ON A PROBLEM IN GEOMETRIC MEASURE THEORY RELATED TO SPHERE AND CIRCLE PACKING

Thesis by

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Our revels are now ended...These our actors,

As I foretold you, were all spirits, and

Are melted into air, into thin air,

And, like the baseless fabric of this vision,

The cloud-capped towers, the gorgeous palaces,

The solemn temples, the great globe itself,

Yea, all which it inherit, shall dissolve,

And, like this insubstantial pageant faded,

Leave not a rack behind: we are such stuff

As dreams are made on; and our little life

Is rounded with a sleep...

William Shakespeare

Acknowledgment

I thank my advisor, professor Thomas Wolff, for his patience, his guidance and his invaluable support throughout all these years.

Abstract

In this thesis we prove that a Borel set which contains spheres centered at all points of a Borel set of Hausdorff dimension greater than 1 must have positive Lebesgue measure, and, using the same method, we rederive a special case of Stein's spherical means maximal inequality. We also prove the corresponding result for circles, provided that the set of centers has Hausdorff dimension greater than 3/2.

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Introduction

By the term packing problem we usually refer to the following type of question: given a collection of geometric objects in a Euclidean space is it possible to find a set of Lebesgue measure zero containing a translate, or a congruent copy of dilates of every object in the collection?

The beginning of the study of packing problems can be traced to Besicovitch who, as early as 1919, solved the "Kakeya problem" by constructing a compact set of plane measure zero containing a line segment in every direction. The monograph by Falconer [**5**] contains a fairly extensive account of the history of the subject, as well as its applications to other areas of mathematics.

In this thesis we are concerned with the problem of investigating the relation between the measure of the union of a set of spheres or circles and the metric properties of the set of their centers.

To begin with, a partial answer to the above question comes from the work of Stein [**14**] on the spherical means maximal operator.

For $f \in C_c(\mathbb{R}^d)$, define

$$\overline{\mathcal{M}}f(x) = \sup_{1 \le r \le 2} \int_{S^{d-1}} f(x - ry) d\sigma(y)$$

where S^{d-1} is the $(d-1)$-dimensional sphere and $d\sigma$ is surface measure. Then we have the following:

THEOREM 0.1. *Suppose that* $d \geq 3$, $p > \frac{d}{d-1}$. *Then there exists a constant* $C > 0$ *that depends only on* d *and* p *such that*

$$\|\overline{\mathcal{M}}f\|_p \leq C\|f\|_p$$

for all $f \in C_c(\mathbb{R}^d)$.

Using this result one can easily prove the following:

THEOREM 0.2. *Let* $F \subset \mathbb{R}^d$, $d \geq 3$, *be a set of positive Lebesgue measure. If* $E \subset \mathbb{R}^d$ *is a set which contains spheres centered at all points of* F, *then* E *has positive Lebesgue measure.*

The two-dimensional case was settled by Bourgain [**2**], who proved that the conclusion of the above theorem still holds if $d = 2$, and, independently, by Marstrand [**9**], who used purely geometric methods.

THEOREM 0.3. *Let* $F \subset \mathbb{R}^2$ *be a set of positive Lebesgue measure. If* $E \subset \mathbb{R}^2$ *is a set that contains circles centered at all points of* F, *then* E *has positive plane measure.*

This should be contrasted with a construction due to Talagrand [**15**].

THEOREM 0.4. *There is a set of plane measure zero containing for each x on a given straight line, a circle centered at x.*

It is, therefore, natural to ask whether one can weaken the condition that the set F in the above theorems should be of postive measure. The main results in this thesis are the following:

THEOREM 1. *Let $F \subset \mathbb{R}^d$, $d \geq 3$, be a Borel set of Hausdorff dimension $s, s > 1$. If $E \subset \mathbb{R}^d$ is a Borel set that contains spheres centered at each point of F, then E has positive Lebesgue measure.*

THEOREM 2. *Let $F \subset \mathbb{R}^2$ be a Borel set of Hausdorff dimension $s, s > 3/2$. If $E \subset \mathbb{R}^2$ is a Borel set that contains circles centered at each point of F, then E has positive Lebesgue measure.*

The thesis is organized as follows:

In Chapter 1 we state some geometric lemmas needed later on, in Chapter 2 we prove Theorem 1 and construct a counterexample related to it, in Chapter 3 we prove Theorem 2, and finally, in Chapter 4, we discuss the possibility of weakening the condition $s > 3/2$ in Theorem 2.

Throughout this thesis, $a \lesssim b$ means $a \leq Ab$ for some absolute constant A, and similarly with $a \gtrsim b$ and $a \sim b$. We will denote Lebesgue measure by $|\cdot|$.

CHAPTER 1

Background

We start with some notation:

$B(x, r)$ is the open disk (or ball) with center x and radius r.

$C(x, r)$ is the circle (or sphere) with center x and radius r.

$C^\delta(x, r)$ is the δ-neighborhood of the circle (or sphere) $C(x, r)$, i.e., the set $\{y \in \mathbb{R}^d : r - \delta < |x - y| < r + \delta\}$.

If $C(x, r)$ and $C(y, s)$ are circles, then define

$$d((x, y), (r, s)) = |x - y| + |r - s|$$

$$\Delta((x, y), (r, s)) = ||x - y| - |r - s||$$

Note that $\Delta = 0$ if and only if the circles are internally tangent, that is, they are tangent and one is contained in the bounded component of the complement of the other. In what follows, we assume that the centers of all circles (or spheres) in question are contained in the disk $B(0, 1/4)$ and that their radii are in the interval $[1/2, 2]$.

The following lemma gives estimates on the size of the intersection of two annuli in terms of their relative position and their degree of tangency. The reader is referred to Wolff [**17**] for a proof.

LEMMA 1.1. *Suppose that* $r \neq s$, $0 < \delta < 1$. *Then there exists an absolute constant* A_0 *such that:*

1. $C^\delta(x,r) \cap C^\delta(y,s)$ *is contained in a* δ*-neighborhood of arc length* $\leq A_0\sqrt{\frac{\Delta+\delta}{|x-y|+\delta}}$ *centered at the point* $x - r \cdot sgn(r-s)\frac{x-y}{|x-y|}$.
2. *The area of intersection satisfies*

$$|C^\delta(x,r) \cap C^\delta(y,s)| \leq A_0 \frac{\delta^2}{\sqrt{(\delta+\Delta)(\delta+d)}}.$$

Furthermore we have the higher dimensional analogue whose proof we omit.

LEMMA 1.2. *Suppose that* $C^\delta(x,r)$, $C^\delta(y,s)$ *are spheres in* $\mathbb{R}^d$, $d \geq 3$, *such that* $r \neq s$. *Then for* $0 < \delta < 1$

$$|C^\delta(x,r) \cap C^\delta(y,s)| \lesssim \frac{\delta^2}{\delta + |x-y|}.$$

The following lemma is essentially Marstrand's three circle lemma [**9**]. It is a quantitative version of the following fact known as the circles of Apollonius:

Given three circles which are not internally tangent at a single point, there are at most two other circles that are internally tangent to the given ones.

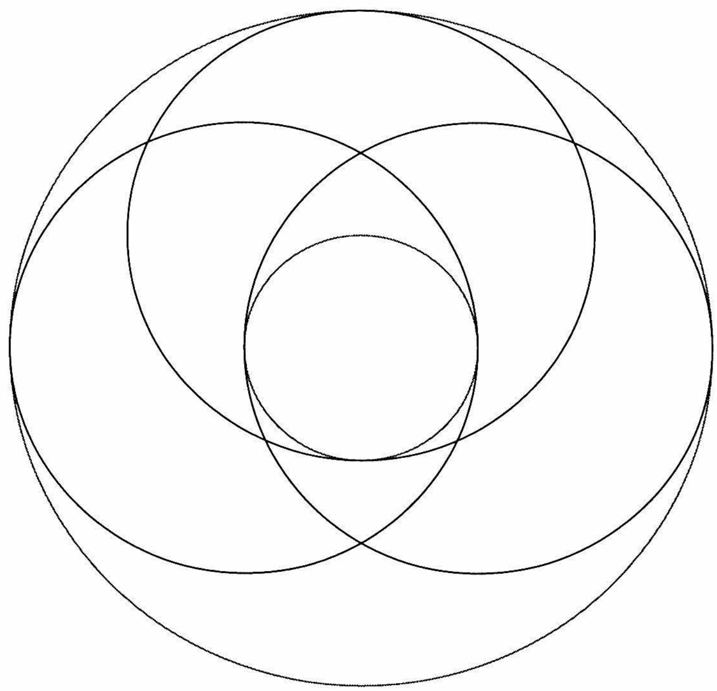

FIGURE 1. The circles of Apollonius.

LEMMA 1.3. *There exists a constant A_1 such that if $\epsilon, t, \lambda \in (0,1)$ satisfy $\lambda \geq A_1\sqrt{\frac{\epsilon}{t}}$ then for three fixed circles $C(x_i, r_i), i = 1,2,3$ and for $\delta \leq \epsilon$ the set*

$$\{(x,r) \in \mathbb{R}^2 \times \mathbb{R} : \Delta((x,r),(x_i,r_i)) < \epsilon \ \forall i,$$

$$d((x,r),(x_i,r_i)) > t \ \forall i,$$

$$C^\delta(x,r) \cap C^\delta(x_i,r_i) \neq \emptyset \ \forall i,$$

$$dist(C^\delta(x,r) \cap C^\delta(x_i,r_i), C^\delta(x,r) \cap C^\delta(x_j,r_j)) \geq \lambda$$

$$\forall i,j : i \neq j\}$$

is contained in the union of two ellipsoids in $\mathbb{R}^3$ *each of diameter* $\lesssim \frac{\epsilon}{\lambda^2}$.

A proof of the preceding result can be found in Wolff [**17**].

We conclude this chapter with some facts from Geometric Measure Theory. We refer the reader to Falconer [**5**] for definitions, proofs and details. In what follows, $\mathcal{H}^s$ denotes s-dimensional Hausdorff measure.

THEOREM 1.1. *Let* E *be a Borel set in* $\mathbb{R}^d$ *and let* $s > 0$. *Assume that* $\mathcal{H}^s(E) > 0$. *Then there exists a nontrivial finite measure* μ *supported on* E *such that* $\mu(B(x,r)) \leq r^s$ *for* $x \in \mathbb{R}^n$ *and* $r > 0$.

If E is an s-set, i.e., $0 < \mathcal{H}^s(E) < \infty$, then a point $x \in E$ is called regular if the upper and the lower densities at x are equal to one; otherwise x is called irregular. An s-set E is said to be irregular if $\mathcal{H}^s$-almost all of its points are irregular. Irregular 1-sets are characterized by the following:

THEOREM 1.2. *A* 1*-set in* $\mathbb{R}^2$ *is irregular if and only if it has projections of linear Lebesgue measure zero in two distinct directions.*

In fact, one can say much more.

THEOREM 1.3. *Let* E *be an irregular* 1*-set in* $\mathbb{R}^2$. *Then* $proj_\theta(E)$ *has linear Lebesgue measure zero for almost all* $\theta \in [0, \pi)$, *where* $proj_\theta$

denotes orthogonal projection from $\mathbb{R}^2$ *onto the line through the origin making angle* θ *with some fixed axis.*

CHAPTER 2

The higher dimensional case

In this chapter we assume that $d \geq 3$.

For $f : \mathbb{R}^d \to \mathbb{R}$, $\delta > 0$ small, we define $\mathcal{M}_\delta : B(0, 1/4) \to \mathbb{R}$ by

$$\mathcal{M}_\delta f(x) = \sup_{1/2 \leq r \leq 2} \frac{1}{|C^\delta(x,r)|} \int_{C^\delta(x,r)} |f(y)| dy$$

Theorem 1 will be a consequence of the following $L^2 \to L^2$ maximal inequality:

PROPOSITION 2.1. *Let $F \subset B(0, 1/4)$ be a compact set in $\mathbb{R}^d$ such that there exist $s > 1$ and a finite measure μ supported on F with $\mu(B(x,r)) \leq r^s$ for $x \in \mathbb{R}^d$ and $r > 0$. Then there exists a constant A that depends only on s such that*

$$\left(\int_F (\mathcal{M}_\delta f(x))^2 \, d\mu(x) \right)^{1/2} \leq A \|f\|_2$$

for small $\delta > 0$ and all f.

In order to prove Proposition 2.1 we need the following weighted version of Schur's Lemma:

LEMMA 2.1. *Let (S, μ) be a measure space, and let $C > 0$. Suppose K is a measurable kernel on $S \times S$ and w is a measurable function on S such that*

$$\int_S |K(x,y)w(y)|d\mu(y) \leq C|w(x)| \ \forall x \in S$$
$$\int_S |K(x,y)w(x)|d\mu(x) \leq C|w(y)| \ \forall y \in S$$

Then for $f \in L^2(S)$ the function Tf defined by

$$Tf(x) = \int_S K(x,y)f(y)d\mu(y)$$

is well defined almost everywhere, is in $L^2(S)$, and satisfies $\|Tf\|_2 \leq C\|f\|_2$.

PROOF. Let

$$I = \int_S |K(x,y)f(y)|d\mu(y)$$

Then by Hölder's inequality

$$\begin{aligned} I &= \int_S |K(x,y)w(y)|^{1/2}|K(x,y)w(y)|^{1/2}\frac{|f(y)|}{|w(y)|}d\mu(y) \\ &\leq \left(\int_S |K(x,y)w(y)|d\mu(y)\right)^{1/2}\left(\int_S |K(x,y)w(y)|\frac{|f(y)|^2}{|w(y)|^2}d\mu(y)\right)^{1/2} \\ &\leq C^{1/2}w^{1/2}(x)\left(\int_S |K(x,y)|\frac{1}{|w(y)|}|f(y)|^2d\mu(y)\right)^{1/2} \end{aligned}$$

Therefore

$$\int_S \left(\int_S |K(x,y)f(y)|d\mu(y)\right)^2 d\mu(x)$$

$$= C \int_S w(x) \left(\int_S |K(x,y)| \frac{1}{|w(y)|} |f(y)|^2 d\mu(y) \right) d\mu(x)$$
$$= C \int_S |f(y)|^2 \frac{1}{|w(y)|} \left(\int_S |K(x,y)w(x)| d\mu(x) \right) d\mu(y)$$
$$\leq C \int_S |f(y)|^2 d\mu(y).$$

□

Proof of Proposition 2.1. Decompose $\mathbb{R}^d$ into disjoint cubes $\{Q_j\}$ of the form

$$\left[k_1 \frac{\delta}{\sqrt{d}}, (k_1+1)\frac{\delta}{\sqrt{d}}\right) \times \cdots \times \left[k_d \frac{\delta}{\sqrt{d}}, (k_d+1)\frac{\delta}{\sqrt{d}}\right)$$

where $k_1, \ldots, k_d \in \mathbb{Z}$. Then

$$\int_F (\mathcal{M}_\delta f(x))^2 \, d\mu(x) = \sum_j \int_{Q_j} (\mathcal{M}_\delta f(x))^2 \, d\mu(x)$$

Note that if $x, y \in Q_j$ then

$$\mathcal{M}_\delta f(x) \lesssim \mathcal{M}_{2\delta} f(y)$$

Therefore, if x_j is the center of the cube Q_j and if we let $a_j = \mu(Q_j)$ then

$$\int_F (\mathcal{M}_\delta f(x))^2 d\mu(x) \lesssim \sum_j a_j (\mathcal{M}_{2\delta} f(x_j))^2 \tag{1}$$

Since $|C^\delta(x,r)| \sim \delta$, we can choose $r_j \in [1/2, 2]$ such that

$$\mathcal{M}_{2\delta} f(x_j) \lesssim \frac{1}{\delta} \int_{C^{2\delta}(x_j, r_j)} f(y) dy$$

It follows that

$$\sum_j a_j (\mathcal{M}_{2\delta} f(x_j))^2 \lesssim \frac{1}{\delta^2} \sum_j a_j \left(\int_{C^{2\delta}(x_j, r_j)} f(y) dy \right)^2 \tag{2}$$

By duality, there exists $\{b_j\}$ with $\sum_j b_j^2 = 1$ such that

$$\begin{aligned}
&\frac{1}{\delta^2} \sum_j a_j \left(\int_{C^{2\delta}(x_j, r_j)} f(y) dy \right)^2 \\
&= \frac{1}{\delta^2} \left(\sum_j b_j a_j^{1/2} \int_{C^{2\delta}(x_j, r_j)} f(y) dy \right)^2 \\
&= \frac{1}{\delta^2} \left(\int \left(f(y) \sum_j b_j a_j^{1/2} \chi_{C^{2\delta}(x_j, r_j)}(y) \right) dy \right)^2 \\
&\leq \frac{1}{\delta^2} \|f\|_2^2 \int \left(\sum_j b_j a_j^{1/2} \chi_{C^{2\delta}(x_j, r_j)}(y) \right)^2 dy \\
&= \frac{1}{\delta^2} \|f\|_2^2 \int \left(\sum_{i,j} b_i b_j a_i^{1/2} a_j^{1/2} \chi_{C^{2\delta}(x_i, r_i) \cap C^{2\delta}(x_j, r_j)}(y) \right) dy \\
&= \frac{1}{\delta^2} \|f\|_2^2 \sum_{i,j} b_i b_j a_i^{1/2} a_j^{1/2} |C^{2\delta}(x_i, r_i) \cap C^{2\delta}(x_j, r_j)| \\
&\lesssim \|f\|_2^2 \sum_{i,j} b_i b_j a_i^{1/2} a_j^{1/2} \frac{1}{|x_i - x_j| + \delta} \qquad (3)
\end{aligned}$$

where the last inequality follows from Lemma 1.2. Now let

$$K(i,j) = \frac{a_i^{1/2} a_j^{1/2}}{|x_i - x_j| + \delta}$$

$$w(i) = a_i^{1/2}$$

Then, by Hölder's inequality,

$$\sum_{i,j} b_i b_j a_i^{1/2} a_j^{1/2} \frac{1}{|x_i - x_j| + \delta}$$
$$\leq \left(\sum_i b_i^2\right)^{1/2} \left(\sum_i \left(\sum_j b_j \frac{a_i^{1/2} a_j^{1/2}}{|x_i - x_j| + \delta}\right)^2\right)^{1/2}$$
$$= \left\| \left(\sum_j b_j K(i,j)\right)_i \right\|_{\ell^2} \tag{4}$$

For each i,j let $\tilde{x}_{i,j} \in \overline{Q}_i$ be such that

$$\frac{1}{|\tilde{x}_{i,j} - x_j| + \delta} = \min_{x \in \overline{Q}_i} \frac{1}{|x - x_j| + \delta}$$

Then for all j we have

$$\begin{aligned}
\sum_i K(i,j) w(i) &= w(j) \sum_i \frac{a_i}{|x_i - x_j| + \delta} \\
&\lesssim w(j) \sum_i \frac{a_i}{|\tilde{x}_{i,j} - x_j| + \delta} \\
&\leq w(j) \sum_i \int_{Q_i} \frac{d\mu(x)}{|x - x_j| + \delta} \\
&= w(j) \int \frac{d\mu(x)}{|x - x_j| + \delta} \\
&\leq C w(j)
\end{aligned}$$

where the last inequality follows from the fact that a measure satisfying the condition in the statement of Theorem 1 has uniformly bounded potential (see [**5**]).

By symmetry

$$\sum_j K(i,j)w(j) \lesssim Cw(i), \ \forall i$$

Hence, by Lemma 2.1, we have

$$\left\| \left(\sum_j b_j K(i,j) \right)_i \right\|_{\ell^2} \lesssim C \left(\sum_j b_j^2 \right)^{1/2} = C \tag{5}$$

It follows from 1, 2, 3, 4, and 5 that

$$\int_F (\mathcal{M}_\delta f(x))^2 d\mu(x) \leq A \|f\|_2^2$$

where A depends only on the measure of F and on s. □

Note that if we let

$$\mathcal{A}f(x) = \sup_{1/2 \leq t \leq 2} \int_{S^{d-1}} |f(x - ty)| d\sigma(y), \ f \in C_c(\mathbb{R}^d)$$

then, using Fatou's Lemma, we obtain the δ-free version of Proposition 2.1 as follows:

$$\begin{aligned}
\int (\mathcal{A}f(x))^p d\mu(x) &= \int \sup_t \sup_\epsilon \inf_{\delta<\epsilon} \left(\frac{1}{|C^\delta(x,t)|} \int_{C^\delta(x,t)} |f(y)| dy \right)^p d\mu(x) \\
&\leq \sup_\epsilon \inf_{\delta<\epsilon} \int \sup_t \left(\frac{1}{|C^\delta(x,t)|} \int_{C^\delta(x,t)} |f(y)| dy \right)^p d\mu(x) \\
&\leq \int (\mathcal{M}_\delta f(x))^p d\mu(x) \\
&\lesssim \|f\|_p^p
\end{aligned}$$

PROOF OF THEOREM 1. We may assume that $F \subset B(0, \frac{1}{4})$. Suppose that $|E|$=0 and choose t so that $1 < t < s$. Then there exist a compact set $E_1 \subset E$, a compact set $F_1 \subset F$ with $\mathcal{H}^t(F_1) > 0$, and a positive number r, such that, for each $x \in F_1$, there is a sphere centered at x with radius $r(x) \in (r, 2r)$ which intersects E_1 in set of surface measure at least $r^{d-1}(x)$. Without loss of generality we may assume that $r = 1$. It follows that for all $x \in F_1$

$$\mathcal{M}_\delta \chi_{E_1^\delta}(x) \gtrsim 1$$

where E_1^δ is the δ-neighborhood of the set E_1.

By Theorem 1.1, there exists a nontrivial finite measure μ supported on F_1 such that $\mu(B(x,r)) \leq r^t$, for $x \in \mathbb{R}^d$, $r > 0$. Therefore, by Proposition 2.1, we have

$$\mu(F_1) \lesssim \int_{F_1} (\mathcal{M}_\delta \chi_{E_1^\delta}(x))^2 d\mu(x) \lesssim |E_1^\delta|$$

The right-hand side of the above inequality tends to zero as $\delta \to 0$, so we get a contradiction. □

We will show that we cannot drop the condition $s > 1$ in Theorem 1.

PROPOSITION 2.2. *There exists a set of d-dimensional Lebesgue measure zero containing for each* $x \in [0,1] \times \underbrace{\{0\} \times \cdots \times \{0\}}_{d-1}$, *a sphere centered at* x.

PROOF. The idea, which goes back to Davies [4], is to parametrize the set of radii using a suitable irregular 1-set.

Divide the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$ into 16 disjoint squares of side 1/4. Let $S_{(i,j)}$ $1 \leq i, j \leq 4$ be those squares (indexed from bottom to top, left to right). Let

$$E_1 = S_{(1,2)} \cup S_{(1,4)} \cup S_{(4,1)} \cup S_{(4,3)}$$

Apply the same procedure to each of $S_{(1,2)}$, $S_{(1,4)}$, $S_{(4,1)}$, $S_{(4,3)}$, and let E_2 be the union of the new squares. Continuing in the same manner we obtain a decreasing sequence of compact sets $\{E_n\}$. Let $E = \bigcap_{n=1}^{\infty} E_n$. Then E is a 1-set such that

$$\mathrm{proj}_0(E) = [0,1], \ |\mathrm{proj}_{\pi/2}(E)| = 0, \ |\mathrm{proj}_{\pi/4}(E)| = 0$$

where $|\cdot|$ is linear Lebesgue measure, and proj_0, $\mathrm{proj}_{\pi/2}$, $\mathrm{proj}_{\pi/4}$ denote, respectively, orthogonal projection onto the x-axis, the y-axis, and the line through the origin making angle $\pi/4$ with the x-axis. It follows from Theorem 1.2 that E is irregular.

Let

$$\begin{aligned} A &= \bigcup_{(a,b)\in E} \{(x_1,\ldots,x_d) : (x_1-a)^2 + x_2^2 + \cdots + x_d^2 = a^2 + b\} \\ &= \bigcup_{(a,b)\in E} \{(x_1,\ldots,x_d) : x_d^2 = 2ax_1 + b - x_1^2 - \cdots - x_{d-1}^2\} \end{aligned}$$

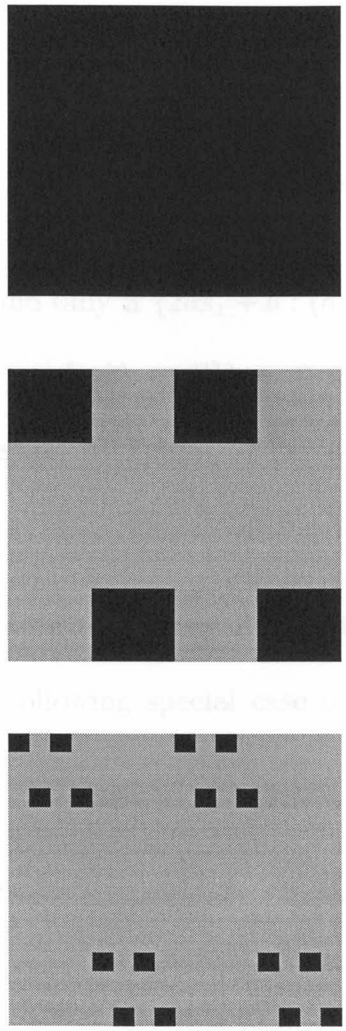

FIGURE 2. Construction of the parameter set E.

Since $\text{proj}_0(E) = [0, 1]$, A contains a sphere centered at each point of $\{(a, 0, \ldots, 0) : a \in [0, 1]\}$.

Fix $s_1, \ldots, s_{d-1}$. Then

$$A \cap \{(x_1, \ldots, x_d) : x_1 = s_1, x_2 = s_2, \ldots, x_{d-1} = s_{d-1}\}$$

$$= \{s_1\} \times \{s_2\} \times \cdots \times \{s_{d-1}\} \times B$$

where

$$B = \{x_d : x_d^2 = 2as_1 + b - s_1^2 - s_2^2 - \cdots - s_{d-1}^2,\ (a,b) \in E\}$$

B has measure zero if and only if $\{2as_1 + b : (a,b) \in E\}$ has measure zero. But $\mathcal{L}^1(\{2as_1 + b : (a,b) \in E\}) = 0$ for almost all $s_1 \in \mathbb{R}$ by Theorem 1.3. Therefore, by Fubini, A has d-dimensional measure zero. □

It is interesting to note that using Proposition 2.1, we obtain a geometric proof of the following special case of the spherical means maximal theorem in $\mathbb{R}^d$:

COROLLARY 2.1. *There exists an absolute constant C such that*

$$\|\mathcal{M}_\delta f\|_{L^2(\mathbb{R}^d)} \leq \|f\|_2$$

for small $\delta > 0$ and all f.

PROOF. Decompose $\mathbb{R}^d$ into disjoint cubes $\{Q_j\}$ of diameter $1/2$. Then by Proposition 2.1, we have that for all j

$$\int_{Q_j} (\mathcal{M}_\delta f(x))^2 dx \lesssim \|f\|_2^2$$

For each j let

$$I(j) = \{k : \text{dist}(Q_j, Q_k) < 3\}$$

Then there exists an integer N such that

$$\text{card}(I(j)) < N \ \forall j$$

$$\sum_j \chi_{I(j)}(k) < N \ \forall k$$

Also, for each k let $f_k = f\chi_{Q_k}$, so that

$$f = \sum_k f_k$$

Note that if $k \notin I(j)$ then $\chi_{Q_j}\mathcal{M}_\delta f_k = 0$. It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\mathcal{M}_\delta f(x))^2 dx \\
&= \sum_j \int_{Q_j} (\mathcal{M}_\delta f(x))^2 dx \\
&\leq \sum_j \int_{Q_j} \left(\sum_k \mathcal{M}_\delta f_k(x) \right)^2 dx \\
&= \sum_j \sum_{k,l} \int_{Q_j} \mathcal{M}_\delta f_k(x) \mathcal{M}_\delta f_l(x) dx \\
&\leq \sum_j \sum_{k,l \in I(j)} \left(\int_{Q_j} (\mathcal{M}_\delta f_k(x))^2 dx \right)^{1/2} \left(\int_{Q_j} (\mathcal{M}_\delta f_l(x))^2 dx \right)^{1/2} \\
&\lesssim \sum_j \sum_{k,l \in I(j)} \|f_k\|_2 \|f_l\|_2 \\
&\leq N \sum_j \sum_{k \in I(j)} \|f_k\|_2^2
\end{aligned}$$

$$= N \sum_k \sum_j \chi_{I(j)}(k) \|f_k\|_2^2$$

$$\leq N^2 \sum_k \|f_k\|_2^2$$

$$= N^2 \|f\|_2^2$$

□

CHAPTER 3

The two-dimensional case

Before we proceed with the proof of Theorem 2, it might be instructive to discuss briefly the underlying ideas. It turns out that the two-dimensional problem can be reduced to estimating the measure of a family of thin annuli. By Lemma 1.1, the measure of the intersection of two annuli is large when the corresponding circles are internally tangent. It is, therefore, essential that we be able to control the total number of such tangencies. To this end, we employ Marstrand's three circle lemma together with a suitable counting argument. This approach was first used in Kolasa and Wolff [**8**], and, subsequently, by Schlag [**10**], [**11**], [**12**]. We should, however, mention that, in contrast with the afforementioned authors, we do not make any cardinality estimates since these are not particularly useful in the case of general Hausdorff measures.

The motivation for the combinatorial part of the proof is the following observation (see [**6**] for more details):

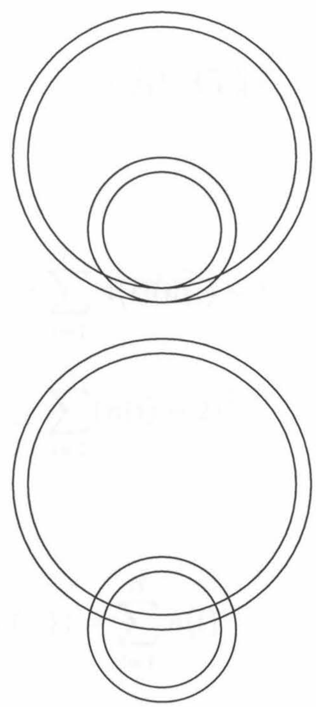

FIGURE 3. Tangential and transversal intersection of two annuli.

PROPOSITION 3.1. *Let $\{C_j\}_{j=1}^N$ be a family of distinct circles such that no three are tangent at a single point. Then*

$$card(\{(i,j) : C_i \parallel C_j\}) \lesssim N^{5/3}$$

where $C_i \parallel C_j$ means that C_i and C_j are internally tangent.

PROOF. Let $Q = \{(i, j_1, j_2, j_3) : C_i \parallel C_{j_k},\ k = 1, 2, 3\}$.

Fix j_1, j_2, j_3. Then, by the circles of Apollonius, there are at most two choices for j. Therefore,

$$\text{card}(Q) \leq 2N(N-1)(N-2) < 2N^3$$

On the other hand, if we let $n(i) = \text{card}(\{j : C_i \parallel C_j\})$, then

$$Q = \bigcup_{i=1}^{N} (\{i\} \times \{(j_1, j_2, j_3) : C_i \parallel C_{j_k},\ k = 1, 2, 3\})$$

Hence

$$\begin{aligned}\text{card}(Q) &\geq \sum_{i=1}^{N} n(i)(n(i)-1)(n(i)-2) \\ &\geq \sum_{i=1}^{N} (n(i)-2)^3\end{aligned}$$

It follows that

$$\begin{aligned}\text{card}(\{(i,j) : C_i \parallel C_j\}) &= \sum_{i=1}^{N} n(i) \\ &= \sum_{i=1}^{N} (n(i)-2) + 2N \\ &\leq \left(\sum_{i=1}^{N} (n(i)-2)^3\right)^{1/3} N^{2/3} + 2N \\ &\leq (\text{card}(Q))^{1/3} N^{2/3} + 2N \\ &\leq (2N^3)^{1/3} N^{2/3} + 2N \\ &\lesssim N^{5/3}\end{aligned}$$

□

The proof of Theorem 2 will be a quantitative version of Proposition 3.1, with Lemma 1.3 playing the role of the restriction imposed by the circles of Apollonius.

PROOF OF THEOREM 2. We may assume that $F \subset B(0, \frac{1}{4})$. Suppose that $|E| = 0$ and choose s_1 so that $3/2 < s_1 < s$. Then there exist a compact set $E_1 \subset E$, a compact set $F_1 \subset F$ with $\mathcal{H}^{s_1}(F_1) > 0$ and a positive number r, such that, for each $x \in F_1$, there is a circle centered at x with radius $r(x) \in (r, 2r)$ which intersects E_1 in a set of angle measure at least π. Without loss of generality we may assume that $r = 1$.

By Theorem 1.1, there exists a nontrivial finite measure μ supported on F_1 such that $\mu(B(x,r)) \leq r^{s_1}$, for $x \in \mathbb{R}^2$, $r > 0$.

Let $\{x_i\}_{i \in I}$ be a maximal δ-separated set of points in F_1, and let $a_i = \mu(B(x_i, \delta))$. Choose $r_i > 0$ such that

$$|C^\delta(x_i, r_i) \cap E_1^\delta| \geq \frac{1}{2}|C^\delta(x_i, r_i)| \tag{6}$$

where E_1^δ is the δ-neighborhood of E_1.

Let κ be the infimum of those $\lambda > 0$ such that there exists $J \subset I$ satisfying

$$\sum_{j \in J} a_j \geq \frac{1}{2}\mu(F_1)$$

and for all $j \in J$

$$\left|\left\{x \in C^\delta(x_j, r_j) \cap E_1^\delta : \sum_{i \in I} a_i \chi_{C^\delta(x_i, r_i)}(x) \leq \lambda\right\}\right| \geq \frac{1}{4}|C^\delta(x_j, r_j)|$$

Choose N large enough so that

$$s_1 - \frac{3}{2} > \frac{7 + 2s_1 - 1}{N}$$

and

$$\frac{N^2 - N}{N^2 - N - 2} < \frac{2s_1 + 1}{3}.$$

Let $C_2 > 1$ be a large constant to be determined later on.

Define $\beta : [\delta, 1] \times [\delta, 1] \longrightarrow \mathbb{R}$ by

$$\beta(t,\epsilon) = \begin{cases} \frac{t^{\frac{1}{2}(s_1-1/2)}}{\epsilon^{1/4}} C_2^{-\frac{3}{2}\frac{N}{N-2}}, & \text{if } t^{\frac{2s_1+1}{3}} < C_2^{2\frac{N}{N-2}}\epsilon \\ \frac{\epsilon^{1/N}}{t^{1/(N+1)}}, & \text{if } t^{\frac{2s_1+1}{3}} \geq C_2^{2\frac{N}{N-2}}\epsilon. \end{cases}$$

Then, for small δ, β has the following properties:

$$\beta(t,\epsilon) \geq C_2\sqrt{\frac{\epsilon}{t}} \Rightarrow \beta(t,\epsilon) = \frac{\epsilon^{1/N}}{t^{1/(N+1)}} \tag{7}$$

$$\beta(t,\epsilon) < C_2\sqrt{\frac{\epsilon}{t}} \Rightarrow \beta(t,\epsilon) = \frac{t^{\frac{1}{2}(s_1-1/2)}}{\epsilon^{1/4}} C_2^{-\frac{3}{2}\frac{N}{N-2}} \tag{8}$$

$$\sum_{\substack{\delta 2^k \leq 1 \\ \delta 2^l \leq 1}} \beta(\delta 2^k, \delta 2^l) < M \tag{9}$$

where M is a constant that depends only on N and on s_1.

Define for all $i, j \in I$ and $t, \epsilon \in [\delta, 1]$

$$\Delta_{ij} = max\{\delta, ||x_i - x_j| - |r_i - r_j||\}$$

$$S_{t,\epsilon}(j) = \{i \in I : C^\delta(x_i, r_i) \cap C^\delta(x_j, r_j) \neq \emptyset,\ t \leq |x_i - x_j| \leq 2t,$$

$$\epsilon \leq \Delta_{ij} \leq 2\epsilon\}$$

$$A_{t,\epsilon}(j) = \Big\{x \in C^\delta(x_j, r_j) : \sum_{i \in S_{t,\epsilon}(j)} a_i \chi_{C^\delta(x_i,r_i)}(x) \geq \frac{1}{M}\beta(t,\epsilon)\frac{\kappa}{2}\Big\}$$

CLAIM 3.1. *There exist $t, \epsilon \in [\delta, 1]$ and a set of indices $\bar{J}$ such that*

$$|A_{t,\epsilon}(j)| \geq \frac{1}{4M}\beta(t,\epsilon)|C^\delta(x_j, r_j)|, \ \forall j \in \bar{J}$$

and

$$\sum_{j \in \bar{J}} a_j \geq \frac{1}{2M}\beta(t,\epsilon)\mu(F_1)$$

PROOF. Let

$$J_0 = \Big\{j \in I : \Big|\Big\{x \in C^\delta(x_j, r_j) \cap E_1^\delta : \sum_{i \in I} a_i \chi_{C^\delta(x_i,r_i)}(x) \leq \frac{\kappa}{2}\Big\}\Big| \geq \frac{1}{4}|C^\delta(x_j, r_j)|\Big\}$$

By the minimality of κ, we have

$$\sum_{j \in J_0} < \frac{1}{2}\mu(F_1)$$

Therefore, if J' is the complement of J_0, then

$$\sum_{j \in J'} a_j \geq \frac{1}{2}\mu(F_1) \tag{10}$$

and for all $j \in J'$

$$\Big|\Big\{C^\delta(x_j, r_j) \cap E_1^\delta : \sum_{i \in I} a_i \chi_{C^\delta(x_i,r_i)}(x) \leq \frac{\kappa}{2}\Big\}\Big| < \frac{1}{4}|C^\delta(x_j, r_j)|$$

Hence, using 6 we obtain

$$\left|\left\{C^\delta(x_j,r_j)\cap E_1^\delta : \sum_{i\in I} a_i\chi_{C^\delta(x_i,r_i)}(x) > \frac{\kappa}{2}\right\}\right| \geq \frac{1}{4}|C^\delta(x_j,r_j)| \tag{11}$$

For each $j \in J'$ let

$$B_j = \left\{x \in C^\delta(x_j,r_j)\cap E_1^\delta : \sum_{i\in I} a_i\chi_{C^\delta(x_i,r_i)}(x) > \frac{\kappa}{2}\right\} \tag{12}$$

Then for all $j \in J'$

$$B_j \subset \bigcup_{k,l} A_{\delta 2^k,\delta 2^l}(j)$$

Indeed, suppose there existed $j \in J'$, $x \in B_j$ such that for all k, l with $\delta 2^k, \delta 2^l \leq 1$ we had $x \notin A_{\delta 2^k,\delta 2^l}(j)$. Then, by 9

$$\begin{aligned}\sum_{i\in I} a_i\chi_{C^\delta(x_i,r_i)}(x) &= \sum_{k,l}\sum_{i\in S_{\delta 2^k,\delta 2^l}(j)} a_i\chi_{C^\delta(x_i,r_i)}(x)\\ &\leq \frac{1}{M}\frac{\kappa}{2}\sum_{k,l}\beta(\delta 2^k,\delta 2^l)\\ &< \frac{\kappa}{2}, \text{ contradicting 12.}\end{aligned}$$

It follows that for all $j \in J'$ there exist k, l such that

$$|A_{\delta 2^k,\delta 2^l}(j)| \geq \frac{1}{4M}\beta(\delta 2^k,\delta 2^l)|C^\delta(x_j,r_j)| \tag{13}$$

In fact, if this were not the case, we would have that for some $j \in J'$

$$\begin{aligned}|B_j| &\leq \left|\bigcup\nolimits_{k,l} A_{\delta 2^k,\delta 2^l}(j)\right|\\ &\leq \sum_{k,l}|A_{\delta 2^k,\delta 2^l}(j)|\end{aligned}$$

$$\leq \frac{1}{4M}|C^{\delta}(x_j, r_j)| \sum_{k,l} \beta(\delta 2^k, \delta 2^l)$$

$$< \frac{1}{4}|C^{\delta}(x_j, r_j)|, \text{ which contradicts 11.}$$

Finally, let

$$J(k,l) = \{j \in J' : |A_{\delta 2^k, \delta 2^l}(j)| \geq \frac{1}{4M}\beta(\delta 2^k, \delta 2^l)|C^{\delta}(x_j, r_j)|\}$$

Then, by 13

$$J' = \bigcup_{k,l} J(k,l)$$

We claim that there exist $t = \delta 2^k$,$\epsilon = \delta 2^l$ such that

$$\sum_{j \in J(k,l)} a_j \geq \frac{1}{2M}\beta(t, \epsilon)\mu(F_1)$$

If not, then we would have

$$\sum_{j \in J'} a_j \leq \sum_{k,l} \sum_{j \in J(k,l)} a_j$$

$$< \frac{1}{2M}\mu(F_1) \sum_{k,l} \beta(\delta 2^k, \delta 2^l)$$

$$< \frac{1}{2}\mu(F_1), \text{ contradicting 10.}$$

□

Fix $t, \epsilon \in [\delta, 1]$ as above. Then there are two cases:

Case 1: $\beta(t,\epsilon) \geq C_2\sqrt{\frac{\epsilon}{t}}$

It follows from the definition of κ that there exists a set of indices $J \subset I$ such that

$$\sum_{j\in J} a_j \geq \frac{1}{2}\mu(F_1)$$

and

$$\left|\left\{x \in C^\delta(x_j,r_j)\cap E_1^\delta : \sum_{i\in I} a_i\chi_{C^\delta(x_i,r_i)}(x) \leq 2\kappa\right\}\right| \geq \frac{1}{4}|C^\delta(x_j,r_j)|$$

for all $j \in J$.

Let

$$Q = \{(j,j_1,j_2,j_3) : j \in \bar{J},\ \ j_1,j_2,j_3 \in J,\ \ j_1,j_2,j_3 \in S_{t,\epsilon}(j)$$

$$\operatorname{dist}(C^\delta(x_j,r_j)\cap C^\delta(x_{j_k},r_{j_k}), C^\delta(x_j,r_j)\cap C^\delta(x_{j_l},r_{j_l})) \geq \frac{\beta(t,\epsilon)}{C_1 M}$$

$$\forall k,l\ k \neq l\}$$

where $C_1 > 1$ is a constant to be determined before C_2.

Further, define the following sets of indices:

$$Q_1 = \{(j_1,j_2,j_3) : \exists j \text{ such that } (j,j_1,j_2,j_3) \in Q\}$$

$$Q_2 = \{j_1 : \exists j_2, j_3 \text{ such that } (j_1,j_2,j_3) \in Q_1\}$$

For $j_1 \in Q_2$ let

$$Q(j_1) = \{j_2 : \exists j_3 \text{ such that } (j_1, j_2, j_3) \in Q_1\}$$

$$= \{j_3 : \exists j_2 \text{ such that } (j_1, j_2, j_3) \in Q_1\}$$

Let

$$R = \sum_{(j, j_1, j_2, j_3) \in Q} a_j a_{j_1} a_{j_2} a_{j_3}$$

Note that if C_2 is large enough, then

$$\frac{\beta(t, \epsilon)}{C_1 M} \geq \frac{C_2}{C_1 M} \sqrt{\frac{\epsilon}{t}} \geq A_1 \sqrt{\frac{\epsilon}{t}}$$

where A_1 is the constant in Lemma 1.3. It follows that if $(j_1, j_2, j_3) \in Q_1$ then the set $\{x_j : (j, j_1, j_2, j_3) \in Q\}$ is contained in the union of two ellipsoids of diameter $\lesssim \frac{\epsilon}{\beta^2(t, \epsilon)}$. Hence

$$R \lesssim \left(\frac{\epsilon}{\beta^2(t, \epsilon)} \right)^{s_1} \sum_{(j_1, j_2, j_3) \in Q_1} a_{j_1} a_{j_2} a_{j_3}$$

Furthermore, if $j_1 \in Q_2$ and $j_2 \in Q(j_1)$ then there exists j such that $j_1, j_2 \in S_{t,\epsilon}(j)$. Therefore,

$$|x_{j_1} - x_{j_2}| \leq |x_{j_1} - x_j| + |x_j - x_{j_2}| \leq 4t$$

It follows that for fixed $j_1 \in Q_2$ the set $\{x_{j_2} : j_2 \in Q(j_1)\}$ is contained in a disk with center x_{j_1} and radius $4t$. Hence

$$\sum_{j_2 \in Q(j_1)} a_{j_2} \lesssim t^{s_1}$$

Therefore,

$$R \lesssim \left(\frac{\epsilon}{\beta^2(t,\epsilon)}\right)^{s_1} \sum_{(j_1,j_2,j_3)\in Q_1} a_{j_1}a_{j_2}a_{j_3}$$
$$\leq \left(\frac{\epsilon}{\beta^2(t,\epsilon)}\right)^{s_1} \sum_{j_1\in Q_2} a_{j_1}\left(\sum_{j_2\in Q(j_1)} a_{j_2}\right)^2$$
$$\lesssim \mu(F_1)\left(\frac{\epsilon}{\beta^2(t,\epsilon)}\right)^{s_1}(t^{s_1})^2 \tag{14}$$

Now fix $j \in \bar{J}$.

Claim 3.2. *There are three subsets D_1, D_2, D_3 of $A_{t,\epsilon}(j)$ such that*

$$dist(D_k, D_l) \geq \frac{2\beta(t,\epsilon)}{C_1 M}, \ \forall k, l \ k \neq l$$

and

$$|D_k| \gtrsim \delta\beta(t,\epsilon), \ \forall k$$

provided that C_1 is large enough.

Proof. We use complex notation. If $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$ let

$$G_{\theta_1,\theta_2} = A_{t,\epsilon}(j) \cap \{x_j + re^{i\theta} \in C^\delta(x_j, r_j) : \theta_1 \leq \theta \leq \theta_2\}$$

Then there exist $0 = \theta_1 < \cdots < \theta_7 = 2\pi$ such that

$$|G_{\theta_k,\theta_{k+1}}| = \frac{|A_{t,\epsilon}(j)|}{6}, \ k = 1, ..., 6$$

Let

$$D_k = G_{\theta_{2k-1},\theta_{2k}}, \ k = 1, 2, 3$$

Note that for all l

$$\frac{\beta(t,\epsilon)}{24M}|C^{\delta}(x_j,r_j)| \leq |G_{\theta_l,\theta_{l+1}}| \lesssim \operatorname{diam}(G_{\theta_l,\theta_{l+1}})\delta$$

Therefore,

$$\operatorname{diam}(G_{\theta_l,\theta_{l+1}}) \gtrsim \beta(t,\epsilon)$$

It follows that if we choose C_1 large enough, then we have

$$\operatorname{dist}(D_k,D_l) \geq \frac{2\beta(t,\epsilon)}{C_1 M}, \text{ and } |D_k| \gtrsim \beta(t,\epsilon)\delta$$

□

For each k let

$$S_k = \{i \in S_{t,\epsilon}(j) : D_k \cap C^{\delta}(x_i,r_i) \neq \emptyset\}$$

Then

$$\begin{aligned}
\kappa\beta^2(t,\epsilon)\delta &\lesssim \int_{D_k} \frac{\beta(t,\epsilon)}{M}\frac{\kappa}{2}dx \\
&\leq \sum_{i\in S_k} a_i|D_k \cap C^{\delta}(x_i,r_i)| \\
&\leq \sum_{i\in S_k} a_i|C^{\delta}(x_j,r_j) \cap C^{\delta}(x_i,r_i)| \\
&\lesssim \sum_{i\in S_k} a_i\frac{\delta^2}{\sqrt{t\epsilon}}
\end{aligned}$$

where the last inequality follows from Lemma 1.1. Therefore,

$$\sum_{i\in D_k} a_i \gtrsim \frac{1}{\delta}\kappa\beta^2(t,\epsilon)\sqrt{t\epsilon}$$

By Lemma 1.1, if $i \in S_{t,\epsilon}(j)$ then

$$\operatorname{diam}(C^\delta(x_j, r_j) \cap C^\delta(x_i, r_i)) \leq A\sqrt{\frac{\epsilon}{t}} \leq \frac{A\beta(t,\epsilon)}{C_2}$$

Therefore, $i_1 \in S_k$, $i_2 \in S_l$, $k \neq l$ implies that

$$\begin{aligned}\operatorname{dist}(C^\delta(x_{i_1}, r_{i_1}) \cap C^\delta(x_j, r_j), C^\delta(x_{i_2}, r_{i_2}) \cap C^\delta(x_j, r_j)) &\geq \frac{2\beta(t,\epsilon)}{C_1 M} \\ &- \frac{2A\beta(t,\epsilon)}{C_2} \\ &\geq \frac{\beta(t,\epsilon)}{C_1 M}\end{aligned}$$

provided that C_2 is sufficiently large. It follows that if $j_k \in S_k$, $k = 1, 2, 3$, then $(j, j_1, j_2, j_3) \in Q$. Hence

$$\begin{aligned}R &\geq \sum_{j \in \bar{J}} a_j \sum_{\substack{j_1 \in S_1 \\ j_2 \in S_2 \\ j_3 \in S_3}} a_{j_1} a_{j_2} a_{j_3} \\ &\gtrsim \beta(t,\epsilon)\left(\frac{1}{\delta}\kappa\beta^2(t,\epsilon)\sqrt{t\epsilon}\right)^3\end{aligned}$$

If we compare the above equation with 14, and then use 7, we obtain

$$\kappa^3 \lesssim \delta^3 \frac{\epsilon^{s_1-3/2} t^{2s_1-3/2}}{\beta^{2s_1+7}(t,\epsilon)} \lesssim \delta^3$$

Case 2: $\beta(t,\epsilon) \leq C_2\sqrt{\frac{\epsilon}{t}}$

Fix $j \in \bar{J}$. Then we have

$$\kappa\delta\beta^2(t,\epsilon) \lesssim \kappa\frac{\beta(t,\epsilon)}{M}|A_{t,\epsilon}(j)|$$

$$
\begin{aligned}
&= \int_{A_{t,\epsilon}(j)} \kappa \frac{\beta(t,\epsilon)}{M} dx \\
&\lesssim \sum_{i \in S_{t,\epsilon}(j)} a_i |C^\delta(x_i, r_i) \cap C^\delta(x_j, r_j)| \\
&\lesssim \frac{\delta^2}{\sqrt{t\epsilon}} \sum_{i \in S_{t,\epsilon}(j)} a_i
\end{aligned}
$$

where we have used Lemma 1.1 and the definition of $A_{t,\epsilon}(j)$.

Note that the set $\{x_i : i \in S_{t,\epsilon}(j)\}$ is contained in a disk of radius $2t$. Therefore,

$$\sum_{i \in S_{t,\epsilon}(j)} a_i \lesssim t^{s_1}$$

It follows that

$$\kappa \lesssim \delta \frac{t^{s_1 - 1/2}}{\epsilon^{1/2} \beta^2(t,\epsilon)}$$

Using 8, we obtain $\kappa \lesssim \delta$. We conclude that, in either case

$$\kappa \lesssim \delta \tag{15}$$

To complete the proof, notice that

$$
\begin{aligned}
\frac{1}{2}\mu(F_1) &\leq \sum_{j \in J} a_j \\
&= \frac{1}{\delta} \sum_{j \in J} a_j \delta \\
&\lesssim \frac{1}{\delta} \sum_{j \in J} a_j \left| \left\{ x \in C^\delta(x_j, r_j) \cap E_1^\delta : \sum_{i \in I} a_i \chi_{C^\delta(x_i, r_i)}(x) \leq 2\kappa \right\} \right| \\
&\leq \frac{1}{\delta} \int_{\left\{x \in E_1^\delta : \sum_{j \in J} a_j \chi_{C^\delta(x_j, r_j)}(x) \leq 2\kappa \right\}} \Big(\sum_{j \in J} a_j \chi_{C^\delta(x_j, r_j)}(x) \Big) dx
\end{aligned}
$$

$$\lesssim \frac{1}{\delta}\kappa|E_1^\delta|$$

$$\lesssim |E_1^\delta| \tag{16}$$

where the last inequality follows from 15.

If we let $\delta \to 0$ then the right-hand side of 16 tends to zero, which is a contradiction. □

CHAPTER 4

Possible improvements

As we discussed at the beginning of Section 3, the proof of Theorem 2 was motivated by a result of combinatorial nature, namely Proposition 3.1, which asserts that if one is given a family of N circles such that no three of them are internally tangent at a point, then there is a bound of the form $CN^{5/3}$ on the total number of tangencies.

This, however, is far from being sharp. Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [**3**] developed a technique which leads to a bound of the form $C_\epsilon N^{3/2+\epsilon}$ $\forall\epsilon > 0$, suggesting that it might be possible to weaken the condition $s > 3/2$ in Theorem 2. Indeed, Wolff [**16**] proved the following $L^3 \to L^3$ maximal inequality:

THEOREM 4.1. *For* $x_1 \in \mathbb{R}$, *let*

$$M_\delta f(x_1) = \sup_{\substack{r\in[1/2,2]\\ x_2\in\mathbb{R}}} \frac{1}{|C^\delta(x,r)|}\int_{C^\delta(x,r)} |f|$$

where $x = (x_1, x_2)$. *Then*

$$\forall\epsilon > 0\ \exists A_\epsilon : \|M_\delta f\|_{L^3(\mathbb{R})} \le A_\epsilon \delta^{-\epsilon}\|f\|_3.$$

Using this, he proved, in the same paper, the following:

THEOREM 4.2. *If $\alpha \leq 1$ and if E is a set in the plane which contains circles centered at all points of a set with Hausdorff dimension at least α, then E has Hausdorff dimension at least $1 + \alpha$.*

The preceding result suggests that a set E as in the statement of Theorem 2 has to be fairly large. In view of this and the analogy between Proposition 2.1 and the spherical means maximal theorem, it seems reasonable to make the following conjecture which would imply that Theorem 2 is true for all $s > 1$.

CONJECTURE 1. *For $\delta > 0$ small, $f : \mathbb{R}^2 \to \mathbb{R}$, define $\mathcal{M}_\delta : B(0, 1/4) \to \mathbb{R}$, by*

$$\mathcal{M}_\delta f(x) = \sup_{1/2 \leq r \leq 2} \frac{1}{|C^\delta(x,r)|} \int_{C^\delta(x,r)} |f(y)| dy$$

Let $F \subset B(0, 1/4)$ be a compact set in $\mathbb{R}^2$ such that there exist $s > 1$ and a finite measure μ supported on F with $\mu(B(x,r)) \leq r^s$, for $x \in \mathbb{R}^2$ and $r > 0$. Then there exists a constant A that depends only on s, such that

$$\left(\int_F (\mathcal{M}_\delta f(x))^{p(s)} d\mu(x) \right)^{1/p(s)} \leq A \|f\|_{p(s)}. \tag{15}$$

Note that in order for the above inequality to hold, it is necessary that $p(s) \geq 4 - s$. To see that, let $I = [-1/8, 1/8]$, and let $E \subset I$ be a Cantor set of Hausdorff dimension $s-1$. Then $\mathcal{H}^{s-1}(E \cap B(0, \delta)) \sim \delta^{s-1}$.

Define

$$F_\delta = I \times (E \cap B(0, \delta^{1/2}))$$

and

$$R_\delta = [1-\delta, 1+\delta] \times [-2\delta^{1/2}, 2\delta^{1/2}]$$

Notice that

$$x \in F_\delta \Rightarrow \mathcal{M}_\delta \chi_{R_\delta}(x) \gtrsim \delta^{1/2}$$

Therefore, using 15

$$\begin{aligned} \delta^{1/2}(\mathcal{H}^s(F_\delta))^{1/p(s)} &\lesssim \left(\int_{F_\delta} (\mathcal{M}_\delta \chi_{R_\delta})^{p(s)}(x) d\mathcal{H}^s(x) \right)^{1/p(s)} \\ &\lesssim \|\chi_{R_\delta}\|_{p(s)} \\ &= \delta^{3p(s)/2} \end{aligned}$$

On the other hand

$$\mathcal{H}^s(F_\delta) \sim \mathcal{H}^{s-1}(E \cap B(0, \delta^{1/2})) \sim \delta^{(s-1)/2}$$

Hence

$$\delta^{\frac{1}{2}} \delta^{\frac{(s-1)}{2p(s)}} \lesssim \delta^{\frac{3}{2p(s)}}$$

which is possible only if $p(s) \geq 4 - s$.

Now we will present a heuristic argument affording evidence that the techniques in [**3**] might be used to prove that Theorem 2 is true for all $s > 1$. Assume that E is compact, $|E| = 0$, and suppose that F

satisfies the following regularity condition: There exists a non-trivial finite measure μ supported on F such that $\mu(B(x,r)) \sim r^s$, for all $x \in F$, $r > 0$. Let $\{x_j\}_{j=1}^N$ be a maximal δ-separated subset of F. Then $N \sim \left(\frac{1}{\delta}\right)^s$. Let $\{r_j\}_{j=1}^N$ be such that $C^\delta(x_j, r_j) \subset E^\delta$. Define

$$A = \{(i,j) : C(x_i, r_i),\ C(x_j, r_j) \text{ are tangent}\}$$

$$B = \{(i,j) : C(x_i, r_i),\ C(x_j, r_j) \text{ intersect at an angle} \gtrsim \pi/100\}$$

Now suppose that if $i \neq j$ then

$$\text{either } (i,j) \in A \text{ in which case } |C^\delta(x_i, r_i) \cap C^\delta(x_j, r_j)| \lesssim \delta^{3/2}$$

$$\text{or } (i,j) \in B \text{ in which case } |C^\delta(x_i, r_i) \cap C^\delta(x_j, r_j)| \lesssim \delta^2$$

Choose $\epsilon > 0$ such that $\epsilon > \frac{1}{2}\left(1 - \frac{1}{s}\right)$. It follows from the remarks at the beginning of this chapter that $|A| \lesssim N^{\frac{3}{2}+\epsilon}$.

By an argument similar to the one used in the proof of Theorem 1, one shows that

$$\begin{aligned}
1 &\lesssim \delta^{s-1}|E^\delta|^{1/2} \left(\sum_{i,j} |C^\delta(x_i, r_i) \cap C^\delta(x_j, r_j)| \right)^{1/2} \\
&\lesssim \delta^{s-1}|E^\delta|^{1/2} \left(\sum_{(i,j)\in A} \delta^{3/2} + \sum_{(i,j)\in B} \delta^2 + \frac{1}{\delta^s}\delta \right)^{1/2} \\
&\lesssim \delta^{s-1}|E^\delta|^{1/2} \left(\left(\frac{1}{\delta^s}\right)^{3/2+\epsilon} \delta^{3/2} + \left(\frac{1}{\delta^s}\right)^2 \delta^2 + \frac{1}{\delta^s}\delta \right)^{1/2} \\
&\lesssim |E^\delta|^{1/2}
\end{aligned}$$

Letting $\delta \to 0$, we get a contradiction.

We conclude by presenting an alternate way to attack the two-dimensional problem (see Schlag[**11**]).

Let $L^p_\alpha(\mathbb{R}^d) = \{f : f = G_\alpha * g,\ g \in L^p(\mathbb{R}^d)\}$, $\alpha \in \mathbb{R}$, $p > 1$, be the space of Bessel potentials, with norm $\|f\|_{\alpha,p} = \|g\|_p$. Here G_α is the Bessel kernel, i.e., the inverse Fourier transform of the function $\hat{G}_\alpha(\xi) = (1 + |\xi|^2)^{-\alpha/2}$.

The Bessel capacity of a set $E \subset \mathbb{R}^d$ is defined as

$$B_{\alpha,p}(E) = \inf\{\|f\|^p_{\alpha,p} : \ f \geq 1 \text{ on } E\}$$

The relation between capacity and Hausdorff dimension is given by the following result due to Havin and Maz'ya [**7**]:

THEOREM 4.3. *Let $E \subset \mathbb{R}^d$ be a Borel set. If $p > 1$, $\alpha p \leq d$, then*

$$B_{\alpha,p}(E) = 0 \Rightarrow \mathcal{H}^{d-\alpha p+\epsilon}(E) = 0, \text{ for every } \epsilon > 0.$$

We refer the reader to the Appendix for a proof of a generalization of the preceding result.

Furthermore, for $f \in C^\infty_c(\mathbb{R}^2)$, define

$$\mathfrak{M}_f(x,t) = \int_{S^1} f(x - ty)d\sigma(y)$$

where $d\sigma$ is arc measure.

Sogge [**13**] made the following conjecture regarding the above operator:

CONJECTURE 2. *For all $\epsilon > 0$, there exists $A_\epsilon > 0$, such that*

$$\|\mathfrak{M}_f\|_{L^4_{1/2-\epsilon}(\mathbb{R}^2\times[1,2])} \leq A_\epsilon \|f\|_{L^4(\mathbb{R}^2)} \tag{18}$$

for all $f \in C_c^\infty(\mathbb{R}^2)$.

We will show how 18 would imply that the conclusion of Theorem 2 holds for all $s > 1$. As before, suppose that $|E|$=0 and choose s_1 and $\delta > 0$, so that $1 < 1 + \delta < s_1 < s$. Then there exist a compact set $E_1 \subset E$ and a compact set $F_1 \subset F$ with $\mathcal{H}^{s_1}(F_1) > 0$ such that for each $x \in F_1$ there is a circle centered at x with radius $r(x) \in (1, 2)$ which intersects E_1 in a set of angle measure at least π. Let $\tilde{E}_1 = \{(x, r(x)) : x \in F_1\}$. Then there exists a sequence $\{f_n\}$ in $C_c^\infty(\mathbb{R}^2)$ such that $f_n \geq 1$ on E_1 and $\|f_n\|_4 \to 0$ as $n \to \infty$. Choose $\psi \in C_c^\infty(\mathbb{R}^2)$ such that $\psi = 1$ on $(1, 2)$ and define

$$g_n(x,t) = \mathfrak{M}_{f_n}(x,t)\psi(t)$$

Then $g_n \gtrsim 1$ on $\tilde{E}_1$. Hence, by 18

$$B_{1/2-\delta/8,4}(\tilde{E}_1) \leq \|g_n\|^4_{L^4_{1/2-\delta/8}(\mathbb{R}^3)} \lesssim \|f_n\|^4_{L^4(\mathbb{R}^2)}$$

Letting $n \to \infty$ we obtain $B_{1/2-\delta/8,4}(\tilde{E}_1) = 0$, and therefore, by Theorem 5.1

$$\mathcal{H}^{3-(1/2-\delta/8)4+\delta/2}(\tilde{E}_1) = 0 \Rightarrow \mathcal{H}^{1+\delta}(\tilde{E}_1) = 0 \Rightarrow \mathcal{H}^{1+\delta}(F_1) = 0$$

contradicting the choice of δ.

Appendix

Here we prove a generalization of Theorem 4.3.

Let $L^{p_1,p_2}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, $p_1 > 1$, $p_2 > 1$ be the space of all functions with finite $\|\cdot\|_{p_1,p_2}$ norm, where

$$\|g\|_{p_1,p_2} = \left(\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} |g(x_1,x_2)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right)^{1/p_2}$$

For $\alpha > 0$, define the space

$$L^{p_1,p_2}_{\alpha}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) = \{f : f = G_\alpha * g,\ g \in L^{p_1,p_2}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})\}$$

with norm $\|f\|_{\alpha,p_1,p_2} = \|g\|_{p_1,p_2}$.

The mixed-norm capacity of $E \subset \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is defined as

$$B_{\alpha,p_1,p_2}(E) = \inf\{\|f\|^{p_2}_{\alpha,p_1,p_2} : f \geq 1 \text{ on } E\}$$

THEOREM. *Let $E \subset \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ be a Borel set.*

If $p_1 \leq p_2$ and $d_2 + d_1\frac{p_2}{p_1} - p_2\alpha \geq 0$ then

$$B_{\alpha,p_1,p_2}(E) = 0 \Rightarrow \mathcal{H}^{d_2+d_1\frac{p_2}{p_1}-p_2\alpha+\epsilon}(E) = 0, \text{ for every } \epsilon > 0.$$

If $p_2 \leq p_1$ and $d_1 + d_2\frac{p_1}{p_2} - p_1\alpha \geq 0$ then

$$B_{\alpha,p_1,p_2}(E) = 0 \Rightarrow \mathcal{H}^{d_1+d_2\frac{p_1}{p_2}-p_1\alpha+\epsilon}(E) = 0, \text{ for every } \epsilon > 0.$$

PROOF. Without loss of generality we may assume that $E \subset [0,1]^d$. Let μ be a finite measure supported on E, and let u be a non-negative C_c^∞ function such that $u \geq 1$ on E. Then

$$\begin{aligned}\mu(E) &\leq \int u(x)d\mu(x) \\ &= \int G_\alpha * D^\alpha u(x)d\mu(x) \\ &= \int D^\alpha u(y) \int G_\alpha(x-y)d\mu(x)dy \\ &\leq \|u\|_{\alpha,p_1,p_2}\|G_\alpha * \mu\|_{q_1,q_2}\end{aligned}$$

where q_1, q_2 are the conjugate exponents of p_1, p_2 respectively, and $D^\alpha u$ is the fractional derivative operator acting on u, defined as the inverse Fourier transform of the function $(1+|\xi|^2)^{\alpha/2}\hat{u}(\xi)$.

For each $n \geq 0$ we subdivide $\mathbb{R}^d$ into disjoint dyadic cubes of side 2^{-n}, so that each cube of side 2^{-k} is split into 2^d cubes of side $2^{-(k+1)}$. If Q is such a dyadic cube then $l(Q)$ denotes its sidelength and $\widetilde{Q}$ the cube with the same center as Q and sidelength $3l(Q)$.

Let

$$\widetilde{I}_\alpha(x) = \begin{cases} |x|^{\alpha-d}, & \text{if } 0 < |x| \leq 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

It follows from the properties of the Bessel kernel (see, e.g., [**1**]) that there exist constants a and A such that

$$G_\alpha(x) \leq A\widetilde{I}_\alpha(x), \ 0 < |x| \leq 1$$

and

$$G_\alpha(x) \leq Ae^{-a|x|}, \ |x| > 1$$

Therefore,

$$\begin{aligned}
&\|G_\alpha * \mu\|_{q_1,q_2} \\
&\lesssim \left(\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} (\widetilde{I}_\alpha * \mu(x_1,x_2))^{q_1} dx_1 \right)^{\frac{q_2}{q_1}} dx_2 \right)^{\frac{1}{q_2}} \\
&+ \left(\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \left(\int_{|(x_1,x_2)-y|>1} e^{-a|(x_1,x_2)-y|} d\mu(y) \right)^{q_1} dx_1 \right)^{\frac{q_2}{q_1}} dx_2 \right)^{\frac{1}{q_2}} \\
&= B + B'
\end{aligned}$$

B' is easy to estimate. By Minkowski's inequality for integrals, we have

$$\begin{aligned}
B' &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} e^{-aq_1|(x_1,x_2)-y|} dx_1 \right)^{q_2/q_1} dx_2 \right)^{1/q_2} d\mu(y) \\
&\leq \mu(E) \left(\int_{\mathbb{R}^{d_2}} e^{-\frac{1}{\sqrt{2}}aq_2|x_2|} dx_2 \left(\int_{\mathbb{R}^{d_1}} e^{-\frac{1}{\sqrt{2}}aq_1|x_1|} dx_1 \right)^{q_2/q_1} \right)^{1/q_2} < \infty
\end{aligned}$$

On the other hand

$$\widetilde{I}_\alpha * \mu(x) = \int_{|x-y|\leq 1} \frac{d\mu(y)}{|x-y|^{d-\alpha}}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}<|x-y|\leq 2^{-n}} \frac{d\mu(y)}{|x-y|^{d-\alpha}} \\
&\leq \sum_{n=0}^{\infty} 2^{(n+1)(d-\alpha)} \mu(B(x, 2^{-n})) \\
&\lesssim \sum_{l(Q)\leq 1} \frac{\mu(\widetilde{Q})}{l(Q)^{d-\alpha}} \chi_Q(x) \\
&\lesssim \left(\sum_{n=0}^{\infty} 2^{-\delta p_1 (n+1)} \right)^{1/p_1} \left(\sum_{l(Q)\leq 1} \frac{\mu(\widetilde{Q})^{q_1}}{l(Q)^{q_1(d-\alpha+\delta)}} \chi_Q(x) \right)^{1/q_1}
\end{aligned}
$$

where δ is a positive number.

Let $\pi_2 : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$ be the usual projection $\pi_2(x_1, x_2) = x_2$. Also, let $s = d_2 + d_1 \frac{p_2}{p_1} - p_2\alpha$ and $t = d_1 + d_2 \frac{p_1}{p_2} - p_1\alpha$.

Suppose that $p_1 \leq p_2$ and that $\mathcal{H}^{s+\epsilon}(E) > 0$ for some $\epsilon > 0$. Then there exists a nontrivial finite measure μ supported on E such that $\mu(B(x,r)) \leq r^{s+\epsilon}$ for all $x \in \mathbb{R}^d$, $r > 0$. It follows that

$$
\begin{aligned}
B^{q_2} &= \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} (\widetilde{I}_\alpha * \mu(x_1, x_2))^{q_1} dx_1 \right)^{q_2/q_1} dx_2 \\
&\leq C \int_{\mathbb{R}^{d_2}} \left(\sum_{l(Q)\leq 1} \frac{\mu(\widetilde{Q})^{q_1}}{l(Q)^{q_1(d-\alpha+\delta)}} l(Q)^{d_1} \chi_{\pi_2(Q)}(x_2) \right)^{q_2/q_1} dx_2 \\
&\leq C \sum_{l(Q)\leq 1} \frac{\mu(\widetilde{Q})^{q_2}}{l(Q)^{q_2(d-\alpha+\delta)-d_1\frac{q_2}{q_1}-d_2}} \\
&= C \sum_{n=0}^{\infty} 2^{n(q_2(d-\alpha+\delta)-d_1\frac{q_2}{q_1}-d_2)} \sum_{l(Q)=2^{-n}} \mu(\widetilde{Q}) \mu(\widetilde{Q})^{q_2-1} \\
&\lesssim C\mu(E) \sum_{n=0}^{\infty} \frac{2^{n(q_2(d-\alpha+\delta)-d_1\frac{q_2}{q_1}-d_2)}}{2^{n(q_2-1)(s+\epsilon)}} < \infty
\end{aligned}
$$

provided that δ has been chosen so that $p_2\delta < \epsilon$.

Now suppose that $p_2 \le p_1$ and that $\mathcal{H}^{t+\epsilon}(E) > 0$ for some $\epsilon > 0$. Then, as above, there exists a nontrivial finite measure supported on E such that $\mu(B(x,r)) \le r^{t+\epsilon}$ for all $x \in \mathbb{R}^d$, $r > 0$. It follows that

$$B^{q_1} \le C \left(\int_{\mathbb{R}^{d_2}} \left(\sum_{l(Q)\le 1} \frac{\mu(\widetilde{Q})^{q_1}}{l(Q)^{q_1(d-\alpha+\delta)}} l(Q)^{d_1} \chi_{\pi_2(Q)}(x_2) \right)^{q_2/q_1} dx_2 \right)^{q_1/q_2}$$

$$\le C \sum_{l(Q)\le 1} \frac{\mu(\widetilde{Q})^{q_1}}{l(Q)^{q_1(d-\alpha+\delta)-d_2\frac{q_1}{q_2}-d_1}}$$

$$= C \sum_{n=0}^{\infty} 2^{n(q_1(d-\alpha+\delta)-d_2\frac{q_1}{q_2}-d_1)} \sum_{l(Q)=2^{-n}} \mu(\widetilde{Q})\mu(\widetilde{Q})^{q_1-1}$$

$$\lesssim C\mu(E) \sum_{n=0}^{\infty} \frac{2^{n(q_1(d-\alpha+\delta)-d_2\frac{q_1}{q_2}-d_1)}}{2^{n(q_1-1)(t+\epsilon)}} < \infty$$

provided that $p_1\delta < \epsilon$.

It follows that $\mu(E) \lesssim \|u\|_{\alpha,p_1,p_2}$. By assumption, $B_{\alpha,p_1,p_2}(E) = 0$. Therefore $\mu(E) = 0$ which is a contradiction. □

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