The perturbation of Hamiltonian systems with a non-Abelian symmetry

Thesis by

Anthony D. Blaom

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To my father,

Anthony E. Perry,

for inspiration.

Abstract

The perturbation theory of non-commutatively integrable systems is revisited from the point of view of non-Abelian symmetry groups. Using a coordinate system intrinsic to the geometry of the symmetry, we generalize well-known estimates of Nekhoroshev (1977) in a class of systems having almost G-invariant Hamiltonians. These estimates are shown to have a natural interpretation in terms of momentum maps and co-adjoint orbits.

Contents

Overture

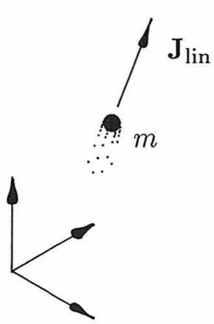
A detailed exposition of the results to be expounded in this dissertation will be offered in the Introduction. While some of these results are sketched here, the main task at hand is to describe, informally and in plain language, the questions and problems motivating our work, as well as the ideas and point of view leading us ultimately to answers. It will be illustrative to keep two examples in mind that serve as good prototypes for the larger class of systems that will be studied later.

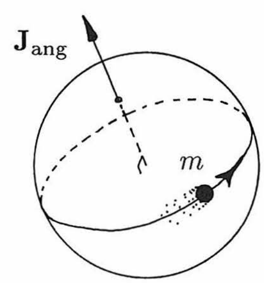
Two simple Hamiltonian systems

Example A. Consider the (classical) motion of a point mass m moving through ordinary three-dimensional space, free of external forces including gravity. The linear momentum $\mathbf{J}_{\text{lin}}$ of the mass is conserved, so that the motion is in a straight line at constant speed.

Example B. Alternatively, suppose that our point mass is constrained to move on the surface of a smooth sphere, the only external force being the normal force necessary to maintain the constraint. In this case the angular momentum $\mathbf{J}_{\text{ang}}$ is conserved and the motion is along great circles of the sphere at constant speed.

A





B

Both Examples A and B possess a *conservation law.* The conservation laws can be 'explained' by the existence of *symmetries* in the underlying equations. This is the content of a well-known theorem of Noether (see, e.g., Marsden (1992)). Example A clearly possesses a three-dimensional *translational* symmetry, while Example B possesses a *rotational* symmetry.

Suppose these symmetries are broken. For instance, suppose that in Example A we add a weak spatially periodic potential, and in Example B we introduce a small aspherical distortion. What is the fate of the above conservation laws?

The complexity of perturbed motions

In the absence of perturbations, Examples A and B are both *integrable* systems, i.e., systems whose solutions can be written down in an explicit or 'closed' form. Unfortunately, integrability is the exception rather than the rule, and generic perturbations to integrable systems create extraordinarily complicated behavior. Numerically such behavior is manifest in spectacular fractal-like phase portraits (see, e.g., Fig. 1), broadband Fourier spectra (Noid, Koszykowski and Marens, 1977), exponential divergence of nearby trajectories (Benettin and Galgani, 1979) and computations of KS entropy (Chirikov, 1979). On the analytical side, we have the existence of homo- and heteroclinic tangles, giving rise to horseshoe maps and chaotic dynamics on Cantor sets (Moser, 1973; Smale, 1967).

According to Poincaré's nonexistence theorem (see, e.g., Benettin, Ferrari, Galgani and Giorgilli (1982)) conserved quantities in an integrable system are irrevocably destroyed by generic perturbations. Despite this and our preceding remarks, there are two important theorems of canonical perturbation theory that establish, under appropriate hypotheses, some kind of 'stability' for the conserved quantities. These are the Kolmogorov-Arnold-Moser (or KAM) theorem (Kolmogorov, 1954; Arnold, 1963; Moser, 1962) and Nekhoroshev's theorem (Nekhoroshev, 1977). For generic perturbations, the KAM theorem does not directly apply to Example B (or to other examples like it), and we discuss it no further[1]. Nekhoroshev's

[1] The interested reader is referred to, e.g., Arnold, Kozlov and Neishtadt (1988).

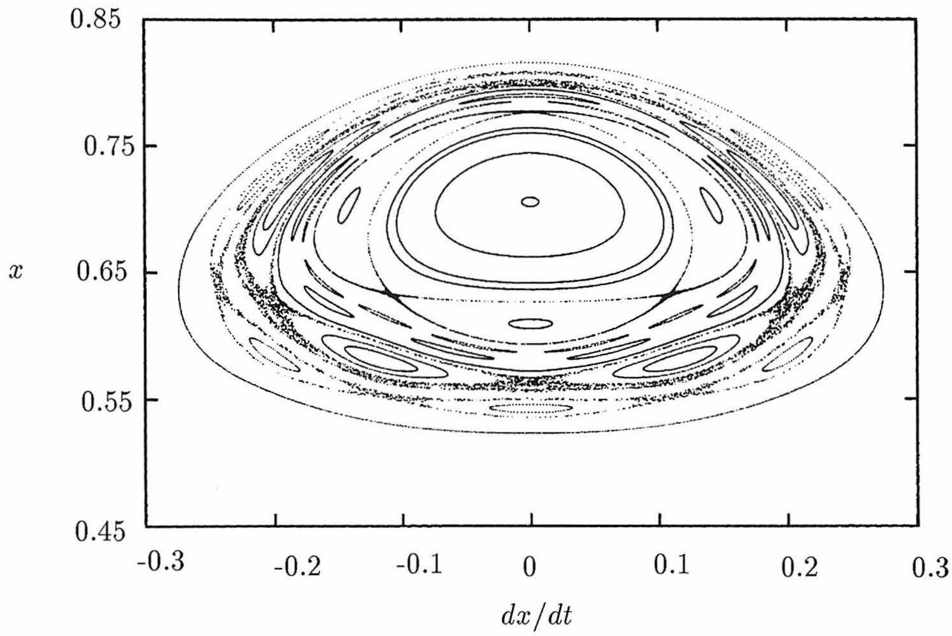


FIGURE 1. A phase portrait made by iterating selected points under a return map for the restricted three-body problem with Jacobi constant $C = 3.1$ and with $\mu = 0.001$, the approximate mass ratio between Jupiter and the Sun.

theorem can be applied to both Examples A and B, but not without significant differences that we now elucidate.

Nekhoroshev's theorem

Nekhoroshev's theorem applies to systems admitting *action-angle coordinates.* Nekhoroshev also stated his result in terms of a generalization called *partial action-angle coordinates*, but his argument in the more general setting was incomplete. A complete argument using partial action-angle coordinates was offered only recently (Fassò, 1995). This will be relevant to our discussion of Example B, but not in Example A, which we describe first.

Action-angle coordinates in Example A are easily constructed. Suppose as before, that attention is restricted to spatially periodic perturbations, and let L be a 'lattice of periodicity.' For simplicity, assume that L has an orthogonal basis, and that the corresponding periods of L are the same in each direction. Non-dimensionalize all lengths by this period, and all masses by the mass of m.

Modulo the lattice L, the position of m is determined by three angles q_1, q_2, q_3. The associated 'actions' are the components p_1, p_2, p_3 of the momentum vector $\mathbf{J}_{\text{lin}}$. What makes these coordinates action-angle coordinates is that: (i) they put the equations of motion into *Hamiltonian form*,

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$
$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \ ,$$

where the appropriate *Hamiltonian* H is in this case the total energy of the system, and (ii) the unperturbed value H_0 of the Hamiltonian (i.e., the kinetic energy) depends only on the action variables:

$$H_0(q, p) = h(p) \equiv \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2 \ .$$

In particular, the unperturbed motions of a system in action-angle coordinates are quasi-periodic:

$$p_j(t) = p_j(0)$$
$$q_j(t) = q_j(0) + t\Omega_j \ ,$$

where $\Omega_j \equiv \partial h/\partial p_j(p(0))$. In our example, we have $\Omega_j = p_j(0)$, and the constancy of the action variables corresponds to the conservation of momentum $\mathbf{J}_{\text{lin}}$.

We are now ready to apply Nekhoroshev's result:

Nekhoroshev's Theorem. (Nekhoroshev (1977)[2]) *Consider a Hamiltonian system in action-angle coordinates* $q_1, \ldots, q_n$, $p_1, \ldots, p_n$ *and consider perturbed Hamiltonians of the form*

$$H(q,p) = h(p) + \epsilon F(q,p) \ .$$

Restrict attention to values of the action vector $p \equiv (p_1, \ldots, p_n)$ *lying in some finite closed ball* B*, and initial conditions* $p(0)$ *lying in the interior of* B*. Assume* h *and* F *are real-analytic, and that the level sets in* B *of the unperturbed Hamiltonian* h *are strictly convex. Then there exist positive constants* a, b, c, t_0 *and* r_0 *such that for all sufficiently small* $\epsilon \geqslant 0$ *all solutions of the perturbed system satisfy the exponential estimate*

$$|t| \leqslant t_0 \exp(c\epsilon^{-a}) \quad \Longrightarrow \quad |p(t) - p(0)| \leqslant r_0 \epsilon^b \ .$$

In Example A we have $p = \mathbf{J}_{\mathrm{lin}}$, so that supposing that the perturbation varies proportionally with some parameter ϵ, Nekhoroshev's theorem predicts that order ϵ^b 'drifts' in the momentum $\mathbf{J}_{\mathrm{lin}}$ of the mass m take times on the order of $\exp(c\epsilon^{-a})$. If ϵ is small, such times can be very large indeed. To demonstrate just how large 'very large' can be in practice, we will relate a vivid example drawn from celestial mechanics.

In Giorgilli and Skokos (1997) Nekhoroshev type estimates are derived for Trojan asteroids in a restricted three-body model of the Jupiter-Sun system. In an appropriate rotating system of coordinates, the model has an equilibrium solution known as the Lagrange point L_4. It is shown that an asteroid beginning at rest (in the rotating coordinates) and lying within $0.127D$ of L_4, where D is the distance from Jupiter to the Sun, will not increase that distance by a factor exceeding 1.05 before the universe has *doubled* its current estimated age! There are, moreover, at least four such Trojan asteroids presently known.

[2]For this formulation see, e.g., Lochak (1992). In Nekhoroshev's original statement, the convexity condition is replaced by a 'steepness' condition. This is a weaker (indeed C^∞-generic) condition we will not attempt to consider here or elsewhere.

Non-commutative integrability

Before turning to Example B, let us summarize some key observations about Example A, which are typical of systems admitting action-angle coordinates:

A1. Unperturbed motions in Example A (which has *three* degrees of freedom) are quasi-periodic with *three* independently controllable[3] frequencies.

A2. The conserved quantity $\mathbf{J}_{\text{lin}}$ is a vector with *three* components.

A3. The underlying translational symmetry is *Abelian*, meaning that the net effect of two successive translations is independent of the order in which they are applied.

We add one final observation which is less obvious but nevertheless important:

A4. The Poisson bracket (see below) $\{J_i, J_j\}$ of any pair of components J_1, J_2, J_3 of $\mathbf{J}_{\text{lin}}$ vanishes.

By definition, the *Poisson bracket* $\{f, h\}$ of two functions f and h is computed by differentiating f along solution curves of the system obtained by taking as Hamiltonian the function h.

If we are to apply Nekhoroshev's theorem as above to Example B, then we shall first need to construct action-angle coordinates. But at odds with this requirement is the disturbing fact that analogues of the observations A1–A4 follow an altogether different pattern in Example B:

B1. Unperturbed motions in Example B (which has *two* degrees of freedom) are periodic, i.e., have only *one* associated frequency.

B2. The conserved quantity $\mathbf{J}_{\text{ang}}$ has *three* components, i.e., one more component than the system has degrees of freedom.

B3. The underlying rotational symmetry is *non-Abelian*, since the net effect of applying successive rotations in three-space depends in general on the order in which these rotations are applied.

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[3]Controllable by varying the initial conditions $p_j(0)$.

B4. The Poisson brackets of components J_1, J_2, J_2 of $\mathbf{J}_{\mathrm{ang}}$ satisfy the cyclic conditions

$$\{J_1, J_2\} = J_3 \ ,$$
$$\{J_2, J_3\} = J_1 \ , \qquad \{J_3, J_1\} = J_2 \ .$$

What is remarkable is that (local) action-angle coordinates *can* be constructed in Example B although, as one might imagine in view of the incongruence of the properties listed above, the construction is substantially more complicated than was the case in Example A. We do not attempt to describe this construction here. At any rate, the unperturbed Hamiltonian $H_0(q,p) = h(p)$ in action-angle coordinates fails to be convex, so that Nekhoroshev's theorem above fails to apply.

Integrable systems whose conserved quantities satisfy nontrivial Poisson bracket relations, as in B4, are known as *non-commutatively integrable* systems. In work proceeding the research on exponential estimates, Nekhoroshev (1972) determined that a large class of non-commutatively integrable systems admit a generalization of action-angle coordinates known as *partial action-angle coordinates*, which are in some sense more natural. Partial action-angle coordinates consist of two sets of variables: The first set consists of k angles $q_1, \ldots, q_k$ and k conjugate actions $p_1, \ldots, p_k$. The second set consists of $(n-k)$ variables $x_1, \ldots, x_{n-k}$ and $(n-k)$ conjugate variables $y_1, \ldots, y_{n-k}$, which are all ordinary (i.e., non-angular) coordinate functions. Here n denotes the total number of degrees of freedom. These coordinates, like conventional action-angle coordinates, are *canonical* in the sense that they put the equations of motion into Hamiltonian form:

$$\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} & \dot{x}_j &= \frac{\partial H}{\partial y_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} & \dot{y}_j &= -\frac{\partial H}{\partial x_j} \ . \end{aligned}$$

In partial action-angle coordinates, a non-commutatively integrable Hamiltonian depends only on the p_j variables.

In Nekhoroshev's work on exponential estimates, he realized that most of his arguments carry over to the non-commutatively integrable case if one uses partial action-angle coordinates, and assumes convexity of the unperturbed Hamiltonian with respect to the p_j variables only. Even in Example B, however (the simplest example of a non-commutatively integrable system imaginable!), constructing partial action-angle coordinates is not trivial. Furthermore, one cannot construct partial action-angle coordinates globally in Example B without coordinate singularities. This is true even if we remove points in phase space corresponding to trivial motions (the mass m at rest), which constitute a 'natural' singularity of the problem. Difficulties posed by coordinate singularities are well-known in celestial mechanics; see, e.g., Coffey, Deprit and Miller (1986) and the references contained therein.

Another problem with Nekhoroshev's generalized setting, as Fassò (1995) has pointed out, is that 'fast' (i.e., order ϵ) motions in the x_j, y_j variables take solutions to the perturbed problem in a non-commutatively integrable system out of locally defined partial action-angle coordinate charts, before the exponential estimates on the p_j variables can be rigorously established. Fassò overcomes this problem by showing how to make intrinsic sense of 'normal forms' for the perturbed Hamiltonian, although to express and compute Nekhoroshev type estimates still requires one to fix an atlas of partial action-angle coordinate charts. The extent to which these estimates depend on the choice of atlas is an open problem (Fassò, 1995, Appendix C).

The symmetry point of view

The reason for the difficulties presented by action-angle or partial action-angle coordinates in non-commutatively integrable systems is that the canonical nature of such coordinates is at odds with the intrinsic non-Abelian symmetry underlying many such systems. *In this thesis we take the point of view that the geometry of an underlying symmetry, as well as its corresponding conservation law, should be built into whatever geometric framework is employed to carry out an analysis of perturbations.* This viewpoint is to take precedence over previous requirements that one work exclusively with *canonical* coordinate systems.

Rather, one should endeavor to understand how non-canonical contributions enter the equations of motion, when these are viewed in a coordinate system intrinsic to the non-Abelian symmetry.

With this kind of understanding in hand, we seek to *geometrize* and *generalize* the commutative version of Nekhoroshev's theorem given above, in such a way that its application to systems with a perturbed non-Abelian symmetry becomes transparent. In particular, this generalization should yield estimates that apply immediately to the conservation law intrinsically associated with such a symmetry.

Action-group coordinates

The simplest coordinate system of the kind we advocate here is one that we shall refer to as action-*group* coordinates. In Example A, and other systems whose integrability arises from the existence of an appropriate Abelian symmetry, action-group coordinates correspond to conventional action-angle coordinates. A discussion of action-group coordinates in general is postponed to Chap. 2. We preview these coordinates here in the special case of Example B, which we shall revisit throughout our exposition of the general theory. A more sophisticated demonstration of this theory (Chap. 11) will be an application to the Euler-Poinsot rigid body.

Nondimensionalize all lengths in Example B by the radius of the sphere, and all masses by that of m. The state of the system is determined by the position q and instantaneous linear momentum vector v of m. We view q and v and vectors in three-space, subject to the conditions $\|q\| = 1$ and $v \cdot q = 0$ consistent with the physical constraints.

Assume the unit sphere is centered on the origin of an inertial orthonormal frame with basis e_1, e_2, e_3. It is convenient to express states (q, v) of m with respect to a *reference state* (q_0, v_0), which we choose arbitrarily to be (e_1, e_2) (see Fig. 2). For an arbitrary state (q, v) there exists a 3×3 orientation preserving rotation matrix g and a number $p \geqslant 0$ such that

$$q = gq_0$$

$$v = pgv_0 \ .$$

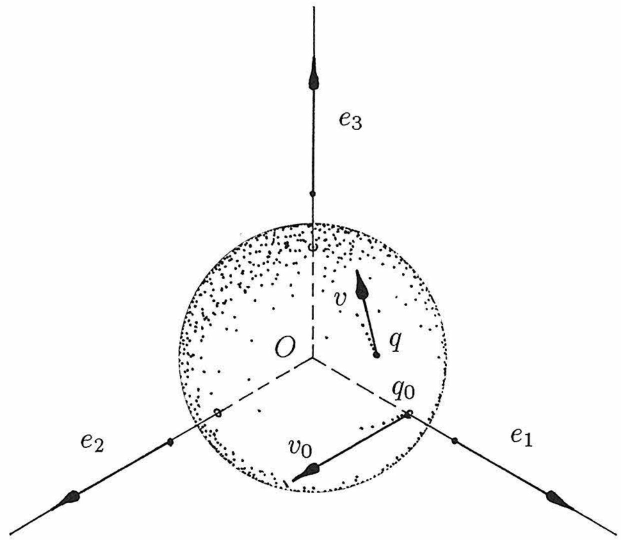

FIGURE 2. The reference state (q_0, v_0) and general state (q, v) of the mass m in Example B. (The basis vectors e_1, e_2, e_3 have been translated for clarity.)

If we neglect trivial states $(v = 0)$, then g and p are determined *uniquely*, and p is strictly positive. Whence g, p constitute 'coordinates,' in the sense that there is a one-to-one correspondence between states (q, v) and values for the pair (g, p).

Notice that the rotational symmetry of the unperturbed system is completely transparent in the coordinates (g, p) as the unperturbed Hamiltonian $H_0(g, p)$ (i.e., the kinetic energy determined by the standard metric on the sphere[4]) is independent of the 'symmetry' coordinate g; it depends only on the 'action' coordinate p, in the spirit of constructions of conventional action-angle coordinates:

$$H_0(g, p) = h(p) \equiv \frac{1}{2}p^2 \ .$$

[4]We realize perturbations in the form of aspherical distortions by perturbing this metric, not by perturbing the sphere with which configurations q are identified.

The equations of motion take the form

$$\dot{g} = g \begin{bmatrix} \frac{1}{p}\frac{\partial H}{\partial g_2} \\ -\frac{1}{p}\frac{\partial H}{\partial g_1} \\ \frac{\partial H}{\partial p} \end{bmatrix}^{\wedge}$$

$$\dot{p} = -\frac{\partial H}{\partial g_3} \ ,$$

where

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}^{\wedge} \equiv \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}$$

and

$$\frac{\partial H}{\partial g_j}(g,p) \equiv \frac{d}{dt} H(ge^{t\hat{e}_j}, p)\big|_{t=0} \qquad (1 \leqslant j \leqslant 3)$$

$$\frac{\partial H}{\partial p}(g,p) \equiv \frac{d}{dt} H(g, p+t)\big|_{t=0} \ .$$

Roughly speaking, the non-canonical contributions to the equations are the terms

$$\frac{1}{p}\frac{\partial H}{\partial g_2} \ , \qquad -\frac{1}{p}\frac{\partial H}{\partial g_1} \ .$$

In the unperturbed case, the equations reduce to

$$\dot{g} = g(pe_3)^{\wedge}$$

$$\dot{p} = 0 \ ,$$

which admit the general solution

$$p(t) = p(0)$$

$$g(t) = g(0)\exp(t\Omega\hat{e}_3) \ ,$$

where $\Omega \equiv p(0)$.

Nekhoroshev estimates in the non-commutative case

The angular momentum of m in Example B is given by

$$\mathbf{J}_{\text{ang}} = q \times v = pg(q_0 \times v_0) = pge_3 \ .$$

If one keeps $p > 0$ constant and runs through all possible values of the symmetry variable g, then the corresponding locus of $\mathbf{J}_{\text{ang}}$ in three-space is a sphere of radius p. In fact these spheres are intrinsic geometric objects associated with the rotational symmetry of the problem known as *co-adjoint orbits.* These orbits (which are zero dimensional in the Abelian case) influence the perturbed motions, and in particular the fate of the conservation law $\dot{\mathbf{J}}_{\text{ang}} = 0$, in a fundamental way. Specifically, Nekhoroshev type estimates for a generic perturbation can only be applied to 'drifts' in the momentum $\mathbf{J}_{\text{ang}}$ in those directions *transverse* to the co-adjoint orbits (spheres). That is, Nekhoroshev estimates, under appropriate hypotheses, apply to the action variable $p = \|\mathbf{J}_{\text{ang}}\|$. Relatively fast motions (order ϵ) can and do appear in the *tangential* directions, being projections onto three-space of fast dynamics in the g variable. These motions correspond to those pointed out by Fassò and discussed above. Unlike partial action-angle coordinate charts however, an action-group coordinate chart completely contains such motions.

Thesis outline

This concludes our informal introduction to the main themes of this dissertation. After a formal exposition of our results detailed in the Introduction, the thesis splits roughly into two parts: In Part 1, we concentrate on dynamics and analytical aspects, and in particular on the derivation of Nekhoroshev estimates in the framework of 'non-canonical' coordinate systems. In Part 2, we describe geometric constructions underlying the theory in detail. The reader primarily interested in geometric aspects is advised to read the Introduction and the first three chapters of Part 1, although one might get away with less. More detailed outlines preface each Part.

Action-group coordinates are in fact just a special case of a more general notion of 'symmetry-intrinsic coordinates' known as *Hamiltonian G-space normal forms.* While the efforts described here apply to some nontrivial systems, such as the Euler-Poinsot rigid body, they are really just the first step in a more extensive program based on such normal forms. We discuss this further in our Concluding Remarks (Chap. 13).

Acknowledgments

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Introduction

The usual starting point of Hamiltonian perturbation theory is the construction of a symplectic diffeomorphism between some open subset of the phase space P and an appropriate open subset of a 'model space' M, such that the unperturbed Hamiltonian is represented by a function on M whose integrability is readily apparent. When the integrability results from the existence of a sufficient number of Poisson-commuting integrals, and the joint level sets of these integrals are compact, the appropriate choice of model space is action-angle coordinates ($M = \mathbb{T}^n \times \mathbb{R}^n$, $n \equiv \frac{1}{2} \dim P$), whose existence is then guaranteed by the Arnold-Liouville theorem (Arnold et al., 1988). Another model space that has appeared in recent practice (Fassò, 1995) is a generalization introduced by Nekhoroshev (1972) known as *partial* (or *generalized*) action-angle coordinates ($M = \mathbb{T}^k \times \mathbb{R}^k \times \mathbb{R}^{2(n-k)}$). These coordinates are applicable when the integrability is 'non-commutative' (see below). In either case the model space is *canonical* in the sense that, up to a covering, it is Euclidean space equipped with its usual symplectic structure.

Non-commutative integrability

In geometric language, a Hamiltonian H_0, defined on some symplectic manifold (P, ω), is *integrable* if H_0 is constant on the leaves of a coisotropic and symplectically complete foliation $\mathcal{F}$ on P (Dazord and Delzant, 1987). A foliation $\mathcal{F}$ is *coisotropic* if its tangent distribution D contains its symplectic orthogonal D^ω. It is *symplectically complete* if D^ω is integrable as a distribution. In that case, the leaves of the corresponding foliation $\mathcal{F}^\omega$ are invariant under the flow of the Hamiltonian vector field X_{H_0}.

If the leaves of $\mathcal{F}^\omega$ are *compact*, then they are in fact k dimensional tori ($k \equiv \dim \mathcal{F}^\omega = 2n - \dim \mathcal{F}$) and $\mathcal{F}^\omega$ is an 'angular fibering' in the sense of Nekhoroshev (1972). In particular, in the case of *commutative* integrability ($k = n$) a neighborhood of each k-torus admits action-angle coordinates, while in the case of *non-commutative* integrability ($k < n$) a neighborhood of each such torus admits partial action-angle coordinates. In either case, the Hamiltonian H_0 is represented by a function depending only on the k actions 'conjugate' to the angles representing the tori. This has the consequence that X_{H_0} restricted to a given k-torus is conjugate to a *linear* vector field (i.e., is 'covered' by a linear vector field on $\mathbb{R}^k$). Proofs of these statements and further details are given in Dazord and Delzant (1987).

Generally, in the non-commutative case, it is not possible to construct a partial action-angle coordinate chart such that it contains a full neighborhood of a leaf of $\mathcal{F}$. Unfortunately, as Fassò (1995) has pointed out, perturbations to H_0 create 'fast' motions (motions of the same order as the perturbation) along the leaves of $\mathcal{F}$. (A detailed analysis of these motions, in the special case of the Euler-Poinsot rigid body, is carried out in Benettin and Fassò (1996).) These motions take trajectories out of a locally defined partial action-angle coordinate chart in a relatively short time. Fassò overcomes this problem by showing how to make intrinsic sense of normal forms for the perturbed Hamiltonian. Such normal forms are used to deduce Nekhoroshev estimates (Nekhoroshev, 1977) on the perturbed motions, although to compute and express such estimates a particular atlas of partial action-angle coordinate charts must be fixed. The extent to which these estimates depend on the choice is an open problem (see Fassò (1995, Appendix C)).

One alternative approach to non-commutative integrability, due to Mishchenko and Fomenko (1978) (see also Arnold et al. (1988)), is to convert non-commutative integrability into *commutative* integrability. This is accomplished, at least locally, by 'gluing' together the k-tori to form n-tori; out of these n-tori one builds conventional action-angle coordinates. There is no escaping the difficulties mentioned above, however, because the unperturbed Hamiltonian will still depend on only $k < n$ actions, leading to 'fast' motions in the perturbed system that take trajectories out of locally defined coordinate charts as before. Moreover, this degeneracy of the Hamiltonian appears mysteriously because its source, namely the non-commutative geometry of the original problem, is concealed in the 'commutative' coordinates employed.

Integrability through symmetry

Non-commutative integrability arises frequently in applications and is often manifest in the form of a non-Abelian symmetry group G. In that case the leaves of $\mathcal{F}$ correspond to orbits of G in the phase space[5]. Generally the geometry of the symmetry is obscured in partial action-angle coordinate charts since, as we remarked above, such charts need not contain full neighborhoods of leaves of $\mathcal{F}$[6]. An alternative that we explore here is to substitute for a canonical model space a *Hamiltonian G-space normal form.* One can view these normal forms as 'non-canonical coordinates' built directly out of the non-Abelian group action. The simplest of these normal forms is a generalization of action-angle coordinates that we shall refer to as *action-group coordinates.* The 'group' in action-group coordinates can be any compact connected Lie group G; when $G = \mathbb{T}^n$, one recovers conventional action-angle coordinates. In action-group coordinates the k-tori discussed above are represented by cosets in G of some maximal torus $T \subset G$.

Action-group coordinates appear in Dazord and Delzant (1987, Section 5). Forerunners of these coordinates have been obtained by Marsden (1981), Gotay (1982), Marle (1983a) and Guillemin and Sternberg (1984, §41). We shall see that in the perturbation analysis of a non-Abelian symmetry, a single action-group coordinate chart suffices, and applies under conditions that are readily verified.

Generalized Nekhoroshev estimates

The main objective of this dissertation is to demonstrate that a Hamiltonian G-space normal form can indeed be used as a geometric framework for perturbation theory. We

[5] A proof of this fact will be recalled in Remark 3.12.

[6] In fact (see Remark 4.4), if G acts freely and is compact connected and non-Abelian, then it is *never* possible for a partial action-angle coordinate chart to contain a neighborhood of a G-orbit (i.e., leaf of $\mathcal{F}$).

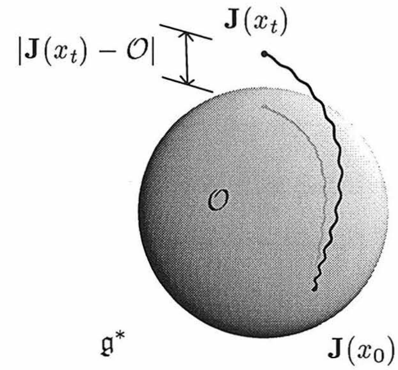

FIGURE 3. Schematic representation of a solution curve $t \mapsto x_t$ of the perturbed Hamiltonian H, projected onto 'momentum space' $\mathfrak{g}^*$ using the momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. The co-adjoint orbit through the initial point $\mathbf{J}(x_0)$ (denoted $\mathcal{O}$) is depicted as a sphere.

do this in Part 1 by generalizing a well-known theorem of Nekhoroshev (1977) to a 'non-canonical' setting that includes action-group coordinates (Theorem 5.9). As a corollary we deduce a Nekhoroshev-type estimate on the evolution of momentum maps, in a class of integrable Hamiltonian systems with nearly G-invariant Hamiltonians (Corollary 7.1). This result may be informally described as follows.

Suppose that a system with Hamiltonian H_0 possesses a symmetry group G. Then, under appropriate hypotheses, the system will possess a corresponding conservation law. This law is embodied in the existence of a vector-valued function $\mathbf{J} : P \to \mathfrak{g}^*$ ($\mathfrak{g}$ denoting the Lie algebra of G) known as a *momentum map* that is constant on solution curves $t \mapsto x_t$ of X_{H_0}:

$$\frac{d}{dt}\mathbf{J}(x_t) = 0 \ .$$

Next, consider a perturbation to H_0 of the form

$$H = H_0 + \epsilon F \ ,$$

where F is arbitrary. Furthermore, assume that the symmetry is sufficiently large so as to enforce integrability in the unperturbed system[7]. Let $t \mapsto x_t$ be a solution of the *perturbed* Hamiltonian H, and let $\mathcal{O} \subset \mathfrak{g}^*$ denote the co-adjoint orbit through the initial point $\mathbf{J}(x_0) \in \mathfrak{g}^*$. Then we show that there exist positive constants a, b, c, t_0 and r_0, such that for all sufficiently small $\epsilon \geqslant 0$, one has (see Fig. 3)

$$|t| \leqslant t_0 \exp(c\epsilon^{-a}) \quad \Longrightarrow \quad |\mathbf{J}(x_t) - \mathcal{O}| \leqslant r_0 \epsilon^b \ .$$

Here $|\mathbf{J}(x_t) - \mathcal{O}|$ denotes the distance of $\mathbf{J}(x_t)$ from the orbit $\mathcal{O}$, measured using some Ad^*-invariant inner product on $\mathfrak{g}^*$. The estimate holds provided the Hamiltonian is real-analytic and satisfies an appropriate 'convexity' condition. One must also assume that action-group coordinates can be constructed in an appropriate neighborhood of x_0. Sufficient conditions will be formulated precisely (see below). The constants appearing in the estimate depend on the unperturbed Hamiltonian H_0, on the magnitude and analyticity properties of F, and on characteristics of the symmetry group G.

In other words, under appropriate conditions the perturbed dynamics, when projected onto 'momentum space' $\mathfrak{g}^*$ by $\mathbf{J}$, evolves exponentially slowly in those directions *transverse* to the co-adjoint orbits. It turns out that in directions tangential to the orbits, relatively 'fast' motions (order ϵ) are possible. These fast motions correspond to those observed by Fassò described above.

On the construction of action-group coordinates

The local existence of action-group coordinates has been proven by Dazord and Delzant (1987, Section 5) for suitable non-commutatively integrable systems. We briefly survey this and related work at the end of Chap. 3. In Part 2 we will offer an alternative construction that we have applied to several mechanical systems. In this approach the construction of action-group coordinates is reduced to the construction of conventional action-*angle* coordinates in an associated lower dimensional phase space known as a *symplectic cross*

[7] To be precise, we require that the Marsden-Weinstein reduced spaces be zero-dimensional; see Chap. 3. In some cases the two-dimensional case can be treated also; see Remark 3.9.

section. This approach may be useful to practitioners of perturbation theory, who are already intimately familiar with conventional action-angle coordinates. The construction may be regarded as a particular application of the so-called 'symplectic cross section theorem' (Guillemin, Lerman and Sternberg, 1996; Guillemin and Sternberg, 1984).

Action-group coordinates can be constructed *globally* using the cross section method precisely when action-angle coordinates can be constructed globally in the symplectic cross section. Necessary and sufficient conditions for the existence of global action-angle coordinates in a Hamiltonian system are already known (Duistermaat, 1980). In Part 2 we will apply the symplectic cross section technique to the axisymmetric Euler-Poinsot rigid body. Perturbations to this problem have been studied already by Benettin and Fassò (1996) using partial action-angle coordinate charts.

Unfortunately to construct action-group coordinates in the neighborhood of a point, one must assume that its image under the momentum map is a *regular* point of the co-adjoint action. In addition, the symmetry group must be acting *freely.* Nevertheless there do exist nontrivial examples for which action-group coordinates can be constructed, as the axisymmetric Euler-Poinsot rigid body demonstrates. Other examples include the problem of geodesics on S^2 (and hence the 'regularized' 2D Kepler problem[8]), the $1:1$ resonance, and the problem of hydrodynamics of three point vortices on S^2. Moreover, it seems likely that techniques such as those outlined here (both the analytic and geometric) will generalize to cases where more sophisticated Hamiltonian G-space normal forms are applicable. We discuss this possibility further in Chap. 13.

[8] See Moser (1970).

Part 1

Dynamics

Outline of Part 1

In this first Part we focus on dynamics. Underlying geometric constructions will be considered in more detail in Part 2. After dispensing in Chap. 1 with some Lie theoretic preliminaries, we begin in Chap. 2 with a résumé of the action-group coordinate framework. We state in Chap. 3 conditions ensuring the existence of action-group coordinates in a given system. We include a description of geodesic motions on S^2, where the existence of action-group coordinates is fairly transparent.

Chap. 4 describes the geometry of the unperturbed dynamics of a system in action-group coordinates, as well as the dynamics obtained after naively applying 'multi-phase averaging' to the perturbation. This will explain, from the symmetry point of view, the origin of the 'fast' motions tangent to the co-adjoint orbits, and motivate the Nekhoroshev estimates for the transverse motions, which we deduce in Sections 5–7. By abstracting Lochak's proof of Nekhoroshev's theorem (Lochak, 1992; Lochak, 1993), we are able to relegate most of the notoriously tedious arguments to an appendix (Appendix A). This abstraction (Theorem 5.9) is also of some independent interest, however (see Chap. 13).

There are opportunities for further investigation along the lines initiated in this thesis. We discuss some of these in Chap. 13.

Contents of Part 1

CHAPTER 1

Preliminaries

The purpose of this chapter is to recall some basic Lie theoretic results and to establish some associated terminology and notation.

Weyl chambers

When one replaces the torus $\mathbb{T}^n$ in the action-angle model space $\mathbb{T}^n \times \mathbb{R}^n$ with a compact connected Lie group G, the natural generalization of the action space $\mathbb{R}^n$ turns out to be a *Weyl chamber* of G. We now recall the standard 'geometric' definition of this object.

1.1 Definition If a group G acts on a manifold X, then an orbit in X is *regular* if there exist no orbits in X of strictly greater dimension. A point $x \in X$ is called *regular* if it lies on a regular orbit. Let $\mathfrak{g}$ denote the Lie algebra of a compact connected Lie group G and let $\mathfrak{t} \subset \mathfrak{g}$ be any maximal Abelian subalgebra. Denote by $\mathfrak{g}_{\mathrm{reg}} \subset \mathfrak{g}$ the regular points of the adjoint action of G, $g \cdot \xi \equiv \mathrm{Ad}_g \, \xi$ $(\xi \in \mathfrak{g})$. A connected component $\mathfrak{t}_0$ of the set $\mathfrak{t} \cap \mathfrak{g}_{\mathrm{reg}}$ is called an (open) *Weyl chamber of G in $\mathfrak{g}$*.

Some related Lie theoretic facts needed in the sequel are summarized below.

1.2 Theorem *Let G be a compact connected Lie group. Then:*

1. *$\mathfrak{g}_{\mathrm{reg}} \subset \mathfrak{g}$ is open and dense.*
2. *All maximal tori of G are conjugate.*
3. *Every $g \in G$ lies in some maximal torus.*
4. *The Lie algebra $\mathfrak{t}$ of any maximal torus $T \subset G$ is a* **maximal** *Abelian subalgebra.*
5. *Every point $\xi \in \mathfrak{g}$ belongs to at least one maximal Abelian subalgebra.*
6. *The map sending a maximal torus to its Lie algebra is a bijection between the maximal tori of G and the maximal Abelian subalgebras of $\mathfrak{g}$. If $\mathfrak{t}$ is a maximal Abelian*

subalgebra, then $\mathfrak{t} \cap \mathfrak{g}_{\text{reg}} \subset \mathfrak{t}$ *is open and dense, and the isotropy group* G_ξ *is identical for all points* $\xi \in \mathfrak{t} \cap \mathfrak{g}_{\text{reg}}$. *This group* G_ξ *is precisely the inverse of* $\mathfrak{t}$ *under the above correspondence, i.e.,* G_ξ *is the unique maximal torus whose Lie algebra is* $\mathfrak{t}$.

7. *Each regular adjoint orbit intersects each Weyl chamber in exactly one point.*
8. *If* $\mathfrak{t}$ *is a maximal Abelian subalgebra and we define* $\mathfrak{t}^\perp \equiv [\mathfrak{g}, \mathfrak{t}]$, *then one has the direct sum decomposition*

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp \ .$$

Proofs of the above facts can be found in, e.g., Bröcker and tom Dieck (1985). The reader may view these facts as generalizations to compact groups of familiar properties of the rotation group SO(3). For example, 1.2.4 corresponds to Euler's theorem that every rotation is a rotation about some fixed axis.

Henceforth G *denotes a compact connected Lie group with Lie algebra* $\mathfrak{g}$.

Let $(\xi, \eta) \mapsto \xi \cdot \eta$ be some Ad-invariant inner product on $\mathfrak{g}$ (recall that such products always exist since G is compact[1]). Then

1.3 $$\mathrm{Ad}_g \xi \cdot \mathrm{Ad}_g \eta = \xi \cdot \eta \qquad \forall \xi, \eta \in \mathfrak{g}, \forall g \in G \ .$$

In particular, we have the infinitesimal version of 1.3,

1.4 $$\mathrm{ad}_\zeta \xi \cdot \eta + \xi \cdot \mathrm{ad}_\zeta \eta = [\zeta, \xi] \cdot \eta + \xi \cdot [\zeta, \eta] = 0 \qquad \forall \xi, \eta, \zeta \in \mathfrak{g}.$$

The inner product on $\mathfrak{g}$ has a unique extension to a (non-degenerate) $\mathbb{C}$-bilinear form on its complexification $\mathfrak{g}^{\mathbb{C}} \equiv \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, which we also denote by $(\xi, \eta) \mapsto \xi \cdot \eta$. The identities 1.3 and 1.4 generalize:

1.5 $$\begin{aligned} &\mathrm{Ad}_g \xi \cdot \mathrm{Ad}_g \eta = \xi \cdot \eta \qquad (\xi, \eta \in \mathfrak{g}^{\mathbb{C}};\ g \in G) \\ &\mathrm{ad}_\zeta \xi \cdot \eta + \xi \cdot \mathrm{ad}_\zeta \eta = [\zeta, \xi] \cdot \eta + \xi \cdot [\zeta, \eta] = 0 \qquad (\xi, \eta, \zeta \in \mathfrak{g}^{\mathbb{C}}) \ . \end{aligned}$$

[1]To obtain one, simply average an arbitrary inner product over G using the Haar probability measure.

Weyl chambers in $\mathfrak{g}^*$

Fix a maximal torus T and corresponding maximal Abelian subalgebra $\mathfrak{t}$. Equation 1.4 says that $\mathrm{ad}_\zeta : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric with respect to the Ad-invariant product. Its image and kernel are therefore orthogonal for any $\zeta \in \mathfrak{t}$. This fact, and the commutativity of $\mathfrak{t}$, easily show that the decomposition 1.2.8 is orthogonal (explaining the notation $\mathfrak{t}^\perp$). If we define $\underline{\mathfrak{t}} \equiv \mathrm{Ann}\, \mathfrak{t}^\perp$ (Ann denotes the annihilator) and $\underline{\mathfrak{t}}^\perp \equiv \mathrm{Ann}\, \mathfrak{t}$, then we obtain the dual decomposition

1.6 $$\mathfrak{g}^* = \underline{\mathfrak{t}} \oplus \underline{\mathfrak{t}}^\perp \ .$$

Let $\varphi : \mathfrak{g} \to \mathfrak{g}^*$ denote the isomorphism induced by the Ad-invariant inner product. Then the orthogonality of the decomposition 1.2.8 implies

1.7 $$\varphi(\mathfrak{t}) = \underline{\mathfrak{t}} \ .$$

The isomorphism φ is G-equivariant if we let G act on $\mathfrak{g}^*$ via the co-adjoint action, $g \cdot \mu \equiv \mathrm{Ad}^*_{g^{-1}}\, \mu$. It thus establishes an equivalence between the adjoint and co-adjoint representations. In particular, by virtue of 1.7, φ maps a Weyl chamber $\mathfrak{t}_0 \subset \mathfrak{t}$ to a connected component of $\underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}}$, which we call a *Weyl chamber in* $\mathfrak{g}^*$. Here $\mathfrak{g}^*_{\mathrm{reg}}$ denotes the regular points of the co-adjoint action. The equivalence of the two representations establishes analogues of 1.2.5, 1.2.6 and 1.2.7 for the co-adjoint action:

1.8 Corollary

1. *For each $\mu \in \mathfrak{g}^*$ there exists a maximal Abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that $\mu \in \underline{\mathfrak{t}} \equiv \mathrm{Ann}[\mathfrak{g}, \mathfrak{t}]$.*
2. *Let $\mathcal{W}$ be a Weyl chamber in $\mathfrak{g}^*$, i.e., a connected component of $\underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}}$, for some maximal Abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Then $G_\mu = T$ for all $\mu \in \mathcal{W}$, where T is the maximal torus with Lie algebra $\mathfrak{t}$.*
3. *Each regular co-adjoint orbit intersects each Weyl chamber in $\mathfrak{g}^*$ in exactly one point.*

CHAPTER 2

Action-group coordinates

In this section we describe the action-group model space. Chap. 3 will state conditions under which this model space is realizable in a particular system, and give a brief historical sketch of its origins.

Let G denote a (real-analytic) compact connected Lie group and let $G^{\mathbb{C}}$ denote its *complexification* (see, e.g., Bröcker and tom Dieck (1985)). For computing estimates later on we need to assume that G is realized as a real-analytic subgroup of $\mathrm{SO}(n_G, \mathbb{R})$ for some integer n_G. Since G is compact and connected this is always possible, by a corollary of the Peter-Weyl theorem (see, e.g., *op. cit*, Theorem 4.1 and Exercise 4.7.1). We may then identify the Lie algebra of G with a subalgebra $\mathfrak{g} \subset \mathbb{R}^{n_G \times n_G}$. The adjoint action can be written as $\mathrm{Ad}_g \xi = g \xi g^{-1}$, and the Lie bracket as $[\xi_1, \xi_2] \equiv \mathrm{ad}_{\xi_1} \xi_2 = \xi_1 \xi_2 - \xi_2 \xi_1$. The complexification $G^{\mathbb{C}}$ of G can be identified with a complex subgroup of $\mathrm{SO}(n_G, \mathbb{C})$. The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $G^{\mathbb{C}}$ is identifiable with the complex subalgebra $\mathfrak{g} \oplus i\mathfrak{g} \subset \mathbb{C}^{n_G \times n_G}$.

2.1 Remark Although a group like $G = \mathrm{SU}(2)$ is a *real* Lie group, it is ordinarily realized as a subgroup of $\mathbb{C}^{2\times 2}$. According to the preceding assumptions, we first need to realize this group as a subgroup of $\mathbb{R}^{4\times 4}$ (using the standard identification $\mathbb{C}^2 \cong \mathbb{R}^4$). The complexification $G^{\mathbb{C}}$ will then be realized as a complex subgroup of $\mathbb{C}^{4\times 4}$.

> *Henceforth $T \subset G$ denotes a fixed maximal torus, $\mathfrak{t}$ its Lie algebra, and $\mathcal{W} \subset \underline{\mathfrak{t}}$ a Weyl chamber in $\mathfrak{g}^*$.*

The model space and its symplectic structure

The natural projection $\mathfrak{g}^* \to \mathfrak{t}^*$ (the dual map of inclusion) has kernel $\underline{\mathfrak{t}}^{\perp}$ and thus restricts, by virtue of 1.6, to an isomorphism $i : \underline{\mathfrak{t}} \to \mathfrak{t}^*$. This map identifies $\mathcal{W}$ with an open set $\mathfrak{t}_0^* \equiv i(\mathcal{W}) \subset \mathfrak{t}^*$. One calls $\mathfrak{t}_0^*$ a Weyl Chamber also.

We define the *action-group model space* for a compact connected Lie group G as $G\times\mathfrak{t}_0^*$. A natural symplectic structure ω_G^* on $G\times\mathfrak{t}_0^*$, generalizing the canonical structure $\sum_j dq_j\wedge dp_j$ on $\mathbb{T}^n\times\mathbb{R}^n$, is given by the following proposition:

2.2 Proposition *Equip* T^*G *with its natural symplectic structure. For each* $\alpha\in\mathfrak{g}^*$, *let* α_G *denote the left-invariant one-form on* G *with* $\alpha_G(\mathrm{id}_G)=\alpha$ *(here viewing one-forms as sections of the cotangent bundle). Then the embedding* $G\times\mathfrak{t}_0^*\hookrightarrow\mathrm{T}^*G$ *that maps* (g,p) *to* $(i^{-1}(p))_G(g)$, *maps* $G\times\mathfrak{t}_0^*$ *onto a* ***symplectic*** *submanifold of* T^*G.

We define ω_G^* to be the symplectic structure on $G\times\mathfrak{t}_0^*$ pulled back by this embedding. Note that ω_G^* does not depend on the choice of Ad-invariant inner product on $\mathfrak{g}$, or on the realization of G as a linear group. The proof of 2.2 will follow from a more general observation we shall make in Part 2.

The injection $\varphi^{-1}\circ i^{-1}:\mathfrak{t}^*\hookrightarrow\mathfrak{g}$ maps $\mathfrak{t}^*$ isomorphically onto $\mathfrak{t}$ (see 1.7), and so identifies $\mathfrak{t}_0^*$ with a Weyl chamber $\mathfrak{t}_0\subset\mathfrak{t}$ in $\mathfrak{g}$. For computing estimates later on, it is convenient for us to make the corresponding identification of $G\times\mathfrak{t}_0^*$ with $G\times\mathfrak{t}_0$. We now describe explicitly the symplectic structure on $G\times\mathfrak{t}_0$ induced by this identification. This symplectic structure of course *does* depend on the choice of Ad-invariant inner product.

Since we need a complexification of the model space to make estimates later on, we develop notation which also makes sense on the complex manifold $G^{\mathbb{C}}\times\mathfrak{t}^{\mathbb{C}}$ containing $G\times\mathfrak{t}_0$. (By $\mathfrak{t}^{\mathbb{C}}$ we mean the complex subspace $\mathfrak{t}\oplus i\mathfrak{t}$ of $\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}\oplus i\mathfrak{g}$.) For $\xi_0\in\mathfrak{g}^{\mathbb{C}}$, $\tau_0\in\mathfrak{t}^{\mathbb{C}}$ and $(g,p)\in G^{\mathbb{C}}\times\mathfrak{t}^{\mathbb{C}}$, define the (complex) vector $(\xi_0,\tau_0)_{g,p}$ tangent to $G^{\mathbb{C}}\times\mathfrak{t}^{\mathbb{C}}$ at (g,p) by

2.3 $$(\xi_0,\tau_0)_{g,p}\equiv\frac{d}{dt}(g\exp(t\xi_0),p+t\tau_0)\ .\Big|_{t=0}$$

Note that every tangent vector in $\mathrm{T}_{(g,p)}(G^{\mathbb{C}}\times\mathfrak{t}^{\mathbb{C}})$ is of this form.

On $G^{\mathbb{C}}\times\mathfrak{t}^{\mathbb{C}}$ define the one-form Θ_G by

2.4 $$\langle\Theta_G,(\xi_0,\tau_0)_{g,p}\rangle\equiv p\cdot\xi_0\ ,$$

and define the two-form ω_G on $G^{\mathbb{C}}\times\mathfrak{t}^{\mathbb{C}}$ by

2.5 $$\omega_G\equiv-d\Theta_G\ .$$

We claim:

Restricted to $G \times \mathfrak{t}_0 \subset G^{\mathbb{C}} \times \mathfrak{t}^{\mathbb{C}}$, ω_G agrees with the (real) symplectic structure ω_G^* on $G \times \mathfrak{t}_0^*$ defined above, after making the identification $G \times \mathfrak{t}_0 \cong G \times \mathfrak{t}_0^*$ discussed above.

Our claim follows from formulas for ω_G^* (analogous to 2.4 and 2.5) we shall derive in Part 2. These formulas also appear in Dazord and Delzant (1987, Section 5) (who obtain it via a different route). For our applications to perturbation theory, we also need explicit equations of motion and an explicit formula for the Poisson bracket. We turn to these next.

Hamiltonian vector fields in action-group coordinates

If $\xi : G^{\mathbb{C}} \times \mathfrak{t}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ and $\tau : G^{\mathbb{C}} \times \mathfrak{t}^{\mathbb{C}} \to \mathfrak{t}^{\mathbb{C}}$ are arbitrary holomorphic maps, we define vector fields $\xi \cdot \frac{\partial}{\partial g}$ and $\tau \cdot \frac{\partial}{\partial p}$ on $G^{\mathbb{C}} \times \mathfrak{t}^{\mathbb{C}}$ by

$$\left(\xi \cdot \frac{\partial}{\partial g}\right)(g,p) \equiv (\xi(g,p), 0)_{g,p}$$
$$\left(\tau \cdot \frac{\partial}{\partial p}\right)(g,p) \equiv (0, \tau(g,p))_{g,p} \ .$$

An arbitrary vector field on $G^{\mathbb{C}} \times \mathfrak{t}^{\mathbb{C}}$ is then of the form $\xi \cdot \frac{\partial}{\partial g} + \tau \cdot \frac{\partial}{\partial p}$ for some vector-valued functions $\xi : G^{\mathbb{C}} \times \mathfrak{t}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ and $\tau : G^{\mathbb{C}} \times \mathfrak{t}^{\mathbb{C}} \to \mathfrak{t}^{\mathbb{C}}$.

Write $\mathfrak{t}^{\perp\mathbb{C}} = \mathfrak{t}^{\perp} \oplus i\mathfrak{t}^{\perp}$ and define $\mathfrak{t}_0^{\mathbb{C}}$ to be the connected component containing $\mathfrak{t}_0$ of the set

$$\{p \in \mathfrak{t}^{\mathbb{C}} \mid \mathrm{ad}_p : \mathfrak{t}^{\perp\mathbb{C}} \to \mathfrak{t}^{\perp\mathbb{C}} \text{ is invertible } \} \ .$$

For $p \in \mathfrak{t}_0^{\mathbb{C}}$, let $\lambda_p : \mathfrak{t}^{\perp\mathbb{C}} \to \mathfrak{t}^{\perp\mathbb{C}}$ denote the inverse of $\mathrm{ad}_p : \mathfrak{t}^{\perp\mathbb{C}} \to \mathfrak{t}^{\perp\mathbb{C}}$. In particular, let us record that

2.6 $$\lambda_p([p,\xi]) = \xi = [p, \lambda_p(\xi)] \qquad (\xi \in \mathfrak{t}^{\perp\mathbb{C}}, p \in \mathfrak{t}_0^{\mathbb{C}})$$

and

2.7 $$\mathfrak{t}_0 = \mathfrak{t} \cap \mathfrak{t}_0^{\mathbb{C}} \ .$$

It is possible to show that the restriction of the (complex) form ω_G to $G^{\mathbb{C}} \times \mathfrak{t}_0^{\mathbb{C}}$ is non-degenerate. If $H : G^{\mathbb{C}} \times \mathfrak{t}_0^{\mathbb{C}} \to \mathbb{C}$ is holomorphic, then the corresponding (complex) *Hamiltonian vector field* X_H is defined by $X_H \lrcorner\, \omega_G = dH$. Indeed

2.8
$$X_H = \xi_H \cdot \frac{\partial}{\partial g} + \tau_H \cdot \frac{\partial}{\partial p}$$

where

2.9
$$\xi_H(g,p) \equiv \frac{\partial H}{\partial p}(g,p) + \lambda_p \sigma^{\perp} \frac{\partial H}{\partial g}(g,p)$$

and

2.10
$$\tau_H(g,p) \equiv -\sigma \frac{\partial H}{\partial g}(g,p) \ .$$

Here $\sigma : \mathfrak{g} \to \mathfrak{t}$ denotes the projection along $\mathfrak{t}^{\perp}$, and $\sigma^{\perp} : \mathfrak{g} \to \mathfrak{t}^{\perp}$ that along $\mathfrak{t}$. These formulas are the complexified version of formulas on $G \times \mathfrak{t}_0 \cong G \times \mathfrak{t}_0^*$ whose derivation will be given in Part 2. The vector valued functions $\frac{\partial H}{\partial g} : G^{\mathbb{C}} \times \mathfrak{t}_0^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ and $\frac{\partial H}{\partial p} : G^{\mathbb{C}} \times \mathfrak{t}_0^{\mathbb{C}} \to \mathfrak{t}^{\mathbb{C}}$ appearing in 2.9 and 2.10 are defined implicitly by

$$\frac{\partial H}{\partial g}(g,p) \cdot \xi_0 = \langle dH, (\xi_0, 0)_{g,p} \rangle = \frac{d}{dt} H(g \exp(t\xi_0), p)\big|_{t=0} \qquad (\xi_0 \in \mathfrak{g}^{\mathbb{C}})$$

and

$$\frac{\partial H}{\partial p}(g,p) \cdot \tau_0 = \langle dH, (0, \tau_0)_{g,p} \rangle = \frac{d}{dt} H(g, p + t\tau_0)\big|_{t=0} \qquad (\tau_0 \in \mathfrak{t}^{\mathbb{C}}) \ .$$

Here a dot denotes the non-degenerate $\mathbb{C}$-bilinear form on $\mathfrak{g}^{\mathbb{C}}$.

Equations of motion in action-group coordinates

Recalling that we are identifying G with a linear group, we may think of elements of G as matrices and use formulas 2.8–2.10 to write the 'equations of motion' corresponding to a Hamiltonian H as

2.11
$$\begin{aligned} \dot{g} &= \ \ g \frac{\partial H}{\partial p} + g \lambda_p \sigma^{\perp} \frac{\partial H}{\partial g} \\ \dot{p} &= -\sigma \frac{\partial H}{\partial g} \end{aligned} \qquad \big((g,p) \in G^{\mathbb{C}} \times \mathfrak{t}_0^{\mathbb{C}} \big) \ .$$

Although one can (abstractly) make sense of these equations without the realization of G as a linear group, our estimates later on will depend on the vector space structure of $\mathbb{C}^{n_G \times n_G}$ in which we have assumed $G^{\mathbb{C}}$ to be embedded.

2.12 Remark It is worth comparing the above equations to those corresponding to Hamiltonian vector fields on $\mathrm{T}^*G \cong G \times \mathfrak{g}^*$ (left trivialization, say). Such equations were already known to Cushman (1977) (see Abraham and Marsden (1978, Proposition 4.4.1)). In the present notation, and identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ as above, these equations take the form

$$\begin{aligned} \dot{g} &= g\frac{\partial H}{\partial \mu} \\ \dot{\mu} &= -\frac{\partial H}{\partial g} + \left[\mu, \frac{\partial H}{\partial \mu}\right] \end{aligned} \qquad \left((g, \mu) \in G \times \mathfrak{g}\right) \ .$$

Can one use the 'coordinates' $G \times \mathfrak{g}$ as a geometric framework for studying perturbations to integrable Hamiltonian systems whose phase space is T^*G (such as the Euler-Poinsot rigid body)? Our (unpublished) investigations suggest that such an approach fails, as far as Nekhoroshev estimates are concerned, unless the Hamiltonian is both left *and* right invariant[1]. This is not the case in many interesting examples.

To help convince the reader that 2.11 indeed generalizes Hamilton's canonical equations, we point out a simple example.

2.13 Example Take $G = S^1$. Then we realize G as SO(2), i.e., the set of matrices of the form

$$g = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \ , \qquad \theta \in \mathbb{R} \ .$$

It is clear that $\mathfrak{t} = \mathfrak{g} \subset \mathbb{R}^{2\times 2}$ consists of matrices of the form

$$p = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \ , \qquad I \in \mathbb{R}$$

[1] i.e., invariant with respect to the cotangent lift of pre- as well as post-multiplication in the group.

and that the only choice for $\mathfrak{t}_0$ is $\mathfrak{t}_0 = \mathfrak{t}$. Also, $\mathfrak{t}^\perp = 0$. Choose for an inner product on $\mathfrak{g}$ the one given by

$$\begin{bmatrix} 0 & -I_1 \\ I_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} \equiv I_1 I_2 \ .$$

Write

$$H'(\theta, I) \equiv H(g,p) = H\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \right) .$$

Then, after making the change of coordinates from (g,p) to (θ, I), one sees that 2.11 (in uncomplexified form) becomes

$$\begin{aligned} \dot\theta &= \frac{\partial H'}{\partial I} \\ \dot I &= -\frac{\partial H'}{\partial \theta} \ , \end{aligned}$$

which are Hamilton's canonical equations on $S^1 \times \mathbb{R}$.

One generalizes the above argument to $G = \mathbb{T}^n \cong \mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)$ by realizing G as the set of block diagonal $2n \times 2n$ matrices whose 2×2 blocks are of the form just described for $n = 1$. One then recovers the familiar form of Hamilton's equations on $\mathbb{T}^n \times \mathbb{R}^n$.

The next example is relevant to the problem of geodesic motions on S^2 (see 3.4).

2.14 Example Take $G = \mathrm{SO}(3)$. Then $\mathfrak{g} = \mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ via the isomorphism $\xi \mapsto \hat\xi : \mathbb{R}^3 \to \mathfrak{so}(3)$ defined by $\hat\xi u = \xi \times u$ ($u \in \mathbb{R}^3$). Note that this isomorphism extends uniquely to a $\mathbb{C}$-linear isomorphism $\mathbb{C}^3 \to \mathfrak{so}(3,\mathbb{C})$. Let $\{e_1, e_2, e_3\}$ denote the standard basis of $\mathbb{R}^3$. Choose $T \subset \mathrm{SO}(3)$ to be the rotations about the e_3 axis, so that $\mathfrak{t} = \mathrm{span}\{e_3\} \cong \mathbb{R}$. Then $\mathfrak{t}^\perp = \mathrm{span}\{e_1, e_2\} \cong \mathbb{R}^2$. The adjoint action is given simply by $g \cdot \xi = g\xi$ ($g \in \mathrm{SO}(3), \xi \in \mathbb{R}^3$), so that the regular adjoint orbits are the spheres centered at the origin of positive radius. Whence $\mathfrak{g}_{\mathrm{reg}} = \mathbb{R}^3 \backslash \{0\}$. The standard inner product $a \cdot b \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$ is Ad-invariant. A connected component $\mathfrak{t}_0$ of $\mathfrak{t} \cap \mathfrak{g}_{\mathrm{reg}}$ is given by $\mathfrak{t}_0 \equiv \{t e_3 \mid t > 0\} \cong (0, \infty)$. So $G \times \mathfrak{t}_0 \cong \mathrm{SO}(3) \times (0,\infty)$.

One has $\mathfrak{g}^{\mathbb{C}} \cong \mathbb{C}^3$ with $\mathrm{ad}_\xi : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ given by $\mathrm{ad}_\xi\,\eta = \xi \times \eta$. Fixing $p \in \mathfrak{t}^{\mathbb{C}} \cong \mathbb{C}$, one finds that $\mathrm{ad}_p : \mathfrak{t}^{\perp\mathbb{C}} \to \mathfrak{t}^{\perp\mathbb{C}}$ ($\mathfrak{t}^{\perp\mathbb{C}} \cong \mathbb{C}^2$) is given by $\mathrm{ad}_p(\xi_1, \xi_2) = (-p\xi_2, p\xi_1)$. Therefore, $\mathfrak{t}_0^{\mathbb{C}} \cong \mathbb{C}\backslash\{0\}$ and $\lambda_p : \mathfrak{t}^{\perp\mathbb{C}} \to \mathfrak{t}^{\perp\mathbb{C}}$ is given by $\lambda_p(\xi_1, \xi_2) = (\xi_2/p, -\xi_1/p)$ ($p \in \mathbb{C}\backslash\{0\}$). The projections $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{t}^{\mathbb{C}}$ and $\sigma^\perp : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{t}^{\perp\mathbb{C}}$ are given by $\sigma(\xi_1, \xi_2, \xi_3) = \xi_3$, $\sigma^\perp(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2)$.

The equations of motion 2.11 can now be written as

$$\dot{g} = g \begin{bmatrix} \frac{1}{p}\frac{\partial H}{\partial g_2} \\ -\frac{1}{p}\frac{\partial H}{\partial g_1} \\ \frac{\partial H}{\partial p} \end{bmatrix}^{\wedge} = g \begin{bmatrix} 0 & -\frac{\partial H}{\partial p} & -\frac{1}{p}\frac{\partial H}{\partial g_1} \\ \frac{\partial H}{\partial p} & 0 & -\frac{1}{p}\frac{\partial H}{\partial g_2} \\ \frac{1}{p}\frac{\partial H}{\partial g_1} & \frac{1}{p}\frac{\partial H}{\partial g_2} & 0 \end{bmatrix} \qquad (g \in \mathrm{SO}(3, \mathbb{C}))$$

$$\dot{p} = -\frac{\partial H}{\partial g_3} \qquad (p \in \mathbb{C}\backslash\{0\}) \ ,$$

where

$$\frac{\partial H}{\partial g_j}(g,p) \equiv \frac{d}{dt} H(g e^{t\hat{e}_j}, p)\big|_{t=0} \qquad (1 \leqslant j \leqslant 3)$$

$$\frac{\partial H}{\partial p}(g,p) \equiv \frac{d}{dt} H(g, p+t)\big|_{t=0} \ .$$

The Poisson bracket in action-group coordinates

Our convention for defining Poisson brackets is $\{u, v\} \equiv X_v \lrcorner X_u \lrcorner \omega$. If $t \mapsto x_t$ is an integral curve of a Hamiltonian vector field X_H, then according to this definition

$$\frac{d}{dt} u(x_t) = \{u, H\}(x_t)$$

for any function u.

2.15 Lemma *The Possion bracket on $G^{\mathbb{C}} \times \mathfrak{t}_0^{\mathbb{C}}$ is given by*

$$\{u, v\}(g,p) = \frac{\partial u}{\partial g}(g,p) \cdot \frac{\partial v}{\partial p}(g,p) - \frac{\partial v}{\partial g}(g,p) \cdot \frac{\partial u}{\partial p}(g,p)$$
$$- p \cdot [\lambda_p \sigma^\perp \frac{\partial u}{\partial g}(g,p), \lambda_p \sigma^\perp \frac{\partial v}{\partial g}(g,p)] \ .$$

PROOF.

$$\begin{aligned}
\{u,v\}(g,p) &= < du, X_v(g,p) > \\
&= \xi_v(g,p)\cdot\frac{\partial u}{\partial g}(g,p) + \tau_v(g,p)\cdot\frac{\partial u}{\partial p}(g,p) \qquad \text{by 2.8} \\
&= \Big(\frac{\partial v}{\partial p}(g,p) + \lambda_p\sigma^{\perp}\frac{\partial v}{\partial g}(g,p)\Big)\cdot\frac{\partial u}{\partial g}(g,p) - \sigma\frac{\partial v}{\partial g}(g,p)\cdot\frac{\partial u}{\partial p}(g,p)
\end{aligned}$$

by 2.9 and 2.10,

$$\begin{aligned}
&= \frac{\partial u}{\partial g}(g,p)\cdot\frac{\partial v}{\partial p}(g,p) - \frac{\partial v}{\partial g}(g,p)\cdot\frac{\partial u}{\partial p}(g,p) \\
&\quad + \lambda_p\sigma^{\perp}\frac{\partial v}{\partial g}(g,p)\cdot\frac{\partial u}{\partial g}(g,p) \ .
\end{aligned}$$

The last term can be written

$$\begin{aligned}
\lambda_p\sigma^{\perp}\frac{\partial v}{\partial g}(g,p)\cdot\frac{\partial u}{\partial g}(g,p) &= \lambda_p\sigma^{\perp}\frac{\partial v}{\partial g}(g,p)\cdot\sigma^{\perp}\frac{\partial u}{\partial g}(g,p) \\
&= \lambda_p\sigma^{\perp}\frac{\partial v}{\partial g}(g,p)\cdot[p,\lambda_p\sigma^{\perp}\frac{\partial u}{\partial g}(g,p)] \qquad \text{by 2.6} \\
&= -p\cdot[\lambda_p\sigma^{\perp}\frac{\partial u}{\partial g}(g,p),\lambda_p\sigma^{\perp}\frac{\partial v}{\partial g}(g,p)] \qquad \text{by 1.5} \ .
\end{aligned}$$

$\square$

CHAPTER 3

On the existence of action-group coordinates

In this section we formulate the main theorem on the existence of action-group coordinates for a Hamiltonian system with a non-Abelian symmetry. A proof of the theorem is postponed to Part 2, where the construction of the coordinates is reduced to the construction of conventional action-angle coordinates in an associated lower-dimensional phase space.

A Hamiltonian system is said to possess a (continuous) *symmetry* when there exists a Lie group G acting on the system's phase space (P, ω), with respect to which the Hamiltonian function H_0 is invariant: $H_0(g \cdot x) = H_0(x)$ for all $g \in G$ and $x \in P$. The group G is said to be acting in a *Hamiltonian fashion* if it acts by symplectic diffeomorphisms, and if the infinitesimal generators ξ_P ($\xi \in \mathfrak{g}$) of the action are (global) Hamiltonian vector fields on P. In that case there exists a map $\mathbf{J} : P \to \mathfrak{g}^*$, called a *momentum map,* with the property that it delivers Hamiltonian functions $J_\xi : P \to \mathbb{R}$ for the generators according to the formula $J_\xi(x) \equiv \langle \mathbf{J}(x), \xi \rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Thus $\xi_P = X_{J_\xi}$, where X_f denotes the Hamiltonian vector field corresponding to a function f, i.e., the vector field defined through $X_f \lrcorner\, \omega = df$.

Noether's theorem states that the functions J_ξ (which Poisson-commute only in the Abelian case) are integrals of motion for the G-invariant Hamiltonian H_0. In particular, $\mathbf{J}(x_t) \in \mathfrak{g}^*$ is constant for all solution curves $t \mapsto x_t$.

See Marsden and Ratiu (1994), Abraham and Marsden (1978, Chapter 4) or Guillemin and Sternberg (1984) for background on momentum maps and for an introduction to the geometric point of view of symmetry in mechanics. For a rapid introduction and recent survey, see Marsden (1992). Unless otherwise indicated notation follows Abraham and Marsden (1978).

System symmetry breaking

Let (P, ω) be a symplectic manifold and let G be a compact connected Lie group acting on P in a Hamiltonian fashion, with momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. We suppose that $\mathbf{J}$ is G-equivariant (with G acting on $\mathfrak{g}^*$ by the co-adjoint action). The quadruple $(P, \omega, G, \mathbf{J})$ (or simply P) will then be called a *Hamiltonian G-space.*

3.1 Example (Action-angle coordinates) Take $P = \mathbb{T}^n \times \mathbb{R}^n$ and $\omega = \Sigma dq_j \wedge dp_j$. Let $G = \mathbb{T}^n \equiv \mathbb{R}^n / 2\pi\mathbb{Z}^n$ act on P according to $\theta \cdot (q, p) \equiv (\theta + q, p)$. This action is Hamiltonian with equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^* \cong \mathbb{R}^n$ given by $\mathbf{J}(q, p) = p$. A Hamiltonian is G-invariant precisely when it depends only on the action coordinates p.

A basic problem of perturbation theory is to understand the dynamics associated with Hamiltonians $H : P \to \mathbb{R}$ of the form

$$H = H_0 + F \ ,$$

where H_0 is G-invariant and F is small in an appropriate sense. Viewing G as a symmetry group for H_0, one sometimes refers to this as a *Hamiltonian system-symmetry breaking problem.*

Orbit type conditions

If a group G acts on a space M, then we denote the point stabilizer (isotropy) subgroup at $x \in M$ by G_x. Recall that the *orbit type* of x is the conjagacy class of subgroups of G with representative G_x. It is denoted (G_x).

Reconsider the case of a Hamiltonian G-space as above. In addition to the orbit type of a point $x \in P$ we shall also, in a slight abuse of terminology, refer to its *co-adjoint orbit type.* This is defined to be the orbit type of $\mathbf{J}(x) \in \mathfrak{g}^*$.

It follows from a general fact about group actions (see, e.g., Bredon (1972)) that if P is connected then there exists an open dense subset of P whose points are of uniform orbit type (called the *maximal* orbit type). From the point of view of understanding the fate of generic initial conditions in the symmetry breaking problem above, it is reasonable to restrict attention to this open G-invariant subspace. In all applications of which this author is aware, one can also find an open dense subset of P with constant *co-adjoint* orbit type. (Whether this holds in general is not immediately clear[1].) Such a subset is also G-invariant, by momentum map equivariance.

The preceding arguments suggest that, from the point of view of perturbation theory, an interesting class of Hamiltonian G-spaces is those spaces with simultaneously a constant orbit and co-adjoint orbit type. The simplest case is a space in which G acts freely and $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$. In this case all points in P have orbit type (id_G) and (by 1.8.1, 1.8.2 and 1.2.2) co-adjoint orbit type (T), where T is any maximal torus of G. These are the spaces we shall consider here and are precisely those spaces for which action-group coordinates can be constructed, under an appropriate integrability hypothesis. Note that if G is Abelian, then $\mathfrak{g}^*_{\mathrm{reg}} = \mathfrak{g}^*$, so that the condition $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$ is always satisfied in that case.

3.2 Example (Action-group coordinates) Take $P \equiv G \times \mathfrak{t}_0$ and $\omega \equiv \omega_G$ (notation as in Chap. 2). The group G acts freely on P according to $g \cdot (h, p) \equiv (gh, p)$. We claim that this action admits an equivariant momentum map.

Writing $\Phi_g(h,p) \equiv g \cdot (h,p)$, one can deduce from 2.4 that Θ_G is invariant under pull-back by Φ_g, for all $g \in G$. As a consequence the Lie derivative of Θ_G along any infinitesimal generator ξ_P $(\xi \in \mathfrak{g})$ vanishes. Therefore, by Cartan's 'magic formula' and 2.5, one has

$$\xi_P \lrcorner\, \omega = d(\xi_P \lrcorner\, \Theta_G) \qquad (\xi \in \mathfrak{g}) \ .$$

Thus the generator ξ_P is the Hamiltonian vector field corresponding to the function $J^G_\xi \equiv \xi_P \lrcorner\, \Theta_G$. The definition of infinitesimal generators leads, in the notation of 2.3, to the explicit formula

$$\xi_P(g,p) = (\mathrm{Ad}_{g^{-1}} \xi, 0)_{g,p} \ .$$

[1]To settle the question seems to require a detailed understanding of the structure of the momentum map image. So far only the case of *compact* P seems sufficiently well-understood; see Kirwan (1984) for a treatment of the compact case and see Hilgert, Neeb and Plank (1994) for partial results in the non-compact case.

One now computes

$$J^G_\xi(g,p) = p \cdot (\mathrm{Ad}_{g^{-1}}\,\xi) = (\mathrm{Ad}_g\,p)\cdot\xi = \langle \varphi^{-1}(\mathrm{Ad}_g\,p), \xi\rangle \ .$$

Whence a momentum map $\mathbf{J}^G : P \to \mathfrak{g}^*$ is given by

3.3 $$\mathbf{J}^G(g,p) = \varphi^{-1}(\mathrm{Ad}_g\,p) = \mathrm{Ad}^*_{g^{-1}}\,\varphi^{-1}(p) \ .$$

Recall that φ maps the Weyl chamber $\mathcal{W}$ in $\mathfrak{g}^*$ bijectively onto the chamber $\mathfrak{t}_0$, so that $\varphi^{-1}(p) \in \mathcal{W} \subset \underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}}$ for all $p \in \mathfrak{t}_0$. In particular, $\mathbf{J}^G(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$, so that $(G \times \mathfrak{t}_0, \omega_G, G, \mathbf{J}^G)$ is a Hamiltonian G-space satisfying our orbit type conditions.

Of course Example 3.1 is just Example 3.2 in the special case $G = \mathbb{T}^n$.

3.4 Example (Geodesic motions on S^2) Consider a point mass M constrained to move on the surface of a smooth sphere of radius L. One identifies the position of M with a point on the unit sphere S^2, so that the phase space is T^*S^2, equipped with its standard symplectic structure. We identify this space with

$$\mathrm{T}S^2 \cong \{(q,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|q\| = 1, q \cdot v = 0\} \ ,$$

using the standard metric on S^2, in which case the symplectic structure is $\omega = -d\Theta$, where

$$\langle \Theta, \frac{d}{dt}(q_t, v_t)\big|_{t=0}\rangle \equiv v_0 \cdot \dot{q}_0 \ .$$

Here $\dot{q}_0 = d/dt|_{t=0}\, q_t$ is to be viewed as an element of $\mathbb{R}^3$. The Hamiltonian is $H_0(q,v) = L^2\|v\|^2/(2M)$, which is G-invariant with respect to the action of $G \equiv \mathrm{SO}(3)$ defined by $g \cdot (q,v) \equiv (gq, gv)$. This action is free and has an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^* \cong \mathbb{R}^3$ given by $\mathbf{J}(q,v) \equiv q \times v$. We restrict attention to the open dense G-invariant subset of phase space obtained by removing the zero section:

$$P \equiv \{(q,v) \in \mathrm{T}S^2 \mid y \neq 0\} \ .$$

Then $\mathbf{J}(P) = \mathbb{R}^3 \backslash \{0\} = \mathfrak{g}^*_{\mathrm{reg}}$, so that $(P, \omega, G, \mathbf{J})$ is a Hamiltonian G-space satisfying our orbit type conditions.

3.5 Remark Bearing in mind Poincaré's model of SO(3),

$$\mathrm{SO}(3) \cong \{(q, v) \in \mathrm{T}S^2 \mid \|v\| = 1\} \ ,$$

one observes in Example 3.4, that P can be identified with $\mathrm{SO}(3) \times (0, \infty)$, the action-group model space for $G = \mathrm{SO}(3)$ (see 2.14). In fact this identification is *symplectic*[2]. Whence it gives a (rather straightforward) realization of action-group coordinates for the problem. Incidentally, in these coordinates the Hamiltonian is given by $H_0(g, p) = L^2 p^2/(2M)$, $(g, p) \in \mathrm{SO}(3) \times (0, \infty)$. Also note that neither conventional action-angle coordinates, nor partial action-angle coordinates, can be constructed globally on P in this problem.

Another simple example satisfying our orbit type conditions is the $1 : 1$ resonance, where $P \equiv \mathbb{R}^4 \backslash \{0\}$ and $G = \mathrm{SU}(2)$. More complicated examples include the problem in hydrodynamics of three point vortices on S^2 ($G = \mathrm{SO}(3) \times S^1$), and the axisymmetric Euler-Poinsot rigid body to be discussed in Part 2 ($G = \mathrm{SO}(3) \times S^1$). In both cases one needs to restrict to some open subset of the phase space to ensure the above orbit type conditions.

In more complicated examples one or both of the above conditions on the orbit types often fail. For example, in the problem of geodesic motions on S^3 (see Chap. 13) the symmetry group SO(4) fails to act freely (although most points do have co-adjoint orbit type (T), $T \subset \mathrm{SO}(4)$ denoting the maximal torus). In the 1:1:1 resonance ($P = \mathbb{R}^6 \equiv \mathbb{C}^3$, $G = \mathrm{SU}(3)$; see Part 2), all non-zero points have orbit type (SU(2)) and co-adjoint orbit (SU(2)), so that both orbit type conditions fail.

These comments not withstanding, the above orbit type conditions are the natural ones to study first.

Integrability

In the case that G acts freely, a reduction in dimension of a Hamiltonian system with symmetry is achieved by *Marsden-Weinstein reduction*:

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[2]For a proof, see Part 2.

3.6 Theorem (Marsden and Weinstein (1974), Meyer (1973))

Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space and assume G acts freely and properly. Then for any $\mu \in \mathbf{J}(P)$, $\mathbf{J}^{-1}(\mu)$ is a G_μ-invariant submanifold of P and $P_\mu \equiv \mathbf{J}^{-1}(\mu)/G_\mu$ admits the structure of a smooth manifold, with respect to which the natural projection $\gamma_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ is a surjective submersion. There is a unique symplectic form ω_μ on P_μ with the property that $\gamma_\mu^* \omega_\mu = i_\mu^* \omega$, where $i_\mu : \mathbf{J}^{-1}(\mu) \to P$ is the inclusion.

The symplectic manifolds P_μ ($\mu \in \mathbf{J}(P)$) are called the *reduced spaces*. Of course, as we assume G is compact, all actions of G are proper.

If H_0 is a G-invariant Hamiltonian, then for each μ in Theorem 3.6, $\mathbf{J}^{-1}(\mu) \subset P$ is an invariant submanifold (by Noether), and there exists a Hamiltonian $H^\mu : P_\mu \to \mathbb{R}$ such that the vector fields $X_{H_0}|\mathbf{J}^{-1}(\mu)$ and X_{H^μ} (defined by $X_{H^\mu} \lrcorner\, \omega_\mu = dH^\mu$) are γ_μ-related. It can be shown that if the integral curves in the reduced space P_μ of X_{H^μ} are known, then the integral curves of X_{H_0} lying in $\mathbf{J}^{-1}(\mu) \subset P$ can be reconstructed by solving *linear* ordinary differential equations with time dependent coefficients (Marsden, Montgomery and Ratiu, 1990). If $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$ then the reduced spaces all have dimension

$$d \equiv \dim P - (\dim G + \operatorname{rank} G) \ .$$

Here $\operatorname{rank} G$ denotes the dimension of the maximal tori of G.

If $d = 0$ or $d = 2$ then Hamilton's equations on the reduced spaces can always be 'solved': In the $d = 0$ case integral curves of X_{H^μ} are trivial (and the reconstruction equations are in fact *autonomous*); in the $d = 2$ case integral curves of X_{H^μ} coincide with level sets of the reduced Hamiltonian H^μ. It is natural to call the original system *integrable* in either case:

3.7 Definition A Hamiltonian G-space $(P, \omega, G, \mathbf{J})$, with G acting freely and $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$, will be called *$G$-integrable* (or *geometrically integrable*) if $d = 0$, and *(G, H_0)-integrable* (or *dynamically integrable*) if $d = 2$. In either case we say that the integrability is *non-commutative* if G is non-Abelian.

There are, of course, more general notions of non-commutative integrability (see, e.g., Dazord and Delzant (1987)). In the present context the above definition is appropriate.

In this paper we will be restricting attention to ***geometric*** *integrability.*

3.8 Examples The problem of geodesic motions on S^2 (Example 3.4) is geometrically integrable. Action-group coordinates (Example 3.2) also constitute a geometrically integrable space.

3.9 Remark Restricting attention to the geometrically integrable case does not entirely rule out treatment of dynamically integrable systems. Under certain conditions one can enlarge the action of G, in a (G, H_0)-integrable space, to an action of $G\times S^1$, in such a way that the space is $G\times S^1$-integrable. The basic idea is to 'compactify' the action of $\mathbb{R}$ generated by X_{H_0}. In that case the G-invariance of the Hamiltonian H_0 extends to $G \times S^1$-invariance, by energy conservation. For example, the Euler-Poinsot rigid body is dynamically integrable with respect to an SO(3) symmetry. If one avoids certain hyperbolic invariant manifolds, then the preceding compactifying procedure delivers $\mathrm{SO}(3)\times S^1$-integrability. In the special axisymmetric case the extra S^1-action corresponds to the symmetry of the body. See Part 2 for details.

The existence theorem

The following result states conditions under which action-group coordinates can be constructed in a Hamiltonian G-space. Recall that $\varphi : \mathfrak{g} \to \mathfrak{g}^*$ denotes the isomorphism corresponding to the fixed Ad-invariant inner product on $\mathfrak{g}$.

3.10 Theorem *Let G be a compact connected Lie group, and let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space on which G is acting freely, and for which $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$. Assume the space is G-integrable in the sense of 3.7. Assume that P is connected, and that each fiber of $\mathbf{J} : P \to \mathbf{J}(P)$ is compact or has a finite number of connected components. Assume that $U \equiv \varphi^{-1}(\mathbf{J}(P)) \cap \mathfrak{t}_0$ (which is open in $\mathfrak{t}_0$) is smoothly contractible. Let G act on $G \times U \subset G \times \mathfrak{t}_0$ as in Example 3.2. Then there exists an equivariant symplectic diffeomorphism $\phi : G \times U \to P$ such that $\mathbf{J} \circ \phi = \mathbf{J}^G$, where $\mathbf{J}^G : G \times U \to \mathfrak{g}^*$ is the momentum map defined by 3.3.*

This theorem holds in either the C^∞ or real-analytic categories. A proof will appear in Part 2.

3.11 Remark The contractability hypothesis can be weakened if the fibres of $\mathbf{J}$ are known to be connected (see Part 2). On the other hand, if U in 3.10 fails to be contractible, then one may always work 'locally' by replacing P by a connected component of $\mathbf{J}^{-1}(G \cdot V)$, for some appropriate contractible set $V \subset U$. This suffices for the applications to perturbation theory in the sequel.

3.12 Remark Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space on which G is acting freely and for which $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$. In that case G_μ is Abelian (a maximal torus in fact) for all $\mu \in \mathbf{J}(P)$. In Part 2 we shall recall the following formula (following from the equivariance of the momentum map):

$$\omega(\xi_P(x), \eta_P(x)) = \langle \mathbf{J}(x), [\xi, \eta] \rangle \qquad (x \in P,\ \xi, \eta \in \mathfrak{g}) \ .$$

Since G_μ is Abelian ($\mu \in \mathbf{J}(P)$), the restriction of ω to G_μ-orbits vanishes. It follows that the distribution D on P defined by $D(x) \equiv (\mathrm{T}_x(G_{\mathbf{J}(x)} \cdot x))^\omega$ is *coisotropic* (see the Introduction for a definition).

Assume P is G-integrable. Then $\dim \mathbf{J}^{-1}(\mu)/G_\mu = 0$ for any $\mu \in \mathbf{J}(P)$, so that $D = (\ker \mathrm{T}\mathbf{J})^\omega$. But from the definition of a momentum map, $\mathfrak{g}^* \xleftarrow{\mathbf{J}} P \xrightarrow{\rho} P/G$ has the 'dual pair' property (ρ denoting the natural projection), i.e. $\ker \mathrm{T}\mathbf{J}$ and $\ker \mathrm{T}\rho$ are ω-orthogonal distributions. In particular, $D = \ker \mathrm{T}\rho$, i.e., D is the distribution tangent to the foliation $\mathcal{F}$ of P by G-orbits. This foliation, as we have just demonstrated, is coisotropic. Furthermore, as $D^\omega = \ker \mathrm{T}\mathbf{J}$, $\mathcal{F}$ is also symplectically complete. A G-invariant Hamiltonian $H_0 : P \to \mathbb{R}$ is therefore integrable in the sense of the Introduction.

Historical remarks

Recall (see the Introduction) that a rather general notion of integrability, geometrically formulated, is that of a symplectically complete coisotropic foliation $\mathcal{F}$. This foliation is to have the property that the Hamiltonian of interest is constant on its leaves. This formulation of integrability has its origins in the work of Alan Weinstein and Paulette Libermann; see Dazord and Delzant (1987) for details. Under the hypotheses of Theorem 3.10, the orbits of G form the leaves of a symplectically complete coisotropic foliation (see Remark 3.12 above).

There are two complementary approaches to studying the geometric structure of a foliation $\mathcal{F}$ as above. One may focus either on neighborhoods of the (coisotropic) leaves of $\mathcal{F}$, or on neighborhoods of leaves of the symplectic orthogonal foliation $\mathcal{F}^\omega$ (which is *isotropic*[3]). The latter problem, as we have described in the Introduction, was solved by Nekhoroshev (1972) (under the hypothesis of compact leaves), and clarified later in Dazord and Delzant (1987). Attempts to address the former problem, which are more recent, begin with independent studies of Gotay (1982) and Marle (1982; 1983b), which classify the neighborhood of an arbitrary *isolated* coisotropic submanifold. The complementary problem of classifying the neighborhood of an isotropic submanifold, and the 'overlapping' problem of classifying the neighborhood of a Lagrangian[4] submanifold, were already solved in Weinstein (1981) and Weinstein (1971) respectively.

In the studies of Gotay and Marle, it is shown that the neighborhood of a coisotropic submanifold $M \subset P$ looks like a neighborhood of the zero section of the dual E^* of its characteristic bundle E, equipped with an appropriate symplectic structure. The *characteristic bundle* of M is defined by

$$E \equiv \{v \in \mathrm{T}M \mid \omega(v, w) = 0 \ \ \forall w \in \mathrm{T}M\}.$$

In the special case of symmetry considered in Theorem 3.10, $E(x) = \mathrm{T}_x(G_\mu \cdot x)$, where $\mu \equiv \mathbf{J}(x)$.

The symplectic diffeomorphism realizing a neighborhood of M in P as a neighborhood of the zero section of E^*, and the associated symplectic structure bestowed on E^*, both

[3]Meaning that the tangent distribution of $\mathcal{F}^\omega$ is contained by its symplectic orthogonal.

[4]A submanifold is *Lagrangian* if it is simultaneously isotropic and coisotropic, or equivalently if spaces tangent to the submanifold coincide with their symplectic orthogonals.

depend on a choice of splitting $TM = E \oplus V$. However, up to neighborhood equivalences, all choices lead to the same answer.

In Marle (1983a) the case where M is the orbit of a free[5] and geometrically integrable Hamiltonian action is studied. Here it is pointed out that the choice of splitting $TM = E \oplus V$ can be reduced to the choice of a splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$, where $\mathfrak{h}$ denotes the Lie algebra of some subgroup $H \subset G$ representing the co-adjoint orbit type (H) of points in M. Marle's description is more general than action-group coordinates (which correspond to the case $H = T$), but lacks the same concreteness in the sense that the bundle E^* remains an abstractly defined object.

Dazord and Delzant (1987, Sect. 5) show that in the special 'regular' case ($H = T$), the bundle E^* trivializes: $E^* \cong G \times U$ ($U \subset \mathbb{R}^k$). Moreover, they show that U is naturally identifiable with an (open) Weyl chamber $\mathfrak{t}_0^*$, the appropriate symplectic structure on $G \times \mathfrak{t}_0^*$ being $-d\Theta_G^*$, where Θ_G^* corresponds, under the identifications discussed in Chap. 2, to the one-form Θ_G defined in 2.4.

The equations of motion 2.11 and the formula of Lemma 2.15 for the Poisson bracket do not appear in Dazord and Delzant (1987), or elsewhere, as far as we are aware. We are also unaware of work, outside that to be presented in Part 2, in which action-group coordinates are constructed in concrete examples.

One other approach to constructing 'coordinates' in the neighborhood of the orbits of a G action, is to apply the technology of *symplectic cross sections* (Guillemin et al., 1996; Guillemin and Sternberg, 1984) alluded to in the Introduction. In particular, this technique is well suited to addressing *global* existence questions. An application of this technology in the special case of action-group coordinates is to be expounded in Part 2.

[5] The case of non locally free actions appears to be unexplored.

CHAPTER 4

Naive averaging

Consider a Hamiltonian G-space P satisfying the hypotheses of Theorem 3.10 and let $H : P \to \mathbb{R}$ be a Hamiltonian of the form

$$H = H_0 + F \ ,$$

where H_0 is G-invariant and F is some perturbation. Then according to the theorem there exists an open set $U \subset \mathfrak{t}_0$ and an equivariant symplectic diffeomorphism $\phi : G \times U \to P$. The equivariance implies that $(H_0 \circ \phi)(g, p) = h(p)$ for some $h : U \to \mathbb{R}$. Thus in action-group coordinates a Hamiltonian system-symmetry breaking problem of the above form translates into the problem of studying a Hamiltonian $H : G \times U \to \mathbb{R}$ of the form

4.1 $$H(g, p) = h(p) + F(g, p) \ .$$

The first task of this section is to describe the dynamics associated with the unperturbed part $H_0(g, p) = h(p)$ of such a Hamiltonian. We will then describe the dynamics associated with an averaged form of the perturbed Hamiltonian H. This will highlight the role played by the symmetry in determining the nature of the perturbed dynamics, as well as motivate the Nekhoroshev estimates developed in subsequent sections.

The unperturbed dynamics

According to 2.11 the equations of motion for a Hamiltonian $H_0(g, p) = h(p)$ are

$$\dot{g} = g\Omega(p) \qquad\qquad \dot{p} = 0 \ ,$$

where $\Omega(p) \equiv \nabla h(p) \in \mathfrak{t}$. We conclude that all integral curves $t \mapsto (g_t, p_t)$ of X_{H_0} are of the form

$$g_t = g_0 e^{t\Omega_0} \qquad\qquad p_t = p_0 \ ,$$

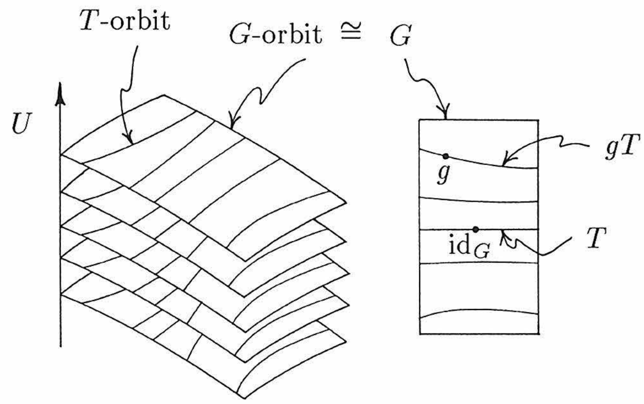

FIGURE 1. Cartoon of the unperturbed motions in $G \times U \subset G \times \mathfrak{t}_0$, depicting the foliation by invariant G-orbits, and the finer foliation by invariant T-orbits.

where $\Omega_0 \equiv \Omega(p_0)$. In particular, each G-orbit $G \times \{p_0\}$ ($p_0 \in U$) is an invariant manifold for the flow, and the restriction of the flow to $G \times \{p_0\} \cong G$ is of the form $g_0 \mapsto g_0 e^{t\Omega_0}$. This generalizes the situation in classically integrable systems where one has a foliation by invariant *tori*, each supporting a quasi-periodic motion whose frequency $\Omega(p)$ is delivered by a similar formula.

Corresponding to the identification $G \times \{p_0\} \cong G$ we have an identification of $X_{H_0}|G \times \{p_0\}$ with the left-invariant vector field on G corresponding to $\Omega_0 \in \mathfrak{t} \subset \mathfrak{g}$. It follows that this vector field on G corresponding to $X_{H_0} \mid G \times \{p_0\}$ is tangent to T and every left coset gT. If we let T act on $G \times U$ according to $q \cdot_T (g, p) \equiv (gq, p)$ ($q \in T$), then these cosets of T in G correspond to T-orbits in the phase space $G \times U$. Thus we have a finer foliation of the phase space $G \times U$ by invariant *tori* (see Fig. 1), but these tori have *strictly less* than half the dimension of phase space in the non-Abelian case.

Let $k = \dim \mathfrak{t}$. The k-dimensional $\mathbb{Z}$-module

4.2 $$I \equiv \{p \in \mathfrak{t} \mid \exp(p) = \mathrm{id}\}$$

is called the *integral lattice* of T in $\mathfrak{t}$. We have:

An integral curve of X_{H_0} lying in $G \times \{p_0\}$ is periodic if and only if $\Omega_0 \in \nu I \equiv \{\nu \mathbf{n} \mid \mathbf{n} \in I\}$ for some $\nu > 0$, in which case a (not necessarily minimal) period is $1/\nu$.

4.3 Definition Identify $\mathfrak{t}$ with $\mathbb{R}^k$, by choosing some $\mathbb{Z}$-basis for I, and call an element ν of $\mathfrak{t} \cong \mathbb{R}^k$ *irrational* if $\sum_j \nu_j n_j \neq 0$ for all non-zero $n \in \mathbb{Z}^k$.

We leave it to the reader to check that the choice of basis in this definition is immaterial. We have:

The integral curves lying on $G \times \{p_0\}$ densely fill each of the T-orbits foliating $G \times \{p_0\}$ if and only if $\Omega_0 \in \mathfrak{t}$ is irrational.

4.4 Remark The action of T on $G \times U$ discussed above is in fact *Hamiltonian*, with equivariant momentum map $\mathbf{j}^G : G \times U \to \mathfrak{t}^*$ given by $\mathbf{j}^G(g,p) \equiv (i \circ \varphi)(p)$. For a proof, see Part 2. The fibers of $\mathbf{j}^G$ are the G-orbits, which constitute the leaves of a coisotropic foliation $\mathcal{F}$ on $G \times U$, by Remark 3.12. Since the space is geometrically integrable, Remark 3.12 also implies that $\mathcal{F}$ is symplectically complete and that the leaves of the associated ω-orthogonal foliation $\mathcal{F}^\omega$ are connected components of fibers of $\mathbf{J}^G$. The fibers of $\mathbf{J}^G$ are in fact connected (indeed they are precisely the orbits of the T action discussed above; see Lemma 4.5.2 below), so that the maps

$$\mathfrak{t}_0^* \xleftarrow{\mathbf{j}^G} P \xrightarrow{\mathbf{J}^G} \mathfrak{g}^*$$

constitute a concrete realization of the foliations $\mathcal{F}$ and $\mathcal{F}^\omega$ as fibers of a dual pair[1].

Since $\mathcal{F}$ is a symplectically complete coisotropic foliation with compact fibers, one may construct (see, e.g., Dazord and Delzant (1987)) an atlas of partial action-angle coordinate charts on $G \times U$. In each chart

$$U_1 \xrightarrow{\sim} \mathbb{T}^k \times U_2 \qquad (U_1 \subset G \times U, \quad U_2 \subset \mathbb{R}^k \times \mathbb{R}^{2(n-k)})$$

[1]A pair of Poisson maps $Q_1 \xleftarrow{\rho_1} P \xrightarrow{\rho_2} Q_2$ (Q_1, Q_2 Poisson manifolds) is a *dual pair* if the tangent maps $T\rho_1$ and $T\rho_2$ have ω-orthogonal kernels.

the foliation by T-orbits will be represented by the trivial foliation

$$\left(\mathbb{T}^k \times \{y\}\right)_{y \in U_2} .$$

But if G is a non-Abelian compact connected Lie group, the bundle $G \to G/T$ is topologically *nontrivial*. Since the T-orbits lying in a G-orbit $G \times \{p\} \cong G$ ($p \in U$) correspond to left cosets of T in G, we conclude that no partial action-angle coordinate chart can be constructed in the non-Abelian case *such that it contains an entire neighborhood of some G-orbit.*

The averaged dynamics

Suppose that a Hamiltonian system with Hamiltonian H_0 admits a foliation $\mathcal{F}$ by closed (compact and boundaryless) invariant manifolds, and assume that the restricted flow on a generic leaf is ergodic[2]. The familiar example is the foliation by invariant tori of an integrable system in action-angle coordinates, provided the Hamiltonian $H_0(q,p) = h(p)$ is *nondegenerate.* Furthermore, assume (as in this example) that each leaf of $\mathcal{F}$ supports a probability measure invariant with respect to the restricted Hamiltonian flow, and varying from leaf to leaf in some smooth fashion. Given a perturbed Hamiltonian H, one can form a 'first order approximation' $\bar{H}$ to H, constructed by averaging H over the leaves of $\mathcal{F}$ using the measures. One then expects the flow of $\bar{H}$ to be a reasonable approximation to that of H (at least better than that of H_0). Turning this heuristic expectation into rigorous theorems is, more or less, the principal preoccupation of the classical perturbation theory[3]. The necessity of ergodicity is well-known.

We now carry out the averaging procedure just outlined in the case of an integrable Hamiltonian system in action-group coordinates. Note that in the example of action-angle coordinates mentioned above the averaged Hamiltonian $\bar{H}$ is itself *integrable*, and therefore not terribly interesting.

[2]By 'generic', we mean with respect to some appropriate Borel measure on the factor space, which we suppose is Hausdorff.

[3]See also the introduction to the book by Lochak and Meunier (1988).

Assume that the unperturbed Hamiltonian $H_0(g,p) = h(p)$ in 4.1 is nondegenerate, i.e. that the map $p \mapsto \Omega(p) : U \to \mathfrak{t}$ is a local diffeomorphism. Then for almost any $p \in U$ (in the sense of Lebesgue measure), the frequency $\Omega(p)$ is irrational (in the sense of 4.3). It follows that we may take the leaves of $\mathcal{F}$ to be the orbits of the T-action $q \cdot_T (g,p) \equiv (gq, p)$ discussed above. We therefore define

$$\bar{H}(g,p) \equiv \int_T H(gq,p)d\mu(q) \ ,$$

where μ is the unique translation-invariant probability measure on the torus T. By construction $\bar{H}$ is T-invariant. Our objective is to Poisson-reduce (see, e.g., Marsden (1992)) the dynamics of $\bar{H}$ to dynamics on an appropriate realization of the quotient space $(G \times U)/T$.

Recall that the action of G on $G \times \mathfrak{t}_0$ admits an equivariant momentum map $\mathbf{J}^G : G \times \mathfrak{t}_0 \to \mathfrak{g}^*$ defined by 3.3.

4.5 Lemma

1. *The fibers of* $\mathbf{J}^G$ *are the* T*-orbits.*
2. *If* $\mathcal{O} \subset \mathfrak{g}^*$ *is a co-adjoint orbit, then* $(\mathbf{J}^G)^{-1}(\mathcal{O})$ *is some* G*-orbit* $G \times \{p\}$ $(p \in \mathfrak{t}_0)$.

PROOF. By 3.3, two points $(g_1,p_1), (g_2,p_2) \in G \times \mathfrak{t}_0$ lie in the same fiber of $\mathbf{J}^G$ if and only if

$$\mathrm{Ad}_{g_1} p_1 = \mathrm{Ad}_{g_2} p_2 \ . \tag{4.6}$$

Since p_1 and p_2 lie in the same Weyl chamber $\mathfrak{t}_0$, it follows from Theorem 1.2.7 that 4.6 is equivalent to

$$p_1 = p_2 \qquad \text{and} \qquad g_2^{-1} g_1 \in G_{p_1} \ .$$

The requirement $g_2^{-1}g_1 \in G_{p_1}$ is equivalent to g_1 and g_2 belonging to the same left coset of G_{p_1} in G. Since $p_1 \in \mathfrak{t}_0 \subset \mathfrak{t} \cap \mathfrak{g}_{\mathrm{reg}}$, Theorem 1.2.6 implies that $G_{p_1} = T$.

The preceding arguments show that the fibers of $\mathbf{J}^G$ consist of those sets of the form $gT \times \{p\}$ $(g \in G,\ p \in \mathfrak{t}_0)$, which proves 1.

By 3.3, $\mathbf{J}^G$ maps a G-orbit $G \times \{p\}$ onto the co-adjoint orbit through $p \in \mathfrak{t}_0$. By the equivariance of $\mathbf{J}^G$, the preimage $\mathbf{J}^{-1}(\mathcal{O})$ is a union of G-orbits in $G \times \mathfrak{t}_0$. To show $\mathbf{J}^{-1}(\mathcal{O})$

is in fact a single orbit it therefore suffices to show that distinct G-orbits cannot be mapped to the same co-adjoint orbit. But $\mathbf{J}^G$ maps an orbit $G \times \{p_1\}$ and an orbit $G \times \{p_2\}$ to the co-adjoint orbits through p_1 and p_2 respectively. By Theorem 1.2.7 these orbits are the same if and only if $p_1 = p_2$. □

Note that Lemma 4.5.1 says that we have a natural way of identifying the abstract quotient $(G \times U)/T$ with $V \equiv \mathbf{J}^G(G \times U) \subset \mathfrak{g}^*$, the momentum map $\mathbf{J}^G : G \times U \to V$ being a realization of the natural projection $G \times U \to (G \times U)/T$.

Being an *equivariant* momentum map, $\mathbf{J}^G$ is also a *Poisson* map, if we equip $\mathfrak{g}^*$ with the positive Lie-Poisson structure $\{\cdot,\cdot\}_+$ (see, e.g., Marsden and Ratiu (1994, Chapter 10)). If u is a smooth function on $\mathfrak{g}^*$, then by definition its corresponding *Hamiltonian vector field* X_u *on* $\mathfrak{g}^*$ is the vector field satisfying the equation $X_u \lrcorner\, df = \{f, u\}_+$ for all smooth $f : \mathfrak{g}^* \to \mathbb{R}$. Hamiltonian vector fields on a Poisson manifold are always tangent to the symplectic leaves, which in this case are the co-adjoint orbits. If a function is constant on a leaf, its Hamiltonian vector field vanishes.

The functions $\bar{H}$, H_0 and $\bar{F}$ on $G \times U$ are T-invariant. They therefore drop, via $\mathbf{J}^G : G \times U \to V$, to functions $\bar{H}'$, H_0' and $\bar{F}'$ on $V \subset \mathfrak{g}^*$. Since H_0 is G-invariant, 4.5.2 guarantees that H_0' is constant on the co-adjoint orbits, and hence that $X_{H_0'} = 0$. Therefore,

$$X_{\bar{H}'} = X_{\bar{F}'} \ .$$

Since $\mathbf{J}^G$ is a Poisson map, the vector fields $X_{\bar{H}}$ and $X_{\bar{H}'}$ are $\mathbf{J}^G$-related. Therefore, $\mathbf{J}^G$ *maps integral curves of* $X_{\bar{H}}$ *onto integral curves of* $X_{\bar{F}'}$.

If $F = 0$, then $X_{\bar{F}'} = 0$. This is consistent with Noether's theorem, which states that $\mathbf{J}^G(x_t) \in \mathfrak{g}^*$ is constant for all integral curves $t \mapsto x_t$ of $X_{\bar{H}}$, if $\bar{H}$ is G-invariant.

If $F \neq 0$, then $X_{\bar{F}'}$ need not be trivial, or even integrable, but must nevertheless remain tangent to the co-adjoint orbits. It follows from 4.5.2 that the G-orbits $G \times \{p\}$ $(p \in U)$ persist as invariant manifolds of $X_{\bar{H}}$. In other words, *the action variables* $p \in U$ *are integrals of motion for the averaged Hamiltonian* $\bar{H}$.

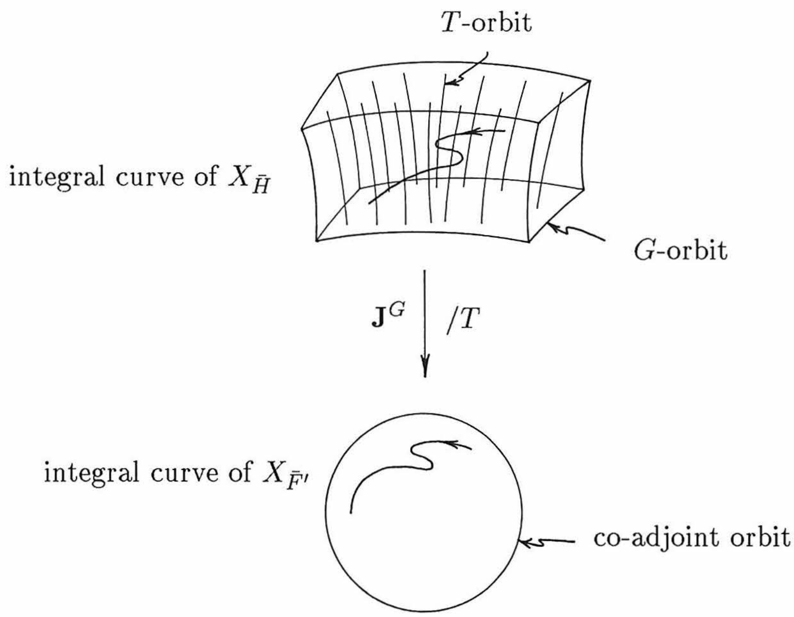

FIGURE 2. Schematic showing an integral curve of the averaged Hamiltonian vector field $X_{\bar{H}}$ (which lies in some G-orbit $G \times \{p\}$) being mapped by $\mathbf{J}^G$ to an integral curve of $X_{\bar{F}'}$ (which will lie in some co-adjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$).

Although one cannot show that the action variables persist as constants of the motion in the fully perturbed (unaveraged) system, one can derive, under certain conditions, exponential bounds on their evolution of Nekhoroshev type. We turn to such estimates in the following chapters.

On the other hand, the non-trivial dynamics on the co-adjoint orbits created by an $F \neq 0$ means that a solution curve $t \mapsto x_t$ of $X_{\bar{H}}$, which must lie on some G-orbit $G \times \{p\} \cong G$, moves from one T-orbit to another, in a fashion determined by $\bar{F}'$, and at a speed of order $1/\epsilon$, if ϵ is the size of the perturbation F (see Fig. 2). These 'fast' motions correspond to those pointed out by Fassò (1995) in the context of symplectically complete isotropic foliations, which we described in the Introduction. In particular (see 4.4), one cannot guarantee that a trajectory of the averaged Hamiltonian will stay in a partial action-angle coordinate chart longer than a time of order $1/\epsilon$. This is far shorter than the exponential time scales one hopes to establish for the action variables (in the fully perturbed system), explaining the shortcoming of partial action-angle coordinate charts described in the Introduction.

CHAPTER 5

An abstract formulation of Nekhoroshev's theorem

When $G = \mathbb{T}^n$, the action-group model space $G \times \mathfrak{t}_0$ is just ordinary action-angle coordinates $\mathbb{T}^n \times \mathbb{R}^n$. In this case Nekhoroshev's theorem (Nekhoroshev, 1977) asserts that for all sufficiently small perturbations F in the Hamiltonian 4.1, the action variables p evolve, in some sense, exponentially slowly. This is provided that the Hamiltonian is real-analytic and satisfies a certain 'convexity' assumption. See, e.g., Lochak (1992) for a precise statement.

Our ultimate objective is to generalize Nekhoroshev's result to the case in which G is a general compact connected Lie group. We refer to this as a problem in *non-canonical* perturbation theory, since the symplectic structure ω_G on $G \times \mathfrak{t}_0$ is non-canonical when G is non-Abelian (in the sense given in the Introduction).

Most of the techniques involved in generalizing Nekhoroshev's theorem to the non-canonical setting are not new. For example, one can still make the standard use of Lie transforms to effect symplectic coordinate changes. Since we have no desire at this point to repeat these standard arguments, it is convenient for us to abstract the setting of the theorem.

The idea is that once one has certain 'basic estimates' (e.g., bounds on Poisson brackets), the proof of Nekhoroshev's theorem (although non-trivial and tedious in detail) requires only very basic properties of the underlying coordinate system and its symplectic structure. These basic estimates and other requirements are formalized as Assumptions A–C below. A generalization of Nekhoroshev's theorem, with these assumptions as hypotheses, is then stated (Theorem 5.9 below). The proof is essentially a rewriting of Lochak's proof of Nekhoroshev's theorem for conventional action-angle coordinates (Lochak, 1992; Lochak, 1993) and is relegated to Appendix A. The more significant task will be to show that

Assumptions A–C indeed apply to action-group coordinates. This will be the subject of Chap. 6.

Throughout this section, the reader may find it helpful to keep the example of conventional action-angle coordinates in mind. In what follows this corresponds to the choice $G = \mathbb{T}^n$ and $\mathfrak{t} = \mathbb{R}^n$.

Notation

Let M be a compact C^0-manifold with (possibly void) boundary whose interior $\operatorname{int} M$ is a complex (resp. real-analytic) manifold, and let V be a complex (resp. real-analytic) vector space. Then we denote by $\mathcal{A}^V(M)$ the space of continuous maps $u : M \to V$ that are holomorphic (resp. real-analytic) on $\operatorname{int} M$. If $|\cdot|$ is a norm on V, then the space $\mathcal{A}^V(M)$ is equipped with the norm

$$\|u\| \equiv \sup_{x \in M} |u(x)| \ .$$

Preliminary notions

An essential ingredient in any proof of Nekhoroshev's theorem is the complexification of the coordinates, and the analytic continuation of the Hamiltonian (which must be assumed real-analytic). The following definition formalizes in a precise way the notion that we require.

5.1 Definition Let G be a real-analytic manifold (without boundary), and suppose that there exists a nested family $(G^\sigma)_{0 \leqslant \sigma \leqslant \bar{\sigma} < \infty}$ of sets ($\sigma < \sigma' \Rightarrow G^\sigma \subset G^{\sigma'}$) with $G^0 = G$ and satisfying the conditions:

1. G^σ is a C^0-manifold with boundary for all $\sigma \neq 0$.
2. $\operatorname{int} G^\sigma$ is a complex manifold for all $\sigma \neq 0$, the inclusion $\operatorname{int} G^\sigma \hookrightarrow \operatorname{int} G^{\bar{\sigma}}$ being holomorphic for all $\sigma \neq 0$.
3. Any real-analytic tensor on $G^0 = G$ has, for some $\sigma > 0$, a holomorphic extension to $\operatorname{int} G^\sigma$ and a continuous extension to G^σ.

Then we call the family $(G^\sigma)_{0 \leqslant \sigma \leqslant \bar{\sigma}}$ a *local complexification* of G.

5.2 Remark Local complexifications exist under rather general conditions, and are unique up to an appropriate equivalence (Whitney and Bruhat, 1959).

See 5.6 below for an example of a local complexification in the above sense. (In Chap. 6 we shall write down a local complexification explicitly in the case that G is an arbitrary compact connected Lie group.)

5.3 Definition Let V be a real (resp. complex) normed vector space and let $U \subset V$ be open. Let a real-analytic (resp. holomorphic) map $h : U \to \mathbb{R}$ (resp. $\mathbb{C}$) be called (m, M)-*convex* if there exists $m, M > 0$ such that

$$\left.\begin{aligned} D^2h(p)(v,v) &\geqslant m|v|^2 \\ |D^2h(p)(u,v)| &\leqslant M|u||v| \end{aligned}\right\} \qquad \forall u,v \in V \ , \ \forall p \in U \ .$$

The abstract set-up

Let G be a compact real-analytic manifold (not necessarily a Lie group) and $(G^\sigma)_{0\leqslant\sigma\leqslant\bar\sigma}$ a local complexification, with G^σ compact for all σ. Let $\mathfrak{t}$ be a finite dimensional real vector space with inner product $(a,b) \mapsto a \cdot b$, and corresponding norm $|\cdot|$.

We assume that for some open set $\mathfrak{t}_0 \subset \mathfrak{t}$ the real-analytic manifold $G \times \mathfrak{t}_0$ is equipped with a symplectic two-form ω.

For $p \in \mathfrak{t}_0$ write

$$B_R(p) \equiv \{p' \in \mathfrak{t} \mid |p'-p| \leqslant R\} \ .$$

Extend the inner product on $\mathfrak{t}$ to a $\mathbb{C}$-bilinear form on $\mathfrak{t}^{\mathbb{C}} \equiv \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$. A norm on $\mathfrak{t}^{\mathbb{C}}$ is then given by $|a \otimes 1 + b \otimes i| \equiv \sqrt{a \cdot a + b \cdot b}$. Write

$$B_R^\rho(p) \equiv \{p' \in \mathfrak{t}^{\mathbb{C}} \mid |p' - B_R(p)| \leqslant \rho\} \ .$$

Fix $\bar p \in \mathfrak{t}_0$ and positive constants $\bar\sigma$ and $\bar\rho$, and write $B \equiv B_{\bar\rho}(\bar p)$ and $B^{\bar\rho} \equiv B_{\bar\rho}^{\bar\rho}(\bar p)$, so that $B^{\bar\rho}$ is a complex neighborhood of the real ball B. Assume that $\bar\sigma$ and $\bar\rho$ are small enough that

$$B_{2\bar\rho} \subset \mathfrak{t}_0 \tag{5.4}$$

and that ω has a holomorphic extension to a (complex) bounded symplectic form on $G^{\bar\sigma}\times B^{\bar\rho}$. By *bounded* we mean that the induced Poisson bracket satisfies the following property: for all u and v in $\mathcal{A}^{\mathbb{C}}(G^{\bar\sigma}\times B^{\bar\rho})$, the bracket $\{u,v\}$ is assumed bounded on $\operatorname{int}(G^{\bar\sigma}\times B^{\bar\rho})$, with a continuous extension to $G^{\bar\sigma}\times B^{\bar\rho}$, i.e.

$$u,v\in\mathcal{A}^{\mathbb{C}}(G^{\bar\sigma}\times B^{\bar\rho})\Rightarrow\{u,v\}\in\mathcal{A}^{\mathbb{C}}(G^{\bar\sigma}\times B^{\bar\rho})\ .$$

Consider a Hamiltonian of the form

5.5 $$H(g,p)=h(p)+F(g,p)\ ,$$

where $h\in\mathcal{A}^{\mathbb{R}}(B)$ and $F\in\mathcal{A}^{\mathbb{R}}(G\times B)$. Assume that h is (m',M')-convex on $\operatorname{int}B$ for some positive constants m' and M', and that $\bar\sigma$ and $\bar\rho$ are small enough that:

1. h and F have holomorphic extensions $h\in\mathcal{A}^{\mathbb{C}}(B^{\bar\rho})$, $F\in\mathcal{A}^{\mathbb{C}}(G^{\bar\sigma}\times B^{\bar\rho})$.
2. There exists $m,M>0$ such that h is (m,M)-convex on $\operatorname{int}B^{\bar\rho}$.

Fundamental assumptions

Make the following crucial assumption about the nature of the Hamiltonian vector fields defined by ω:

Assumption A (Existence of a 'period lattice') *Let X_u denote the (complex) Hamiltonian vector field corresponding to a function $u\in\mathcal{A}^{\mathbb{C}}(G^{\bar\sigma}\times B^{\bar\rho})$, i.e., $X_u\lrcorner\,\omega=du$. Assume that there exists a lattice $\mathfrak{t}^{\mathbb{Z}}\subset\mathfrak{t}$, whose dimension (over $\mathbb{Z}$) equals the dimension of $\mathfrak{t}$ (over $\mathbb{R}$), with the following property. Suppose $W\in\mathcal{A}^{\mathbb{C}}(G^{\bar\sigma}\times B^{\bar\rho})$ is of the form $W(g,p)=w(p)$. Then if $\nabla w(p)\in\nu\mathfrak{t}^{\mathbb{Z}}\equiv\{\nu\mathbf{n}\mid\mathbf{n}\in\mathfrak{t}^{\mathbb{Z}}\}$ for some $p\in B^{\bar\rho}$ and $\nu>0$, we require that all integral curves of X_W beginning in $G^{\sigma}\times\{p\}$ ($\sigma\in[0,\bar\sigma]$ arbitrary) remain in $G^{\sigma}\times\{p\}$ for all time and are* periodic *with (not necessarily minimal) period $1/\nu$. (If $\nabla w(p)=0$, this condition implies that $G^{\bar\sigma}\times\{p\}$ is a manifold of equilibria for X_W.)*

5.6 Example (action-angle coordinates) Take $G=\mathbb{T}^n\equiv\mathbb{R}^n/2\pi\mathbb{Z}^n$, $\mathfrak{t}=\mathfrak{t}_0=\mathbb{R}^n$ and $\omega=\Sigma dq_j\wedge dp_j$. Identifying $\mathbb{T}^n$ with $\{(z_1,\dots,z_n)\in\mathbb{C}^n\mid|z_j|=1\ \forall j\}$, we see that a local

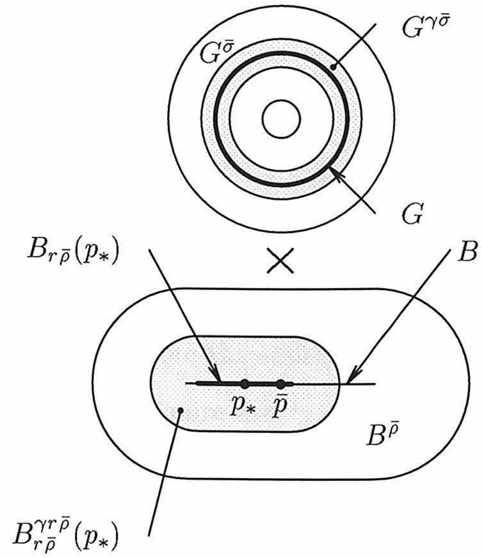

FIGURE 1. Schematic of the domain $D_\gamma(p_*, r) \equiv G^{\gamma\bar{\sigma}} \times B_{r\bar{\rho}}^{\gamma r\bar{\rho}}(p_*)$ (shaded).

complexification $(G^\sigma)_{0\leqslant\sigma\leqslant\bar{\sigma}}$ of G is given by $G^\sigma \equiv \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_j| \in [1-\sigma, 1+\sigma]\ \forall j\}$, $\bar{\sigma} \equiv \frac{1}{2}$. We satisfy Assumption A if we take $\mathfrak{t}^{\mathbb{Z}} = 2\pi\mathbb{Z}^n$.

Let $k \equiv \dim \mathfrak{t}$ and fix a basis $\{\beta_1, \ldots, \beta_k\}$ for $\mathfrak{t}^{\mathbb{Z}}$ (which we *do not* require to be orthogonal with respect to the inner product on $\mathfrak{t}$). Let $c_1 > 0$ be a constant (depending only on $\mathfrak{t}^{\mathbb{Z}}$, the inner product on $\mathfrak{t}$, and the choice of basis) such that

5.7 $$|\tau_1\beta_1 + \cdots + \tau_k\beta_k| \leqslant c_1\sqrt{k} \max_{0\leqslant j\leqslant k} |\tau_j| \qquad (\tau_j \in \mathbb{R}) \ .$$

It is clear that such a constant exists. (It will be estimated explicitly in the case of action-group coordinates in Chap. 6.)

Let $p_* \in \operatorname{int} B$ and $0 \leqslant r \leqslant 1$ be free parameters and define the following family of (complex) neighborhoods of $G \times \{p_*\}$ (see Fig. 1):

$$D_\gamma(p_*, r) \equiv G^{\gamma\bar{\sigma}} \times B_{r\bar{\rho}}^{\gamma r\bar{\rho}}(p_*) \qquad (0 \leqslant \gamma \leqslant 1) \ .$$

So that these domains are well-defined subsets of $G^{\bar\sigma} \times B^{\bar\rho}$, the parameters p_* and r must be subject to the condition

5.8
$$r \leqslant 1 - \frac{|p_* - \bar p|}{\bar\rho} \ .$$

Note that $D_0(p_*, r) = G \times B_{r\bar\rho}(p_*) \subset G \times B$ is a *real* neighborhood of $G \times \{p_*\} = D_0(p_*, 0)$, and that $D_1(\bar p, 1) = G^{\bar\sigma} \times B^{\bar\rho}$ is the complex domain on which the Hamiltonian H is defined by hypothesis. We denote the supremum norm on $\mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ by $\|\cdot\|_\gamma^{p_*,r}$.

The remaining hypotheses we require are:

Assumption B (On 'times of validity') *There exists $c_2 > 0$, independent of m, M, γ, p_* and r, such that for all $u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$, $0 < \delta \leqslant \gamma$ and $|t| \leqslant t_0 \equiv c_2\bar\sigma\bar\rho\delta^2 r/\|u\|_\gamma^{p_*,r}$, every integral curve $\tau \mapsto (g_\tau, p_\tau)$ of X_u beginning at a point $(g_0, p_0) \in D_{\gamma-\delta}(p_*, r)$ is well-defined and satisfies*

$$(g_t, p_t) \in D_{\gamma-\delta/2}(p_*, r) \text{ with } |p_t - p_0| \leqslant \left(\frac{|t|}{t_0}\right)\left(\frac{\delta}{2}\right) r\bar\rho \ .$$

Assumption C (The existence of a 'Cauchy inequality' for Poisson brackets) *There exist constants $c_3 > 0$ and $c_4 > 0$, independent of m, M, γ, p_* and r, such that for all $0 < \delta \leqslant \gamma$ and all $u, v, W \in \mathcal{A}^{\mathbb{C}}((D_\gamma(p_*, r))$ with W of the form $W(g,p) = w(p)$ one has the estimates*

$$\|\{u, v\}\|_{\gamma-\delta}^{p_*,r} \leqslant \left(\frac{c_3}{\bar\sigma\bar\rho}\right) \frac{\|u\|_\gamma^{p_*,r}\,\|v\|_\gamma^{p_*,r}}{\delta^2 r}$$

$$\|\{W, v\}\|_{\gamma-\delta}^{p_*,r} \leqslant c_4 \left(\sup_{p \in B_{r\bar\rho}^{\gamma r\bar\rho}(p_*)} \left|\nabla w(p)\right|\right) \frac{\|v\|_\gamma^{p_*,r}}{\delta^2} \ .$$

(Recall that $\{u, v\} \equiv X_v \lrcorner X_u \lrcorner \omega$.)

Nekhoroshev's theorem

Write $\Omega(p) = \nabla h(p) \in \mathfrak{t}$ and define the 'time scales'

$$T_m \equiv \frac{1}{\bar\rho m} \qquad T_M \equiv \frac{1}{\bar\rho M}$$

$$T_\Omega \equiv \left(\sup_{p\in B^{\bar\rho}} |\Omega(p)|\right)^{-1} \qquad T_{h'''} \equiv \left(\bar\rho^2 \sup_{p\in B}\sup_{u,v,w\in\mathfrak{t}} |D^3 h(p)(u,v,w)|\right)^{-1}$$

and the associated 'dimensionless constants'

$$c_5 \equiv \frac{T_{h'''}}{T_m} \qquad c_6 \equiv \frac{T_M}{T_\Omega} \qquad c_7 \equiv \frac{T_M}{T_m} = \frac{m}{M} \ .$$

We allow $T_{h'''} = \infty$ (in which case $c_5 = \infty$). A constant with the physical dimensions of the Hamiltonian H_0 is

$$E \equiv \frac{\bar\sigma\bar\rho}{T_M} \ .$$

Recall that k denotes the dimension of the real vector space $\mathfrak{t}$.

5.9 Theorem (Nekhoroshev-Lochak (abstract form))

Consider the scenario desribed in this section with Hamiltonian H of the form 5.5, under the Assumptions A, B and C above. Let $\|\cdot\|$ denote the supremum norm on $\mathcal{A}^{\mathbb{C}}(G^{\bar\sigma}\times B^{\bar\rho})$ and define the 'perturbation parameter'

$$\epsilon \equiv \frac{\|F\|}{E} \ .$$

Then there exist positive constants $a = a(k)$, $b = b(k)$, c, $\epsilon_0 = \epsilon_0(k, c_1, \dots, c_7)$, $t_0 = t_0(c_1,\dots,c_7)$ and $r_0 = r_0(c_1,\dots,c_7)$, such that for all $\epsilon \leqslant \epsilon_0$ every (real) solution curve $t \mapsto (g_t, p_t) \in G\times B$ of X_H with $p_0 = \bar p$ obeys the exponential stability estimate

$$\frac{|t|}{T_\Omega} \leqslant t_0 \exp(c\epsilon^{-a}) \ \Rightarrow\ \frac{|p_t - p_0|}{\bar\rho} \leqslant r_0\epsilon^b \ .$$

A proof of 5.9, based largely on Lochak (1992) and Lochak (1993), is outlined in Appendix A. Estimates for the constants $a, b, c, \epsilon_0, t_0, r_0$ obtained in Appendix A are summarized below. For pedagogical reasons, we have not attempted to optimize the exponents a and b (these *are* optimized in Lochak and Neishtadt (1992) and Lochak (1993)). Preliminary investigations of ours suggest that $a = \frac{1}{2k}$ and $b = \frac{1}{2k}$ should be possible (c.f. $a = \frac{1}{2n}$ and $b = \frac{1}{2n}$ in *op. cit*, n the degrees of freedom). For more information see Appendix A.

5.10 Addendum *In Theorem 5.9 above one may take*

$$a = \frac{1}{4(1+k)} \ ,$$

$$b = \frac{1}{2(1+k)} \ ,$$

$$c = 1$$

$$\epsilon_0 = \min\left\{ \left(\frac{8}{{c_7'}^3 l_1 l_2} \right)^{1/b}, \left(\frac{1}{2{c_7'}^2 l_1 l_2} \right)^{1/b} \right\} \ ,$$

$$t_0 = \frac{3{c_7'}^2}{8ec_4} \ ,$$

$$r_0 = 2\left(1 + \frac{1}{c_7'}\right) \frac{{c_7'}^3 l_1 l_2}{16} \ ,$$

where

$$l_4 \equiv \frac{{c_7'}^2 l_2}{4} \left(\frac{c_7 {c_7'}^3 l_2}{16 c_1 \sqrt{k}} \right)^k \ ,$$

$$l_3 \equiv \min\left\{ c_7 c_5', 2c_1\sqrt{k} \right\} \ ,$$

$$l_2 \equiv \min\left\{ c_2, \frac{e}{2c_4 + 5c_3} \right\} \ ,$$

$$l_1 \equiv \max\left\{ 1, \frac{192\bar{\sigma}}{c_7}, \left(\frac{1}{l_4}\right)^{k+1/2} \right\} \ ,$$

$$c_7' \equiv \frac{\sqrt{5}-2}{2} c_7 \ ,$$

$$c_5' \equiv \min\{\frac{c_5}{2}, 1\} \ .$$

CHAPTER 6

Applying the general theorem to action-group coordinates

In this section we show that the abstract formulation of Nekhoroshev's theorem given in the previous section applies to action-group coordinates (Proposition 6.1 below). The corresponding estimates will be combined with the local existence theorem for action-group coordinates (Theorem 3.10) to deduce Nekhoroshev estimates for momentum maps in Chap. 7.

6.1 Proposition *Nekhoroshev's theorem, as formulated in 5.9, applies in the case that G is a compact connected Lie group and where $\mathfrak{t}$, $\mathfrak{t}_0$ and $\omega = \omega_G$ have the meanings given in Chap. 2.*

Recall (see, e.g., Bröcker and tom Dieck (1985, V8.1)) that up to a finite covering any compact connected Lie group is the direct sum of a semisimple and Abelian (toral) factor. Although 6.1 holds for any compact connected Lie group G, we shall only prove it in the case that G is semisimple. The Abelian case is already covered by Nekhoroshev's original statement. Furthermore, it turns out that there is essentially no 'coupling' between the Abelian and semisimple factors in the general case. To simultaneously keep track of both an Abelian and semisimple factor just introduces uninstructive bookkeeping to the calculations. We leave these for the more energetic reader to work out.

> *For the remainder of this section G denotes a compact connected* ***semisimple*** *Lie group.*

In this section we shall make use of basic properties of semisimple Lie algebras, as well as their *root* and *inverse root* systems. We refer the reader to Bröcker and tom Dieck (1985) for a discussion of the relationship between these roots and the Weyl chambers, and for further background. Here a root will always mean a *real* root, in the sense of, e.g., V1.3, *op. cit.*

The local complexification of G

It is now convenient to fix the choice of Ad-invariant inner product on $\mathfrak{g}$ introduced in Chap. 1, which until now has been completely arbitrary. Define

$$\xi \cdot \eta \equiv -\operatorname{Trace}(\mathrm{ad}_\xi \circ \mathrm{ad}_\eta) \qquad (\xi, \eta \in \mathfrak{g}) \ .$$

Minus one times this product is known as the *Killing form* of $\mathfrak{g}$; it is symmetric, Ad-invariant, and known to be nondegenerate precisely when $\mathfrak{g}$ is semisimple. Remember that we have been denoting the unique $\mathbb{C}$-bilinear extension of our product to $\mathfrak{g}^{\mathbb{C}}$ by $\xi \cdot \eta$ also. We continue to denote the norm on $\mathfrak{g}$ induced by the inner product (and its obvious extension to $\mathfrak{g}^{\mathbb{C}}$) by $|\cdot|$.

Recall that we assume that G is realized as a subgroup of $\mathrm{SO}(n_G, \mathbb{R})$, so that we may view $\mathfrak{g}^{\mathbb{C}}$ as a subalgebra of $\mathbb{C}^{n_G \times n_G}$. The latter Lie algebra is equipped with a natural symmetric $\mathbb{C}$-bilinear form, namely the *Schur-Hadamard product*

$$S(A, B) \equiv \frac{1}{n_G} \operatorname{Trace}(AB^T) \qquad (A, B \in \mathbb{C}^{n_G \times n_G}) \ .$$

An associated norm on $\mathbb{C}^{n_G \times n_G}$ is given by

$$|A|_S \equiv \sqrt{S(A, \bar{A})} = \sqrt{\frac{1}{n_G} \operatorname{Trace}(A\bar{A}^T)} \ ,$$

where a bar denotes complex conjugation. Note that as $G \subset \mathrm{SO}(n_G, \mathbb{R})$,

6.2 $$|g|_S = 1 \qquad (g \in G)$$

and

6.3 $$|gA|_S = |A|_S = |Ag|_S \qquad (g \in G, A \in \mathbb{C}^{n_G \times n_G}) \ .$$

By writing things out with respect to the standard basis of $\mathbb{C}^{n_G}$ and applying the Schwartz inequality several times, one can derive the estimate

6.4 $$|AB|_S \leqslant \sqrt{n_G}|A|_S|B|_S \ .$$

If $\bar{\sigma} > 0$ is chosen sufficiently small, then a well-defined local complexification $(G^\sigma)_{0\leqslant\sigma\leqslant\bar{\sigma}}$ of G is given by

6.5 $$G^\sigma \equiv \{g \in G^{\mathbb{C}} \mid |g - G|_S \leqslant \sigma\} \ .$$

It will be convenient to assume

$$\bar{\sigma} \leqslant 1 \ .$$

We need to establish the relation between $|\cdot|$ (the norm on $\mathfrak{g}^{\mathbb{C}}$ determined by the Killing form) and the restriction to $\mathfrak{g}^{\mathbb{C}}$ of $|\cdot|_S$. Suppose that $A, B \in \mathbb{C}^{n_G\times n_G}$ and $g \in G$. Then since $\mathrm{Ad}_g A = gAg^{-1}$,

$$\begin{aligned} S(\mathrm{Ad}_g A, \mathrm{Ad}_g B) &= \frac{1}{n_G}\,\mathrm{Trace}(gAg^{-1}(g^{-1})^T B^T g^T) \\ &= \frac{1}{n_G}\,\mathrm{Trace}(gAB^T g^{-1}) \qquad (\text{since } g \in \mathrm{SO}(n_G, \mathbb{R})) \\ &= \frac{1}{n_G}\,\mathrm{Trace}(AB^T) \qquad (\text{since Trace is invariant under conjugation}) \\ &= S(A, B) \ . \end{aligned}$$

This shows that $S(\cdot,\cdot)$ is invariant with respect to the adjoint action of G. In particular its restriction to $\mathfrak{g}$ is Ad-invariant. But as G is semisimple, there is up to a constant only one Ad-invariant inner product on $\mathfrak{g}$, namely the Killing form (see *op. cit.*). By the uniqueness of extensions of $\mathbb{R}$-bilinear forms on $\mathfrak{g}$ to $\mathbb{C}$-bilinear forms on $\mathfrak{g}^{\mathbb{C}}$, it follows that

6.6 $$S(\xi, \eta) = k_1 \xi \cdot \eta \qquad (\xi, \eta \in \mathfrak{g}^{\mathbb{C}}) \ ,$$

for some $k_1 > 0$.

The constant k_1 depends on the realization of $G \subset \mathrm{SO}(n_G, \mathbb{R})$ and is the only constant we leave implicitly defined. It is easily computed on a case-by-case basis.

From 6.6 it follows that

6.7 $$|\xi|_S = \sqrt{k_1}|\xi| \qquad (\xi \in \mathfrak{g}^{\mathbb{C}}) \ .$$

In particular we deduce from 6.4 and 6.7 the estimate

6.8 $$|g\xi|_S \leqslant k_2|g|_S|\xi| \qquad (g \in G^{\mathbb{C}}, \xi \in \mathfrak{g}^{\mathbb{C}}) \ ,$$

where

6.9 $$k_2 \equiv \sqrt{n_G k_1} \ .$$

Existence of a period lattice (Assumption A)

Define

$$\bar{k} \equiv (\dim G - \operatorname{rank} G)/2 \geqslant k \equiv \dim \mathfrak{t} = \operatorname{rank} G \ .$$

Then with respect to the fixed chamber $\mathfrak{t}_0$, $\mathfrak{g}$ has $\bar{k}$ positive real roots

$$\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{\bar{k}} \in \mathfrak{t}^* \ .$$

We assume that these have been ordered so that $\{\alpha_1, \ldots, \alpha_k\}$ is the basis of $\mathfrak{t}^*$ consisting of the indecomposable elements of $\{\alpha_1, \ldots, \alpha_{\bar{k}}\}$. The corresponding inverse roots will be denoted $\alpha_1^*, \ldots, \alpha_{\bar{k}}^* \in \mathfrak{t}$. They are given by

6.10 $$\alpha_j^* = \frac{2\kappa^{-1}(\alpha_j)}{B(\alpha_j, \alpha_j)} \qquad (1 \leqslant j \leqslant \bar{k}) \ ,$$

where $\kappa : \mathfrak{t} \to \mathfrak{t}^*$ is the isomorphism $\langle \kappa(a), b \rangle \equiv a \cdot b$, and $B(\,\cdot\,,\,\cdot\,)$ is the induced inner product on $\mathfrak{t}^*$. *It is a fact that the inverse roots lie on the integral lattice I of T in* $\mathfrak{t}$ (defined by 4.2).

Suppose $W \in \mathcal{A}^{\mathbb{C}}(G^{\bar{\sigma}} \times B^{\bar{\rho}})$ is of the form $W(g,p) = w(p)$. Then by the equations of motion 2.11, an integral curve $t \mapsto (g_t, p_t)$ of X_W beginning in $G^{\sigma} \times \{p_0\}$ $(0 \leqslant \sigma \leqslant \bar{\sigma}$, $p_0 \in B^{\bar{\rho}})$ is given by $(g_t, p_t) = (g_0 \exp(t\nabla w(p_0)), p_0)$. Suppose that there exists a $\nu > 0$

such that $\nabla w(p) \in \nu I$. In that case $t\nabla w(p_0)$ lies in $\mathfrak{t}$ (i.e., is *real*) for any $t \in \mathbb{R}$ so that $\exp(t\nabla w(p_0)) \in G$. Now $|g_0 - g_0^{\mathbb{R}}|_S \leqslant \sigma$ for some $g_0^{\mathbb{R}} \in G$ (as $g_0 \in G^\sigma$). By 6.3, we have

$$|g_0 \exp(t\nabla w(p_0)) - g_0^{\mathbb{R}} \exp(t\nabla w(p_0))|_S = |g_0 - g_0^{\mathbb{R}}|_S \leqslant \sigma \ .$$

Since $g_0^{\mathbb{R}} \exp(t\nabla w(p_0)) \in G$, this shows that $g_0 \exp(t\nabla w(p_0)) \in G^\sigma$ for all $t \in \mathbb{R}$, i.e., $(g_t, p_t) \in G^\sigma \times \{p_0\}$ for all $t \in \mathbb{R}$. Furthermore, as $\nabla w(p_0) \in \nu I$, this solution curve has $1/\nu$ as a period. We therefore satisfy Assumption A of Chap. 5 if we take $\mathfrak{t}^{\mathbb{Z}}$ to be any k-dimensional sublattice of I.

While the 'optimal' choice for $\mathfrak{t}^{\mathbb{Z}}$ is I itself, future computations are simplified by choosing $\mathfrak{t}^{\mathbb{Z}}$ to the sublattice generated by the inverse roots:

$$\mathfrak{t}^{\mathbb{Z}} \equiv \mathrm{span}_{\mathbb{Z}}\{\alpha_1^*, \dots, \alpha_k^*\} \ .$$

Note that $\mathfrak{t}^{\mathbb{Z}} = I$ if G is simply connected.

Estimating c_1

A basis $\{\beta_1, \dots, \beta_k\}$ for $\mathfrak{t}^{\mathbb{Z}}$ is given by $\beta_j \equiv \alpha_j^*$. We will estimate the constant c_1 of 5.7 in terms the Cartan integers n_{ij} of the root system. These are defined by

$$n_{ij} \equiv \frac{2B(\alpha_i, \alpha_j)}{B(\alpha_i, \alpha_i)} \in \mathbb{Z} \qquad (1 \leqslant i, j \leqslant \bar{k}) \ .$$

It is a fact that the restriction to $\mathfrak{t}$ of -1 times the Killing form can be expressed in terms of the positive real roots according to

$$a \cdot b = 8\pi^2 \sum_{m=1}^{\bar{k}} \langle \alpha_m, a \rangle \langle \alpha_m, b \rangle \qquad (a, b \in \mathfrak{t})$$

It follows from 6.10 that

6.11 $$\alpha_i^* \cdot \alpha_j^* = 8\pi^2 \sum_{m=1}^{\bar{k}} n_{im} n_{jm} \equiv m_{ij} \qquad (1 \leqslant i, j \leqslant k) \ .$$

Let $\lambda_1, \dots, \lambda_k > 0$ denote the eigenvalues of the (symmetric and positive definite) matrix $(m_{ij})_{1 \leqslant i,j \leqslant k}$. Then for some orthogonal transformation $A : \mathbb{R}^k \to \mathbb{R}^k$ and any $\tau \equiv$

$(\tau_1, \ldots, \tau_k) \in \mathbb{R}^k$,

$$\begin{aligned}
|\tau_1\beta_1 + \ldots + \tau_k\beta_k|^2 &= \sum_{i,j=1}^{k} \tau_i m_{ij} \tau_j \\
&= \sum_{j=1}^{k} (A\tau)_j \lambda_j (A\tau)_j \\
&\leqslant (\sup_j \lambda_j) \sum_{j=1}^{k} (A\tau)_j^2 = (\sup_j \lambda_j) \sum_{j=1}^{k} |\tau_j|^2 \\
&\leqslant (\sup_j \lambda_j) k (\sup_j |\tau_j|)^2 \ .
\end{aligned}$$

We therefore satisfy 5.7 if we choose c_1 to be the square root of the maximum eigenvalue of the matrix (m_{ij}), which is defined in terms of the Cartan integers by 6.11.

6.12 Example $G = \mathrm{SU}(3)$ (see Fig. 1). In this case $\mathfrak{t}^{\mathbb{Z}} = I$, $k = 2$ and $\bar{k} = 3$. The matrix of Cartan integers is

$$(n_{ij})_{1 \leqslant i,j \leqslant \bar{k}} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} ,$$

so that

$$(m_{ij})_{1 \leqslant i,j \leqslant k} = \left(8\pi^2 \sum_{m=1}^{\bar{k}} n_{im} n_{jm} \right)_{1 \leqslant i,j \leqslant k} = 24\pi^2 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} .$$

The largest eigenvalue of the right-hand side is $72\pi^2$, so that for $G = \mathrm{SU}(3)$ we may take $c_1 = 6\pi\sqrt{2}$.

Bounds on λ_p

Recall (Chap. 2) that λ_p ($p \in \mathfrak{t}_0^{\mathbb{C}}$) is defined as the inverse of $\mathrm{ad}_p : \mathfrak{t}^{\perp\mathbb{C}} \to \mathfrak{t}^{\perp\mathbb{C}}$. Since λ_p appears in the equations of motion 2.11 and in the formula for the Poisson bracket 2.15, we will need to estimate the operator norm of λ_p before proceeding to verify Assumptions B and C.

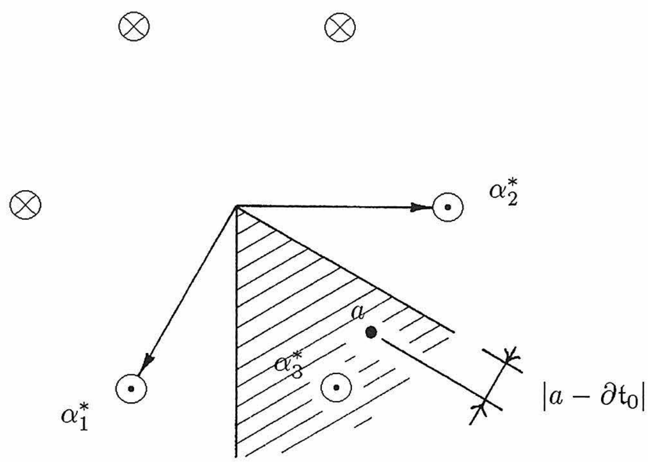

FIGURE 1. The system of inverse roots for $G = \mathrm{SU}(3)$ $(k = 2, \bar{k} = 3)$. Here $\odot$ indicates a positive inverse root, and $\otimes$ a negative inverse root. Arrows indicate the location of the basis roots. The shaded region indicates the Weyl chamber $\mathfrak{t}_0$.

6.13 Remark Since the operator $\mathrm{ad}_p : \mathfrak{t}^{\perp\mathbb{C}} \to \mathfrak{t}^{\perp\mathbb{C}}$ becomes singular as p approaches the walls $\partial\mathfrak{t}_0$ of $\mathfrak{t}_0$, the norm of λ_p is unbounded as $p \to \partial\mathfrak{t}_0$. We will see later that this results in a deterioration in the estimates of Assumptions B and C as $\bar{p} \to \partial\mathfrak{t}_0$. This phenomenon is particular to the non-Abelian case.

Fix $p \in \mathfrak{t}_0^{\mathbb{C}}$. Note that $\dim \mathfrak{t}^{\perp\mathbb{C}} = 2\bar{k}$. For $1 \leqslant j \leqslant \bar{k}$ let ξ_j (resp. ξ_{-j}) denote an element of $\mathfrak{t}^{\perp\mathbb{C}}$ spanning the weight space corresponding to the root α_j (resp. $-\alpha_j$). Then $\{\xi_{-\bar{k}}, \ldots, \xi_{-1}, \xi_1, \ldots, \xi_{\bar{k}}\}$ is a basis for $\mathfrak{t}^{\perp\mathbb{C}}$ (over $\mathbb{C}$), and

$$\mathrm{ad}_p\, \xi_{\pm j} = \pm 2\pi i \langle \alpha_j, p\rangle \xi_{\pm j} \ .$$

(Here the roots α_j have been extended to $\mathbb{C}$-linear functionals on $\mathfrak{t}^{\mathbb{C}}$.) So with respect to the above basis, ad_p diagonalizes with diagonal

$$(-2\pi i\langle \alpha_{\bar{k}}, p\rangle, \ldots, -2\pi i\langle\alpha_1, p\rangle, 2\pi i\langle\alpha_1, p\rangle, \ldots 2\pi i\langle\alpha_{\bar{k}}, p\rangle) \ .$$

Consequently $\lambda_p = \mathrm{ad}_p^{-1}$ diagonalizes with diagonal

$$\left(\frac{-1}{2\pi i\langle\alpha_{\bar{k}},p\rangle},\dots,\frac{-1}{2\pi i\langle\alpha_1,p\rangle},\frac{1}{2\pi i\langle\alpha_1,p\rangle},\dots\frac{1}{2\pi i\langle\alpha_{\bar{k}},p\rangle}\right) .$$

It follows that for any $\xi \in \mathfrak{t}^{\perp\mathbb{C}}$,

$$|\lambda_p\xi| \leqslant \frac{1}{2\pi}\left(\sup_{1\leqslant j\leqslant \bar{k}}\frac{1}{|\langle\alpha_j,p\rangle|}\right)|\xi| .$$

Since $\langle\alpha_j,a\rangle \in \mathbb{R}$ for all $a \in \mathfrak{t}$, $|\langle\alpha_j,p\rangle| \geqslant |\langle\alpha_j,\mathrm{Re}\,p\rangle|$. Also, every positive root α_j $(1 \leqslant j \leqslant \bar{k})$ can be written as a linear combination of the basis roots $\alpha_1,\dots,\alpha_k$ with *positive* coefficients. The above estimate can therefore be written

6.14
$$|\lambda_p\xi| \leqslant \frac{1}{2\pi}\left(\sup_{1\leqslant j\leqslant k}\frac{1}{|\langle\alpha_j,\mathrm{Re}\,p\rangle|}\right)|\xi| \qquad (\xi\in\mathfrak{t}^{\perp\mathbb{C}}) .$$

Manipulating 6.10, one shows that

6.15
$$\alpha_j = \frac{2\kappa(\alpha_j^*)}{|\alpha_j^*|^2} .$$

Recall that $p \in \mathfrak{t}_0^{\mathbb{C}}$. The diagonal matrix representation of ad_p above, Equation 2.7, and the definition of $\mathfrak{t}_0^{\mathbb{C}}$ show that

$$\mathrm{Re}\,p \in \mathfrak{t}_0 .$$

If $a \in \mathfrak{t}_0$, then an elementary geometric property of root systems is (see Fig. 1) that

$$|a-\partial\mathfrak{t}_0| = \inf_{1\leqslant j\leqslant k}\frac{\alpha_j^*}{|\alpha_j^*|}\cdot a .$$

So for any j with $1 \leqslant j \leqslant k$ one has

$$|a-\partial\mathfrak{t}_0| \leqslant \frac{\alpha_j^*}{|\alpha_j^*|}\cdot a = \frac{\langle\kappa^{-1}(\alpha_j^*),a\rangle}{|\alpha_j^*|} = \frac{1}{2}|\alpha_j^*|\langle\alpha_j,a\rangle ,$$

using 6.15. This shows that

$$|\langle\alpha_j,a\rangle| \geqslant \frac{2|a-\partial\mathfrak{t}_0|}{|\alpha_j^*|} \qquad (a\in\mathfrak{t}_0, 1\leqslant j\leqslant k) .$$

In particular we can apply this to 6.14, with $a \equiv \mathrm{Re}\,p$, to conclude that

$$|\lambda_p\xi| \leqslant \frac{\sup_{1\leqslant j\leqslant k}|\alpha_j^*|}{4\pi|\,\mathrm{Re}\,p-\partial\mathfrak{t}_0|}|\xi| ,$$

i.e.,

6.16 $$|\lambda_p \xi| \leqslant \frac{k_3|\xi|}{|\operatorname{Re} p - \partial \mathfrak{t}_0|} \qquad (\xi \in \mathfrak{t}^{\perp \mathbb{C}}) \ ,$$

where

6.17 $$k_3 \equiv \frac{1}{4\pi} \sup_{1 \leqslant j \leqslant k} |\alpha_j^*| \ .$$

Note that k_3 can be expressed purely in terms of Cartan integers: k_3^2 is the maximum diagonal entry in the matrix $(m_{ij})_{1 \leqslant i,j \leqslant k}$ defined by 6.11. For example if $G = \mathrm{SU}(3)$, then by 6.12, $k_3 = 4\pi\sqrt{3}$.

Bounds on directional derivatives

In the remainder of this section the reader may find it helpful to refer frequently to Fig. 1. In the notation of Chap. 5, we now prove:

6.18 Lemma *Let $u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$, $\xi \in \mathfrak{g}^{\mathbb{C}}$ and $\tau \in \mathfrak{t}^{\mathbb{C}}$ be given. If $0 < \delta \leqslant \gamma$, then for all $(g, p) \in D_{\gamma-\delta}(p_*, r)$,*

1. $$\left| \xi \cdot \frac{\partial u}{\partial g}(g,p) \right| \leqslant 2k_2(e-1) \frac{|\xi| \|u\|_\gamma^{p_*,r}}{\delta \bar{\sigma}} \ ,$$
2. $$\left| \tau \cdot \frac{\partial u}{\partial p}(g,p) \right| \leqslant \frac{|\tau| \|u\|_\gamma^{p_*,r}}{r \delta \bar{\rho}} \ .$$

PROOF. Fix $(g,p) \in D_{\gamma-\delta}(p_*, r)$. By definition

$$\xi \cdot \frac{\partial u}{\partial g}(g,p) = f'(0) \ ,$$

where $f(t) \equiv u(g \exp(t\xi), p)$. We seek $\nu > 0$ sufficiently small that $|t| \leqslant \nu$ ensures that

6.19 $$(g \exp(t\xi), p) \in D_\gamma(p_*, r) \ .$$

Then $|t| \leqslant \nu$ ensures that $f(t) \in \mathbb{C}$ is well-defined, and by Cauchy's inequality for a holomorphic function of a complex variable,

6.20 $$\left| \xi \cdot \frac{\partial u}{\partial g}(g,p) \right| \leqslant \frac{1}{\nu} \sup_{|t| \leqslant \nu} |f(t)| \leqslant \frac{\|u\|_\gamma^{p_*,r}}{\nu} \qquad (t \in \mathbb{C}) \ .$$

Since $(g,p) \in D_{\gamma-\delta}(p_*, r)$ (and because G is compact) there exists $g^{\mathbb{R}} \in G$ such that

6.21 $$|g - g^{\mathbb{R}}|_S \leqslant (\gamma - \delta)\bar{\sigma} \ .$$

We have

$$\begin{aligned}
|g \exp(t\xi) - G|_S &\leqslant |g\exp(t\xi) - g^{\mathbb{R}}|_S \\
&\leqslant |g\exp(t\xi) - g|_S + |g - g^{\mathbb{R}}|_S \\
&\leqslant \sqrt{n_G}|g|_S\,|\exp(t\xi) - \mathrm{id}|_S + |g - g^{\mathbb{R}}|_S \qquad \text{by 6.4} \\
&\leqslant \sqrt{n_G}\big(\,|g^{\mathbb{R}}|_S + |g - g^{\mathbb{R}}|_S\,\big)|\exp(t\xi) - \mathrm{id}|_S + |g - g^{\mathbb{R}}|_S \\
&\leqslant \sqrt{n_G}\big(\,1 + (\gamma-\delta)\bar{\sigma}\,\big)|\exp(t\xi) - \mathrm{id}|_S + (\gamma-\delta)\bar{\sigma} \qquad \text{by 6.2 and 6.21} \\
&\leqslant 2\sqrt{n_G}|\exp(t\xi) - \mathrm{id}|_S + (\gamma-\delta)\bar{\sigma} \qquad \text{Since } \bar{\sigma} \leqslant 1 \ . \qquad (6.22)
\end{aligned}$$

Combining the Taylor series expansion of $\exp(t\xi) \in \mathbb{C}^{n_G \times n_G}$ with 6.4, 6.7 and 6.9:

$$\begin{aligned}
|\exp(t\xi) - \mathrm{id}|_S &\leqslant \frac{1}{\sqrt{n_G}}\left((k_2|t||\xi|) + \frac{1}{2!}(k_2|t||\xi|)^2 + \frac{1}{3!}(k_2|t||\xi|)^3 + \cdots\right) \\
&= \frac{1}{\sqrt{n_G}}(e^{k_2|t||\xi|} - 1) \\
&\leqslant \frac{k_2}{\sqrt{n_G}}(e-1)|t||\xi| \ ,
\end{aligned}$$

assuming that

6.23 $$k_2|t||\xi| \leqslant 1 \ .$$

So (assuming 6.23) 6.22 implies

$$|g\exp(t\xi) - G|_S \leqslant 2(e-1)k_2|t||\xi| + (\gamma-\delta)\bar{\sigma} \ .$$

Therefore $|g\exp(t\xi) - G|_S \leqslant \gamma\bar{\sigma}$ (so that 6.19 holds) provided

6.24 $$k_2|t||\xi| \leqslant \frac{\delta\bar{\sigma}}{2(e-1)} \ .$$

In summary, 6.19 holds if 6.23 and 6.24 do. Since 6.24 is stronger than 6.23 ($\delta, \bar{\sigma} \leqslant 1$), we may take

$$\nu \equiv \frac{\delta \bar{\sigma}}{2k_2(e-1)|\xi|} .$$

With this choice, 6.20 becomes 6.18.1, the first estimate of the lemma.

The proof of 6.18.2 is simpler and left to the reader. □

We write

$$\left\| \frac{\partial u}{\partial g} \right\|_{\gamma}^{p_*,r} \equiv \sup_{(g,p) \in D_\gamma(p_*,r)} \left| \frac{\partial u}{\partial g}(g,p) \right| .$$

One has

$$\left| \frac{\partial u}{\partial g}(g,p) \right| = \sup_{|\xi|=1} \left| \xi \cdot \frac{\partial u}{\partial g}(g,p) \right| \qquad (\xi \in \mathfrak{g}^{\mathbb{C}}) .$$

An analogous statement holds for $|(\partial u/\partial p)(g,p)|$. The following corollary of 6.18 is therefore clear.

6.25 Corollary *Let $u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ be given and suppose $0 < \delta \leqslant \gamma$. Then*

1. $$\left\| \frac{\partial u}{\partial g} \right\|_{\gamma-\delta}^{p_*,r} \leqslant 2k_2(e-1) \frac{\|u\|_{\gamma}^{p_*,r}}{\delta \bar{\sigma}} ,$$

2. $$\left\| \frac{\partial u}{\partial p} \right\|_{\gamma-\delta}^{p_*,r} \leqslant \frac{\|u\|_{\gamma}^{p_*,r}}{r \delta \bar{\rho}} .$$

Times of validity of Hamiltonian flows (Assumption B)

6.26 Lemma (Bounds on the components of a Hamiltonian vector field) *Let $u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ be given and suppose $0 < \delta \leqslant \gamma$. Then*

1. $$\|\xi_u\|_{\gamma-\delta}^{p_*,r} \leqslant \left(1 + \frac{2k_2 k_3 (e-1)\bar{\rho}}{|B_{2\bar{\rho}} - \partial \mathfrak{t}_0|\bar{\sigma}}\right) \frac{\|u\|_{\gamma}^{p_*,r}}{r \delta \bar{\rho}} ,$$

2. $$\|\tau_u\|_{\gamma-\delta}^{p_*,r} \leqslant 2k_2(e-1) \frac{\|u\|_{\gamma}^{p_*,r}}{\delta \bar{\sigma}} .$$

Recall that ξ_u and τ_u are defined by 2.9 and 2.10. Note also that as $B_{2\bar{\rho}}$ is closed, 5.4 ensures $|B_{2\bar{\rho}} - \partial \mathfrak{t}_0| > 0$.

PROOF. The definition 2.9 of ξ_u gives, with the help of 6.16,

$$\|\xi_u\|_{\gamma-\delta}^{p_*,r} \leqslant \left\|\frac{\partial u}{\partial p}\right\|_{\gamma-\delta}^{p_*,r} + k_3 \left(\inf_{p \in B_{r\bar{\rho}}^{(\gamma-\delta)r\bar{\rho}}(p_*)} |\mathrm{Re}\, p - \partial \mathfrak{t}_0| \right)^{-1} \left\|\frac{\partial u}{\partial g}\right\|_{\gamma-\delta}^{p_*,r} .$$

Recalling that 5.8 ensures that $B_{r\bar{\rho}}^{(\gamma-\delta)r\bar{\rho}} \subset B^{\bar{\rho}}$ $(0 \leqslant \gamma \leqslant 1,\ 0 < \delta \leqslant \gamma)$, we have

$$\left(\inf_{p \in B_{r\bar{\rho}}^{(\gamma-\delta)r\bar{\rho}}(p_*)} |\mathrm{Re}\, p - \partial \mathfrak{t}_0| \right)^{-1} \leqslant |B_{2\bar{\rho}} - \partial \mathfrak{t}_0|^{-1} .$$

Applying 6.25, and exploiting the fact that $r \leqslant 1$, one easily deduces 6.26.1.

The inequality 6.26.2 follows similarly from 2.10 and 6.25.1. □

We are now ready to turn to the verification of Assumption B. Let $u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ be given and suppose that $0 < \delta \leqslant \gamma$. Let $t \mapsto (g_t, p_t)$ be an integral curve of X_u, and suppose that the initial condition satisfies $(g_0, p_0) \in D_{\gamma-\delta}(p_*, r)$, i.e.,

$$g_0 \in G^{(\gamma-\delta)\bar{\sigma}} \qquad\qquad p_0 \in B_{r\bar{\rho}}^{(\gamma-\delta)r\bar{\rho}}(p_*) .$$

According to the equations of motion 2.11,

6.27 $$\dot{g}_t = g_t \xi_u(g_t, p_t) ,$$

6.28 $$\dot{p}_t = \tau_u(g_t, p_t) .$$

Because $D_{\gamma-\delta/2}(p_*, r)$ is compact, the solution $(g_t, p_t) \in D_{\gamma-\delta/2}(p_*, r)$ is well-defined until it reaches the boundary of $D_{\gamma-\delta/2}(p_*, r)$. That is, there exists a time st_0 $(s = \pm 1,\ 0 < t_0 \leqslant \infty)$ such that

6.29 $$|t| \leqslant t_0 \qquad \Rightarrow \qquad (g_t, p_t) \in D_{\gamma-\delta/2}(p_*, r)$$

and such that at least one of the following holds:

6.30 $$|g_{st_0} - G|_S = (\gamma - \delta/2)\bar{\sigma} ,$$

6.31 $$|p_{st_0} - B_{r\bar{\rho}}(p_*)| = (\gamma - \delta/2) r\bar{\rho} .$$

According to 6.27,

$$g_{st_0} - g_0 = \int_0^{st_0} g_t \xi_u(g_t, p_t) dt .$$

From 6.29 and 6.8 we deduce

$$
\begin{aligned}
|g_{st_0} - g_0|_S &\leqslant k_2 \left(\sup_{g \in G^{(\gamma-\delta/2)\bar{\sigma}}} |g|_S \right) \|\xi_u\|^{p_*,r}_{\gamma-\delta/2}\, t_0 \\
&\leqslant k_2(1 + (\gamma - \delta/2)\bar{\sigma})\, \|\xi_u\|^{p_*,r}_{\gamma-\delta/2}\, t_0 \qquad \text{by 6.2} \\
&\leqslant 2k_2\, \|\xi_u\|^{p_*,r}_{\gamma-\delta/2}\, t_0 \qquad (\bar{\sigma} \leqslant 1)\ .
\end{aligned}
$$

Applying 6.26.1:

$$
|g_{st_0} - g_0|_S \leqslant 4k_2\zeta \frac{\|u\|^{p_*,r}_{\gamma}}{r\delta\bar{\rho}} t_0\ ,
$$

where

6.32 $$\zeta \equiv 1 + \frac{2k_2k_3(e-1)\bar{\rho}}{|B_{2\bar{\rho}} - \partial \mathfrak{t}_0|\bar{\sigma}}\ .$$

6.33 Remark Notice that $\zeta \to \infty$ as the real ball $B_{2\bar{\rho}} = B_{2\bar{\rho}}(\bar{p})$ approaches the walls of $\mathfrak{t}_0$.

Since $g_0 \in G^{(\gamma-\delta)\bar{\sigma}}$, there exists $g_0^{\mathbb{R}} \in G$ such that $|g_0 - g_0^{\mathbb{R}}|_S \leqslant (\gamma - \delta)\bar{\sigma}$. We compute

$$
\begin{aligned}
|g_{st_0} - G|_S \leqslant |g_{st_0} - g_0^{\mathbb{R}}|_S &\leqslant |g_{st_0} - g_0|_S + |g_0 - g_0^{\mathbb{R}}|_S \\
&\leqslant 4k_2\zeta \frac{\|u\|^{p_*,r}_{\gamma}}{r\delta\bar{\rho}} t_0 + (\gamma - \delta)\bar{\sigma}\ .
\end{aligned}
$$

So, supposing 6.30 holds, we have

6.34 $$t_0 \geqslant \left(\frac{1}{8k_2\zeta}\right) \frac{\bar{\sigma}\bar{\rho}\delta^2 r}{\|u\|^{p_*,r}_{\gamma}}\ .$$

According to 6.28

$$
p_{st_0} = p_0 + \int_0^{st_0} \tau_u(g_t, p_t)\,dt\ .
$$

From 6.29 and 6.26.2 (with δ replaced by $\delta/2$) we deduce

$$
|p_{st_0} - p_0| \leqslant 4k_2(e-1)\frac{\|u\|^{p_*,r}_{\gamma}}{\delta\bar{\sigma}} t_0\ .
$$

Since $p_0 \in B_{r\bar{\rho}}^{(\gamma-\delta)r\bar{\rho}}(p_*)$, there exists $p_0^{\mathbb{R}} \in B_{r\bar{\rho}}(p_*)$ such that $|p_0 - p_0^{\mathbb{R}}| \leqslant (\gamma-\delta)r\bar{\rho}$. We compute

$$\begin{aligned} |p_{st_0} - B_{r\bar{\rho}}| \leqslant |p_{st_0} - p_0^{\mathbb{R}}| &\leqslant |p_{st_0} - p_0| + |p_0 - p_0^{\mathbb{R}}| \\ &\leqslant 4k_2(e-1)\frac{\|u\|_\gamma^{p_*,r}}{\delta\bar{\sigma}}t_0 + (\gamma-\delta)r\bar{\rho} \ . \end{aligned}$$

So, supposing 6.31 holds, we have

6.35 $$t_0 \geqslant \left(\frac{1}{8k_2(e-1)}\right)\frac{\bar{\sigma}\bar{\rho}\delta^2 r}{\|u\|_\gamma^{p_*,r}} \ .$$

Since 6.30 and/or 6.31 holds, 6.34 and/or 6.35 holds. Therefore

$$t_0 \geqslant \frac{c_2\bar{\sigma}\bar{\rho}\delta^2 r}{\|u\|_\gamma^{p_*,r}} \ ,$$

if we take

6.36 $$c_2 \equiv \frac{1}{8k_2\max\{\zeta, e-1\}} \ .$$

Arguments like the above also show that

$$|p_t - p_0| \leqslant \left(\frac{|t|}{t_0}\right)\left(\frac{\delta}{2}\right) r\bar{\rho} \qquad (|t| \leqslant t_0) \ .$$

With the above choice of c_2, Assumption B of Chap. 5 is thus established.

Inequalities for Poisson brackets (Assumption C)

Let $u, v \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ be given and suppose $0 < \delta \leqslant \gamma$. Fixing some $(g,p) \in D_{\gamma-\delta}(p_*, r)$, we have

$$\begin{aligned} \{u,v\}(g,p) &= \langle du, X_v(g,p)\rangle \\ &= \xi_v(g,p)\cdot\frac{\partial u}{\partial g}(g,p) + \tau_v(g,p)\cdot\frac{\partial u}{\partial p}(g,p) \qquad \text{by 2.8} \\ &= f'(0) \ , \end{aligned}$$

where

$$f(t) \equiv u(g\exp(t\xi_v(g,p)), p + t\tau_v(g,p)) \ .$$

We seek $\nu > 0$ sufficiently small that $|t| \leqslant \nu$ guarantees

6.37
$$(g\exp(t\xi_v(g,p)), p + t\tau_v(g,p)) \in D_\gamma(p_*, r) \ .$$

Then $|t| \leqslant \nu$ ensures that $f(t) \in \mathbb{C}$ is well-defined and, by Cauchy's inequality,

6.38
$$|\{u,v\}(g,p)| \leqslant \frac{1}{\nu} \sup_{|t|\leqslant\nu} |f(t)| \leqslant \frac{\|u\|_\gamma^{p_*,r}}{\nu} \ .$$

To satisfy 6.37 we need

6.39
$$|g\exp(t\xi_v(g,p)) - G|_S \leqslant \gamma\bar{\sigma}$$

6.40
$$\text{and} \qquad |(p + t\tau_v(g,p)) - B_{r\bar{\rho}}(p_*)| \leqslant \gamma r\bar{\rho} \ .$$

Arguing as in the proof of 6.18.1, we satisfy 6.39 if

$$|t| \leqslant \frac{\delta\bar{\sigma}}{2k_2(e-1)\,|\xi_v(g,p)|} \ .$$

Since $(g,p) \in D_{\gamma-\delta}(p_*, r)$, it suffices, by 6.26.1, to ensure

6.41
$$|t| \leqslant \frac{\bar{\sigma}\bar{\rho}\delta^2 r}{2k_2(e-1)\zeta\,\|v\|_\gamma^{p_*,r}} \ ,$$

where ζ is the constant defined by 6.32.

Arguing similarly, one sees that 6.40 holds provided

$$|t| \leqslant \frac{r\delta\bar{\rho}}{|\tau_v(g,p)|} \ .$$

By 6.26.2, it suffices to ensure

6.42
$$|t| \leqslant \frac{\bar{\sigma}\bar{\rho}\delta^2 r}{2k_2(e-1)\,\|v\|_\gamma^{p_*,r}} \ .$$

To summarize, if 6.41 and 6.42 hold, then so do 6.39 and 6.40, and hence 6.37. It follows that we may take

$$\nu \equiv \frac{\bar{\sigma}\bar{\rho}\delta^2 r}{2k_2(e-1)\max\{1,\zeta\}\,\|v\|_\gamma^{p_*,r}} \ .$$

With this choice of ν, 6.38 becomes

$$|\{u,v\}(g,p)| \leqslant \left(\frac{2k_2(e-1)\max\{1,\zeta\}}{\bar{\sigma}\bar{\rho}}\right) \frac{\|u\|_\gamma^{p_*,r}\,\|v\|_\gamma^{p_*,r}}{\delta^2 r} \ .$$

Since $(g, p) \in D_{\gamma-\delta}(p_*, r)$ was arbitrary, the first estimate of Assumption C (Chap. 6) holds if we take

6.43 $$c_3 \equiv 2k_2(e-1)\max\{1, \zeta\} \ .$$

If $W \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ is of the form $W(g,p) = w(p)$, then (by, e.g., 2.15),

$$\begin{aligned}\{u, W\}(g,p) &= \frac{\partial u}{\partial g}(g,p) \cdot \nabla w(p) \qquad ((g,p) \in D_{\gamma-\delta}(p_*, r)) \\ &= f'(0) \ ,\end{aligned}$$

where $f(t) \equiv u(g\exp(t\nabla k(p)), p)$. Cauchy's inequality and the by now familiar argument leads to the estimate

$$\|\{u, W\}\|_\gamma^{p_*, r} \leqslant \frac{2k_2(e-1)}{\delta\bar{\sigma}} \sup_{p \in B_{r\bar{\rho}}^{(\gamma-\delta)r\bar{\rho}}(p_*)} |\nabla w(p)| \ .$$

Since $\delta \leqslant 1$, the second estimate of Assumption C certainly holds if we take

6.44 $$c_4 \equiv 2k_2(e-1) \ .$$

This completes the proof of Proposition 6.1 in the case of a semisimple G.

CHAPTER 7

Nekhoroshev-type estimates for momentum maps

In this section we combine Theorem 3.10, Theorem 5.9 and Proposition 6.1 to obtain the Nekhoroshev type estimates on momentum maps discussed in the Introduction.

Let G be a compact connected Lie group and $(P, \omega, G, \mathbf{J})$ a Hamiltonian G-space. Consider a Hamiltonian of the form

$$H = H_0 + F \ ,$$

where H_0 is G-invariant, and F is arbitrary.

Existence of action-group coordinates. Assume that the hypotheses of Theorem 3.10 apply (working 'locally' à la Remark 3.11 if necessary), so that there exists a diffeomorphism $\phi : G \times U \to P$ ($U \equiv \varphi^{-1}(\mathbf{J}(P)) \cap \mathfrak{t}_0$) such that $(H_0 \circ \phi)(g, p) = h(p)$, for some smooth $h : U \to \mathbb{R}$. Recall that $\varphi : \mathfrak{g} \to \mathfrak{g}^*$ is the isomorphism corresponding to the fixed Ad-invariant inner product on $\mathfrak{g}$.

Complexification of the coordinates. Fix some $\bar{x} \in P$ and define $(\bar{g}, \bar{p}) \equiv \phi^{-1}(\bar{x})$. Fix a positive constant $\bar{\rho}$ small enough that the real closed ball $B_{2\bar{\rho}}(\bar{p})$ is contained in U and the complex closed ball $B_{\bar{\rho}}^{\bar{\rho}}(\bar{p})$ (defined in Chap. 5) is contained in $\mathfrak{t}_0^{\mathbb{C}}$ (defined in Chap. 2). Fix a positive constant $\bar{\sigma} \leqslant 1$ small enough that $(G^\sigma)_{0 \leqslant \sigma \leqslant \bar{\sigma}}$, as defined by 6.5, is a local complexification of G in the sense of 5.1.

Analyticity of the Hamiltonian. Assume that $h \in \mathcal{A}^{\mathbb{R}}(B_{\bar{\rho}}(\bar{p}))$ and $F \circ \phi \in \mathcal{A}^{\mathbb{R}}(G \times B_{\bar{\rho}}(\bar{p}))$ (notation as in Chap. 5). Furthermore assume that $\bar{\sigma}$ and $\bar{\rho}$ are sufficiently small that h and $F \circ \phi$ have holomorphic extensions $h \in \mathcal{A}^{\mathbb{C}}(B_{\bar{\rho}}^{\bar{\rho}}(\bar{p}))$ and $F \circ \phi \in \mathcal{A}^{\mathbb{C}}(G^{\bar{\sigma}} \times B_{\bar{\rho}}^{\bar{\rho}}(\bar{p}))$.

Convexity of the unperturbed Hamiltonian. Assume that h is (m, M)-convex on $\operatorname{int} B_{\bar{\rho}}^{\bar{\rho}}(\bar{p})$ in the sense of 5.3, for some $m, M > 0$.

Constants. Write $\Omega(p) = \nabla h(p) \in \mathfrak{t}$ and define the 'time scales'

$$T_m \equiv \frac{1}{\bar{\rho} m} \qquad T_M \equiv \frac{1}{\bar{\rho} M}$$

$$T_\Omega \equiv \left(\sup_{p \in B^{\bar{\rho}}} |\Omega(p)| \right)^{-1} \qquad T_{h'''} \equiv \left(\bar{\rho}^2 \sup_{p \in B} \sup_{u,v,w \in \mathfrak{t}} |D^3 h(p)(u,v,w)| \right)^{-1}$$

and the associated 'dimensionless constants'

$$c_5 \equiv \frac{T_{h'''}}{T_m} \qquad c_6 \equiv \frac{T_M}{T_\Omega} \qquad c_7 \equiv \frac{T_M}{T_m} = \frac{m}{M} \ .$$

Recall that we allow $T_{h'''} = \infty$.

The perturbation parameter. Let $\|\cdot\|$ denote the supremum norm on $\mathcal{A}^{\mathbb{C}}(G^{\bar{\sigma}} \times B^{\bar{\rho}})$ and define the 'perturbation parameter'

$$\epsilon \equiv \frac{\|F\|}{E} \ ,$$

where $E \equiv \bar{\sigma}\bar{\rho}/T_M$. We have the following corollary of Theorem 5.9 and Proposition 6.1. (Recall that k denotes the rank of G.)

7.1 Corollary *There exist positive constants c_1, c_2, c_3, c_4 independent of h and the perturbation F, and positive constants $a = a(k)$, $b = b(k)$, $c = c(c_1, \ldots, c_7)$, $\epsilon_0 = \epsilon_0(k, c_1, \ldots, c_7)$, $t_0 = t_0(c_1, \ldots, c_7)$ and $r_0 = r_0(c_1, \ldots, c_7)$, such that for all $\epsilon \leqslant \epsilon_0$ and every integral curve $t \mapsto x_t \in P$ of X_H with $x_0 = \bar{x}$ one has*

$$\frac{|t|}{T_\Omega} \leqslant t_0 \exp(c\epsilon^{-a}) \ \Rightarrow \ \frac{|\mathbf{J}(x_t) - \mathcal{O}|}{\bar{\rho}} \leqslant r_0 \epsilon^b \ ,$$

where $\mathcal{O} \subset \mathfrak{g}^*$ is the co-adjoint orbit through the point $\mathbf{J}(x_0)$.

Estimates for the constants $a, b, c, \epsilon_0, t_0, r_0$ (which are not optimal) appear in Addendum 5.10.

If G is semisimple (and one takes -1 times the Killing form as the fixed inner product on $\mathfrak{g}$), then c_1 depends only on the Cartan integers of G (according to the explicit computation *presented in Chap. 6) and one may take*

$$c_2 \equiv \frac{1}{8k_2 \max\{\zeta, e-1\}}$$

$$c_3 \equiv 2k_2(e-1)\max\{1,\zeta\}$$

$$c_4 \equiv 2k_2(e-1)$$

$$\zeta \equiv 1 + \frac{2k_2k_3(e-1)\bar{\rho}}{|B_{2\bar{\rho}}(\bar{p}) - \partial \mathfrak{t}_0|_{\bar{\sigma}}} \ .$$

The constant k_3 (see 6.17) depends only on the Cartan integers. The constant k_2 (see 6.9) depends on the realization of G as a linear group.

The reader should keep Remark 6.33 in mind.

PROOF. That one arrives at the following estimate for the action variables is clear:

$$\frac{|t|}{T_\Omega} \leqslant \exp(c\epsilon^{-a}) \ \Rightarrow \ \frac{|p_t - p_0|}{\bar{\rho}} \leqslant r_0\epsilon^b \qquad \big((g_t, p_t) \equiv \phi^{-1}(x_t)\big) \ .$$

According to Theorem 3.10 and 3.3, we have $\mathbf{J}(x_t) = (\mathbf{J}^G \circ \phi^{-1})(x_t) = g_t \cdot \varphi^{-1}(p_t)$ and compute

$$\begin{aligned}|\mathbf{J}(x_t) - \mathcal{O}| &\leqslant \left|\mathbf{J}(x_t) - g_t g_0^{-1} \cdot \mathbf{J}(x_0)\right| = \left|g_t \cdot \varphi^{-1}(p_t) - g_t \cdot \varphi^{-1}(p_0)\right| \\ &= \left|\varphi^{-1}(p_t) - \varphi^{-1}(p_0)\right| = |p_t - p_0| \ ,\end{aligned}$$

since the isomorphism $\varphi : \mathfrak{t} \to \underline{\mathfrak{t}}$ is isometric. This proves that the estimate claimed in the corollary indeed holds. □

Part 2

Geometry

Outline of Part 2

Here we describe geometric constructions underlying the analyses given in 'Part 1: Dynamics.' We assume that the reader has at least read the Introduction and the historical remarks at the end of Chap. 3. The exposition is otherwise relatively self-contained.

In Chap. 8 we study a special class of Hamiltonian G-spaces, namely those possessing points of 'regular co-adjoint orbit type.' It is shown (Theorem 8.14) that essentially all geometric information describing such a space is encoded in a *symplectic cross section*, which is a lower dimensional Hamiltonian T-space, where $T \subset G$ denotes some maximal torus (we assume that G is compact and connected). This result may be viewed as an application of the philosophy of Guillemin and Sternberg (1984, §41) (see also Guillemin et al. (1996)).

In Chap. 9 we show how *action-group coordinates*, a generalization of action-angle coordinates that was applied to perturbation theory in Part 1, can be realized as a symplectic submanifold of T^*G. We derive explicit expressions for the symplectic structure and Hamiltonian vector fields in these coordinates.

In Chap. 10 we use the results of Chap. 8 to reduce the problem of constructing action-group coordinates in an integrable system, to the construction of action-*angle* coordinates in an associated symplectic cross section (Theorem 10.2). We tackle the problem of *global* existence by relating work of Duistermaat (1980). We also prove 'semi-global' results, which are more convenient to apply in concrete examples. This includes a proof of Theorem 3.10 of Part 1 (here stated as Corollary 10.12).

In Chap. 11 we apply the preceding theory by constructing explicit action-group coordinates in the axisymmetric Euler-Poinsot rigid body.

The constructions of Chap. 10 apply to systems that have zero dimensional Marsden-Weinstein reduced spaces, with respect to some known symmetry group G. In Chap. 12 we show how one can sometimes enlarge G in a system with *two* dimensional reduced spaces to $G' \equiv G \times S^1$, in such a way as to render the G'-reduced spaces zero dimensional. We demonstrate this in the case of the rigid body in Appendix D.

In Appendix E we include a detailed proof of Weinstein's Symplectic Correspondence Theorem, which we need in Chap. 12.

Contents of Part 2

CHAPTER 8

On Hamiltonian G-spaces with regular momenta

Preliminaries

Before turning to our main subject, we review and supplement terminology used in Chap. 3 of Part 1. We already assume some familiarity with the structure of compact Lie groups, say with what has been summarized in Chap. 1 of Part 1. We work throughout in the category of smooth manifolds and maps, where 'smooth' means C^∞ or real-analytic.

If a Lie group G acts on a symplectic manifold (P, ω) by symplectic diffeomorphisms, and the infinitesimal generators ξ_P ($\xi \in \mathfrak{g}$) are *global* Hamiltonian vector fields, then one says that G is acting in a *Hamiltonian fashion*, or that the action is *Hamiltonian.* In that case there exists a map $\mathbf{J} : P \to \mathfrak{g}^*$, called a *momentum map*, with the property that it delivers Hamiltonian functions $J_\xi : P \to \mathbb{R}$ for the generators according to the formula $J_\xi(x) \equiv \langle \mathbf{J}(x), \xi \rangle$. So, if we denote the Hamiltonian vector field corresponding to a Hamiltonian H by X_H, we have $\xi_P = X_{J_\xi}$. If furthermore the momentum map is G-equivariant (with G acting on $\mathfrak{g}^*$ by the co-adjoint action), then we shall refer to $(P, \omega, G, \mathbf{J})$ (or simply P) as a *Hamiltonian G-space.*

For background on Hamiltonian actions and momentum maps see Marsden and Ratiu (1994), Abraham and Marsden (1978, Chapter 4) or Guillemin and Sternberg (1984). *We limit ourselves throughout to the case of G compact and connected.*

By the *co-adjoint orbit type* of a point $x \in P$ we shall mean the orbit type of $\mathbf{J}(x) \in \mathfrak{g}^*$. The space P has uniform co-adjoint orbit type (T) ($T \subset G$ denoting a maximal torus) if and only if $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$. Here $\mathfrak{g}^*_{\mathrm{reg}}$ denotes the regular points of the co-adjoint action (not to be confused with regular *values* of $\mathbf{J}$). In that case we will call P a *Hamiltonian G-space with regular momenta.* Our interest in such spaces is motivated by the fact that, under an appropriate integrability hypothesis, they admit action-group coordinates which, as we showed in Part 1, can be used as a geometric framework for perturbation theory. A simple example is geodesic motions on S^2 (provided we exclude the trivial motions); see, e.g., Chap. 3 of Part 1.

For us the construction of 'coordinates' in a Hamiltonian system will mean the construction of an appropriate *equivalence* between Hamiltonian G-spaces:

8.1 Definition Two Hamiltonian G-spaces $(P_1, \omega_1, G, \mathbf{J}^1)$ and $(P_2, \omega_2, G, \mathbf{J}^2)$ will be said to be *equivalent* if there exists a G-equivariant symplectic diffeomorphism $\phi : P_1 \to P_2$ such that $\mathbf{J}^2 \circ \phi = \mathbf{J}^1$.

> *Henceforth $T \subset G$ denotes a fixed maximal torus, $\mathfrak{t} \subset \mathfrak{g}$ its Lie algebra, and $\mathcal{W} \subset \underline{\mathfrak{t}} \equiv \mathrm{Ann}[\mathfrak{g}, \mathfrak{t}]$ a fixed (open) Weyl chamber in $\mathfrak{g}^*$ (terminology and notation as in Chap. 1 of Part 1); $\mathcal{O}$ will denote a fixed regular co-adjoint orbit, and μ_0 the unique point of intersection of $\mathcal{O}$ with $\mathcal{W}$.*

The natural projection $\mathfrak{g}^* \to \mathfrak{t}^*$ (dual map of inclusion) restricts to an isomorphism $i : \underline{\mathfrak{t}} \to \mathfrak{t}^*$, which identifies $\mathcal{W}$ with an open set $\mathfrak{t}_0^* \equiv i(\mathcal{W})$. We refer to $\mathcal{W}$ as a Weyl chamber also.

We now turn to a systematic study of Hamiltonian G-spaces with regular momenta. While we make no explicit assumptions on the orbit type (G_x) of points $x \in P$, the following is worth noting.

8.2 Proposition *If P is a Hamiltonian G-space with regular momenta, then $\dim G_x \leqslant \operatorname{rank} G$ for all $x \in P$.*

This follows from the equivariance of $\mathbf{J}$ and the identity

8.3 $$\mathfrak{g}_x = \mathrm{Ann}\,\mathrm{Im}\,\mathrm{T}_x\mathbf{J} \ ,$$

which follows from the definition of a momentum map.

For any Hamiltonian G-space P, the open G-invariant subset $P' \equiv \mathbf{J}^{-1}(\mathfrak{g}^*_{\mathrm{reg}})$ is a Hamiltonian G-space with regular momenta. If, however, $\dim G_x > \operatorname{rank} G$ for all $x \in P$, then

Proposition 8.2 implies that P' is empty. The following example (relevant to the so-called 1:1:1 resonance) is a case in point.

8.4 Example Equip $P \equiv \mathbb{R}^6$ with the standard symplectic structure, and let $G \equiv \mathrm{SU}(3)$ act (faithfully) on P by identifying $\mathbb{R}^6$ with $\mathbb{C}^3$. Then G acts symplectically, as is well-known, and so admits a momentum map, on account of the Poincaré lemma. This momentum map is equivariant since SU(3) is semisimple (Guillemin and Sternberg, 1984). One finds that for all $x \in P$, dim $G_x \geq 3 >$rank $G = 2$, and that accordingly $\mathbf{J}(P) \cap \mathfrak{g}^*_{\mathrm{reg}} = \emptyset$. (In the 1:1:1 resonance one studies cubic and higher order perturbations of the G-invariant Hamiltonian $H(z_1, z_2, z_3) = \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2 + \frac{1}{2}|z_3|^2$.)

Henceforth $(P, \omega, G, \mathbf{J})$ *denotes a Hamiltonian* G*-space with* $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$.

On the geometry of the co-adjoint action

Our first observation is that there exists a natural G-equivariant trivialization $\mathfrak{g}^*_{\mathrm{reg}} \xrightarrow{\sim} \mathcal{O} \times \mathcal{W}$ (with G acting on $\mathcal{O} \times \mathcal{W}$ according to $g \cdot (\nu, w) \equiv (g \cdot \nu, w)$). This trivialization is given by $\mu \mapsto (\pi_{\mathcal{O}}(\mu), \pi_{\mathcal{W}}(\mu))$, where $\pi_{\mathcal{O}} : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O}$ and $\pi_{\mathcal{W}} : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{W}$ are defined implicitly by

8.5
$$\begin{aligned} \pi_{\mathcal{O}}(g \cdot w) &\equiv g \cdot \mu_0 \\ \pi_{\mathcal{W}}(g \cdot w) &\equiv w \end{aligned} \qquad (g \in G, w \in \mathcal{W}) \; ,$$

every element of $\mathfrak{g}^*_{\mathrm{reg}}$ being of the form $g \cdot w$ for some $g \in G$ and $w \in \mathcal{W}$. Indeed $\mathcal{W}$ is a global slice at μ_0 for the co-adjoint action of G on $\mathfrak{g}^*_{\mathrm{reg}}$, so that there exists a natural G-equivariant isomorphism $\mathfrak{g}^*_{\mathrm{reg}} \cong G \times_T \mathcal{W}$ (see Appendix B). Since all points in $\mathcal{W}$ have isotropy group T, we have $G \times_T \mathcal{W} \cong G/T \times \mathcal{W} \cong \mathcal{O} \times \mathcal{W}$.

Note that the fibers of $\pi_{\mathcal{O}} : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O}$ are the Weyl chambers in $\mathfrak{g}^*$; the fibers of $\pi_{\mathcal{W}} : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{W}$ are the regular co-adjoint orbits.

8.6 Example Take $G = \mathrm{SO}(3)$ and let $T \subset G$ be the rotations about the e_3 axis. The Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ via the isomorphism $\xi \mapsto \hat{\xi} : \mathbb{R}^3 \to \mathfrak{so}(3)$ defined by $\hat{\xi} u = \xi \times u$ $(u \in \mathbb{R}^3)$. We have a corresponding identification $\mathfrak{g}^* \cong \mathbb{R}^3$. The co-adjoint action is then represented by the standard action of SO(3) on $\mathbb{R}^3$. We have $\underline{t} = \mathrm{span}\{e_3\}$. For a Weyl chamber in $\mathfrak{g}^*$ choose $\mathcal{W} \equiv \{te_3 \mid t > 0\}$. For a regular co-adjoint orbit $\mathcal{O}$ choose the unit sphere $S^2 \subset \mathbb{R}^3$. The unique intersection point of $\mathcal{W}$ and $\mathcal{O}$ is $\mu_0 \equiv e_3$. The projection $\pi_{\mathcal{O}} : \mathbb{R}^3 \backslash \{0\} \to S^2$ is given by $\pi_{\mathcal{O}}(\mu) \equiv \mu / \|\mu\|$. The projection $\pi_{\mathcal{W}} : \mathbb{R}^3 \backslash \{0\} \to \mathcal{W} \cong (0, \infty)$ is given by $\pi_{\mathcal{W}}(\mu) \equiv \|\mu\|$.

Since $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$, it is natural to pull the projections $\pi_{\mathcal{O}}$ and $\pi_{\mathcal{W}}$ back using the momentum map: Define

8.7
$$\begin{aligned} \pi &\equiv \pi_{\mathcal{O}} \circ \mathbf{J} : P \to \mathcal{O} \ , \\ \mathfrak{j} &\equiv i \circ \pi_{\mathcal{W}} \circ \mathbf{J} : P \to \mathfrak{t}_0^* \subset \mathfrak{t}^* \ . \end{aligned}$$

In the following paragraphs we shall see that each fiber of π encodes the geometric structure of the full space P. Later, we shall see that $\mathfrak{j} : P \to \mathfrak{t}^*$ is the momentum map for a natural Hamiltonian action of T on P. Unlike the action of T as a subgroup of G, this new T-action leaves the fibers of π invariant. Furthermore the orbits of this action do not depend on the choice of maximal torus T.

The fibering of P by Hamiltonian T-spaces

The map π is G-equivariant. Since G acts transitively on $\mathcal{O}$, it follows that $\pi : P \to \mathcal{O}$ is a surjective submersion.

8.8 Theorem (Guillemin-Sternberg) *Each fiber $F_\nu \equiv \pi^{-1}(\nu)$ of π is a symplectic submanifold of P invariant with respect to the action of the maximal torus G_ν. Furthermore, G_ν acts on F_ν in a Hamiltonian fashion (with respect to the symplectic structure $\omega_\nu \equiv \omega | F_\nu$) with an equivariant momentum map $\mathbf{J}^\nu : F_\nu \to \mathfrak{g}_\nu^*$ given by $\mathbf{J}^\nu(x) \equiv \mathbf{J}(x) | \mathfrak{g}_\nu$ $(x \in F_\nu)$. This makes F_ν a Hamiltonian G_ν-space $(F_\nu, \omega_\nu, G_\nu, \mathbf{J}^\nu)$.*

Each element $g \in G$ maps each fiber of π symplectomorphically onto another fiber, and this induced action on the fibers is transitive.

Theorem 8.8 follows, for example, from the symplectic cross section theorem (Guillemin and Sternberg, 1984, §41). For a statement of this theorem the reader is referred to Guillemin et al. (1996), from whom we also borrow the following terminology:

8.9 Definition Write $F \equiv F_{\mu_0}$, $\omega_F \equiv \omega_{\mu_0}$ and $\mathbf{J}^F \equiv \mathbf{J}^{\mu_0}$. The Hamiltonian T-space $(F, \omega_F, T, \mathbf{J}^F)$ will be referred to as the *symplectic cross section* of $(P, \omega, G, \mathbf{J})$. Note that $\mathbf{J}^F = i \circ \mathbf{J}|F$.

In fact, the symplectic cross section theorem gives more information than is contained in the statement of Theorem 8.8. It shows that $\pi : P \to \mathcal{O}$ is naturally isomorphic to the bundle $G \times_T F \to G/T$. In particular, $\pi : P \to \mathcal{O}$ is a *symplectic fibration*, meaning that it is a locally trivial fiber bundle, with fibers modeled on F, whose structure group is contained in the symplectomorphism group of F. The symplectic cross section theorem also gives a characterization of the symplectic connection on $\pi : P \to \mathcal{O}$ (see 8.11 below for a definition). This characterization allows one to recover the symplectic structure ω on P from: (i) its restriction to F, and (ii) the natural co-adjoint orbit symplectic structure of the base $\mathcal{O}$. The interested reader is referred to *op. cit.* for details. For our purposes, Theorem 8.8 and the rather straightforward observation in Lemma 8.12 below will suffice.

Since all maximal tori of G are conjugate, we can replace the action of G_ν on a fiber F_ν ($\nu \in \mathcal{O}$) by a Hamiltonian action of the fixed torus $T = G_{\mu_0}$. (In fact, a slightly stronger statement is true; see Corollary 8.18.) This makes each fiber a Hamiltonian T-space. By Theorem 8.8 there exists for any fibers F_{ν_1} and F_{ν_2} an element of G mapping F_{ν_1} symplectomorphically onto F_{ν_2}. This map is in fact an *equivalence* of Hamiltonian T-spaces, in the sense of 8.1. In particular, each fiber of π is equivalent as a Hamiltonian T-space to the symplectic cross section.

8.10 Remark (On G-invariant dynamics) If $H : P \to \mathbb{R}$ is a G-invariant Hamiltonian, then the fibers of $\pi \equiv \pi_{\mathcal{O}} \circ \mathbf{J}$ will be X_H-invariant, by Noether's theorem (which states that the preimage of *any* set under the momentum map is X_H-invariant). Furthermore, the dynamics in any two fibers will be conjugate because any fiber can be mapped symplectomorphically onto any other by a group element. So any feature of the dynamics in a given fiber is repeated in *all* the fibers. For example, the existence of an equilibrium point for X_H in fact implies the existence of an entire manifold of equilibria diffeomorphic to $\mathcal{O}$. Borrowing language from perturbation theory (where one is interested in non-invariant perturbations to H), we say that such a Hamiltonian is more 'resonant[1].' This built-in resonance is particular to the non-Abelian case, for otherwise $\mathcal{O}$ is just a point and π has only one fiber, viz. P.

The symplectic connection

8.11 Definition The *symplectic connection* on $\pi : P \to \mathcal{O}$ is the assignment to each $x \in P$ of a space $\mathrm{Hor}_x \equiv (\ker \mathrm{T}_x\pi)^\omega$ called the *horizontal space* at x which, on account of $F_{\pi(x)} \subset P$ being a symplectic submanifold, is complementary to $\ker \mathrm{T}_x\pi$ in T_xP.

The restriction of ω to the horizontal spaces is easily characterized:

8.12 Lemma 1. $\mathrm{Hor}_x \subset \mathrm{T}_x(G \cdot x) \qquad (x \in p)$.

2. $\omega(\xi_P(x), \eta_P(x)) = \langle \mathbf{J}(x), [\xi, \eta] \rangle \qquad (x \in P;\ \ \xi, \eta \in \mathfrak{g})$.

PROOF. Since $\pi = \pi_{\mathcal{O}} \circ \mathbf{J}$, we have $\ker \mathrm{T}_x\pi \supset \ker \mathrm{T}_x\mathbf{J}$. Taking ω-orthogonal complements, we obtain $\mathrm{Hor}_x \subset (\ker \mathrm{T}_x\mathbf{J})^\omega$. By the definition of a momentum map, $\mathfrak{g}^* \xleftarrow{\mathbf{J}} P \to P/G$ has the 'dual pair' property, i.e. $(\ker \mathrm{T}_x\mathbf{J})^\omega = \mathrm{T}_x(G \cdot x)$. So 1 holds.

To prove 2, we observe that as $\mathbf{J}$ is equivariant, it is also infinitesimally equivariant, from which it follows that $\{J_\xi, J_\eta\} = J_{[\xi,\eta]}$, where $\{f, h\} \equiv X_h \lrcorner\, X_f \lrcorner\, \omega$ (see, e.g., Abraham and Marsden (1978, Theorem 4.2.8) for a proof). We can now compute

$$\begin{aligned}\omega(\xi_P(x), \eta_P(x)) &= \omega(X_{J_\xi}(x), X_{J_\eta}(x)) = \{J_\xi, J_\eta\}(x) \\ &= J_{[\xi,\eta]}(x) = \langle \mathbf{J}(x), [\xi, \eta] \rangle \ .\end{aligned}$$

□

[1]A perturbation theorist might also coin the term 'degenerate.' From the symmetry point of view, this terminology is misleading, as the phenomenon we have just described applies to *any* G-invariant Hamiltonian.

A simple example: Geodesic motions on the sphere

We now describe a simple example for which one can give an explicit realization of the symplectic cross section.

Recall (see, e.g., Chap. 3 of Part 1) that the phase space for geodesic motions on S^2 can be identified with a Hamiltonian G-space $(\mathrm{T}S^2, \omega, \mathrm{SO}(3), \mathbf{J})$ where $\mathbf{J} : \mathrm{T}S^2 \to \mathfrak{so}(3)^* \cong \mathbb{R}^3$ is given by $\mathbf{J}(q, v) \equiv q \times v$. (Here we are viewing $\mathrm{T}S^2$ as $\{(q, v) \in S^2 \times \mathbb{R}^3 \mid q \cdot v = 0\}$.) If we restrict attention to the open dense invariant subset $P \subset \mathrm{T}S^2$ obtained by removing the zero section, we obtain a Hamiltonian G-space $(P, \omega, \mathrm{SO}(3), \mathbf{J})$ with regular momenta.

Choose T, $\mathcal{W}$ and $\mathcal{O}$ as in Example 8.6. Then the symplectic fibration $\pi : P \to S^2$ is given by $\pi(q, v) \equiv \pi_{\mathcal{O}}(\mathbf{J}(q, v)) = v/\|v\|$.

The symplectic cross section $F = \pi^{-1}(e_3)$ is symplectomorphic to the cylinder $S^1 \times (0, \infty)$, equipped with the standard symplectic structure $d\theta \wedge dI$. Indeed a diffeomorphism $\varphi : S^1 \times (0, \infty) \to F$ is given by

$$\varphi(\theta, I) \equiv (R^3_\theta e_1, I R^3_\theta e_2) \ ,$$

where $R^3_\theta \equiv \exp(\theta \hat{e}_3)$. To show that φ is symplectic, recall that $\omega = -d\Theta$ where

8.13 $$\langle \Theta, \frac{d}{dt}(q_t, v_t)\big|_{t=0} \rangle \equiv v_0 \cdot \dot{q}_0 \ .$$

From this one computes $\frac{\partial}{\partial \theta} \lrcorner \varphi^* \Theta = I$ and $\frac{\partial}{\partial I} \lrcorner \varphi^* \Theta = 0$, so that $\varphi^* \Theta = I d\theta$, giving $\varphi^* \omega = d\theta \wedge dI$ as required.

The action of $T \cong S^1$ on $S^1 \times (0, \infty)$ that makes the symplectomorphism φ a T-equivariant map is given by $\theta' \cdot (\theta, I) \equiv (\theta + \theta', I)$. Identify the Lie algebra of S^1 (and its dual) with $\mathbb{R}$ using the generator $\frac{\partial}{\partial \theta}(0)$. Then the momentum map $\mathbf{J}^F \circ \varphi$ of this action is given by the coordinate function $I : S^1 \times (0, \infty) \to \mathbb{R}$.

To summarize, the symplectic cross section of the Hamiltonian G-space applying to geodesic motions on S^2 is just the cylinder $S^1 \times (0, \infty)$ equipped with the standard S^1 Hamiltonian action.

The Hamiltonian for geodesic motions on S^2 is (up to a rescaling) $H(q, v) = \frac{1}{2}\|v\|^2$, which is SO(3)-invariant. This Hamiltonian restricts to a function on F that is pulled back by $\varphi : S^1 \times (0,\infty) \xrightarrow{\sim} F$ to $\frac{1}{2}I^2$. This is the Hamiltonian for geodesic motions on the *circle*, and is S^1-invariant. Thus geodesic motions on the sphere can be imagined as a family of subsystems F_ν ($\nu \in \mathcal{O}$), each identifiable with geodesic motions on the circle. This corresponds precisely to the following familiar fact: A point mass constrained to move on the surface of a smooth sphere moves as if it were constrained to move on a great circle. This circle is given by the intersection of the sphere with the plane whose normal is aligned with the initial (and subsequently conserved) angular momentum. These normals live in the space $S^2 = \mathcal{O}$, the 'unit momentum sphere.'

Reduction to the Abelian case

Here is the central result of this chapter.

8.14 Theorem

Two Hamiltonian G-spaces $(P_1, \omega_1, G, \mathbf{J}^1)$ and $(P_2, \omega_2, G, \mathbf{J}^2)$ with regular momenta are equivalent (in the sense of 8.1) if and only if their symplectic cross sections are equivalent. Explicitly, if $\varphi : F_1 \to F_2$ is an equivalence between the symplectic cross sections, then a well-defined equivalence $\phi : P_1 \to P_2$ is given by

$$\phi(g \cdot y) \equiv g \cdot \varphi(y) \qquad (g \in G, y \in F_1) \ .$$

PROOF. Recall that as G acts transitively on the fibers of $\pi_1 : P_1 \to \mathcal{O}$ (the symplectic fibration associated with $(P_1, \omega_1, G, \mathbf{J}^1)$), every element of P_1 is of the form $g \cdot y$ for some $g \in G$ and $y \in F_1 = \pi_1^{-1}(\mu_0)$. Furthermore, $g \cdot y = g' \cdot y'$ ($g' \in G$, $y' \in F_1$) if and only if $g' = gq$ and $y' = q^{-1} \cdot y$, for some $q \in T$ (by the G-equivariance of π_1). In that case $g' \cdot \varphi(y') = gq \cdot \varphi(q^{-1} \cdot y) = g \cdot \varphi(y)$, by the T-equivariance of φ. This shows that ϕ is well-defined.

That ϕ is smooth is clear. It is a diffeomorphism since it has a well-defined and smooth inverse $\phi^{-1} : P_2 \to P_1$ given by

$$\phi^{-1}(g \cdot y) \equiv g \cdot \varphi^{-1}(y) \qquad (g \in G, y \in F_2) \ .$$

By construction ϕ is G-equivariant.

We show next that $\mathbf{J}^2 \circ \phi = \mathbf{J}^1$. Let $g \cdot y \in P_1$ be arbitrary ($g \in G$, $y \in F_1$). Then $\mathbf{J}^1(y) \in \mathcal{W} \subset \underline{\mathfrak{t}}$, so that $\mathbf{J}^1(y) = i^{-1}(\mathbf{J}^1(y)|\mathfrak{t}) = i^{-1}(\mathbf{J}^{F_1}(y))$. Similarly, as $\varphi(y) \in F_2$, we have $\mathbf{J}^2(\varphi(y)) = i^{-1}(\mathbf{J}^{F_2}(\varphi(y)))$. But $\mathbf{J}^{F_2} \circ \varphi = \mathbf{J}^{F_1}$, since $\varphi : F_1 \to F_2$ is an equivalence, from which we deduce

8.15 $$\mathbf{J}^2(\varphi(y)) = \mathbf{J}^1(y) \ .$$

Using this fact we now compute

$$(\mathbf{J}^2 \circ \phi)(g \cdot y) = \mathbf{J}^2(g \cdot \varphi(y)) = g \cdot \mathbf{J}^2(\varphi(y)) = g \cdot \mathbf{J}^1(y) = \mathbf{J}^1(g \cdot y) \ .$$

Since $g \cdot y \in F_1$ was arbitrary, this shows that $\mathbf{J}^2 \circ \phi = \mathbf{J}^1$.

To prove that ϕ is an equivalence it remains to show that ϕ is symplectic. Since $\varphi : F_1 \to F_2$ is symplectic, it follows from the definition of ϕ, and the last statement in Theorem 8.8, that ϕ maps fibers of π_1 symplectomorphically onto fibers of π_2. To check the symplecticity of ϕ, it therefore suffices to check that the tangent map $\mathrm{T}\phi$ maps horizontal spaces of the symplectic connection on $\pi_1 : P_1 \to \mathcal{O}$ symplectically onto horizontal spaces of the symplectic connection on $\pi_2 : P_2 \to \mathcal{O}$. From the definition of ϕ and infinitesimal generators, it follows that

8.16 $$\mathrm{T}\phi \cdot \xi_{P_1}(g \cdot y) = \xi_{P_2}(\phi(g \cdot y)) \qquad (\xi \in \mathfrak{g},\ g \in G,\ y \in F) \ .$$

Since $\mathbf{J}^2(\phi(g \cdot y)) = \mathbf{J}^1(g \cdot y)$ (proven above), the symplecticity of $\mathrm{T}\phi$ on horizontal spaces follows immediately from 8.16 and Lemma 8.12. □

The π-invariant T-action

The remaining results of this chapter play a minor role in the sequel and may be skipped on a first reading. We call an action of a group H on P *π-invariant* if the fibers of π are H-invariant submanifolds. The action of T (as a subgroup of G) is not π-invariant, but it does leave the symplectic cross section F invariant. We now show how to extend this action on F to a π-invariant action. It is convenient to begin with a slightly more general result:

8.17 Lemma (Extension Lemma) *Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space with regular momenta, $\pi : P \to \mathcal{O}$ the associated symplectic fibration, and $(F, \omega_F, T, \mathbf{J}^F)$ the symplectic cross section. Suppose that an Abelian group H acts on F in a Hamiltonian fashion and let $\mathbf{K} : F \to \mathfrak{h}^*$ be an associated equivariant momentum map. Assume furthermore that:*

1. *The actions of H and T commute, and*
2. *$\mathbf{K}$ is T-invariant.*
3. *$\mathbf{J}|F$ is H-invariant.*

Then, denoting the action of H on F by $(h, y) \mapsto h \cdot_H y$, there exists a well-defined extension to P given by

$$h \cdot_H (g \cdot y) \equiv g \cdot (h \cdot_H y) \qquad (h \in H, g \in G, y \in F) \ .$$

The extended action of H leaves the fibers of π invariant and is Hamiltonian. An associated momentum map is the well-defined extension of $\mathbf{K}$ to a map $\mathbf{K} : P \to \mathfrak{h}^*$ given by

$$\mathbf{K}(g \cdot y) \equiv \mathbf{K}(y) \qquad (g \in G, y \in F) \ .$$

Moreover, the properties 1 and 2 above extend as follows:

4. *The extended action of H commutes with the action of G, and*
5. *The extended momentum map $\mathbf{K} : P \to \mathfrak{h}^*$ is G-invariant.*
6. *$\mathbf{J}$ is invariant with respect to the extended action of H.*

To prove the lemma (see Appendix C) one appeals to Theorem 8.8 and Lemma 8.12.

Taking $H \equiv T$, we obtain an extension of the action of T on F to a π-invariant action on P, with momentum map $\mathbf{K} : P \to \mathfrak{t}^*$ given by

$$\begin{aligned}
\mathbf{K}(g \cdot y) \equiv \mathbf{J}^F(y) = \mathbf{J}(y)|\mathfrak{t} &= i(\mathbf{J}(y)) \\
&= i(\pi_{\mathcal{W}}(g \cdot \mathbf{J}(y))) \qquad \text{by } 8.5 \\
&= (i \circ \pi_{\mathcal{W}} \circ \mathbf{J})(g \cdot y) \qquad \text{by the equivariance of } \mathbf{J} \\
&= \mathbf{j}(g \cdot y) \qquad (\mathbf{j} \text{ is defined in } 8.7) \ .
\end{aligned}$$

Notice also that we can write $q\cdot_T(g\cdot y) = gqg^{-1}\cdot(g\cdot y)$ $(q \in T,\ g \in G,\ y \in F)$. Summarizing:

8.18 Corollary *If $(P, \omega, G, \mathbf{J})$ is a Hamiltonian G-space with regular momenta, and π the associated symplectic fibration, then there exists a well-defined π-invariant action $(q, x) \mapsto q\cdot_T x$ of T on P given by*

$$q\cdot_T(g\cdot y) \equiv gq\cdot y \qquad (q \in T, g \in G, y \in F)\ .$$

This action is Hamiltonian with momentum map $\mathbf{j} : P \to \mathfrak{t}_0^* \subset \mathfrak{t}^*$ defined by $\mathbf{j} \equiv i \circ \pi_{\mathcal{W}} \circ \mathbf{J}$. The actions of T and G_ν on a fiber F_ν $(\nu \in \mathcal{O})$ are conjugate in the following sense: Let $g \in G$ be such that $\nu = g\cdot\mu_0$ and define the isomorphism $\psi : T \to G_\nu$ by $\psi(q) \equiv gqg^{-1}$. (Recall that $T = G_{\mu_0}$, so that $gTg^{-1} = G_{g\cdot\mu_0} = G_\nu$.) Then $q\cdot_T x = \psi(q)\cdot x$ for any $x \in F_\nu$.

CHAPTER 9

Action-group coordinates as a symplectic cross section

One can use symplectic cross sections as a way of generating new symplectic manifolds. In this chapter we show that the action-group model space $G \times \mathfrak{t}_0^*$ described in Part 1 can be realized as the symplectic cross section of T^*G (with its 'irregular' points removed). We study the space $G \times \mathfrak{t}_0^*$ in some detail, supplying proofs of some facts stated in Part 1.

The symplectic cross section of T^*G

Let Θ denote the *canonical one-form* on T^*G. This is defined by

9.1 $$\langle \Theta, \zeta \rangle = \langle \tau_{\mathrm{T}^*G}(\zeta), \mathrm{T}\tau_G^* \cdot \zeta \rangle \qquad (\zeta \in \mathrm{T}(\mathrm{T}^*G)) \ .$$

The maps $\tau_{\mathrm{T}^*G} : \mathrm{T}(\mathrm{T}^*G) \to \mathrm{T}^*G$ and $\tau_G^* : \mathrm{T}^*G \to G$ denote the canonical projections. A natural symplectic structure on T^*G is $\omega \equiv -d\Theta$.

A (left) action of G on itself is given by

$$g \cdot h \equiv R_{g^{-1}}(h) \qquad (g, h \in G) \ ,$$

where $R_g(h) \equiv hg$. Recall that the (covariant) *cotangent lift* $\mathrm{T}_*\phi : \mathrm{T}^*G \to \mathrm{T}^*G$ of a diffeomorphism $\phi : G \to G$ is defined by

$$\langle \mathrm{T}_*\phi \cdot \alpha, v \rangle \equiv \langle \alpha, \mathrm{T}\phi^{-1} \cdot v \rangle \qquad (\alpha \in \mathrm{T}_g^*G,\ v \in \mathrm{T}_{\phi(g)}G,\ g \in G) \ .$$

We make G act on T^*G by cotangent-lifting the above action of G on itself:

$$g \cdot x \equiv \mathrm{T}_*R_{g^{-1}} \cdot x \qquad (g \in G, x \in \mathrm{T}^*G) \ .$$

It follows from general results concerning cotangent-lifted actions (see, e.g., Marsden and Ratiu (1994)) that the above action of G on T^*G is Hamiltonian, with an equivariant momentum map $\mathbf{J} : \mathrm{T}^*G \to \mathfrak{g}^*$ given by

$$\mathbf{J}(x) \equiv \mathrm{T}_* L_{g^{-1}} \cdot x \qquad (x \in \mathrm{T}^*G,\ g \equiv \tau_G^*(x)) \ ,$$

where $L_g(h) \equiv gh$ $(g, h \in G)$. The open dense G-invariant subset $P \equiv \mathbf{J}^{-1}(\mathfrak{g}^*_{\mathrm{reg}}) \subset \mathrm{T}^*G$ is a Hamiltonian G-space with regular momenta. Each element of P is of the form $\mathrm{T}_* L_g \cdot \mu$ for some $g \in G$ and $\mu \in \mathfrak{g}^*_{\mathrm{reg}}$. (Indeed $(g, \mu) \mapsto \mathrm{T}_* L_g \cdot \mu : G \times \mathfrak{g}^*_{\mathrm{reg}} \to P$ is a diffeomorphism.) The symplectic fibration $\pi : P \to \mathcal{O}$ is given by

$$\pi(\mathrm{T}_* L_g \cdot \mu) = \pi_{\mathcal{O}}(\mu) \qquad (g \in G,\ \mu \in \mathfrak{g}^*_{\mathrm{reg}}) \ .$$

Therefore, the symplectic cross section of $(P, \omega, G, \mathbf{J})$ is given by

$$F \equiv \{\mathrm{T}_* L_g \cdot \mu \mid g \in G,\ \mu \in \mathcal{W}\} \ .$$

For each $\alpha \in \mathfrak{g}^*$, let α_G denote the left-invariant one-form on G with $\alpha_G(\mathrm{id}_G) = \alpha$ (here viewing one-forms on G as sections of $G \to \mathrm{T}^*G$). Then the embedding $\phi : G \times \mathfrak{t}_0^* \hookrightarrow \mathrm{T}^*G$ defined by $\phi(g, p) \equiv (i^{-1}(p))_G(g) = \mathrm{T}_* L_g \cdot i^{-1}(p)$ is a diffeomorphism onto F. (Since Theorem 8.8 says that $F \subset P$ is a symplectic submanifold, this proves Proposition 2.2 stated in Part 1.) We define $\omega_G^* \equiv \phi^* \omega$, and call $(G \times \mathfrak{t}_0^*, \omega_G^*)$ the *action-group model space* of G.

In deriving an explicit formula for the symplectic structure ω_G^*, it will be convenient to have a concrete way of expressing vectors tangent to $G \times \mathfrak{t}_0^*$. To this end, define for each $(\xi, \tau) \in \mathfrak{g} \times \mathfrak{t}^*$ the vector field $(\xi, \tau)_{\mathrm{vf}}$ on $G \times \mathfrak{t}_0^*$ by

9.2 $$(\xi, \tau)_{\mathrm{vf}}(g, p) \equiv \frac{d}{dt}(g \exp(t\xi), p + t\tau)\Big|_{t=0} \ .$$

Since $(\xi, \tau) \mapsto (\xi, \tau)_{\mathrm{vf}}(g, p) : \mathfrak{g} \times \mathfrak{t}^* \to \mathrm{T}_{(g,p)}(G \times \mathfrak{t}_0^*)$ is an isomorphism at every $(g, p) \in G \times \mathfrak{t}_0^*$, every vector tangent to $G \times \mathfrak{t}_0^*$ is uniquely expressible in the form $(\xi, \tau)_{\mathrm{vf}}(g, p)$. One verifies that[1]

9.3 $$[(\xi_1, \tau_1)_{\mathrm{vf}}, (\xi_2, \tau_2)_{\mathrm{vf}}] = ([\xi_1, \xi_2], 0)_{\mathrm{vf}} \ .$$

[1] We identify the Lie algebra of a Lie group with the *left* invariant vector fields, equipped with Jacobi-Lie bracket; our sign convention for the latter is the same as Abraham, Marsden and Ratiu (1988, §4.2.20).

9.4 Lemma *Define* $\Theta_G^* = \phi^*\Theta$, *so that* $\omega_G^* = -d\Theta_G^*$. *Then*

$$\langle \Theta_G^*, (\xi,\tau)_{\mathrm{vf}}(g,p)\rangle = \langle p, \sigma(\xi)\rangle \ ,$$

where $\sigma : \mathfrak{g} \to \mathfrak{t}$ *is the projection onto* $\mathfrak{t}$ *along* $\mathfrak{t}^\perp \equiv [\mathfrak{g},\mathfrak{t}]$. *Furthermore,*

$$\omega_G^*\big(\,(\xi_1,\tau_1)_{\mathrm{vf}}(g,p),(\xi_2,\tau_2)_{\mathrm{vf}}(g,p)\,\big) = \langle \tau_2,\sigma(\xi_1)\rangle - \langle \tau_1,\sigma(\xi_2)\rangle + \langle p, \sigma[\xi_1,\xi_2]\rangle \ .$$

The above expression for Θ_G^* also appears in Dazord and Delzant (1987, Section 5) (who obtain it via a different route).

PROOF. One computes using 9.1

$$\begin{aligned}
\langle \Theta_G^*, (\xi,\tau)_{\mathrm{vf}}(g,p)\rangle &= \langle \Theta, \mathrm{T}\phi\cdot(\xi,\tau)_{\mathrm{vf}}(g,p)\rangle \\
&= \Big\langle \mathrm{T}_*L_g\cdot i^{-1}(p), \frac{d}{dt}\tau_G^*\Big(\phi(g\exp(t\xi),p+t\tau)\Big)\Big|_{t=0}\Big\rangle \\
&= \Big\langle i^{-1}(p), \mathrm{T}L_{g^{-1}}\cdot\frac{d}{dt}\tau_G^*(\mathrm{T}_*L_{g\exp(t\xi)}\cdot i^{-1}(p+t\tau))\Big|_{t=0}\Big\rangle \\
&= \Big\langle i^{-1}(p), \frac{d}{dt}g^{-1}g\exp(t\xi)\Big|_{t=0}\Big\rangle \\
&= \langle i^{-1}(p),\xi\rangle = \langle p,\sigma(\xi)\rangle \ . \qquad (9.5)
\end{aligned}$$

From this we compute

$$\big(\,(\xi_1,\tau_1)_{\mathrm{vf}} \lrcorner\, d((\xi_2,\tau_2)_{\mathrm{vf}} \lrcorner\, \Theta_G^*)\,\big)(g,p) = \frac{d}{dt}\langle p+t\tau_1,\sigma(\xi_2)\rangle\Big|_{t=0} = \langle \tau_1,\sigma(\xi_2)\rangle \ . \qquad (9.6)$$

Applying the well-known identity

$$v \lrcorner\, u \lrcorner\, d\beta = u \lrcorner\, d(v \lrcorner\, \beta) - v \lrcorner\, d(u \lrcorner\, \beta) - [u,v] \lrcorner\, \beta \ ,$$

with $\beta = \Theta_G^*$, we obtain from 9.3, 9.5 and 9.6 the formula for ω_G^* stated in the lemma. ☐

9.7 Remark If G is a torus $\mathbb{T}^k$, then $G\times\mathfrak{t}_0^* \cong \mathbb{T}^k\times\mathbb{R}^k$ and ω_G^* is identifiable with the canonical symplectic structure $\sum_j dq_j \wedge dp_j$. In this sense $G\times\mathfrak{t}_0^*$ (G an arbitrary compact connected Lie group) is a non-Abelian generalization of action-angle coordinates.

Action-group coordinates as a Hamiltonian G-space

Since we have realized $G \times \mathfrak{t}_0^*$ as a symplectic cross section, the maximal torus T acts on $G \times \mathfrak{t}_0^*$. More significant for us, however, is the observation that G acts on $G \times \mathfrak{t}_0^*$:

9.8 Lemma *The action of G on $G \times \mathfrak{t}_0^*$ defined by $g \cdot (h,p) \equiv (gh,p)$ is Hamiltonian, with equivariant momentum map $\mathbf{J}^G : G \times \mathfrak{t}_0^* \to \mathfrak{g}^*$ given by $\mathbf{J}^G(g,p) \equiv g \cdot i^{-1}(p) = \mathrm{Ad}^*_{g^{-1}}\, i^{-1}(p)$. This action makes $(G \times U, \omega_G^*, G, \mathbf{J}^G)$ a Hamiltonian G-space with regular momenta for any open set $U \subset \mathfrak{t}_0^*$.*

PROOF. Imitate the construction in Example 3.2 of Part 1. □

9.9 Remark The action of T on the Hamiltonian G-space $G \times \mathfrak{t}_0^*$ given by Corollary 8.18 is defined by $q \cdot_T (g,p) \equiv (gq,p)$. Corollary 8.18 states that this action is Hamiltonian, with a momentum map $\mathbf{j}^G : G \times \mathfrak{t}_0^* \to \mathfrak{t}_0^*$ given by $\mathbf{j}^G(g,p) = (i \circ \pi_{\mathcal{W}} \circ \mathbf{J}^G)(g,p) = p$.

The following proposition, whose easy proof is left to the reader, shows that the symplectic cross section of the action-group model space of G is the action-group model space of T, i.e., conventional action-*angle* coordinates (by Remark 9.7).

9.10 Proposition
The symplectic cross section of $(G \times U, \omega_G^*, G, \mathbf{J}^G)$ is $(T \times U, \omega_T^*, T, \mathbf{J}^T)$.

Hamiltonian vector fields in action-group coordinates

In our applications to perturbation theory described in Part 1, it was essential to have a concrete way of writing down the equations of motion in action-group coordinates. These equations were derived from an expression for Hamiltonian vector fields that was stated without proof. We now derive that expression explicitly.

In Part 1 it was convenient to identify $G \times \mathfrak{t}_0^*$ with $G \times \mathfrak{t}_0$ using some Ad-invariant inner product on $\mathfrak{g}$. Here we work throughout with $G \times \mathfrak{t}_0^*$ and leave it to the interested reader to translate our results into the form used in Part 1.

9.11 Lemma *Write* $\mathfrak{t}^\perp \equiv [\mathfrak{g}, \mathfrak{t}]$ *and* $\underline{\mathfrak{t}}^\perp \equiv \mathrm{Ann}\, \mathfrak{t}$ *(notation as in Chap. 1 of Part 1). Then for all* $p \in \mathfrak{t}_0^*$ *the map*

$$\xi \mapsto \mathrm{ad}^*_\xi (i^{-1}(p)) : \mathfrak{t}^\perp \to \underline{\mathfrak{t}}^\perp$$

is an isomorphism.

PROOF. Since $\dim \mathfrak{t}^\perp = \dim \underline{\mathfrak{t}}^\perp$, it suffices to show that $\mathrm{ad}^*_\xi\, i^{-1}(p) = 0$ implies $\xi = 0$. Write $\mu \equiv i^{-1}(p) \in \mathcal{W} \subset \underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}}$, so that $\mathfrak{t} = \mathfrak{g}_\mu$ (by, e.g., Corollary 1.8.2 of Part 1). Supposing that $\xi \in \mathfrak{t}^\perp$ satisfies $\mathrm{ad}^*_\xi\, i^{-1}(p) = 0$, we have $\mathrm{ad}^*_\xi\, \mu = 0$, i.e., $\xi \in \mathfrak{g}_\mu = \mathfrak{t}$. But $\mathfrak{t} \cap \mathfrak{t}^\perp = \{0\}$ (by, e.g., Theorem 1.2.8 of Part 1), so that $\xi = 0$. $\square$

For any $p \in \mathfrak{t}_0^*$, we define $\lambda_p : \underline{\mathfrak{t}}^\perp \to \mathfrak{t}^\perp$ to be the inverse of the map described in 9.11 above. For future reference, let us record that this means

9.12
$$\lambda_p \left(\mathrm{ad}^*_\xi\, i^{-1}(p)\right) = \xi \qquad (\xi \in \mathfrak{t}^\perp,\ p \in \mathfrak{t}_0^*)\ .$$

For any smooth function $f : G \times \mathfrak{t}_0^* \to \mathbb{R}$ define the vector-valued functions $\frac{\partial f}{\partial g} : G \times \mathfrak{t}_0^* \to \mathfrak{g}^*$ and $\frac{\partial f}{\partial p} : G \times \mathfrak{t}_0^* \to \mathfrak{t}$ by

$$\langle \frac{\partial f}{\partial g}(g,p), \xi \rangle \equiv \langle df, (\xi, 0)_{\mathrm{vf}}(g,p) \rangle = \frac{d}{dt} f(g \exp(t\xi), p)\Big|_{t=0} \qquad (\xi \in \mathfrak{g})\ ,$$
$$\langle \tau, \frac{\partial f}{\partial p}(g,p) \rangle \equiv \langle df, (0, \tau)_{\mathrm{vf}}(g,p) \rangle = \frac{d}{dt} f(g, p + t\tau)\Big|_{t=0} \qquad (\tau \in \mathfrak{t}^*)\ .$$

9.13 Proposition (Hamiltonian vector fields on $G \times \mathfrak{t}_0^*$)

The Hamiltonian vector field X_f *corresponding to a function* f *is given by*

$$X_f(g,p) = (\xi, \tau)_{\mathrm{vf}}(g,p)\ ,$$

where

$$\xi \equiv \frac{\partial f}{\partial p}(g,p) + \Lambda_p \frac{\partial f}{\partial g}(g,p)$$
$$\tau \equiv -i\sigma^* \frac{\partial f}{\partial g}(g,p)$$
$$\Lambda_p \equiv \lambda_p \circ (\mathrm{id} - \sigma^*)\ .$$

Here $\sigma^* : \mathfrak{g}^* \to \underline{\mathfrak{t}}$ *denotes the projection along* $\underline{\mathfrak{t}}^\perp \equiv \operatorname{Ann}\mathfrak{t}$, *i.e.*, $\langle \sigma^*\mu, \xi\rangle = \langle \mu, \sigma\xi\rangle$ $(\mu \in \mathfrak{g}^*,$ $\xi \in \mathfrak{g})$.

Proof. Define $\xi \in \mathfrak{g}$ and $\tau \in \mathfrak{t}^*$ as in the statement of the lemma. It suffices to verify that the one-forms $(\xi,\tau)_{\mathrm{vf}} \lrcorner\, \omega_G^*$ and df agree on elements of $\mathrm{T}_{(g,p)}(G \times \mathfrak{t}_0^*)$. Let $x \in \mathfrak{g}$ and $y \in \mathfrak{t}^*$ be given. Then, by Lemma 9.4,

$$\begin{aligned}
\langle (\xi,\tau)_{\mathrm{vf}} \lrcorner\, \omega_G^*, (x,y)_{\mathrm{vf}}(g,p)\rangle &= \langle y, \sigma\xi\rangle - \langle \tau, \sigma x\rangle + \langle p, \sigma[\xi, x]\rangle \\
&= \langle i\sigma^*\frac{\partial f}{\partial g}(g,p), \sigma x\rangle + \langle y, \frac{\partial f}{\partial p}(g,p)\rangle + \langle i^{-1}(p), [\Lambda_p \frac{\partial f}{\partial g}(g,p), x] + [\frac{\partial f}{\partial p}(g,p), x]\rangle \\
&= \langle \frac{\partial f}{\partial g}(g,p), \sigma x\rangle + \langle y, \frac{\partial f}{\partial p}(g,p)\rangle + \langle i^{-1}(p), \Lambda_p \frac{\partial f}{\partial g}(g,p), x]\rangle \ . \qquad (9.14)
\end{aligned}$$

We have used the fact $\langle i^{-1}(p), [\frac{\partial f}{\partial p}(g,p), x]\rangle = 0$, which is true because $i^{-1}(p) \in \operatorname{Ann}[\mathfrak{t}, \mathfrak{g}]$ and $\frac{\partial f}{\partial p}(g,p) \in \mathfrak{t}$. Noting that $\Lambda_p \frac{\partial f}{\partial g}(g,p) \in \mathfrak{t}^\perp$ and using 9.12,

$$\begin{aligned}
\langle i^{-1}(p), [\Lambda_p \frac{\partial f}{\partial g}(g,p), x]\rangle &= \langle \lambda_p^{-1}\Lambda_p \frac{\partial f}{\partial g}(g,p), x\rangle \\
&= \langle (\mathrm{id} - \sigma^*)\frac{\partial f}{\partial g}(g,p), x\rangle \\
&= \langle \frac{\partial f}{\partial g}(g,p), (\mathrm{id} - \sigma)x\rangle \ ,
\end{aligned}$$

so that 9.14 becomes

$$\begin{aligned}
\langle (\xi,\tau)_{\mathrm{vf}} \lrcorner\, \omega_G^*, (x,y)_{\mathrm{vf}}(g,p)\rangle &= \langle \frac{\partial f}{\partial g}(g,p), x\rangle + \langle y, \frac{\partial f}{\partial p}(g,p)\rangle \\
&= \langle df, (x,y)_{\mathrm{vf}}(g,p)\rangle \ .
\end{aligned}$$

Since $x \in \mathfrak{g}$ and $y \in \mathfrak{t}^*$ were arbitrary, this proves our assertion. □

The Poisson bracket in action-group coordinates

The Poisson bracket on $G \times \mathfrak{t}_0^*$ is defined by $\{f, h\} \equiv X_h \lrcorner\, X_f \lrcorner\, \omega_G^*$.

9.15 Corollary (The Poisson bracket on $G \times \mathfrak{t}_0^*$)

Dropping the '(g,p)' argument from '$\frac{\partial f}{\partial g}(g,p)$,' etc., the Poisson bracket on $G \times \mathfrak{t}_0^*$ *is given* *by*

$$\{f,h\}_N(g,p) = \langle\frac{\partial f}{\partial g}, \frac{\partial h}{\partial p}\rangle - \langle\frac{\partial h}{\partial g}, \frac{\partial f}{\partial p}\rangle - \langle p, \sigma[\Lambda_p\frac{\partial f}{\partial g}, \Lambda_p\frac{\partial h}{\partial g}]\rangle,$$

where $\Lambda_p : \mathfrak{g}^* \to \mathfrak{g}$ *is the map defined in Lemma 9.13 above.*

PROOF. Imitate the proof of Lemma 2.15 of Part 1. □

CHAPTER 10

Constructing action-group coordinates

Noticing that equivalent Hamiltonian G-spaces (see 8.1) have identical momentum map images, we introduce the following terminology:

10.1 Definition Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space and define $V \equiv \mathbf{J}(P)$ and $U \equiv i(V \cap \mathcal{W})$. (If G is Abelian, then $U = V$.) We say that P *admits G-compatible action-group coordinates* if $(P, \omega, G, \mathbf{J})$ is equivalent to $(G \times U, \omega_G, G, \mathbf{J}^G)$. We sometimes distinguish the Abelian case (G, a torus) by saying that *P admits G-compatible action-**angle** coordinates.*

The non-Abelian case

Here is our main result. It follows immediately from Theorem 8.14 and Proposition 9.10.

10.2 Theorem

A Hamiltonian G-space $(P, \omega, G, \mathbf{J})$ with regular momenta admits G-compatible action-group coordinates $\Leftrightarrow$ its symplectic cross section $(F, \omega_F, T, \mathbf{J}^F)$ admits T-compatible action-angle coordinates. Indeed if U denotes the image of $\mathbf{J}^F$, and $\varphi : T \times U \xrightarrow{\sim} F$ an equivalence, then a well-defined equivalence $\phi : G \times U \xrightarrow{\sim} P$ is given by

$$\phi(g \cdot y) \equiv g \cdot \varphi(y) \qquad (g \in G,\ y \in F) \ .$$

The existence of action-*angle* coordinates can be characterized as follows.

10.3 Proposition *Let T be a torus and $(F, \omega, T, \mathbf{J})$ a Hamiltonian T-space. Then F admits T-compatible action-angle coordinates $\Leftrightarrow$ the following conditions hold:*

1. *T acts freely and the space is geometrically integrable (see below).*

2. $\mathbf{J} : F \to U \equiv \mathbf{J}(F)$ *has connected fibers.*

3. $\mathbf{J} : F \to U$ *admits a (global) isotropic section* $s : U \to F$.

Recall that a section $s : U \to F$ is *isotropic* if $s^*\omega = 0$. As in Part 1, when a group G acts freely on a Hamiltonian G-space $(P, \omega, G, \mathbf{J})$, we call the space *geometrically integrable* if the Marsden-Weinstein reduced spaces are zero dimensional. If $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$, then geometric integrability occurs precisely when $\dim P = \dim G + \operatorname{rank} G$ (see Chap. 3 of Part 1 for details). In particular, in Proposition 10.3 (where $G = T$) this means $\dim F = 2 \dim U$. In that case the image of a section $s : U \to F$ has half the dimension of the ambient space. A section whose image has this dimension and is isotropic is usually called a *Lagrangian* section.

PROOF OF $\Rightarrow$. That conditions 1 and 2 hold is obvious. Condition 3 follows since the map $p \mapsto (q, p) : U \to T \times U$ is an isotropic section (for any $q \in T$). $\square$

PROOF OF $\Leftarrow$. Since T acts freely, $\mathbf{J} : F \to U$ is a surjective submersion (apply identity 8.3). By the pre-image theorem the fibres of $\mathbf{J}$ are submanifolds of F of dimension $\dim F - \dim T$. Since the space is geometrically integrable this dimension is $\dim T$. By momentum map equivariance — which amounts to *invariance* since T is Abelian — the connected components of a fibre are therefore T-orbits. It follows from condition 2 that each fibre is a *single* T-orbit. Consequently, as T acts freely, the map $\phi : T \times U \to F$ defined by $\phi(q, p) \equiv q \cdot s(p)$ is a diffeomorphism.

Now ϕ is clearly T-equivariant, and we have $(\mathbf{J} \circ \phi)(q, p) = p = \mathbf{J}^T(q, p)$, since $\mathbf{J}$ is T-invariant and s is a section. So to show that ϕ is an equivalence, it remains to show that ϕ is symplectic. This means showing that $\phi^*\omega = \omega_T^*$ where, according to Lemma 9.4, ω_T^* is given by

10.4
$$\omega_T^* \left((\xi_1, \tau_1)_{\mathrm{vf}}(q, p), (\xi_2, \tau_2)_{\mathrm{vf}}(q, p) \right) = \langle \tau_2, \xi_1 \rangle - \langle \tau_1, \xi_2 \rangle$$
$$\left((q, p) \in T \times U;\ \xi_1, \xi_2 \in \mathfrak{t};\ \tau_1, \tau_2 \in \mathfrak{t}^* \right) .$$

We will compare $\phi^*\omega$ and ω_T^* on pairs of vectors of the form $(\xi, \tau)_{\mathrm{vf}}(q,p)$. We start with a simple computation:

$$\begin{aligned}
\mathrm{T}\phi \cdot (\xi,\tau)_{\mathrm{vf}}(q,p) &= \frac{d}{dt}\phi(q\exp(t\xi), p+t\tau)\big|_{t=0} \\
&= \frac{d}{dt}\exp(t\xi)\cdot(q\cdot s(p+t\tau))\big|_{t=0} \qquad (T \text{ is Abelian}) \\
&= \xi_F(\phi(q,p)) + (\mathrm{T}\Phi_q \circ \mathrm{T}s)\cdot \frac{d}{dt}(p+t\tau)\big|_{t=0}\ ,
\end{aligned}$$

where the diffeomorphism $\Phi_q : F \to F$ is defined by $\Phi_q(x) \equiv q\cdot x$. Using this we compute

10.5 $\phi^*\omega(\,(\xi_1,\tau_1)_{\mathrm{vf}}(q,p), (\xi_2,\tau_2)_{\mathrm{vf}}(q,p)\,) =$

$$\begin{aligned}
&\omega(\,(\xi_1)_F(\phi(q,p)), (\xi_2)_F(\phi(q,p))\,) \\
&+\omega\left(\,(\xi_1)_F(\phi(q,p)), (\mathrm{T}\Phi_q\circ\mathrm{T}s)\cdot\frac{d}{dt}(p+t\tau_2)\big|_{t=0}\right) \\
&-\omega\left(\,(\xi_2)_F(\phi(q,p)), (\mathrm{T}\Phi_q\circ\mathrm{T}s)\cdot\frac{d}{dt}(p+t\tau_1)\big|_{t=0}\right) \\
&+\omega\left(\,(\mathrm{T}\Phi_q\circ\mathrm{T}s)\cdot\frac{d}{dt}(p+t\tau_1)\big|_{t=0}, (\mathrm{T}\Phi_q\circ\mathrm{T}s)\cdot\frac{d}{dt}(p+t\tau_2)\big|_{t=0}\right)\ .
\end{aligned}$$

The first term on the right-hand side vanishes by 8.12.1, since T is Abelian. The last term vanishes because Φ_q is symplectic and $s : U \to F$ is isotropic. The remaining terms are of the form

$$\begin{aligned}
&\omega\left(\,\xi_F(\phi(q,p)), (\mathrm{T}\Phi_q\circ\mathrm{T}s)\cdot\frac{d}{dt}(p+t\tau)\big|_{t=0}\right) \\
&\quad= \omega\left(\,\xi_F(q,p), \mathrm{T}s\cdot\frac{d}{dt}(p+ts)\big|_{t=0}\right) \text{ since } \Phi_q \text{ is symplectic and } \Phi_q^*\xi_F = \xi_F \\
&\quad= \langle dJ_\xi, \mathrm{T}s\cdot\frac{d}{dt}(p+t\tau)\rangle \qquad \text{since } \xi_F = X_{J_\xi} \\
&\quad= \langle\frac{d}{dt}(\mathbf{J}(s(p+t\tau)), \xi\rangle = \langle\tau,\xi\rangle \qquad \text{since } \mathbf{J}\circ s = \mathrm{id}\ (s \text{ is a section}).
\end{aligned}$$

From this we see that 10.5 and 10.4 have the same right hand sides, so that $\phi^*\omega = \omega_T^*$ as claimed. □

Of course our main interest in Proposition 10.3 is the case in which F is the symplectic cross section of some Hamiltonian G-space P (G non-Abelian). In that case, some of the conditions listed in 10.3 can be cast as conditions on the space P. This will be useful later on.

10.6 Lemma

Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space with regular momenta and $(F, \omega_F, T, \mathbf{J}^F)$ its symplectic cross section. Then

1. *T acts freely and $(F, \omega_F, T, \mathbf{J}^F)$ is geometrically integrable*
   $\Leftrightarrow$ G acts freely and $(P, \omega, G, \mathbf{J})$ is geometrically integrable.
2. *Every fiber of $\mathbf{J}^F : F \to U \equiv \mathbf{J}^F(F)$ is homeomorphic to a fiber of $\mathbf{J} : P \to V \equiv \mathbf{J}(P)$ and conversely. (In particular, $\mathbf{J}^F$ has connected fibers $\Leftrightarrow$ $\mathbf{J}$ has connected fibers.)*

PROOF. That $T \subset G$ acts freely on $F \subset P$ when G acts freely on P is trivial. Conversely, suppose T acts freely on F, and let $g \in G$ be such that $g \cdot x = x$ for some $x \in P$. Then by the equivariance of the symplectic fibration $\pi : P \to \mathcal{O}$, we have $g \in G_\nu$, where $\nu \equiv \pi(x)$. Since G acts transitively on $\mathcal{O}$, there is $h \in G$ such that $\nu = h \cdot \mu_0$. Then $G_\nu = G_{h\cdot\mu_0} = hG_{\mu_0}h^{-1} = hTh^{-1}$, so that $g = hqh^{-1}$ for some $q \in T$. Then $g \cdot x = x$ implies $q \cdot (h^{-1} \cdot x) = h^{-1} \cdot x$. Since $h^{-1} \cdot x \in F_{\mu_0} = F$ (again by the equivariance of π), and we are supposing that T acts freely on F, it follows that $q = \mathrm{id}$, so that $g = hh^{-1} = \mathrm{id}$. This shows that G acts freely. Since $\dim P = \dim \mathcal{O} + \dim F = (\dim G - \operatorname{rank} G) + \dim F$, and $\operatorname{rank} G = \dim T$, we have

$$\dim F = 2 \dim T \quad \Leftrightarrow \quad \dim P = \dim G + \operatorname{rank} G \ .$$

This shows that F is geometrically integrable if and only if P is geometrically integrable. So 1 is true.

Each fiber of $\mathbf{J}$ lies entirely in a fiber of the symplectic fibration $\pi : P \to \mathcal{O}$, and is therefore (by, e.g., Theorem 8.8) homeomorphic to a fiber of $\mathbf{J}$ lying entirely in the fiber $F = F_{\mu_0}$. Now by definition $\mathbf{J}^F = i(\mathbf{J}|F)$, where $i : \underline{\mathfrak{t}} \to \mathfrak{t}^*$ is an isomorphism. So every fiber of $\mathbf{J}^F$ is literally a fiber of $\mathbf{J}$, and every fiber of $\mathbf{J}$ is homeomorphic to a fiber of $\mathbf{J}^F$. So 2 is true. $\square$

The following observation is useful in constructing explicit action-group coordinates in concrete examples:

10.7 Proposition *Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space with regular momenta and $(F, \omega_F, T, \mathbf{J}^F)$ its symplectic cross section. Assume that F admits T-compatible action-angle coordinates (so that P admits G-compatible action-group coordinates, by Theorem 10.2) and let $s : U \to F$ be a Lagrangian section (whose existence is guaranteed by Proposition 10.3). Then a realization of G-compatible action-group coordinates is given by the equivalence $\phi : G \times U \xrightarrow{\sim} P$ defined explicitly by $\phi(g, p) \equiv g \cdot s(p)$.*

PROOF. Let $s : U \to F$ be a Lagrangian section. As in the proof of Proposition 10.3, we have an equivalence $\varphi : T \times U \xrightarrow{\sim} F$ given by $\varphi(q, p) \equiv q \cdot s(p)$. Applying Theorem 10.2, a well-defined equivalence $\phi : G \times U \to P$ is given by

$$\phi(g \cdot (q, p)) \equiv g \cdot \varphi(q, p) \qquad (g \in G,\ q \in T \subset G,\ p \in U) \ .$$

But this means $\phi(gq, p) = (gq) \cdot s(p)$ for any $g \in G$, $q \in T$, $p \in U$. In particular $\phi(g, p) = g \cdot s(p)$, as claimed. □

10.8 Example (Geodesic motions on S^2) In the example of geodesic motions on S^2 discussed in Chap. 8, the map $\varphi : S^1 \times (0, \infty) \to F$ defines an equivalence between the space

$$(T \times U, \omega_T, T, \mathbf{J}^T) = (S^1 \times (0, \infty), d\theta \wedge dI, S^1, I)$$

and the cross section $(F, \omega_F, T, \mathbf{J}^F)$. In other words, F admits T-compatible action-angle coordinates. The map $I \mapsto (\theta, I) : (0, \infty) \to S^1 \times (0, \infty)$ is a Lagrangian section for any θ, so that a Lagrangian section $s : (0, \infty) \to F$ is given by

$$s(I) \equiv \varphi(0, I) = (e_1, Ie_2) \ .$$

The action-group model space for G is (up to the obvious identifications) $\mathrm{SO}(3) \times (0, \infty)$ (see, e.g., Example 2.14, Part 1). Applying Proposition 10.7, a realization of G-compatible action-group coordinates for P is given by the equivalence $\phi : \mathrm{SO}(3) \times (0, \infty) \to P$ defined by

$$\phi(g, p) \equiv g \cdot s(p) = (ge_1, pge_2) \ .$$

The SO(3)-invariant Hamiltonian $H(q, v) = \frac{1}{2}\|v\|^2$ is pulled back to $H \circ \phi(g, p) = h(p) \equiv \frac{1}{2}p^2$. (That this problem admits action-group coordinates was observed more or less directly in Remark 3.5, Part 1.)

The Abelian case

Theorem 10.2 reduces the construction of action-group coordinates to that of action-angle coordinates. The remainder of this chapter revisits in some detail the classical problem of constructing action-angle coordinates, from the present Hamiltonian G-space point of view. (The necessary and sufficient conditions given in Proposition 10.3 are not always very practical.)

Suppose then that $(F, \omega, T, \mathbf{J})$ is a Hamiltonian T-space on which a torus T is acting freely, and assume that the space is geometrically integrable. These assumptions are easy to check. Then, as we observed in the proof of Proposition 10.3, $\mathbf{J} : F \to U \equiv \mathbf{J}(F)$ is a surjective submersion, and the connected components of the fibers of $\mathbf{J}$ are T-orbits. Now unless one assumes that the fibers are *compact*, serious technical difficulties impede further progress[1]. Similar difficulties arise in attempts to generalize momentum map 'convexity theorems' to the non-compact case (see, e.g., Hilgert et al. (1994)[2]).

If the fibers of $\mathbf{J}$ are compact, or equivalently, have a finite number of connected components[3], then by a corollary of the Reeb stability theorem (see, e.g., Camacho and Neto (1985)), $\mathbf{J} : F \to U$ is a locally trivial fiber bundle. We consider two scenarios:

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[1]Unless one works 'locally'. See, e.g., Remark 3.11, Part 1.

[2]Note that these results do not apply immediately here since we cannot assume that $\mathbf{J}$ is proper as a map *into* $\mathfrak{t}^*$.

[3]Remember that these fibers are *regular* submanifolds, by the preimage theorem.

Scenario 1. Suppose that $\mathbf{J} : F \to U$ has connected components. (Checking this is not always trivial.) Then $\mathbf{J} : F \to U$ is a Lagrangian fibration with connected fibers, to which the study of Duistermaat (1980) is applicable. Recall that a submanifold $L \subset F$ is *Lagrangian* if the inclusion $\iota : L \hookrightarrow F$ is isotropic ($\iota^*\omega = 0$) and if $\dim F = 2 \dim L$. That the fibers of $\mathbf{J}$ are Lagrangian follows from 8.12.2 and our integrability assumption. Duistermaat characterizes the obstruction to the existence of a (not necessarily Lagrangian) section $s : U \to F$ in terms of an algebraic invariant ν of the bundle known as a *Chern class.* A section exists if and only if $\nu = 0$. In that case the de Rham cohomology class $[s^*\omega] \in H^2(U)$ is the same *for all sections* s, and a *Lagrangian* section exists if and only if this class vanishes. A sufficient condition is that ω is exact. See *op. cit.* for details.

Scenario 2. In this case we weaken the connectedness hypothesis but enforce a topological assumption on the momentum map image. We formulate our conclusions in the form of a theorem, whose proof is constructive, up to the existence of a Riemannian metric.

10.9 Theorem *Let $(F, \omega, T, \mathbf{J})$ be a Hamiltonian T-space on which a torus T is acting freely, and assume this space is geometrically integrable (i.e., $\dim F = 2 \dim T$). Assume F is connected and that each fiber of $\mathbf{J} : F \to U \equiv \mathbf{J}(F)$ is compact, or has a finite number of connected components. Assume that U is smoothly contractible. Then each fiber is in fact connected, and there exists a Lagrangian section $s : U \to F$ of $\mathbf{J} : F \to U$. In particular, F admits T-compatible action-angle coordinates.*

Note that in examples it is often easy to check the compactness of the fibers of $\mathbf{J}$. The Euler-Poinsot rigid body (see Chap. 11) is a case in point.

PROOF. The last statement follows from the preceding ones and Proposition 10.3. We begin by showing that the fibers of $\mathbf{J}$ are connected. By the hypotheses and our earlier remarks, $\mathbf{J} : F \to U$ is a locally trivial fiber bundle, each fiber being a finite number of T-orbits. Proceed by constructing a covering space $\tilde{\mathbf{J}} : \tilde{U} \to U$ as follows: For each $p \in U$ let Q_p denote the set of connected components of $\mathbf{J}^{-1}(p)$, and for each $x \in F$ let $[x]$ denote the element of $Q_{\mathbf{J}(x)}$ that contains x. Define $\tilde{U} \equiv \cup_{p \in U} Q_p$ and $\tilde{\mathbf{J}} : \tilde{U} \to U$ by

$\tilde{\mathbf{J}}([x]) \equiv \mathbf{J}(x)$. Using the local triviality of $\mathbf{J} : F \to U$ and the connectedness of F, it is not difficult to show that $\tilde{U}$ admits the structure of a smooth *connected* manifold with respect to which $\tilde{\mathbf{J}} : \tilde{U} \to U$ is a (smooth) covering map. The multiplicity (number of sheets) of this cover is the number of connected components in each fibre $\mathbf{J}^{-1}(p)$. But since $U \subset \mathfrak{t}_0^*$ is contractible, it is simply connected. From covering space theory it follows that $\tilde{\mathbf{J}} : \tilde{U} \to U$ is a diffeomorphism. The multiplicity of the cover is therefore one, proving that the fibres of $\mathbf{J}$ are connected.

The next step is to construct a section for $\mathbf{J} : F \to U$. Equip the bundle $\mathbf{J} : F \to U$ with an Ehresmann connection (see, e.g., Kobayashi and Nomizu (1963)). This is always possible by giving F the structure of a (smooth) Riemannian manifold. That one can do so in the C^∞ category is a standard partition of unity argument. In the real-analytic category this follows from the Whitney-Morrey-Grauert Embedding Theorem (Grauert, 1952). Now U contracts to some point $p_0 \in U$. That is, we have a smooth map $(p,t) \mapsto K_t(p) : U \times [0,1] \to U$ with $K_0 = c_{p_0}$ and $K_1 = \mathrm{id}_U$, where $c_{p_0} : U \to U$ is the constant map $c_{p_0}(p) \equiv p_0$. Fix any $x_0 \in \mathbf{J}^{-1}(p_0)$ and define a smooth section $s : U \to F$ as follows: For any point $p \in U$ define a path $\gamma : [0,1] \to U$ from p_0 to p by $\gamma(t) \equiv K_t(p)$. Use the connection to lift γ to a path $\tilde{\gamma} : [0,1] \to F$ with $\tilde{\gamma}(0) = x_0$, and define $s(p) \equiv \tilde{\gamma}(1)$.

Since the fibers of $\mathbf{J}$ are precisely the T-orbits, and T acts freely, the smooth map $\phi : T \times U \to F$ defined by $\phi(q,p) \equiv q \cdot s(p)$ is a diffeomorphism.

Our final step will be to exhibit a *Lagrangian* section $\tilde{s} : U \to F$. This section will be constructed by modifying the existing section s using a 'Lie transform' argument appearing in Guillemin and Sternberg (1984, §44). This argument is in the spirit of proofs and generalizations of Darboux's theorem given by Moser (1965) and Weinstein (1971). To this end, we first show that ω is *exact*. Indeed, as $\mathbf{J} : F \to U$ is globally trivial and U is contractible, there exists a smooth deformation retract of F onto some fiber T'. The fibres are just T-orbits, which are Lagrangian since T is Abelian (apply the formula in 8.12.2). Because of the deformation retraction, the inclusion $\iota : T' \to F$ induces an isomorphism in de Rham cohomology. Since T' is Lagrangian $\iota^*\omega = 0$, implying that the cohomology class of ω vanishes, i.e., $\omega = d\Theta$ for some one-form Θ.

We are now ready to construct the Lagrangian section. Suppose α is a one-form on U. Then at each point $p \in U \subset \mathfrak{t}^*$ one may associate to α an element $\hat{\alpha}(p) \in \mathfrak{t}$ defined through

$$\langle \nu, \hat{\alpha}(p) \rangle = \langle \alpha, \nu \rangle \qquad (\nu \in \mathrm{T}_p U \cong \mathfrak{t}^*) \ .$$

One may next associate to α a vector field X^α on F, defined by

$$X^\alpha(x) \equiv (\hat{\alpha}(p))_F(x) \qquad (x \in F,\ p \equiv \mathbf{J}(x)) \ ,$$

where η_F denotes the infinitesimal generator of the T action associated with an element $\eta \in \mathfrak{t}$. In particular, by the definition of a momentum map

10.10 $$\langle X^\alpha \lrcorner\, \omega, v \rangle = \langle dJ_{\hat{\alpha}(p)}, v \rangle \qquad (v \in \mathrm{T}_x F,\ x \in F,\ p \equiv \mathbf{J}(x)) \ .$$

The vector field X^α is tangent to the T-orbits which, as shown above, are precisely the fibers of $\mathbf{J}$. These fibers are compact, so that we have a well-defined flow associated with X^α, which we denote by $(x,t) \mapsto \alpha_t(x) : F \times (-\infty, \infty) \to F$. The time-one map α_1 is a diffeomorphism of F preserving the fibers of $\mathbf{J} : F \to U$. In general α_1 is *not* symplectic. The strategy is to define $\tilde{s} \equiv \alpha_1 \circ s$ and determine how α might be chosen such that $\tilde{s}^*\omega = 0$. We have

$$\begin{aligned} \tilde{s}^*\omega &= s^*\alpha_1^*\omega \\ &= s^*\omega + s^*\int_0^1 \left(\frac{d}{dt}\alpha_t^*\omega\right) dt \\ &= s^*\omega + s^*\int_0^1 (\alpha_t^*(X^\alpha \lrcorner\, d\omega) + \alpha_t^* d(X^\alpha \lrcorner\, \omega))\, dt \\ &= s^*\omega + \int_0^1 s^*\alpha_t^* d(X^\alpha \lrcorner\, \omega)\, dt \end{aligned}$$

10.11 $$= s^*\omega + \int_0^1 d(s^*\alpha_t^*(X^\alpha \lrcorner\, \omega))\, dt \ .$$

For any $u \in \mathrm{T}_p U \cong \mathfrak{t}^*$ we have

$$
\begin{aligned}
\langle s^* \alpha_t^* (X^\alpha \lrcorner\, \omega), u \rangle &= \langle X^\alpha \lrcorner\, \omega, \mathrm{T}(\alpha_t \circ s) \cdot u \rangle \\
&= \langle dJ_{\hat{\alpha}(p)}, \mathrm{T}(\alpha_t \circ s) \cdot u \rangle \qquad \text{by 10.10} \\
&= \frac{d}{d\tau} J_{\hat{\alpha}(p)}((\alpha_t \circ s)(p + \tau u)) \Big|_{\tau=0} \\
&= \frac{d}{d\tau} \langle (\mathbf{J} \circ \alpha_t \circ s)(p + \tau u), \hat{\alpha}(p) \rangle \Big|_{\tau=0} \\
&= \langle \alpha, \frac{d}{d\tau}(p + \tau u) \Big|_{\tau=0} \rangle \\
&= \langle \alpha, u \rangle \ .
\end{aligned}
$$

Therefore $s^* \alpha_t^* (X^\alpha \lrcorner\, \omega) = \alpha$. Using this in 10.11 gives $\tilde{s}^* \omega = s^* \omega + d\alpha$. Choosing $\alpha = s^* \Theta$ gives $\tilde{s}^* \omega = 0$, so that $\tilde{s}$ is a Lagrangian section. □

The following corollary was stated without proof as Theorem 3.10 in Part 1. The only difference here is that we have not made the identification $G \times \mathfrak{t}_0^* \cong G \times \mathfrak{t}_0$ that was convenient in our applications to perturbation theory. Recall that $i : \underline{\mathfrak{t}} \xrightarrow{\sim} \mathfrak{t}^*$ is the restriction to $\underline{\mathfrak{t}} \equiv \mathrm{Ann}[\mathfrak{g}, \mathfrak{t}]$ of the natural projection $\mathfrak{g}^* \to \mathfrak{t}^*$.

10.12 Corollary *Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space on which G is acting freely, and for which $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$. Assume the space is geometrically integrable (i.e., $\dim P = \dim G + \operatorname{rank} G$), that P is connected, and that each fiber of $\mathbf{J} : P \to \mathbf{J}(P)$ is compact or has a finite number of connected components. Assume that $U \equiv i(\mathbf{J}(P) \cap \mathcal{W})$ (which is open in $\mathfrak{t}_0^*$) is smoothly contractible. Then the spaces $(P, \omega, G, \mathbf{J})$ and $(G \times U, \omega_G^*, G, \mathbf{J}^G)$ are equivalent, i.e., P admits G-compatible action-group coordinates.*

10.13 Remark Since $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$ is G-invariant, $i(\mathbf{J}(P) \cap \mathcal{W}) = \mathbf{j}(P)$, where $\mathbf{j} : P \to \mathfrak{t}^*$ is the map defined by 8.7. This fact is useful in computations.

PROOF. Let $(F, \omega_F, T, \mathbf{J}^F)$ denote the symplectic cross section. Since G acts freely on P and $(P, \omega, G, \mathbf{J})$ is geometrically integrable, T acts freely on F and $(F, \omega_F, T, \mathbf{J}^F)$ is geometrically integrable, by Lemma 10.6.1. By the definition of $\mathbf{J}^F$, we have $i(\mathbf{J}(P) \cap \mathcal{W}) = \mathbf{J}^F(F)$, so that $U = \mathbf{J}^F(F)$ is contractible by hypothesis. By Lemma 10.6.2 and the hypothesis on the fibers of $\mathbf{J}$, each fiber of $\mathbf{J}^F$ is compact or has a finite number of connected components. We have therefore established that the hypotheses of Theorem 10.9 apply to the symplectic cross section, which therefore admits T-compatible action-angle coordinates. Theorem 10.2 finishes off the proof. $\square$

CHAPTER 11

The axisymmetric Euler-Poinsot rigid body

In this chapter we apply the general results of the preceding chapter to a nontrivial example, the Euler-Poinsot rigid body.

Problem prescription

The *Euler-Poinsot rigid body* is a rigid body fixed at but free to rotate about a point O that is motionless in some inertial frame of reference. It is well-known that the dynamics of such a body is described by integral curves of an appropriate Hamiltonian vector field on $\mathrm{T}^*\,\mathrm{SO}(3)$. We refer the reader to, e.g., Marsden (1992, p. 87) for details. Note that one ordinarily identifies the phase space $\mathrm{T}^*\,\mathrm{SO}(3)$ with $\mathrm{T}\,\mathrm{SO}(3)$, using an appropriate invariant metric (see below).

Let $\boldsymbol{\lambda}, \boldsymbol{\rho} : \mathrm{SO}(3) \times \mathbb{R}^3 \xrightarrow{\sim} \mathrm{T}\,\mathrm{SO}(3)$ denote the usual left and right trivializations:

$$\boldsymbol{\lambda}(\Lambda, m) \equiv \frac{d}{dt}\Lambda e^{t\hat{m}}\big|_{t=0} \ ,$$
$$\boldsymbol{\rho}(\Lambda, n) \equiv \frac{d}{dt}e^{t\hat{n}}\Lambda\big|_{t=0} \ .$$

Here $\xi \mapsto \hat{\xi} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is the isomorphism defined by $\hat{\xi}u = \xi \times u$ $(u \in \mathbb{R}^3)$. We think of tangent vectors as equivalence classes of curves and use '$\frac{d}{dt}\Lambda_t|_{t=0}$' to denote the class represented by $t \mapsto \Lambda_t$. In what follows, the vector m will correspond to the *body* angular momentum of the body about O, while n will correspond to the *spatial* angular momentum about O. Note that

11.1 $$\boldsymbol{\lambda}(\Lambda, m) = \boldsymbol{\rho}(\Lambda, n) \Leftrightarrow n = \Lambda m \ .$$

Define a one-form Θ on $\mathrm{T}\,\mathrm{SO}(3)$ as follows. An arbitrary vector tangent to $\mathrm{T}\,\mathrm{SO}(3)$ is of the form $\frac{d}{dt}\boldsymbol{\rho}(\Lambda_t, n_t)|_{t=0}$. Let n' be the element of $\mathbb{R}^3$ uniquely determined by

$$\frac{d}{dt}\Lambda_t\Big|_{t=0} = \boldsymbol{\rho}(\Lambda_0, n') \ .$$

Then by decree,

11.2 $$\langle\Theta, \frac{d}{dt}\boldsymbol{\rho}(\Lambda_t, n_t)\big|_{t=0}\rangle \equiv n_0 \cdot n' \ ,$$

where the dot denotes the usual Euclidean dot product.

An invariant Riemannian metric on $\mathrm{SO}(3)$ is given by

$$\langle\langle \boldsymbol{\rho}(\Lambda, n_1), \boldsymbol{\rho}(\Lambda, n_2)\rangle\rangle \equiv n_1 \cdot n_2 \ .$$

The differential form Θ above is just the canonical one-form on $\mathrm{T}^*\,\mathrm{SO}(3)$ (see Chap. 9), viewed as a one-form on $\mathrm{T}\,\mathrm{SO}(3)$ using the $\langle\langle\cdot\,,\cdot\rangle\rangle$-induced identification $\mathrm{T}^*\,\mathrm{SO}(3) \cong \mathrm{T}\,\mathrm{SO}(3)$. In particular, $\omega = -d\Theta$ is a symplectic two-form on $\mathrm{T}\,\mathrm{SO}(3)$.

Define $H : \mathrm{T}\,\mathrm{SO}(3) \to \mathbb{R}$ by

$$H(\boldsymbol{\lambda}(\Lambda, m)) \equiv \frac{1}{2I_1}m_1^2 + \frac{1}{2I_2}m_2^2 + \frac{1}{2I_3}m_3^2 \ .$$

Then:

> *The dynamics of an Euler-Poinsot rigid body with moments of inertia* I_1, I_2, I_3 *about the fixed point* O *is described by the Hamiltonian system* $(\mathrm{T}\,\mathrm{SO}(3), \omega, H)$ *above.*

For a proof, see *op. cit.*

Assume that the moment of inertia ellipsoid is axisymmetric, i.e., that two of the moments of inertia are equal: $I_1 = I_2 \equiv I$, say. In that case the Hamiltonian is invariant with respect to the following action of $G \equiv \mathrm{SO}(3) \times S^1$ on $\mathrm{T}\,\mathrm{SO}(3)$: for $(A, \theta) \in \mathrm{SO}(3) \times S^1$, define

$$(A, \theta) \cdot \boldsymbol{\lambda}(\Lambda, m) \equiv \boldsymbol{\lambda}(A\Lambda R_\theta^3, R_\theta^3 m) \ ,$$

where $R^3_\theta \equiv \exp(\theta \hat{e}_3)$ is the rotation about the e_3-axis through angle θ. Alternatively, we may write this action as

$$(A, \theta) \cdot \boldsymbol{\rho}(\Lambda, n) = \boldsymbol{\rho}(A \Lambda R^3_\theta, An) \ .$$

The SO(3) part of the symmetry group corresponds to the familiar rotational 'spatial' symmetry of the system, while the S^1 action corresponds to the axisymmetry of the inertia ellipsoid (i.e., a 'body' symmetry). The action is Hamiltonian with equivariant momentum map $\mathbf{J} : \mathrm{T}\,\mathrm{SO}(3) \to \mathfrak{g}^* \cong \mathbb{R}^3 \times \mathbb{R}$ given by

$$\mathbf{J}(\boldsymbol{\lambda}(\Lambda, m)) \equiv (\Lambda m, m_3) \ ,$$

where $m_3 \equiv m \cdot e_3$. Alternatively,

$$\mathbf{J}(\boldsymbol{\rho}(\Lambda, n)) = (n, (\Lambda^{-1} n) \cdot e_3) \ .$$

These formulas are consistent with Noether's theorem (stating that $\mathbf{J}$ is constant on solution curves) and the familiar fact that in an axisymmetric rigid body the spatial angular momentum and the component of body angular momentum along the symmetry axis are conserved.

Our objective is to construct G-compatible action-group coordinates in the space $(\mathrm{T}\,\mathrm{SO}(3), \omega, G, \mathbf{J})$.

Existence

Before constructing action-group coordinates explicitly, we convince ourselves that such coordinates must exist. In the first place, we claim:

J *has* ***compact*** *fibers.*

Indeed, suppose that $y \equiv (\mu, a) \in \mathbf{J}(P) \subset \mathbb{R}^3 \times \mathbb{R}$. Then certainly

$$\mathbf{J}^{-1}(y) \subset \boldsymbol{\lambda}\left(\mathrm{SO}(3) \times \overline{B_{\|\mu\|}(0)}\right) \ ,$$

where $\overline{B_r(0)} \subset \mathbb{R}^3$ denotes the closed ball of radius r. Since $\boldsymbol{\lambda}$ is a diffeomorphism, this shows that $\mathbf{J}^{-1}(y)$ is contained in a compact set. Since $\mathbf{J}^{-1}(y)$ is closed in $\mathrm{T}\,\mathrm{SO}(3)$ (by the continuity of $\mathbf{J}$), $\mathbf{J}^{-1}(y)$ itself must be compact.

So that G acts freely and the image of the momentum map is contained in $\mathfrak{g}^*_{\text{reg}} = \mathbb{R}^3\backslash\{0\} \times \mathbb{R}$, we restrict attention to the open dense G-invariant set $P \subset \mathrm{T\,SO}(3)$ defined by

$$P \equiv \{\boldsymbol{\lambda}(\Lambda, m) \mid \Lambda \in \mathrm{SO}(3) \text{ and } m \in \mathbb{R}^3 \text{ and } (m_1 \neq 0 \text{ or } m_2 \neq 0)\} \ .$$

That is, we remove from phase space those points whose body angular momentum vector m lies on the e_3-axis.

For a maximal torus $T \subset G$ we choose

$$T \equiv \{(R^3_\phi, \theta) \in \mathrm{SO}(3) \times S^1 \mid \phi \in [0, 2n), \theta \in S^1\} \cong \mathbb{T}^2 \ .$$

Then $\underline{\mathfrak{t}} = \mathrm{span}\{e_3\} \times \mathbb{R}$. For a Weyl chamber $\mathcal{W}$ in $\mathfrak{g}^*$ choose

$$\mathcal{W} \equiv \{(te_3, \tau) \in \mathbb{R}^3 \times \mathbb{R} \mid t \in (0, \infty), \tau \in \mathbb{R}\} \ .$$

For a regular co-adjoint orbit choose $\mathcal{O} \equiv S^2 \times \{0\}$. The unique intersection point of $\mathcal{W}$ and $\mathcal{O}$ is $\mu_0 \equiv (e_3, 0)$.

The projection $\pi_{\mathcal{O}} : \mathbb{R}^3\backslash\{0\} \times \mathbb{R} \to \mathcal{O} \cong S^2$ is given by $\pi_{\mathcal{O}}(\mu, a) = \mu / \|\mu\|$. The projection $\pi_{\mathcal{W}} : \mathbb{R}^3\backslash\{0\} \times \mathbb{R} \to \mathcal{W} \cong (0, \infty) \times \mathbb{R}$ is given by $\pi_{\mathcal{W}}(\mu, a) \equiv (\|\mu\|, a)$.

Now $\dim P = 6 = 4 + 2 = \dim G + \mathrm{rank}\, G$. So $(P, \omega, G, \mathbf{J})$ is geometrically integrable. Since $\mathbf{J}$ has compact fibers, we can apply Corollary 10.12, provided we can show that $U \equiv i(\mathbf{J}(P) \cap \mathcal{W})$ is smoothly contractible. By Remark 10.13, $U = \mathbf{j}(P)$, where $\mathbf{j} \equiv i \circ \pi_{\mathcal{W}} \circ \mathbf{J}$. With our identifications, i is just the identity, so that we obtain $\mathbf{j}(\boldsymbol{\lambda}(\Lambda, m)) = (\|m\|, m_3)$. It follows that

$$U = \{(p_1, p_2) \in (0, \infty) \times \mathbb{R} \mid |p_2| < p_1\}$$

(see Fig. 1). In particular, U is indeed contractible. By Corollary 10.12, G-compatible action-group coordinates exist.

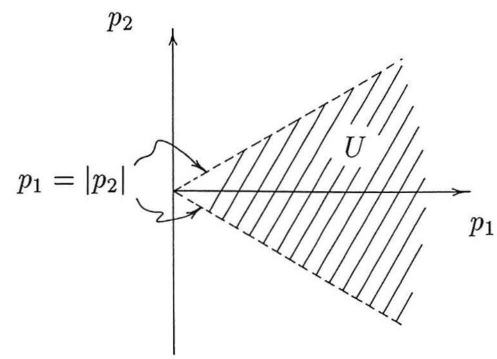

FIGURE 1. The set $U \equiv i(\mathbf{J}(P) \cap \mathcal{W}) = \mathbf{j}(P) = \mathbf{J}^F(F)$ in the axisymmetric Euler-Poinsot rigid body.

The symplectic cross section

Since P admits G-compatible action-group coordinates, the symplectic cross section F admits T-compatible action-angle coordinates (Theorem 10.2). We shall construct action-group coordinates explicitly by applying Proposition 10.7. To this end, we now construct a concrete realization of F, its symplectic structure ω_F, and the momentum map $\mathbf{J}^F : F \to U$.

The symplectic fibration $\pi : P \to \mathcal{O} \cong S^2$ (see Theorem 8.8) is given by $\pi(\boldsymbol{\rho}(\Lambda, n)) = n/\|n\|$. So the symplectic cross section is

$$\begin{aligned} F \equiv \pi^{-1}(e_3) = \{\boldsymbol{\rho}(\Lambda, n) \mid \Lambda \in \mathrm{SO}(3) \text{ and } m \in \mathbb{R}^3 \\ \text{and } (\Lambda^{-1}n \cdot e_1 \neq 0 \text{ or } \Lambda^{-1}n \cdot e_2 \neq 0) \text{ and } n/\|n\| = e_3\} \\ = \{\boldsymbol{\rho}(\Lambda, p_1 e_3) \mid \Lambda \in \mathrm{SO}(3)\backslash Z,\ p_1 > 0\}\ , \\ \text{where} \qquad Z \equiv \{\Lambda \in \mathrm{SO}(3) \mid |\Lambda e_3 \cdot e_3| = 1\}\ . \end{aligned}$$

Notice that if one constructs Euler angles on SO(3), according to the appropriate sign convention, then $Z \subset \mathrm{SO}(3)$, which is just the disjoint union of two circles, corresponds to the coordinate singularities. In particular, if we write $\mathrm{SO}(3)' \equiv \mathrm{SO}(3)\backslash Z$, then $\mathrm{SO}(3)' \cong \mathbb{T}^2 \times (0, \pi)$.

The map $\psi : \mathrm{SO}(3)' \times (0,\infty) \to F$ defined by $\psi(\Lambda, p_1) \equiv \boldsymbol{\rho}(\Lambda^{-1}, p_1 e_3)$ is a diffeomorphism onto F. Let $i_F : F \hookrightarrow P$ denote the inclusion. Then $\psi^*\omega_F = -d\Theta'$, where $\Theta' \equiv \psi^* i_F^* \Theta$. A vector tangent to $\mathrm{SO}(3)' \times (0,\infty)$ is of the form $\frac{d}{dt}(\Lambda \exp(t\hat{\xi}), p_1 + t\tau)\big|_{t=0}$. One computes using our earlier definition of Θ (equation 11.2),

$$11.3 \quad \langle \Theta', \frac{d}{dt}(\Lambda \exp(t\hat{\xi}), p_1 + t\tau)\big|_{t=0} \rangle = \langle \Theta, \frac{d}{dt}\boldsymbol{\rho}(\exp(-t\hat{\xi})\Lambda^{-1}, (p_1 + t\tau)e_3)\big|_{t=0} \rangle = -p(e_3 \cdot \xi) \ .$$

The map $\mathbf{J}^F : F \to \mathbb{R}^2$ is given by $\mathbf{J}^F(\boldsymbol{\rho}(\Lambda, n)) = (n_3, (\Lambda^{-1} n) \cdot e_3)$ for all $\boldsymbol{\rho}(\Lambda, n) \in F$. So $(\mathbf{J}^F \circ \psi)(\Lambda, p_1) = (p_1, (\Lambda e_3 \cdot e_3) p_1)$. As in the proof of 10.12, $\mathbf{J}^F(F) = U$ (computed above).

Let $T \cong \mathbb{T}^2$ act on $\mathrm{SO}(3) \times (0,\infty)$ in the way that makes ψ a $\mathbb{T}^2$-equivariant map. (We will not need to compute this action explicitly.) Then ψ establishes an equivalence between the space

$$(\mathrm{SO}(3)' \times (0,\infty), -d\Theta', \mathbb{T}^2, \mathbf{J}') \ ,$$

and the symplectic cross section $(F, \omega_F, \mathbb{T}^2, \mathbf{J}^F)$, where Θ' is the one-form computed in 11.3, and $\mathbf{J}' \equiv \mathbf{J}^F \circ \psi : \mathrm{SO}(3)' \times (0,\infty) \to U$ is given by $\mathbf{J}'(\Lambda, p_1) = (p_1, (\Lambda e_3 \cdot e_3) p_1)$.

Explicit action-group coordinates

To construct action-group coordinates by applying 10.7, it remains to exhibit a Lagrangian section for $\mathbf{J}^F$.

Define $s : U \to SO(3)' \times (0,\infty)$ by

$$s(p_1, p_2) \equiv (R^1_{\arccos(p_2/p_1)}, p_1) \ ,$$

where $R^1_\alpha \equiv \exp(\alpha \hat{e}_1)$ is the rotation about the e_1-axis through angle α. By convention, arccos is to take values in $(0, \pi)$.

One verifies immediately that $\mathbf{J}' \circ s = \mathrm{id}_U$, i.e., s is a section of $\mathbf{J}' : \mathrm{SO}(3)' \times (0, \infty) \to U$. We claim that s is in fact Lagrangian. Indeed, for any $\tau_1, \tau_2 \in \mathbb{R}$ we have

$$\begin{aligned}\langle s^*\Theta', \frac{d}{dt}(p_1 + t\tau_1, p_2 + t\tau_2)\Big|_{t=0}\rangle &= \langle \Theta', \frac{d}{dt}(R^1_{\arccos(\frac{p_2+t\tau_2}{p_1+t\tau_1})}, p_1 + t\tau_1)\Big|_{t=0}\rangle \\ &= \langle \Theta', \frac{d}{dt}(R^1_{\arccos(p_2/p_1)} \exp(tf(p_1, p_2, \tau_1, \tau_2)\hat{e_1}), p_1 + t\tau_1)\Big|_{t=0}\rangle \ ,\end{aligned}$$

for some real-valued function f. It is irrelevant what f actually is since, applying the formula 11.3 for Θ', we obtain

$$\langle \Theta', \frac{d}{dt}(p_1 + t\tau_1, p_2 + t\tau_2)\Big|_{t=0}\rangle = p_1[e_3 \cdot f(p_1, p_2, \tau_1, \tau_2)e_1] = 0 \ .$$

Since τ_1 and τ_2 were arbitrary, $s^*\Theta' = 0$, implying $s^*(-d\Theta') = 0$. So s is indeed Lagrangian. Since $\psi : \mathrm{SO}(3)' \times (0, \infty) \to F$ is an equivalence, $\psi \circ s : U \to F$ is a Lagrangian section for $\mathbf{J}^F : F \to U$. One computes

$$(\psi \circ s)(p_1, p_2) \equiv \boldsymbol{\rho}\left(R^1_{-\arccos(p_2/p_1)}, p_1 e_3 \right) \ .$$

We may now apply Proposition 10.7 as follows: Define $\phi : G \times U = SO(3) \times S^1 \times U \to P \subset \mathrm{T}SO(3)$ by

$$\begin{aligned}\phi((A, \theta), (p_1, p_2)) &\equiv (A, \theta) \cdot ((\psi \circ s)(p_1, p_2)) \\ &= \boldsymbol{\rho}(AR^1_{-\arccos(p_2/p_1)} R^3_\theta, \ p_1 A e_3) \\ &= \boldsymbol{\lambda}(AR^1_{-\arccos(p_2/p_1)} R^3_\theta, \ p_1 R^3_{-\theta} R^1_{\arccos(p_2/p_1)} e_3) \\ &= \boldsymbol{\lambda}(AR^1_{-\arccos(p_2/p_1)} R^3_\theta, \ m_{p_1,p_2,\theta}) \ ,\end{aligned}$$

$$\text{where} \quad m_{p_1,p_2,\theta} \equiv (-\sqrt{p_1^2 - p_2^2} \sin\theta, -\sqrt{p_1^2 - p_2^2} \cos\theta, p_1) \ .$$

Then ϕ is an equivalence realizing G-compatible action-group coordinates in the space P. The Hamiltonian ϕ^*H pulled back to $G \times U$ is given by $\phi^*H(g, p) = h(p)$, where

$$h(p_1, p_2) \equiv \frac{1}{2I} p_1^2 + \frac{1}{2}(\frac{1}{I_3} - \frac{1}{I}) p_2^2 \ .$$

Notice that h is convex (in the sense of Definition 5.3, Part 1) in the *prolate* ($I_3 < I$) case.

CHAPTER 12

Converting dynamic integrability to geometric integrability

Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space with $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$, and $H : P \to \mathbb{R}$ a fixed G-invariant Hamiltonian. Assume that G acts freely. Then all the Marsden-Weinstein reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ ($\mu \in \mathbf{J}(P)$) have the same dimension, viz. $d \equiv \dim P - \operatorname{rank} G - \dim G$. Recall that we call the space *geometrically integrable* (or *G-integrable*) if $d = 0$. We call the space *dynamically integrable* (or *(G, H)-integrable*) if $d = 2$. These definitions are motivated in, e.g., Chap. 3 of Part 1.

We have already shown (Lemma 10.6) that P is geometrically integrable if and only if its symplectic cross section F is geometrically integrable. An analogous statement holds for dynamic integrability, which one proves in the same way.

To construct action-group coordinates (see, e.g., Corollary 10.12) one requires geometric integrability. In examples, however, it can happen that the symmetry group that presents itself is only large enough to enforce dynamic integrability. The objective in this chapter is to show how to extend the action of G in the dynamically integrable case to a geometrically integrable Hamiltonian action of $G \times S^1$, with respect to which H is $G \times S^1$-invariant. (Of course we can always extend the symmetry group by a factor $\mathbb{R}$ by considering the flow of X_H, assuming that X_H is complete. The point is that we seek an extension that is *compact.*) The extra S^1 action is to be built in some way from the reduced dynamics of H, and consequently depends on the particular H of interest (a *different* G-invariant Hamiltonian will not necessarily be $G \times S^1$-invariant). We shall not attempt great generality but focus on the case where the reduced dynamics of H in each (two-dimensional) reduced space is composed of *an equilibrium point surrounded by periodic trajectories.* This is true for the asymmetric Euler-Poinsot rigid body, if we restrict attention to an appropriate open subset of the phase space. For simplicity, we illustrate our algorithm in the axisymmetric rigid body, viewed as a $(\mathrm{SO}(3), H)$-integrable space (see Appendix D). It turns out that the extra S^1 action (which our algorithm constructs *ab initio*) corresponds to the familiar S^1 symmetry of the body, as exploited, for example, in the previous chapter. We shall make assumptions in the general theory that lead to the fewest technicalities, even if these are inconvenient to check in applications.

Setup

With Lemma 8.17 in hand one can easily reduce the above S^1 extension problem to the Abelian case by passing to the symplectic cross section. We therefore begin by restricting attention to the case of a toral symmetry.

> *In this chapter T denotes a torus, $(F, \omega, T, \mathbf{J})$ a Hamiltonian T-space on which T is acting freely, and $H : F \to \mathbb{R}$ a fixed T-invariant Hamiltonian. We assume that $(F, \omega, T, \mathbf{J})$ is* **dynamically** *integrable (i.e., $\dim F = 2\dim T + 2$). For simplicity, we assume that $\mathbf{J} : F \to \mathbf{J}(F)$ has the property that preimages of connected sets are connected. This is true, for example, if $\mathbf{J} : F \to \mathbf{J}(F)$ is a locally trivial fiber bundle with connected fibers. (Note that $\mathbf{J}$ is already a surjective submersion, on account of 8.3).*

Denote by $\rho : F \to F/T$ the canonical projection. Since we assume that T acts freely and properly, F/T admits a differential structure with respect to which ρ is a surjective submersion. Since T is compact, $\rho : F \to F/T$ is a principal T-bundle (by a corollary of the Reeb stability theorem; see, e.g., Camacho and Neto (1985)). We equip F/T with the unique Poisson structure with respect to which ρ is a Poisson map. Since $\mathbf{J}$ is T-equivariant (by the definition of a Hamiltonian T-space) and T is Abelian, it follows that $\mathbf{J}$ is *T-invariant* and therefore factors through ρ. That is, there exists a (unique) map $\mathbf{j} : F/T \to \mathfrak{t}^*$ with the property that the following diagram commutes:

$$
\begin{tikzcd}
 & F \arrow[dl, "\mathbf{J}"'] \arrow[dr, "\rho"] & \\
\mathfrak{t}^* & & F/T \arrow[ll, "\exists \mathbf{j}"]
\end{tikzcd}
$$

Since the Hamiltonian H is T-invariant, it drops to a function on F/T that we shall denote by h.

To formulate and prove our claims, it will be convenient to realize all the Marsden-Weinstein reduced spaces as the symplectic leaves of F/T. This is possible (since T is Abelian) with the help of the following fact.

12.1 Lemma *Let $\mathfrak{t}^*_{\mathbf{J}}$ denote the image of $\mathbf{J}$. Then under the assumptions above, $\mathbf{j} : F/T \to \mathfrak{t}^*_{\mathbf{J}}$ is a surjective submersion whose fibers are the symplectic leaves in F/T.*

That $\mathbf{j} : F/T \to \mathfrak{t}^*_{\mathbf{J}}$ is a surjective submersion follows from the fact that $\mathbf{J} : F \to \mathfrak{t}^*_{\mathbf{J}}$ is a surjective submersion. The other conclusion of the lemma follows from the Symplectic Leaf Correspondence Theorem for dual pairs. See Appendix E.

Let $\Sigma_\mu \equiv \mathbf{j}^{-1}(\mu)$ be a symplectic leaf in F/T ($\mu \in \mathfrak{t}^*_{\mathbf{J}}$). Then as $\mathbf{j}^{-1}(\mu) = \rho(\mathbf{J}^{-1}(\mu)) = \mathbf{J}^{-1}(\mu)/T = \mathbf{J}^{-1}(\mu)/T_\mu$, *the leaf Σ_μ is literally a Marsden-Weinstein reduced space.* Of course each leaf $\Sigma_\mu \subset F/T$ is X_h-invariant.

The S^1 extension

Let $\mathfrak{t}^{\mathbb{Z}} \subset \mathfrak{t}$ denote the integral lattice of T in $\mathfrak{t}$ (i.e., the kernel of $\exp : \mathfrak{t} \to T$) and write $\frac{1}{2\pi}\mathfrak{t}^{\mathbb{Z}} \equiv \{\frac{1}{2\pi}\nu \mid \nu \in \mathfrak{t}^{\mathbb{Z}}\}$. Here is the central result of the chapter:

12.2 Theorem (S^1 Extension Theorem) *Assume in addition to the hypotheses outlined above the following (replacing F with an appropriate T-invariant subset if necessary):*

1. *Each symplectic leaf Σ_μ ($\mu \in \mathfrak{t}^*_{\mathbf{J}}$) is diffeomorphic to an open disk in $\mathbb{R}^2$.*

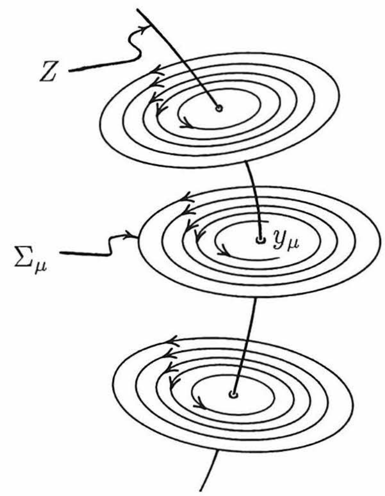

FIGURE 1. The reduced dynamics in F/T, as described in the hypotheses of Theorem 12.2.

2. *There exists a submanifold $Z \subset F/T$ of codimension two that is transversal to each leaf, and having the following property (see Fig. 1): Z intersects Σ_μ at a single point y_μ that is an equilibrium point of $X_h|\Sigma_\mu$ of elliptic type.*
3. *Each 'punctured disk' $\Sigma'_\mu \equiv \Sigma_\mu \backslash \{y_\mu\}$ is foliated by nontrivial periodic orbits of X_h.*

Next, let $\xi_0 \in \frac{1}{2\pi}\mathfrak{t}^{\mathbb{Z}}$ and $c_0 \in \mathbb{R}$ be arbitrary, and let $I : F/T \to \mathbb{R}$ be the unique function satisfying:

4. $I(y_\mu) = c_0 + \langle \mu, \xi_0 \rangle$ $(\mu \in \mathfrak{t}^*_{\mathbf{J}})$.
5. *Each punctured disk Σ'_μ is foliated by (minimally) 2π-periodic orbits of X_I, which, up to orientation preserving reparametrizations of time, coincide with the orbits of X_h. In other words, $I|\Sigma'_\mu$ is a classical 'action' for the Hamiltonian $h|\Sigma'_\mu$ (see Remark 12.3 below).*

Then:

6. *$X_{I\circ\rho}$ has a well-defined 2π-periodic flow $(t, x) \mapsto \Phi_t(x) : \mathbb{R} \times F \to F$.*

7. *The action of S^1 defined by*

$$(\theta \bmod 2\pi) \cdot_{S^1} x \equiv \Phi_\theta(x) \qquad (x \in F)$$

commutes with that of T.

8. *The Hamiltonian H is S^1-invariant.*

9. *The S^1 action is Hamiltonian, with momentum map $I \circ \rho : F \to \mathrm{Lie}(S^1)^* \cong \mathbb{R}$.*

10. *The momentum map $I \circ \rho$ is both S^1- and T-invariant.*

11. *The momentum map $\mathbf{J} : F \to \mathfrak{t}^*$ is S^1-invariant.*

12. *The S^1 action is* **free** *on $F' \equiv F \backslash \rho^{-1}(Z)$.*

In particular, one can extend the action of T to an action of $T' \equiv T \times S^1$ by defining

$$(q, s) \cdot x \equiv s \cdot_{S^1} (q \cdot x) = q \cdot (s \cdot_{S^1} x) \qquad \big((q, s) \in T',\, x \in F \big) \ .$$

This action has T'-equivariant momentum map $\mathbf{J}' : F \to \mathfrak{t}'^* \cong \mathfrak{t}^* \times \mathbb{R}$ given by $\mathbf{J}'(x) \equiv (\mathbf{J}(x), (I \circ \rho)(x))$. Furthermore, H is T'-invariant, T' acts freely on F', and the space

$$(F', \omega, T', \mathbf{J}')$$

is ***geometrically*** *integrable.*

12.3 Remark Once the value of I on Z is fixed à la condition 4, its (unique) extension to a function satisfying condition 5 can be described explicitly: One takes

$$I(y) \equiv I(y_\mu) + \frac{1}{2\pi} \int_{\Pi_y} \omega_\mu \qquad \big(y \in (F/T) \backslash Z,\, \mu \equiv \mathbf{j}(y) \big) \ ,$$

where ω_μ denotes the symplectic structure on the leaf Σ_μ, and $\Pi_y \subset \Sigma_\mu$ denotes the oriented two-manifold whose boundary is the X_h-orbit through y.

The nontrivial part of the theorem is conclusion 6. Specifically, we show that the $X_{I \circ \rho}$-invariant submanifold $\rho^{-1}(Z) \subset P$ is foliated by (possibly trivial) 2π-periodic orbits of $X_{I \circ \rho}$, and that the other integrable curves of $X_{I \circ \rho}$ are *minimally* 2π-periodic. The latter fact is *not* obvious a priori. Indeed if we relax condition 4 and let $t \mapsto \gamma(t)$ be an integral curve of $X_{I \circ \rho}$, with $\gamma(0) \in F'$, then although $\rho(\gamma(0)) = \rho(\gamma(2\pi))$ (γ is mapped to an integral curve of X_I since ρ is Poisson), it can happen that $\gamma(2\pi) = q \cdot \gamma(0)$ for some nontrivial $q \in T$. This

q, which depends only on the curve $\gamma' \equiv \rho \circ \gamma$, is known as the *phase change* (or *holonomy*) along the curve γ'. It can be computed using the technology of Ehresmann connections on principal bundles (see below).

Our task splits into three steps. In Step 1 below, we show by appealing to condition 4 that $\rho^{-1}(Z)$ is foliated by 2π-periodic orbits of $X_{I \circ \rho}$. In Step 2, we show that all integral curves of X_I lying in the *same* leaf Σ_μ have identical associated phase changes. In Step 3, we use the results of Step 1, and a certain continuity argument, to show that these phase changes are in fact trivial. This will prove 6, and the remaining conclusions of the theorem will follow easily. Before beginning the proof, we need to make two preliminary observations. The first concerns phases, and the second relative equilibria (i.e., equilibria in F/T).

Aside: On computing phases

We assume the reader is familiar with the concept of an *Ehresmann connection* on a principal T-bundle. The reader in unfamiliar territory is directed to Kobayashi and Nomizu (1963).

Let F and B be arbitrary smooth manifolds, and suppose that $\rho : F \to B$ is a principal T-bundle, with T acting on F from the left. If X is a T-invariant vector field on F, then there exists a vector field X_B on B such that X and X_B are ρ-related. If $t \mapsto \gamma_B(t)$ is an integral curve of X_B, then every integral curve $t \mapsto \gamma(t)$ of X with initial condition $\gamma(0) \in \rho^{-1}(\gamma_B(0))$ covers γ_B in the sense that $\rho(\gamma(t)) = \gamma_B(t)$. If γ_B is *periodic* with (minimal) period $t_0 > 0$, then $\gamma(t_0) = q \cdot \gamma(0)$ for some uniquely defined $q \in T$. The element q is independent of the initial condition $\gamma(0) \in \rho^{-1}(\gamma_B(0))$ and will be called the *phase change of X along γ_B*. For a discussion of phases in Hamiltonian mechanics, and their origins, see Marsden et al. (1990).

Let α be a $\mathfrak{t}$-valued connection one-form on the bundle $\rho : F \to B$. A vector field X on F is *horizontal* if $X \lrcorner \alpha = 0$. If Y is a vector field on B, then we denote by Y^h the *horizontal lift* of Y, with respect to α. This is the unique horizontal vector field Y^h ρ-related to Y. This vector field is T-invariant.

The following fact is well-known. For a proof see, e.g. Marsden et al. (1990, pp. 39–49).

12.4 Theorem

1. *There exists a* $\mathfrak{t}$*-valued two-form* σ *on* B*, called the* **curvature**[1] *of* α*, well-defined by*

$$\sigma(v,w) \equiv -\alpha([V^h, W^h](x)) \qquad (v, w \in \mathrm{T}_y B,\ y \in B,\ x \in \rho^{-1}(y)) \ ,$$

where V *and* W *are any locally defined vector fields on* B *such that* $V(y) = v$ *and* $W(y) = w$.

2. *Let* X *be a horizontal* T*-invariant vector field on* B*, and* X_B *the corresponding vector field on the base* B *(i.e., such that* $X_B^h = X$*). Let* $t \mapsto \gamma_B(t) \in B$ *be a periodic integral curve of* X_B*, whose image is the oriented boundary of some compact two-dimensional submanifold* $\Sigma \subset B$*. Then the phase change* q *of* X *along* γ_B *is given by*

$$q = \exp \int_\Sigma \sigma \ .$$

Aside: Relative equilibria and 'limiting phases'

If $s : M \to N$ is a submersion, we write $s_* : \operatorname{Ann}\ker \mathrm{T}s \to \mathrm{T}^*N$ for the map that is $\mathbb{R}$-linear on each space $\operatorname{Ann}\ker \mathrm{T}_x s$ ($x \in M$), and that sends $d_x(f \circ s)$ to $d_{s(x)} f$, for any locally defined function f on N. Here Ann denotes annihilator. Let $(F, \omega, T, \mathbf{J})$ be a Hamiltonian T-space as before, and let $\xi : \operatorname{Ann}\ker \mathrm{T}\mathbf{j} \to \mathfrak{t}$ be the composition of $\mathbf{j}_* : \operatorname{Ann}\ker \mathrm{T}\mathbf{j} \to \mathrm{T}^*\mathfrak{t}^*_{\mathbf{J}}$ with the map $\mathrm{T}^*\mathfrak{t}^*_{\mathbf{J}} \to \mathfrak{t}^{**} \cong \mathfrak{t}$ that 'forgets the base point'.

12.5 Proposition *If* $y \in F/T$ *is an equilibrium point of* X_h *(so that* $d_y h \in \operatorname{Ann}\ker \mathrm{T}\mathbf{j}$*), then*

$$X_H(x) = \eta_F(x) \qquad (x \in \rho^{-1}(y))$$

$$\text{where} \qquad \eta \equiv \xi(d_y h) \ .$$

[1] Conventionally *curvature* refers to the two-form $d\alpha - \alpha \wedge \alpha$ on F. In the Abelian case considered here this form is the pull-back by ρ of σ.

Recall that η_F denotes the infinitesimal generator on F corresponding to $\eta \in \mathfrak{t}$.

12.6 Remark If F is a locally defined function on $\mathfrak{t}^*_{\mathbf{J}}$ such that $d_y h = d_y(F \circ \mathbf{j})$ (such a function will always exist since $d_y h \in \operatorname{Ann}\ker \mathrm{T}\mathbf{j}$), then $\xi(d_y h) = \frac{\delta F}{\delta \mu}$, where $\mu \equiv \mathbf{j}(y)$.

Proof of 12.5. Let F and μ be as in the remark. Then, recalling that $\mathbf{J} = \mathbf{j} \circ \rho$, we have $d_x H = d_x(F \circ \mathbf{J})$ for all $x \in \rho^{-1}(y)$. In that case,

$$X_H(x) = X_{F \circ \mathbf{J}}(x) = \left(\frac{\delta F}{\delta \mu}\right)_F (x) \ ,$$

by the collective Hamiltonian theorem (see, e.g., Marsden and Ratiu (1994, Theorem 12.5.2)). □

12.7 Corollary (On limiting phases) *Let $\epsilon \mapsto \gamma_\epsilon$ be a smooth[2] family of periodic orbits of X_h, limiting on some equilibrium point $y \in F/T$, in the sense that*

$$\lim_{\epsilon \to 0} \gamma_\epsilon(t) = y \ ,$$

for all t. Let t_ϵ denote the (minimal) period of γ_ϵ and assume that the limit

$$t_0 \equiv \lim_{\epsilon \to 0} t_\epsilon$$

exists. Then

$$\lim_{\epsilon \to 0} q_\epsilon = \exp(t_0 \xi(d_y h)) \ ,$$

where $q_\epsilon \in T$ denotes the phase change of X_H along γ_ϵ.

Proof. Apply the preceding proposition and the continuity of integral curves with respect to initial data. □

[2] $\gamma_\epsilon(t)$ should be smooth with respect to ϵ (with t fixed) and t (with ϵ fixed).

Proof of the S^1 Extension Theorem

We are now equipped to prove conclusion 6 of Theorem 12.2 in the three steps outlined above.

Step 1. Each $y_\mu \in F/T$ ($\mu \in \mathfrak{t}^*_{\mathbf{J}}$) is an elliptic equilibrium point of X_I, so that $d_{y_\mu} I \in \operatorname{Ann}\ker T\mathbf{j}$. It follows from hypothesis 4 of the theorem that $d_{y_\mu} I = d_{y_\mu}(F \circ \rho)$ where $F(\mu) \equiv c_0 + \langle \mu, \xi_0 \rangle$. By Remark 12.6, we have $\xi(d_{y_\mu} I) = \xi_0$, so that $X_{I\circ\rho}(x) = (\xi_0)_F(x)$ for all $x \in \rho^{-1}(y_\mu)$, by Proposition 12.5 (with H and h in 12.5 replaced by $I \circ \rho$ and I). Since $\mu \in \mathfrak{t}^*_{\mathbf{J}}$ was arbitrary, and $\xi_0 \in \frac{1}{2\pi}\mathfrak{t}^{\mathbb{Z}}$, we conclude that the integral curves of $X_{I\circ\rho}$ lying in the $X_{I\circ\rho}$-invariant submanifold $\rho^{-1}(Z) \subset P$ are 2π-periodic (although these orbits may have a smaller — even trivial — period).

Step 2. Fix some $\mu \in \mathfrak{t}^*_{\mathbf{J}}$. By Noether's theorem, $X_{I\circ\rho}$ is tangent to $\mathbf{J}^{-1}(\mu) = \rho^{-1}(\Sigma_\mu)$. Write $\mathbf{J}^{-1}(\mu)' \equiv \rho^{-1}(\Sigma'_\mu)$ and $(F/T)' \equiv (F/T)\backslash Z$. Since $\rho : F \to F/T$ is a principal T-bundle, and $\Sigma_\mu \subset F/T$ is a (regular) submanifold, the restriction

$$\rho_\mu \equiv \rho|\mathbf{J}^{-1}(\mu)' : \mathbf{J}^{-1}(\mu)' \to \Sigma'_\mu$$

is also a principal T-bundle (remember $\mathbf{J}^{-1}(\mu)$ is T-invariant since T is Abelian). We shall compute phase changes of $X_{I\circ\rho}$ along curves lying in Σ'_μ by constructing a connection on $\rho_\mu : \mathbf{J}^{-1}(\mu)' \to \Sigma'_\mu$. See Fig. 3.

Observe that action-angle coordinates (in the purely classical sense) can be constructed in the two-dimensional system $(\Sigma'_\mu, \omega_\mu, h|\Sigma'_\mu)$. Here ω_μ denotes the symplectic structure on the symplectic leaf Σ_μ, restricted to Σ'_μ. By hypothesis 5 of the theorem, we may take the 'action' to be $I_\mu \equiv I|\Sigma'_\mu : \Sigma'_\mu \to \mathbb{R}$.

A conjugate 'angle' $\theta_\mu : \Sigma'_\mu \to S^1$ can be obtained as the restriction to Σ'_μ of an S^1-valued map θ, defined on some open neighborhood $\mathbf{j}^{-1}(U) \cap (F/T)'$ of Σ'_μ as follows ($U \subset \mathfrak{t}^*_{\mathbf{J}}$ denoting some open neighborhood of μ). There exists (for sufficiently small U) a codimension one submanifold $S \subset \mathbf{j}^{-1}(U) \cap (F/T)'$ that is transversal to each orbit of X_I in the (X_I-invariant) set $\mathbf{j}^{-1}(U) \cap (F/T)'$, and intersecting each such orbit at precisely one point (see Fig. 2). The flow of X_I is 2π-periodic, inducing an action of S^1 on F/T. We

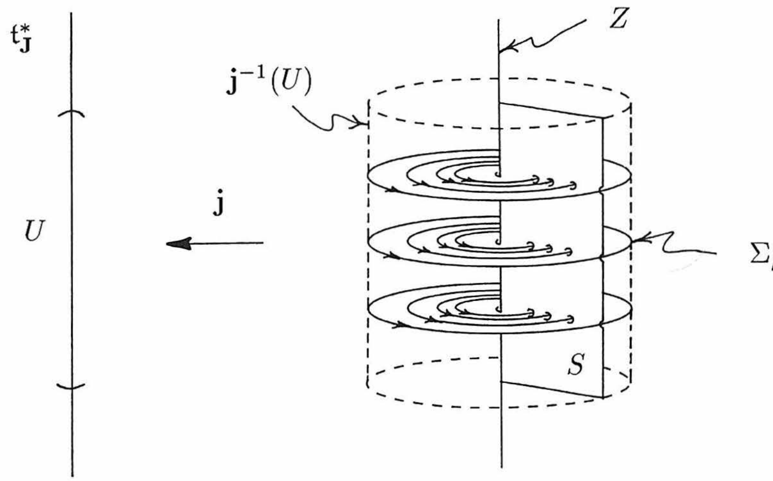

FIGURE 2. Constructing $\theta : \mathbf{j}^{-1}(U) \cap (F/T)' \to S^1$.

define $\theta : \mathbf{j}^{-1}(U) \cap (F/T)' \to S^1$ by declaring $\theta(y)$ to be the uniquely defined element of S^1 such that $y = \theta(y) \cdot_{S^1} s$, for some (uniquely determined) $s \in S$.

For the sake of a lucid notation, we shall commit the usual sin by pretending that θ is an $\mathbb{R}$-valued function. We leave it to the reader to verify at each step that this sin can be forgiven. Note that by construction $I_{\mu'}$ and $\theta_{\mu'} \equiv \theta | \Sigma'_{\mu'}$ constitute action-angle coordinates in every leaf ($\mu' \in U$). Therefore, they are conjugate in the sense that

12.8 $$\omega_{\mu'} = d\theta_{\mu'} \wedge dI_{\mu'} \qquad (\mu' \in U) \ .$$

The preimage under ρ of the open set $\mathbf{j}^{-1}(U) \cap (F/T)'$, on which θ is defined, is the open set $\mathbf{J}^{-1}(U) \cap F'$. This latter set is an open neighborhood of $\mathbf{J}^{-1}(\mu)'$ in F'. A T-invariant Ehresmann connection on the bundle $\rho_\mu : \mathbf{J}^{-1}(\mu)' \to \Sigma'_\mu$ is obtained by declaring the horizontal space Hor_x at $x \in \mathbf{J}^{-1}(\mu)'$ to be the space

$$\mathrm{Hor}_x \equiv \mathrm{span}\{X_{\theta \circ \rho}(x), X_{I \circ \rho}(x)\}$$

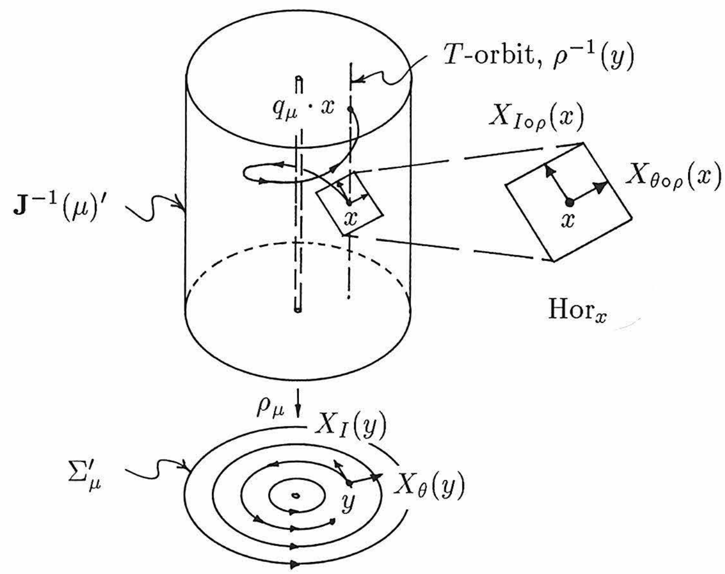

FIGURE 3. Computing phases using the connection on $\rho_\mu : \mathbf{J}^{-1}(\mu) \to \Sigma'_\mu$.

(see Fig. 3). This choice is valid since: (i) $X_{\theta\circ\rho}$ and $X_{I\circ\rho}$ are T-invariant, (ii) $X_{\theta\circ\rho}(x)$ and $X_{I\circ\rho}(x)$ are tangent to $\mathbf{J}^{-1}(\mu)'$ (by Noether), and (iii) The image of Hor_x under $\mathrm{T}\rho$ is

$$\mathrm{span}\{X_\theta(\rho(x)), X_I(\rho(x))\} = \mathrm{T}_{\rho(x)}\Sigma'_\mu \ .$$

We let α_μ denote the connection one-form on $\rho_\mu : \mathbf{J}^{-1}(\mu)' \to \Sigma'_\mu$ that is compatible with the above choice of horizontal spaces. Note that we cannot extend α_μ to a connection one-form on $\mathbf{J}^{-1}(\mu) \to \Sigma_\mu$. This is because the vector field $X_\theta|\Sigma'_\mu$ cannot be extended to a vector field on Σ_μ without a discontinuity at y_μ. (This is why we need the 'limiting phase' argument to be expounded in Step 3 below.)

By construction, the vector field $X_{I\circ\rho}|\mathbf{J}^{-1}(\mu)'$ is horizontal with respect to α_μ. We may therefore employ Theorem 12.4 to compute its phase changes. A simple observation is

$$[X^h_{\theta\circ\rho}, X^h_{I\circ\rho}](x) = [X_{\theta\circ\rho}, X_{I\circ\rho}](x) = [X_\theta, X_I](\rho(x)) = X_{\{\theta,I\}}(\rho(x))$$

$$(x \in \mathbf{J}^{-1}(U) \cap F') \ .$$

Since θ and I are conjugate (in the sense of 12.8), $\{\theta, I\} = 0$ on $\mathbf{j}^{-1}(U) \cap (F/T)'$. Therefore, by 12.4.1, the curvature of α_μ is zero. *It follows from 12.4.2 that the phase change of $X_{I \circ \rho}$ has the same value along every orbit of X_I lying in the puncture disk Σ'_μ.* We denote this constant phase by $q_\mu \in T$ (as in Fig. 3).

Step 3. Keep $\mu \in \mathfrak{t}^*_{\mathbf{J}}$ fixed as in Step 2. The nontrivial periodic orbits of X_I lying in Σ'_μ can be parameterized by their energy relative to the energy at y_μ: Let γ_ϵ denote the orbit with $\gamma_\epsilon(0) \in S$ whose image is the $(I(y_\mu) + \epsilon)$-level set of I_μ. Then the family $\epsilon \mapsto \gamma_\epsilon$ limits on the equilibrium point y_μ as in the hypotheses of Corollary 12.7. The orbit γ_ϵ has period $t_\epsilon = 2\pi$, for all ϵ. Since $\xi(d_{y_\mu} I) = \xi_0$ (see Step 1) and $\xi_0 \in \frac{1}{2\pi}\mathfrak{t}^{\mathbb{Z}}$, we conclude from Corollary 12.7, and the results of Step 2, that $q_\mu = \mathrm{id}_T$. This completes the proof of conclusion 6 of Theorem 12.2.

If X is any T-invariant vector field on F, and $t \mapsto \Phi_t$ the corresponding flow map, then $\Phi_t(q \cdot x) = q \cdot \Phi_t(x)$ for all $q \in T$, and for all $t \in \mathbb{R}$ for which $\Phi_t(x)$ is defined ($x \in F$). Since $X_{I \circ \rho}$ is T-invariant, this proves 7.

Since $X_{I \circ \rho}$ and X_I are ρ-related, it follows that ρ is S^1-equivariant, where S^1 acts on F as in 7, and on F/T as described in Step 2. Therefore

$$H(s \cdot_{S^1} x) = h \circ \rho(s \cdot_{S^1} x) = h(s \cdot_{S^1} \rho(x)) = h(\rho(x)) = H(x) \ ,$$

where we have used the fact that h is constant on the orbits of X_I, which follows from condition 5. This proves conclusion 8.

Conclusion 9 is obvious. That $I \circ \rho$ is T-invariant is trivial. That it is S^1-invariant follows from the same argument used to prove the S^1-invariance of $H = h \circ \rho$ above. So conclusion 10 holds true. Conclusion 11 follows from the S^1-equivariance of ρ, the X_I-invariance of the leaves Σ_μ ($\mu \in \mathfrak{t}^*_{\mathbf{J}}$), and the formula $\mathbf{J} = \mathbf{j} \circ \rho$. Condition 12 follows from the S^1-equivariance of ρ.

The only nontrivial fact left to check is the T'-equivariance of $\mathbf{J}'$. Since T' is Abelian, this amounts to checking that $\mathbf{J}'$ is *T'-invariant.* This follows from the fact that both $\mathbf{J}$ and $I \circ \rho$ are both T- and S^1-invariant.

This completes the proof of Theorem 12.2.

The non-Abelian case

12.9 Corollary *Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space and $H : P \to \mathbb{R}$ a G-invariant Hamiltonian. Assume the symplectic cross section $(F, \omega_F, T, \mathbf{J}^F)$ satisfies the hypotheses of Theorem 12.2, with the role of H in 12.2 being played by $H|F$. (In particular, assume that G acts freely and that P is dynamically integrable). Let F' and $I \circ \rho$ be as in the theorem and define $\mathbf{K} \equiv I \circ \rho$ and $P' \equiv G(F') \subset P$. Then there is a Hamiltonian action of S^1 on P, with respect to which H is invariant, and which has as an S^1-equivariant momentum map $\mathbf{K} : P \to \mathbb{R}$ the well-defined extension of $\mathbf{K} : F \to \mathbb{R}$ given by*

$$\mathbf{K}(g \cdot y) \equiv \mathbf{K}(y) \qquad (g \in G,\, y \in F)\ .$$

Moreover, this action of S^1 commutes with that of G (so that $G \times S^1$ acts on P), is free on P', and P' is $G \times S^1$-integrable (i.e., geometrically integrable with respect to the extended symmetry group $G \times S^1$).

Proof. Combine Lemma 8.17 (taking $H \equiv S^1$) with Theorem 12.2. □

CHAPTER 13

Concluding remarks

Generalizations

We conjecture that the estimates on the evolution of $\mathbf{J}(x_t)$ described in the Introduction (and in detail in Chap. 7) hold under weaker hypotheses than the existence of action-group coordinates requires. Of course one still needs some kind of integrability assumption, such as 'reduced spaces are zero dimensional,' and one may need to assume uniform orbit and co-adjoint orbit types. Also, a convexity (or quasi-convexity) assumption will be necessary. To generalize our results to other orbit and co-adjoint orbit types will require more general Hamiltonian G-space normal forms. Moreover, one will need very concrete realizations of these spaces, as well as a good handle on the symplectic structure, equations of motion, etc. One will also need explicit (local) complexifications of the space and its associated structures.

The classification of Hamiltonian G-spaces satisfying various conditions is still the subject of considerable research. See Guillemin et al. (1996) for a recent bibliography. While more general Hamiltonian G-space normal forms are known, realizations as concrete as action-group coordinates do not seem to have been worked out. In the best scenario, these realizations will be of the form considered in our abstract formulation of Nekhoroshev's theorem (Chap. 5), so that deriving Nekhoroshev estimates will again reduce to verifying our Assumptions A–C. This hope is not entirely fanciful, as a couple of examples will demonstrate.

Recall that to apply the abstract Nekhoroshev-Lochak theorem one needs 'coordinates' of the form $G \times U$, for some open subset U of a vector space $\mathfrak{t}$ and some compact real-analytic manifold G. In particular, G does not need to be a Lie group. For example, G could be a quotient of groups G_1/G_2. A complexification of G in that case would be $G_1^{\mathbb{C}}/G_2^{\mathbb{C}}$.

Consider for instance geodesic motions on S^3 (relevant to the regularized Kepler problem). The phase space is T^*S^3 and a symmetry group is SO(4), acting by the cotangent lift of the standard action on $S^3 \subset \mathbb{R}^4$. If we remove the zero-section $\mathbf{0}$ from T^*S^3, then all points in $P \equiv \mathrm{T}^*S^3\backslash\mathbf{0}$ have orbit type (SO(2)) and co-adjoint orbit type (T), where $T \subset \mathrm{SO}(4)$ is a maximal torus (so that we still have $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$ but G does not act freely on P). The orbit space $P/\mathrm{SO}(4)$ is identifiable with $(0,\infty)$ and the orbit space projection $P \to (0,\infty)$ is in fact a trivial bundle: $P \cong \mathrm{SO}(4)/\mathrm{SO}(2)\times(0,\infty)$. So we may take $G_1 \equiv \mathrm{SO}(4)$, $G_2 \equiv \mathrm{SO}(2)$, and $U \equiv (0,\infty)$.

In the 1:1:1 resonance the phase space is $\mathbb{R}^6 \cong \mathbb{C}^3$, and a symmetry group is SU(3). Removing the origin, one obtains a uniform orbit type (SU(2)) and co-adjoint orbit type (SU(2)), so that *both* our orbit type conditions fail. Nevertheless one has diffeomorphisms $\mathbb{R}^6\backslash\{0\} \cong S^5 \times (0,\infty) \cong \mathrm{SU}(3)\,/\,\mathrm{SU}(2)\times(0,\infty)$, which is again of the form proposed above.

In both the above examples one can prove that Assumption A of Chap. 5 (the existence of a 'period lattice') holds. It may be worth trying to verify the remaining Assumptions B and C. It is worth observing that in both examples the space U can be identified with some open wall of the Weyl chamber of the symmetry group.

Fourier series

Lochak's approach to Nekhoroshev estimates, as exploited here, conveniently avoids the use of Fourier series. For other kinds of perturbation analysis (e.g., proving the existence of 'whiskered tori' in the perturbed system) Fourier series may be unavoidable. In principle there should be no obstacle to implementing Fourier series in such cases. After all, the generalization of Fourier series on tori to series on compact connected groups is well-known (see, e.g., Bröcker and tom Dieck (1985)).

APPENDIX A

Proof of the Nekhoroshev-Lochak Theorem (abstract form)

This appendix is devoted to proving Theorem 5.9. To a large extent we follow Lochak's original arguments (Lochak (1992), Lochak (1993)). With the exception of some comments of our own on the role of convexity, we offer minimal motivation. For deeper insight, the reader is referred to Lochak's original papers, especially Lochak (1993).

To keep our exposition as simple as possible, we have restrained from employing two 'technical tricks' that would have improved our estimates. The first, discovered by Neishtadt (1984) and implemented in Lochak and Neishtadt (1992), concerns the normal form calculation. The idea is that one should make the first coordinate change larger than the others. The second trick we could have used is to rescale the frequency (in the simultaneous approximation argument) so that its largest component is unity. This allows one to lower the dimension by one. See Lochak (1992) for details.

Consider the setup described in Chap. 5 and restrict attention to a value of the parameter p_* with the property that $\Omega(p_*) = \frac{1}{T_*}\mathbf{n}$, for some $T_* > 0$ and $\mathbf{n} \in \mathfrak{t}^{\mathbb{Z}}$ ($\Omega \equiv \nabla h$). Then according to Assumption A, $G^{\bar{\sigma}} \times \{p_*\}$ is foliated by T_*-periodic solution trajectories of X_{H_0} ($H_0(g,p) \equiv h(p)$). One of our first objectives is to derive a normal form for the perturbed Hamiltonian $H = H_0 + F$ in a neighborhood of $G \times \{p_*\}$ (Proposition A.16 below). We begin with some preliminaries:

Averaging over periodic orbits

Define

A.1 $$W(g,p) \equiv w(p) \equiv \Omega(p_*) \cdot p \ ,$$

so that $\nabla w(p) = \frac{1}{T_*}\mathbf{n}$ *for all* $p \in B^{\bar{\rho}}$. By Assumption A, X_W has a well-defined flow map $(t,x) \mapsto \Phi_W^t(x) : \mathbb{R} \times (G^{\bar{\sigma}} \times B^{\bar{\rho}}) \to G^{\bar{\sigma}} \times B^{\bar{\rho}}$ satisfying $\Phi_W^{t+T_*} = \Phi_W^t$, and such that Φ_W^t

maps a set of the form $G^\sigma \times \{p\}$ $(0 \leqslant \sigma \leqslant \bar{\sigma},\ p \in B^{\bar{\rho}})$ onto itself. In particular Φ_W^t maps $D_\gamma(p_*, r)$ onto $D_\gamma(p_*, r)$ for any values of the parameters γ and r, and is the identity map when $t = T_*$.

For $u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$, we define $\bar{u}, \tilde{u}, \mathcal{I}u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ by

$$\bar{u} \equiv \frac{1}{T_*} \int_0^{T_*} u \circ \Phi_W^t dt \ ,$$

$$\tilde{u} \equiv u - \bar{u} \ ,$$

$$\mathcal{I}u \equiv \frac{1}{T_*} \int_0^{T_*} t u \circ \Phi_W^t dt \ .$$

Clearly $\|\bar{u}\|_\gamma^{p_*,r} \leqslant \|u\|_\gamma^{p_*,r}$ and $\|\tilde{u}\|_\gamma^{p_*,r} \leqslant 2\,\|u\|_\gamma^{p_*,r}$.

A.2 Lemma

1.
$$u + \{W, \mathcal{I}u\} = \bar{u}$$

2.
$$\|\mathcal{I}u\|_\gamma^{p_*,r} \leqslant \frac{1}{2} T_* \|u\|_\gamma^{p_*,r}$$

PROOF. The estimate A.2.2 is obvious. Regarding A.2.1, one computes

$$\begin{aligned}
\{W, \mathcal{I}u\} &= -\frac{d}{d\tau} \mathcal{I}u \circ \Phi_W^\tau\big|_{\tau=0} \\
&= -\frac{1}{T_*}\frac{d}{d\tau} \int_0^{T_*} t(u \circ \Phi_W^t) \circ \Phi_W^\tau dt\big|_{\tau=0} \\
&= -\frac{1}{T_*} \int_0^{T_*} t(\frac{d}{dt} u \circ \Phi_W^t) d\tau \\
&= -\frac{1}{T_*}\left[t u \circ \Phi_W^t\right]_0^{T_*} + \frac{1}{T_*}\int_0^{T_*} u \circ \Phi_W^t dt \\
&= -u + \bar{u}
\end{aligned}$$

□

Lie transforms and the Iterative Lemma

In what follows we shall be interested in Hamiltonians of the form

$$\mathcal{H} \equiv H_0 + Z + N \qquad (Z, N \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))) \ ,$$

with $\tilde{Z} = 0$. The Hamiltonian 5.5 is of this form if we define $\mathcal{H} \equiv H$, $Z \equiv \bar{F}$ and $N \equiv \tilde{F}$.

In the method known as *Lie transforms*, one makes symplectic changes of coordinates using the time-one map associated with the Hamiltonian flow generated by some 'auxiliary' Hamiltonian. Indeed let $\chi \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ be given, and suppose that X_χ has a well-defined flow map $(t, x) \mapsto \Phi_\chi^t(x) : [-1, 1] \times D_{\gamma-\delta}(p_*, r) \to D_\gamma(p_*, r)$, for some $\delta \in (0, \gamma]$. Then $\phi \equiv \Phi_\chi^1$ is symplectic. The following computations make use of Taylor's formula with integral remainder:

$$\begin{aligned}
\mathcal{H} \circ \phi =&(H_0 + Z + N) + (H_0 \circ \phi - H_0) + \big((Z + N) \circ \phi - (Z + N) \big) \\
=&(H_0 + Z) + N + (W \circ \phi - W) \\
&+ \big((H_0 - W) \circ \phi - (H_0 - W) \big) + \big((Z + N) \circ \phi - (Z + N) \big) \\
=&(H_0 + Z) + (N + \{W, \chi\}) \\
&+ \int_0^1 (1 - \tau)\{\{W, \chi\}, \chi\} \circ \Phi_\chi^\tau d\tau + \Delta_1(H_0 - W) + \Delta_1(Z + N) \ ,
\end{aligned}$$

where

A.3
$$\Delta_1 u \equiv u \circ \phi - u = \int_0^1 \{u, \chi\} \circ \Phi_\chi^\tau d\tau \qquad \big(u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r)) \big) \ .$$

Choosing $\chi \equiv \mathcal{I}N$ and applying A.2.1 (with $u \equiv N$), we obtain

$$\begin{aligned}
\mathcal{H} \circ \phi = (H_0 + Z') + &\int_0^1 (1 - \tau)\{\{W, \chi\}, \chi\} \circ \Phi_\chi^\tau d\tau \\
&+\Delta_1(H_0 - W) + \Delta_1(Z + N) \ ,
\end{aligned}$$

where $Z' \equiv Z + \bar{N}$ (so that $\tilde{Z}' = 0$). By A.2.1, and our choice $\chi \equiv \mathcal{I}N$,

$$\{W, \chi\} = \bar{N} - N = -\tilde{N} \ ,$$

so that

$$\int_0^1 (1 - \tau)\{\{W, \chi\}, \chi\} \circ \Phi_\chi^\tau d\tau = - \int_0^1 (1 - \tau)\{\tilde{N}, \chi\} \circ \Phi_\chi^\tau d\tau \ .$$

We may therefore write $\mathcal{H} \circ \phi = H_0 + Z' + N'$, if we define

$$N' \equiv \Delta_1(H_0 - W) + \Delta_1(Z + N) - \Delta_2 \tilde{N}$$

where

A.4 $$\Delta_2 u \equiv \int_0^1 (1-\tau)\{u,\chi\} \circ \Phi_\chi^\tau d\tau \qquad \left(u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r)) \right) .$$

Summarizing:

A.5 Lemma *Let $Z, N \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ and $0 < \delta \leqslant \gamma$ be given, with $\tilde{Z} = 0$, and assume that the Hamiltonian vector field associated with $\chi \equiv \mathcal{I}N$ has a well-defined flow map*

$$(t,x) \mapsto \Phi_\chi^t : [-1,1] \times D_{\gamma-\delta}(p_*, r) \to D_\gamma(p_*, r) .$$

Then the Hamiltonian

$$\mathcal{H} \equiv H_0 + Z + N$$

is pulled back by the symplectic coordinate transformation $\phi \equiv \Phi_\chi^1$ to

$$\mathcal{H} \circ \phi = H_0 + Z' + N' ,$$

where

1. $$Z' \equiv Z + \bar{N} \qquad (\textit{so that } \tilde{Z}' = 0) ,$$
2. $$N' \equiv \Delta_1(H_0 - W) + \Delta_1(Z + N) - \Delta_2 \tilde{N}$$

and where $\Delta_1, \Delta_2 : \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r)) \to \mathcal{A}^{\mathbb{C}}(D_{\gamma-\delta}(p_*, r))$ are the operations defined by equations A.3, A.4.

A.6 Remark The term $\Delta_1(H_0 - W)$ corresponds to what Lochak (1992) refers to as the 'frequency shift' contribution to the transformed Hamiltonian.

Lemma A.5 is the basis for proving the following 'iterative lemma':

A.7 Lemma (Iterative Lemma) *Suppose $Z, N \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$, $0 \leqslant r \leqslant 1$, $0 < \delta \leqslant \gamma$ and $\epsilon \geqslant 0$ are given, with $\tilde{Z} = 0$, and assume that*

1. $$\begin{aligned} \|Z + N\|_\gamma^{p_*, r} &\leqslant 3\epsilon E \\ \|N\|_\gamma^{p_*, r} &\leqslant 2\epsilon E \end{aligned} \qquad \left(E \equiv \frac{\bar{\sigma}\bar{\rho}}{T_M} \right) .$$

Furthermore, assume that for some $l_1 \geqslant 1$

2. $$r \geqslant \sqrt{l_1 \epsilon}$$

3. and $$\frac{1}{\delta^2}\frac{T_*}{T_M} r \leqslant l_1 l_2 \ ,$$

where

4. $$l_2 \equiv \min\left\{c_2, \frac{e}{2c_4 + 5c_3}\right\} \ .$$

Then there exists a symplectic diffeomorphism ϕ *from* $D_{\gamma-\delta}(p_*, r)$ *into* $D_\gamma(p_*, r)$ *such that the Hamiltonian*

$$\mathcal{H} \equiv H_0 + Z + N$$

is pulled back to

$$\mathcal{H} \circ \phi = H_0 + Z' + N' \ ,$$

for some $Z', N' \in \mathcal{A}^{\mathbb{C}}(D_{\gamma-\delta}(p_*, r))$ *with* $\tilde{Z}' = 0$ *and*

5. $$\|Z' + N'\|_{\gamma-\delta}^{p_*,r} \leqslant \|Z + N\|_{\gamma}^{p_*,r} + \|N'\|_{\gamma-\delta}^{p_*,r} \ ,$$

6. $$\|N'\|_{\gamma-\delta}^{p_*,r} \leqslant \frac{1}{e}\|N\|_{\gamma}^{p_*,r} \ .$$

Furthermore, for all $(g', p') \in D_{\gamma-\delta}(p_*, r)$

7. $$|p - p'| \leqslant \frac{1}{2}\delta r \bar{\rho} \qquad ((g, p) \equiv \phi(g', p')) \ .$$

The main point is that the fluctuating part N' of the transformed Hamiltonian $\mathcal{H} \circ \phi$ is less than the fluctuating part N of the untransformed Hamiltonian $\mathcal{H}$, by a factor of e. (We call N the *fluctuating part* of $\mathcal{H} = H_0 + Z + N$ because $\widetilde{H_0 + Z} = 0$.)

Note also that A.7.2 and A.7.3 are the first conditions involving the parameter r, which has been free until now (apart from the modest requirement 5.8).

PROOF. In preparation for applying Lemma A.5, define $\chi \equiv \mathcal{I}N \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$. Assume that δ satisfies

A.8 $$\frac{c_2 \bar{\sigma} \bar{\rho} \delta^2 r}{\|\chi\|_\gamma^{p_*,r}} \geqslant 1 \ .$$

Then Assumption B ensures that X_χ has a well-defined flow map $(t, x) \mapsto \Phi_\chi^t(x)$ as in the hypotheses of Lemma A.5, and that this map satisfies

A.9 $$\Phi_\chi^t \left(D_{\gamma-\delta}(p_*, r) \right) \subset D_{\gamma-\delta/2}(p_*, r) \qquad (-1 \leqslant t \leqslant 1) \ .$$

Furthermore, if $\phi \equiv \Phi_\chi^1$, then Assumption B also tells us that for all $(g', p') \in D_{\gamma-\delta}(p_*, r)$ the estimate A.7.7 holds.

Applying A.2.2:

A.10 $$\begin{aligned} \|\chi\|_\gamma^{p_*,r} &\leqslant \frac{1}{2} T_* \|N\|_\gamma^{p_*,r} \\ &\leqslant T_* \epsilon E = \bar{\sigma} \bar{\rho} \frac{T_*}{T_M} \epsilon \ . \end{aligned}$$

The requirement A.8 is therefore met if

$$\frac{1}{c_2} \left(\frac{\epsilon}{r^2} \right) \left(\frac{1}{\delta^2} \right) \left(\frac{T_*}{T_M} \right) r \leqslant 1 \ .$$

But this is guaranteed by the hypotheses A.7.2 and A.7.3.

For $u \in \mathcal{A}^{\mathbb{C}}(D_\gamma(p_*, r))$ one computes using the definition A.3, A.9 and Assumption C,

A.11 $$\|\Delta_1 u\|_{\gamma-\delta}^{p_*,r} \leqslant \int_0^1 \|\{u, \chi\}\|_{\gamma-\delta/2}^{p_*,r} \, d\tau \leqslant \left(\frac{4c_3}{\bar{\sigma}\bar{\rho}} \|u\|_\gamma^{p_*,r} \right) \frac{\|\chi\|_\gamma^{p_*,r}}{\delta^2} \ .$$

Similarly one computes from A.4, A.9 and Assumption C,

A.12 $$\|\Delta_2 u\|_{\gamma-\delta}^{p_*,r} \leqslant \left(\frac{2c_3}{\bar{\sigma}\bar{\rho}} \|u\|_\gamma^{p_*,r} \right) \frac{\|\chi\|_\gamma^{p_*,r}}{\delta^2} \ .$$

Also, by A.3, A.9 and Assumption C,

$$\|\Delta_1(H_0 - W)\|_{\gamma-\delta}^{p_*,r} \leqslant \|\{H_0 - W, \chi\}\|_{\gamma-\delta/2}^{p_*,r} \leqslant 4c_4 \left(\sup_{p \in B_{r\bar{\rho}}^{\gamma r \bar{\rho}}(p_*)} |\Omega(p) - \Omega(p_*)| \right) \frac{\|\chi\|_\gamma^{p_*,r}}{\delta^2} \ .$$

Since

$$\Omega(p) - \Omega(p_*) = \int_0^1 D^2 h\big(\, p + \tau(p_* - p)\,\big)(p - p_*)d\tau \ ,$$

and h is (m, M)-convex on $\operatorname{int} B^{\bar\rho} \supset \operatorname{int} B^{\gamma r \bar\rho}_{r\bar\rho}(p_*)$ (see 5.1), we have

$$\sup_{p \in B^{\gamma r\bar\rho}_{r\bar\rho}(p_*)} |\Omega(p) - \Omega(p_*)| \leqslant M(\gamma r\bar\rho + r\bar\rho) \leqslant 2Mr\bar\rho = \frac{2r}{T_M} \ ,$$

so that we arrive at the estimate

A.13
$$\|\Delta_1(H_0 - W)\|^{p_*,r}_{\gamma-\delta} \leqslant \frac{8c_4 r}{T_M} \frac{\|\chi\|^{p_*,r}_\gamma}{\delta^2} \ .$$

Lemma A.5 applies, and can now use A.5.2, A.13, A.11 and A.12 to compute

$$\begin{aligned}
\|N'\|^{p_*,r}_{\gamma-\delta} &\leqslant \left(\frac{8c_4 r}{T_M} + \frac{4c_3}{\bar\sigma\bar\rho}\|Z+N\|^{p_*,r}_\gamma + \frac{2c_3}{\bar\sigma\bar\rho}\|\tilde N\|^{p_*,r}_\gamma\right)\frac{\|\chi\|^{p_*,r}_\gamma}{\delta^2} \\
&\leqslant 2\left(\frac{2c_4 r}{T_M} + \frac{5c_3\epsilon E}{\bar\sigma\bar\rho}\right)\frac{T_*\,\|N\|^{p_*,r}_\gamma}{\delta^2} \qquad \text{using A.7.1 and A.10} \\
&= 2(2c_4 r + 5c_3\epsilon)\frac{1}{\delta^2}\frac{T_*}{T_M}\,\|N\|^{p_*,r}_\gamma \\
&\leqslant \left(2c_4 + \frac{5c_3}{l_1} r\right)\frac{1}{\delta^2}\frac{T_*}{T_M} r\,\|N\|^{p_*,r}_\gamma \qquad \text{using A.7.2} \\
&\leqslant 2\,(2c_4 + 5c_3)\,\frac{1}{\delta^2}\frac{T_*}{T_M} r\,\|N\|^{p_*,r}_\gamma \qquad \text{since } l_1 \geqslant 1 \text{ and } r \leqslant 1 \ .
\end{aligned}$$

Combining this with A.7.3 gives the estimate A.7.6.

Using A.5.1, one also computes

$$\begin{aligned}
\|Z' + N'\|^{p_*,r}_{\gamma-\delta} = \|Z + \bar N + N'\|^{p_*,r}_{\gamma-\delta} &\leqslant \|Z + \bar N\|^{p_*,r}_{\gamma-\delta} + \|N'\|^{p_*,r}_{\gamma-\delta} \\
&= \|\overline{Z+N}\|^{p_*,r}_{\gamma-\delta} + \|N'\|^{p_*,r}_{\gamma-\delta} \leqslant \|Z+N\|^{p_*,r}_\gamma + \|N'\|^{p_*,r}_{\gamma-\delta} \ ,
\end{aligned}$$

which proves A.7.5. □

The Hamiltonian in normal form

We now return our attention to the Hamiltonian

$$H = H_0 + F \qquad (H_0(g,p) \equiv h(p))$$

of Chap. 5, and in particular to its restriction to the domain $D_\gamma(p_*, r)$. We continue to leave γ and r as free parameters. Let $s \geqslant 1$ be given and denote by $[s] \in \mathbb{Z}$ its integer part. We seek to compose $[s]$ coordinate transformations $\phi_1, \dots, \phi_{[s]}$ using the Iterative Lemma A.7 (with $\epsilon \equiv \|F\|/E$). At every step we will take $\delta \equiv \gamma/(2s)$ so that, assuming

$$r \geqslant \sqrt{l_1 \epsilon}$$

$$\text{and} \qquad \frac{T_*}{T_M} s^2 r \leqslant \frac{\gamma^2}{4} l_1 l_2 \ ,$$

the hypotheses A.7.2 and A.7.3 will always be satisfied.

Write $\mathcal{H}^0 \equiv H$, $Z^0 \equiv \bar{F}$ and $N^0 \equiv \tilde{F}$. A successful jth application of the Iterative Lemma will deliver a symplectic diffeomorphism ϕ^j from $D_{\gamma - j\gamma/(2s)}(p_*, r)$ into $D_{\gamma-(j-1)\gamma/(2s)}(p_*, r)$ and a new Hamiltonian $\mathcal{H}^j \equiv \mathcal{H}^{j-1} \circ \phi^j = H_0 + Z^j + N^j$, for some Z^j and N^j with $\tilde{Z}^j = 0$. In that case, writing $\alpha_j \equiv \|Z^j + N^j\|^{p_*, r}_{\gamma - j\gamma/(2s)}$ and $\beta_j \equiv \|N^j\|^{p_*, r}_{\gamma - j\gamma/(2s)}$, the inequalities A.7.5 and A.7.6 will imply

A.14
$$\alpha_j \leqslant \alpha_{j-1} + \beta_j \ ,$$
$$\beta_j \leqslant \frac{1}{e} \beta_{j-1} \ .$$

The composition $\phi \equiv \phi_1 \circ \cdots \circ \phi_{[s]} : D_{\gamma - [s]\gamma/(2s)}(p_*, r) \to D_\gamma(p_*, r)$ will restrict to a map $\phi : D_{\gamma/2}(p_*, r) \to D_\gamma(p_*, r)$ (since $[s]/(2s) \leqslant 1/2$).

What needs to be checked is that the hypothesis A.7.1 of the Iterative Lemma holds at every step. That is, we must inductively prove that

A.15
$$\alpha_{j-1} \leqslant 3\epsilon E \ ,$$
$$\beta_{j-1} \leqslant 2\epsilon E \ ,$$

for all $1 \leqslant j \leqslant [s]$. By construction $\alpha_0 \leqslant \|F\| = \epsilon E$ and $\beta_0 \leqslant \|\tilde{F}\| \leqslant 2\|F\| = 2\epsilon E$. This anchors the induction. Suppose that A.15 (and hence A.14) holds for all $j \leqslant n$ $(n < [s])$. Then A.14 implies $\beta_n \leqslant e^{-n}\beta_0 \leqslant 2e^{-n}\epsilon E \leqslant 2\epsilon E$ and

$$\alpha_n \leqslant \alpha_0 + \left(\frac{1}{e} + \frac{1}{e^2} + \cdots + \frac{1}{e^n}\right)\beta_0 \leqslant \left(1 + \frac{2}{e-1}\right)\epsilon E \leqslant 3\epsilon E \ .$$

So A.15 also holds in the case $j = n + 1$, which completes the induction.

We have $\alpha_{[s]} \leqslant \alpha_0 + (e^{-1} + \cdots + e^{-[s]})\beta_0 \leqslant 3\epsilon E$ and $\beta_{[s]} \leqslant e^{-[s]}\beta_0 \leqslant e^{-s}(2e\epsilon E)$, and summarize the preceding arguments as follows:

A.16 Proposition (The Hamiltonian in normal form) *Assume that* $0 \leqslant r \leqslant 1$ *and* $s \geqslant 1$ *satisfy*

$$r \leqslant 1 - \frac{|p_* - \bar{p}|}{\bar{\rho}} \ ,$$
$$r \geqslant \sqrt{l_1 \epsilon} \ ,$$
$$\left(\frac{T_*}{T_M}\right) s^2 r \leqslant \frac{\gamma^2}{4} l_1 l_2 \ ,$$

for some $0 \leqslant \gamma \leqslant 1$ *and* $l_1 > 0$ *(*l_2 *is defined by A.7.4). Then there exists a symplectic change of coordinates* $\phi : D_{\gamma/2}(p_*, r) \to D_\gamma(p_*, r)$ *such that*

$$H \circ \phi = H_0 + Z + N \ ,$$

for some $Z, N \in \mathcal{A}^{\mathbb{C}}(D_{\gamma/2}(p_*, r))$*, with* $\tilde{Z} = 0$ *and*

$$\|Z + N\|_{\gamma/2}^{p_*,r} \leqslant 3\epsilon E \ ,$$
$$\|N\|_{\gamma/2}^{p_*,r} \leqslant e^{-s}(2e\epsilon E) \ .$$

Furthermore, for all $(g', p') \in D_{\gamma/2}(p_*, r)$

$$|p - p'| \leqslant \frac{\gamma}{4} r \bar{\rho} \qquad \big((g, p) \equiv \phi(g', p')\big) \ .$$

The first of the three hypotheses is just 5.8, repeated here for completeness.

Moving coordinate systems and reduction to the zero frequency case

To expose the role played by convexity[1] in establishing bounds on the perturbed dynamics, we prefer over Lochak's argument one based on moving coordinate transformations that we describe next.

Proposition A.16 says that there exists a change of coordinates ϕ such that $H \circ \phi = H_0 + Z + N$, for some Z and N with $\tilde{Z} = 0$ and such that N is exponentially small with

[1] Here we are thinking of the *first* estimate in 5.3, which we have yet to employ.

respect to the parameter s. Restrict attention to the real domain $D_0(p_*, r) = G \times B_{r\bar{\rho}}(p_*)$, and suppose that $\Omega(p_*) = 0$ ($\Omega \equiv \nabla h$, $H_0(g,p) = h(p)$). Then the convexity of h and the Morse lemma imply that in a neighborhood of $G \times \{p_*\}$ the level sets of H_0 are diffeomorphic to $G \times S^k$, where S^k denotes the unit sphere in $\mathfrak{t}$, centered at p_*. In particular these level sets are *compact.* Now $\|Z+N\|_0^{p_*,r} \leqslant 3\epsilon E$ (by A.16), so for ϵ sufficiently small the level sets of $H \circ \phi$ are also compact (in some neighborhood of $G \times \{p_*\}$) and energy conservation implies that integral curves of $X_{H\circ\phi}$, beginning sufficiently close to $G \times \{p_*\}$, remain $\sqrt{\epsilon}$-close *for all time.*

One attempts to convert the case $\Omega(p_*) \neq 0$ to the zero frequency case just described by making an appropriate moving coordinate transformation. The following lemma shows how to compute solution curves in a Hamiltonian system, as they appear in a moving coordinate system, by computing solutions to an associated *nonautonomous* system. The proof of the lemma is elementary and left to the reader.

A.17 Lemma (On moving coordinate transformations) *Let $\mathcal{H}$ and W be differentiable functions on $D_0(p_*, r)$ and assume X_W has a well-defined flow map $(t,x) \mapsto \Phi_W^t(x) : \mathbb{R} \times D_0(p_*, r) \to D_0(p_*, r)$. Suppose $t \mapsto x_t \in D_0(p_*, r)$ is an integral curve of $X_{\mathcal{H}}$ and define $x_t' \equiv (\Phi_W^t)^{-1}(x_t)$ (i.e., x_t' is the point x_t as seen by the 'moving coordinates' $(\Phi_W^t)_{t\in\mathbb{R}}$). Then $t \mapsto x_t'$ is an integral curve of the nonautonomous system $(X_{\mathcal{H}_t'})_{t\in\mathbb{R}}$, where $\mathcal{H}_t'$ is the time-dependent Hamiltonian defined by*

$$\mathcal{H}_t' \equiv \mathcal{H} \circ \Phi_W^t - W \ .$$

By $t \mapsto x_t'$ being an integral curve of the nonautonomous system $(X_{\mathcal{H}_t'})_{t\in\mathbb{R}}$, we mean that

$$\frac{d}{dt} x_t' = X_{\mathcal{H}_t'}(x_t') \qquad (t \in \mathbb{R}) \ .$$

Taking $W(g,p) \equiv w(p) = \Omega(p_*) \cdot p$ and $\mathcal{H} = H \circ \phi = H_0 + Z + N$ (with $\Omega(p_*)$ possibly non-zero), we have

$$\mathcal{H}_t' = (H_0 - W) + Z + N \circ \Phi_W^t$$

(because H_0 and Z are Φ_W^t-invariant). Or, defining $H_0' \equiv H_0 - W$ and $N_t \equiv N \circ \Phi_W^t$, we may write

$$\mathcal{H}_t' = H_0' + Z + N_t \ .$$

We can write $H_0'(g,p) \equiv h'(p)$ if we define $h' \equiv h - w$. Then if $\Omega' \equiv \nabla h'$, we have $\Omega'(p_*) = 0$. We are therefore in the zero frequency situation already described, with the important exception that we are dealing with a *time-dependent* perturbation $Z + N_t$.

Time-dependence destroys energy conservation, and therefore stability, as we argued above. However, the smaller the magnitude of the time-dependence (i.e., of $\partial N_t/\partial t$), the closer to stability one expects to get. Indeed this is the case:

A.18 Proposition *Let $Z, N_t : D_0(p_*, r) \to \mathbb{R}$ be differentiable functions ($t \in \mathbb{R}$) such that $\tilde{Z} = 0$ and such that $t \mapsto N_t(x)$ is differentiable for all x in the interior of $D_0(p_*, r)$. Let $\alpha \geqslant 0$ be any number such that*

$$\|Z + N_0\|_0^{p_*,r} \leqslant \alpha E \ ,$$

and assume that for some positive $r_1 \leqslant r$ one has

1. $$r_1^2 \geqslant \frac{16\bar{\sigma}\alpha}{c_7} \ .$$

Consider the time-dependent Hamiltonian

$$\mathcal{H}_t \equiv H_0' + Z + N_t \ , \qquad (t \in \mathbb{R})$$

where $H_0' \equiv H_0 - W$, and an integral curve $t \mapsto (g_t, p_t)$ of the nonautonomous system $(X_{\mathcal{H}_t})_{t \in \mathbb{R}}$. Then one has

$$\left(\frac{|p_0 - p_*|}{\bar{\rho}} \leqslant c_7' r_1 \quad \text{and} \quad |t| \leqslant t_0 \right) \implies \frac{|p_t - p_*|}{\bar{\rho}} \leqslant r_1 \ ,$$

where

$$c_7' \equiv \frac{\sqrt{5} - 2}{2} c_7 \qquad \text{and} \qquad t_0 \equiv \frac{\alpha}{\left(\sup_{t \in \mathbb{R}} \left\| \frac{\partial N_t}{\partial t} \right\|_0^{p_*,r} \right) / E} \ .$$

Note that $0 < c_7' < 1$, since $c_7 = m/M \leqslant 1$. Notice also of course that the 'stability time' t_0 approaches ∞ as $\|\partial N_t/\partial t\|_0^{p_*,r} \to 0$.

PROOF. Suppose then that $|p_0 - p_*| \leqslant c_7'\bar{\rho}r_1$, with c_7' as above. Then by the compactness of $D_0(p_*, c_7' r_1) = G \times B_{c_7'\bar{\rho}r_1}(p_*)$, the curve $t \mapsto (g_t, p_t) \in D_0(p_*, r_1)$ is well-defined until it reaches the boundary of $D_0(p_*, r_1)$. That is, there exists a time st_0 ($s = \pm 1, 0 < t_0 \leqslant \infty$) such that

$$|p_{st_0} - p_*| = \bar{\rho}r_1$$

$$\text{and} \qquad |t| \leqslant t_0 \implies (g_t, p_t) \in D_0(p_*, r_1) \ .$$

Write $x_t \equiv (g_t, p_t)$. Then the analogue of energy conservation for nonautonomous Hamiltonians is the identity

$$\text{A.19} \qquad \frac{d}{dt}\left(\mathcal{H}_t(x_t)\right) = \frac{\partial \mathcal{H}_t}{\partial t}(x_t) \ .$$

Recall that one proves this by applying the chain rule and computing

$$\frac{d}{d\tau}\mathcal{H}_t(x_\tau)|_{\tau=t} = \langle d\mathcal{H}_t, \frac{d}{dt}x_t\rangle = \omega(X_{\mathcal{H}_t}(x_t), X_{\mathcal{H}_t}(x_t)) = 0 \ .$$

Substituting $\mathcal{H}_t = H_0 + Z + N_t$ into A.19 and integrating leads to

$$h'(p_{st_0}) - h'(p_0) = \int_0^{st_0} \frac{\partial N_\tau}{\partial \tau}(x_\tau)d\tau - \left((Z + N_{st_0})(x_{st_0}) - (Z + N_0)(x_0) \right) \ ,$$

where $h' \equiv h - w$. Since $D^2 h' = D^2 h$, we can apply Taylor's formula with integral remainder to conclude

$$\text{A.20} \qquad \int_0^1 (1-\tau)D^2 h(p_0 + \tau\delta p)(\delta p, \delta p)d\tau = \int_0^{st_0} \frac{\partial N_\tau}{\partial \tau}(x_\tau)d\tau$$
$$- \left((Z + N_{st_0})(x_{st_0}) - (Z + N_0)(x_0) \right) - Dh'(p_0)\delta p \ ,$$

where $\delta p \equiv p_{st_0} - p_0$. We will arrive at a lower bound on t_0 proving our proposition by bounding the LHS of A.20 from below and the terms on the RHS from above. Indeed, by the (m, M)-convexity of h, we have

$$\text{A.21} \qquad \int_0^1 (1-\tau)D^2 h(p_0 + \tau\delta p)(\delta p, \delta p)d\tau \geqslant \frac{1}{2}m|\delta p|^2 = \frac{c_7 E|\delta p|^2}{2\bar{\rho}^2\bar{\sigma}} \ .$$

Furthermore, we have

A.22
$$\left|\int_0^{st_0} \frac{\partial N_\tau}{\partial \tau}(x_\tau)d\tau\right| \leqslant t_0 \sup_{t\in\mathbb{R}} \left\|\frac{\partial N_t}{\partial t}\right\|_0^{p_*,r} ,$$

and

$$\begin{aligned}
|(Z+N_{st_0})(x_{st_0}) - (Z+N_0)(x_0)| &\leqslant |(Z+N_0)(x_{st_0}) - (Z+N_0)(x_0)| \\
&\quad + |N_{st_0}(x_{st_0}) - N_0(x_{st_0})| \\
&\leqslant 2\,\|Z+N_0\|_0^{p_*,r} + \left|\int_0^{st_0} \frac{\partial N_\tau}{\partial \tau}(x_{st_0})d\tau\right| \\
&\leqslant 2\alpha E + t_0 \sup_{t\in\mathbb{R}} \left\|\frac{\partial N_t}{\partial t}\right\|_0^{p_*,r} .
\end{aligned} \qquad \text{A.23}$$

Finally, as $D^2h = D^2h'$ and $Dh'(p_*) = \Omega(p_*) - \Omega(p_*) = 0$, we have

$$\begin{aligned}
\left|Dh'(p_0)\delta p\right| &= \left|\int_0^1 Dh^2(p_* + \tau(p_0 - p_*))(p_0 - p_*, \delta p)d\tau\right| \\
&\leqslant Mc_7'\bar{\rho}r_1|\delta p| = \frac{c_7' E r_1 |\delta p|}{\bar{\rho}\bar{\sigma}} .
\end{aligned} \qquad \text{A.24}$$

Applying the estimates A.21–A.24 to A.20, we obtain

$$\frac{c_7 E|\delta p|^2}{2\bar{\rho}^2\bar{\sigma}} \leqslant 2\alpha E + 2t_0 \sup_{t\in\mathbb{R}} \left\|\frac{\partial N_t}{\partial t}\right\|_0^{p_*,r} + \frac{c_7' r_1 E|\delta p|}{\bar{\rho}\bar{\sigma}} .$$

Since $\delta p = p_{st_0} - p_0$, $|p_{st_0} - p_*| = \bar{\rho}r_1$ and $|p_0 - p_*| \leqslant c_7'\bar{\rho}r_1$, we have

$$(1 - c_7')\bar{\rho}r_1 \leqslant |\delta p| \leqslant (1 + c_7')\bar{\rho}r_1 ,$$

so that the preceding estimate yields

$$t_0 \geqslant \frac{E}{\sup_{t\in\mathbb{R}} \left\|\frac{\partial N_t}{\partial t}\right\|_0^{p_*,r}} \left(\left(\frac{c_7(1-c_7')^2}{4} - \frac{c_7'(1+c_7')}{2} \right) \frac{r_1^2}{\bar{\sigma}} - \alpha \right) .$$

The reader can check (recalling that $c_7' \equiv (\sqrt{5}-2)c_7/2$ and $0 < c_7 \leqslant 1$) that the coefficient of $r_1^2/\bar{\sigma}$ is bounded from below by $c_7/8$. Together with our hypothesis on r_1^2, this yields

$$t_0 \geqslant \frac{\alpha}{\left(\sup_{t\in\mathbb{R}} \left\|\frac{\partial N_t}{\partial t}\right\|_0^{p_*,r}\right)/E} .$$

□

Nekhoroshev estimates in the neighborhood of the periodic orbits

We now combine our preceding arguments to deduce Nekhoroshev-type estimates on motions in a neighborhood of $G \times \{p_*\}$, which by assumption is foliated by periodic orbits of X_H.

Let $r \geqslant 0$ be given and define $R \equiv 4r/c_7' \geqslant 4r$ ($c_7' \in (0,1)$ being the constant defined in A.18). Assume

A.25
$$R \leqslant 1 - \frac{|p_* - \bar{p}|}{\bar{\rho}} \ ,$$
$$R \geqslant \sqrt{l_1 \epsilon} \ ,$$
$$\frac{T_*}{T_M} s^2 R \leqslant \frac{\gamma^2}{4} l_1 l_2 \ ,$$

for some $l_1 > 0$. Then Proposition A.16 applies, with r replaced with R, delivering some symplectic coordinate change $\phi : D_{\gamma/2}(p_*, R) \to D_\gamma(p_*, R)$.

Let $({}^1p_0, {}^1g_0) \in D_0(p_*, r) \subset D_0(p_*, R)$ be given. Then

$$|{}^1p_0 - p_*| \leqslant \bar{\rho} r \ .$$

For t in some open interval of zero, there exists an integral curve $t \mapsto ({}^1g_t, {}^1p_t) \in D_0(p_*, R)$ of X_H with initial condition $({}^1g_0, {}^1p_0)$ (remember $r \leqslant R/4 < R$). Our objective is to derive an estimate on the evolution of ${}^1p_t$.

Define

$$({}^2g_t, {}^2p_t) \equiv \phi^{-1}({}^1g_t, {}^1p_t) \ .$$

Then $t \mapsto ({}^2g_t, {}^2p_t)$ is an integral curve of $X_{H\circ\phi}$. According to the last estimate in Proposition A.16,

$$|{}^2p_0 - {}^1p_0| \leqslant \frac{\gamma}{4} \bar{\rho} R \ ,$$

from which we deduce

$$|{}^2p_0 - p_*| \leqslant \bar{\rho} \left(\frac{\gamma}{4} R + r \right) = \bar{\rho} r \left(\frac{\gamma}{c_7'} + 1 \right) \ .$$

Choose $\gamma \equiv c_7'$. Then

$$|{}^2\!p_0 - p_*| \leqslant 2\bar{\rho}r \ .$$

Define

$$({}^3\!g_t, {}^3\!p_t) \equiv (\Phi_W^t)^{-1}({}^2\!g_t, {}^2\!p_t) \ .$$

By properties of Φ_W^t established at the beginning of this appendix, we known that ${}^3\!p_0 = {}^2\!p_0$, so that the above estimate implies

$$|{}^3\!p_0 - p_*| \leqslant 2\bar{\rho}r \ .$$

By Lemma A.17 and the arguments that followed, $t \mapsto ({}^3\!g_t, {}^3\!p_t)$ is a solution curve for the nonautonomous Hamiltonian

$$\mathcal{H}_t \equiv H_0' + Z + N_t \ ,$$

where $H_0' \equiv H_0 - W$, $N_t \equiv N \circ \Phi_W^t$ and where $Z, N \in D_{\gamma/2}(p_*, R) = D_{c_7'/2}(p_*, R)$ are the functions delivered by Proposition A.16 (with r replaced with R and $\gamma \equiv c_7'$). In particular, we know from A.16 that

$$\begin{aligned} \|Z + N\|_{c_7'/2}^{p_*,R} &\leqslant 3\epsilon E \ , \\ \|N\|_{c_7'/2}^{p_*,R} &\leqslant e^{-s}(2e\epsilon E) \ . \end{aligned}$$

Since $N_0 = N$ and $\|N_t\|_{c_7'/2}^{p_*,R} = \|N \circ \Phi_W^t\|_{c_7'/2}^{p_*,R} = \|N\|_{c_7'/2}^{p_*,R}$ (by forementioned properties of Φ_W^t), we deduce

$$\begin{aligned} \|Z + N_0\|_0^{p_*,R} &\leqslant 3\epsilon E \ , \\ \left\|\frac{\partial N_t}{\partial t}\right\|_0^{p_*,R} &= \|\{N_t, W\}\|_0^{p_*,R} \\ &\leqslant \frac{4c_4}{c_7'^2}|\Omega(p_*)|\,\|N_t\|_{c_7'/2}^{p_*,R} \qquad \text{by Assumption C} \\ &\leqslant e^{-s}\left(\frac{8ec_4}{c_7'^2 T_\Omega}\epsilon E\right) \ . \end{aligned}$$

We now apply Proposition A.18 with $\alpha \equiv 3\epsilon$, $r_1 \equiv 2r/c_7' = R/2 < R$, and with r in A.18 replaced by R. To satisfy A.18.1 we require $r \geqslant \sqrt{12{c_7'}^2\bar{\sigma}\epsilon/c_7}$. The condition A.25 already guarantees this if the free parameter $l_1 \geqslant 1$ is subject to the additional condition

$$l_1 \geqslant 192\bar{\sigma}/c_7 \ .$$

Since $|{}^3p_0 - p_*| \leqslant 2\bar{\rho}r = \bar{\rho}c_7' r_1$, we can apply A.18 to the trajectory $t \mapsto ({}^3g_t, {}^3p_t)$ to deduce

$$|t| \leqslant t_0 \implies \frac{|{}^3p_t - p_*|}{\bar{\rho}} \leqslant \frac{2r}{c_7'} \ ,$$

where

$$t_0 \equiv \frac{\alpha}{\left(\sup_{t\in\mathbb{R}} \left\|\frac{\partial N_t}{\partial t}\right\|_0^{p_*,R}\right)/E} \geqslant \frac{3{c_7'}^2 T_\Omega}{8ec_4} e^s \ .$$

Now ${}^3p_t = {}^2p_t$, and by the last estimate in Proposition A.16 (with r replaced with R and $\gamma \equiv c_7'$),

$$|{}^2p_t - {}^1p_t| \leqslant \frac{c_7'}{4} R\bar{\rho} = \bar{\rho}r \ .$$

In particular,

$$|{}^1p_t - p_*| \leqslant |{}^3p_t - p_*| + \bar{\rho}r \ ,$$

so that

$$|t| \leqslant t_0 \implies \frac{|{}^1p_t - p_*|}{\bar{\rho}} \leqslant \left(1 + \frac{2}{c_7'}\right) r \ .$$

Note that $(1 + 2/c_7')r \leqslant R/4 + R/2 < R$.

Summarizing:

A.26 Theorem (c.f. Theorem 1, Lochak (1993))

Consider the Hamiltonian $H = H_0 + F$ considered in Chap. 5 ($H_0(g,p) = h(p)$). Assume *that* $\Omega(p_*) = \mathbf{n}/T_*$, *for some* $p_* \in \operatorname{int} B$, $\mathbf{n} \in \mathfrak{t}^{\mathbb{Z}}$ *and* $T_* > 0$ $(\Omega \equiv \nabla h)$. *Furthermore assume that for some* $r \geqslant 0$, $s \geqslant 1$ *and* $l_1 \geqslant \max\{1, 192\bar{\sigma}/c_7\}$ *we have*

$$\begin{aligned} r &\leqslant \frac{c_7'}{4}\left(1 - \frac{|p_* - \bar{p}|}{\bar{\rho}}\right) , \\ r &\geqslant \frac{c_7'}{4}\sqrt{l_1 \epsilon} , \\ \frac{T_*}{T_M} s^2 r &\leqslant \frac{{c_7'}^3}{16} l_1 l_2 , \end{aligned}$$

where $c_7' \equiv (\sqrt{5} - 2)c_7/2$ *and* l_2 *is given by A.7.4. Then (real) integral curves* $t \mapsto (g_t, p_t) \in G \times B$ *of* X_H *satisfy the estimate*

$$\left(\frac{|p_0 - p_*|}{\bar{\rho}} \leqslant r \quad \text{and} \quad \frac{|t|}{T_\Omega} \leqslant \frac{3{c_7'}^2}{8ec_4} e^s\right) \implies \frac{|p_t - p_*|}{\bar{\rho}} \leqslant \left(1 + \frac{2}{c_7'}\right) r .$$

Taking $s \equiv \epsilon^{-b/2}$ and $r \equiv ({c_7'}^3/16) l_1 l_2 (T_M/T_*)\epsilon^b$ for some $b > 0$:

A.27 Corollary (c.f. Corollary 3, Lochak (1993)) *Suppose* $l_1 \geqslant \max\{1, 192\bar{\sigma}/c_7\}$ *and* $b > 0$ *are given such that*

$$\frac{{c_7'}^2 l_1 l_2}{A_*}\epsilon^b \leqslant \frac{T_*}{T_M} \leqslant \frac{{c_7'}^2 l_1^{1/2} l_2}{4}\epsilon^{b-1/2} ,$$

where

$$A_* \equiv \left(1 - \frac{|p_* - \bar{p}|}{\bar{\rho}}\right)$$

and l_2 *is given by A.7.4. Then*

$$\left(\frac{|p_0 - p_*|}{\bar{\rho}} \leqslant \frac{{c_7'}^3 l_1 l_2}{16}\frac{\epsilon^b}{T_*/T_M} \quad \text{and} \quad \frac{|t|}{T_\Omega} \leqslant \frac{3{c_7'}^2}{8ec_4}\exp(\epsilon^{-b/2})\right) \implies$$

$$\frac{|p_t - p_*|}{\bar{\rho}} \leqslant \left(1 + \frac{2}{c_7'}\right)\frac{{c_7'}^3 l_1 l_2}{16}\frac{\epsilon^b}{T_*/T_M} .$$

Estimates for an arbitrary initial condition

We now seek an estimate on solutions $t \mapsto (g_t, p_t)$ with $p_0 \equiv \bar{p}$. Recall that $\bar{p}$ is a point in $\mathfrak{t}_0$, subject to the condition that the Hamiltonian is defined in some complex neighborhood $G^{\bar{\sigma}} \times B^{\bar{\rho}}$ of $G \times \{\bar{p}\}$.

Lochak's marvelous observation is that a stability time for any initial condition can be deduced from those of the periodic orbits studied above. To do so requires understanding how the periodic orbits are distributed, and the relevant tool is purely arithmetic.

View $\{\beta_1, \ldots, \beta_k\}$ (the $\mathbb{Z}$-basis for $\mathfrak{t}^{\mathbb{Z}}$) as an $\mathbb{R}$-basis for $\mathfrak{t}$, and let $\gamma : \mathbb{R}^k \to \mathfrak{t}$ denote the associated isomorphism. The l^∞-norm on $\mathbb{R}^k$ induces a norm $|\cdot|_\infty$ on $\mathfrak{t}$:

$$|\gamma(\tau_1, \ldots, \tau_k)|_\infty \equiv \max_{1 \leqslant j \leqslant k} |\tau_j| \qquad (\tau_j \in \mathbb{R}) .$$

We have $|\xi| \leqslant c_1\sqrt{k}|\xi|_\infty$ ($\xi \in \mathfrak{t}$), courtesy of 5.7.

Since $\gamma(\mathbb{Z}^k) = \mathfrak{t}^{\mathbb{Z}}$, Dirichlet's simultaneous approximation theorem (see, e.g., Schmidt (1980)) tells us that

$$(\forall \varpi \in \mathfrak{t})(\forall Q > 1)(\exists q \in [1, Q) \cap \mathbb{Z}) : \; |q\varpi - \mathfrak{t}^{\mathbb{Z}}|_\infty \leqslant \frac{1}{Q^{1/k}} .$$

Taking $\varpi \equiv \Omega(\bar{p})$ and writing $T_* \equiv T_M q$, one deduces:

A.28 Theorem (On the proximity of periodic orbits)
For all $Q > 1$ there exists $T_* > 0$ and $\Omega_* \in \frac{1}{T_*}\mathfrak{t}^{\mathbb{Z}}$ such that

$$1 < \frac{T_*}{T_M} \leqslant Q \quad \textit{and} \quad |\Omega(\bar{p}) - \Omega_*| \leqslant \frac{c_1\sqrt{k}}{T_* Q^{1/k}} .$$

Because of the convexity of h, the frequency map is locally invertible:

A.29 Lemma *Define $c_5' \equiv \min\{c_5/2, 1\}$. The map $\Omega : B \to \mathfrak{t}$ has a locally defined inverse Ω^{-1} mapping $B_{c_7 c_5'/(2T_M)}(\Omega(\bar{p}))$ diffeomorphically onto an open subset of $B = B_{\bar{\rho}}(\bar{p})$.*

Proof. Apply the inverse function theorem, as formulated in, e.g., Proposition 2.5.6 of Abraham et al. (1988)[2]. □

[2] Their statement contains a typographical error. Correct values for the constants given are $P \equiv \min\{(2KM)^{-1}, R\}$, $Q \equiv \min\{(2NL)^{-1}, P/(2M)\}$, $S \equiv \min\{(2KM)^{-1}, Q/(2L)\}$.

If $\Omega_* \in B_{c_7 c_5'/(2T_M)}(\Omega(\bar{p}))$ and $p* \equiv \Omega^{-1}(\Omega_*)$, then one computes

$$\begin{aligned}
|\bar{p} - p_*| &= |\Omega^{-1}(\Omega(\bar{p})) - \Omega^{-1}(\Omega_*)| \\
&= \left|\int_0^1 D\Omega^{-1}(\Omega_* + t\delta\Omega)\delta\Omega dt\right| \qquad (\,\delta\Omega \equiv \Omega(\bar{p}) - \Omega_*\,) \\
&= \left|\int_0^1 [D^2 h(\Omega^{-1}(\Omega_* + t\delta\Omega))]^{-1}\delta\Omega dt\right| \\
&\leqslant \frac{1}{m}|\delta\Omega| = \bar{\rho} T_m |\Omega(\bar{p}) - \Omega_*| \ .
\end{aligned}$$

With the aid of this estimate, the reader may readily verify that A.28 and A.29 combine to yield:

A.30 Corollary *Assume*

$$Q^{1/k} \geqslant \max\{\frac{2c_1\sqrt{k}}{c_7 c_5'}, 1\} \ .$$

Then there exists $T_* > 0$ and $p_* \in B$ with $\Omega(p_*) \in \frac{1}{T_*}\mathfrak{t}^{\mathbb{Z}}$ such that

$$1 < \frac{T_*}{T_M} \leqslant Q \quad \textit{and} \quad \frac{|\bar{p} - p_*|}{\bar{\rho}} \leqslant \frac{c_1\sqrt{k}}{c_7 Q^{1/k}(T_*/T_M)} \ .$$

Let $b > 0$ and $l_1 \geqslant \max\{1, 192\bar{\sigma}/c_7\}$ be given. If we choose

$$Q^{1/k} \equiv \frac{16c_1\sqrt{k}}{c_7 {c_7'}^3 l_1 l_2}\epsilon^{-b} \ ,$$

then the hypothesis of A.30 is satisfied if

$$\epsilon \leqslant \left(\frac{c_7 {c_7'}^3 l_1 l_2}{8 l_3}\right)^{1/b} \ ,$$

where $l_3 \equiv \min\{c_7 c_5', 2c_1\sqrt{k}\}$. In that case, for some $T_* > 0$, with

A.31 $$1 < \frac{T_*}{T_M} \leqslant \left(\frac{16c_1\sqrt{k}}{c_7 {c_7'}^3 l_1 l_2}\right)^k \epsilon^{-kb} \ ,$$

there exists $p_* \in B$ such that $\Omega(p_*) \in \frac{1}{T_*}\mathfrak{t}^{\mathbb{Z}}$ and

A.32 $$\frac{|\bar{p} - p_*|}{\bar{\rho}} \leqslant \frac{{c_7'}^3 l_1 l_2}{16}\frac{\epsilon^b}{T_*/T_M} \ .$$

If we can satisfy the hypothesis on T_*/T_M in Corollary A.27, then for any solution $t \mapsto (g_t, p_t)$ with $p_0 = \bar{p}$ we will be able to conclude from A.32 (using $|p_t - \bar{p}| \leqslant |p_t - p_*| + |\bar{p} - p_*|$ and $T_*/T_M > 1$) that

A.33 $$\frac{|t|}{T_\Omega} \leqslant \frac{3{c_7'}^2}{8ec_4}\exp(\epsilon^{-b/2}) \implies \frac{|p_t - \bar{p}|}{\bar{\rho}} \leqslant \left(2 + \frac{2}{c_7'}\right)\frac{{c_7'}^3 l_1 l_2}{16}\epsilon^b \ .$$

By A.31 the hypothesis on T_*/T_M in A.27 is met provided

A.34 $$\epsilon \leqslant \left(\frac{A_*}{{c_7'}^2 l_1 l_2}\right)^{1/b}$$

A.35 $$\text{and} \quad \epsilon^{1/2-(1+k)b} \leqslant l_4 l_1^{k+1/2} \ ,$$

where

$$l_4 \equiv \frac{{c_7'}^2 l_2}{4}\left(\frac{c_7 {c_7'}^3 l_2}{16 c_1 \sqrt{k}}\right)^k \ .$$

We observe from A.31, A.32 and the definition of A_*, that

$$A_* \geqslant 1 - \frac{{c_7'}^3 l_1 l_2}{16}\epsilon^b \ .$$

So we ensure $A_* \geqslant 1/2$ if we assume

$$\epsilon \leqslant \left(\frac{8}{{c_7'}^3 l_1 l_2}\right)^{1/b} \ .$$

The inequality A.34 is guaranteed in that case provided

$$\epsilon \leqslant \left(\frac{1}{2{c_7'}^2 l_1 l_2}\right)^{1/b} \ .$$

What remains is to ensure A.35. Remember that l_1 is still a free parameter, apart from the requirement $l_1 \geqslant \max\{1, 192\bar{\sigma}/c_7\}$. Choose

$$l_1 \equiv \max\left\{1, \frac{192\bar{\sigma}}{c_7}, \left(\frac{1}{l_4}\right)^{k+1/2}\right\} \ .$$

Then to satisfy the condition A.35 it suffices to ensure $\epsilon^{1/2-(k+1)b} \leqslant 1$, which we achieve with the choice

$$b = \frac{1}{2(1+k)} \ .$$

To summarize, the exponential estimate A.33 holds, with l_1 and b as above, provided

$$\epsilon_0 = \min\left\{\left(\frac{8}{{c'_7}^3 l_1 l_2}\right)^{1/b}, \left(\frac{1}{2{c'_7}^2 l_1 l_2}\right)^{1/b}\right\} .$$

This completes the proof of Theorem 5.9 and its Addendum 5.10.

APPENDIX B

Proof that $\mathcal{W}$ is a slice

Suppose G acts smoothly on a smooth manifold M. Recall that S is a *slice* at $x \in M$ if S is a G_x-invariant submanifold of M ($G_x \subset G$ the isotropy subgroup at x) containing x and such that

$$G \times_{G_x} S \to M \ , \qquad [g, x] \mapsto g \cdot s$$

is a diffeomorphism onto some open neighborhood of $G \cdot x$. Here $G \times_{G_x} S$ is the quotient $(G \times S)/G_x$, where G_x acts on $G \times S$ according to $h \cdot (g, s) \equiv (gh^{-1}, h \cdot x)$, and $[g, s] \equiv (g, s)$ mod G_x. We call S a *global* slice if the image of the map above is M. Sufficient conditions for a G_x-invariant submanifold $S \ni x$ to be a slice at x are: (i) S intersects G-orbits in a 'complementary fashion,' i.e., $\mathrm{T}_s M = \mathrm{T}_s S \oplus \mathrm{T}_s(G \cdot s)$ $(s \in S)$, and (ii) $s \in S$, $g \in G$ and $g \cdot s \in S$ together imply $g \in G_x$.

Let us now verify that the (open) Weyl chamber $\mathcal{W}$ in $\mathfrak{g}^*$ is a global slice for the co-adjoint action of G on $\mathfrak{g}^*_{\mathrm{reg}}$. Since each regular co-adjoint orbit intersects $\mathcal{W}$ in a unique point (see, e.g., Corollary 1.8.3, Part 1), it follows that $G(\mathcal{W}) = \mathfrak{g}^*_{\mathrm{reg}}$ (so that if $\mathcal{W}$ is indeed a slice, then it is a global slice) and

$$(w \in \mathcal{W} \ , g \in G \ , \text{ and } g \cdot w \in \mathcal{W}) \Rightarrow g = \mathrm{id} \ .$$

By the above, it remains only to show that $\mathcal{W}$ intersects co-adjoint orbits in a complementary fashion, i.e.,

B.1 $$\mathrm{T}_w \mathfrak{g}^* = \mathrm{T}_w(G \cdot w) \oplus \mathrm{T}_w \mathcal{W} \qquad (w \in \mathcal{W}) \ .$$

We claim, identifying the various spaces with subspaces of $\mathfrak{g}^*$, that

B.2 $$\mathrm{T}_w(G\cdot w) = \underline{\mathfrak{t}}^{\perp} \equiv \mathrm{Ann}\,\mathfrak{t}$$

B.3 $$\mathrm{T}_w\mathcal{W} = \underline{\mathfrak{t}} \equiv \mathrm{Ann}[\mathfrak{g},\mathfrak{t}] \ ,$$

for all $w \in \mathcal{W}$. Then B.1 follows immediately from the direct sum decomposition

$$\mathfrak{g}^* = \underline{\mathfrak{t}}^{\perp} \oplus \underline{\mathfrak{t}} \ ;$$

see, e.g., Sect. 1, Part 1.

Since $\mathcal{W} \subset \underline{\mathfrak{t}}$ is open, B.3 is obvious.

PROOF OF B.2. Since regular co-adjoint orbits are of type (T) (by, e.g., 1.8.1, 1.8.2 and 1.2.2 of Part 1), we know that $\dim \mathrm{T}_w(G\cdot w) = \dim \underline{\mathfrak{t}}$ $(w \in \mathcal{W} \subset \underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}})$. To prove B.2 it therefore suffices to show that $\mathrm{T}_w(G\cdot w) \subset \underline{\mathfrak{t}}^{\perp} = \mathrm{Ann}\,\mathfrak{t}$. Elements β of $\mathrm{T}_w(G\cdot w)$ are of the form $\beta = \mathrm{ad}^*_\xi\, w$ for some $\xi \in \mathfrak{g}$. But then for all $\tau \in \mathfrak{t}$ we compute

$$\langle \beta, \tau\rangle = \langle \mathrm{ad}^*_\xi\, w, \tau\rangle = -\langle \mathrm{ad}^*_\tau\, w, \xi\rangle = 0 \ ,$$

since $\mathfrak{g}_w = \mathfrak{t}$ (by, e.g., Corollary 1.8, Part 1). This shows that $\beta \in \mathrm{Ann}\,\mathfrak{t} = \underline{\mathfrak{t}}^{\perp}$. Since $\beta \in \mathrm{T}_w(G\cdot w)$ was arbitrary, this proves $\mathrm{T}_w(G\cdot w) \subset \underline{\mathfrak{t}}^{\perp}$, as required. $\square$

APPENDIX C

Proof of the Extension Lemma

This appendix is devoted to pay proof of Lemma 8.8. Recall the fact (following from the last statement in Theorem 8.8) that every point of P is of the form $g \cdot y$ for some $g \in G$ and $y \in F$. We shall use this fact repeatedly in the sequel without further comment.

Let us check that the extended action of H is well-defined. As in the proof of Theorem 8.14, we have $g' \cdot y' = g \cdot y$ ($g, g' \in G$; $y, y' \in F$) if and only if $g' = gq$ and $y' = q^{-1} \cdot y$ for some $q \in T$. In that case, for any $h \in H$, we obtain $g' \cdot (h \cdot_H y') = (gq) \cdot (h \cdot_H (q^{-1} \cdot y)) = g \cdot (h \cdot_H y)$, since the actions of T and H on F commute by hypothesis. This shows that the extended action is well-defined.

That the extension of $\mathbf{K}$ is well-defined follows similarly, appealing to hypothesis 2.

Next, we argue that the extended action of H is symplectic. Fix some $h \in H$ and define $\phi : P \to P$ by $\phi(x) \equiv h \cdot_H x$. We need to show that ϕ is symplectic. Since G acts symplectically on P, and H acts symplectically on F, it follows from the definition of the extended action that ϕ maps fibers of $\pi : P \to \mathcal{O}$ symplectically onto themselves. It therefore remains only to show that $\mathrm{T}\phi$ maps horizontal spaces of the symplectic connection on $\pi : P \to \mathcal{O}$ symplectically onto horizontal spaces. From the definition of ϕ, the extended action of H, and infinitesimal generators, it follows that

C.1 $$\mathrm{T}\phi \cdot \xi_P(g \cdot y) = \xi_P(\phi(g \cdot y)) \qquad (\xi \in \mathfrak{g},\ g \in G,\ y \in F) \ .$$

Since $\mathbf{J}$ is G-equivariant and $\mathbf{J}|F$ is H-invariant (hypothesis 3), we compute

C.2 $$\mathbf{J}(\phi(g \cdot y)) = \mathbf{J}(h \cdot_H (g \cdot y)) = \mathbf{J}(g \cdot (h \cdot_H y))$$
$$= g \cdot \mathbf{J}(h \cdot_H y) = g \cdot \mathbf{J}(y) = \mathbf{J}(g \cdot y) \ .$$

That $\mathrm{T}\phi$ is symplectic on horizontal spaces now follows from C.1 and Lemma 8.12.

Conclusion 4 follows by construction. So does conclusion 5. Conclusion 6 was proven in C.2 above. It remains to prove that $\mathbf{K} : P \to \mathfrak{h}^*$ is a momentum map for the extended action of H.

Write η_P^H (resp. η_F^H) for the infinitesimal generator of the H action on P (resp. F) corresponding to $\eta \in \mathfrak{h}$. Let $\eta \in \mathfrak{h}$ be arbitrary. Then one computes using the definition of the extended action,

$$\text{C.3} \qquad \eta_P^H(g \cdot y) = \mathrm{T}\Phi_g \cdot \eta_F^H(y) \qquad (g \in G,\, y \in F) \ ,$$

where $\Phi_g(x) \equiv g \cdot x$ $(x \in P)$. Consider the one-form $\alpha \equiv \eta_P^H \lrcorner\, \omega - dK_\eta$. Then as $\eta \in \mathfrak{h}$ is arbitrary, proving that $\mathbf{K} : P \to \mathfrak{h}^*$ is a momentum map for the extended action amounts to proving that α vanishes. It suffices to check

$$\text{C.4} \qquad \alpha | \mathrm{T}_{g\cdot y} F_{g\cdot\mu_0} = 0$$

$$\text{C.5} \qquad \text{and} \qquad \alpha | \mathfrak{g}_P(g \cdot y) = 0 \ ,$$

where $g \in G$ and $y \in F$ are arbitrary, and $\mathfrak{g}_P(x) \equiv \mathrm{T}_x(G \cdot x)$ $(x \in P)$.

Suppose $v \in \mathrm{T}_{g\cdot y} F_{g\cdot\mu_0}$. Then $v = \mathrm{T}\Phi_g \cdot w$ for some $w \in \mathrm{T}_y F$. In that case we compute

$$\begin{aligned}
\langle \alpha, v \rangle &= \omega\left(\eta_P^H(g \cdot y),\, \mathrm{T}\Phi_g \cdot w\right) - \langle dK_\eta, \mathrm{T}\Phi_g \cdot w \rangle \\
&= \omega\left(\mathrm{T}\Phi_g \cdot \eta_F^H(y),\, \mathrm{T}\Phi_g \cdot w\right) - \langle dK_\eta, w \rangle \qquad \text{using C.3 and the } G\text{-invariance of } \mathbf{K} \\
&= \omega(\eta_F^H(y), w) - \langle dK_\eta, w \rangle \qquad \text{since } G \text{ acts symplectically} \\
&= 0 \ ,
\end{aligned}$$

since $\mathbf{K} : F \to \mathfrak{h}^*$ is a momentum map for the action of H on F, by hypothesis. This proves C.4.

Let $\xi \in \mathfrak{g}$ be arbitrary. Then, using C.3, the G-invariance of $\mathbf{K}$, and the symplecticity of Φ_g, one computes

$$\begin{aligned}
\langle \alpha, \xi_P(g \cdot y) \rangle &= \langle \alpha, \mathrm{T}\Phi_g \cdot (g^{-1} \cdot \xi)_P(y) \rangle \\
\text{C.6} \qquad &= \omega\left(\eta_F^H(y),\, (g^{-1} \cdot \xi)_P(y)\right) - \langle dK_\eta, (g^{-1} \cdot \xi)_P(y) \rangle \ .
\end{aligned}$$

By Lemma 8.12.1, we can write $g^{-1} \cdot \xi = \xi^1 + \xi^2$ for some $\xi^1, \xi^2 \in \mathfrak{g}$ such that $\xi^1_P(y) \in \mathrm{Hor}_y$ and $\xi^2_P(y) \in \mathrm{T}_y F$. Then the right-hand side of C.6 becomes

$$\omega(\eta^H_F(y), \xi^1_P(y)) - \langle dK_\eta, \xi^1_P(y)\rangle + \left(\omega(\eta^H_F(y), \xi^2_P(y)) - \langle dK_\eta, \xi^2_P(y)\rangle\right) \ .$$

The first term vanishes, by the definition of Hor_x. The second term vanishes because $\mathbf{K}$ is G-invariant. The term in parentheses vanishes because $\mathbf{K} : F \to \mathfrak{h}^*$ is the momentum map of the action of H on F, and because $\xi^2_P(y) \in \mathrm{T}F$. Therefore, $\langle \alpha, \xi_P(g \cdot y)\rangle = 0$. Since $\xi \in \mathfrak{g}$ was arbitrary, equation C.5 must hold.

APPENDIX D

An application of the S^1 Extension Theorem: The axisymmetric Euler-Poinsot rigid body revisited

The purpose of this appendix is to give an application of the results of Chap. 12. We show how the body symmetry of the axisymmetric Euler-Poinsot rigid body (viewed as a $(\mathrm{SO}(3), H)$-integrable space) can be constructed *ab initio* by applying the S^1 extension theorem (Theorem 12.2). In principle, the same technique could be applied to obtain $\mathrm{SO}(3) \times S^1$ symmetry in the *asymmetric* rigid body, provided one restricts to appropriate open subsets of the phase space. We do not attempt this here.

In Chap. 11 we recalled that the Euler-Poinsot rigid body is described by the system $(\mathrm{T}\,\mathrm{SO}(3), -d\Theta, H)$, where

D.1 $$H(\boldsymbol{\lambda}(\Lambda, m)) \equiv \frac{1}{2I_1}m_1^2 + \frac{1}{2I_2}m_2^2 + \frac{1}{2I_3}m_3^2$$

and Θ is the one-form on $\mathrm{T}\,\mathrm{SO}(3)$ defined by 11.2. Irrespective of the values of the moments of inertia I_1, I_2, I_3, the Hamiltonian H is invariant with respect to the action of $G \equiv \mathrm{SO}(3)$ defined by

D.2 $$A \cdot \boldsymbol{\lambda}(\Lambda, m) \equiv \boldsymbol{\lambda}(A\Lambda, m) \ ,$$

or equivalently by $A \cdot \boldsymbol{\rho}(\Lambda, n) \equiv \boldsymbol{\rho}(A\Lambda, An)$. ($\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ are defined in Chap. 11.) This is just the action of $\mathrm{SO}(3) \times S^1$ of Chap. 11 with the S^1 part left out. A momentum map $\mathbf{J} : \mathrm{T}\,\mathrm{SO}(3) \to \mathbb{R}^3$ is given by

D.3 $$\mathbf{J}(\boldsymbol{\lambda}(\Lambda, m)) \equiv \Lambda m \ ,$$

or equivalently by $\mathbf{J}(\boldsymbol{\rho}(\Lambda, n)) \equiv n$.

Let $\mathrm{T}\,\mathrm{SO}(3)\backslash\mathbf{0}$ denote $\mathrm{T}\,\mathrm{SO}(3)$ with its zero section removed. Then G acts freely on $\mathrm{T}\,\mathrm{SO}(3)\backslash\mathbf{0}$ and $\mathbf{J}(\mathrm{T}\,\mathrm{SO}(3)\backslash\mathbf{0}) \subset \mathfrak{g}^*_{\mathrm{reg}} \cong \mathbb{R}^3\backslash\{0\}$. The Hamiltonian G-space

$$(\mathrm{T}\,\mathrm{SO}(3)\backslash\mathbf{0}, -d\Theta, \mathrm{SO}(3), \mathbf{J})$$

is $(\mathrm{SO}(3), H)$- (i.e., dynamically) integrable.

Before turning to the axisymmetric case, let us study the symplectic cross section of $\mathrm{T}\,\mathrm{SO}(3)\backslash\mathbf{0}$ in some detail.

The symplectic cross section

Choose T, $\mathcal{W}$ and $\mathcal{O}$ as in Example 8.6. Then $\mathcal{W}$ and $\mathcal{O}$ intersect at the point $\mu_0 \equiv e_3$. The symplectic fibration $\pi : \mathrm{T}\,\mathrm{SO}(3)\backslash\mathbf{0} \to \mathcal{O} \cong S^2$ (see Theorem 8.8) is given by $\pi(\boldsymbol{\rho}(\Lambda, n)) = n/\,\|n\|$. The map $\psi : \mathrm{SO}(3) \times (0,\infty) \to \mathrm{T}\,\mathrm{SO}(3)\backslash\mathbf{0}$ defined by

D.4 $$\psi(\Lambda, p) \equiv \boldsymbol{\rho}(\Lambda^{-1}, pe_3)$$

is a diffeomorphism onto the symplectic cross section $F \equiv \pi^{-1}(e_3)$ of the space

$$(\mathrm{T}\,\mathrm{SO}(3)\backslash\mathbf{0}, -d\Theta, \mathrm{SO}(3), \mathbf{J})$$

(see Definition 8.9). The map ψ is symplectic if we equip $\mathrm{SO}(3) \times (0,\infty)$ with the symplectic structure $-d\Theta'$, where Θ' is the one-form on $\mathrm{SO}(3) \times (0,\infty)$ defined by the formula 11.3. The action of $T \cong S^1$ on $\mathrm{SO}(3) \times (0,\infty)$ that makes $\psi : \mathrm{SO}(3) \times (0,\infty) \xrightarrow{\sim} F$ a T-equivariant map is given by

D.5 $$(\theta \bmod 2\pi) \cdot (\Lambda, p) \equiv \boldsymbol{\rho}(e^{\theta \hat{e}_3}\Lambda^{-1}, pe_3)\ .$$

The action of T on F has momentum map $\mathbf{J}^F : F \to \mathfrak{t}^* \cong \mathbb{R}$ given by $\mathbf{J}^F(\boldsymbol{\rho}(\Lambda, n)) = n \cdot e_3$. So $(\mathbf{J}^F \circ \psi)(\Lambda, p) = p$.

To summarize, $\psi : \mathrm{SO}(3) \times (0,\infty) \xrightarrow{\sim} F$ is an equivalence between the space

$$\big(\mathrm{SO}(3) \times (0,\infty), -d\Theta', S^1, \mathbf{J}^F \circ \psi\big)$$

and the symplectic cross section F, where Θ' and $\mathbf{J}^F \circ \psi$ are as described above.

Our next task is to obtain a concrete realization of the Poisson reduced space $\mathrm{SO}(3) \times (0,\infty)/S^1$. First, observe that the symplectic structure $-d\Theta'$ on $\mathrm{SO}(3) \times (0,\infty)$ is $-\omega^*_{\mathrm{SO}(3)}$, where $\omega^*_{\mathrm{SO}(3)}$ is the symplectic structure that $\mathrm{SO}(3) \times (0,\infty)$ carries by virtue of being (up to the obvious identifications) the action-group model space $G \times \mathfrak{t}_0^*$ for $G = \mathrm{SO}(3)$ (see Lemma 9.4). This is no surprise since the symplectic structure on $G \times \mathfrak{t}_0^*$ (G arbitrary) was *defined* precisely by realizing $G \times \mathfrak{t}_0^*$ as a symplectic cross section of T^*G (Chap. 9). The reason for the difference in sign observed here is the following: In the rigid body phase space $\mathrm{T}^*\,\mathrm{SO}(3)$ (which we have been identifying with $\mathrm{T}\,\mathrm{SO}(3)$) the symmetry group $\mathrm{SO}(3)$ of H acts by cotangent lifting the left action on $\mathrm{SO}(3)$ defined by $g \cdot h \equiv gh$, while in Chap. 9 we cotangent lifted the left action on G defined by $g \cdot h \equiv hg^{-1}$.

In Example 10.8 we constructed action-group coordinates for geodesic motions on S^2, i.e., we constructed an equivalence of Hamiltonian $\mathrm{SO}(3)$-spaces $\phi : \mathrm{SO}(3) \times (0,\infty) \xrightarrow{\sim} \mathrm{T}S^2\backslash\mathbf{0}$. This map, which was defined by

$$\phi(\Lambda, p) \equiv (\Lambda e_1, p\Lambda e_2) \ ,$$

is symplectic, when $\mathrm{SO}(3) \times (0,\infty)$ is equipped with the symplectic structure $\omega^*_{\mathrm{SO}(3)}$. But as we are using $\mathrm{SO}(3) \times (0,\infty)$ as a realization of the symplectic cross section of $\mathrm{T}\,\mathrm{SO}(3)\backslash\mathbf{0}$, we are equipping $\mathrm{SO}(3) \times (0,\infty)$ with the symplectic structure $-\omega^*_{\mathrm{SO}(3)}$. To make ϕ *symplectic* we therefore redefine the symplectic structure on $\mathrm{T}S^2\backslash\mathbf{0}$ to be $+d\Theta$, where Θ is the one-form on $\mathrm{T}S^2\backslash\mathbf{0}$ defined in 8.13.

We leave it to the reader to verify that the action of S^1 on $\mathrm{T}S^2\backslash\mathbf{0}$ with respect to which Ψ is S^1-equivariant is given by

D.6 $$(\theta \bmod 2\pi) \cdot (q,v) \equiv (e^{-\theta\hat{\mu}}q, e^{-\theta\hat{\mu}}v) \ , \qquad \text{where} \qquad \mu \equiv q \times v/\,\|v\| \ .$$

The key observation at this point is that the orbits of this S^1 action are the fibers of the map $\mathbf{J} : \mathrm{T}S^2\backslash\mathbf{0} \to \mathbb{R}^3\backslash\{0\}$ defined by $\mathbf{J}(q,v) \equiv q \times v$ (not to be confused with the momentum map above of the same name). But we have already seen in Chap. 8 that $\mathbf{J}$ is the momentum map for a Hamiltonian action of $\mathrm{SO}(3)$ on $\mathrm{T}S^2\backslash\mathbf{0}$. Actually, since we have redefined the sign of the symplectic structure on $\mathrm{T}S^2\backslash\mathbf{0}$, the momentum map of this action is, in the current context, $-\mathbf{J}$. In particular, since $-\mathbf{J}$ is equivariant, it is *Poisson* if we equip $\mathbb{R}^3 \cong \mathfrak{so}(3)^*$ with the positive Lie-Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_+$ (see, e.g., Marsden and Ratiu (1994, Chap. 10)). It follows that $\mathbf{J}$ is Poisson if we equip $\mathbb{R}^3$ with the negative Lie-Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_- \equiv -\{\,\cdot\,,\,\cdot\,\}_+$. This structure is given by

D.7
$$\{f,h\}_-(y) \equiv -y \cdot \nabla f(y) \times \nabla h(y) \ .$$

Since $\phi : \mathrm{SO}(3) \times (0,\infty) \to \mathrm{T}S^2\backslash\mathbf{0}$ is symplectic, the composite

$$\rho \equiv \mathbf{J} \circ \phi : \mathrm{SO}(3) \times (0,\infty) \to \mathbb{R}^3\backslash\{0\}$$

is also a Poisson map. This map is also a surjection, whose fibers are the S^1-orbits in $\mathrm{SO}(3) \times (0,\infty)$ (by the S^1-equivariance of ϕ). Thus $(\mathbb{R}^3\backslash\{0\}, \{\,\cdot\,,\,\cdot\,\}_-)$ is a realization of the Poisson manifold $\mathrm{SO}(3) \times (0,\infty)/S^1$, and $\rho : \mathrm{SO}(3) \times (0,\infty) \to \mathbb{R}^3\backslash\{0\} \cong \mathrm{SO}(3) \times (0,\infty)/S^1$ is a realization of the natural projection.

Since $H|F$ is T-invariant ($T \cong S^1$), the Hamiltonian $\psi^* H$ (which represents the restriction of H to the symplectic cross section) is S^1-invariant, dropping via ρ to a function on $\mathbb{R}^3\backslash\{0\}$. Indeed, denoting the standard coordinate functions on $\mathbb{R}^3$ by y_1, y_2, y_3, we have $\psi^* H = h \circ \rho$, where $h : \mathbb{R}^3\backslash\{0\} \to \mathbb{R}$ is given by

$$h = \frac{1}{2I_1} y_1^2 + \frac{1}{2I_2} y_2^2 + \frac{1}{2I_3} y_3^2 \ .$$

The reader will readily verify this fact after observing that

$$\rho(\Lambda, p) = \mathbf{J}(\Lambda e_1, p\Lambda e_2) = \Lambda e_1 \times (p\Lambda e_2) = p\Lambda e_3 \ .$$

Hamiltonian vector fields on $(\mathbb{R}^3\backslash\{0\}, \{\,\cdot\,,\,\cdot\,\}_-)$ take the form

$$X_f(y) = -y \times \nabla f(y) \ ,$$

so that the equations of motion on $\mathbb{R}^3\backslash\{0\}$ corresponding to Hamiltonian h are

D.8
$$\begin{aligned}\dot{y}_1 &= \left(\frac{1}{I_2}-\frac{1}{I_3}\right) y_2 y_3\\ \dot{y}_2 &= \left(\frac{1}{I_3}-\frac{1}{I_1}\right) y_3 y_1\\ \dot{y}_3 &= \left(\frac{1}{I_1}-\frac{1}{I_2}\right) y_1 y_2 \ .\end{aligned}$$

These are precisely the familiar Lie-Poisson reduced rigid body equations (see, e.g., *op. cit.*), albeit obtained via a rather convoluted route. Rather than Poisson reduce the full system directly, we have obtained these equations by doing Poisson reduction in (a realization of) the symplectic cross section.

The axisymmetric case

Before summarizing the above results, we need to restrict attention to an appropriate open subset of $T\,SO(3)\backslash\mathbf{0}$, in anticipation of applying Theorem 12.2.

Assume the body is axisymmetric, $I_1 = I_2 \equiv I$ say. Then, writing $z(t) \equiv y_1(t)+iy_2(t) \in \mathbb{C}$, the general solution to the equations D.8 is

D.9
$$y_3(t) = y_3(0)$$

D.10
$$z(t) = e^{i\sigma y_3(0)t} z(t) \ ,$$

D.11
$$\text{where} \qquad \sigma \equiv \left(\frac{1}{I_3}-\frac{1}{I}\right) \ .$$

Points on the y_3-axis or on the y_1-y_2 coordinate plane are equilibria. The remaining solutions are periodic with period $(\sigma y_3(0))^{-1}$. The symplectic leaves of $(\mathbb{R}^3\backslash\{0\},\{\,\cdot\,,\,\cdot\,\}_-)$ are the spheres centered on the origin. The solution trajectories on a typical sphere are shown in Fig. 1.

To apply Theorem 12.2 we need to restrict attention to the half-space lying above or below the y_1-y_2 coordinate plane. For concreteness, we choose the upper half-space $\mathbb{R}^3_+ \equiv \{y \in \mathbb{R}^3 \mid y_3 > 0\}$. Let us now summarize our previous results, with the various constructions restricted to the appropriate open subsets:

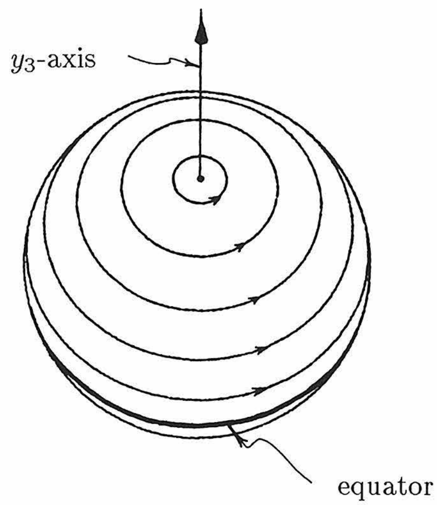

FIGURE 1. The reduced dynamics of an axisymmetric rigid body on a typical symplectic leaf (sphere). Points on the equator are equilibria.

D.12 Proposition (Poisson reduction in the symplectic cross section) *Define the open set $P \subset \mathrm{T\,SO}(3)\backslash\mathbf{0}$ by*

$$P \equiv \{\boldsymbol{\lambda}(\Lambda, m) \mid m_3 > 0\}$$

and equip P with the symplectic structure $\omega \equiv -d\Theta$, where Θ is the one-form on $\mathrm{T\,SO}(3)$ defined by 11.2. Then P is invariant with respect to the action of $\mathrm{SO}(3)$ defined by D.2, which has momentum map $\mathbf{J} : P \to \mathbb{R}^3$ defined by D.3.

Define the open set $\mathrm{SO}(3)^\circ \subset \mathrm{SO}(3)$ by

$$\mathrm{SO}(3)^\circ \equiv \{\Lambda \in \mathrm{SO}(3) \mid \Lambda_{33} > 0\} ,$$

where $\Lambda_{33} \equiv \Lambda e_3 \cdot e_3$. Equip $F' \equiv \mathrm{SO}(3)^\circ \times (0, \infty)$ with the symplectic structure $\omega' \equiv -\omega^*_{\mathrm{SO}(3)}$, where $\omega^*_{\mathrm{SO}(3)}$ is the symplectic structure carried by $\mathrm{SO}(3) \times (0,\infty) \supset \mathrm{SO}(3)^\circ \times (0, \infty)$ by virtue of being (up to the obvious identifications) the action-group model space of $\mathrm{SO}(3)$ (see Lemma 9.4). Explicitly, $\omega' = d\Theta'$, where Θ' is the one-form on $\mathrm{SO}(3) \times (0, \infty)$ given in Equation 11.3. Let $T \cong S^1$ act on F' according to D.5. This action has momentum map $\mathbf{J}' : F' \to \mathbb{R}$ defined by $\mathbf{J}'(\Lambda, p) \equiv p$. Then the map $\psi : F' \to P$ defined by

D.4 is an equivalence from $(F', \omega', T, \mathbf{J}')$ *onto the symplectic cross section* $(F, \omega_F, T, \mathbf{J}^F)$ *of* $(P, \omega, G, \mathbf{J})$. *The map* $\rho : F' \to \mathbb{R}^3_+$ *defined by*

$$\rho(\Lambda, p) \equiv p\Lambda e_3$$

is a Poisson map onto $(\mathbb{R}^3_+, \{\cdot, \cdot\}_-)$, *where* $\{\cdot, \cdot\}_-$ *is defined by D.7. Moreover, the fibers of* ρ *are the* T*-orbits in* F', *so that* $(\mathbb{R}^3_+, \{\cdot, \cdot\}_-)$ *is a realization of the Poisson manifold* F'/T *and* ρ *is a realization of the natural projection* $F' \to F'/T$. *Assuming* $I_1 = I_2 \equiv I$, *we have* $H' \equiv \psi^* H = h \circ \rho$, *where* $h : \mathbb{R}^3_+ \to \mathbb{R}$ *is defined by*

$$h \equiv \frac{1}{2I}(y_1^2 + y_2^2) + \frac{1}{2I_3} y_3^2 \ .$$

Constructing the S^1 extension in the symplectic cross section

Let $(F', \omega', T, \mathbf{J}')$ be the realization of the symplectic cross section F of $P \subset \mathrm{T\,SO}(3)$ given in the proposition above. The momentum map $\mathbf{J}'$ has image $\mathfrak{t}^*_{\mathbf{J}} \equiv (0, \infty)$, and $\mathbf{J}' : F' \to (0, \infty)$ factors through $\rho : F' \to \mathbb{R}^3_+$, delivering a map $\mathbf{j}' : \mathbb{R}^3_+ \to (0, \infty)$ such that $\mathbf{J}' = \mathbf{j}' \circ \rho$. Indeed $\mathbf{j}'(y) = \|y\|$. Each symplectic leaf $\Sigma_\mu = (\mathbf{j}')^{-1}(\mu)$ ($\mu \in (0, \infty)$) is an open hemisphere of radius μ.

If we take

$$Z \equiv \{(0, 0, t) \mid t > 0\} \ ,$$

then the Hamiltonian T-space $(F', \omega', \mathbb{T}, \mathbf{J}')$ satisfies hypotheses 1–3 of Theorem 12.2. The function $I : F'/T \cong \mathbb{R}^3_+ \to \mathbb{R}$ defined by

$$I(y) \equiv y_3 - n\,\|y\|$$

satisfies conditions 4 and 5 of 12.2 for any choice of integer n. We choose $n \equiv 0$ (because this choice will lead to the S^1 extension corresponding to the body symmetry; a different choice leads to an equally valid, but different, S^1 action[1]). With this choice, we have $I \circ \rho(\Lambda, p) =$

[1]For example, if we choose $n = 1$, then $I(y) = A/(2\pi\,\|y\|)$, where A denotes the area enclosed by the X_h-orbit parking through y (see Remark 12.3). In that case, the function $I \circ \rho \circ \psi^{-1}$ (extended by the SO(3) action to a function on P) corresponds to an 'action integral' for the rigid body appearing in Tantalo (1994).

$p\Lambda_{33}$. The corresponding function on the symplectic cross section F is $\mathbf{K} \equiv I \circ \rho \circ \psi^{-1}$. One computes

$$\mathbf{K}(\boldsymbol{\rho}(\Lambda, pe_3)) = p\Lambda_{33} \qquad (\Lambda \in \mathrm{SO}(3)^\circ,\ p \in (0,\infty)) \ ,$$

where $\boldsymbol{\rho}$ is the map defined at the beginning of Chap. 8 (not to be confused with the map ρ giving us a realization of the quotient map $F' \to F'/T$).

By Corollary 12.9, we obtain a momentum map for the sought after S^1 extension by extending $\mathbf{K}$ to a function on P in the prescribed manner. Let $A \in \mathrm{SO}(3)$ and $m \in \mathbb{R}^3_+$ be arbitrary, and let us determine what the value of $\mathbf{K}(\boldsymbol{\lambda}(A,m))$ should be, according to the prescription of 12.9. Define $\Lambda \equiv \exp((m \times e_3/\,\|m\|)\hat{})$ and $p \equiv \|m\|$, so that $\Lambda m = pe_3$. Then if $g \equiv A\Lambda^{-1}$, we compute using D.2 and 11.1,

$$\boldsymbol{\lambda}(A,m) = g \cdot \boldsymbol{\lambda}(\Lambda, m) = g \cdot \boldsymbol{\rho}(\Lambda, \Lambda m) = g \cdot \boldsymbol{\rho}(\Lambda, pe_3) \ .$$

Therefore, we set

$$\mathbf{K}(\boldsymbol{\lambda}(A,m)) \equiv \mathbf{K}(\boldsymbol{\rho}(\Lambda, pe_3)) = p\Lambda_{33} = \|m\| \left(\frac{m_3}{\|m\|}\right) = m_3 \ .$$

But this is precisely the momentum map for the familiar S^1 action corresponding to the body symmetry (which in our case is symmetry about the e_3-axis). Since a Hamiltonian S^1 action is determined by its momentum map, this proves our initial claim.

APPENDIX E

The Symplectic Leaf Correspondence Theorem

Throughout this appendix 'smooth' means C^∞.

Foliations with singularities

The following discussion of generalized distributions, and their associated foliations, closely follows Libermann and Marle (1987, Appendix 3). These notions were introduced and studied by Sussmann (1973), Stefan (1973), and others[1].

A *generalized distribution* on a smooth manifold M is a subset D of the tangent bundle $\mathrm{T}M$ with the property that $D(x) \equiv D \cap \mathrm{T}_xM$ is a subspace of T_xM at every $x \in M$. The adjective 'generalized' (which we will omit in the sequel) refers to the fact that we allow the dimension of $D(x)$ to be x dependent. We write $\hat{D}$ for the set of locally defined smooth vector fields X on M satisfying $X(x) \in D(x)$ wherever $X(x)$ is defined. A distribution D is deemed *smooth* if for all $x \in M$ there exist smooth vector fields $X_1, \ldots, X_k \in \hat{D}$ defined in a neighborhood of x such that $\{X_1(x), \ldots, X_k(x)\}$ is a basis for $D(x)$.

We are ultimately interested in the following special case:

E.1 Example Let Q be a Poisson manifold and B the associated Poisson tensor, viewed as a vector bundle morphism $B : \mathrm{T}^*Q \to \mathrm{T}Q$. Next, define the *characteristic distribution* D on Q by $D \equiv B(\mathrm{T}^*Q)$. If $x_1, \ldots, x_n$ are local coordinate functions defined in a neighborhood of a point $q \in Q$, then $D(q)$ is generated by $\{X_{x_1}(q), \ldots, X_{x_j}(q)\}$, where $X_f(x) \equiv B(d_xf)$. By removing appropriate vector fields if necessary, we may suppose that $\{X_{x_1}(q), \ldots, X_{x_j}(q)\}$ is in fact a basis for $D(q)$. Since the X_{x_j} are smooth and belong to $\hat{D}$, the characteristic distribution D is smooth.

[1] A vast bibliography on the subject of foliations has been supplied by Tondeur (1997).

An *integral manifold* of a distribution D on a manifold M is an immersed connected submanifold $\Sigma \subset M$ such that $\mathrm{T}_x\Sigma = D(x)$ at all $x \in \Sigma$. A distribution D is *integrable* if through every point $x \in M$ there exists an integral manifold of D.

E.2 Theorem and Definition *Let D be a smooth integrable distribution on a smooth manifold M. Given $x, y \in M$, write $x \sim y$ if one can find an integral manifold of D containing both x and y. Then $\sim$ is an equivalence relation on M, defining a partition of M called the* **(generalized) foliation** *associated with D. Equivalence classes of $\sim$ are immersed submanifolds of M and are called* **leaves** *of the foliation associated with D. Each leaf is maximal in the sense that: (i) The leaf through a point contains all integral manifolds passing through that point; and (ii) No integral manifold properly contains any leaf.*

For a proof of E.2, see Libermann and Marle (1987, p. 385).

If $\rho : P \to Q$ is a smooth map, and D is a distribution on Q, then we define a distribution ρ^*D on P by declaring $v \in \rho^*D(x)$ if $\mathrm{T}\rho \cdot v \in D(\rho(x))$. We call ρ^*D the *pull-back* of D. Smooth integrable distributions, and their associated foliations, are well-behaved under appropriate pull-backs:

E.3 Theorem *Let $\rho : P \to Q$ be a submersion and assume that the preimage of connected sets under ρ are always connected. Let D be a smooth integrable distribution on Q. Then the pull-back distribution ρ^*D is smooth and integrable, and the leaves of the foliation on P associated with ρ^*D are precisely those nonempty sets of the form $\rho^{-1}(\Sigma)$, where Σ is a leaf of the foliation associated with D.*

The proof of E.3 is postponed to the end of this appendix. Theorem E.3 will be crucial in the sequel.

Symplectic leaves and dual pairs

E.4 Theorem (Symplectic Stratification Theorem)

The characteristic distribution on a Poisson manifold Q (see Example E.1 above) is integrable. Furthermore, each leaf of the associated foliation inherits a symplectic structure *from the Poisson structure on Q in the following way: Given a realization $i : S \hookrightarrow Q$ of Σ (meaning that i is an injective immersion with $i(S) = \Sigma$), there exists a unique symplectic structure on S with respect to which i is a Poisson map.*

Recall that a smooth map of Poisson manifolds $\rho : P \to Q$ is *Poisson* if $\{f, h\} \circ \rho = \{f \circ \rho, h \circ \rho\}$ for all locally defined functions f, h on Q. The leaves of the foliation in Theorem E.4 are called the *symplectic leaves* of Q. The above formulation of the Symplectic Stratification Theorem appears in Libermann and Marle (1987, p. 130). Its earliest incarnation is due to Kirillov (1976).

E.5 Definition Let P be a symplectic manifold and Q_1, Q_2 Poisson manifolds. A pair of Poisson maps $Q_1 \xleftarrow{\rho_1} P \xrightarrow{\rho_2} Q_2$ is called a *dual pair* if $\ker T\rho_1$ and $\ker T\rho_2$ are symplectically orthogonal distributions. This pair is a *full* dual pair if ρ_1 and ρ_2 are surjective submersions.

E.6 Example Suppose a Lie group G acts on P in a Hamiltonian fashion, and let $\mathbf{J} : P \to \mathfrak{g}^*$ be a (not necessarily equivariant) momentum map. Then $\mathfrak{g}^* \xleftarrow{\mathbf{J}} P \to P/G$ is a dual pair. If G acts freely and properly, then the image $\mathfrak{g}^*_{\mathbf{J}}$ of $\mathbf{J}$, is an open subset of $\mathfrak{g}^*$, and $\mathfrak{g}^*_{\mathbf{J}} \xleftarrow{\mathbf{J}} P \to P/G$ is a *full* dual pair, with respect to an appropriate differentiable structure on P/G.

The Symplectic Leaf Correspondence Theorem

The main result we wish to recall in this appendix is a natural one-to-one correspondence between the symplectic leaves in each leg of a dual pair $Q_1 \xleftarrow{\rho_1} P \xrightarrow{\rho_2} Q_2$. This result has been stated by Alan Weinstein, who sketched a proof under a slight variation of our hypotheses (Weinstein, 1983). We have been unable to fill in the details of Weinstein's original proof, however, and moreover we are unaware of a complete and correct proof in the literature. Although detailed proofs are probably known, we supply here for the record a detailed proof, which we hope is also correct. With Theorem E.3 in hand, the result follows quite naturally. We begin by understanding how ρ_1 and ρ_2 pull back the characteristic distributions on Q_1 and Q_2.

E.7 Lemma *Let $\rho : P \to Q$ be a Poisson submersion and let D denote the characteristic distribution on Q. If P is symplectic, and ω denotes its symplectic structure, then*

$$\rho^* D = \ker \mathrm{T}\rho + (\ker \mathrm{T}\rho)^\omega \ .$$

PROOF. Let $B_P : \mathrm{T}^*P \to \mathrm{T}P$ and $B_Q : \mathrm{T}^*Q \to \mathrm{T}Q$ denote the Poisson tensors. Let $\rho_* : \operatorname{Ann}\ker \mathrm{T}\rho \to \mathrm{T}^*Q$ denote the map sending $d_x(f \circ \rho)$ to $d_{\rho(x)}f$, for any locally defined function f on Q (Ann denotes annihilator). Then $\rho : P \to Q$ being Poisson means $B_Q \circ \rho_* = \mathrm{T}\rho \circ B_P$. Fix $x \in P$. Then

$$\begin{aligned} \mathrm{T}\rho\,(\,(\ker \mathrm{T}_x\rho)^\omega\,) &= \mathrm{T}\rho\,(\,B_P(\operatorname{Ann}\ker \mathrm{T}_x\rho)\,) \\ &= B_Q\,(\,\rho_*(\operatorname{Ann}\ker \mathrm{T}_x\rho)\,) = B_Q(\mathrm{T}^*_{\rho(x)}Q) = D(\rho(x)) \ . \end{aligned}$$

Here we have used the fact that ρ_* maps $\operatorname{Ann}\ker \mathrm{T}\rho$ *onto* (in fact isomorphically onto) $\mathrm{T}^*_{\rho(x)}Q$. This follows immediately from its definition.

The above calculation demonstrates that $(\mathrm{T}_x\rho)^{-1}(D(\rho(x))) = (\ker \mathrm{T}_x\rho) + (\ker \mathrm{T}_x\rho)^\omega$ $(x \in P)$, from which the claim of the lemma follows. □

If $Q_1 \xleftarrow{\rho_1} P \xrightarrow{\rho_2} Q_2$ is a full dual pair, and D_1, D_2 denote the characteristic distributions on Q_1, Q_2, then $\rho_1^* D_1 = \rho_2^* D_2$, by Lemma E.7. Let us suppose that ρ_1 and ρ_2 satisfy the connectedness hypothesis of Theorem E.2. Then by that theorem, each leaf of the generalized foliation associated with $\rho_1^* D_1 = \rho_2^* D_2$ is simultaneously the preimage of a symplectic leaf in Q_1 and a symplectic leaf in Q_2. This establishes a one-to-one correspondence between the symplectic leaves in $\rho_1(P) \subset Q_1$ and the symplectic leaves in $\rho_2(P) \subset Q_2$. Since we assume the dual pair is full, $\rho_1(P) = Q_1$ and $\rho_2(P) = Q_2$. This furnishes a proof of the result we are after:

E.8 Theorem (Symplectic Leaf Correspondence Theorem) *Let P be a symplectic manifold and $Q_1 \xleftarrow{\rho_1} P \xrightarrow{\rho_2} Q_2$ a full dual pair. Assume that each leg $\rho_j : P \to Q_j$ $(i = 1, 2)$ satisfies the property that preimages of connected sets are connected. Let $\mathcal{F}_j$ denote the* *set of symplectic leaves in Q_j . Then there exists a bijection $\mathcal{F}_1 \to \mathcal{F}_2$ given by*

$$\Sigma_1 \mapsto \rho_2(\rho_1^{-1}(\Sigma_1)) \ ,$$

having inverse

$$\Sigma_2 \mapsto \rho_1(\rho_2^{-1}(\Sigma_2)) \ .$$

Keeping Example E.6 in mind, an immediate corollary is Lemma 12.1 stated in our discussion on converting dynamic integrability into geometric integrability.

Details

We now turn to the proof of Theorem E.3. We make use of the following observation:

E.9 Lemma *Let $\rho : P \to Q$ be a surjective submersion of smooth manifolds, and let $S \subset P$ be a compact pathwise connected set. Then S has a connected open neighborhood U with the property that the restriction $\rho : U \to \rho(U)$ is a locally trivial fiber bundle.*

PROOF. Since S is compact and smooth finite dimensional manifolds are locally compact, we can cover S by a finite number of open subsets $U_1, \dots, U_N$, each having compact closure in P. Since S is pathwise connected, there exists an (open) connected component U of the union $\cup_j U_j$ that contains S. The component U will have compact closure, so that the restriction $\rho : U \to \rho(U)$ will be *proper*. This finishes the proof, since any proper surjective submersion is a locally trivial fiber bundle; see, e.g., Bates and Śniatycki (1992). □

A PROOF OF THEOREM E.3. We first verify that $\rho^* D$ is smooth. Let $x_0 \in P$ be given and let U be a connected neighborhood of x_0 such that the restriction $\rho : U \to \rho(U)$ is a locally trivial fiber bundle. The existence of such a U is guaranteed by Lemma E.9 above. In fact, restricting U if necessary, we may suppose that U is a globally trivial fiber bundle. It follows that there exist vector fields $X_1, \dots X_k$ defined on U (restricting U if necessary) that are tangent to the fibers of $\rho|U$ (and therefore belonging to $\widehat{\rho^* D}$) and are such that $\{X_1(x), \dots, X_k(x)\}$ is a basis for the tangent space of the fiber through x, for all x in some open neighborhood of x_0. Since D is smooth, there exist (restricting U if necessary) vector fields $Y_1, \dots, Y_m$ $(k+m = \dim P)$ on $\rho(U)$ such that $\{Y_1(\rho(x_0)), \dots, Y_m(\rho(x_0))\}$ is a basis for $D(\rho(x_0))$. Equip the locally trivial bundle $\rho : U \to \rho(U)$ with the structure of an Ehresmann connection, and use this connection to lift the vector fields $Y_1, \dots, Y_m$ to horizontal vector fields $\tilde{Y}_1, \dots, \tilde{Y}_m$ defined on U. Since $\mathrm{T}\rho \cdot \tilde{Y}_j(x) = Y_j(\rho(x))$ $(x \in U)$, it follows that $\tilde{Y}_j \in \widehat{\rho^* D}$. By construction $\{X_1(x_0), \dots, X_k(x_0), \tilde{Y}_1(x_0), \dots, \tilde{Y}_m(x_0)\}$ is a basis of $\rho^* D(x_0)$. Since $x_0 \in P$ was arbitrary, this proves that $\rho^* D$ is smooth.

We next demonstrate that $\rho^* D$ is integrable. Let $x_0 \in P$ be given, and let $\Sigma \subset Q$ be an integral manifold of D passing through $\rho(x_0)$. We may suppose that Σ is a regular submanifold. (If not, replace Σ by $i(U)$ where $U \subset S$ is an appropriate open set and $i : S \hookrightarrow Q$ denotes some realization of the immersed submanifold Σ.) By the preimage theorem, the set $\rho^{-1}(\Sigma)$ is a (regular) submanifold of P. This submanifold (which is connected since Σ is, by our hypotheses on ρ) is an integral manifold of $\rho^* D$. Since it contains $x_0 \in P$, which was arbitrary, this establishes the integrability of $\rho^* D$.

It remains to show that the leaves of the foliation on P associated with $\rho^* D$ are preimages of leaves of the foliation on Q associated with D. Let $Z \subset P$ be a leaf, and let $x_0 \in Z$ be arbitrary. Denote by Σ the leaf of the foliation on Q containing $y_0 \equiv \rho(x_0)$. We need to show that $Z = \rho^{-1}(\Sigma)$. We begin by showing $\rho^{-1}(\Sigma) \subset Z$.

Let $x \in \rho^{-1}(\Sigma)$ be given and define $y \equiv \rho(x) \in \Sigma$. Let $i : S \hookrightarrow Q$ be a realization of the immersed submanifold Σ, and let $s_0, s \in S$ be the points such that $i(s_0) = y_0$ and $i(s) = y$. Since S is connected (Σ is an integral manifold) there exists a continuous curve in S, defined on some closed interval, joining s_0 and s. Since the image of this curve is compact, there exists a connected open neighborhood U of this image having compact closure in S (the relevant argument already appearing in our proof of Lemma E.9). In particular, $\Sigma' \equiv i(U) \subset \Sigma$ is an *embedded* integral manifold of D containing y_0 and y. By the preimage theorem, $\rho^{-1}(\Sigma')$ is a regular submanifold of P. This submanifold (which is connected since Σ' is, by our hypotheses on ρ) is an integral manifold of $\rho^* D$, and contains both x_0 and x. Therefore x_0 and x belong to the same leaf, namely Z. In particular, $x \in Z$. Since $x \in \rho^{-1}(\Sigma)$ was arbitrary, this establishes the inclusion $\rho^{-1}(\Sigma) \subset Z$.

Finally, let us prove the reverse inclusion $Z \subset \rho^{-1}(\Sigma)$. Let $x \in Z$ be arbitrary. Let $i : S \hookrightarrow P$ be a realization of the immersed submanifold Z, and let $\gamma : [0,1] \to S$ be a continuous curve joining s_0 and s, where $i(s_0) = x_0$ and $i(s) = x$. By Lemma E.9, there exists a connected neighborhood U of $i(\gamma([0,1]))$ in P such that the restriction $\rho : U \to \rho(U) \subset Q$ is a locally trivial fiber bundle. Let $V \subset S$ be an open connected neighborhood of the compact set $\gamma([0,1])$ that is small enough to guarantee that the integral manifold $Z' \equiv s(V)$ of $\rho^* D$ is a regular submanifold of P. For example, take V to be a neighborhood with compact closure, so that the restriction $i : V \to P$ is proper. The set Z' contains the image of the curve $\gamma' \equiv i \circ \gamma$, which joins x_0 and x. There exist numbers $t_0, \ldots, t_m$ with $0 = t_0 < t_1 < \cdots < t_m = 1$ such that each curve segment $\gamma'([t_{j-1}, t_j])$ $(1 \leqslant j \leqslant n)$ is contained in the domain $U_j \subset P$ of a fiber bundle chart $\varphi_j : U_j \to W_j \times F$ $(W_j \subset Q$, $\rho^{-1}(W_j) = U_j$, $F \equiv \rho^{-1}(x_0))$.

We claim that $x = \gamma'(t_m)$ is contained in $\rho^{-1}(\Sigma)$. We prove this by induction on the integer indexing the curve segments. Since $\gamma'(t_0) = x_0$ belongs to $\rho^{-1}(\Sigma)$, the induction is anchored. Let $1 \leqslant j \leqslant m$ be given and suppose that $\gamma'(t_{j-1}) \in \rho^{-1}(\Sigma)$. We need to show that $\gamma'(t_j) \in \rho^{-1}(\Sigma)$. The distribution $\rho^* D$ is represented on the connected chart image $W_j \times F$ by the distribution $D'(w,f) \equiv D(w) \oplus \mathrm{T}_f F$. One easily argues that the leaves of this distribution are of the form $\Sigma' \times F$, where Σ' is a leaf of the foliation on $W_j \subset Q$ associated with $D|W_j$. It follows that the leaves of the foliation on $U_j \subset P$ associated with $\rho^* D|U_j$ are of the form $\rho^{-1}(\Sigma')$, where Σ' is again some leaf of the foliation associated with $D|W_j$. Some connected component of the set $Z' \cap U_j$ contains $\gamma'([t_{j-1}, t_j])$. Let us call this component Z_j. Then Z_j is clearly an integral manifold of the foliation on U_j associated with $\rho^* D$. Therefore Z_j is contained in some leaf of this foliation (by the maximality of the leaves), i.e., in some set of the form $\rho^{-1}(\Sigma')$, where Σ' is a leaf of the foliation on W_j associated with $D|W_j$. In particular, $\rho(\gamma'(t_{j-1}))$ and $\rho(\gamma'(t_j))$ must lie on the same leaf Σ' of the foliation on W_j associated with $D|W_j$. But this leaf is certainly an integral manifold of the foliation on Q associated with D. Therefore, $\rho(\gamma'(t_j))$ and $\rho(\gamma'(t_{j-1}))$ lie on the same symplectic leaf in Q. This leaf is Σ since $\rho(\gamma'(t_{j-1})) \in \Sigma$, by the inductive hypothesis. So

$\gamma'(t_j) \in \rho^{-1}(\Sigma)$, which completes the induction. It follows that $x = \gamma'(t_m)$ lies in $\rho^{-1}(\Sigma)$. Since $x \in Z$ was arbitrary, this shows that $Z \subset \rho^{-1}(\Sigma)$. $\square$

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