1 & -6 & 0 \\ 5 & -1 & 4 & -2 & 3 & -3 & 2 & -4 & 1 & -5 & 0 \\ 4 & 8 & 1 & 5 & -2 & 2 & -5 & -1 & -8 & -4 & 0 \\ 3 & 6 & -2 & 1 & 4 & -4 & -1 & 2 & -6 & -3 & 0 \\ 2 & 4 & 6 & 8 & -1 & 1 & -8 & -6 & -4 & -2 & 0 \\ 2 & 4 & 6 & -3 & -1 & 1 & 3 & -6 & -4 & -2 & 0 \\ 1 & 2 & 3 & 4 & 5 & -5 & -4 & -3 & -2 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 4 & 2 & 0 & 4 & -1 & 0 & 1 & -4 & 0 & -2 & -4 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & -1 & -2 & 0 & -1 & -2 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 & -5 & -4 & -3 & -2 & -1 & 0 \\ 1 & 2 & 3 & 4 & -1 & 0 & 1 & -4 & -3 & -2 & -1 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 & -1 & 0 & -3 & -2 & -1 & 0 \\ 1 & 2 & 3 & 0 & -1 & 0 & 1 & 0 & -3 & -2 & -1 & 0 \\ 1 & 2 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & -2 & -1 & 0 \\ 1 & 2 & 0 & 1 & 2 & 0 & -2 & -1 & 0 & -2 & -1 & 0 \\ 1 & 2 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & -2 & -1 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -2 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix}
15 & 4 & 6 & 8 & -3 & 12 & -12 & 3 & -8 & -6 & -4 & -15 & 0 \\
10 & 20 & 4 & 1 & -2 & 8 & -8 & 2 & -1 & -4 & -20 & -10 & 0 \\
10 & 20 & 4 & 1 & -2 & -5 & 5 & 2 & -1 & -4 & -20 & -10 & 0 \\
10 & 7 & 4 & 14 & -2 & 8 & -8 & 2 & -14 & -4 & -7 & -10 & 0 \\
10 & 7 & 4 & 1 & -2 & 8 & -8 & 2 & -1 & -4 & -7 & -10 & 0 \\
10 & 7 & 4 & 1 & -2 & -5 & 5 & 2 & -1 & -4 & -7 & -10 & 0 \\
9 & 18 & 1 & -3 & 6 & 2 & -2 & -6 & 3 & -1 & -18 & -9 & 0 \\
9 & 5 & 1 & -3 & 6 & 2 & -2 & -6 & 3 & -1 & -5 & -9 & 0 \\
8 & 16 & -2 & 6 & 1 & -4 & 4 & -1 & -6 & 2 & -16 & -8 & 0 \\
8 & 3 & -2 & 6 & 1 & -4 & 4 & -1 & -6 & 2 & -3 & -8 & 0 \\
7 & 1 & 8 & 2 & -4 & 3 & -3 & 4 & -2 & -8 & -1 & -7 & 0 \\
6 & 12 & 18 & -2 & 4 & -3 & 3 & -4 & 2 & -18 & -12 & -6 & 0 \\
6 & 12 & 5 & -2 & 4 & 10 & -10 & -4 & 2 & -5 & -12 & -6 & 0 \\
6 & 12 & 5 & -2 & 4 & -3 & 3 & -4 & 2 & -5 & -12 & -6 & 0 \\
6 & -1 & 5 & -2 & 4 & -3 & 3 & -4 & 2 & -5 & 1 & -6 & 0 \\
5 & 10 & 2 & 7 & -1 & 4 & -4 & 1 & -7 & -2 & -10 & -5 & 0 \\
4 & 8 & 12 & 3 & -6 & -2 & 2 & 6 & -3 & -12 & -8 & -4 & 0 \\
4 & 8 & -1 & 3 & 7 & -2 & 2 & -7 & -3 & 1 & -8 & -4 & 0 \\
3 & 6 & 9 & -1 & 2 & 5 & -5 & -2 & 1 & -9 & -6 & -3 & 0 \\
2 & 4 & 6 & 8 & 10 & -1 & 1 & -10 & -8 & -6 & -4 & -2 & 0 \\
2 & 4 & 6 & 8 & -3 & -1 & 1 & 3 & -8 & -6 & -4 & -2 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix},$$

$$A_{14} = \begin{bmatrix}
9 & 4 & -1 & 8 & 3 & -2 & 0 & 2 & -3 & -8 & 1 & -4 & -9 & 0 \\
8 & 2 & 3 & 4 & -2 & 6 & 0 & -6 & 2 & -4 & -3 & -2 & -8 & 0 \\
5 & 10 & 1 & 6 & -3 & 2 & 0 & -2 & 3 & -6 & -1 & -10 & -5 & 0 \\
5 & 3 & 1 & 6 & -3 & 2 & 0 & -2 & 3 & -6 & -1 & -3 & -5 & 0 \\
5 & 3 & 1 & -1 & 4 & 2 & 0 & -2 & -4 & 1 & -1 & -3 & -5 & 0 \\
4 & 8 & 12 & 2 & -1 & -4 & 0 & 4 & 1 & -2 & -12 & -8 & -4 & 0 \\
4 & 8 & 5 & 2 & -1 & -4 & 0 & 4 & 1 & -2 & -5 & -8 & -4 & 0 \\
4 & 1 & 5 & 2 & -1 & 3 & 0 & -3 & 1 & -2 & -5 & -1 & -4 & 0 \\
3 & 6 & 9 & -2 & 1 & 4 & 0 & -4 & -1 & 2 & -9 & -6 & -3 & 0 \\
3 & 6 & 2 & -2 & 1 & 4 & 0 & -4 & -1 & 2 & -2 & -6 & -3 & 0 \\
2 & 4 & 6 & 1 & 3 & -2 & 0 & 2 & -3 & -1 & -6 & -4 & -2 & 0 \\
2 & 4 & -1 & 1 & 3 & -2 & 0 & 2 & -3 & -1 & 1 & -4 & -2 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 0 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 4 & 5 & -1 & 0 & 1 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 4 & -2 & -1 & 0 & 1 & 2 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix},$$

$$A_{15} = \begin{bmatrix}
16 & 2 & 3 & 4 & 5 & 6 & -8 & 8 & -6 & -5 & -4 & -3 & -2 & -16 & 0 \\
11 & 7 & 3 & -1 & 10 & 6 & 2 & -2 & -6 & -10 & 1 & -3 & -7 & -11 & 0 \\
8 & 16 & 9 & 2 & 10 & 18 & -4 & 4 & -18 & -10 & -2 & -9 & -16 & -8 & 0 \\
8 & 1 & 9 & 2 & 10 & 3 & -4 & 4 & -3 & -10 & -2 & -9 & -1 & -8 & 0 \\
8 & 1 & 9 & 2 & 0 & 3 & -4 & 4 & -3 & 0 & -2 & -9 & -1 & -8 & 0 \\
8 & 1 & 4 & 2 & 0 & 3 & -4 & 4 & -3 & 0 & -2 & -4 & -1 & -8 & 0 \\
7 & 14 & 6 & -2 & 5 & 12 & 4 & -4 & -12 & -5 & 2 & -6 & -14 & -7 & 0 \\
7 & 14 & 6 & -2 & 5 & -3 & 4 & -4 & 3 & -5 & 2 & -6 & -14 & -7 & 0 \\
7 & -1 & 6 & -2 & 5 & -3 & 4 & -4 & 3 & -5 & 2 & -6 & 1 & -7 & 0 \\
6 & 12 & 3 & -1 & 0 & 6 & 2 & -2 & -6 & 0 & 1 & -3 & -12 & -6 & 0 \\
6 & 7 & 3 & -1 & 0 & 6 & 2 & -2 & -6 & 0 & 1 & -3 & -7 & -6 & 0 \\
6 & 2 & 8 & 4 & 0 & 6 & -3 & 3 & -6 & 0 & -4 & -8 & -2 & -6 & 0 \\
6 & 2 & 3 & 4 & 0 & 6 & 2 & -2 & -6 & 0 & -4 & -3 & -2 & -6 & 0 \\
6 & 2 & 3 & 4 & 0 & 6 & -3 & 3 & -6 & 0 & -4 & -3 & -2 & -6 & 0 \\
4 & 8 & 12 & 1 & 5 & 9 & -2 & 2 & -9 & -5 & -1 & -12 & -8 & -4 & 0 \\
4 & 8 & 12 & 1 & 5 & -6 & -2 & 2 & 6 & -5 & -1 & -12 & -8 & -4 & 0 \\
4 & 8 & 12 & 1 & 0 & 4 & -2 & 2 & -4 & 0 & -1 & -12 & -8 & -4 & 0 \\
4 & 8 & 12 & 1 & 0 & -1 & -2 & 2 & 1 & 0 & -1 & -12 & -8 & -4 & 0 \\
4 & 8 & 2 & 6 & 0 & -1 & -2 & 2 & 1 & 0 & -6 & -2 & -8 & -4 & 0 \\
4 & 8 & 2 & 1 & 0 & 4 & -2 & 2 & -4 & 0 & -1 & -2 & -8 & -4 & 0 \\
4 & 8 & 2 & 1 & 0 & -1 & 3 & -3 & 1 & 0 & -1 & -2 & -8 & -4 & 0 \\
4 & 8 & 2 & 1 & 0 & -1 & -2 & 2 & 1 & 0 & -1 & -2 & -8 & -4 & 0 \\
4 & 8 & 0 & 1 & 5 & 0 & -2 & 2 & 0 & -5 & -1 & 0 & -8 & -4 & 0 \\
4 & 8 & 0 & 1 & 2 & 0 & -2 & 2 & 0 & -2 & -1 & 0 & -8 & -4 & 0 \\
4 & 5 & 0 & 1 & 5 & 0 & -2 & 2 & 0 & -5 & -1 & 0 & -5 & -4 & 0 \\
4 & 3 & 7 & 1 & 0 & 4 & -2 & 2 & -4 & 0 & -1 & -7 & -3 & -4 & 0 \\
4 & 3 & 7 & 1 & 0 & -1 & -2 & 2 & 1 & 0 & -1 & -7 & -3 & -4 & 0 \\
4 & 3 & 2 & 6 & 0 & 4 & 3 & -3 & -4 & 0 & -6 & -2 & -3 & -4 & 0 \\
4 & 3 & 2 & 6 & 0 & 4 & -2 & 2 & -4 & 0 & -6 & -2 & -3 & -4 & 0 \\
4 & 3 & 2 & 6 & 0 & -1 & 3 & -3 & 1 & 0 & -6 & -2 & -3 & -4 & 0 \\
4 & 3 & 2 & 6 & 0 & -1 & -2 & 2 & 1 & 0 & -6 & -2 & -3 & -4 & 0 \\
4 & 3 & 2 & 1 & 5 & 4 & -2 & 2 & -4 & -5 & -1 & -2 & -3 & -4 & 0 \\
4 & 3 & 2 & 1 & 5 & -1 & -2 & 2 & 1 & -5 & -1 & -2 & -3 & -4 & 0 \\
4 & 3 & 2 & 1 & 0 & 4 & 3 & -3 & -4 & 0 & -1 & -2 & -3 & -4 & 0 \\
4 & 3 & 2 & 1 & 0 & 4 & -2 & 2 & -4 & 0 & -1 & -2 & -3 & -4 & 0 \\
4 & 3 & 2 & 1 & 0 & -1 & 3 & -3 & 1 & 0 & -1 & -2 & -3 & -4 & 0 \\
4 & 3 & 2 & 1 & 0 & -1 & -2 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & 0 \\
4 & 2 & 0 & 1 & 2 & 0 & -2 & 2 & 0 & -2 & -1 & 0 & -2 & -4 & 0 \\
3 & 1 & 4 & 2 & 0 & 3 & 1 & -1 & -3 & 0 & -2 & -4 & -1 & -3 & 0 \\
3 & 1 & 4 & 2 & 0 & -2 & 1 & -1 & 2 & 0 & -2 & -4 & -1 & -3 & 0 \\
2 & 4 & 6 & 8 & 10 & 12 & -1 & 1 & -12 & -10 & -8 & -6 & -4 & -2 & 0 \\
2 & 4 & 6 & 8 & 10 & -3 & -1 & 1 & 3 & -10 & -8 & -6 & -4 & -2 & 0 \\
2 & 4 & 6 & 8 & 0 & 2 & -1 & 1 & -2 & 0 & -8 & -6 & -4 & -2 & 0 \\
2 & 4 & 6 & 8 & 0 & -3 & -1 & 1 & 3 & 0 & -8 & -6 & -4 & -2 & 0 \\
2 & 4 & 6 & 3 & 0 & -3 & -1 & 1 & 3 & 0 & -3 & -6 & -4 & -2 & 0 \\
2 & 4 & 1 & 3 & 0 & 2 & 4 & -4 & -2 & 0 & -3 & -1 & -4 & -2 & 0 \\
2 & 4 & 1 & 3 & 0 & 2 & -1 & 1 & -2 & 0 & -3 & -1 & -4 & -2 & 0 \\
2 & 4 & 0 & 2 & 4 & 0 & -1 & 1 & 0 & -4 & -2 & 0 & -4 & -2 & 0 \\
2 & 4 & 0 & 2 & 1 & 0 & -1 & 1 & 0 & -1 & -2 & 0 & -4 & -2 & 0 \\
2 & 1 & 0 & 2 & 1 & 0 & 2 & -2 & 0 & -1 & -2 & 0 & -1 & -2 & 0 \\
2 & 1 & 0 & 2 & 1 & 0 & -1 & 1 & 0 & -1 & -2 & 0 & -1 & -2 & 0 \\
2 & 1 & 0 & 2 & 0 & 0 & 1 & -1 & 0 & 0 & -2 & 0 & -1 & -2 & 0 \\
2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & -2 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 4 & 0 & 1 & 2 & -2 & -1 & 0 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & -1 & 0 & 1 & 2 & -2 & -1 & 0 & 1 & -3 & -2 & -1 & 0 \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & -1 & 0 & -2 & -1 & 0 & -2 & -1 & 0 \\
1 & 2 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & -2 & -1 & 0 \\
1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & -1 & 0 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & -1 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix},$$

$$
A_{16} = \begin{bmatrix}
11 & 6 & 1 & 12 & 7 & 2 & -3 & 0 & 3 & -2 & -7 & -12 & -1 & -6 & -11 & 0 \\
11 & 6 & 1 & 12 & -1 & 2 & -3 & 0 & 3 & -2 & 1 & -12 & -1 & -6 & -11 & 0 \\
9 & 2 & 11 & 4 & -3 & 6 & -1 & 0 & 1 & -6 & 3 & -4 & -11 & -2 & -9 & 0 \\
9 & 2 & 3 & 4 & -3 & 6 & -1 & 0 & 1 & -6 & 3 & -4 & -3 & -2 & -9 & 0 \\
7 & 6 & 1 & 0 & 7 & 2 & -3 & 0 & 3 & -2 & -7 & 0 & -1 & -6 & -7 & 0 \\
5 & 10 & 15 & 4 & 1 & -2 & -5 & 0 & 5 & 2 & -1 & -4 & -15 & -10 & -5 & 0 \\
5 & 10 & 7 & 4 & 1 & -2 & -5 & 0 & 5 & 2 & -1 & -4 & -7 & -10 & -5 & 0 \\
5 & 10 & -1 & 4 & 9 & -2 & 3 & 0 & -3 & 2 & -9 & -4 & 1 & -10 & -5 & 0 \\
5 & 10 & -1 & 4 & 1 & -2 & 3 & 0 & -3 & 2 & -1 & -4 & 1 & -10 & -5 & 0 \\
5 & 2 & 7 & 4 & 1 & -2 & 3 & 0 & -3 & 2 & -1 & -4 & -7 & -2 & -5 & 0 \\
5 & 2 & -1 & 4 & 1 & -2 & 3 & 0 & -3 & 2 & -1 & -4 & 1 & -2 & -5 & 0 \\
3 & 6 & 9 & 12 & -1 & 2 & 5 & 0 & -5 & -2 & 1 & -12 & -9 & -6 & -3 & 0 \\
3 & 6 & 9 & 12 & -1 & 2 & -3 & 0 & 3 & -2 & 1 & -12 & -9 & -6 & -3 & 0 \\
3 & 6 & 9 & 4 & -1 & 2 & -3 & 0 & 3 & -2 & 1 & -4 & -9 & -6 & -3 & 0 \\
3 & 6 & 9 & 0 & -1 & 2 & 5 & 0 & -5 & -2 & 1 & 0 & -9 & -6 & -3 & 0 \\
3 & 6 & 9 & 0 & -1 & 2 & 1 & 0 & -1 & -2 & 1 & 0 & -9 & -6 & -3 & 0 \\
3 & 6 & 5 & 0 & -1 & 2 & 5 & 0 & -5 & -2 & 1 & 0 & -5 & -6 & -3 & 0 \\
3 & 6 & 1 & 4 & 7 & 2 & -3 & 0 & 3 & -2 & -7 & -4 & -1 & -6 & -3 & 0 \\
3 & 6 & 1 & 4 & -1 & 2 & 5 & 0 & -5 & -2 & 1 & -4 & -1 & -6 & -3 & 0 \\
3 & 6 & 1 & 4 & -1 & 2 & -3 & 0 & 3 & -2 & 1 & -4 & -1 & -6 & -3 & 0 \\
3 & 6 & 1 & 0 & 3 & 2 & -3 & 0 & 3 & -2 & -3 & 0 & -1 & -6 & -3 & 0 \\
3 & 6 & 1 & 0 & -1 & 2 & 1 & 0 & -1 & -2 & 1 & 0 & -1 & -6 & -3 & 0 \\
3 & 2 & 5 & 0 & -1 & 2 & 1 & 0 & -1 & -2 & 1 & 0 & -5 & -2 & -3 & 0 \\
3 & 2 & 1 & 4 & -1 & 2 & 1 & 0 & -1 & -2 & 1 & -4 & -1 & -2 & -3 & 0 \\
3 & 2 & 1 & 4 & -1 & 2 & -1 & 0 & 1 & -2 & 1 & -4 & -1 & -2 & -3 & 0 \\
3 & 2 & 1 & 0 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & 0 & -1 & -2 & -3 & 0 \\
3 & 2 & 1 & 0 & 3 & 2 & -1 & 0 & 1 & -2 & -3 & 0 & -1 & -2 & -3 & 0 \\
3 & 2 & 1 & 0 & 3 & 0 & -1 & 0 & 1 & 0 & -3 & 0 & -1 & -2 & -3 & 0 \\
3 & 2 & 1 & 0 & -1 & 2 & 1 & 0 & -1 & -2 & 1 & 0 & -1 & -2 & -3 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & -1 & 0 & 1 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 4 & 5 & -2 & -1 & 0 & 1 & 2 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 4 & 1 & -2 & -1 & 0 & 1 & 2 & -1 & -4 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & -3 & -2 & -1 & 0 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 0 & 1 & 2 & -1 & 0 & 1 & -2 & -1 & 0 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -3 & -2 & -1 & 0 \\
1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & -1 & 0 \\
1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & -1 & -2 & -1 & 0 & -1 & -2 & -1 & 0 \\
1 & 2 & 1 & 0 & 1 & 2 & -1 & 0 & 1 & -2 & -1 & 0 & -1 & -2 & -1 & 0 \\
1 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & -2 & -1 & 0 \\
1 & 2 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & -2 & -1 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & -2 & -1 & 0 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & -1 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 \\
1 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -2 & -1 & -1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}.
$$

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