Conformal Laminations

Thesis by

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Abstract

A lamination on a circle is an equivalence relation on the points of the circle. Laminations can be induced on a circle by a map that is continuous on the closed disc and injective in the interior. Such laminations are characterized topologically, as being flat and closed. In this paper we investigate the conditions under which a closed, flat lamination is induced by a conformal mapping. We show that if the set of multiple points of the lamination form a Cantor set, whose end points are identified under the lamination, then the lamination is conformal. More generally, the union of such laminations is also conformal. We also show conjecture that any closed, flat lamination, such that the set of multiple points is of logarithmic capacity zero, is conformal.

Contents

Chapter 1

Introduction

1.1 Notation and Definitions

A lamination on the unit circle, $\mathbb{T}$, is an equivalence relationship on the points of $\mathbb{T}$. So a lamination L on $\mathbb{T}$ can be thought of as a subset of $\mathbb{T} \times \mathbb{T}$ that is reflexive, symmetric and transitive.

Definition 1.1.1. *A lamination L on $\mathbb{T}$ is closed if L is a closed set in $\mathbb{T} \times \mathbb{T}$, i.e.,*

$$\lambda_n, \nu_n \in \mathbb{T}, \lambda_n \sim \nu_n \textit{and } \lambda_n \to \lambda, \nu_n \to \nu \textit{ as } n \to \infty \implies \lambda \sim \nu.$$

Definition 1.1.2. *A lamination on $\mathbb{T}$ is flat if $\lambda_1 \sim \lambda_2, \nu_1 \sim \nu_2$,and λ_1 and λ_2 lie in different components of $\mathbb{T} \backslash (\nu_1, \nu_2) \implies \lambda_1 \sim \nu_1$.*

We will denote the complex plane by $\mathbb{C}$ and the extended complex plane by $\hat{\mathbb{C}}$. A lamination on $\mathbb{T}$ can be introduced by considering a continuous map $\phi : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$, where ϕ is injective on $\mathbb{D}$. All points that mapped to the same point are then identified under the lamination.

Definition 1.1.3. *Let $\phi : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$,be a continuous map that is injective on $\mathbb{D}$. The lamination L_ϕ, induced by ϕ on $\mathbb{T}$ is defined as follows:*

$$\lambda \sim \nu \Leftrightarrow \phi(\lambda) = \phi(\nu), \forall \lambda, \nu \in \mathbb{T}.$$

The proof of the following theorem follows from the continuity of the map, ϕ and the separation properties of the Riemann Sphere.

Theorem 1.1.1. *If $\phi : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$,is a continuous map that is injective on $\mathbb{D}$, then L_ϕ is a closed, flat lamination.*

Let $\mathbb{D}^*$ be the exterior disc $\{z : z > 1\} \cup \infty$. Then, we can also define L_ϕ such that $\phi : \overline{\mathbb{D}^*} \to \hat{\mathbb{C}}$ where ϕ is continuous in $\overline{\mathbb{D}^*}$ and injective in $\mathbb{D}^*$.

Definition 1.1.4. *L is a conformal lamination if there exists a mapping $\phi : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$, such that ϕ is conformal in $\mathbb{D}$ and $L = L_\phi$.*

Definition 1.1.5. *If L is a lamination on $\mathbb{T}$ such that $x \sim y$under L and $x \neq y$, then x is called a multiple point of L.*

If L is a lamination on $\mathbb{T}$, then we shall denote the set of multiple points of L by $mult(L)$ and the closure by $\overline{mult(L)}$. Note that according to our definitions $L \subset \mathbb{T} \times \mathbb{T}$ while $mult(L) \subset \mathbb{T}$.

We can also define a lamination on any closed curve which is the boundary of a simply connected domain, Ω. If $\gamma \in \hat{\mathbb{C}}$ is a curve and $C \subset \gamma$ is a closed set, then we can write $\gamma \backslash C = \cup_{j=1}^{\infty} I_j$, where each I_j is a component of $\gamma \backslash C$. We shall call the $I_j's$ the complementary intervals of C in γ. In our discussion the closed set C will usually be $\overline{mult(L)}$. If the end points of I_j are (not) equivalent under L, we shall call I_j a complementary interval with (non)equivalent end points of L.

A subset $L' \subset L$ is called a sublamination if L' defines a lamination on $\mathbb{T}$. Note if L is flat then L' is not flat in general. A simple example would be the flat lamination L which identifies the 4 points $\{1, -1, i, -i\}$ on $\mathbb{T}$. The sublamination L' such that, $1 \sim -1$, and $i \sim -i$, form 2 separate equivalence classes, is not flat.

1.2 Totally Disconnected, Cantor-Type and Julia-Type Laminations

We first give a brief review of quotient spaces.

Definition 1.2.1. *Let X be a topological space, Y a set, and $f : X \to Y$ an onto map. Then the quotient topology on Y is defined by specifying $V \subset Y$ to be open $\Leftrightarrow f^{-1}(V)$ is open in X.*

Note that the quotient topology on Y is the largest topology on Y which makes fa continuous map.

Definition 1.2.2. *Let X be a topological space and $\sim$ be an equivalence relation on X. Let $Y = X/\sim$ be the set of equivalence classes and $\pi : x \to Y$ be the canonical map taking $x \in X$ to its equivalence class $[x] \in X/\sim$. Then Y with the topology induced by π is called the quotient space of X.*

If $X \subset \mathbb{T}$ and L is a lamination on X, then we shall denote the set of equivalence classes by X/L.

We now define 3 types of laminations that will play an important role in this paper.

Definition 1.2.3. *A closed, flat lamination L on $\mathbb{T}$ is called a totally disconnected lamination if $\overline{mult(L)}/L$ is a totally disconnected set.*

See Fig. 1.1 for examples of totally disconnected laminations. In the left-hand side figures, the straight lines join equivalent points of the lamination. The right-hand side figures represent the quotient space, $\mathbb{D}^*/L$, which can be thought of as being obtained by collapsing each convex hull of an equivalence class to a single point. The mapping $\pi(z)$ in the figures is the quotient map corresponding to L.

A perfect, nowhere dense set is called a Cantor set.

Definition 1.2.4. *Let $C \subset \mathbb{T}$ be a Cantor set and $\{I_j\}_{j=1}^{\infty}$ be the set of complementary intervals. Let $I_j = (a_j, b_j)$. Then L is a Cantor-type lamination if*

$$(a, b) \in L \text{ iff } a = a_i, b = b_i, \text{ for some } i \in \mathbb{N},$$

It is clear that a Cantor-type lamination is closed and flat. See Fig.1.2 for an example of a Cantor-type lamination. The arcs in the left hand side figure join the

Figure 1.1: Totally Disconnected Laminations.

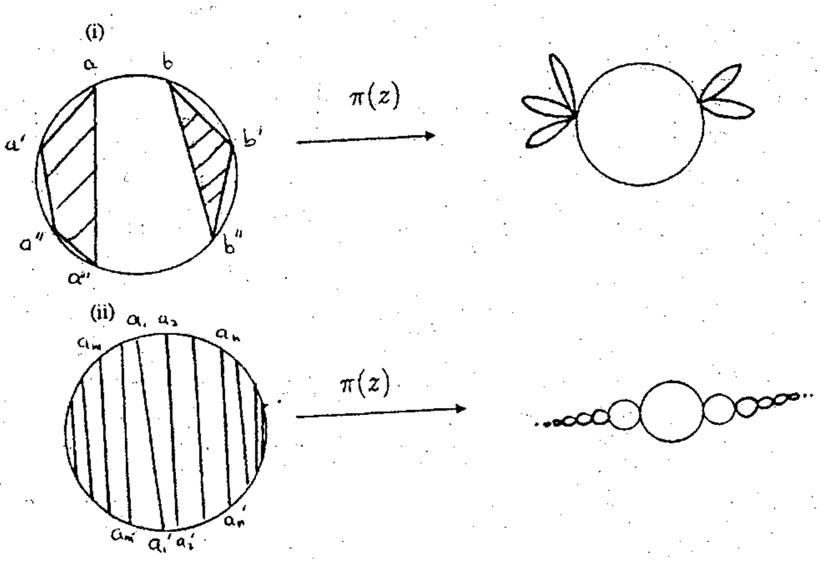
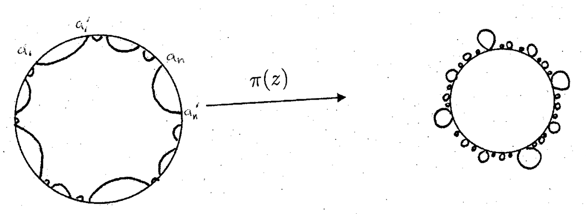


Figure 1.2: Cantor-Type Laminations



equivalent points of L, which are the end points of the complementary intervals of C labelled as $(a_i, a_{i'})$. The right hand figure represents the quotient space $\mathbb{D}^*/L$.

We now define the completion of a subset of $\mathbb{T} \times \mathbb{T}$.

Definition 1.2.5. *Let L be any non empty subset of $\mathbb{T} \times \mathbb{T}$. Then the completion of L is the intersection of all the closed, flat laminations that contain L.*

We will denote the completion of L by L^c.

Claim. *L^c is the minimal closed, flat lamination containing L.*

Proof. It suffices to show that L^c is a closed, flat lamination. Let $L^c = \cap_{\lambda \in \Lambda} L_\lambda$, where L_λ are closed, flat laminations containing L. We first check that L^c is a lamination: It is trivial to check that L^c is reflexive and symmetric. We check that L^c is transitive:

$$\text{Assume } (a,b),(b,c) \in L^c$$

$$\Rightarrow (a,b),(b,c) \in L_\lambda, \forall \lambda \in \Lambda$$

$$\Rightarrow (a,c) \in L_\lambda, \forall \lambda \in \Lambda$$

$$\Rightarrow (a,c) \in L^c.$$

Hence L^c is a lamination. L^c is a closed set in $\mathbb{T} \times \mathbb{T}$ since L_λ is a closed set in $\mathbb{T} \times \mathbb{T}$ and the intersection of closed sets is closed.

We now show that L^c is a flat subset of $\mathbb{T} \times \mathbb{T}$.

Let $a_1 \sim a_2, b_1 \sim b_2$, under L^c and a_1 and a_2 lie in different components of $\mathbb{T} \backslash (b_1, b_2)$. Then , $\forall \lambda \in \Lambda, a_1 \sim a_2, b_1 \sim b_2$ under L_λ. Since a_1 and a_2 lie in different components of $\mathbb{T} \backslash (b_1, b_2)$, it follows that $\forall \lambda \in \Lambda, a_1 \sim b_1$ under L_λ. Hence $a_1 \sim b_1$ under L^c. This proves the claim.

□

We prove the following lemma for the completion of a subset of $\mathbb{T} \times \mathbb{T}$:

Lemma 1.2.1. *Let L^c be the completion of $L \subset \mathbb{T} \times \mathbb{T}$. Then*

$$L^c = (\overline{L})^c \tag{1.2.1}$$

and

$$(1.2.2) \qquad \Rightarrow \overline{mult(L)} = \overline{mult(L^c)}$$

Proof. Since L^c is closed in $\mathbb{T} \times \mathbb{T}$ it follows that

$$L^c \supset \overline{L}$$

$$\Rightarrow (L^c)^c \supset (\overline{L})^c$$

$$\Rightarrow L^c \supset (\overline{L})^c$$

Since $L^c \subset (\overline{L})^c$ it follows that

$$L^c = (\overline{L})^c$$

which proves (1.2.1).

To prove (1.2.2) note that L^c is a subset of the closed, flat lamination L^* which identifies $\overline{mult(L)}$ to a single point. But

$$\overline{mult(L^*)} = \overline{mult(L)}$$

It follows that

$$\overline{mult(L^c)} \subset \overline{mult(L^*)} = \overline{mult(L)}.$$

Since

$$\overline{mult(L^c)} \supset \overline{mult(L)}$$

$$\Rightarrow \overline{mult(L)} = \overline{mult(L^c)}$$

which proves (1.2.2).

□

Definition 1.2.6. *Let* $L = (\cup_{\lambda \in \Lambda} L_\lambda)$ *be a flat subset of* $\mathbb{T} \times \mathbb{T}$, *where for* $\lambda \in \Lambda, L_\lambda$

is a Cantor-type lamination such that

$$(i) mult(L_\lambda)\text{'s are pairwise disjoint sets}$$

(ii) If $z \in mult(L)$*,and* $I = (z, z')$ *is the associated complementary interval with equivalent end points , then there are points, arbitrarily close to* z *on both sides which do not belong to the closure of any complementary interval of any* L_λ*, except* I*. And if* $z \in \overline{mult(L)} \backslash multL$*, then there are points, arbitrarily close to* z *on both sides which do not belong to the closure of any complementary interval of any* L_λ*.*

Then L *is called a Julia-type lamination.*

We need to check that L as defined above is a lamination. It is clear that L is reflexive and symmetric. Furthermore, L is transitive since it consists only of double points by condition (i) of the definition. It is easy to check that L is, in fact, a closed lamination. Let $a_n \in mult(L_{\lambda_n})$ and $a_n \sim b_n$. Assume that $(a_n, b_n) \to (a, b)$. Since both a_n and $b_n \in mult(L)$ we have by (ii) of **definition 1.2.6** that if $a_n \to a$ then $b_n \to a$. Hence $a = b$ and since $a \sim a$ it follows that a Julia-type lamination is closed. We shall say that L is generated by $\{L_\lambda\}$.

Remark: We need condition (ii) in the definition of Julia-type lamination to exclude the following trivial laminations: Let L identify all the points of a Cantor set of positive capacity. Then $L = (\cup_{\lambda\in\Lambda} L_\lambda)$, where each L_λ is a Cantor-type lamination and L_λ do not obey (ii). But L is not a conformal lamination. Condition (ii) is crucial in our proof of **Theorem 1.31** that every Julia-type lamination is conformal. Consider any closed flat lamination L that identifies the upper semi circle of $\mathbb{D}$ to the lower semi circle in a $1-1$ manner. Then L is generated by $\{L_\lambda\}$, where the L_λ are the degenerate Cantor sets, which consist of exactly 2 points. However, such a lamination does not obey condition (ii) and so is not a Julia-type lamination. In our definition of a Julia-type lamination we have not assumed that Λ is a countable set. We now show that (ii) of **Definition 1.2.6** implies that a Julia-type lamination has to be generated by a countable number of Cantor sets.

Claim. *Any Julia-type lamination is generated by a countable number of Cantor laminations.*

Proof. Assume $L = \cup_{\lambda\in\Lambda} L_\lambda$, where L_λ is a Cantor lamination. Let $S \subset \cup_{\lambda\in\Lambda} L_\lambda \subset \mathbb{T}\times\mathbb{T}$ be a countable dense subset of $\cup_{\lambda\in\Lambda} L_\lambda$ and let $S = \cup_{n=1}^{\infty}(a_n, b_n)$

Let $(a_n, b_n) \in L_{\lambda_n}$.

Then

$$\cup_{n=1}^{\infty} L_{\lambda_n} \supset \cup_{n=1}^{\infty}(a_n, b_n) = S$$

$$\Rightarrow \overline{\cup_{n=1}^{\infty} L_{\lambda_n}} \supset \overline{S}$$

But

$$\overline{S} \supset \cup_{\lambda\in\Lambda} L_\lambda$$

Hence

$$\Rightarrow \overline{\cup_{n=1}^{\infty} L_{\lambda_n}} \supset \cup_{\lambda\in\Lambda} L_\lambda$$

$$\Rightarrow (\overline{\cup_{n=1}^{\infty} L_{\lambda_n}})^c \supset (\cup_{\lambda\in\Lambda} L_\lambda) = L$$

But by **1.2.1**

$$(\overline{\cup_{n=1}^{\infty} L_{\lambda_n}})^c = (\cup_{n=1}^{\infty} L_{\lambda_n})^c$$

$$\Rightarrow (\cup_{n=1}^{\infty} L_{\lambda_n})^c \supset L$$

Since the opposite inclusion is obvious, we have

$$\Rightarrow (\cup_{n=1}^{\infty} L_{\lambda_n})^c = L$$

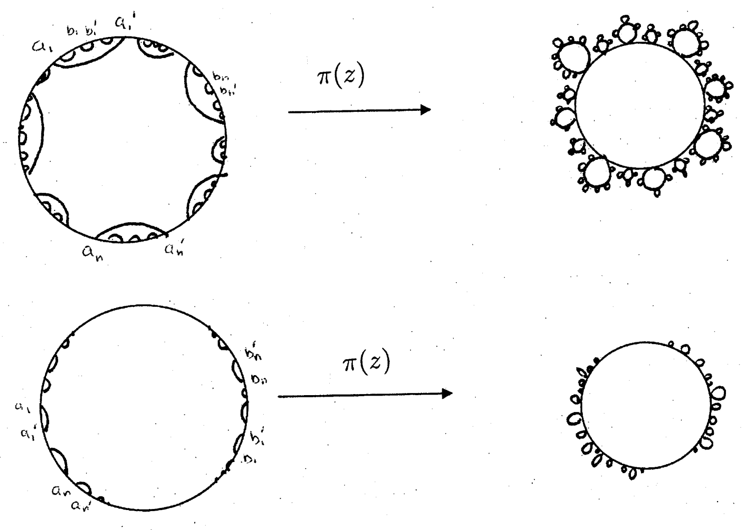
But as we showed earlier, $(\cup_{n=1}^{\infty} L_{\lambda_n})$ is a closed flat lamination. Hence

$$\Rightarrow (\cup_{n=1}^{\infty} L_{\lambda_n}) = L$$

□

Note that, in fact, if L is a Julia-type lamination then $L/\cup_{z\in\mathbb{T}}\{z,z\}$ is a discrete set and $mult(L)$ is a countable set which consists only of double points. We give two

Figure 1.3: Julia-type Laminations Generated by 2 Cantor Sets



examples of a Julia-type lamination generated by two Cantor sets in Fig. 1.3. The complementary intervals of the two Cantor sets are labelled as $(a_i, a_{i'})$ and $(b_i, b_{i'})$, respectively. The arcs in $\overline{\mathbb{D}}$ represent the closed convex hulls of the equivalence classes and $\pi(z)$ is the quotient map.

Fig. 1.4 gives the Julia-type lamination and the filled in Julia set for $P(z) = z^2 - 1$. Again, the arcs joining the end points of the Cantor sets represent the closed convex hulls of the equivalence classes and $\pi(z)$ is the quotient map.

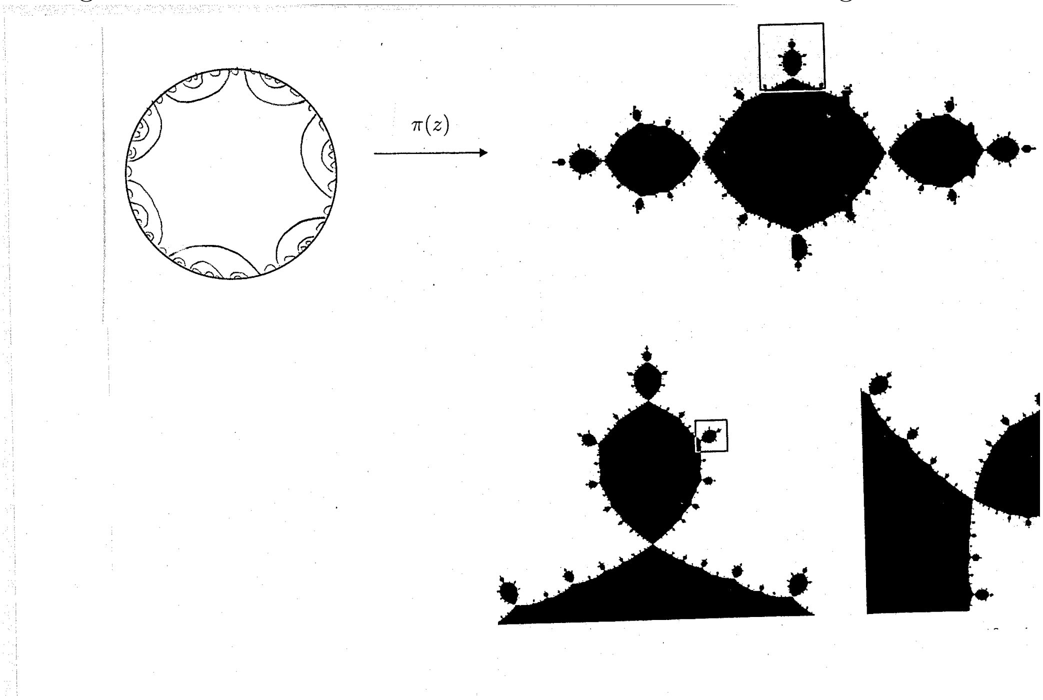
Using these definitions we can define a sublamination, L' of L, to be totally disconnected, Cantor-type, or Julia-type.

1.3 Main Results

In this paper we shall investigate the conditions under which a lamination on $\mathbb{T}$ is conformal. This problem has also been investigated by Leung ([**15**]). The main result which we will prove in **Chapter 4**, is as follows:

Theorem 1.3.1. *Let L be any Julia-type lamination on $\mathbb{T}$. Then L is a conformal lamination.*

Figure 1.4: The Filled in Julia Set of P_{-1} and Several Magnifications



To prove this theorem, we will introduce a new way to measure the oscillation of a Jordan curve which we will call the K-oscillation of the curve.

In **Chapter 5** we will use an approach employing potential theory to prove the following:

Theorem 1.3.2. *Let L be any totally disconnected lamination on $\mathbb{T}$, such that the closure of the set of multiple points, $\overline{mult(L)}$, is of zero logarithmic capacity. Then L is a conformal lamination.*

The zero capacity condition is necessary since a set of positive capacity on $\mathbb{T}$ cannot be mapped to a single point by a mapping that is conformal on $\mathbb{D}$ and continuous on $\mathbb{T}$. In **Chapter 6** we shall prove **Theorem 6.0.1** that union of a totally disconnected and Julia-type laminations, which are disjoint in the sense that their intersection contains only points along the diagonal of $\mathbb{T} \times \mathbb{T}$, is also conformal. We also conjecture that any lamination such that the logarithmic capacity of the set of multiple points is zero is conformal. We describe a possible approach to proving this conjecture.

Chapter 2

Overview of the Welding and Lamination Problem

We first give a brief introduction to quasiconformal and quasisymmetric mappings, which play an important role in conformal weldings and laminations. We then define weldings and laminations and gives some examples.

2.1 Quasiconformal Mappings

Let $\mathbb{C}$ be the complex plane. A quasiconformal mapping, of a plane domain $\Omega \subset \mathbb{C}$, is a sense preserving homeomorphism, such that infinitesimal circles are mapped into infinitesimal ellipses whose ratio of axis is uniformly bounded by a constant. A geometric definition can be given in terms of the module of a quadrilateral. A quadrilateral consists of a Jordan domain, Q and a sequence of boundary points, z_1, z_2, z_3, z_4, called the vertices of Q. By the Riemann Mapping Theorem and the Schwartz-Christoffel transformation there is a conformal mapping of $Q(z_1, z_2, z_3, z_4)$ onto a rectangle $R(w_1, w_2, w_3, w_4)$, where the z_i are mapped onto $w_i, i = 1, 2, 3, 4$. This rectangle is canonical, in the sense that a conformal mapping between any two such rectangles is a similarity transform. Hence all the canonical rectangles of a given $Q(z_1, z_2, z_3, z_4)$ have the same ratio of sides $\frac{a}{b} := M(Q)$, where a denotes the length of the side between w_1 and w_2, and b denotes the length of the side between w_2 and w_3. The conformally invariant number, $M(Q)$ is called the Module of the quadrilateral(c.f.[**2**]).

Definition 2.1.1. *A sense-preserving homeomorphism* $\phi : \Omega \to \mathbb{C}$ *is called a* K *quasi-conformal mapping if for any quadrilateral* $Q(z_1, z_2, z_3, z_4)$*, whose closure is contained in* Ω*,*

$$\sup_Q \frac{M(\phi(Q)}{M(Q)} \leq K < \infty.$$

The number $\frac{M(\phi(Q)}{M(Q)}$ is often called the dilatation of Q under $\phi(z)$. Since the dilatations of $Q(z_1, z_2, z_3, z_4)$ and $Q(z_2, z_3, z_4, z_1)$ are reciprocal numbers, it follows that $K \geq 1$.

The analytic formulation of this definition is as follows:

Definition 2.1.2. *A sense preserving homeomorphism* $\phi : \Omega \to \mathbb{C}$ *is called a* K *quasiconformal mapping if*

1)$\phi(x + iy)$ *is absolutely continuous in* x *for almost all* y *and in* y *for almost all* x *and*

(2) The partial derivatives are locally square integrable and satisfy the Beltrami equation

$$\frac{\partial \phi}{\partial \overline{z}} = \mu(z) \frac{\partial \phi}{\partial z} \textit{for almost all } z \in \Omega,$$

where $\mu(z)$ *is a measurable function with*

$$|\mu(z)| \leq \frac{K-1}{K+1} < 1$$

The complex number $\mu(z)$ is often called the complex dilatation of $\phi(z)$. The following existence-uniqueness theorem, proved in [**15**], plays a fundamental role in the theory of conformal weldings and laminations :

Theorem 2.1.1. *Given an arbitrary measurable complex function* $\mu(z)$ *on a domain* $\Omega \subset \mathbb{C}$*, with* $\sup |\mu(z)| < 1$*, there exists a unique (upto a conformal mapping) quasi-conformal mapping* $\phi : \Omega \to \mathbb{C}$ *with complex dilatation* $\mu(z)$ *for almost all* $z \in \Omega$*.*

2.2 Quasisymmetric Maps

Alfhors and Beurling were the first to describe the boundary behavior of quasiconformal maps in terms of a property called quasisymmetry, (see [**4**]).

Definition 2.2.1. *A sense preserving homeomorphism $\phi : \mathbb{T} \to \mathbb{C}$ is called a quasisymmetric map if there exists a constant M, such that for $z_1, z_2, z_3 \in \mathbb{T}$,*

$$|z_1 - z_2| = |z_2 - z_3| \Rightarrow |\phi(z_1) - \phi(z_2)| \leq M|\phi(z_2) - \phi(z_3)|.$$

A formally stronger but equivalent definition is given by Väisälä (see [**29**], [**30**]):

Definition 2.2.2. *The map $\phi : \mathbb{T} \to \mathbb{C}$ is called a quasisymmetric map if it is injective and if there exists a strictly increasing function $\lambda(x), (0 \leq x < \infty)$with $\lambda(0) = 0$ such that:*

$$\frac{|\phi(z_1) - \phi(z_2)|}{|\phi(z_2) - \phi(z_3)|} \leq \lambda \frac{|z_1 - z_2|}{|z_2 - z_3|}, \forall z_1, z_2, z_3 \in \mathbb{T}.$$

Alfhors also introduced the concept of a quasicircle in $\mathbb{C}$ which is defined as a Jordan curve J such that

$$\text{diam } J(a, b) \leq M|a - b|, \forall a, b \in J$$

where $J(a, b)$ is the smaller arc of J with endpoints a and b(c.f.[**9**]). The inner domain of a quasicircle is called a quasidisc.

The following theorem, proved in [**25**], describes the relation between quasiconformal and quasisymmetric maps:

Theorem 2.2.1. *Let J be a Jordan domain in $\mathbb{C}$ and let f map $\mathbb{D}$ conformally onto the inner domain of J. Then the following conditions are equivalent:*

(a) J is a quasicircle;

(b) f is quasisymmetric on $\mathbb{T}$;

(c) f has a quasiconformal extension to $\mathbb{C}$;

(d) there is a quasiconformal map of $\mathbb{C}$ onto $\mathbb{C}$ that maps $\mathbb{T}$ onto J.

2.3 The Welding Problem

2.3.1 Conformal Weldings

Two plane domains or bordered Riemann surfaces can be conformally welded (or sown) together into a single Riemann surface by an identification of two boundary arcs, (c.f.[**1**]). More formally, let $\overline{W_1}$ and $\overline{W_2}$ be two bordered Riemann surfaces with J_1 and J_2 two respective boundary arcs. Let $\alpha(J_1) = J_2$, where α is a homeomorphism. Then $\overline{W_1}$ and $\overline{W_2}$ are said to be welded together if the topological sum, $\overline{W_1} \cup_\alpha \overline{W_2}$, where x is identified with $\alpha(x)$ for $x \in J_1$, can be given a conformal structure which is compatible with the original structures of $\overline{W_1}$ and $\overline{W_2}$. The homeomorphism α is then called a welding homeomorphism.

If α is analytic and orientation-reversing, then it is certainly a welding homeomorphism. If the welding takes place along the unit circle (or the real axis) then analyticity can be replaced by weaker conditions, (see [**22**],[**23**],[**14**],[**24**],[**6**]). The problem, in this case can be restated as follows: Let $\mathbb{D}^*$ be the exterior unit disc. A homeomorphism, $\alpha : \mathbb{T} \to \mathbb{T}$, is a welding homeomorphism if there exists a Jordan domain J (with J^* exterior) and conformal mappings:

$$\phi : \mathbb{D} \to J$$

and

$$\phi^* : \mathbb{D}^* \to J^*$$

such that

$$\phi \circ \alpha = \phi^*.$$

Note that the definition makes sense, since if the welding solutions ϕ and ϕ^* exist then they are well defined and injective on $\mathbb{T}$ by Caratheodory's theorem.

A straightforward argument, (c.f. [**22**]), based on the existence and uniqueness theorem for quasiconformal mappings with a given complex dilatation, shows that it is sufficient to consider quasiconformal welding solutions instead of conformal solutions.

Besides the existence of welding solutions, the other issue is of the uniqueness of these solutions. This is equivalent to the conformal removability of the Jordan curve, ∂J.

The case where the homeomorphism α is a quasisymmetric function was completely solved by Pfluger and Oikawa, (see [**15**],[**22**], [**24**]).

Theorem 2.3.1. *Let $\alpha : \mathbb{T} \to \mathbb{T}$ be a sense preserving homeomorphism. The α is a welding homeomorphism with a unique quasidisc J iff α is quasisymmetric.*

Using a generalized version of the existence-uniqueness theorem for quasiconformal mappings, Lehto [**14**] has shown that a weakened quasisymmetry condition is sufficient for a welding along the real axis. Lehto and Virtanen [**15**], Pfluger [**24**], and David [**6**] prove conformal weldings for other classes of homeomorphisms. Oikawa [**22**] considers the case where α is quasisymmetric except at isolated singular points. He shows that in this case the problem reduces to the Type problem of Riemann surfaces. In particular, welding is possible iff the resulting Riemann surface $R := \overline{\mathbb{H}^+}\backslash\{0\} \cup_\alpha \overline{\mathbb{H}^-}\backslash\{0\}$ is a parabolic surface, where H^+ and H^- are the upper and lower half planes respectively.

This criteria can be used to give examples of homeomorphisms that do not admit a welding solution.

Example 1: Let k and m be positive real numbers, with $k \neq m$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a homeomorphism defined as follows:

$$\alpha(t) = t^k, \forall t \geq 0$$

and

$$\alpha(t) = -|t|^m, \forall t \leq 0$$

Using extremal length methods, Oikawa [**22**] has shown that in this case the resulting Riemann surface, R is hyperbolic and hence a welding solution does nor exist.

Example 2: Let

$$g_1 : \{x + iy : x > 0, y < sin\frac{1}{x}\} \to \{x + iy : x > 0, y < 0\}$$

and

$$g_2 : \{x + iy : x > 0, y > sin\frac{1}{x}\} \to \{x + iy : x > 0, y > 0\}$$

be conformal. Then, Vainio [**28**] has shown that

$$\alpha(x) = \begin{cases} g_2 \circ g_1^{-1}(x) & x > 0 \\ 0 & x = 0 \\ -g_2 \circ g_1^{-1}(-x) & x < 0 \end{cases}$$

is a homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$. However, the resulting Riemann surface is again hyperbolic, and hence does not admit a welding solution.

2.3.2 Caratheodory Weldings

Example 1 and **2** can be considered as examples of Generalized or Caratheodory weldings. Generalized weldings were introduced by Hamilton (see [**10**],[**11**]). A homeomorphism $\alpha : \mathbb{T} \to \mathbb{T}$ is called regular if for every $E \subset \mathbb{T}$ with dim $E = 0$, $m(\alpha(E)) = m(\alpha^{-1}(E)) = 0$, where m is the Hausdorff measure. Hamilton proves the following theorem for regular homeomorphisms (see [**11**]):

Theorem 2.3.2. *Let $\alpha : \mathbb{T} \to \mathbb{T}$ be a regular homeomorphism. The there exist conformal mappings ϕ and ϕ^* on $\mathbb{D}$ and $\mathbb{D}^*$ respectively, such that,*

$$(i) \phi(\mathbb{D}) \cap \phi^*(\mathbb{D}^*) = \emptyset$$

$$(ii) \phi \circ \alpha(z) = \phi^*(z)$$

for all $z \in \mathbb{T} \backslash E$ where $m(E) = 0$.

2.3.3 Kleinian Groups

An important source of examples of weldings comes from the theory of simultaneous uniformization of Kleinian Groups. Two Riemann surfaces R and R^* are said to be conjugate if there is an orientation reversing homeomorphism from R to R^*. Bers [**5**] proved that if R and R^* are compact hyperbolic Riemann surfaces then there is a Kleinian group Γ, simultaneously uniformizing R and R^*. In this case the conjugating homeomorphism is quasiconformal, and hence lifts to a welding function $\alpha : \mathbb{T} \to \mathbb{T}$. The welding solutions are

$$\phi : \mathbb{D} \to W$$

and

$$\phi^* : \mathbb{D} \to W^*$$

where W and W^* are Γ invariant components, so that R is conformally equivalent to W/Γ and R^* is conformally equivalent to W^*/Γ. The limit set $L(\Gamma)$ is the Jordan curve separating W and W^*. Hamilton [**12**] proves a generalized welding version of this theorem for conjugate Riemann surfaces of the first kind.

2.4 Laminations

2.4.1 Topological Laminations

Thurston [**26**] gives a simple criterion for a continuous lamination to be flat:

Theorem 2.4.1. *Let L be a continuous lamination on $\mathbb{T}$. Then L is flat iff the convex hulls of the equivalence classes of L are disjoint.*

It is simple to show that if a lamination L is induced by a continuous mapping $\phi : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$ which is injective in $\mathbb{D}$, then L is continuous and flat. The converse is also true by a theorem of Moore (see [**17**],[**18**]). Hence we have the following equivalence:

Theorem 2.4.2. *Let L be a lamination on $\mathbb{T}$. Then L is induced by a continuous mapping $\phi : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$ which is injective in $\mathbb{D}$, iff L is flat and continuous.*

Moore's Triod theorem (see [**19**],[**25**]) also restricts the cardinality of the equivalence classes of L with more than 2 multiple points to be countable.

Theorem 2.4.3. *Let f be a homeomorphism of $\mathbb{T}$ into $\mathbb{C}$. Then there are atmost countably many points $a \in \mathbb{C}$ such that*

$$f(r\zeta_j) \to a, \ \text{as } r \to 1^-(j = 1, 2, 3)$$

for three distinct points $\zeta_1, \zeta_2, \zeta_3$ on $\mathbb{T}$.

2.4.2 Conformal Laminations

A lamination L on $\mathbb{T}$ that is induced by ϕ which is conformal on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ is called a conformal lamination. There are some simple analogies between a conformal lamination and conformal weldings. The proof of the following result is a simple argument based on the existence and uniqueness theorem for quasiconformal maps with a prescribed complex dilatation, completely analogous to the welding case.

Theorem 2.4.4. *If L is a lamination on $\mathbb{T}$ induced by a continuous mapping ϕ : $\overline{\mathbb{D}} \to \hat{\mathbb{C}}$ which is quasiconformal in $\mathbb{D}$, then L is a conformal lamination.*

The following corollary shows that conformal laminations are invariant under quasisymmetric transformations.

Corollary 2.4.1. *If L is a conformal lamination on $\mathbb{T}$induced by ϕ and $\alpha : \mathbb{T} \to \mathbb{T}$ is quasisymmetric, then the lamination L' is also conformal, where L' is defined as:*

$$x \sim y \Leftrightarrow \phi \circ \alpha(x) = \phi \circ \alpha(y).$$

Proof. Since $\alpha : \mathbb{T} \to \mathbb{T}$ is quasisymmetric, by **Theorem 1.2.1**, there exists a quasiconformal extension of α:

$$\phi_1 : \mathbb{C} \to \mathbb{C}$$

Since ϕ_1 maps $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$, we have that the continuous mapping

$$\phi \circ \phi_1|_{\overline{\mathbb{D}}} \to \hat{\mathbb{C}}$$

is quasiconformal in $\mathbb{D}$ and induces the lamination L'. □

2.4.3 Laminations Induced by Welding Homeomorphisms

We shall denote the upper (lower) half unit disc by $\mathbb{D}^+(\mathbb{D}^-)$, and the upper (lower) semi circle by $I^+(I^-)$.

A homeomorphism of I^+ to I^-, which fixes 1 and -1, induces a lamination L on $\mathbb{T}$, where every point on I^+ is identified with its image in I^-. There is a simple equivalence between the conformality of this lamination and the welding properties of the homeomorphism.

Theorem 2.4.5. *Let $\psi : I^+ \to I^-$ be a homeomorphism that fixes the points* 1 *and* -1*. Then the lamination L induced by ψ on $\mathbb{T}$ is conformal iff $\alpha : \partial\mathbb{D}^+ \to \partial\mathbb{D}^-$ is a conformal welding homeomorphism, where $\alpha : \partial\mathbb{D}^+ \to \partial\mathbb{D}^-$ is defined as follows:*

$$\alpha(z) = \begin{cases} \psi(z) & \forall z \in I^+ \\ z & \forall z \in [1, -1] \end{cases}$$

Proof. Assume α is a conformal welding homeomorphism. Let J be a Jordan domain and let

$$\phi_1 : \mathbb{D}^+ \to J$$

and

$$\phi_2 : \mathbb{D}^- \to J^*$$

be a conformal welding solution. Hence,

$$\phi_2 \circ \alpha(z) = \phi_1(z), \forall z \in \partial\mathbb{D}^+.$$

But

$$\alpha(z) = z, \forall z \in [1, -1]$$

Hence

$$\phi_2(z) = \phi_1(z), \forall z \in [1, -1] \tag{2.4.1}$$

Define

$$\phi(z) = \begin{cases} \phi_1(z) & \forall z \in \overline{\mathbb{D}^+} \\ \phi_2(z) & \forall z \in \overline{\mathbb{D}^-} \end{cases}$$

By (2.4.1) $\phi(z)$ is well defined and a homeomorphism on $\overline{\mathbb{D}}$. Also $\phi(z)$ is conformal in $\mathbb{D}^+$ and $\mathbb{D}^-$ Since the segment $[1, -1]$ is conformally removable, $\phi(z)$ is conformal in $\mathbb{D}$ and induces the lamination L.

Now assume that L is induced by a continuous mapping $\phi : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$, which is conformal in $\mathbb{D}$. Then $J := \phi(\mathbb{D}^+)$ is a Jordan domain Define $\phi_1 : \mathbb{D}^+ \to \hat{\mathbb{C}}$ as

$$\phi_1(z) = \phi(z)|_{\mathbb{D}^+}$$

And define $\phi_2 : \mathbb{D}^- \to \hat{\mathbb{C}}$ as

$$\phi_2(z) = \phi(z)|_{\mathbb{D}^-}$$

Hence

$$\phi_1(\mathbb{D}^+) = J$$

and

$$\phi_2 \circ \alpha(z) = \phi_1(z), \forall z \in \partial\mathbb{D}^+$$

are a conformal welding solution.

$\square$

2.4.4 Complex Dynamics

Laminations on the circle play a fundamental role in the study of dynamics of complex polynomials. Douady and Hubbard [**8**] were the first to introduce the idea of external rays to study complex dynamics. They showed that each rational external ray of a polynomial, f, with a connected Julia set, has a well-defined limit as it approaches the Julia set, (see [**17**]). Using the Riemann Mapping theorem this gives rise to a lamination on $\mathbb{T}$ where any multiple point of the lamination has a rational argument. This lamination is not necessarily closed. However, if the Julia set is locally connected then every external rays 'lands 'and the resulting lamination is closed. **Example 1:** Consider the filled in Julia set of the polynomial $f : z \to z^2 + e^{2\pi it}z$, with $t = \frac{(\sqrt{5}-1)}{2}$. Then the lamination induced by this set on $\mathbb{T}$ is continuous and flat, and the Julia set is locally connected.

Example 2: Let $\Gamma \subset \mathbb{C}$ be the quadruple comb: $\{[-1,1]\times 0\} \cup \{A \times [-1,1]\}$ where A contains $\frac{1}{n}$ and $\frac{-1}{n}, n \in \mathbb{N}$ and 0. This set is compact and simply connected but not locally connected. Since every external ray of this set 'lands, 'it defines a lamination on $\mathbb{T}$. However, this lamination is not continuous. For instance, the external rays with argument $\frac{1}{4}$ and $\frac{3}{4}$ land on the points $(0,1)$ and $(0,-1)$, respectively. But there exists a sequence ϵ_n with $\epsilon_n \to 0$, such that the external rays with arguments $\frac{1}{4} - \epsilon_n$ and $\frac{3}{4} + \epsilon_n$ land on the same point of the x-axis of Γ. Hence we get a sequence of points x_n and y_n such that, $x_n \sim y_n$ and $x_n \to x, y_n \to y$, but $x \not\sim y$.

Chapter 3

Topology of Laminations

In this chapter we describe some topological properties of Julia-type laminations and totally disconnected laminations. We show that Cantor-type laminations and totally disconnected laminations can be thought of as the two topological 'extremes' of laminations.

3.1 Topology of Cantor-Type Laminations

Lemma 3.1.1. *If L is a Cantor-type lamination on $\mathbb{T}$, then $\overline{mult(L)}/L$ is homeomorphic to $\mathbb{T}$.*

Proof. Consider the usual Cantor function, $f : \mathbb{T} \to \mathbb{T}$, which maps the closure of every complementary interval with equivalent end points of $\overline{mult(L)}$, onto a distinct point of $\mathbb{T}$. Clearly, by a slight variation of this Cantor function, there exists a continuous function $f' : \mathbb{T} \to \mathbb{C}$, which maps the closure of every complementary interval of $\overline{mult(L)}$ to a distinct Jordan curve with an endpoint on $\mathbb{T}$. See Fig. 1.2. This is the required quotient map. □

There is a partial converse of this lemma, which we shall not use in this paper.

Lemma 3.1.2. *If L is any closed lamination on $\mathbb{T}$, such that $\overline{mult(L)}/L$ is homeomorphic to $\mathbb{T}$, then L contains a Cantor-type sublamination.*

Proof. Consider any equivalence class E of L. Since L is closed E is a closed set of $\mathbb{T}$. Let $\{I_j\}_{j=1}^{\infty}$ be the set of complementary intervals of E in $\mathbb{T}$. Note that there is only one complementary interval, $I_k \in \{I_j\}_{j=1}^{\infty}$, which contains points of $mult(L)$. If more than one complementary interval contained points of $mult(L)$, then the image of these points would either lie in different components of $\overline{mult(L)}/L$ or E/L would be a cut point of $\overline{mult(L)}/L$. Let a_k and b_k be the end points of I_k. Then the closure of the set of end points of all such I_k, constitutes a Cantor-type lamination. □

3.2 Topology of Totally Disconnected Laminations

Consider a Cantor set $C \subset \mathbb{T}$. Define a lamination L which identifies all the points of C. Then since $\overline{mult(L)}/L$ is a singleton, L is a totally disconnected lamination. We will show in **Chapter 5** that L is conformal iff logarithmic capacity of C is zero. However, note that L contains a Cantor-type sublamination L', which identifies the end points of the complementary intervals of C. We now show that this is the only situation in which a totally disconnected lamination can contain a Cantor-type sublamination.

Lemma 3.2.1. *If L is a totally disconnected lamination on $\mathbb{T}$ and L' is a Cantor-type sublamination of L, then L identifies all the points of L'.*

Proof. Let L' be a Cantor-type sublamination of L . Let the continuous mapping $\phi_1 : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$, which is injective on $\mathbb{D}$, induce L', such that $\overline{M_{l'}}/L'$ is $\mathbb{T}$ or an arc of $\mathbb{T}$. Then $\overline{mult(L)}/L$ contains the continuous image of $\overline{M_{l'}}/L'$ under the quotient map. If $\overline{mult(L)}/L$ is totally disconnected it follows that the continuous image of $\overline{M_{l'}}/L'$ has to be a singleton under the quotient map.

□

There is a converse of this lemma.

Lemma 3.2.2. *If L is a closed, flat lamination on $\mathbb{T}$such that the set of multiple points is nowhere dense and any Cantor-type sublamination is identified to a single point, then L is a totally disconnected lamination.*

Proof. If L is not a totally disconnected lamination, then one of the components of $\overline{mult(L)}/L$ must contain more than one point. But $\overline{mult(L)}/L$ is a closed subset of $\mathbb{T}/L$. Hence every component of $\overline{mult(L)}/L$ is compact. So the component with more than one point contains a continuum. This means that the preimage of this component must contain a closed subset of $\overline{mult(L)}$ which is a perfect set, with the end points of the complementary intervals identified. Since $mult(L)$ is nowhere dense, L contains a Cantor-type sublamination which is not identified to a single point.

□

If L is a totally disconnected lamination, we show that the nonequivalent multiple points can be 'separated' by complementary intervals with nonequivalent end points of L. We shall need this separation property to prove the conformality of totally disconnected laminations of capacity zero, in **Chapter 5**. We now state this property precisely:

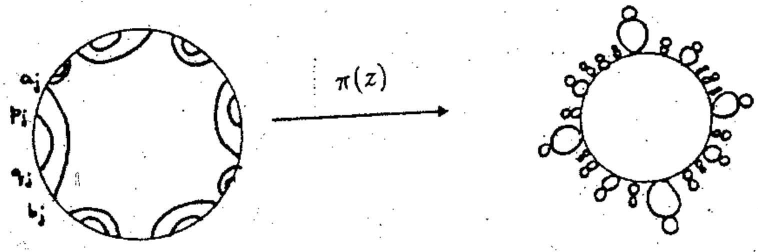
Theorem 3.2.1. *Let L be a totally disconnected lamination. Let $a, b \in mult(L), a \not\sim b$, and C_1, C_2 be the 2 components of $\mathbb{T}\backslash\{a, b\}$. Then C_1 contains a complementary interval with nonequivalent end points, $I_1 = (p_1, q_1)$, and C_2 contains a complementary interval with nonequivalent end points ,$I_2 = (p_2, q_2)$, and any equivalence class of L is either contained completely in J_1 or J_2, where J_1 and J_2 are the two components of $\mathbb{T}\backslash\{I_1 \cup I_2\}$.*

To prove the theorem we first show that a lamination L, where every complementary interval with nonequivalent end points is contained in an arc with equivalent end points, is not totally disconnected.

More precisely, we prove the following lemma:

Lemma 3.2.3. *Let L be a closed, flat lamination on $\mathbb{T}$. Let $a, b \in mult(L), a \not\sim b$. Let C be a component of $\mathbb{T}\backslash\{a, b\}$. Assume that if I is a complementary interval of $\overline{mult(L)}$ contained in C, then there exists an open arc $(e, e') \in C$, such that $e, e' \in mult(L)$ and $e \sim e'$. Then L is not totally disconnected.*

Figure 3.1: L is not totally disconnected.



Before proving the lemma, we give an example of such a lamination in Fig 3.1. Here $C \in \mathbb{T}$ is a Cantor set and $\{I_j\}_{j=1}^{\infty}$ are the complementary intervals. If $I_j = (a_j, b_j)$, then $\exists p_j, q_j \in I_j$ and the lamination L is defined as

$$a_j \sim b_j$$

and

$$p_j \sim q_j.$$

The complementary intervals with nonequivalent end points are of the form (a_j, p_j) or (q_j, b_j) and they are both contained in the arc (a_j, b_j), which has equivalent end points.

We now prove **Lemma 3.2.3.**

Proof. Let $\{I_j\}_{j=1}^{\infty}$, be the complementary intervals of $\overline{mult(L)}$ contained in C. For each j let A_j be the maximal arc that contains I_j and has equivalent end points. By the flatness condition any two such arcs A_k and A_l are either disjoint or equal. Let $A = \cup_{j=1}^{\infty} A_j$. Since the end points of A_j are equivalent we have that $\gamma = (C \backslash A)/L$ is a connected set that contains more than one point. Since $\gamma \subset \mathbb{T}/L$ we have that $\mathbb{T}/L$ is not totally disconnected.

□

Note: It follows that if L is a totally disconnected lamination, then if $a, b \in mult(L), a \not\sim b$ and for $i = 1, 2$ C_i are the two components of $\mathbb{T} \backslash \{a, b\}$, then there exist complementary intervals with nonequivalent end points $I_i, I_i \subset C_i$, such that for any open arc $(e, f) \in C_i$ containing I_i, with $e, f \in mult(L)$, $e \not\sim f$.

We now prove **Theorem 3.2.1**.

Proof. By considering the disjoint, closed convex hulls of a and b we can log assume that $a = -1, b = 1$ are the equivalence classes of a and b. Let C_1 be the upper semi circle and C_2 be the lower semi circle, the two components of $\mathbb{T}\backslash\{a, b\}$. By the note following the definition of type 2 laminations $\exists I_1 = (p_1, q_1), p_1 \not\sim q_1$ such that for any open arc $(e, f) \in C_1$, containing I_1,with $e, f \in \overline{mult(L)}$, then $e \not\sim f$. So for any $e \in \overline{mult(L)} \cap [a, p_1]$, the equivalence class of $e \subset C_2 \cup [a, p_1]$ and for any $e \in \overline{mult(L)} \cap [q_1, b]$, the equivalence class of $e \subset C_2 \cup [q_1, b]$. We now consider the following 3 cases separately:

1) Assume for any $e \in \overline{mult(L)} \cap [a, p_1]$, the equivalence class of $e \subset [a, p_1]$ and for any $e \in \overline{mult(L)} \cap [q_1, b]$, the equivalence class of $e \subset [q_1, b]$. Then again by the note following the definition $\exists I_2 = (p_2, q_2), p_2 \not\sim q_2$ such that for any open arc $(e, f) \in C_2$,containing I_2, with $e, f \in \overline{mult(L)}$, then $e \not\sim f$. Hence, For any $e \in \overline{mult(L)} \cap [a, p_2]$, the equivalence class of $e \subset [a, p_2]$ and for any $e \in \overline{mult(L)} \cap [q_2, b]$, the equivalence class of $e \subset [q_2, b]$. So I_1 and I_2 are the required intervals.

2) Now assume $\exists e \in \overline{mult(L)} \cap [a, p_1]$, such that an equivalent point $e' \in C_2$. Pick the e closest to p_1 and let e' be the corresponding equivalent point closest to b. Assume for any $e \in \overline{mult(L)} \cap [q_1, b]$, the equivalence class of $e \subset [q_1, b]$. Then since $e' \not\sim b$, by the note again we have that $\exists I_2 = (p_2, q_2), p_2 \not\sim q_2$ such that for any open arc $(e, f) \in (e'b)$, containing I_2,with $e, f \in \overline{mult(L)}$, then $e \not\sim f$. And so I_1 and I_2 are the required intervals.

3) For the final case we assume $\exists e_1 \in \overline{mult(L)} \cap [a, p_1]$, such that an equivalent point $e_1' \in C_2$. Pick the e_1 closest to p_1 and let e_1' be the corresponding equivalent point closest to b. And assume $\exists e_2 \in \overline{mult(L)} \cap [q_1, b]$, such that an equivalent point $e_2' \in c_2$. Pick the e_2 closest to q_1 and let e_2' be the corresponding equivalent point closest to a. Then since $e_1' \not\sim e_2' \exists I_2 = (p_2, q_2), p_2 \not\sim q_2$ such that for any open arc $(e, f) \in (e_1', e_2')$, containing I_2, with $e, f \in \overline{mult(L)}$, then $e \not\sim f$. And so I_1 and I_2 are

the required intervals.

□

The converse of **Theorem 3.2.1** is also true.

Theorem 3.2.2. *Let L be a closed, flat lamination on $\mathbb{T}$. Let $a, b \in mult(L), a \not\sim b$, and C_1, C_2 be the 2 components of $\mathbb{T}\backslash\{a, b\}$. Assume that C_1 contains a complementary interval with nonequivalent end points $I_1 = (p_1, q_1)$, and C_2 contains a nonequivalent complementary interval $I_2 = (p_2, q_2)$, and any equivalence class of L is either contained completely in J_1 or J_2, where J_1 and J_2 are the two components of $\mathbb{T}\backslash\{I_1 \cup I_2\}$. Then L is a totally disconnected lamination.*

Proof. Let z_1 and $z_2 \in \overline{mult(L)}$ and $z_1 \not\sim z_2$. Let D_1 and D_2 be the two components of $\mathbb{T}\backslash\{z_1, z_2\}$ Then, by assumption ,$\exists I_1 \subset D_1, I_2 \subset D_2$, where I_1 and I_2 are complementary intervals with nonequivalent end points, and any equivalence class of L is either contained completely in J_1 or J_2, where J_1 and J_2 are the two components of $\mathbb{T}\backslash\{I_1 \cup I_2\}$. Let a and b be the mid point of I_1 and I_2, respectively. Let C_1 and C_2 be the two components of $\mathbb{T}\backslash\{a, b\}$. Then for $i = 1, 2$ the images of $C_i$under the quotient map are open disjoint sets of $\mathbb{T}/L$ containing the images of z_i, whose union is $\mathbb{T}/L$. Since any 2 distinct points of $\overline{mult(L)}/L$ lie in different components, $\overline{mult(L)}/L$ is a totally disconnected set.

□

Chapter 4

Julia-Type Laminations

In this section we shall prove **Theorem 1.3.1**, that a Julia-type lamination is conformal.

The proof involves defining a global measure of the oscillation of a Jordan arc, which we shall call the K-oscillation. The proof will proceed inductively where at each step i, we will define a conformal map ϕ_i that will join the end points of a single complementary interval. We will show that if, at end of the $(i-1)th$ step, the complementary intervals are K-oscillating, then the complementary intervals at the end of the ith step are $K+\epsilon$-oscillating, for any $\epsilon > 0$. Thus, the oscillations of the complementary intervals will be bounded by a constant less than 1 at every step of the induction. Furthermore, we will show that the distortion of ϕ_i is controlled by the diameter of the complementary intervals at the end of the $(i-1)th$ step, which in turn is controlled by the oscillation of the complementary intervals at the end of the $(i-2)th$ step. Thus, the induction will converge to the required function.

4.1 Oscillation of a Curve

Definition 4.1.1. *Let J be a Jordan arc in $\mathbb{C}$ with diameter d. We will say J is K-oscillating if there is point $z \in J$ such that if C_1 and C_2 are the 2 components of $J \backslash \{z\}$, then for $i = 1, 2$:*

$$diam\ C_i \leq Kd$$

We shall say z is a K point of J.

Figure 4.1: K-oscillating curves and K points

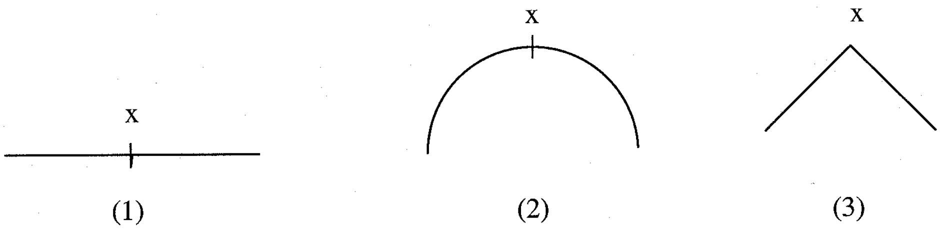
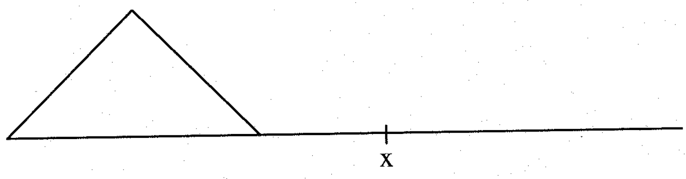


Figure 4.2: A curve that is not uniformly $\frac{1}{2}$ oscillating



The following observations follow directly form the definition:

1) For any Jordan arc $J, K \geq \frac{1}{2}$ and we can choose $K \leq 1$.

2) If J is K-oscillating, then J is K' oscillating $\forall K' \geq K$.

Examples (see fig 4.1):

1) straight line of length d is $\frac{1}{2}$ oscillating, with the mid point as the $\frac{1}{2}$-point.

2) semicircle of diameter d is $\frac{1}{\sqrt{2}}$ oscillating, with the mid point as the $\frac{1}{\sqrt{2}}$-point.

3) V-curve of diameter d, which consists of 2 straight line segments of length d meeting at an angle of 60 degrees is 1 oscillating, with the vertex as the 1-point.

We shall say J is uniformly K-oscillating if J and any subarc of J is K-oscillating. All the examples above are uniformly K-oscillating. Consider a V-curve of diameter d and vertex of 60 degrees, with a straight line segment of length $3d$ attached to one leg at an angle of 120 degrees. This arc is $\frac{1}{2}$ oscillating, with the $\frac{1}{2}$ point on the straight line segment at a distance of $2d$ from the end point. However it is not uniformly $\frac{1}{2}$ oscillating since the V-curve is not $\frac{1}{2}$ oscillating (see fig 4.2).

Claim. *Let J be a Jordan arc in $\hat{C}$ which is uniformly K-oscillating. Let h be a bilipchitz mapping of J onto J' such that $a|z_1 - z_2| \leq |h(z_1) - h(z_2)| \leq b|z_1 - z_2|, \forall z_1, z_2 \in J$. Then J' is uniformly $K\frac{b}{a}$-oscillating.*

Proof. Pick any Jordan arc $I' \subset J'$. Let $I = h^{-1}I'$. Since I is K-oscillating, let z be the Kpoint of I and Let C_1 and C_2 be the 2 components of $I\backslash\{z\}$. If $z' = h(z)$ we claim that z' is a $K\frac{b}{a}$ point for J'. Let C_1' and C_2' be the 2 components of $I'\backslash\{z'\}$. Let

$$d' := \text{diam } (I')$$

and

$$d := \text{diam } I$$

Then By the Lipchitz condition

$$a|z_1 - z_2| \leq |h(z_1) - h(z_2)|$$

we have that

$$d \leq \frac{d'}{a} \tag{4.1.1}$$

Then for $i = 1, 2$ by the Lipchitz condition $|h(z_1) - h(z_2)| \leq b|z_1 - z_2|$ and the fact that I is uniformly K-oscillating we have

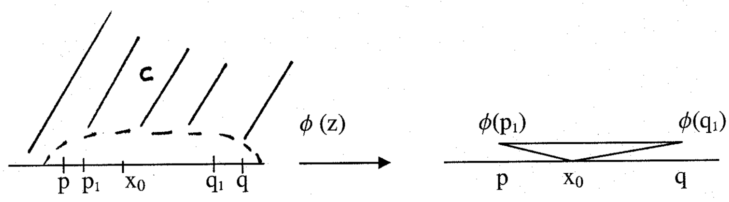
$$\text{diam } (C_i') \leq b \text{ diam } C_i \leq Kbd$$

By (4.1.1) we have

$$\text{diam } (C_i') \leq K\frac{b}{a}d'$$

□

Figure 4.3: Mapping a single interval on $\mathbb{R}$



4.2 Julia-Type Laminations on the Circle

In this section we will prove that a Julia-type lamination on $\mathbb{T}$ is conformal. The proof will involve a sequence of lemmas, where we shall first consider the case of a single complementary interval on the real axis (**Lemma 4.2.1**). We use the result to consider a single complementary interval on a Jordan curve (**Lemma 4.2.2**). In **Lemma 4.2.3** we will consider a single complementary interval on a curve $\gamma \simeq \mathbb{T}/L'$, where L' is a sublamination with a finite number of complementary intervals. Finally, we will use the result of **Lemma 4.2.3** inductively, to prove **Theorem 1.3.1.**

Definition 4.2.1. *: Let I be a Jordan arc or curve in $\hat{\mathbb{C}}$ which is the boundary of a domain Ω. We shall call the set of point $\{z : z \in \Omega,\ dist\ (z, I) < \epsilon\}$, an open ϵ neighborhood of I in Ω and denote it either by $N_\epsilon(I)$,or $B_\epsilon(I)$.*

Now we state and prove **Lemma 4.2.1** (see fig 4.3):

Lemma 4.2.1. *Let $\gamma = [p, q]$ be a closed segment $\subset \mathbb{R} \subset \overline{\mathbb{H}^+}$. Let $x_0 \in [p, q]$be given. Let $\epsilon^* > 0$ be given. Let C be any compact set $\subset \overline{\mathbb{H}^+}$disjoint from $[p, q]$. Then there exists a neighborhood $N([p, q])$of $[p, q]$ disjoint from C and there exists a conformal mapping $\phi : \mathbb{H}^+ \to \mathbb{H}^+$, continuous upto the boundary, and points $p_1, q_1 \in [p, q]$ with p_1, q_1 arbitrarily close to p, q respectively, such that*

$$(i)\phi(\infty) = \infty.$$

$$(ii)\phi(p) = \phi(q) = x_0$$

$$(iii)\mathbb{R} \subset \phi(\mathbb{R}) and\ \phi is\ injective\ on\ \mathbb{R}\backslash\{p, q\}$$

$$(iv)\|\phi(z) - z\| \leq \epsilon^*, \|\phi'(z) - z\| \leq \epsilon^*, \forall z \notin N([p, q]).$$

$$(v)\|\phi(z) - z\| \leq \epsilon^*, \forall z \in [p_1, q_1].$$

$$(vi)\Im\phi(z) \leq \epsilon^*, \forall z \in [p, q].$$

Proof. Consider the mapping:

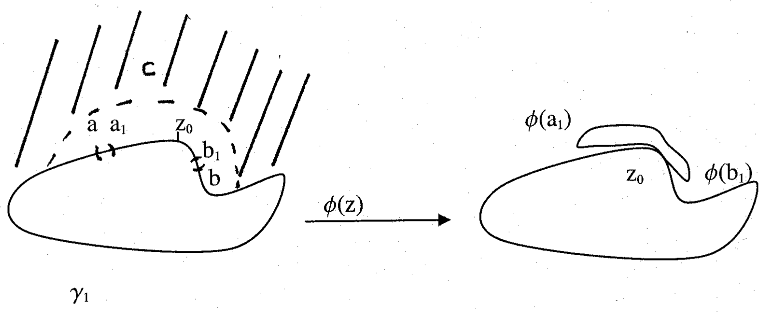
$$f_{x,\delta}(z) = \sqrt{(z-x)^2 - \delta^2} + x, x \in \mathbb{R}$$

which is conformal in $\mathbb{H}^+$and continuous upto the boundary. On the real axis, it maps the point x to $x + i\delta$ and the points $x \pm \delta$ to x. Let x_1 be the midpoint of $[p, q]$ and $\delta = \frac{|q-p|}{2}$. Let $\phi_1 = f_{x_1,\delta}(z)$. Let ϕ_2 be the map as described in **Cor 3.2** with $\gamma = \phi_1[p, q]$. Then $\phi_2 \circ \phi_1[p, q]$ is a Jordan curve with $\phi_2 \circ \phi_1(p) = \phi_2 \circ \phi_1(q) \in \mathbb{R}$ and $\phi_2 \circ \phi_1\mathbb{R}\backslash(p, q) = \mathbb{R}$. Let $\Omega = \phi_2 \circ \phi_1(\mathbb{H}^+\backslash(p, q))$.

Pick any points $z_1, z_2 \in \mathbb{H}^+$. Let γ_1 be a Jordan curve in $\mathbb{H}^+$, containing the points (z_1, z_2, x_0), such that γ_1 is contained in the triangle with vertices at z_1, z_2 and x_0. Let ϕ_3 be the Riemann map that maps Ω onto $\mathbb{H}^+\backslash\gamma_1$. Now define $\phi := \phi_3 \circ \phi_2 \circ \phi_1$. Note we can choose ϕ such that $\phi(p_1) = z_1$ and $\phi((q_1) = z_2$. So $\phi : \mathbb{H}^+ \to \mathbb{H}^+$, is conformal and continuous upto the boundary and satisfies $(i), (ii)$ and (iii). Since C is a closed set disjoint from $[p, q]$ there is an open neighborhood $N([p, q])$of $[p, q]$ disjoint from C. We can pick the 2 vertices z_1, z_2 of γ_1 to be in $N([p, q])$ and furthermore we can choose z_1 to be arbitrarily close to p and z_2 to be arbitrarily close to q.

We now show that we can choose ϕ to be uniformly close to the identity outside of $N[p, q]$. Let $N_1[p, q] \subset N[p, q]$ and $N_1[p, q] \cap \mathbb{R} = [p_0, q_0]$. Now we choose a sequence of curves γ_1^n, as above, which converge to the interval $[p, q]$. Let ϕ_n be the associated conformal maps. Note that we can reflect the conformal map $\phi_n|_{\mathbb{H}^+\backslash[p_0,q_0]}$ by reflection to $\phi_n(\hat{\mathbb{C}}\backslash[p_0, q_0])$. Since ϕ_n are a normal family, they contain a a subsequence that converges on compact sets of $\hat{\mathbb{C}}\backslash[p_0, q_0])$. Let f be the limit function. Note that as

Figure 4.4: Mapping a single interval on a Jordan curve



n gets larger the harmonic measure with respect to ∞ of the arc $(\phi(p), z_1)$ goes to zero. Hence for any $x, p' \in \mathbb{R}$ with $x < p' < p$ we have that picking n large enough, the harmonic measure of $[\phi_n(x), \phi_n(p')]$ is arbitrarily close to $|\phi_n(x) - \phi_n(p')|$. Also for any $\epsilon > 0$, there exists an n and p' with $p' < p$ such that $|\phi_n(p') - p'| < \epsilon$. Hence $\lim_{n\to\infty} \phi_n(x) = x$. Hence the limit function f has to be the identity function. This proves (iv) and and a similar argument proves (v). Furthermore we can choose $\Im(z_1) < \epsilon^*$ and $\Im(z_2) < \epsilon^*$ and so (vi) holds.

□

Note: In the proof of the lemma, ϕ maps the segment $[p, q]$ onto an 'almost degenerate' triangle γ_1, with vertices z_1, z_2 and x_0. We can choose γ_1 to be any Jordan curve as long as it contains the points z_1, z_2 and x_0 and γ_1 is contained in the triangle with vertices z_1, z_2 and x_0.

We now generalize the results of **Lemma 4.2.1** to a Jordan curve (see fig 4.4).

Lemma 4.2.2. *Let γ be a Jordan curve in $\hat{C}$, which does not contain the point ∞. Let Ω be the unbounded component of $\hat{C}\backslash\gamma$. Let γ_1 be a compact subarc of γ with a, b as the endpoints, which is uniformly K-oscillating. Let $d =$diam (γ_1) Let z_0 be a K point of γ_1. Let ϵ^* with $1 > \epsilon^* > 0$ be given. Let C be any compact set disjoint from γ_1. Then there exists a neighborhood $N(\gamma_1)$of γ_1 disjoint from C, and a conformal mapping continuous upto the boundary, $\phi : \Omega \to \hat{C}$, such that*

$$(i)\phi(a) = \phi(b) = z_0, \phi(\infty) = \infty$$

$$(ii)\phi is\ injective\ on\ \gamma\backslash\{a,b\}$$

$$(iii)\phi(\gamma\backslash\gamma_1)=\gamma$$

$$(iv)\phi(\gamma_1)\subset N(\gamma_1)$$

Further more there exist points $a_1, b_1 \in \gamma_1$ *such that if* C_1, C_2, C_3 *are the three components of* $\phi(\gamma_1)\backslash\{\phi(a_1),\phi(b_1),\phi(a)$*:*

$$(v)\|\phi(z)-z\|\leq d, \forall z\in N(\gamma_1)$$

$$(vi)\|\phi(z)-z\|\leq \epsilon^*, \forall z\in \overline{\Omega}\backslash N(\gamma_1).$$

$$(vii)(1-\epsilon^*)|z_1-z_2|\leq \|\phi(z_1)-\phi(z_2)\|\leq (1+\epsilon^*)|z_1-z_2|, \forall z_1,z_2\in \overline{\Omega}\backslash N(\gamma_1).$$

$$(viii) For\ i=1,2,3, C_i are\ uniformly\ K+\epsilon^* oscillating.$$

$$(ix) For\ i=1,2,3,\ for\ any\ subarc\ I\subset \phi^{-1}(C_i)\ diam\ \phi(I)\leq \max\{diam\ I, Kd\}+\epsilon^*.$$

Proof. Let $R:\Omega\to\mathbb{H}^+$ be the Riemann map that takes ∞ to ∞. Since γ is a Jordan curve R can be extended continuously and injectively to the boundary. Since both $R^{-1}$and $(R^{-1})'$ are continuous on the compact set $\overline{\mathbb{H}^+}$, they are both equicontinuous. Hence for any $\epsilon_1>0, \exists\delta>0$ such that if for any $x,y\in\overline{\mathbb{H}^+}, |x-y|<\delta$, then

$$|R^{-1}(x)-R^{-1}(y)|<\epsilon_1 \tag{4.2.1}$$

and

$$|(R^{-1})'(x)-(R^{-1})'(y)|<\epsilon_1 \tag{4.2.2}$$

Let $R(a)=p\in\mathbb{R}, R(b)=q\in\mathbb{R}, R(z_0)=x_0\in\mathbb{R}$. Let ϕ_1 be the map described in **Lemma 4.2.1**, with $\delta=\epsilon^*$. Pick $N([p,q])$ as described in **Lemma 4.2.1** such that $R^{-1}N_[p,q]\cap C=\emptyset$. Let $N(\gamma_1)=\subset R^{-1}N([p,q])$ with diameter $N(\gamma_1)<\epsilon^*$. Let $R^{-1}(p_1)=a_1$ and $R^{-1}(q_1)=b_1$. Now define $\phi=R^{-1}\circ\phi_1\circ R$.

It is clear that ϕ satisfies $(i)-(iv)$. (v) follows from (iii) and (iv) of **Lemma**

4.2.1, since the diam$N_{\gamma_1)} < \epsilon^*$. To see that (vi) holds note that $\forall z \notin N(\gamma_1)$,

$$\|\phi(z) - z\| = \|R^{-1} \circ \phi_1 \circ R(z) - z\| = \|R^{-1} \circ \phi_1(w) - R^{-1}(w)\| \tag{4.2.3}$$

But

$$\|\phi_1(w) - w\| \le \delta$$

by construction of ϕ_1. Hence by (4.2.1) and (4.2.3) it follows that

$$\|\phi(z) - z\| \le \epsilon_1^*.$$

Since this is true for any arbitrary $\epsilon_1^* > 0$, it follows that $\phi(z)$ can be chosen arbitrarily close to the identity in $\overline{\Omega} \backslash N(\gamma_1)$. This proves (vi). To show that (vii) holds, we replace R^{-1} by $(R^{-1})'$ in the above argument. Hence by (4.2.2) and (4.2.3) it follows that $\phi'(z)$ can be chosen to be arbitrarily close to 1 in $\overline{\Omega} \backslash N(\gamma_1)$. Hence (vii) holds.

We now show that $(viii)$ holds. We denote by C_1 the Jordan arc with endpoints $\phi(a)$ and $\phi(a_1)$, by C_2 the Jordan arc with endpoints $\phi(a_1)$ and $\phi(b_1)$, and by C_3 the Jordan arc with endpoints $\phi(b_1)$ and $\phi(a)$. Let $i = 1, 2, 3$. By the **Note** following **Lemma 4.2.1** we are free to choose the Jordan arcs $R(C_i)$, as long as they are contained in the triangle with vertices $\phi_1 \circ R(a)$, $\phi_1 \circ R(a_1)$ and $\phi_1 \circ R(b_1)$. In particular we can choose them such that C_i are uniformly $K + \epsilon^*$ oscillating. It remains to show that (ix) holds. Since diam $N(\gamma_1) < \epsilon^*$ and $\phi(\gamma_1) \subset N_{(\gamma_1)}$we have that the dist$(C_i, \gamma_1) < \epsilon^*$. Since z_0 is a K point of γ_1 it follows that

1) for $i = 1, 3$ diam $(C_i) \le Kd + \epsilon^*$ and

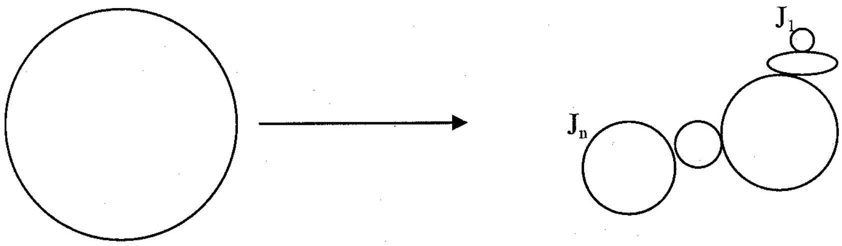
2) diam $(C_2) \le d + \epsilon^*$.

Hence (ix) holds.

□

We now describe the setup for the statement of **Lemma 4.2.3**. Let L be a Julia-type lamination on $\mathbb{T}$ that is generated by a countable number of Cantor sets $\{C_i\}_{i=1}^{\infty}$.

Figure 4.5: A finite number of Jordan curves



Let $\{I_j\}_{j=1}^{\infty}$, be the set of complementary intervals with equivalent end points of L. By the definition of a Julia-type lamination, for any two such complementary intervals, either one is strictly contained in the other, or their closures are disjoint. Now consider the sublamination L', such that the set of complementary intervals with equivalent end points of L' is finite, say $I_1^*, ...I_n^*$. Note that the curve $\Gamma = \mathbb{T}/L'$ is homeomorphic to $\cup_{i=1}^{n} J_i$, where each J_i is a Jordan curve and is the image under the quotient map of some $\overline{I_k}$. Furthermore for any r, s either $J_r \cap J_s = \{\emptyset\}$ or $J_m \cap J_n = \{z\}$. Also, since $\mathbb{T}/L$ is defined upto a homeomorphism we can assume that $\infty \notin \mathbb{T}/L'$. Then $\mathbb{T}/L'$ is the boundary of a domain containing the point ∞, (see fig. 4.5).

We now state and prove **Lemma 4.2.3**.

Lemma 4.2.3. *Let L be a Julia-type lamination on $\mathbb{T}$ generated by the Cantor sets $\{C_i\}_{i=1}^{\infty}$ and let $\{I_j\}_{i=1}^{\infty}$, be the set of complementary intervals with equivalent end points of $\{C_i\}_{i=1}^{\infty}$.. Let L', be a sublamination of L with complementary intervals with equivalent end points $\{I_k^*\}_{k=1}^{n}$.*

Let $\Gamma \simeq \mathbb{T}/L' = \cup_{i=1}^{n} J_i$, where each J_i is a Jordan curve. Let $I'_j \subset \Gamma$ be the image of I_j under the quotient map. Let Ω be the unbounded component of $\hat{\mathbb{C}} \backslash \Gamma$.

Assume that if $I_j \notin \{I_k^*\}_{k=1}^{n}$, then$\exists$ an open neighborhood $B(I_j')$ such that $\overline{B}(I_j') \cap \Gamma$ is uniformly K-oscillating, for $K < 1$.

Let $\hat{I} \in \{I_j\}_{i=1}^{\infty}$ and $\hat{I} \notin \{I_k^*\}_{k=1}^{n}$ be given. Then the image of $\hat{I}$ under the quotient map, $\hat{I}'$ is a Jordan arc with end points, say a and b, and $\hat{I}'$ is contained in a Jordan curve, say J'. Let d be the diameter of $\hat{I}'$. Let $1 > \epsilon > 0$ be given. Then there exists a conformal mapping:

$$\phi : \Omega \to \hat{\mathbb{C}}$$

which is continuous on Γ and:

$$(i)\phi \textit{ is injective on } \Gamma \backslash \{a, b\} \textit{and } \phi(a) = \phi(b).$$

$$(ii)\forall I \notin \{I_k^*\}_{k=1}^{n}, \exists \textit{ an open nhbd } B(\phi(I'))\textit{such that } B(\phi(I'))\cap\phi(\Gamma) \textit{is} K+\epsilon \textit{ uniformly oscillating.}$$

$$(iii)\forall I \notin \{I_k^*\}_{k=1}^{n}, \textit{diam } \phi(I') \leq \textit{max}\{\textit{diam}I', K\textit{diam } d \}.$$

$$(iv)\exists N(\hat{I}')\textit{such that,}\forall z \in \Gamma\cap N(\hat{I}'), \|\phi(z)-z\| \leq d+\epsilon \textit{ and } \forall z \in \Gamma\backslash N(\hat{I}'), \|\phi(z)-z\| \leq \epsilon.$$

$$(v)\phi(\Gamma) = \cup_{i=1}^{n+1} J_i', \textit{where each } J_i' \textit{is a Jordan curve, and } J_k' \cap J_l' = \{\emptyset\} \textit{or} \{z_{kl}\}.$$

Proof. First note that by the definition of a Julia-type lamination, no two complementary intervals with equivalent end points can have a common end point. Hence there exists a neighborhood, $B^*(\hat{I}')$ of $\hat{I}'$ such that

$$1) B^*(\hat{I}') \cap \Gamma = B^*(\hat{I}') \cap J'$$

$$2)\text{For any } I' \in \{I_j'\}, \text{ if } I' \cap B^*(\hat{I}') \neq \emptyset \text{ and } I' \cap \hat{I}' = \emptyset, \text{ then diam } (I') \leq \frac{\epsilon}{2}$$

$$3) B^*(\hat{I}') \subset B(\hat{I}')$$

Now pick ϕ as described in **Lemma 4.2.2** with

$$J' = \gamma \tag{4.2.1}$$

$$\hat{I}' = \gamma_1 \tag{4.2.2}$$

Furthermore pick and $N(\hat{I}'), a_1, b_1 \in \hat{I}'$ and $\epsilon^* > 0$ as defined in **Lemma 4.2.2** such that

(4.2.3) $$N(\hat{I}') \subset B^*(\hat{I}')$$

(4.2.4). $$\frac{1+\epsilon^*}{1-\epsilon^*}K < K+\epsilon \text{ and } \epsilon^* < \frac{\epsilon}{2}$$

(4.2.5) $a_1, b_1$are not in the closure of any complementary interval

(4.2.6) The points $\partial N(\hat{I}') \cap \Gamma$are not in the closure of any complementary interval

Observe that 4.2.5 and 4.2.6 are true by (ii) of definition 1.2.6 of a Julia-type lamination. Also note that the diam $N(\hat{I}') < \epsilon^* < \frac{\epsilon}{2}$. We now show that the statements $(i) - (v)$ hold for ϕ

(i) follows directly from (i) and (ii) of **Lemma 4.2.2**.

To prove (ii) first note that by **definition 1.2.6** and(4.2.6) the closure of any complementary interval, $\overline{I'}$, either contains $\hat{I}'$, or is contained in $\overline{N(\hat{I}')}$ or disjoint from it. If $\overline{I'}$, contains $\hat{I}'$ then $\phi(I') \subset (I')$. Hence (ii) is true. It remains to consider the following two cases:

Case 1:First consider the case of a complementary interval, I' such that the closure is disjoint from $\overline{N(\hat{I}')}$. Then, there is an open neighborhood $B(I')$ disjoint from $N(\hat{I}')$ such that $B(I') \cap \Gamma$ is uniformly K-oscillating. By (vii) of **Lemma 4.2.2**, we have that $\forall z_1, z_2 \in B(I')$,

$$(1-\epsilon^*)|z_1 - z_2| \leq \|\phi(z_1) - \phi(z_2)\| \leq (1+\epsilon^*)|z_1 - z_2|$$

Since $B(I') \cap \Gamma$ is uniformly K-oscillating, it follows by claim 4.1, that $\phi(B(I')) \cap \phi(\Gamma)$ is $\frac{1+\epsilon^*}{1+\epsilon^*}K$ uniformly oscillating. And by (4.3.4) it follows that $\phi(B(I')) \cap \phi(\Gamma)$ is $K+\epsilon$

uniformly oscillating. With $B(\phi(I')) := \phi(B(I'))$ the statement holds.

Case 2: Now consider the other case where $\overline{I}' \subset \overline{N(\hat{I}')}$. Then there exists an open neighborhood $B(I') \subset \overline{N(\hat{I}')}$. If $I' \not\subset \hat{I}'$, then $\phi(B(I')) \cap \phi(\Gamma) \subset \overline{N(\hat{I}')} \cap \Gamma$, by (iii) of **Lemma 4.2.2**. But $\overline{N(\hat{I}')} \cap \Gamma$ is uniformly K-oscillating by 4.3.3. and so it follows that $\phi(B(I')) \cap \phi(\Gamma)$ is uniformly K-oscillating. With $B(\phi(I')) := \phi(B(I'))$ the statement holds.

If $I' \subset \hat{I}'$, then there exists an open neighborhood $B(I')$ such that $\phi(B(I')) \subset C_i$, for $i = 1, 2, 3$ by our choice of a_1 and $b_1$in (4.2.5). It then follows from $(viii)$ of **Lemma 4.2.2** that $\phi(B(I'))$ is uniformly $K + \epsilon$ oscillating. With $B(\phi(I')) := \phi(B(I'))$ the statement holds.

To see that (iii) holds we consider the following two cases separately:

Case 1: Let $I' \cap N(\hat{I}') = \{\emptyset\}$. Then by (vi) of **Lemma 4.2.2** and (4.2.4) we have that $\forall z \in I'$

$$\|\phi(z) - z\| \leq \epsilon^* \leq \frac{\epsilon}{2}$$

Hence diam$\phi(I') \leq$diam $(I') + \epsilon$.

Case 2: Let $I' \cap N(\hat{I}') \neq \emptyset$. Then either I' is disjoint from $\hat{I}'$ or $I' \subset \hat{I}'$

Subcase 1: If $I' \cap \hat{I}' = \emptyset$ then by (4.2.2) it follows that diam $(I') \leq \frac{\epsilon}{2}$. Since z_0 is a K point of $\hat{I}'$ and the diameter of $N(\hat{I}') \leq \epsilon$, it follows that diam $\phi(I') \leq K$ diam $\hat{I}' + \epsilon$.(see fig 4.6)

Subcase 2:

If $I' \subset \hat{I}'$, then (iv) follows directly from (ix) of **Lemma 4.2.2** .

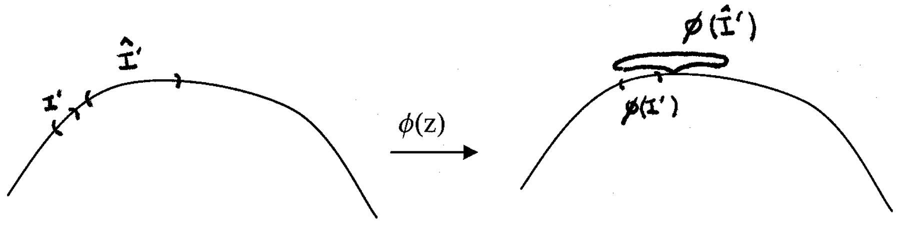
hence (iii) holds.

The statement (iv) follows directly from (v) and (vi) of **Lemma 4.2.2**.

To see that (v) holds, note that ϕ is conformal off J'. Hence for $i = 1, 2..n$ since J_i is by assumption, $\phi(J_i)$ is , except for $\phi(J')$. But $\phi(J' \backslash \hat{I}') = J'$, by (iii) of **Lemma 4.2.2**, which is by assumption.And $\phi(\hat{I}')$ is a Jordan arc.

□

Figure 4.6: diam $\phi(I') \leq K$ diam $\hat{I}' + \epsilon$



Before proving **Theorem 1.3.1**, we prove the following lemma which shows that to prove the uniform convergence of a normal family of functions in $\mathbb{D}^*$ it suffices to show uniform convergence on the boundary of $\mathbb{D}^*$.

Lemma 4.2.4. *Let $\{f_i\}_{i=1}^{\infty}$ be a normal family in $\mathbb{D}^*$ such that for each i*

$$(i) f_i : \mathbb{D}^* \to \hat{\mathbb{C}} \textit{ extends continuously in } \overline{\mathbb{D}^*} \textit{to } \overline{f_i}$$

$$(ii) f_i(\infty) = \infty$$

$$(iii) N(\infty, R) \subset D_i \subset N(\infty, R_0), where D_i = f_i(\mathbb{D}^*)$$

And $\exists g(z) : \mathbb{T} \to \hat{C}$ such that

$$(iv) \lim \overline{f_i}|_{\mathbb{T}} \to g(\mathbb{T}) \textit{ uniformly}$$

Then the limit function $f(z)$ of $\{f_i\}_{i=1}^{\infty}$ extends continuously in $\overline{\mathbb{D}^*}$ to $\overline{f}$.

Proof. Using the conformal mapping $z \to \frac{1}{z}$ we can replace $\mathbb{D}^*$ by $\mathbb{D}$ and ∞ by 0 in the lemma. Let f_{i_k} be a subsequence which converges uniformly on each compact set of $\mathbb{D}$ to $f(z)$. Let $D_{i_k} = f_{i_k}(\mathbb{D}$. We show that $\partial D_{i_k} = \overline{f_{i_k}}|_{\mathbb{T}}$ are uniformly locally connected closed sets. This will be sufficient to prove the lemma, since Pommerenke, (Cor 2.4, [**24**]) has shown that pointwise convergence in $\mathbb{D}$ and uniform local connectedness of ∂D_{i_k} implies uniform convergence in $\overline{\mathbb{D}}$.

Let $\overline{f_i}|_{\mathbb{T}} \to g(\mathbb{T})$. Let $\epsilon > 0$ be given. Since $g(z)$ is the uniform limit of continuous functions, it is a continuous function. Hence $g(\mathbb{T})$ is a locally connected closed set. So $\exists \delta$ such that $\forall a, b \in g(\mathbb{T}, \|a - b\| < \delta$, there exists a continuum $B \subset g(\mathbb{T})$ such that diam $(B) < \frac{\epsilon}{2}$ and $a, b \in B$. Since $\overline{f_{i_k}}|_{\mathbb{T}}$ converge uniformly to $g(\mathbb{T})$, there exists a K such that $\forall k \geq K$:

$$\|\overline{f_{i_k}}(z) - g(z)\| < \max(\frac{\epsilon}{4}, \frac{\delta}{4}), \forall z \in \mathbb{T}$$

For any $k \geq K$, consider $a_k, b_k \in \overline{f_{i_k}}|_{\mathbb{T}}$ such that $\|a_k - b_k\| < \frac{\delta}{4}$. Then pick $x_k \in \overline{f_{i_k}}|_{\mathbb{T}}^{-1}(a_k)$ and $y_k \in \overline{f_{i_k}}|_{\mathbb{T}}^{-1}(b_k)$. Then $\|g(x_k) - g(y_k)\| < \frac{3\delta}{4}$. By assumption there exists a continuum $B \subset g(\mathbb{T})$ such that diam $(B) < \frac{\epsilon}{2}$ and $x_k, y_k \in B$. Let $C = g^{-1}(B)$ and $\overline{f_{i_k}}|_{\mathbb{T}}(C) = B_k$. Then since $x_k, y_k \in C$,we have that$a_k, b_k \in B_k$. But diam $B_k \leq$diam $B + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $k \geq K, \partial D_{i_k}$ are uniformly locally connected closed sets. Since there are only a finite number of D_{i_k} such that $k < K$ and each is a closed locally connected set, we have that ∂D_{i_k} are closed uniformly locally connected sets. It follows that $f_{i_k}(z)$ converge uniformly to $f(z), z \in \overline{\mathbb{D}}$. Hence we can extend $f(z)$ continuously to $\overline{\mathbb{D}}$ by defining $f(z) = g(z), \forall z \in \mathbb{T}$.

□

We now prove **Theorem 1.3.1**:

Proof. To prove the theorem we will define a sequence of functions $\{\phi_i\}_{i=1}^{\infty}$, where $\phi_i(\Omega_i) \to \hat{\mathbb{C}}$ will be conformal in Ω and continuous in $\overline{\Omega}$. we will show that the sequence $k_n := \phi_n \circ \phi_{n-1} \circ ... \phi_1$ is a normal family in $\mathbb{D}^*$ and each k_n extends continuously to $\mathbb{T}$. We will show that the sequence converges on $\mathbb{T}$ and hence by **Lemma 4.2.4** on all of $\mathbb{D}^*$. We will then show that the limit function $k : \mathbb{D}^* \to \hat{\mathbb{C}}$ is the required function.

First note that it suffices to prove the theorem, where k identifies the end points of all but one complementary interval, say $\tilde{\mathrm{I}}$. Then $\mathbb{T}\backslash\, \tilde{\mathrm{I}}$ is a Jordan arc, which is K-oscillating, with $K < 1$. Pick K^*,such that $K < K^* < 1$. Let $\delta = K^* - K$. From now on L will refer to this modified lamination, which does not have $\tilde{\mathrm{I}}$ as a complementary interval. Let S be the countable set of all the complementary intervals of all the L_j. Let $d = \max diam(I), I \in S$.

Assume ϕ_{i-1} has been defined. Let $k_{i-1} = \phi_{i-1} \circ ...\phi_1$. Assume that:

A1) There exists a finite subset $S_{i-1} \subset S$ which we denote by $\{I_j^*\}_{j=1}^{k_{i-1}}$. If L' is the sublamination of L with S_{i-1} as the set of complementary intervals then $k_{i-1}(\mathbb{T}) \simeq \mathbb{T}/L'$. So $k_{i-1}(\mathbb{T})$ is a finite union of Jordan curves, and the intersection of any two such curves is either empty or a singleton.

A2)If $I \in S$ and $I \notin S_{i-1}$ then $k_{i-1}(I)$ is uniformly $K + \Sigma_{j=1}^{i-1} \frac{\delta}{2^j}$ oscillating.

A3) If $I \in S$ and $I \notin S_{i-1}$ then diam $k_{i-1}(I) \leq (K^*)^{i-1} d$

We now define ϕ_i as follows:

Consider all $I \in S\backslash S_{i-1}$ such that diam$k_{i-1}(I) \geq (K^*)^i d - \frac{\delta}{2^i}$. There are only a finite number of such I. We denote them by $S^i = \{I_1^i, ...I_{l_i}^i\}$. For $t = 2, 3...l_i$, we denote the image of these intervals under the mapping $h_{t-1} \circ ...h_1 \circ k_{i-1}$ by $\{{I'}_1^i, ...{I'}_{l_i}^i\}$ We will use **Lemma 4.2.3** l_i times to define the function

$$\phi_i = h_{l_i} \circ ...h_1$$

For $t = 2, 3...l_i$, we define h_t using **Lemma 4.2.3**:

$$h_t : h_{t-1} \circ ...h_i \circ k_{i-1}(\mathbb{D}^*) \to \hat{\mathbb{C}}$$

is conformal and continuous up to the boundary and

1)J' is the Jordan curve in $h_{t-1} \circ ...h_i \circ k_{i-1}(\mathbb{T})$ that contains ${I'}_t^i$

2)$\hat{I}' = {I'}_t^i$

3)$\epsilon = \frac{\delta}{l_i \times 2^{i+1}}$

4)$N({I'}_t^i)$ is disjoint from $N({I'}_{t-1}^i) \cup ... \cup N({I'}_1^i)$

We now show that assumptions A1-A3 are valid.

A1 follows directly from (i) and (ii) of **Lemma 4.2.3**.

We define $S_i = S_{i-1} \cup S^i$. So A3 is valid.

By our choice of ϵ and (iii) of **Lemma 4.2.3**, we have that if $I \in S \backslash S_i$ then $k_i(I)$ is uniformly $K + \Sigma_{j=1}^{i} \frac{\delta}{2^j}$ oscillating. Hence A2 is valid.

To prove the theorem we first show that the sequence $\{k_i\}_{i=1}^{\infty}$ converges uniformly in $\overline{\mathbb{D}^*}$. Since the range of each k_i omits the unit disc, by Montel's theorem it is a normal family. By **Lemma 4.2.4** it suffices to show that $\{k_i\}_{i=1}\infty$ converges uniformly on $\mathbb{T}$.

We claim that $\forall i \in \mathbb{N}$:

$$\|\phi_i(z) - z\| \leq (K^*)^{i-1} d + \frac{\delta}{2^i}, \forall z \in \mathbb{T}$$

We denote the union of the neighborhoods around the complementary intervals of S_i constructed in the proof of the theorem, by N. If $z \notin N$ then by (iv) of **Lemma 4.2.3** we have

$$\|\phi_i(z) - z\| \leq \frac{\delta}{2^i}$$

If $z \in N$ then by our choice of these neighborhoods z can belong to atmost one of them. Hence

$$\|\phi_i(z) - z\| \leq (K^*)^{i-1} d + \frac{\delta}{2^i}$$

which proves the claim.

Now $\forall n, m \in \mathbb{N}, n > m$

$$\|k_n(z) - k_m(z)\| = \|\phi_n \circ \phi_{n-1} ... \phi_{m+1}(z) - z\| \leq \Sigma_{j=m}^{n-1} (K^*)^j d + \Sigma_{j=m+1}^{n} \frac{\delta}{2^j}$$

Since the R.H.S. can be made arbitrarily small, it shows the sequence converges uniformly on the unit circle.

We now show that picking the neighborhoods $N(I'^i_1)...N(I'^i_{l_i})$ small enough we can ensure that if $a, b \in \overline{\cup_{j=1}^{\infty} mult(L_j)}$ and $a \not\sim b$ under L, then $k(a) \neq k(b)$. Let

$a_j, j = 1..n$ be the image of the end points under k_{i-1} of I^*_j, which are not equivalent to each other under L. Note that we can pick the neighborhoods $N(I'^i_1), ...N(I'^i_{l_i})$ small enough so that the distortion in the distance between $k_{i-1}(I^*_j)$ under ϕ_i is arbitrarily small. Since this is true for all $\phi_m, m \geq i$, we have that the sum of the distortion in the distances at every step can be made arbitrarily small. Hence if a and b are contained in the closures of disjoint complementary intervals with equivalent end points, then $k(a) \neq k(b)$. In particular if $a, b \in mult(L)$ and $a \not\sim b$, then $k(a) \neq k(b)$. Furthermore, we can pick the neighborhoods $N(I'^i_1), ...N(I'^i_{l_i})$ small enough so that the distortion in the distance between $a, b \in k_{i-1}\overline{(I^*_j)}$, under ϕ_i is arbitrarily small. In particular, $\|\phi_i(a) - \phi_i(b)\| \leq \frac{\|a-b\|}{4}$. If a and b are not contained in the closures of disjoint complementary intervals with equivalent end points, then by (ii) of **definition 1.2.6** there exists a smallest complementary interval I(this could be $\mathbb{T}$ also) which contains both a and b. Now there are two possibilities: In the first case, no complementary interval contained in I contains either a or b. In this case by our choice of $N(I'^i_1), ...N(I'^i_{l_i}), k(\mathbb{T})$ contains a Jordan arc containing both $k(a)$ and $k(b)$. It follows that $k(a) \neq k(b)$. In the other case we have that no complementary interval contained in I contains a, but b is contained in a smallest complementary interval $I' \subset I$. In this case since $k(I) \cap k(I')$ is at most one point, which is the pre-image of a multiple point, it follows that $k(a) \neq k(b)$.

It remains to show that the limit function $k(z)$induces the lamination L. Since $k(z) : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$ is a continuous function that is injective on $\mathbb{D}$ then by **Theorem 1.1.1**, L_k is a closed, flat lamination. Note that by construction, if for any $j \in \mathbb{N}$, $(a, b) \in L_j$, then $k(a) = k(b)$. Hence $L_k \supset L$. Hence, $(a, b) \in L \Rightarrow k(a) = k(b)$.

We now show that $k(a) = k(b) \Rightarrow (a, b) \in L$.

We have already shown that if $a, b \in \overline{mult(L)}$ and $a \not\sim b$ under L, then $k(a) \neq k(b)$. Now let $a \in (\overline{mult(L)})^c$. Then there is a neighborhood $N(a) \in (\overline{mult(L)})^c$ of a in $\mathbb{C}$, such that every k_i is conformal on this neighborhood. It follows that if $z \neq a$ then $k(a) \neq k(z)$. This completes the proof that $L_k = L$.

$\square$

Note that if there is a curve $\gamma = \overline{\cup_{i=1}^{\infty} J_i}$ where each J_i is a Jordan curve and the intersection of any two J_i is either disjoint or a singleton, then we can define a Julia-type lamination L on γ as follows: Conside a Cantor set on γ such that each complementary interval with equivalent end points is contained in a unique J_i. Hence we can define a Cantor-type lamination on γ as in **Definition 1.2.4**. The Julia-type lamination is then well defined as in **Definition 1.2.6**.

The following corollary is a direct consequence of the proof of **Theorem 1.3.1**:

Corollary 4.2.1. *: Let L' be a totally disconnected lamination on $\mathbb{T}$. Let $\gamma = \mathbb{T}/L'$. Then γ is homeomorphic to $\overline{\cup_{i=1}^{\infty} J_i}$, where each J_i is a Jordan curve and the intersection of any 2 such Jordan curves is disjoint or a singleton. Let L be a lamination of Julia-type on γ. Then L is a conformal lamination.*

Proof. The topology of γ is clear from the topology of the L'. Note that γ is compact and the conformal map that induces L' maps infinity to infinity. For any Jordan curve J_i, the unbounded component of J_i can be mapped conformally to D^*, and the mapping extends continuously and injectively to J_i. Now we can just follow the proof of **Theorem 1.3.1** to prove the lemma.

□

Chapter 5

Totally Disconnected Laminations

In this chapter we will prove **Theorem 1.3.2** that any totally disconnected lamination on $\mathbb{T}$, such that the closure of the set of multiple points is of logarithmic capacity zero, is conformal.

5.1 Logarithmic Capacity and Fekete Points

In this section we will show that using the Fekete points, we can collapse a closed set of logarithmic capacity zero to a single point. Let E be a compact set in $\mathbb{C}$ and G be its unbounded component. For $n = 2, 3...$ consider

$$\triangle_n(E) = \max_{z_1,..z_n \in E} \Pi_{k=1,k\neq j}^{n} \Pi_{j=1}^{n} |z_k - z_j|$$

The maximum is assumed for the *Fekete points*:

$$z_k = z_{nk} \in E(k = 1, ...n)$$

It is not hard to show that the quantity $\triangle_n(E)^{\frac{1}{n(n-1)}}$ is decreasing in n (see [**24**]) . The quantity

$$\lim_{n\to\infty} \triangle_n(E)^{\frac{1}{n(n-1)}}$$

thus converges and is called the logarithmic capacity or the transfinite diameter of E. We first prove the following result:

Theorem 5.1.1. *Let E be any closed set of logarithmic capacity zero, on $\mathbb{T}$. Let L be the lamination defined as follows:*

$$a \sim b, \textit{if and only if } a, b \in E$$

Then L is conformal.

Before proving **Theorem 1.3.2** we prove 2 lemmas. **Lemma 5.1.1** is a slight modification of a result proven by Pommerenke (Prop 9.16, [**24**]).

Lemma 5.1.1. *Let $E \subset \mathbb{T}$ be a closed set of capacity zero. Then there is a starlike function $h(z) = z + ...(z \in \mathbb{D})$ such that*

$$(i)|h(z)| \to \infty, as z \to \zeta, z \in \mathbb{D} \textit{ for each } \zeta \in E.$$

$$(ii) h(z) \textit{ extends to a continuous function, } \overline{h(z)} : \overline{\mathbb{D}} \to \hat{\mathbb{C}}.$$

Proof. Let H_ν be an open set containing E, ($\nu = 0, 1, 2...$), and q_n denote the nth Fekete polynomial of E, such that, n (which depends on ν) is chosen to satisfy:

$$|q_{n_\nu}(z)|^{\frac{1}{n_\nu}} < e^{-4^\nu} for z \in H_\nu$$

Let $z_{n_\nu k}, k = 1, 2..n_\nu$ be the n_ν Fekete points. Then

$$h(z) = z \prod_{\nu=0}^{\infty} \prod_{k=1}^{n_\nu} (1 - \overline{z}_{n_\nu k} z)^{\frac{-1}{n_\nu 2^\nu}} (z \in \mathbb{D})$$

is a starlike function and (i) holds as shown in Pommerenke.

To prove (ii) define the extension of $h(z)$ as follows:

$$\overline{h}(z) = z \prod_{\nu=0}^{\infty} \prod_{k=1}^{n_\nu} (1 - \overline{z}_{n_\nu k} z)^{\frac{-1}{n_\nu 2^\nu}}, (z \in \mathbb{D} \backslash E)$$

and

$$\overline{h}(z) = \infty, (z \in E)$$

To show that this function is continuous on $\overline{\mathbb{D}}$, it suffices to show that the infinite product $\prod_{\nu=0}^{\infty}\prod_{k=1}^{n_\nu}(1-\overline{z}_{n_\nu k}z)^{\frac{-1}{n_\nu 2^\nu}}$ converges uniformly on compact sets of $\overline{\mathbb{D}}\backslash E$. This is equivalent to showing the infinite series $\sum_{\nu=0}^{\infty}\sum_{k=1}^{n_\nu}\frac{1}{n_\nu 2^\nu}log(1-\overline{z}_{n_\nu k}z)$ converges uniformly on compact sets of $\overline{\mathbb{D}}\backslash E$. We will show that the absolute values of the series converges normally. Let S be any compact set of $\overline{\mathbb{D}}\backslash E$ and d be the distance between S and E. Then,

$$|log(1-\overline{z}_{n_\nu k}z)| = log|1-\overline{z}_{n_\nu k}z| = log|z_{n_\nu k} - z| \leq log|d|$$

Hence by picking $\nu = N$ large enough, we have that for any $\epsilon > 0$,

$$\sum_{\nu=N}^{\infty}\sum_{k=1}^{n_\nu}\frac{1}{n_\nu 2^\nu}|log(1-\overline{z}_{n_\nu k}z)| \leq \frac{1}{2^{N-1}}log|d| \leq \epsilon$$

□

5.2 Conformal Maps That Separate Points

A compact curve γ in $\hat{\mathbb{C}}$ is given by a parametric representation $\gamma : \phi(t), \alpha \leq t \leq \beta$, with ϕ continuous on $[\alpha, \beta]$. Note that since C is the continuous image of a locally connected set, it is locally connected.

We will denote the spherical metric on $\hat{\mathbb{C}}$ by $\| \|_s$.

Lemma 5.2.1. *Let γ be a compact curve, not containing the point infinity. Let $\phi(\alpha) = 0$,not be a cut point. Let Ω be the unbounded component of $\hat{\mathbb{C}}\backslash\gamma$. Then there is a conformal map $g(z) : \Omega \to \hat{\mathbb{C}}$, such that:*

(i) $g(\Omega)$ is a Jordan domain, and g can be extended continuously to include the point 0.

(ii) $\|g(z) - z\|_s \leq \epsilon, \forall z \in \Omega.$

(iii) $g(\infty) = \infty, g(0) = 0$.

Proof. :Let $\epsilon \geq 0$ be given. Since g maps infinity to infinity, it is sufficient to prove the lemma with the Euclidean metric, instead of the spherical metric. Let M be the Riemann map from $\mathbb{D} \to \Omega$, which maps ∞ to ∞. Let C_r be the contraction $C_r : z \longrightarrow \frac{z}{r}, r \geq 1$. The mapping $g_r = MC_rM^{-1} : \Omega \to \hat{C}$ is a conformal mapping of Ω onto a Jordan domain,J, and maps ∞ to ∞. Since $\gamma = \partial\Omega$ is locally connected, M extends continuously to $\overline{\mathbb{D}}$, which is a compact set. Hence this extension, call it $\overline{M}$ is equicontinuous. So we can pick r, such that $|\overline{M}(\frac{w}{r}) - \overline{M}(w)| \leq \frac{\epsilon}{2}, \forall w \in \overline{\mathbb{D}}$. So, if $M^{-1}(z) = w$,

$$|g_r(z) - z| = |MC_rM^{-1}(z) - z| = |\overline{M}(\frac{w}{r}) - \overline{M}(w)| \leq \frac{\epsilon}{2}, \forall z \in \Omega$$

Since $\phi(\alpha) = 0$ is not be a cut point , we can extend g_r continuously to 0 and $\|g_r(0) - 0\| \leq \frac{\epsilon}{2}$

Let $g_r(0) = w$, with $|w| \leq \frac{\epsilon}{2}$. Then the mapping $g(z) = g_r(z) - w$, satisfies the requirements of the lemma.

□

The proof of the following corollary is a direct application of the lemma.

Corollary 5.2.1. *Let $\gamma_1, \gamma_2..\gamma_n$ be Jordan arcs in $\mathbb{H}^+$ which are disjoint except for a common end point, $x_0 \in \mathbb{R}$. Let $\gamma = \cup_{i=1}^{n}\gamma_i$ and $\mathbb{H}^+\backslash\gamma = \Omega$. Let $\epsilon > 0$ be given. Then there is a conformal map $g(z) : \Omega \to \mathbb{H}^+$, which extends continuously to $\mathbb{R}$ such that:*

(i) $g(\mathbb{R}) = \mathbb{R}$ and $g(\Omega)$ is bounded by $\mathbb{R}$ and a Jordan curve which meets $\mathbb{R}$ at exactly one point.

(ii) $\|g(z) - z\|_s \leq \epsilon, \forall z \in \Omega$.

(iii) $g(\infty) = \infty, g(x_0) = x_0$.

We now give the proof of **Theorem 5.2.1**.

Proof. Let $h(z)$ be the function described in **Lemma 5.1.1** , and $g(z) = \frac{1}{h(z)}$. Note that $g(E) = 0$ and $g(\infty) = \infty$. We will now define a sequence of functions $\{f_i\}_{i=1}^{\infty}$, inductively, and show the sequence converges uniformly on $\mathbb{D}$ to the required function $f(z)$.

Let $\{I_i\}_{i=1}^{\infty}$ be the set of complementary intervals of $\overline{mult(L)}$ in $\mathbb{T}$. We first define $f_1(z) : \mathbb{D} \to \hat{\mathbb{C}}$ as

$$f_1(z) = h_1 \circ g(z)$$

where h_1 is as defined in **Corollary 5.2.2** with the curve $\gamma = g(\overline{I_1})$ and $\epsilon = \frac{1}{2}$.

Assume f_{n-1} has been defined. Let

$$f_n(z) = h_n \circ f_{n-1}$$

where h_n is as described in **Corollary 5.2.2**, with the curve $\gamma = k_{n-1}(\overline{I_n})$ and $\epsilon = \frac{1}{2^n}$. Then, given $\epsilon > 0$, by picking N large enough we have $\forall n, m \geq N$

$$\|f_n - f_m\| = \sum_{i=N}^{\infty} \frac{1}{2^i} \leq \epsilon, \forall z \in \mathbb{D}.$$

Let $f(z) = \lim_{i\to\infty} f_i(z)$.

We now show that the convergence is uniform in $\overline{\mathbb{D}}$. Let $\epsilon > 0$ be given. Since $\{f_i\}_{i=1}^{\infty}$, converge uniformly in $\mathbb{D} \exists N$ such that, $\forall n, m > N$,

$$\|f_n(w) - f_m(w)\| =\leq \frac{\epsilon}{3}, \forall w \in \mathbb{D}.$$

Since f_n and f_m can be extended continuously to the compact set $\overline{\mathbb{D}}$, the extended functions, $\overline{f_n}$ and $\overline{f_m}$ are uniformly equicontinuous in $\overline{\mathbb{D}}$. Hence, $\exists w^* \in \mathbb{D}$ such that

$$\|\overline{f}_n(z) - \overline{f}_n(w^*)\| =\leq \frac{\epsilon}{3}, \forall z \in \overline{\mathbb{D}}$$

and

$$\|\overline{f}_m(z) - \overline{f}_m(w^*)\| =\leq \frac{\epsilon}{3}, \forall z \in \overline{\mathbb{D}}$$

Hence $\forall n, m > N$,and $z \in \overline{\mathbb{D}}$,

$$\|\overline{f}_n(z)-\overline{f}_m(z)\| \leq \|\overline{f}_n(z)-\overline{f}_n(w^*)\|+\|\overline{f}_n(w^*)-\overline{f}_m(w^*)\|+\|\overline{f}_m(w^*)-\overline{f}_m(z)\| \leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.$$

Hence $f(z)$ can be extended continuously to $\overline{f}(z) : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Since each f_n is conformal in $\mathbb{D}$, the same holds for $f(z)$.

And $f(E) = \lim_{i\to\infty} f_i(E) = 0$. It see that f is injective on $D\backslash E$, note that if $f(\mathbb{D}) = G$ then ∂G has only one cut point, namely 0 and $f^{-1}(0) = E$.

□

5.3 Totally Disconnected Laminations of Capacity Zero

We now prove **Theorem 1.3.2**.

Proof. : Let $\{I_i\}_{i=1}^{\infty}$ be the set of complementary intervals with nonequivalent end points of $\overline{mult(L)}$ in $\mathbb{T}$. We define an equivalence relation on the set $\{I_i\}_{i=1}^{\infty}$ as follows: Consider I_j and I_k. Let J_1 and J_2 be the two components of $\mathbb{T}\backslash\{I_j \cup I_k\}$.

$I_j \sim I_k$ iff any equivalence class of L is completely contained in either J_1 or J_2.

The only thing we need to check is transitivity. Let $I_j \sim I_k$ and $I_k \sim I_m$. Let J_1 and J_2 be the two components of $\mathbb{T}\backslash\{I_j \cup I_k\}$. Without loss of generality, assume $I_m \in J_2$. And Let J_3 and J_4 be the two components of $\mathbb{T}\backslash\{I_k \cup I_m\}$. So the components of $\mathbb{T}\backslash\{I_j \cup I_k \cup I_m\}$ are J_1, J_3 and $J_2 \cap J_4$. Hence transitivity follows. Now consider the equivalence classes of $\{I_i\}_{i=1}^{\infty}$, which have more than 1 member. These classes are disjoint and are at most countable. Call them $\{E_i\}_{i=1}^{\infty}$. Note that by **Theorem 3.2.1**, if $a, b \in mult(L)$ and $a \not\sim b$, then there exists I_j, I_k such that a and b lie in different components of $\mathbb{T}\backslash\{I_j \cup I_k\}$.

To prove the theorem we will define a sequence of functions $\{f_i\}_{i=1}^{\infty}$, inductively, and show the sequence $k_n = f_n \circ f_{n-1} ... \circ f_1 \circ g^*(z), n \in \mathbb{N}$, converges uniformly on $\mathbb{D}$ to the required function $f(z)$.

Let $f(z)$ be the mapping defined in **Theorem 5.1.1** which maps $\overline{mult(L)}$ to 0 and ∞ to ∞. We first define $k_1(z) : \mathbb{D} \to \hat{\mathbb{C}}$ as

$$k_1(z) = h_1 \circ f(z)$$

where h_1 is as defined in **Lemma 5.2.1** with the curve $\gamma = f(E_1)$ and $\epsilon = \frac{1}{2}$.

Assume k_{n-1} has been defined Let

$$f_n(z) = h_n \circ k_{n-1}$$

where h_n is as described in **Lemma 5.2.1**, with the curve $\gamma = k_{n-1}(E_n)$ and $\epsilon = \frac{1}{2^n}$. Then,as in the proof of **Theorem 5.2.1**, given $\epsilon > 0$, by picking N large enough we have $\forall n, m \geq N$

$$\|k_n - k_m\| = \sum_{i=N}^{\infty} \frac{1}{2^i} \leq \epsilon, \forall z \in \overline{\mathbb{D}}$$

Hence $k_n(z)$ converge uniformly on $\mathbb{D}$ to a limit function $f(z)$ and by **Theorem 3.2.1** $f(z)$ separates all nonequivalent points. Hence $f(z)$ satisfies the requirements of the theorem.

□

Chapter 6

Union of Julia-type and totally disconnected Laminations

In this chapter we shall consider some examples of conformal laminations which can be represented as a union of a totally disconnected and a Julia-type lamination.

Theorem 6.0.1. *Let L be a closed, flat lamination on $\mathbb{T}$. Let $\mathbb{T}/L = (\mathbb{T}/L'')/L'$, where L'' is a totally disconnected lamination and L' is a sublamination of Julia-type on $\mathbb{T}/L''$. If $\overline{mult(L'')}$ is of logarithmic capacity zero, then L is a conformal lamination.*

Note that the Julia-type lamination on $\mathbb{T}/L''$ is defined as in **Corollary 4.2.1**.

Proof. : By **Theorem 1.3.2** L'' is conformal. Let $\phi_1(z)$ be the required conformal mapping. By **Cor 4.2.1** there exists a conformal mapping ϕ_2 which induces L'. Then since $\phi_2 \circ \phi_1(\mathbb{T}) = (\mathbb{T}/L'')/L' = \mathbb{T}/L$, we have that $\phi_2 \circ \phi_1$ induces L. □

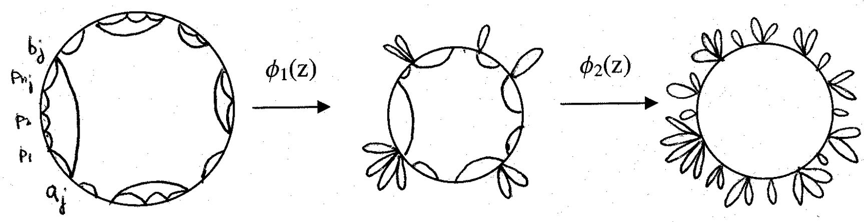
We give some examples of laminations which are conformal by **Theorem 6.0.1**.

Example 1: Let $C \subset \mathbb{T}$ be a Cantor set. Let $\{I_j\}_{j=1}^{\infty}$ be the set of complementary intervals of C, and let $I_j = (a_j, b_j)$. For each j let $\{p_1, p_2 ... p_{n_j}\} \subset (a_j, b_j)$ be a finite set of points. Let L be the lamination whose equivalence classes $\{E_j\}_{j=1}^{\infty}$, are defined as follows:

$$E_j = \{a_j, p_1, p_2 ... p_{n_j}, b_j\}.$$

The L is a conformal lamination. Note that sublamination of Julia-type S' of L is the Cantor-type lamination that identifies a_j and b_j and the lamination L'' that identifies

Figure 6.1: Example 1



the points $a_j, p_1, p_2...p_{n_j}$, for each $j \in \mathbb{N}$, is a totally disconnected lamination of logarithmic capacity zero. Note that we do not include the point b_j, in the equivalence class. See Fig 6.1. Let ϕ_1 induces L'' and $L' = \phi_1(S')$ and $\mathbb{T}/L = (\mathbb{T}/L'')/L'$.

Hence by **Theorem 6.0.1**, L is conformal.

Example 2: The lamination L, induced by the quadratic polynomial $P_c : z \to z^2 + c$ such that 0 is periodic of period 3 and $Im(c) > 0$ is not nowhere dense. But each equivalence class E_i of L consists of exactly three points $\{a_i, b_i, c_i\}$, see Fig. 6.2. In Fig. 6.2, a typical equivalence E_i is the equivalence class denoted by the points $\{\frac{1}{56}, \frac{53}{56}, \frac{51}{56}\}$. Let S' be the Julia-type lamination where each equivalence class consists of the points $\{a_i, c_i\}$ and L'' be the totally disconnected lamination of logarithmic capacity zero, where each equivalence class consists of the points $\{b_i, c_i\}$. If ϕ_1 induces L'' then let $L' = \phi_1(S')$. Since $\mathbb{T}/L = (\mathbb{T}/L'')/L'$, it then follows by **Theorem 6.0.1**, L is conformal.

Example 3: Let $C_1 \subset I_1$ be a Cantor set of logarithmic capacity zero on the upper semicircle. Let $\phi : I_1 \to I_2$ be a homeomorphism of the upper semicircle to the lower, which fixes 1 and -1 such that $\phi(C_1) \subset I_2$ is a Cantor set of logarithmic capacity zero. Define a closed, flat lamination L which identifies the end points of each complementary interval of C_1 and $\phi(C_1)$, and identifies the endpoints of the corresponding homeomorphic complementary intervals, see Fig 6.3.

Figure 6.2: Example 2:The lamination and the Julia set defined by the Rabbit

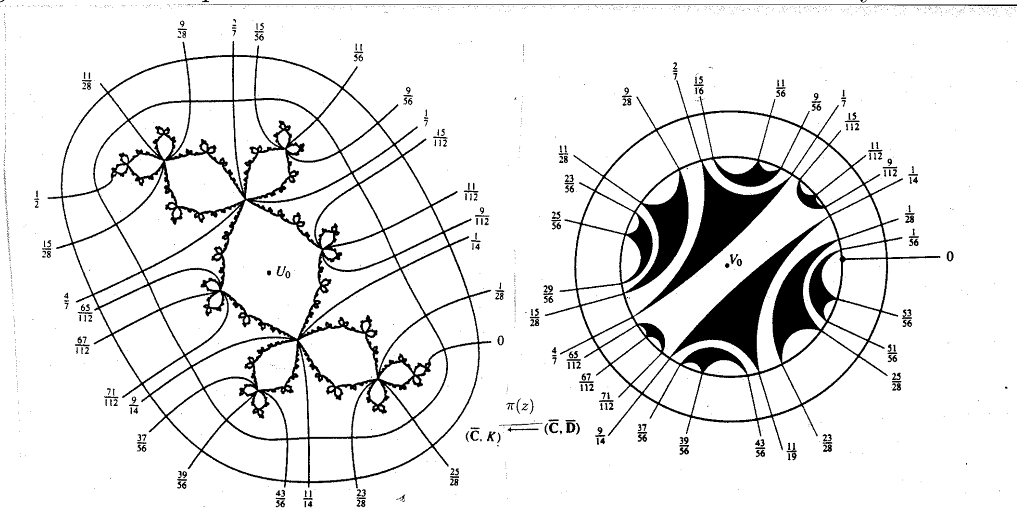


Figure 6.3: Example 3

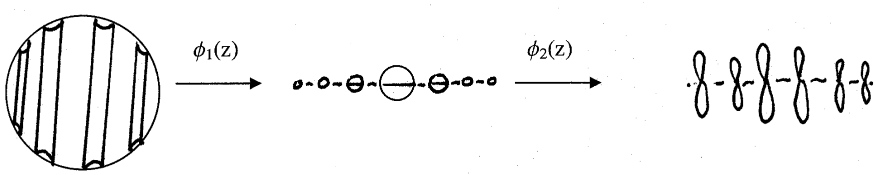
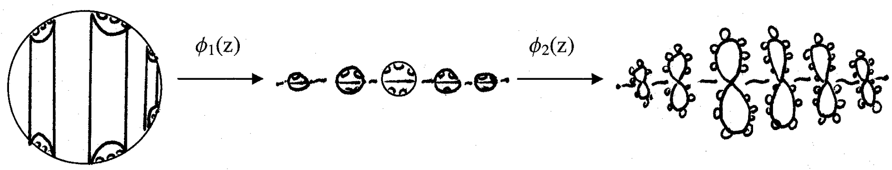


Figure 6.4: Example 4



Let S' be the Cantor-type sublamination that identifies the end points of each complementary interval of C_1 and $\phi(C_1)$. Hence L'' be the sublamination that identifies the endpoints of the corresponding homeomorphic complementary intervals. This is a closed lamination that is totally disconnected and the log cap$(\overline{mult(L'')}) = 0$. Then if ϕ_1 induces L'', let $\phi_1(S') = L'$ and since $\mathbb{T}/L = (\mathbb{T}/L'')/L'$, L is a conformal lamination by **Theorem 6.0.1**.

Example 4: We consider the setup of **Example 2**, but now for each complementary interval of C_1 and $\phi(C_1)$ we introduce a Cantor-type lamination contained in the complementary interval, see fig 6.4. Then the sublamination of Julia-type, L' is a 2 generation Julia-type sublamination. L'' is the same as **Example 2**. And so L is conformal, by **Theorem 6.0.1**.

We conjecture that any closed, flat lamination on $\mathbb{T}$ such that log cap $(mult(L)) = 0$ is conformal. A possible approach to proving this conjecture would be to show that any such lamination admits a decomposition as described in **Theorem 6.0.1**.

The idea is to show that any such lamination contains a 'maximal' Julia-type sublamination which is not properly contained in any Julia-type lamination. If L is totally disconnected then we are done by taking $L = L''$ and L' as the trivial lamination which has no multiple points. If not, then since L is nowhere dense, by **lemma 3.2.2** it contains a Cantor-type lamination L_1, which is not identified to a single point. Consider the set of all sublaminations of Julia-type of L that contain L_1. This is a nonempty set and since each element of the set is a subset of

$\mathbb{T} \times \mathbb{T}$, we can partially order this set by inclusion. By Zorn's Lemma there exists maximal totally ordered set Ω. Let S be the union of all members of Ω. Note that $S = \cup_{\lambda \in \Lambda} L_\lambda$, where each L_λ is a Cantor-type lamination. We show that S satisfies (i) of **Definition 1.2.6**. If $L_j, L_k \in S$, then $L_j \in A_1, L_k \in A_2$ for some $A_1, A_2 \in \Omega$. Since Ω is totally ordered, $A_1 \subset A_2$(or $A_2 \subset A_1$), so that $L_j, L_k \in A_2$. Since A_2 is a Julia-type lamination, $mult(L_j)$ and $mult(L_k)$ are pairwise disjoint. Hence (i) is satisfied. Also, by the maximality of S, it is not properly contained in a Julia-type lamination. However, in general S does not satisfy (ii) of **Definition 1.2.6**. If (ii) does not hold for S, then there exist points $(z, z') \in S$,with $z \neq z'$and (z, z') is not discrete. Let Z be the set of such points. Since L is nowhere dense, we can find a countable number of complementary intervals with equivalent end points $(p_n, q_n) \in L_{\lambda_n}$ $S' := S \backslash \{\cup_n (p_n, q_n) \cup Z\}$satisfies (ii). We can do this by choosing the p_n and q_n which are the end points of the countable complementary intervals of $\overline{multS}$ in $\mathbb{T}$. Furthermore, we can choose the (p_n, q_n) such that each point belongs to only one L_{λ_n}. Then S' satisfies (i) and (ii) of **Definition 1.2.6**. S' is a Julia-type lamination containing L_1, that is not properly contained in any Julia-type sublamination of L.

Now if C is a Cantor-type sublamination of L that is disjoint from S' then for every complementary interval with equivalent end points $I_j = (a_j, b_j)$ of C, either a_j or b_j, but not both, is an end point of a complementary interval with equivalent end points of a Cantor-type lamination that generates S'. To see this, note that both a_j and b_j cannot be end points of such a complementary interval since L' and C are disjoint. However if neither a_j and b_j are such end points, then since L is closed there must be a Cantor-type sublamination for which this is true. Note these points cannot be any of the points p_n and q_n since they are the end points of complementary intervals with equivalent end points of L_{λ_n}. Hence it follows that S is properly contained in a Julia-type lamination, which is a contradiction.

Consider the set $L_2 := \{L \backslash S'\} \cup \{(x, x) : x \in \mathbb{T}\}$. Since L is a flat lamination, and S' is a flat sublamination of L, it follows that L_2 is a flat set that is symmetric. By definition, it is also reflexive. It is not in general a lamination since it might not be transitive. For instance, if $(a, b), (b, c), (c, a) \in L$ and only $(a, b) \in L'$,then $L \backslash S'$

would not be transitive. However, it contains a transitive subset $L'' \subset L_2$, such that $\mathbb{T}/L = (\mathbb{T}/(L'')^c)/L'$. To see this, consider an equivalence class, E of L which contains multiple points of S' and L_2. Now define the equivalence class of L'' to be the set of points E without the multiple points of S' except one. L'' defined in this manner is clearly a transitive subset of L_2 such that $(L'' \cup S')^c = L$. In the above example, this would mean choosing either (c, a) or (b, c) in the equivalence class of L'' depending on whether we choose a or b as the multiple point of S'.

Now if C is a Cantor-type sublamination of L, which is disjoint from S', then by the flatness of L, any complementary interval with equivalent end points (a, b) of C, is contained in, or contains an equivalent complementary interval,(a, c) of a Cantor-type sublamination that generates S'. The end point a is common for the two complementary intervals as explained earlier. Now by transitivity of $L, (b, c) \in L$ and since S' is transitive $(b, c) \notin S'$. In particular we can consider L'', which contains only (b, c) and not (a, b). Then, L'' is a lamination that does not contain a Cantor-type sublamination. However, it is not closed in general. Since L'' and S' are disjoint, it follows that $L' := \pi(S')$, where π is the projection map, is a Julia-type lamination and $\mathbb{T}/L = (\mathbb{T}/(L'')^c)/L'$. We would need to show that $(L'')^c$ which is of log capacity zero is conformal and then the result would follow.

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