PROBLEMS IN NETWORK THEORY

Thesis by

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I am also indebted to Nelson Upthegrove, who has assisted with the preparation of the figures.

ABSTRACT

After a brief philosophical consideration of the status and function of the theoretician in modern technology, the central objectives of the paper are stated; to investigate some of the restrictions on the design and synthesis of linear physical systems which are inherent in the mathematical constructs and methods by which such systems are studied. The principal tools are the Laplace transformation and the theory of functions of a complex variable.

It is shown that many of the commonly encountered generalizations to linear distributed-parameter systems of familiar lumped-parameter-system ideas are valid. These generalizations are perhaps intuitively obvious, but the details are gone through here once and for all.

The implications in regard to these matters of the principle of analytic continuation are considered. Tests are derived to enable one to decide whether or not there is any chance of realizing a prescribed transfer characteristic $T(jw)$, $w_1 < w < w_2$ (analytically expressed data), and what would be the consequences in terms of $T(s)$ outside this range. The paradox of the idealized low-pass filter is examined in this light. The questions are shown to be unanswerable in the case of graphically expressed data.

It is shown that the results of the study are in agreement with the allied work of others, and a problem of filter realization posed by Wallman is solved. The results of the investigations are summarized and reviewed in terms of what theoreticians can accomplish in general.

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I INTRODUCTION AND INFORMAL STATEMENT OF THE PROBLEMS TO BE CONSIDERED

In the time of Heaviside, it was really proper to think of engineers as being essentially "practicians" (R-1,2)*. The popular conception of an engineer (aside from an erroneous association with locomotive and hoist operators, which present-day professional engineers are still trying to live down) has generally been that of a vigorous young man in boots and rough-country clothing, busily engaged in supervising the construction of highways, bridges, dams, and power plants. Practical he had to be, and was, and is.

The emergence of the slide rule as an accepted symbol of engineering in mass-media advertising is very significant. It indicates the growing public awareness of the importance of a class of engineers who own no boots, don't always look busy, and do their work on paper or at the blackboard. If one inquires as to their function, one learns that they are concerned with the theoretical aspects of engineering problems; it seems only fair to dub them "theoreticians."

These classes of engineers are not intended to be mutually exclusive; it is abundantly evident that the individual gains in professional competence and effectiveness as he possesses the good features of both. The practician who ignores theory soon descends to the level of routine mediocrity; the theoretician who never concerns himself with

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* (R-1) refers to Reference #1 in the References following the text of this paper.

practice becomes intoxicated by his reveries and loses contact with reality.

The title of this paper being "Problems in Network Theory" and the contents being strictly non-experimental, it is appropriate to say a little bit about what theoreticians do, or try to do. Briefly, they seek to disprove the truth of the old adage "You never can tell till you try."

It is most certainly not their objective to revive discredited attempts to describe the world of observation solely by reasoning about it. Rather do they seek to expedite the solution of practical problems by rational deduction from the applicable body of theory, bearing always in mind the implicit restrictions on the theory which were imposed when it was itself induced from experimental observations. The "trying" has already taken place; the theoretician just makes use of this systematically stockpiled experience.

Some people might say he isn't really an engineer at all; if he is inclined to put on airs, he may call himself an applied physicist! Whatever he may be, he occupies an increasingly important place in present-day technology; the scope and complexity of modern technical enterprises are such that step-by-step experimental development is criminally uneconomical of time, manpower, and materials, not to mention money.

The successful theoretician finds that he is regarded by his practician colleagues with an attitude which (though certainly not reverence) partakes a little of awe. If he is wise, he will not pretend omniscience; this will make infinitely more bearable the seemingly inescapable day when he is caught with his integrals down. History records that the Delphic Oracle enjoyed a long period of popularity and prestige despite the fact that its predictions were couched in such evasive terms as to admit almost any interpretation whatsoever (R-3). Not so today's theoretician; his services will be esteemed or despised according as he is or is not willing and able to give definite, understandable answers which are both useful and right.

The questions which a theoretician may be asked are strange and diverse. This paper will consider how he might go about answering certain inquiries of the type "Can I do thus-and-so?"

It has been a long time since practicians have expended any serious effort on attempts to construct perpetual-motion machines. It is realized that such a device would function in complete contradiction of the laws of physics as we know them; it is gradually becoming acknowledged that many other apparently rosy prospects are inherently unattainable. Just what will be the effect of this disillusioning information on the popular mind, conditioned by Sunday-supplement science and hucksters' engineering to expect a never-ending stream of surpassing marvels, would be both difficult and interesting to surmise, but this will not be attempted here.

Ideally, the theoretician employs in his analysis the very best theory at his disposal; that which gives the most realistic picture of the world of observation. Unfortunately, this ultimate elegance must frequently be foregone. It often happens that the detailed, high-quality analysis that represents the theoretician's best work must be judged difficult or impossible of accomplishment with relation to the practical problem at hand.

Then the theoretician must make judicious approximations to get his analysis going again. The additional task of interpreting the results of the approximate solution may be difficult, but it is not really a novelty; even his very finest theory has uncertainties in it, of which he will take account if he is meticulous.

In studying the behavior of electrical systems of finite spatial extent, for example, one often supposes that the physical constitution of the circuit can be characterized by certain discrete (or "lumped") parameters (finite in number) which do not vary with time. The dependent variables are taken to be a finite set of more-or-less well-behaved functions of time; a finite number of specified "forcing functions" (of time) may be present, representing the influence on the network of the external universe. When the behavior of the circuit has been specified mathematically by application of the physics involved through Kirchhoff's Laws, one has a finite set of linear ordinary integrodifferential equations with constant coefficients. The boundary conditions are generally initial conditions, in which the values of certain of the dependent variables and their derivatives of various orders are specified at time $t = 0^+$ (R-4).

No realistically minded electrical engineer, practician or theoretician, will claim that these equations are an adequate description of all that can happen in the network. The equations are certainly wrong for currents too large or too small in magnitude, for example. At one extreme, the dissipation of Joule heat in the network's connections could be large enough to melt them. On the other hand, there is always present a background of minute "noise" currents in the circuit elements (due to the thermal agitation of the constituent molecules, for example) which would mask very small currents.

Yet it often happens that the equations are meaningful over an extensive intermediate range of current magnitudes; a vast amount of analysis has been carried out with just this supposition of "linearity," and its application has been most fruitful. This paper will be based exclusively on this assumption and its generalizations.

Once the theoretician has decided to work with the system of equations discussed above, he has many methods of mathematical analysis at his disposal. By the nature of the problem's mathematical formulation, it is very well suited to study by means of the Laplace transformation (R-5, R-6). The integrodifferential equations transform into linear algebraic ones; the boundary conditions are systematically and almost effortlessly introduced; and if the parameter s in the transform

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

is regarded as a complex variable $s = \sigma + jw$ ($j^2 = -1$, σ and w real), he can bring to bear on the problem the extraordinarily elegant and powerful theory of functions of a complex variable (R-7, R-8, R-9).

One of the principal results to be obtained by this sort of study (R-10) may be stated as follows. Suppose that a voltage source v(t) supplies a current i(t) when applied to a linear network without initial storage of energy in the magnetic fields of the inductors or the electric fields of the capacitors (that is, no such fields exist at $t = 0^+$). Furthermore, let v(t) and i(t) be sufficiently well-behaved functions of time so that they possess transforms V(s) and I(s) respectively. Then the function Z(s) defined by

$$Z(s) = \frac{V(s)}{I(s)}$$

is a quotient of polynomials in s with real coefficients which are determined only by the constant parameters of the network and the way its elements are connected.*

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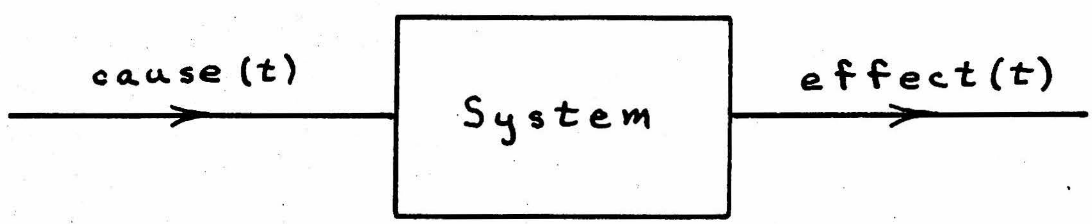
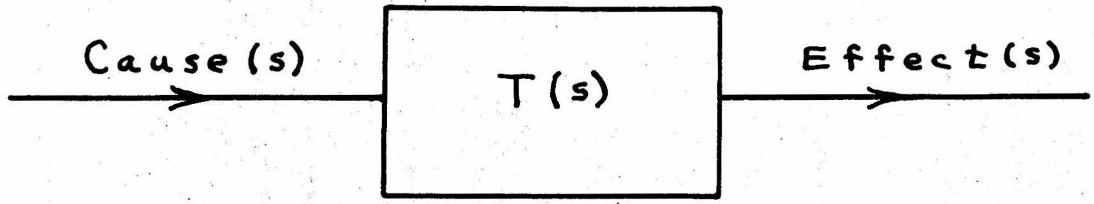
* An equivalent development of this result in a strictly mathematical formulation is given in (R-11).

In precise terminology, Z(s) is a rational function of the complex variable s; it is (with the possible exception of a finite number of poles) regular throughout the entire s-plane. For the present problem, the network is completely characterized by the function Z(s); this circumstance has given rise to a very extensive literature relating to the design and synthesis of finite linear lumped-parameter electrical networks to fulfill prescribed conditions (R-12).

Similar so-called "impedance" functions Z(s) can be found which relate the (transformed) current in one branch of the network to the (transformed) voltage drop across some other branch or chain of branches. The method is applicable to the study of many mechanisms and combined electro-mechanical systems; after a while, one acquires the habit of mentally replacing the time-domain block diagram



by the s-domain (or complex-frequency-domain) block diagram



where

$$T(s) = \frac{\text{Effect}(s)}{\text{Cause}(s)} .$$

This viewpoint is very helpful, but its limitations and qualifications must be borne in mind. An example of a very simple linear lumped-parameter mechanism not subject to it is given in (A-1).*

The first major investigation of this paper is carried out in Part II. It is an attempt to generalize the impedance notion (and most especially the function-theoretical aspects of the matter) to linear distributed-parameter systems. Such systems are by definition described by systems of linear partial integrodifferential equations with time-constant (but not necessarily space-constant) coefficients. The specification of admissible boundary conditions will be made in more detail in Part II; for the present, let it be noted that the physical system must be of finite spatial extent. The principal endeavor in Part II is to establish that such systems are described by impedances and other "network"** functions which are meromorphic functions of s; that is, that

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*(A-1) refers to Appendix #1, which is to be found following the text of this paper.

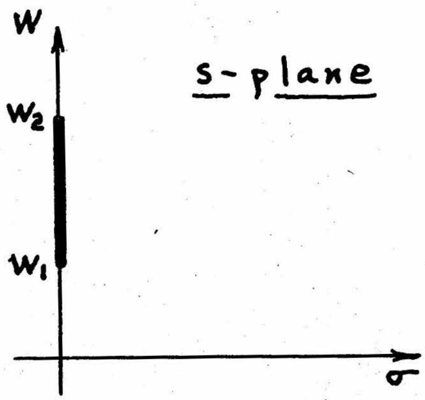
**Topologically, a macroscopically continuous physical system (such as an electrical transmission line) is scarcely a network. Nevertheless, this terminology will be applied to it here.

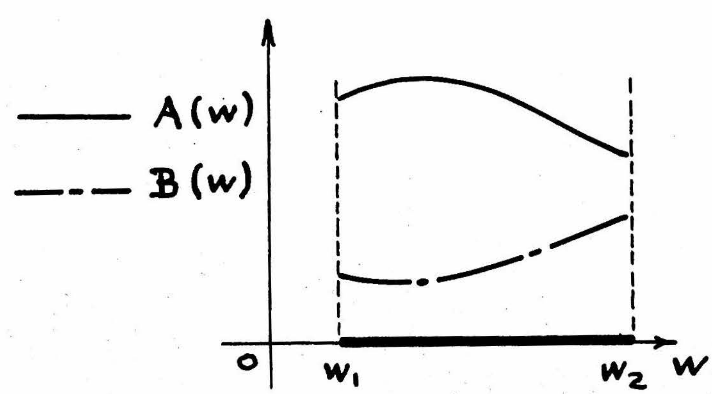
they are regular in the finite s-plane, with the possible exception of isolated poles.

The second major investigation of this paper is carried out in Part III. It is an evaluation of the significance and consequences of the application of the principle of analytic continuation to the generalized network functions developed and hypothesized in Part II.

The principle will be stated with care in Part III; for the present, let it be noted that specification of a meromorphic function in-the-small leads inevitably to a unique determination of the function's values in-the-large.

The results of Part II are immediately applicable to questions of the type "Can I do thus-and-so?" which the theoretician occasionally encounters. For example, suppose that it is desired to build a device which will have some transfer characteristic $T(jw) = A(w) + jB(w)$ for $w_1 < w < w_2$ as shown below $[A(w)$ and $B(w)$ are real functions of $w]$. Can $T(jw)$ be chosen with absolute arbitrariness?





The answer is "No!" The theoretician can point out that analytic continuation of the (meromorphic) transfer characteristic T(jw) (based on, say, the proposed functional values along the lower half of the line segment $\overline{w_1 w_2}$) leads to unique functional values along the upper half of the segment. If these values do not agree with the proposed ones, something is fishy. What really is wrong is that the T(jw) sought is in the nature of things impossible, at least insofar as things are described by the equations studied in Part II. In Part III, tests to exploit this criterion are c onsidered

Suppose, now, that the proposed T(jw) has passed its self-consistency tests with flying colors. What consequences will the practician let himself in for at frequencies w outside the range $w_1 < w < w_2$ in the event that he is successful in contriving a system to give him the desired T(jw) in this range?

The functional value of the (meromorphic) transfer characteristic T(jw) is uniquely determined at any point in the s-plane by the specification of functional values along the line segment $\overline{w_1 w_2}$. Part III considers means to exploit this relationship.

Finally, in Part IV, the results of the investigations are considered from the standpoint of practical applicability and in comparison with related tests and procedures previously developed.

II WHAT ARE GENERALIZED NETWORK FUNCTIONS LIKE?

A. Premonitory Introduction

Despite the asserted "theoreticality" of this paper, the attentive reader will soon encounter occasional defections from the strictest of mathematical standards. Rigor will fall by the wayside; proofs will degenerate into plausibility arguments; indeed, the principal investigation of Part II proving intractable in the general case, the result will remain only an hypothesis.

Another feature of this work probably quite galling to a mathematician is that what <u>does</u> get done is achieved under far stricter assumptions than are absolutely required. It is very likely that much or all of the ensuing development can be carried through with regard to a wider class of functions than those which are considered. To do so would probably be more difficult, though, and the extended validity of the results would not be of any interest in practical problems.

B. The Mathematical Functions to be Considered

A prominent contemporary mathematician (R-13) once defined a well-behaved function as one in which a physicist might be interested. Engineers have less curiosity, however; in considering a given problem, we shall restrict out attention to its <u>E-functions</u>.

Suppose that the equations of motion of the physical system under study involve at the highest the nth time derivative of some dependent variable, n = 0, 1, 2, ... (if n = 0, no time derivative appears). Then the E-functions of this problem possess the following properties:

(1) They are real-valued functions of the real variable t (time), defined "almost everywhere"* in $0 \leqq t < \infty$.

(2) They possess continuous time derivatives of the first (n - 1) orders throughout $0 \leqq t < \infty$ (significant only for $n > 0$).

(3) They possess nth-order time derivatives which are right-semicontinuous at t = 0, are sectionally continuous in every finite interval in $0 \leqq t < \infty$**, and possess only a finite number of maxima and minima in every finite interval in $0 \leqq t < \infty$.

(4) They and their time derivatives of the first n orders are of exponential order as $t \to \infty$***.

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*"Almost everywhere" here means everywhere except for a denumerable set of points having no finite point of accumulation.

**That is, in any finite interval $[T_1, T_2]$ such that $0 \leqq T_1 \leqq t \leqq T_2 < \infty$, an E-function's nth-order time derivative is continuous except for possibly a finite number of simple jump discontinuities (and is therefore bounded in this interval, by the way).

***That is, for any E-function (and each of its first n time derivatives) f(t), there exist finite, real, non-negative constants M_f, a_f, and T_f such that $|f(t)| \leqq M_f \exp(a_f t)$ for all $t \geqq T_f$ {written "$f(t) = 0\,[\exp(a_f t)]$"}. With regard to the consequences of (3) above, T_f can be taken equal to zero for all the f(t) of present interest. That the same M_f will not suffice for any E-function and <u>all</u> of its derivatives is apparent upon consideration of $f(t) = \exp(10t)$, for $f^{(m)}(0) = 10^m$. One might surmise that the same minimal a_f would suffice for any E-function and <u>all</u> of its derivatives, but it has not been possible to prove this.

It is to be noted that a given function may be an E-function relative to one problem but not relative to others. For instance, $k(t) = t^{\frac{1}{2}}$ possesses a derivative which is unbounded as $t \to 0^+$. $k(t)$ is not an E-function if the problem under consideration involves $k'(t)$ or higher time derivatives, but $k(t)$ __is__ an E-function if the problem is free of such derivatives. Of course, functions possessing suitably well-behaved derivatives of __all__ orders are E-functions relative to __any__ problem.

This class of relatively docile functions is intended to include adequate descriptions of the members of what might be called the class of pointer readings in the given problem. These P-variables are the measures of the (macroscopic) physical observables in the engineer's (and scientist's) world.

The class of E-functions falls well within Doetsch's class of L-functions and, indeed, within his class of L_a-functions (R-14). Much use will be made of his results.

It is worthwhile to look at some L-functions which cannot possibly be E-functions of any problem. $q(t) = t^{-\frac{1}{2}}$ is an L-function having the transform $Q(s) = \Gamma(\frac{1}{2})/s^{\frac{1}{2}}$ (R-15). $q(t)$ is unbounded as $t \to 0^+$, and it is perfectly certain that no such P-variable has meaning. Considered from the standpoint of its operational definitions, no physical theory can tolerate infinities, even though they are improperly absolutely integrable.

There are apparent exceptions to this principle in both mechanics and electrical engineering, however. Many problems involving large forces acting for short times are most expeditiously handled by considering momentum changes under the application of instantaneous impulses, not mentioning forces at all. Gardner and Barnes (R-16) emphasize the definitive character of a network's Green's function, or its response to an impulsive (or Kirchhoff-Dirac) input. While this can only be approximated in practice, the concept is undeniably of great utility.

The face-saving way out of the theoretical difficulty here is only now becoming generally known (R-17, R-18, R-19). The physical impossibility of performing true impulsive testing still remains, though. All things considered, there is no loss of useful generality in excluding from attention L- (and "quasi-L-") functions which are unbounded in the neighborhood of any finite t.

Another L-function which is excluded from the family of possible E-functions is that old reliable horror $g(t) = \sin(1/t)$, $t > 0$ (R-20). No matter how g(0) is defined, this bounded function fails to have right-semi-continuity at $t = 0$; its derivatives are unbounded near $t = 0^+$; and it possesses an infinite number of maxima and minima in this vicinity. A good physical reason for feeling that no P-variable can correspond to g(t) is that the oscillations of the function in the vicinity of $t = 0^+$ occur at "frequencies" so outlandishly high as not only to outstrip the classical theory but to transcend present-day quantum theory as well. Again, no deprivation is felt in excluding such L-functions from attention here.

The two exceptional functions above appear to represent the principal classes of L-functions which cannot possibly be E-functions. There is one important aspect of the correspondence between the E-functions and the P-variables of a given problem which requires attention, though. An E-function need be only sectionally continuous in any finite interval (if n = 0). Can one reasonably expect to observe pointer readings which are discontinuous in time?

This question cannot be answered merely by mentioning the mechanical inertia characteristic of existing indicating elements such as galvanometer mirrors and cathode-ray-tube electron streams. The query really relates to the underlying physical observables themselves.

A meaningful answer to the question seems to call for a more extensive and profound philosophical investigation than can be attempted here. There is every inclination to reply in the negative. From this standpoint, discontinuous E-functions must be regarded as only idealized approximate descriptions of their associated P-variables; introduced because they greatly facilitate analysis, and legitimized only by the useful fruit of their employment. The problem of degradation-by-approximation arising here is not a new one to the theoretician, though, as was brought out in Part I.

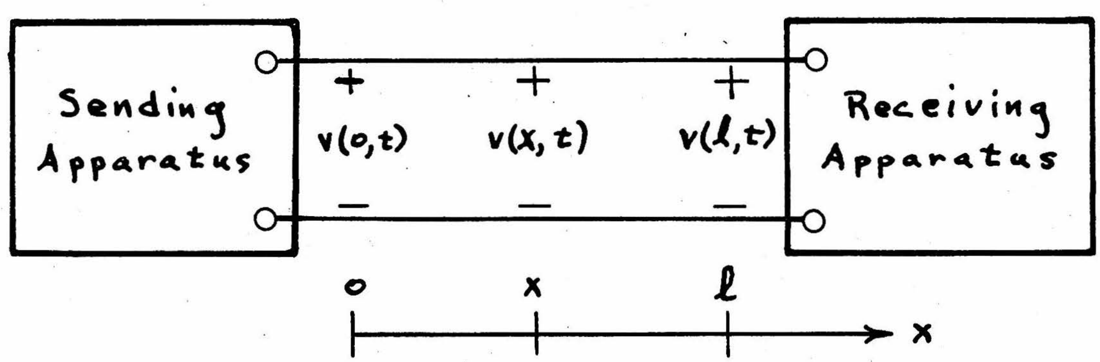
Some doubt can be raised regarding the descriptive adequacy of the family of possible E-functions. One has no difficulty in imagining a F-variable to be expressed by $h(t) = \exp(t^2)$. Yet $h(t)$ is not of exponential order as $t \to \infty$, and it is not possible to carry out the process of Laplace transformation on it (R-21). What is to be done?

The only way out seems to be through "mutilation" of the function (R-22). One introduces an E-function which is equal to $h(t)$ for $0 \leqq t \leqq T$ and has some convenient value (perhaps zero) for $T < t < \infty$ which is of exponential order as $t \to \infty$. If the problem under consideration is one to which classical cause-and-effect relationships (with their sequence in time) apply, this E-function will be a perfectly suitable representation of the troublesome F-variable above for $0 \leqq t \leqq T$. One need only choose T sufficiently large at the start to cover the period of interest.

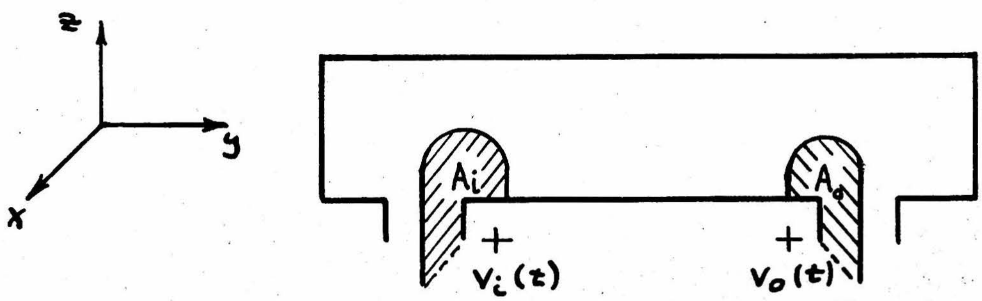
C. The Mathematical Specification of Physical Systems

All linear physical systems are described by systems of linear integrodifferential equations (ordinary or partial) with time-constant coefficients, as stated in Part I. If the system is composed entirely of lumped-parameter elements, both the forcing function (or "cause") and the response function (or "effect") occur directly in the ordinary integro-differential equations of motion. The boundary conditions are generally initial conditions (at $t = 0^+$), often taken so as to correspond to no initial storage of energy.

For distributed-parameter systems, however, the situation is usually different. In the most involved case [which we shall now consider in detail, leaving the particulars of some simpler instances to (A-7)], neither the forcing function nor the response function occurs in the homogeneous linear partial integrodifferential equations of motion. They appear instead as (spatial) boundary values of the dependent variables, or as other restrictions on these (spatial) boundary conditions. As an example of the former, one can cite the one-dimensional electrical transmission line (to be studied in detail in Section (F) following), where the "cause" may be the voltage at the sending end and the "effect" may be the voltage at the receiving end.



The second situation can come about if the transmission line is replaced with a waveguide excited and loaded by coupling loops.



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If one supposes that the loops and the adjacent portions of the guide walls are perfectly conducting, then

$$v_i(t) = \frac{d}{dt}\int_{A_i} \underset{\sim}{B}(x_A, y_A, z_A, t)\cdot d\underset{\sim}{A} = \int_{A_i} \frac{\partial \underset{\sim}{B}(x_A, y_A, z_A, t)}{\partial t}\cdot d\underset{\sim}{A},$$

where A_i is the area enclosed by the coupling loop, the guide wall, and the voltage source, and $\underset{\sim}{B}(x_A, y_A, z_A, t)$ is the magnetic-flux density inside the guide at the element $d\underset{\sim}{A}$ of A_i. A similar expression holds for $v_o(t)$.

More generally, the forcing function $f(t)$ enters the mathematical formulation of the physical problem as

$$f(t) = \int_R \mathfrak{f}(x_R, y_R, z_R, t)\, dR,$$

where R is the finite domain (line, surface, or volume) concerned and $\mathfrak{f}(x_R, y_R, z_R, t)$ is the value of some field variable at the element of integration dR [the integrand may have to be written $\underset{\sim}{\mathfrak{f}}(x_R, y_R, z_R, t)\cdot d\underset{\sim}{R}$ in some instances, as above]. It is in accordance with the work of Section (B) above to suppose that $f(t)$ and $\mathfrak{f}(x, y, z, t)$ [for all (x, y, z) of interest; not just in R] are E-functions of the problem concerned*, so that their transforms $F(s)$ and $\mathfrak{F}(x, y,$

*This will not be the case if the specification of the problem is self-contradictory, as, for example, if one assumes that a non-zero voltage drop is applied between two points of a perfectly conducting body.

z, s) exist. Upon transforming and reversing the order of integration with respect to space and time, one gets

$$F(s) = \int_R \mathcal{F}(x_R, y_R, z_R, s)\, dR.$$

The mathematical justification for this step is given in (A-2).

One is tempted to identify the function-theoretical characteristics of $\mathcal{F}(x_R, y_R, z_R, s)$ with those of F(s), but this would be most improper. A physically significant counter-example is given in (A-3). It is not even correct to suppose that $\mathcal{F}(x_R, y_R, z_R, s)$ has as extensive a half-plane of holomorphy* (R-24) as F(s), as is shown by a lumped-parameter counterexample in (A-4). It can be shown, however (A-5) that the half-plane of holomorphy of F(s) is at least as extensive as that of any $\mathcal{F}(x_R, y_R, z_R, s)$.

D. The Existence of the Transfer Characteristic T(s)

It is now possible to introduce the principal feature of Part II. Suppose that for $t \geqq 0$ a forcing function (an E-function not $\equiv 0$ of the problem) f(t) is applied to a linear distributed-parameter system of finite spatial extent by means of a boundary condition over a domain R_1, and a response

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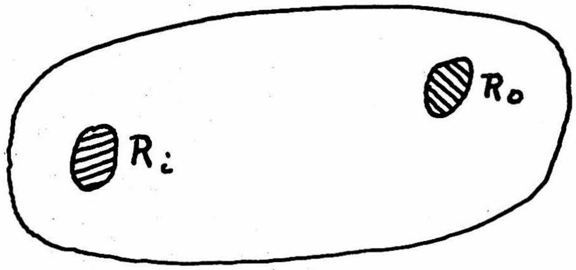
*A function V(s) of a complex variable s is said to be holomorphic in a certain s-domain if it is analytic and single-valued throughout any finite subregion of that domain.

function g(t) is observed over a domain R_o.



The initial conditions ($t = 0^+$) in the system are such that if the forcing function f(t) had not been applied, the system would never have changed its state from that at $t = 0$. The dependent variables in the problem are so chosen that they all vanish in this particular state of the system; they represent the departures of the field variables from quiescent (or "DC") values which may be different from zero.

This implies, for example, that if the linear homogeneous partial integrodifferential equations of motion of the system contain $\frac{\partial f(x, y, z, t)}{\partial t}$ [$f(x, y, z, t)$ being some field variable], then $f(x, y, z, 0^+) = 0$, so that $f(x, y, z, t) = 0$ holds for all $t > 0$. Physically, this seems to represent the requirement that there be no initial storage of energy in the system, just as in the study of linear lumped-parameter networks.

Carrying out the process of Laplace transformation on the equations of motion,* one gets a system of homogeneous linear

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*One feels some confidence that this can be done for a system of bounded spatial extent, since [by the arguments of (A-2) and (A-5)] the field variables throughout the system are uniformly majorized by an E-function whose transform possesses a half-plane of holomorphy. One has no reason to feel that this is so for a spatially unbounded system.

ordinary or partial integrodifferential equations in the field-variable transforms and the space variables. Partial derivatives and integrals of the field variables with respect to space variables go over into the corresponding derivatives and integrals of the field-variable transforms. The partial derivative of a field variable with respect to time goes over into the product of the complex variable s and the field-variable transform by virtue of the initial conditions mentioned above.* The partial integral of a field variable with respect to time (from 0 to t) goes over into the field-variable transform divided by s.

As regards the remaining spatial boundary conditions on the system's behavior, it will be assumed that they transform into linear homogeneous combinations of the field-variable transforms and their spatial derivatives. The coefficients in these expressions may involve the space variables in any reasonable way, but they are specifically assumed to be at most meromorphic functions of s**. A discussion of the implications of an assumption of this sort is given by Doetsch (R-26).

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*It is here assumed that any field variable whose time derivative is under transformation is itself continuous for $0 \leqq t < \infty$. Exactly this qualification was included in the definition of the E-functions of the problem (so that the time derivative concerned would indeed exist throughout $0 \leqq t < \infty$, and the equations of motion would not blow up).

**A function V(s) of a complex variable s is said to be meromorphic if it is analytic and single-valued except possibly for isolated poles throughout any finite subregion of the s-plane.

Let us suppose that a unique solution exists to the problem as stated*. That is, the equations of motion and all the boundary conditions are satisfied by values of the field variables. Remembering what f(t) and g(t) are, we see that

$$f(t) = \int_{R_i} \mathfrak{f}(x_R, y_R, z_R, t)\, dR$$

and

$$g(t) = \int_{R_o} \mathfrak{g}(x_R, y_R, z_R, t)\, dR$$

imply

$$F(s) = \int_{R_i} \mathcal{F}(x_R, y_R, z_R, s)\, dR$$

and

$$G(s) = \int_{R_o} \mathfrak{G}(x_R, y_R, z_R, s)\, dR.$$

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*It is beyond the scope of this paper to consider existence theorems in detail. The crux of the matter seems to lie in imposing neither too many nor too few boundary conditions, and this is generally taken care of in a satisfactory manner in practical applications.

The quantity of present interest is the transfer characteristic of the system, defined by

$$T(s) = \frac{G(s)}{F(s)} \cdot$$

That this quantity is determined only by the constitution of the system itself is fairly obvious, but a proof is given in (A-6).

That the transfer characteristic T(s) can be discussed for linear distributed-parameter systems in which the "cause" and "effect" functions are introduced in ways somewhat different from that above is shown in (A-7).

It must be stressed that the actual existence of T(s) has not been proved. If the system can be studied by Laplace-transform methods, though, T(s) will indeed exist in accordance with the above definition; we shall now examine its properties under this assumption.

E. General Statements About the Transfer Characteristic T(s)

The first thing to note is that T(s) possesses a half-plane of meromorphy.* G(s) is holomorphic for $\sigma > \sigma_H\{g(t)\}$, and F(s) for $\sigma > \sigma_H\{f(t)\}$, by (R-24). F(s) (not $\equiv 0$) does not

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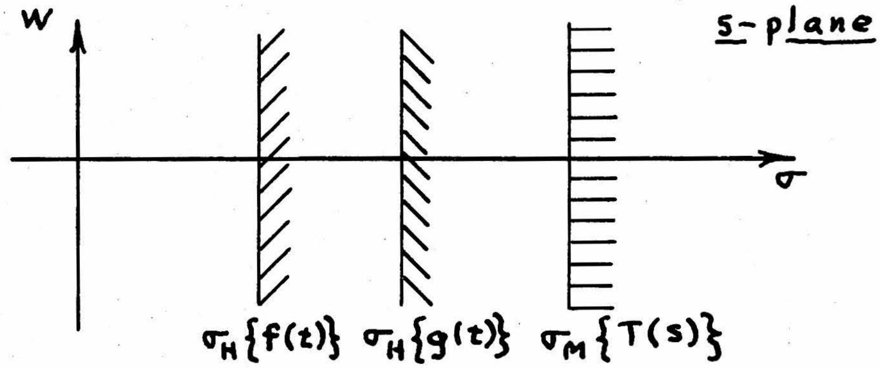
*The use of this term is not in strict agreement with the definition of a meromorphic function given above; the meaning here is that T(s) possesses the property of being analytic and single-valued throughout the interior of a half-plane except for possible isolated poles.

have a finite limit point of zeros in its half-plane of holomorphy (R-27), so that if $\sigma_M\{T(s)\} \geqq \sigma_H\{f(t)\}$, $\sigma_M\{T(s)\} \geqq \sigma_H\{g(t)\}$, then T(s) possibly has poles at the (isolated) zeros of F(s), but no other singularities for $\sigma > \sigma_M\{T(s)\}$. Elsewhere in the finite part of this region it is an analytic function, so that it [T(s)] is meromorphic in the half-plane $\sigma > \sigma_M\{T(s)\}$.*



One can even say something about the half-plane of holomorphy of T(s); this development is pursued in (A-8).

f(t) (by assumption an E-function of the problem under consideration) is by definition a real-valued function of the real variable t. From its definition, F(s) is a real function** of the complex variable s; if $s = \sigma + j0$ is real, $F(\sigma)$ is real too. The same considerations hold for g(t) and G(s); we can conclude that $T(s) = G(s)/F(s)$ is a real function of s as well.

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*The half-plane of meromorphy of T(s) can, of course, be much more extensive than the half-plane determined by this existence argument.

**To be distinguished from "real-valued function."

By the principle of reflection (R-30), $T(\bar{s}) = \overline{T(s)}$ throughout the half-plane of meromorphy of T(s). That is, if

$$T(s) \equiv T(\sigma + jw) \equiv A(\sigma, w) + jB(\sigma, w),$$

where $A(\sigma, w)$ and $B(\sigma, w)$ are real, then

$$A(\sigma, w) = A(\sigma, -w)$$
$$B(\sigma, w) = -B(\sigma, -w).$$

If the half-plane of meromorphy [or a more extensive symmetrical region of meromorphy obtained by analytic continuation of T(s)] contains the so-called "real-frequency" axis $s = 0 + jw$, one sees that the resistive and reactive components of an AC transfer impedance are even and odd functions of frequency respectively, for example. This is the linear-distributed-parameter-system generalization of a well-known result in the study of linear lumped-parameter networks.

Of especial interest, though, is another significance of the half-plane of meromorphy of T(s). Throughout this region (and its extension by analytic continuation) the functional value of T(s) is uniquely determined by its values in the neighborhood of some interior point; in particular, along a segment of the axis $s = \sigma + j0$. This means that the transfer characteristic in all its universality is fully determined by the values of a <u>single</u> real function $[A(\sigma, 0)]$ of a real variable (σ) along a segment of this axis. This is a considerable simplification of the traditional viewpoint, according to which T(s) is determined by the values of <u>two</u> real functions [A(0, w) and B(0, w) or their equivalent] of a real variable (w) along a segment (generally the entirety) of the real-frequency axis. This single-function specification of T(s) would not be well adapted to study of the system by the use of the Fourier integral, but it might be of interest in network design, synthesis, and test. This same property obtains for linear lumped-parameter networks, of course.

The fact that the linear homogeneous algebraic equation $G(s) = T(s) \cdot F(s)$ holds true enables us at once to generalize to linear distributed-parameter systems the well-known superposition theorems of linear lumped-parameter networks. It is no longer surprising that an electrical transmission line can be characterized by its ABCD parameters (R-31); this is just a consequence of the application of the superposition principle.

T(s) is a quotient of transforms, but it is not generally a transform itself. If we take $g(t) \equiv f(t)$ (which is perfectly permissible), then $G(s) \equiv F(s)$ holds true, and we find that $T(s) \equiv 1$, which is not a transform in the usual sense (R-19).

Nevertheless, transform considerations impose limitations on the rate of growth of the magnitude of T(s) as $s \to \infty$ along the σ-axis, for example. Suppose that the equations of motion of the physical system involve at most nth-order time derivatives (n = 0, 1, 2, ...). An admissible forcing function is, then,

$$f(t) = 0, \qquad t < 0$$

$$f(t) = t^n, \qquad 0 < t.$$

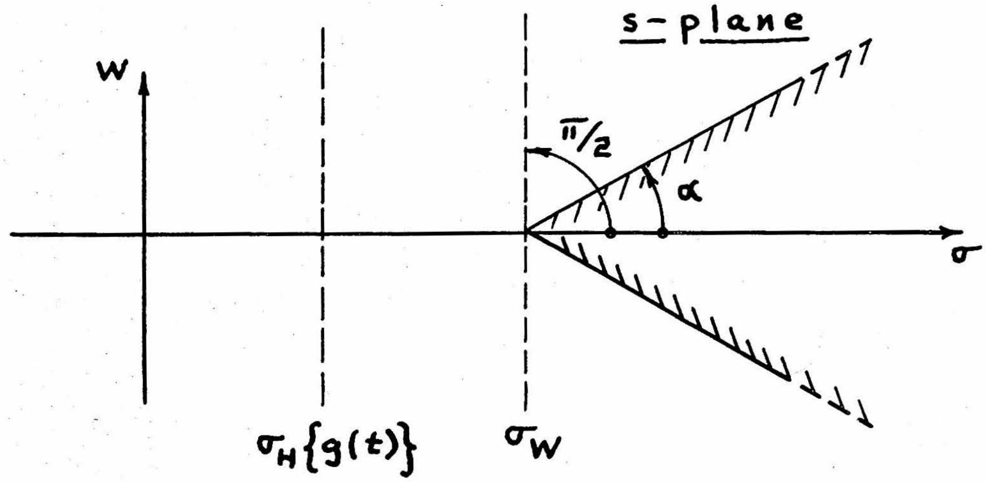
By (R-15), the transform of f(t) is $F(s) = \underline{|n}/s^{n+1}$. Now,

$$G(s) = T(s) \cdot F(s) = \frac{\underline{|n}\, T(s)}{s^{n+1}}.$$

G(s) is holomorphic for $\sigma > \sigma_H\{g(t)\}$, and by (R-32), $G(s) \to 0$ as $s \to \infty$ with complete two-dimensional freedom in any infinite wedge-region described by

$$|\arg(s - \sigma_W)| \leqq \alpha < \pi/2,$$

where $\sigma_W > \sigma_H\{g(t)\}$.



This imposes a restriction on the order of magnitude of T(s) as $s \to \infty$ in the wedge-region. If $T(s) \sim As^{n+1}$, $A \neq 0$, as $s \to \infty$ there, then $G(s) \to A\underline{|n} \neq 0$, in contradiction to the theorem cited above. A similar contradiction arises if T(s) increases in magnitude faster than s^{n+1}. Thus we conclude that T(s) must not increase in magnitude faster than s^k, where $k < n + 1$.

This estimate could probably be sharpened, but it is sufficient to make us realize that T(s), which is invariant with respect to inputs f(t), must not increase in magnitude faster than some power of s as $s \to \infty$ in such a wedge-region for which $\sigma_W > \sigma_H\{T(s)\}$. This holds true for any physical system subject to our study, since a definite indicial integer $n \geqq 0$ exists for any such system.

It would suffice for the development to come in Part III to know only that T(s) possesses no natural boundaries (unconquerable obstacles to analytic continuation). Branch points, essential singularities, singular lines, and such like can be overcome, as will be seen. Nevertheless, one is tempted to hypothesize vastly more: to wit; that T(s) is meromorphic (in the sense of the original definition).

It has not been possible to prove either of these contentions for general physical systems of the type studied above. Considering only the first (which the second must necessarily include), its demonstration can scarcely be expected to come from function-theoretical principles alone, for functional elements possessing natural boundaries are quite common, perhaps the rule rather than the exception (R-33). Neither is the transform theory alone of much help, for Doetsch calls to our attention an E-function (for a problem in which $n = 0$) whose transform has a natural boundary, the axis $s = 0 + jw$ (R-34). Any proof of either of these notions must rest on the introduction of physical considerations which have been quite elusive up to the present time.

The hypotheses will be introduced for what they are worth. Consideration of a large number of solved examples in both study and practice (see Carslaw and Jaeger, R-35, for example) has failed to disclose a single contrary instance.

One aspect of this matter must be stressed. There are many exercises in (R-35) in which T(s) does indeed have branch points. Without exception, these arise in connection with "semi-infinite" electrical cables, heat-transfer structures, or other such devices. One feels no qualms whatsoever in ruling out of consideration any and all physical systems which are unbounded in space, for they are certainly unattainable in practice. Not even the Federal Government can afford to build a semi-infinite transmission line.

Having done this, one is faced with an interesting philosophical problem. If we refuse to think about physical systems unbounded in space, how can we properly discuss physical systems unbounded in time, as required by the $(0,\infty)$ integration in performing the Laplace transformation?

A pragmatic answer is possible, of course. After we have undone the transformation, the resulting time functions satisfy the requirements placed on them by the equations of motion and the boundary conditions. Who could ask for more?

F. T(s) for the Non-uniform Transmission Line

Lest it seem that Part II has degenerated completely into mere intuitive speculation, a problem of great practical importance will be worked through in support of the hypotheses of Section (E).

Consider an electrical transmission line of length ℓ, possessing distributed parameters R, L, G, and C per unit length which are continuous functions of x in $0 \leqq x \leqq \ell$.* The line is terminated at $x = \ell$ by a spatially finite linear lumped-parameter network whose two-terminal input impedance

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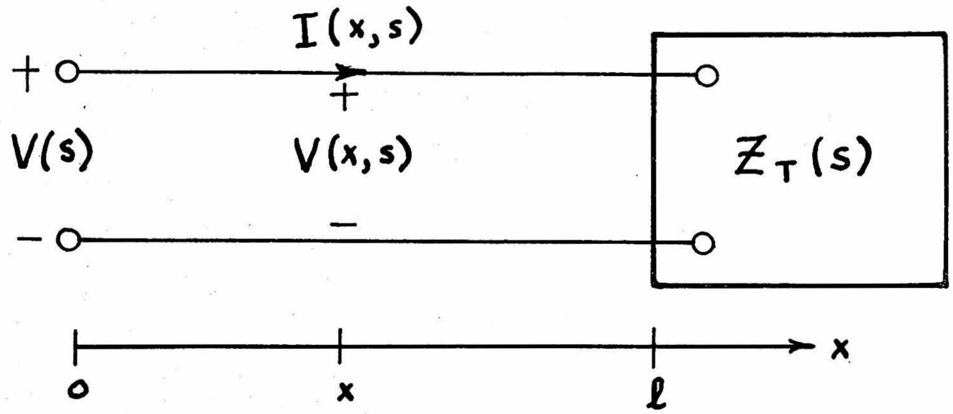
*This implies that the parameters are bounded in $0 \leqq x \leqq \ell$. That is reasonable, since "infinite" R and/or L imply an open circuit in the line; "infinite" G and/or C a short circuit. Either of these faults can best be charged up to a terminating impedance. R, L, G, and C are the series resistance and inductance and the shunt conductance and capacitance per unit length respectively of the transmission line, of course.

$Z_T(s)$ is at most a rational function of s [as stated in Part I and proved in (R-10)]. If the line is short-circuited, $Z_T(s) = 0$; if open-circuited, $1/Z_T(s) = 0$.



It will be supposed that a voltage source v(t) is applied to the line at x = 0 for t ≧ 0, and that the initial distributions of voltage and current along the line and in the terminating network vanish identically. Then the well-known time-domain equations of motion

$$\left.\begin{aligned} \frac{\partial v(x,t)}{\partial x} &= -R(x)\,i(x,t) - L(x)\,\frac{\partial i(x,t)}{\partial t} \\ \frac{\partial i(x,t)}{\partial x} &= -G(x)\,v(x,t) - C(x)\,\frac{\partial v(x,t)}{\partial t} \end{aligned}\right\} \begin{array}{l} 0 < x < \ell \\ 0 < t \end{array}$$

transform into the s-domain equations

$$\frac{dV(x,s)}{dx} = -\left[R(x) + sL(x)\right] I(x,s) = -Z(x,s)\, I(x,s) \qquad (0 < x < \ell)$$

$$\frac{dI(x,s)}{dx} = -\left[G(x) + sC(x)\right] V(x,s) = -Y(x,s) V(x,s) \qquad (0 < x < \ell)$$

where

$$Z(x,s) = R(x) + s\,L(x)$$

and

$$Y(x,s) = G(x) + s\,C(x)$$

are the line's series impedance and shunt admittance per unit length at the point x.

The boundary condition at $x = 0$ is

$$V(0,s) = V(s),$$

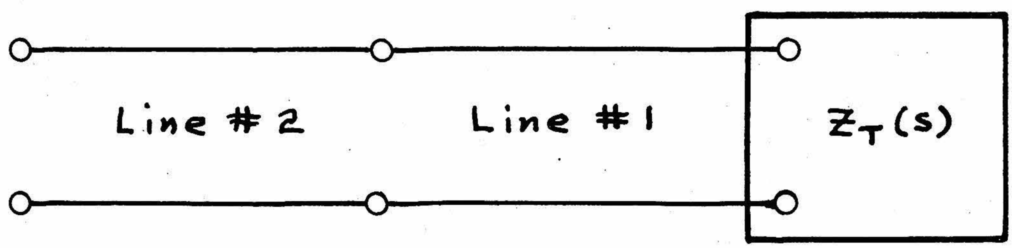
while that at $x = \ell$ is

$$V(\ell, s) - Z_T(s)\, I(\ell, s) = 0.$$

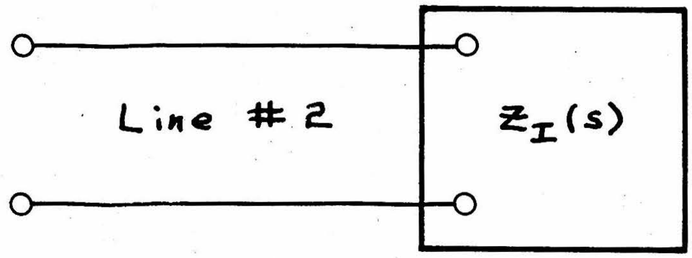
This mathematical specification of the problem satisfies all the requirements of Sections (C) and (D) above; the function-theoretical nature of some of the transfer characteristics is investigated in (A-9).

It is there demonstrated, for example, that the input impedance $Z_I(s) = V(0, s)/I(0, s)$ is indeed a meromorphic function.

Generalizing the physical problem, we may now imagine two different lines connecting in tandem as shown below. Line #1 (which we have just studied) is terminated in $Z_T(s)$ as before.



We may replace the combination of Line #1 and $Z_T(s)$ by the generalized impedance $Z_I(s)$ found above; now only one line is in view.



$Z_I(s)$ has been shown to be a meromorphic function; it is not difficult to see that the manipulation at the start of (A-9) can be carried through again for Line #2. We thus conclude that $Z_{II}(s)$, the input impedance at the terminals of Line #2, is again a meromorphic function. One can generalize this conclusion to the tandem connection of any number of segments of lines, and thus to lines having parameters R, L, G, and C which are only sectionally continuous.

The parallel combination of a finite number of individually terminated, non-coupled lines must yield a meromorphic resultant input impedance, and we can extend this conclusion to "trees" of such lines.

The present theory is not applicable to systems of lines connected at both sending and receiving ends except as such a combination can be interpreted as a single line with new parameters R, L, G, and C.

Some similar remarks could be made about other transfer characteristics of non-uniform lines; the conclusion of functional meromorphicity continually presents itself.

We can now understand why many of the examples in (R-35) led to meromorphic transfer characteristics, for the transformed equations of motion in some instances can be interpreted as applying to transmission lines whose parameters vary in prescribed geometrical fashions. The analysis of (A-9) applies, then, to these exercises.

With this enlightening success behind us, we shall pass from the realm of meta-analysis, endeavoring in Part III to put our conclusions to work in the service of the theoretician, who serves the practician, who in turn serves the world.

III THE CONTINUATION PROBLEM

A. Underlying Mathematical Considerations

The extraordinary elegance and fertility of the principles of the identity, uniqueness, and continuability of analytic functions are reflected in the existence of several alternative fundamental statements in regard to these matters. Perhaps best suited for our present purposes is this one (Churchill, R-39):

> "If a function is single-valued and analytic throughout a region, it is uniquely determined by its values over an arc, or throughout a subregion, within the given region."

From a strictly logical standpoint, this circumstance is simply the consequence of the application to certain postulates and definitions of other arbitrarily established rules of manipulation. It is worthwhile, though, to consider the matter in another light; we can scarcely do better than hear what

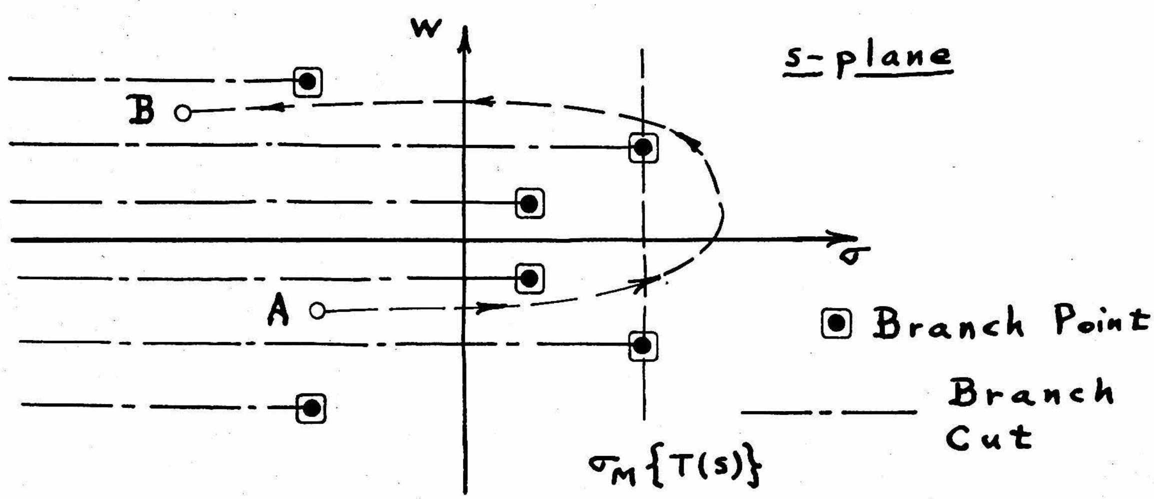
Knopp has to say on the subject (Bagemihl's translation, R-40):

> "Now it is exceedingly remarkable that by means of the single requirement of differentiability, that is, the requirement of regularity, a class of functions having (these) properties is selected from the totality of the most general functions of a complex variable. On the one hand, this class is still very general and includes almost all functions arising in applications. On the other hand, a function belonging to this class possesses such a strong inner bond, that from its behavior in a region, however small, of the (s)-plane one can deduce its behavior in the entire remaining part of the plane . . .
>
> "Since natural phenomena themselves possess an intrinsic regularity, it is clear that, above all, those functions which possess such an inner structure will appear in applications in the natural sciences."

It was shown in Section (E), Part II, that the transfer characteristic T(s) of a linear physical system of the type discussed there possesses a half-plane of meromorphy. If we exclude from the finite portion of this half-plane small neighborhoods of all the singularities (which are poles), the resulting domain is one in which T(s) is analytic and single-valued, and the principle of analytic continuation is immediately applicable throughout it.

It is desirable, though, to extend this domain to include as much of the entire s-plane as possible. This can easily be done if T(s) is meromorphic in the original sense (that is, throughout the s-plane). Indeed, essential singularities and other isolated pathological points which do not introduce multivaluedness can be quarantined in an exactly similar way, leaving virtually the entire s-plane as a region where T(s) is single-valued and analytic.

If we allow the situation to become more complicated, so that T(s) possesses isolated branch points in addition to its other troubles, it is still possible to fix things up satisfactorily. Let all the branch cuts be made in the negative σ-direction, away from T(s)'s half-plane of meromorphy. Then in the simply connected region* remaining, T(s) is analytic and single-valued except for possible isolated irregularities such as poles and essential singularities. One can easily proceed from any point (A) in the cut plane to any other (B) by detouring into T(s)'s half-plane of meromorphy.



By the principle of analytic continuation, then, T(s) is uniquely determined in-the-large by its values in-the-small, even in this trying situation.

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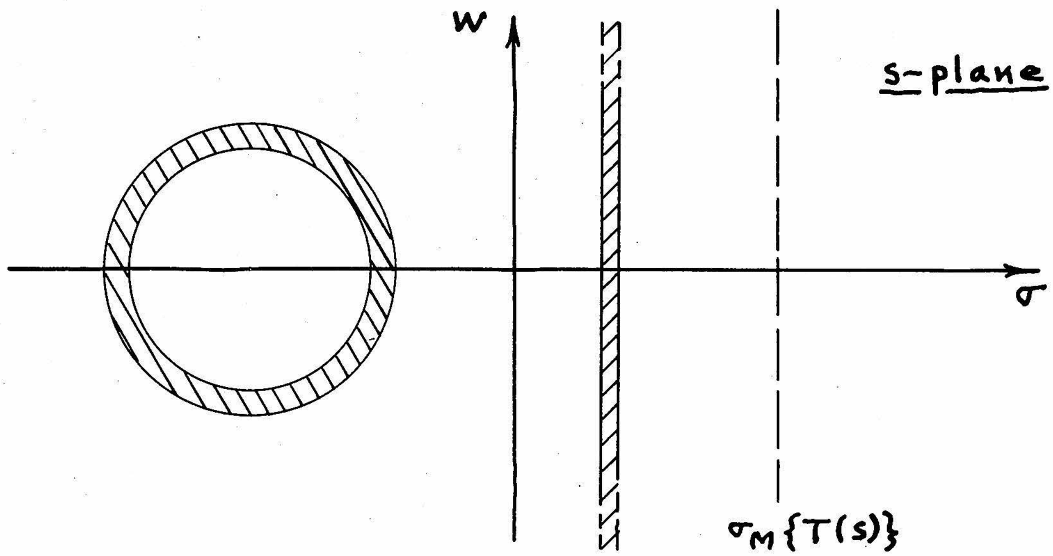
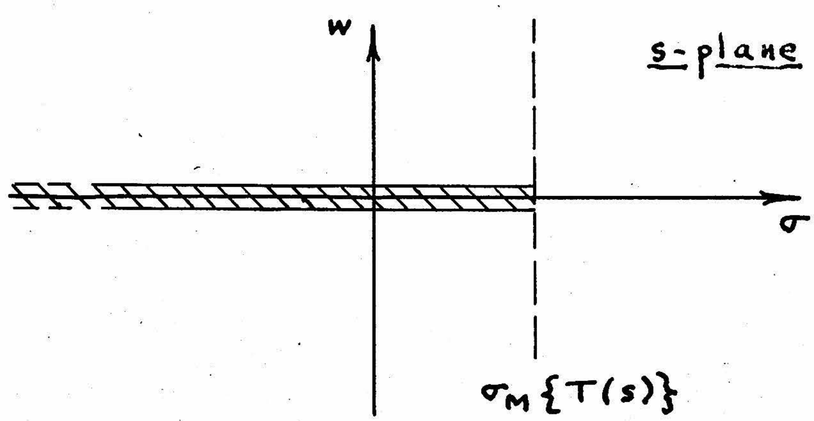
*A two-dimensional region R is said to be simply connected if any closed curve within it encloses only points of R. This implies that any two curves C_1 and C_2 connecting two points A and B in R and lying entirely within R can be continuously deformed into one another without leaving R.

The occurrence of a natural boundary to the continuation of T(s) quite effectively halts our progress. Presence of either of the natural boundaries shown below renders a substantial portion of the s-plane inaccessible from T(s)'s half-plane of meromorphy.



The occurrence of singular lines (non-closed curves across which continuation is impossible*) need not interfere with the annexation of "almost all" of the s-plane to T(s)'s half-plane of meromorphy. A singular line of this sort



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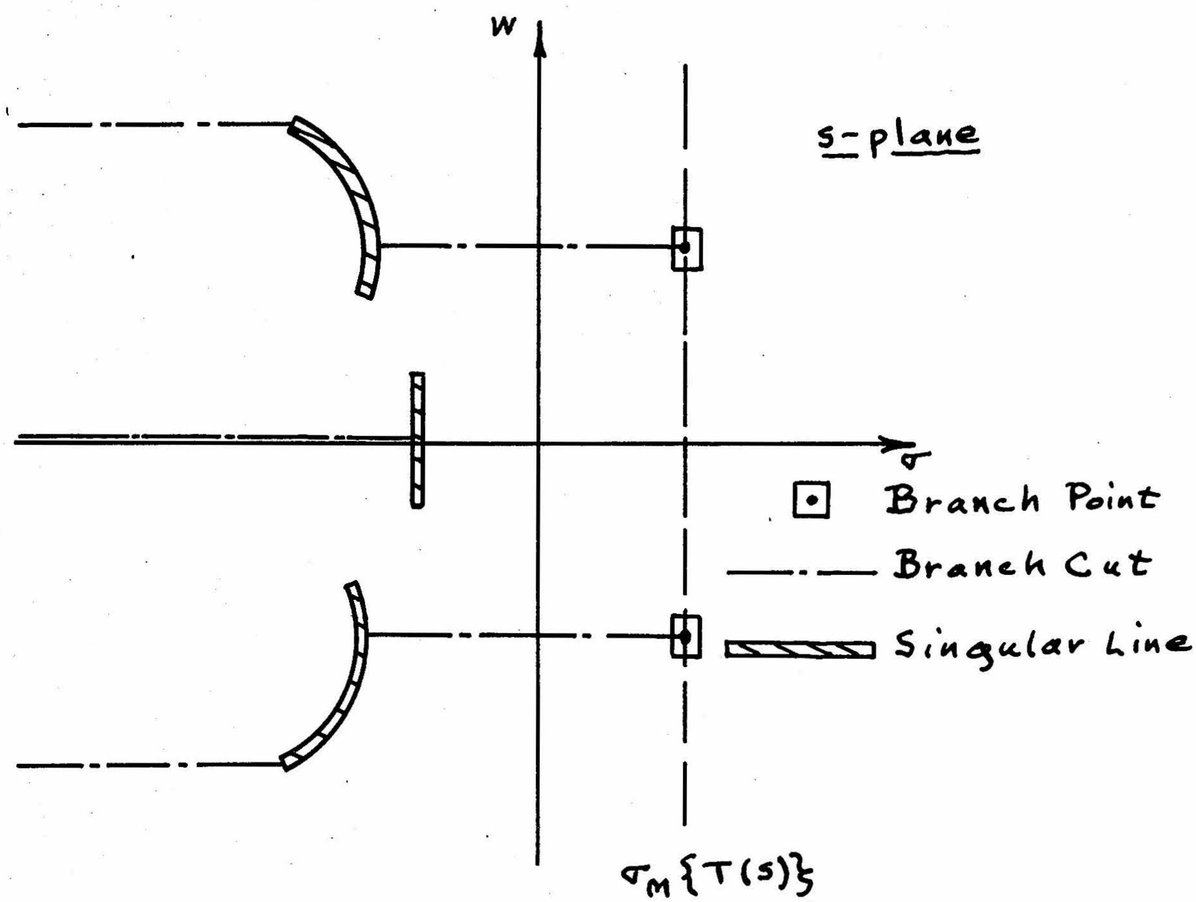
* It must be remembered that continuation is possible across a branch cut, leading onto the next sheet of the Riemann surface for the function concerned.

is no worse to handle than a branch cut (as treated above). Other singular lines can be tamed if we systematically cut the s-plane in this way.



A branch cut goes off to the left (in the negative σ-direction) from some symmetrical point* of each singular line, and any branch cut which impinges on a singular line from the right is

*The points are chosen symmetrically with respect to the σ-axis so that the resulting cut plane will be symmetrical and the principle of reflection (R-30) can be applied to T(s).

diverted so as to coincide with the singular line's own cut in going off to the left. The addition of more singular lines and branch points may complicate the picture, but the cut s-plane will remain simply connected and essentially intact.

The most important hypothesis made about T(s) in Part II (apart from the question of its actual existence) is that it possesses no natural boundaries. Supposing this to be true, the principle of analytic continuation guarantees that T(s) is uniquely determined throughout a suitably cut s-plane by its values in the vicinity of any interior point of that cut plane. In the following two Sections we shall consider means to exploit this fact.

B. The Continuation of Analytically Expressed Data

We are at last in a position to deal with the problem discussed in Part I. Suppose that, as stated there, a theoretician is approached by a practician who desires to realize a certain transfer characteristic

$$T(jw) = A(w) + jB(w) \equiv A(0,w) + jB(0,w)$$

throughout a frequency range $w_1 < w < w_2$. The real functions A(0, w) and B(0, w) (or their equivalent) are for the present supposed given in analytical form; graphical presentation will be considered in Section (C) following.

It is not necessary that the specification be made throughout the w-range by just one expression; T(jw) may be given in each of a finite number of sections of the segment $\overline{w_1 w_2}$. The case of just two sections is sufficiently general for this exposition; the practician desires to have

$$T(jw) = \mathcal{T}_1(w), \quad w_1 < w < \overline{w}$$
$$T(jw) = \mathcal{T}_2(w), \quad \overline{w} < w < w_2,$$

where $\mathcal{T}_1(w)$ and $\mathcal{T}_2(w)$ are complex-valued functions of the real variable w.

It is not difficult to see what must be done. One replaces w by s/j in the analytical expression for $\mathcal{T}_1(w)$. This $\mathcal{T}_1(s/j)$ must at least have the properties deduced for the general transfer characteristic T(s) in Section (E), Part II. Briefly,

(1) T(s) possesses a half-plane of holomorphy, $\sigma > \sigma_H\{T(s)\}$.

(2) T(s) is real for s real in this half-plane.

(3) T(s) does not increase in magnitude faster than s^k as $s \to \infty$ in any wedge-region described by

$$|\arg(s - \sigma_W)| \leqq \alpha < \pi/2,$$

where $\sigma_W > \sigma_H\{T(s)\}$.

(4) T(s) has no natural boundaries (an hypothesis; not

proved).

If $\mathcal{T}_1(s/j)$ satisfies these requirements, then (taking note of possible branch cuts, which were not definitely ruled out in Part II) one can examine the values of $\mathcal{T}_1(w)$ in the range $\bar{w} < w < w_2$. If there is agreement, fine! If not, it would seem that the desired transfer characteristic $T(jw)$, $w_1 < w < w_2$, cannot be attained with any physical system of the sort to which we have restricted out attention.

If this self-consistency test is passed, though, one can proceed to determine the behavior of $T(s)$ throughout a suitably cut simply connected s-plane by examining the behavior of $\mathcal{T}_1(s/j)$ [or $\mathcal{T}_2(s/j)$ now], including as a special case $T(jw)$ for w outside the segment $\overline{w_1w_2}$. This "over-all" view of the transfer characteristic $T(s)$ enables one to discover what problems of stability, active-/passive-network realizability, etc., are implied by the specification of $T(jw)$ in the segment $\overline{w_1w_2}$. Thus the objectives of Part I are attainable in this case.

It must be remembered, though, that these criteria are <u>necessary</u>. It is not known whether or not they are <u>sufficient</u>. No general synthesis procedure is offered here.

The procedure will be better appreciated after working through an example. Particularly informative is that of the idealized low-pass filter with time delay $t_d > 0$ (R-41).*

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*It was this example which first aroused the writer's interest in the topics which have developed into this paper.

This hypothetical device has a transfer characteristic

$$T(jw) = K \exp[-st_d], \quad |w| < w_1$$

$$T(jw) = 0, \quad |w| > w_1$$

which supposedly gives distortionless transmission with a time delay t_d in the pass band, but complete rejection outside it.

The continued transfer characteristic T(s) [based upon the values of T(jw) for $|w| < w_1$] is plainly $T(s) = K \exp(-st_d)$. This function has the properties of a transfer characteristic listed above. Indeed, it can be seen to correspond to the voltage transfer ratio of a distortionless uniform transmission line ($R/L = G/C$) terminated in its characteristic impedance ($R_T = \sqrt{L/C}$), for instance.

This T(s) is an entire function, and it decidedly does <u>not</u> agree with the desired values of T(jw) for $|w| > w_1$. One feels quite certain that the originally specified T(jw) is unattainable, and this belief is strengthened when one recalls that the idealized low-pass filter can be shown by analysis (R-41) to have the whimsical habit of presenting an output signal $\neq 0$ <u>before</u> the input signal (a unit-step function here) is applied!

It is both amusing and discouraging to see to what lengths some authors have gone in attempting to palliate this abomination. As long as physical systems are studied within the framework of classical physics, cause-and-effect relationships (with their sequence in time) must apply without exception; any departure from them is a shame and a disgrace and renders suspect of grossest error the entirety of the associated analysis! Or so a "high-church" theoretician would say. A gentler view of the matter might be to the effect that here is just another instance where the theoretician may find it convenient to approximate and compromise, evaluating the significance of his results in the light of his assumptions and in comparison with experiment. This is not a new problem for him.

Some additional examples of the continuation of analytically expressed data are given in Sections (A) and (B), Part IV.

C. The Continuation of Graphically Expressed Data

Suppose, now, that our theoretician is again approached by a practician who desires to realize a certain transfer characteristic

$$T(jw) = A(w) + jB(w) \equiv A(0,w) + jB(0,w)$$

throughout a frequency range $w_1 < w < w_2$. This time, the real functions $A(0, w)$ and $B(0, w)$ (or their equivalent) are given in graphical form.* How can the theoretician answer the questions posed in Part I?

If the graphs of A(O, w) and B(O, w) were mathematically precise, one could (in principle, at least) determine the values of all the derivatives $\left[\frac{d^n A(o,w)}{dw^n}\right]_{w=w_o}$ and $\left[\frac{d^n B(o,w)}{dw^n}\right]_{w=w_o}$ (n = 0, 1, 2, ...) from them at some convenient point w_o, $w_1 < w_o < w_2$. Supposing $s = jw_o$ to be a point of analyticity of T(s), the Taylor's-series expansion of the transfer characteristic about that point would be

$$T(s) = \sum_{n=0}^{\infty} a_n (s - jw_o)^n \equiv \sum_{n=0}^{\infty} \left[\frac{d^n T(s)}{ds^n}\right]_{s=jw_o} \frac{(s-jw_o)^n}{\underline{\lfloor n}}$$

$$= \sum_{n=0}^{\infty} \left\{ \left[\frac{d^n A(o,w)}{dw^n}\right]_{w=w_o} + j \left[\frac{d^n B(o,w)}{dw^n}\right]_{w=w_o} \right\} \frac{(s-jw_o)^n}{j^n \underline{\lfloor n}}$$

since each of the derivatives of T(s) can be determined at a point of analyticity by (repeated) differentiation in any direction. This functional element could then serve as a basis for the analytic continuation of T(s) by at least two methods.

The classical circle-chain method (R-42) is adequate in

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*We can reasonably assume that these graphs will be sectionally continuous (and therefore bounded) with only a finite number of maxima and minima in $w_1 \leqq w \leqq w_2$, since it is impossible in practice to draw graphs of any other kind.

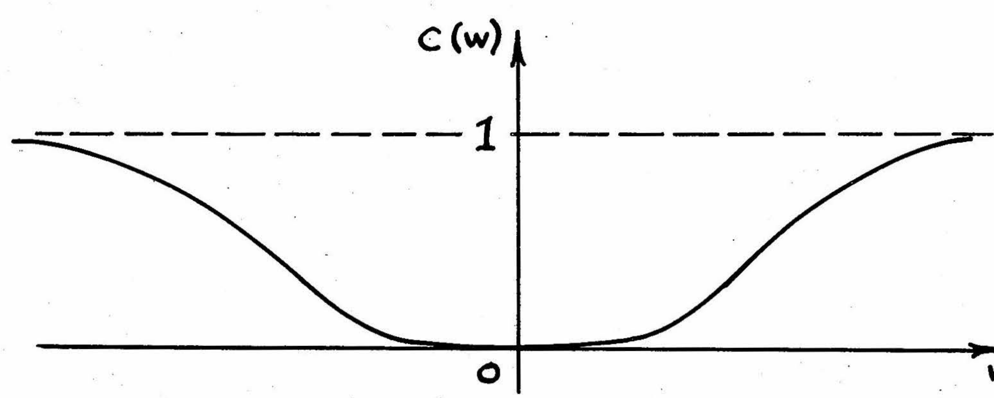
principle, since one does not expect T(s) to have any natural boundaries. The sheer computational bulk of this method, with its sequences of successively determined power series, is discouraging, though.

Another method of continuation in-the-large (based on knowledge of the coefficients a_n) utilizes a process of summation developed by Borel. Since this, too, turns out to be impractical, the details are relegated to (A-11).

It was assumed above that $s = jw_0$ was a point of analyticity of T(s). There may be some difficulty in deciding this *a priori*. One might be fooled by a graph of the continuous, everywhere-arbitrarily-often-differentiable function

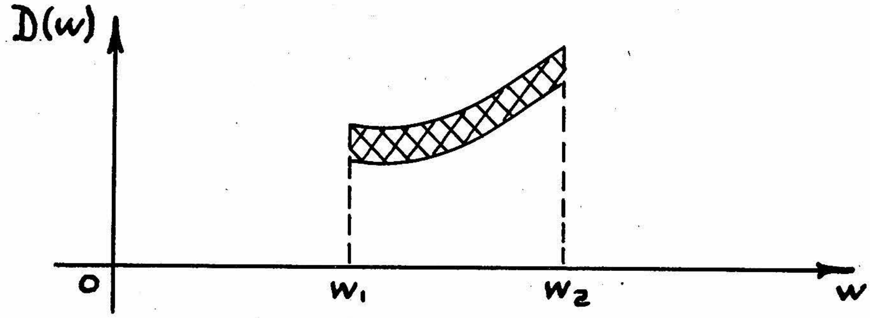
$$C(w) = \exp(-1/w^2), \quad w \neq 0$$

$$C(w) = 0 \qquad\qquad , \quad w = 0.$$



It can be shown (R-43) that $C^{(n)}(0)$ vanishes for all positive integers n. Thus a Taylor's-series expansion of C(w) about $w = 0$ makes no sense.*

These difficulties are of only limited interest, however, since it is certain that the practician will not present graphical data which are mathematically precise. Any curve $D(w)$, $w_1 < w < w_2$, will appear as a "smear," or extended two-dimensional region, rather than as a mathematical curve, since it will have been drawn by a pen, pencil, or other instrument which lays down a strip-mark.



Thus the functions $A(0, w)$ and $B(0, w)$ [and hence $T(jw)$] can be approximated uniformly within the widths of the "smears" throughout the finite range $w_1 < w < w_2$ by any of an indefinitely large number of functions $\left[T_i(s)\right]_{s=jw}$. Unique analytic continuation in-the-large is manifestly impossible.

The question of approximate continuation in-the-small will not be considered here. Some work on this matter has been going on elsewhere, however (R-44).

An alternative explanation for the negative conclusion of this Section can be given by means of the modern theory of

---

*The underlying function-theoretical reason for this state of affairs is, of course, that $C(s/j) = \exp(1/s^2)$, $s \neq 0$, possesses an essential singularity at $s = 0$.

information and communication (R-45, R-46, R-47). In this new and exceedingly interesting discipline, "information" is determined by the relative restrictiveness of choices made between members of sets of alternatives. The simplest and most appealing scheme of information measurement operates on a binary basis and deals with Yes-No decisions, or bits.*

If we expand any (finite) positive real number A in powers of 2, we find that

$$A = \sum_{m=-\infty}^{p} b_m 2^m ,$$

where p is some integer $\gtreqless 0$, $b_p = 1$, and all the other b_m are either 0 or 1 (R-48). There is a possibility of ambiguity in the b_m due to alternative expansions such as

$$A = \frac{1}{2} = \frac{1}{2} + \frac{0}{2^2} + \frac{0}{2^3} + \cdots , \quad b_m = 0, \; m < p = -1$$

$$A = \frac{1}{2} = \frac{1}{2^2} + \frac{1}{2^3} + \cdots , \quad b_m = 1, \; m \leq p = -2,$$

but this does not relieve us from having to specify a denumerable infinity [or $\aleph_0$ (Aleph-Null), Cantor's first transfinite cardinal (R-49)] of the b_m. Thus the number A requires $\aleph_0$ bits of information for its exact specification.

*From "binary digits."

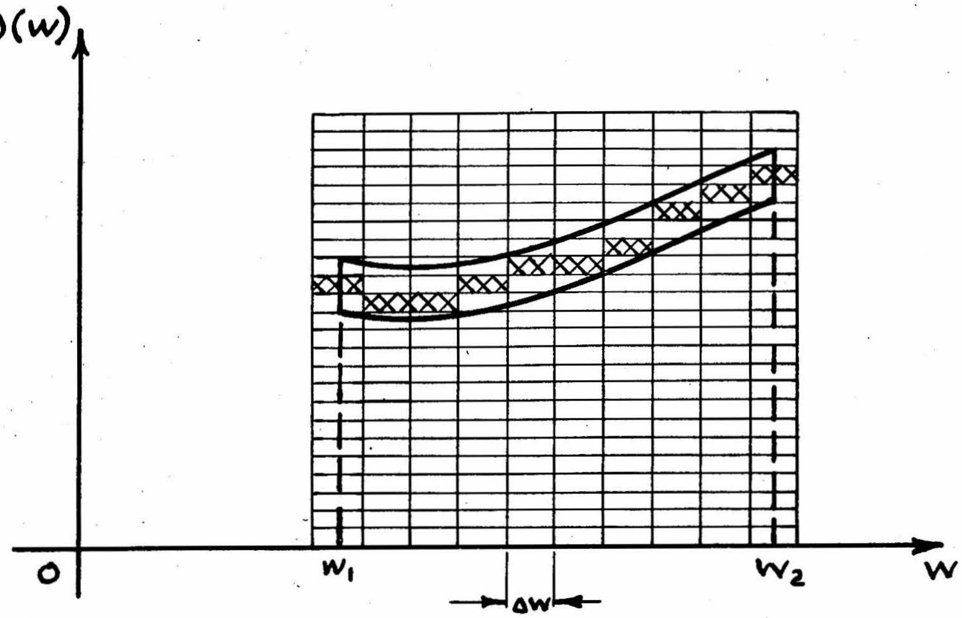
Adding another bit gives a choice between "+" and "-", so that any non-zero real number requires $\aleph_0$ bits. Adding one more bit gives a choice between zero and not-zero, so that <u>any</u> real number requires $\aleph_0$ bits. Doubling $\aleph_0$ results in just $\aleph_0$ (R-49), so that any complex number (which is essentially just an ordered pair of real numbers) requires $\aleph_0$ bits of information. A denumerable collection of denumerables is denumerable (R-50), so that the collection of coefficients a_n of a Taylor's or Laurent's series requires "only" $\aleph_0$ bits of information for its complete specification.* Thus a functional element of T(s) (from which analytic continuation could take place) would represent $\aleph_0$ bits of information.

It is pretty obvious that the amount of information represented by the graphical presentation of $T(jw) = A(0, w) + jB(0, w)$ in the finite range $w_1 < w < w_2$ is finite, since the curves A(0, w) and B(0, w) could be quantized in their ordinates and abscissas without departing from the strip-regions which are the physical curves. For example, the curve D(w), $w_1 < w < w_2$, drawn above contains <u>less</u> information than

---

*This is probably the underlying reason why an analytic function can be at most denumerably infinitely multiply valued at any point, as is the case for $W(s) = \log s$ (R-51). Similarly suggestive is the fact that the classical examples of functional elements possessing natural boundaries [such as $H(s) = \sum_{n=0}^{\infty} s^{n!}$, (R-52)] rely on misbehavior of the functions at a denumerably infinite set of points [in this example the points $s = \exp(j2\pi p/q)$, p and q positive integers, p/q rational]. This is not true of non-analytic functions such as $I(s) = I(\sigma + jw) = 1/\sigma$, which misbehaves at every point of the w-axis (whose equation is $\sigma = 0$).

the set of shaded rectangles in the sketch below.



The abscissas of these rectangles increase by suitably small equal steps Δw, and their ordinates are similarly quantized. Each of these areas represents only a finite amount of information,* and the same thing can be said for the whole finite collection

Thus there is just not available enough information to enable one to carry out unique continuation in-the-large.

If one endeavors to get out of this predicament of impotency by introducing analytic expressions for A(0, w) and B(0, w) over sections of the range $w_1 < w < w_2$, one must prepare to deal with the considerations of Section (B) above.

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*In the sketch above, a choice of one rectangle out of the 24 in each column.

IV A COMPARISON OF THE RESULTS WITH THE WORK OF OTHERS, AND AN EVALUATION OF THE SIGNIFICANCE OF THE CONCLUSIONS

A. The Work of Bode

It is possible to interpret much of the work of Bode as constituting applications of the principle of analytic continuation. A problem with which in different guises he deals repeatedly (R-56) is that of determining an impedance function $Z(jw)$ given either $R(w) = \operatorname{Re} Z(jw)$ or $X(w) = \operatorname{Im} Z(jw)$ for all w. The correspondence between his notation and ours is as follows:

$$Z(jw) \longleftrightarrow T(jw)$$

$$R(w) \longleftrightarrow A(o,w)$$

$$X(w) \longleftrightarrow B(o,w).$$

Suppose that we are given $R(w)$ for all w by analytical expressions (graphical presentation will be discussed presently). If we replace w by s/j, the resulting analytically continued function $R(s/j)$, defined throughout the s-plane except at its singularities, certainly reduces to $R(w)$ for $s = jw$, thus satisfying the data.

The problem is by no means solved, though. The impedance function $R(s/j)$, while perfectly acceptable from the standpoint of the requirements of Section (E), Part II, may possess properties which render it physically unrealizable (R-57). The only way out of this practical difficulty is to devise a function $X(w)$ such that

(1) $jX(s/j)$ satisfies the requirements of Section (E), Part II,

(2) $X(w)$ is real, and

(3) $Z(s) = R(s/j) + jX(s/j)$ is physically realizable. That is, $jX(s/j)$ must also fail to be physically realizable, in such a way that its troubles just cancel those of $R(s/j)$.

The solution of this problem, if it exists, is not unique. It is obvious that the addition of any purely reactive two-terminal network N_2 in series with a two-terminal network N_1 having $Re\left[Z_1(jw)\right] = R(w)$ will yield a two-terminal network N, the real part of whose input impedance will still equal $R(w)$. Alternatively, the fact that we are not given both $R(w)$ and $X(w)$ over any range $w_1 < w < w_2$ implies that we do not have available enough information to compute any of the derivatives $\left[Z^{(n)}(s)\right]_{s = jw}$ $(n = 0, 1, 2, \ldots)$; we have not specified a functional element of $Z(s)$, and its (unique) analytic continuation is not possible.

This is one formulation of the mathematical reason for Bode's restriction of his attention to minimum-reactance (or minimum-susceptance, or minimum-phase-shift) networks, since for such the function $X(w)$ (if it exists) is unique.

The method is well illustrated by this simple example. Suppose that we are given

$$R(w) = \frac{1}{1+w^2}, \quad -\infty < w < \infty.$$

The extended function R(s/j) is not physically realizable, since

$$R(s/j) = \frac{1}{1-s^2} = \frac{1/2}{s+1} - \frac{1/2}{s-1}$$

has a simple pole at s = 1 (in the right half of the s-plane). To get rid of this outrage, we put

$$j\,X(s/j) = \frac{1/2}{s-1} + \phi(s),$$

in which $\phi(s)$ is physically realizable and satisfies the conditions of Section (E), Part II. Since

$$X(w) = \frac{1/2}{j(jw-1)} + \frac{\phi(jw)}{j}$$

and we want X(w) to be real, we must have

$$\frac{\phi(jw)}{j} = \overline{\left(\frac{1/2}{-w-j}\right)} = \frac{1/2}{-w+j}$$

or

$$\phi(jw) = \frac{1/2}{jw+1} .$$

Hence

$$\phi(s) = \frac{1/2}{s+1}$$

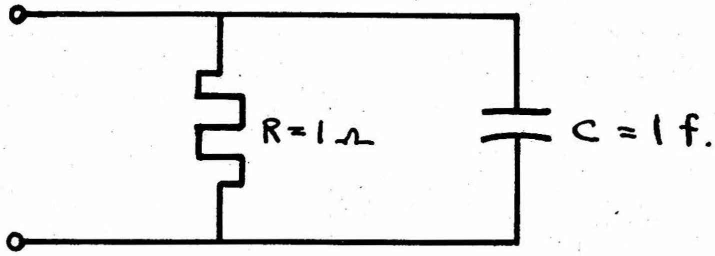
and

$$j\,X(s/j) = \frac{1/2}{s-1} + \frac{1/2}{s+1} .$$

Thus

$$Z(s) = R(s/j) + j\,X(s/j) = \frac{1}{s+1} ,$$

which satisfies all the requirements of the problem and corresponds, in fact, to this minimum-reactance network.



Many of Bode's other results (R-58) are simply changes rung on this same theme.

Bode's treatment of graphically presented data (R-59) is very interesting. His results are admittedly only approximate, in conformity with the pessimistic conclusions of Section (C), Part III. Unfortunately, his methods are not applicable to the central problems of Part III [in which, for example, both R(w) and X(w) are given in some range $w_1 < w < w_2$], since the straight-line approximations he makes would amount to the introduction of analytical expressions for Z(jw), with the attendant difficulties discussed in Section (B), Part III.

We must take issue with Bode for the tenor of his remarks at an earlier point (R-60). Even allowing for the difference between his and our uses of the term "network characteristic," his position seems unreasonable. By virtue of the principle of analytic continuation, the connection between the values of a network's impedance characteristic in the extreme left of the p(or s)-plane and along the real-frequency axis is not tenuous, but of the very strongest! If no natural boundaries intervene, the values in either region can be used to determine those in the other exactly. Since the offensive subject matter is only a digression, however, the remainder of Bode's work stands unimpeached.

B. The Work of Paley-Wiener-Wallman

The only other known investigation of the questions stated in Part I and investigated in Part III is due to Paley, Wiener, and Wallman (R-61, R-62, R-63). The criterion introduced by them can be stated in these words:*

> "Suppose that $M(w) = |T(jw)|$ is an arbitrary amplitude characteristic, i.e., an even non-negative function of frequency, having a Fourier transform. Let us call $M(w)$ 'realizable' if it is possible to associate with the amplitude function $M(w)$ a phase-lag function $\phi(w)$ (not necessarily linear) such that the combined frequency function $T(jw) = M(w) \exp[-j\,\phi(w)]$ yields zero transient response for $t < 0$ [to any input applied for $t \geqq 0$]. This is clearly a very general and non-restricted conception of 'realizability.' Then a necessary and sufficient condition for the amplitude function $M(w)$ to be realizable is that
>
> $$\int_{-\infty}^{\infty} \frac{|\operatorname{Log} M(w)|}{1+w^2}\, dw$$
>
> be finite."

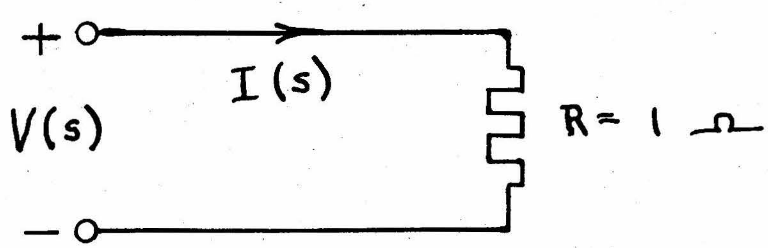
This criterion is not equivalent to the restrictions on T(s) developed in Section (E), Part II. The assumption that $M(w) = |T(jw)|$ possesses a Fourier transform rules out transfer

---

*Taken from (R-63), with slight changes in notation and the addition of the bracket.

characteristics such as $T(s) = 1$, since $M(w) = |T(jw)| = 1$ is not Fourier-transformable.

One must admit that this $T(jw)$ can scarcely be attained in practice for arbitrarily large w. If the physical system concerned is this one,



in which $T(s) = V(s)/I(s) = 1$, the resistor will fail to obey Ohm's Law at very high frequencies owing to failure of the conduction process in the substance. Parasitic shunt capacity across its terminals will become predominant, and the lumped-parameter description of the system will be found inadequate. It is probably true (as Wallman says) that any physical amplitude characteristic $M(w)$ is "inevitably" Fourier-transformable.

An interesting parallel between the Paley-Wiener-Wallman criterion and our restrictions on $T(s)$ is displayed in (A-12).

It is very illuminating to interpret Wallman's examples (R-63) in the light of the work done in this paper.

(1) The idealized low-pass filter. For this device,

$$M(w) = 1, \quad |w| < 1$$

$$M(w) = 0, \quad |w| > 1.$$

$\phi(w)$ is unspecified. Wallman concludes that no $\phi(w)$ exists for which $M(w)$ is realizable in his sense. In Section (B), Part III, we considered a similar device with $K = 1$, $\phi(w) = wt_d$, and $w_1 = 1$ from the standpoint of the principle of analytic continuation and came to a similar pessimistic conclusion.

(2) The Gaussian-error-curve filter. For this device, $M(w) = \exp(-w^2)$; $\phi(w)$ is unspecified. Wallman concludes that no $\phi(w)$ exists for which $M(w)$ is realizable in his sense.

Let us analyze by analytic continuation attempts to realize this filter. Since $M(w) = |T(jw)| = \exp(-w^2)$, $T(s) = \exp(s^2)$ will certainly satisfy the given data. This $T(s)$ satisfies all the requirements of Section (E), Part II, with one significant exception. $T(\sigma) = \exp(\sigma^2)$ increases faster than any σ^K as $\sigma \to \infty$. It has not been possible to "doctor up" this $T(s)$ by applying correction factors exp $[\theta(s)]$ which do not alter $|T(jw)|$. This is in agreement with Wallman's conclusion, though we are unable to give a proof that no such $\theta(s)$ exists, as he does.

(3) The semi-idealized low-pass filter. For this device,

$$M(w) = 1, \quad |w| < 1$$

$$M(w) = \epsilon, \quad |w| > 1.$$

Wallman finds that this amplitude characteristic satisfies the criterion, and he displays a phase-lag characteristic $\phi(w)$ which corresponds to it. These conclusions can be checked by the principle of analytic continuation; the somewhat messy details are given in (A-13).

(4) A filter whose amplitude characteristic is $M(w) = |\sin(w)/w|$. Wallman concludes that this amplitude characteristic is realizable, and he offers a network (composed of an infinite number of lumped-parameter elements, however) due to Guillemin which may possess this amplitude characteristic.

It is possible by means of analytic continuation and allied considerations developed in this paper to synthesize an appealing distributed-parameter system which has $|T(jw)| = M(w)$; this is perhaps the most spectacular result obtained. The procedure is as follows.

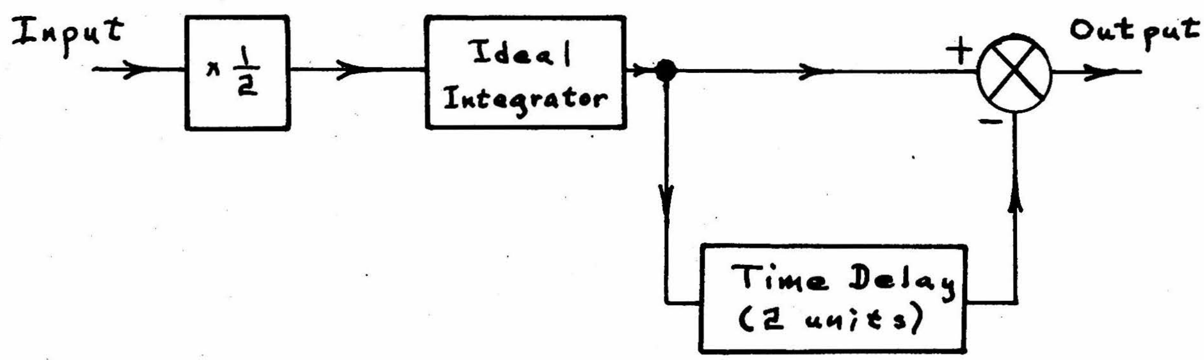
Since $|T(jw)| = |\sin(w)/w|$, we may start by replacing w by s/j. Then

$$T_1(s) = \frac{\sin(s/j)}{(s/j)} = \frac{e^s - e^{-s}}{2s} = \frac{\sinh(s)}{s}.$$

However, this $T_1(s)$ grows "too fast" in magnitude as s along the σ-axis, for example [compare (A-12)]. We must apply a factor exp(-Rs), $R \geqq 1$, to tame the term exp(s)/2s.* Taking R = 1,

$$T(s) = T_1(s)\, e^{-s} = \frac{1 - e^{-2s}}{2s}.$$

Recalling the significance of the transform operators 1/s (integration from 0 to t) and exp(-s) (unit time delay), we see that the desired transfer characteristic can be realized with this system:



The delay unit can be realized with an electrical or acoustical transmission line, for example. The integrator may

*The fact that R is not definitely determined is just a reflection of the fact that we do not specify a functional element of T(s) by the given data for M(w), so that no unique T(s) exists. This is similar to the lack of uniqueness of results obtained by Bode's methods, discussed in Section (A) above.

be a little harder to obtain in practice.

(5) A filter whose amplitude characteristic is $M(w) = [\sin(w)/w]^2$. This is plainly realizable with two units of (4) in tandem.

C. General Electromagnetic Systems

The distributed-parameter systems which have been most studied up to the present time and which have been most extensively employed in applications are transmission lines and wave guides. It is of interest to see what are the implications in this connection of the investigations of Part II and Part III.

Maxwell's equations (rationalized MKS units) are well known; any electromagnetic system considered must operate subject to

$$\nabla \times \underset{\sim}{E} = -\mu \frac{\partial \underset{\sim}{H}}{\partial t}$$

$$\nabla \times \underset{\sim}{H} = \gamma \underset{\sim}{E} + \rho \underset{\sim}{v} + \epsilon \frac{\partial \underset{\sim}{E}}{\partial t}$$

$$\nabla \cdot (\epsilon \underset{\sim}{E}) = \rho$$

$$\nabla \cdot (\mu \underset{\sim}{H}) = 0$$

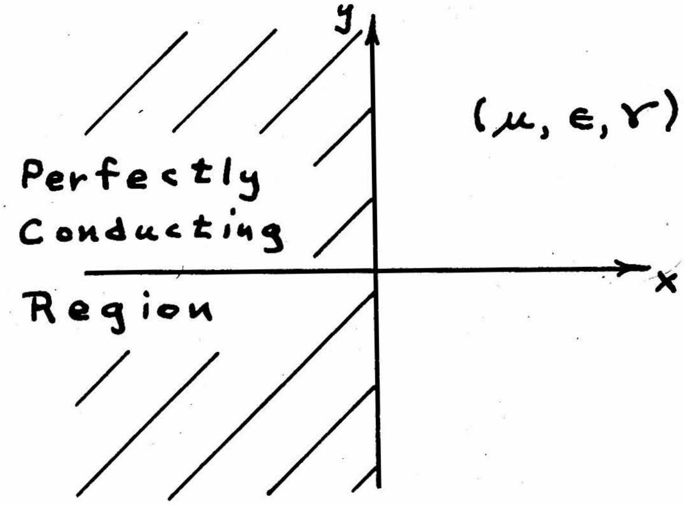
at all its interior points. It is here assumed that μ, ϵ, and γ are constant in time, though not necessarily uniform in space.

In order to render these partial differential equations homogeneous in the components of $\underset{\sim}{E}$, $\underset{\sim}{H}$, and their space and time derivatives, we shall have to put $\rho = 0$. This means that we cannot allow any loose charges to wander around in the interior of our electromagnetic system. This limitation removes from our purview such interesting devices as the traveling-wave tube, for example, in which the essential phenomenon is the interaction between charged particles and the electromagnetic field.

It need not be assumed that the media of which the system is constructed are isotropic. One simply replaces μH_x by $\mu_{11}H_x + \mu_{12}H_y + \mu_{13}H_z$, for example; the equations remain homogeneous. All the μ_{ij}, ϵ_{ij}, and γ_{ij} are supposed constant in time, though not necessarily uniform in space.

Since Maxwell's equations involve $\frac{\partial \underset{\sim}{E}}{\partial t}$ and $\frac{\partial \underset{\sim}{H}}{\partial t}$, we must study the electromagnetic system in terms of dependent variables $\underset{\sim}{E}$ and $\underset{\sim}{H}$ which vanish for $t = 0^+$. They can, of course, be departures from some quiescent state.

It remains only to specify the spatial boundary conditions. It will be supposed that the system is of finite spatial extent; radiating structures are not considered. If one supposes that the electromagnetic system under study is surrounded by a perfectly conducting surface except possibly at the places where the forcing function is applied and the response function is observed, things come out very well. Suppose that part of this boundary is a portion of the yz-plane, the interior of the electromagnetic system lying in the positive x-direction. The positive z-direction is out of the page.



The tangential components of $\underset{\sim}{E}$ must vanish on the boundary, by the definition of a perfect conductor. So we have

$$\left.\begin{aligned} E_y &= 0 \\ E_z &= 0 \end{aligned}\right\} x = 0^+$$

as the first two boundary conditions.

Writing out the scalar equation that comes from the x-component of the first Maxwellian equation above, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} .$$

At $x = 0^+$, E_y and E_z must vanish for all y and z concerned, by the boundary conditions above. Thus $\frac{\partial H_x}{\partial t}$ must vanish, and H_x must be a constant. Since $H_x = 0$ at $t = 0^+$, this constant must be zero. So,

$$H_x = 0, \quad x = 0^+.$$

Writing out the scalar equations that come from the y- and z-components of the second Maxwellian equation above, we have

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \gamma E_y + \epsilon \frac{\partial E_y}{\partial t}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \gamma E_z + \epsilon \frac{\partial E_z}{\partial t} .$$

At $x = 0^+$, H_x, E_y, and E_z must vanish for all y, z, and t concerned, by the boundary conditions above. So,

$$\left.\begin{aligned} \frac{\partial H_y}{\partial x} &= 0 \\ \frac{\partial H_z}{\partial x} &= 0 \end{aligned}\right\} \quad x = 0^+.$$

Writing out the third Maxwellian equation above,

$$\nabla \cdot (\epsilon \underset{\sim}{E}) \equiv \underset{\sim}{E} \cdot (\nabla \epsilon) + \epsilon (\nabla \cdot \underset{\sim}{E}) = 0$$

or

$$E_x \frac{\partial \epsilon}{\partial x} + E_y \frac{\partial \epsilon}{\partial y} + E_z \frac{\partial \epsilon}{\partial z} + \epsilon\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = 0.$$

At $x = 0^+$, E_y and E_z must vanish for all y and z concerned. So,

$$E_x \frac{\partial \epsilon}{\partial x} + \epsilon \frac{\partial E_x}{\partial x} = 0, \quad x = 0^+.$$

This is the sixth and final boundary condition on the scalar components of $\underset{\sim}{E}$ and $\underset{\sim}{H}$.*

In some analyses of electromagnetic systems, one is content to suppose that the system is indeed surrounded by a perfectly conducting surface (the walls of a waveguide or cavity, for instance). For such a study, the homogeneous spatial boundary conditions found above are of the type specified in Section (D), Part II. So, the transfer characteristic T(s) can be introduced here, and the conclusions of Part III are applicable.

A more realistic analysis takes into account the finite conductivity of the guide walls, however. One can extend the

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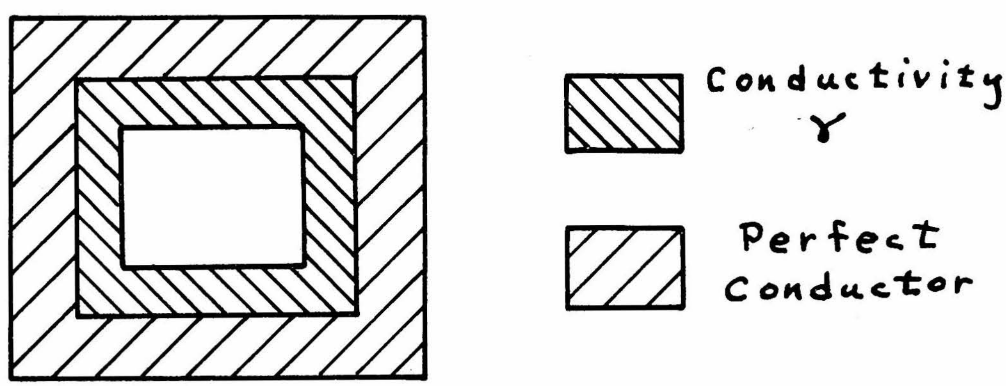
*In the work above, it was implicitly assumed that the medium immediately adjacent to the perfectly conducting boundary was isotropic. The boundary conditions on the components of $\underset{\sim}{E}$ and $\underset{\sim}{H}$ at $x = 0^+$ for the contrary case are much more complicated and will not be given here. They are, however, admissible in the sense of Section (D), Part II.

discussion just concluded to cover cases of this sort in the following way.

At the frequencies of present practical interest, the field variables $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are sharply attenuated as one passes out through an imperfectly conducting guide wall. Their values outside the guide, while in principle mathematically definite, are so small by comparison with the least measurable values of the field variables that they fall outside the scope of Maxwellian theory and can be neglected with good approximation. From this point of view, then, we can imagine that the imperfectly conducting guide walls are themselves surrounded by a perfectly conducting surface. The situation inside the guide is not appreciably



changed, and the previous analysis can now be applied.

From another standpoint, we should find it very difficult to restrict the electromagnetic system under study to a finite region in space unless we introduced a perfectly conducting envelope. As Heaviside said (R-65), "A perfect conductor is a a perfect obstructor . . ."

In the usual definition and calculation of the parameters

R, L, G, and C of an open-wire transmission line, it is supposed that the space lateral to the line is unbounded. Such an assumption is in considerable disagreement with the doctrine of "finite spatial extent" expounded in this paper. However, the approximate equations usually adopted for engineering solutions of transmission-line problems are homogeneous linear partial differential equations [see Section (F), Part II]. If we require that the length of the line be finite, the analysis of Part II and Part III is applicable.

The foregoing development supplies a basis for the introduction of transfer characteristics into the study of linear distributed-parameter electromagnetic systems. This "linearity" is perhaps intuitively obvious; the principal accounts of the network-theoretical aspects of waveguides (R-66, R-67) assume it without question and proceed from there.

D. What Has Been Accomplished?

The investigations of this paper have now been completed; it remains only to decide what has been accomplished and to evaluate the significance of the results.

This is a step which all engineers, practicians and theoreticians alike, would do well to include in their research programs. Such a procedure is certainly necessary to render the work done comprehensible to persons lacking the time, inclination, or ability to wrestle with the inner complexities of the investigation. Indeed, the investigator himself often profits from the preparation of such a summing-up of his achievements.

Very briefly, the investigations have yielded answers to most of the questions and problems posed in Part I (to a surprisingly large proportion of them, in fact).

The field to be considered was limited at the start to that of linear systems of finite spatial extent. This must be acknowledged to be an onerous restriction. Non-linear devices and effects are all about us; furthermore, it is extremely difficult, for example, to analyze an electromagnetic radiating system without making approximations such as those corresponding to placing the system in an infinite space. Nevertheless, within the range of applicability of the mathematical constructs studied, a number of interesting and important matters were dealt with.

Much of Part II is the detailed investigation of general linear distributed-parameter systems; the results can be briefly summarized by saying that many of the commonly encountered generalizations to linear distributed-parameter systems of familiar lumped-parameter-system ideas are valid. This is something which is perhaps intuitively obvious to many people, but a theoretician always views intuitive results with some skepticism until they are checked by analysis or experiment. If he is a cynic, he may even see in them a liberal proportion of wishful thinking.

It was not possible to answer all the questions originally posed in regard to these matters, but in Part III considerable success was achieved in studying the implications of the principle of analytic continuation. The classical paradox of the idealized low-pass filter was shown in a new light, and a general method was given for answering questions of the type "Can I do thus-and-so?" when the data are presented in analytical form.* Questions posed in the case of graphically presented data turned out to be unanswerable if absolute exactness is required in the answers, and reasons why this is so were adduced.**

Finally, in Part IV, it was shown that the results of the present investigations are in essential agreement with the related work of others in this field. In Section (B), Part IV, it was found possible by using the results of Part II and Part III to solve a problem posed by Wallman. One simple realization of a filter possessing a prescribed amplitude characteristic $M(w) = |T(jw)| = |\sin(w)/w|$ was displayedthere.

Surveying this survey, one sees that the achievements are perhaps typical of what theoreticians can accomplish "in practice." The questions of interest to the practician are

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*An affirmative answer here does not guarantee that the desired objective can be accomplished, of course. It just asserts that there is no bar to achieving the desired transfer characteristic, for example, so far as the present studies are concerned.

**Of course, one is generally content with something less than such an ideal answer, but the difficult and important problems of approximation arising here are outside the scope of this paper.

quite easy to state,* but their systematic investigation brings in a host of complications.

The theoretician does not as a rule come right out and say "Yes" or "No" without any qualifications; his answers are couched in pseudo-legal language and do not apply to a host of exceptional instances.

What purpose, then, does the theoretician serve? One can reply only that sometimes his answers are helpful; if both correct and properly understood, they do shed some light on the problem at hand. Furthermore, the quality of his product has improved substantially over what it used to be and bids fair to continue to do so.

Only a theoretician motivated by a great and abiding faith would maintain that the theory will ultimately give all the answers; but, with all its imperfections and shortcomings, it seems to be our best hope if we are not content just to dabble.

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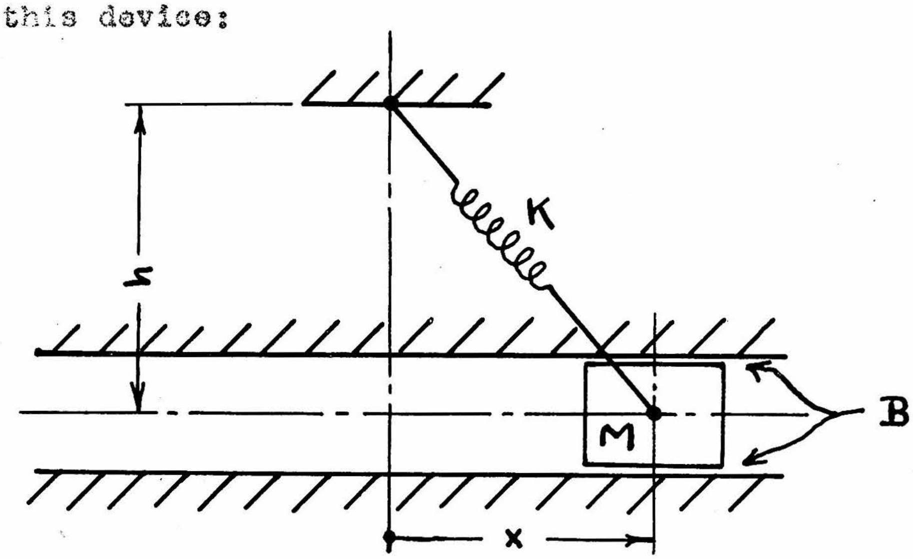
*"Can I do thus-and-so?" for example.

APPENDIX #1

An Example of a Simple Linear Lumped-Parameter Mechanism Ill-Suited to Analysis by the Laplace-Transformation method

Consider this device:



It has only one degree of freedom. It is composed of a rigid mass M and a linear (Hooke's-Law) spring of stiffness parameter K, whose unstretched length is h. Viscous friction (retarding force proportional to relative velocity) affects the mass, with coefficient B. Its equation of motion is (by Newton's Second Law):

$$M\ddot{x} = -B\dot{x} - K\left[1 - \frac{h}{\sqrt{x^2 + h^2}}\right]x.$$

This is not a linear differential equation; attempts to solve it by the Laplace-transformation method lead to failure. Yet the components of the mechanism are linear lumped-parameter mechanical elements.

It hardly seems fair to blame the spring (whose stiffness parameter K is associated with the troublemaking term) for this misfortune; it is doing its linear best. One has some difficulty in putting into words the distinction between mechanical and electrical networks by virtue of which embarrassing mechanisms such as this exist.

Embarrassing it is, for any statements made about the general properties of mechanisms as a result of Laplace-transform analysis must be qualified with the acknowledgment that these statements do not necessarily apply to the members of a vast class of relatively simple devices.

APPENDIX #2

Reversing the Order of Time and Space Integration

It will be here supposed that R is a one-dimensional region; the extension to areas and volumes seems plausible. It is desired to show that

$$\int_0^\infty \int_0^\ell f(x,t) e^{-st} \, dx \, dt = \int_0^\ell \int_0^\infty f(x,t) e^{-st} \, dt \, dx .$$

The complex variable s can be replaced by its real part σ in what follows if desired.

Expressing some classical <u>sufficient</u> conditions (R-23) in the terminology of the present problem, the reversal of integration order will be permissible provided

(1) $f(x,t) \exp(-st) = \phi(x,t)\ \psi(t)$, where $\phi(x,t)$ is continuous in $0 \leqq x \leqq \ell$, $0 \leqq t \leqq T$ (T arbitrary positive), and $\psi(t)$ is bounded and integrable in $0 \leqq t \leqq T$.

(2) $\int_0^\infty f(x,t) e^{-st} dt$ converges uniformly in $0 \leqq x \leqq \ell$.

As regards (1), since $f(x,t)$ is by assumption an E-function of the problem, any discontinuities t-wise it possesses can be charged up to an otherwise well-behaved function $\psi(t)$, so that $\phi(x,t)$ is continuous in $0 \leqq t \leqq T$. It is quite unreasonable to suppose continuity of $\phi(x,t)$ in x, however; in electromagnetic theory, for example, discontinuities in the field vectors at transitions between media are quite common.

Nevertheless, it is physically plausible to assume that $f(x, t)$ is sectionally continuous in $0 \leqq x \leqq \ell$. It is certainly bounded there, and "extraordinary" discontinuities [such as at $x = 0$ for $g(x) = \sin (1/x)$] quite plainly imply spatial variation of the field variable concerned in a fashion not encompassed by the macroscopic theory. From this standpoint, then, let us regard $f(x, t)$ [and thus $\phi(x, t)$] as continuous in $0 \leqq x \leqq \ell_1$, one of the finite number of sections of continuity of $f(x, t)$ in this one-dimensional R. The following argument need only be repeated for each of the other sections.

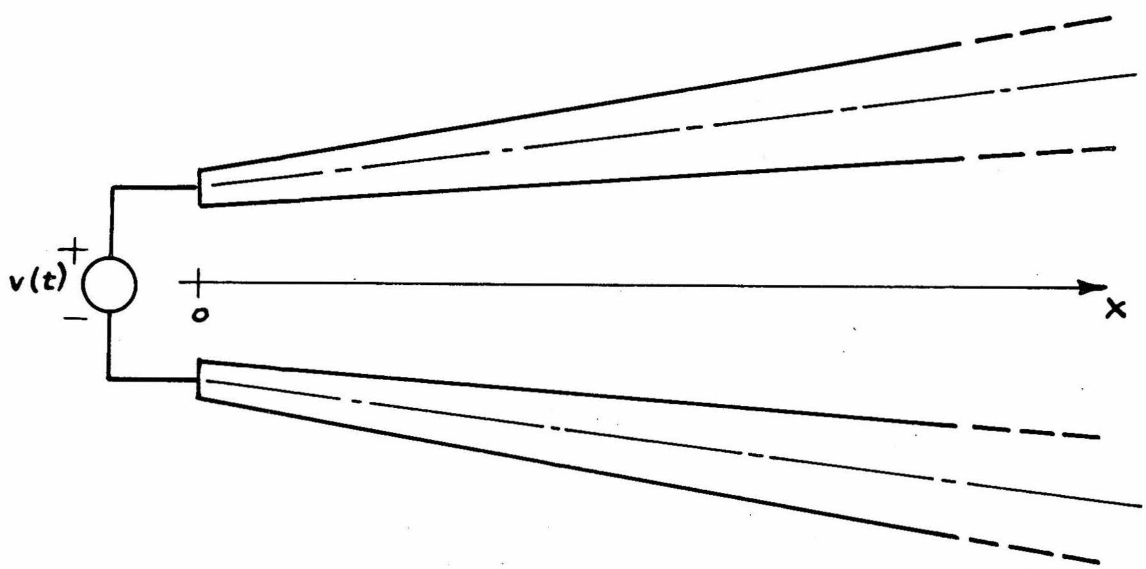
As regards (2), the requirement will be met in $0 \leqq x \leqq \ell_1$ provided there exists a function $m(t) \geqq 0$ such that $|f(x,t)e^{-st}| \leqq m(t)$ for $0 \leqq x \leqq \ell_1$, $0 \leqq t$, and $\int_0^\infty m(t)dt$ exists (R-23).

By virtue of the assumptions about $f(x, t)$ made above, it is certain that a function $M(t) \geqq 0$ exists such that $|f(x, t)| \leqq M(t)$ for $0 \leqq x \leqq \ell_1$, $0 \leqq t$. $M(t)$ corresponds to a P-variable that is in a sense a physical observable, so that it is not unreasonable to suppose that $M(t)$, too, is an E-function of the problem. Thus

$$|f(x,t)e^{-st}| = e^{-\sigma t}|f(x,t)| \leqq e^{-\sigma t}M(t) \leqq e^{-\sigma t}M_M e^{a_M t} = m(t)$$

for $0 \leqq x \leqq \ell_1$, $t \geqq 0$. One has only to take $\sigma > a_M$ to assure the existence of $\int_0^\infty m(t)dt$. Thus the demonstration is completed.

This is an appropriate place to call attention to one implication of the assumption of finite physical extent which is expressly made regarding all systems studied in this paper. One quite reasonably supposes that $f(x, y, z, t)$ is sectionally continuous in space throughout the system, so that an E-function $M(t) \geqq 0$ exists such that $|f(x, y, z, t)| \leqq M(t)$ throughout the system. This is not necessarily true for an unbounded system; as a counterexample, one might cite a certain semi-infinite transmission line.

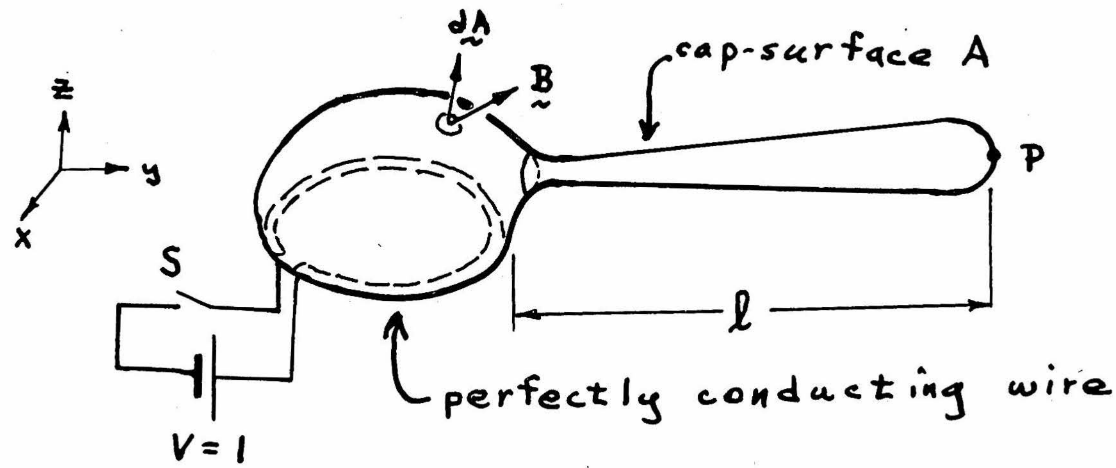


One can perhaps imagine a transient situation in which $v(t)$ is finite but the voltage between conductors increases without bound with x. The diameters of the conductors and their spacing grow with x in such a way that the magnitude of the electric field strength $\underset{\sim}{E}$ never exceeds any assigned value. Clearly, then, no majorant $M(t)$ can exist for the voltage between conductors.

APPENDIX #3

A Counterexample to Depose a Certain Hypothesis

Consider this system.



By classical electromagnetic theory, if the switch S is closed at $t = 0$, then

$$1 = \int_A \frac{\partial \underset{\sim}{B}(x_A, y_A, z_A, t)}{\partial t} \cdot d\underset{\sim}{A} \quad \text{for } t > 0.$$

Suppose that $\underset{\sim}{B}(x, y, z, t) = 0$ for $t < 0$. Transforming,

$$\frac{1}{s} = \int_A s\,\underset{\sim}{\mathcal{B}}(x_A, y_A, z_A, s) \cdot d\underset{\sim}{A},$$

since $\underset{\sim}{B}(x_A, y_A, z_A, 0^+) = 0$. If $s\underset{\sim}{\mathcal{B}}(x_A, y_A, z_A, s)$ is to have the same function-theoretical characteristics as $1/s$, then $\underset{\sim}{\mathcal{B}}(x_A, y_A, z_A, s)$ must be of the form $\underset{\sim}{\mathcal{G}}(x_A, y_A, z_A)/s^2$, and $\underset{\sim}{B}(x_A, y_A, z_A, t) = \underset{\sim}{G}(x_A, y_A, z_A) \cdot t$ for $t > 0$.

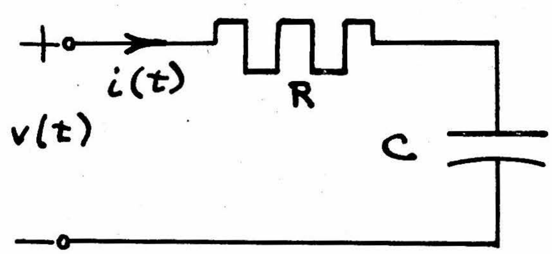
By taking a rather special cap-surface A having a protuberance as shown, one concludes that $\underset{\sim}{B}$ at point P is different from zero arbitrarily soon after $t = 0$. But this is in direct disagreement with the well-known electromagnetic theorem to the effect that $\underset{\sim}{B}$ at P will not change from zero until after at least a time interval $T = \ell/c$, where c is the free-space velocity of electromagnetic-wave propagation. One must, therefore, discard the hypothesis on the function-theoretical properties of $\underset{\sim}{\mathfrak{B}}(x_A, y_A, z_A, s)$.

It is parenthetically noted that this result rests on a firm mathematical foundation, although the discussion above does not recapitulate it. The deposed hypothesis may be true for a different class of equations (those which imply "action-at-a-distance"), but it seems doubtful that such equations describe systems of physical interest.

APPENDIX #4

Another Instructive Counterexample

Consider this electrical circuit, initially without storage of energy.



Its transform equation of motion is

$$V(s) = I(s)\left[R + \frac{1}{Cs}\right]$$

or

$$I(s) = \frac{s\,V(s)}{R\left[s + \frac{1}{RC}\right]} .$$

Suppose that $v(t) = V_o \exp(-at)$, $a > 1/RC$. Then $V(s) = V_o/(s + a)$, and

$$I(s) = \left(\frac{V_o}{R}\right)\frac{s}{(s+a)\left[s + \frac{1}{RC}\right]} .$$

$V(s)$ is holomorphic for $\sigma > -a$. $I(s)$ is holomorphic for $\sigma > -1/RC > -a$, a less extensive half-plane.

APPENDIX #5

The Half-Planes of Holomorphy of F(s) and $\mathcal{F}(x, y, z, s)$

Claim: The half-plane of holomorphy of F(s) is at least as extensive as that of $\mathcal{F}(x_R, y_R, z_R, s)$ for any (x_R, y_R, z_R) in R.

The demonstration will be made for the case of R a one-dimensional region, as in (A-2). By definition,

$$\mathcal{F}(x,s) = \int_0^\infty f(x,t)\, e^{-st}\, dt$$

in $0 \leqq x \leqq \ell$, and to each x there corresponds a minimal (or most efficient) $\sigma(x)$, so that the half-plane of holomorphy of $\mathcal{F}(x, s)$ is $\sigma > \sigma(x)$. Now, since $f(x, t)$ is by assumption sectionally continuous in $0 \leqq x \leqq \ell$, there exists a function $M(t) \geqq 0$ such that $|f(x, t)| \leqq M(t)$, $0 \leqq x \leqq \ell$, $0 \leqq t$.

Just as in (A-2), it is not unreasonable to suppose that M(t) is an E-function of the problem under study. Thus

$$|f(x,t)e^{-st}| = e^{-\sigma t}|f(x,t)| \leqq e^{-\sigma t} M(t) \leqq e^{-\sigma t} M_M e^{a_M t}$$

for $0 \leqq x \leqq \ell$, $0 \leqq t$. It is plain that none of the (minimal) $\sigma(x)$ exceeds a_M, since $\int_0^\infty f(x,t)e^{-st}\,dt$ exists for $\sigma > a_M$ throughout $0 \leqq x \leqq \ell$. Thus the $\sigma(x)$ are bounded above, and a least upper bound $\bar{\sigma}$ exists such that

(1) $\mathcal{F}(x, s)$ is holomorphic for $\sigma > \bar{\sigma}$, $0 \leqq x \leqq \ell$, and

(2) Given any $\epsilon > 0$, there is at least one x_1, $0 \leqq x_1 \leqq \ell$, such that $\mathcal{F}(x_1, s)$ is not holomorphic for $\sigma > \bar{\sigma} - \epsilon$.

Just as in (A-2), we can study the transform of $f(x, t)$ (which is sectionally continuous in $0 \leqq x \leqq \ell$) in $0 \leqq x \leqq \ell_1$, where $f(x, t)$ is continuous x-wise. By classical criteria (R-23), $\mathcal{F}(x, s)$ is continuous x-wise in $0 \leqq x \leqq \ell_1$ for $\sigma > \bar{\sigma}$. Putting

$$F_1(s) = \int_0^{\ell_1} \mathcal{F}(x, s)\, dx,$$

it is seen (R-25) that $F_1(s)$ is holomorphic at least for $\sigma > \bar{\sigma}$. The same process is carried out for each of the finite number (say, n-1) of other sections of continuity of $f(x, t)$ in $0 \leqq x \leqq \ell$; the conclusion is that

$$F(s) = \int_0^{\ell} \mathcal{F}(x, s)\, dx = F_1(s) + F_2(s) + \cdots + F_n(s)$$

is holomorphic at least for $\sigma > \bar{\sigma}$, which was to be proved.

APPENDIX #6

Invariance of the Transfer Characteristic T(s)

Suppose that any other admissible forcing function $f_1(t)$ (not $\equiv 0$) is applied to the system in the same fashion as in Section (D), Part II, giving rise to a similar response function $g_1(t)$. Just as before, the transfer characteristic is

$$T_1(s) = \frac{G_1(s)}{F_1(s)}.$$

Now, put $W(s) = F_1(s)/F(s)$. Then

$$F_1(s) = W(s).F(s) = \int_{R_i} W(s)\,\mathcal{F}(R,s)\,dR = \int_{R_i} \mathcal{F}_1(R,s)\,dR.$$

Multiplying the original linear homogeneous transformed equations of motion and remaining boundary conditions (which are all appropriately linear and homogeneous) by $W(s)$, it is apparent by inspection that $\mathcal{G}_1(R, s) \equiv W(s)\cdot\mathcal{G}(R, s)$ satisfies all of them. So,

$$T_1(s) = \frac{G_1(s)}{F_1(s)} = \frac{\int_{R_o} \mathcal{G}_1(R,s)\,dR}{\int_{R_i} \mathcal{F}_1(R,s)\,dR} = \frac{\int_{R_o} W(s)\,\mathcal{G}(R,s)\,dR}{\int_{R_i} W(s)\,\mathcal{F}(R,s)\,dR}$$

$$= \frac{\int_{R_o} \mathcal{G}(R,s)\,dR}{\int_{R_i} \mathcal{F}(R,s)\,dR} = \frac{G(s)}{F(s)} = T(s).$$

This establishes the fact that the transfer characteristic $T(s)$ is independent of the forcing function $f(t)$, and is thus determined solely by the composition of the physical system itself.

APPENDIX #7

Other Ways of Introducing "Cause" and "Effect" Functions

In Sections (C) and (D) of Part II, extensive consideration was given to linear distributed-parameter systems of finite spatial extent in which the linear partial integro-differential equations of motion were homogeneous. In addition, it was supposed that the initial values of certain field variables and some of their derivatives were zero; that the forcing and response functions entered the mathematical specification of the problem via integrals of field variables over domains in space; and that all other spatial boundary conditions transformed into linear homogeneous algebraic equations of a certain specified type. The transfer characteristic T(s) was defined for a system of this sort, and we must now show that such a function can be discussed in the case of physical systems differing in some particulars from those such as were described above.

First of all, it matters but little how the response function enters the problem. It may be simply the value of a field variable at some point in space, for example. All that counts is for it to be a sort of sum of field-variable values, so that the linearity of the equation $G(s) = T(s) \cdot F(s)$ will be preserved.

Much the same thing is true of how the forcing function enters the problem. It may determine the value of a field variable at some point in space, or it may even enter the partial integrodifferential equations of motion themselves, rendering them inhomogeneous though still linear. As long as the dependent variables of the problem are chosen to give zero initial conditions, and the remaining transformed spatial boundary conditions are linear and homogeneous with coefficients as specified, the development of the system invariant T(s) will follow without difficulty. The extension of (A-6) to the present case offers no great problem.

With this background, then, completely general applicability will be claimed for the notion of the transfer characteristic T(s).

APPENDIX #8

The Half-Plane of Holomorphy of T(s)

As stated in Section (D) and (E) of Part II, a "cause" f(t) applied to a member of a certain class of physical systems gives rise to an "effect" g(t), and the transfer characteristic of the system is defined to be $T(s) \equiv G(s)/F(s)$. Suppose that the system is one whose equations of motion involve at most nth-order time derivatives, so that it can tolerate an nth-order input. That is,

$$f(t) = 0, \quad t < 0$$

$$f(t) = t^n, \quad t > 0, \quad n = 0, 1, 2, \cdots .$$

Then $F(s) \equiv \frac{\lfloor n}{s^{n+1}}$ (R-15), and $T(s) \equiv \frac{s^{n+1} G(s)}{\lfloor n}$. G(s) is holomorphic for $\sigma > \sigma_H\{g(t)\}$ by (R-24). It is plain that T(s) is also holomorphic for $\sigma > \sigma_H\{g(t)\}$, some real number.

This, then, displays a half-plane of holomorphy of T(s), since T(s) is invariant with respect to inputs f(t) by (A-6).*

A related strictly-transform-theoretical result may be of interest. Suppose that d(t) [not a null function (R-28)] is transformable, and $d(0^+) \neq 0$. D(s) (not $\equiv 0$) is holomorphic

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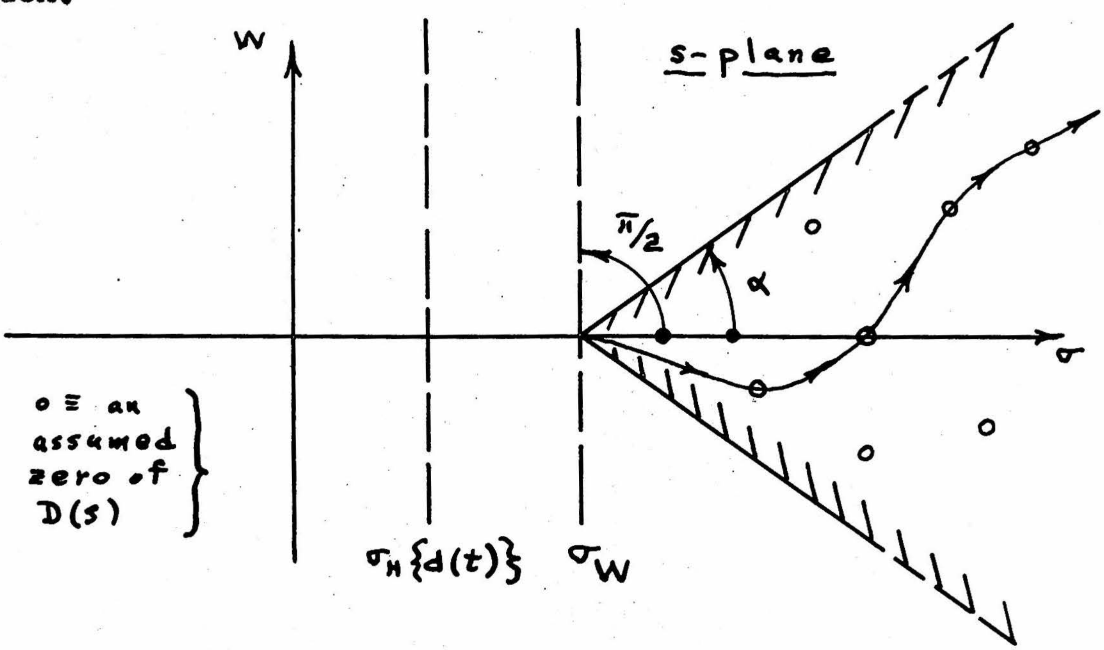
*The half-plane of holomorphy of T(s) can, of course, be much more extensive than the half-plane determined by this existence argument.

in some half-plane $\sigma > \sigma_H\{d(t)\}$ by (R-24). Then D(s) has no more than a finite number of zeros in any infinite wedge-region described by

$$|\arg(s - \sigma_W)| \leqq \alpha < \pi/2 ,$$

where $\sigma_W > \sigma_H\{d(t)\}$.

Proof: Suppose the contrary. The zeros cannot have a finite point of accumulation in the wedge-region, since that would contradict the holomorphy of D(s) for $\sigma > \sigma_H\{d(t)\}$ (R-27). The zeros must, then, be "strung out" to the right in the wedge-region.



Consider Lim sD(s) as $s \rightarrow \infty$ with complete two-dimensional freedom in the wedge-region. By (R-29), this limit exists and is equal to $d(0^+)$. Now, let us perform the limiting operation by traveling out along a path which goes through indefinitely many of the (hypothetical) zeros of D(s). This causes the product sD(s) to vanish occasionally; no matter how far the journey has proceeded, sD(s) will vanish indefinitely many times during the remainder of the trip.

But this implies $\lim_{s \to \infty} sD(s) = 0$, since this limit is known to exist. However, $\lim_{s \to \infty} sD(s) = d(0^+) \neq 0$ by initial assumption. So, the original hypothesis must be false, and the theorem is proved.

APPENDIX #9

T(s) for the Non-Uniform Transmission Line

Let us investigate the solutions of the equations

$$\left.\begin{aligned} \frac{dV(x,s)}{dx} &= -Z(x,s)\,I(x,s) \\ \frac{dI(x,s)}{dx} &= -Y(x,s)\,V(x,s) \end{aligned}\right\} 0 < x < \ell$$

subject to the boundary conditions

$$V(0,s) = V(s)$$

$$V(\ell,s) - Z_T(s)\,I(\ell,s) = 0.$$

$Z(x, s)$ and $Y(x, s)$ are entire functions of s,* and $Z_T(s)$ is a rational function of s.**

If we suppose that the line is neither short- nor open-circuited $\left[V(\ell, s) \not\equiv 0,\ I(\ell, s) \not\equiv 0\right]$, then the problem can be restated in this way:

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*A function Q(s) of a complex variable s is said to be an entire function if it is analytic and single-valued throughout the finite s-plane.

**This example illustrates the manner in which linear lumped-parameter sub-systems are associated with linear distributed-parameter systems, and indicates a reason for the specification of admissible boundary conditions made in Section (D), Part II.

$$\frac{d}{dx}\left[\frac{V(x,s)}{V(\ell,s)}\right] = -\frac{Z(x,s)}{V(\ell,s)} I(x,s) = -\frac{Z(x,s)}{Z_T(s)}\left[\frac{I(x,s)}{I(\ell,s)}\right] \quad (0<x<\ell)$$

$$\frac{d}{dx}\left[\frac{I(x,s)}{I(\ell,s)}\right] = -\frac{Y(x,s)}{I(\ell,s)} V(x,s) = -Y(x,s)Z_T(s)\left[\frac{V(x,s)}{V(\ell,s)}\right] \quad (0<x<\ell)$$

$$\left[\frac{V(x,s)}{V(\ell,s)}\right]_{x=\ell} = 1$$

$$\left[\frac{I(x,s)}{I(\ell,s)}\right]_{x=\ell} = 1.$$

Here is a (translated) theorem on differential equations by Horn (H-36):

"In the system of differential equations with parameters μ_1, μ_2, ...

$$\frac{dy_i}{dx} = f_i(x, y_1, \cdots y_n, \mu_1, \mu_2, \cdots) \quad (i=1,2,\cdots n)$$

let the functions f_i as well as the derivatives $\frac{\partial f_i}{\partial y_1}$, ... $\frac{\partial f_i}{\partial y_n}$, $\frac{\partial f_i}{\partial \mu_1}$, $\frac{\partial f_i}{\partial \mu_2}$, ... be continuous with respect to all the arguments x, y_1, ... y_n, μ_1, μ_2, ... in a certain domain. Then the differential-equation-satisfying functions y_i (i = 1, 2, ... n) which are determined by the initial conditions x = a, $y_1 = b_1$, ... $y_n = b_n$ (independent of μ_1, μ_2, ...) as well as the derivatives $\frac{\partial y_i}{\partial \mu_1}$, $\frac{\partial g_i}{\partial \mu_2}$, ... are continuous functions of the variable x and the parameters μ_1, μ_2, ..."

This is just a reflection of the fact that the integrals of linear ordinary differential equations can have no other singular points than those of the coefficients. The applicability of the theorem to our restated problem is evident; we need only identify μ with s, x = a with x = ℓ, $[V(x, s)/V(\ell, s)]$ with y_1, $[I(x, s)/I(\ell, s)]$ with y_2, b_1 and b_2 both with unity. Since the only singularities of the f_i in the finite s-plane are poles arising from the poles and zeros of $Z_T(s)$, we see that the functions $[V(x, s)/V(\ell, s)]$ and $[I(x, s)/I(\ell, s)]$ cannot have singularities at other than these same points in the finite s-plane!

The function-theoretical aspects of the matter are emphasized by Goursat's statement of an equivalent theorem (R-37):

> "We often have occasion to study systems of linear equations whose coefficients are analytic functions of certain parameters. Let us suppose, for definiteness, that the coefficients ... of the equations ... are continuous functions of x in the interval (a, b), and that they depend also upon a parameter λ of which they are analytic functions in a region D.
>
> "The integrals of this system which take on given initial values for a value x_0 of x included between a and b are represented in the whole interval (a, b) by uniformly convergent series, and from the very manner in which we obtain them it is clear that all the terms of this series are analytic functions of the parameter λ in D. These integrals are therefore themselves analytic functions of λ in the region D."

Some doubt as to the applicability of this theorem may be occasioned by Goursat's plainly stated requirement that the point x_0 (where the values of the dependent variables are specified) is included between the end points of the interval (a, b). In the problem at hand we identify x_0 with ℓ, and (at first sight) (a, b) with (0, ℓ), since that is the location of the transmission line. This procedure obviously does not meet Goursat's conditions, and probably violates some of Horn's implicit assumptions as well.

To get out of this predicament, we may imagine that the restated equations above apply in an extended range $0 < x < 2\ell$. The values of $Z(x, s)$ and $Y(x, s)$ in $\ell < x \leqq 2\ell$ can be taken to be equal to their values at $x = \ell$, thus maintaining the continuity of coefficients which is so important in the theorems. It might be possible to give a physical interpretation of this mathematical artifice, but there is no necessity to do so.

If (referring to Goursat's statement) we take as the region D the finite s-plane exclusive of small neighborhoods of the poles and zeros of $Z_T(s)$, then $\left[V(x, s)/V(\ell, s)\right]$ and $\left[I(x, s)/I(\ell, s)\right]$ must be analytic throughout D. None of their singularities can be branch points (the required branch cuts would trespass on D).

It is not so easy, however, to decide that no essential singularities of finite affix can occur in these functions. This aspect of the problem is investigated in (A-10), where it is shown that the singularities of $\left[V(x, s)/V(\ell, s)\right]$ are poles agreeing in position and order with the zeros of $Z_T(s)$; the singularities of $\left[I(x, s)/I(\ell, s)\right]$ are poles agreeing in position and order with the poles of $Z_T(s)$.

Since the zeros and poles of $Z_T(s)$ are finite in number, $\left[V(x, s)/V(\ell, s)\right]$ and $\left[I(x, s)/I(\ell, s)\right]$ are both, then, meromorphic functions of s.*

Suppose that the transfer characteristic of interest is the input impedance $Z_I(s)$ at the input terminals $(x = 0)$ of the line. By definition,

$$Z_I(s) = \frac{V(0,s)}{I(0,s)} = Z_T(s)\,\frac{\left[\dfrac{V(0,s)}{V(\ell,s)}\right]}{\left[\dfrac{I(0,s)}{I(\ell,s)}\right]},$$

since $V(\ell, s) = Z_T(s)\cdot I(\ell, s)$. $Z_T(s)$ is a rational function, while $[V(0, s)/V(\ell, s)]$ and $1/[I(0, s)/I(\ell, s)]$ are meromorphic functions. The last part of this conclusion follows since the zeros of $[I(0, s)/I(\ell, s)]$, although isolated, are not necessarily finite in number. We are led to conclude, therefore, that $Z_I(s)$, the product of these functions, is a meromorphic function itself, Q. E. D. Note that this transfer characteristic, being an invariant of the system, is found without explicit knowledge of the forcing-function transform $V(s)$.

For the special case in which R, L, G, and C are uniform in $0 \leqq x \leqq \ell$, it can be shown (R-38, compare A-10) that

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*We cannot assert that they are rational functions of s, since the theorems of Horn and Goursat do not give directly any information about the exact behavior of the functions in the vicinity of "$s = \infty$," where $Z(x, s)$ and $Y(x, s)$ have simple poles. Indeed, the example considered in (A-10) shows that the solutions can have essential singularities at "$s = \infty$."

$$Z_I(s) = Z_T(s)\left\{\frac{\cosh\left(\sqrt{Z(s)Y(s)}\,\ell\right) + \left[\frac{Z(s)}{Z_T(s)}\right]\left[\frac{\sinh\left(\sqrt{Z(s)Y(s)}\,\ell\right)}{\sqrt{Z(s)Y(s)}}\right]}{Z_T(s)Y(s)\left[\frac{\sinh\left(\sqrt{Z(s)Y(s)}\,\ell\right)}{\sqrt{Z(s)Y(s)}}\right] + \cosh\left(\sqrt{Z(s)Y(s)}\,\ell\right)}\right\}$$

agreeing with the general expression for $Z_I(s)$.

It is interesting to note that the singularities of $Z_I(s)$ are not necessarily those of $Z_T(s)$, as is proved by considering the case $Z_T(s) = R_T \neq 0$, a constant (resistive termination of the line).

This $Z_T(s)$ is an entire function which never vanishes. Any singularities of $Z_I(s)$ come about, therefore, by the vanishing of its bracketed denominator above. If we specialize to $R = 0$, $G = 0$, $L \neq 0$, $C \neq 0$ (dissipationless line) so that $Z(s) = sL$, $Y(s) = sC$, then

$$Z_I(s) = R_T\left[\frac{\cosh\left(\sqrt{LC}\,\ell s\right) + \frac{1}{R_T}\sqrt{\frac{L}{C}}\,\sinh\left(\sqrt{LC}\,\ell s\right)}{R_T\sqrt{\frac{C}{L}}\,\sinh\left(\sqrt{LC}\,\ell s\right) + \cosh\left(\sqrt{LC}\,\ell s\right)}\right]$$

If $R_T = \sqrt{L/C}$, then $Z_I(s) = R_T$ for all s; the line is terminated in its "characteristic" impedance. If $R_T > \sqrt{L/C}$, then $Z_I(s)$ has a (simple) pole at $s = \sigma = -\left[\operatorname{artanh}\left(\sqrt{L/CR_T^2}\right)\right] / \sqrt{LC}\,\ell$, and perhaps other singularities elsewhere as well.

For completeness we shall consider the open-circuited line, specifically excluded from the foregoing development. Noting that $V(\ell, s) \not\equiv 0$, we can restate the problem in this way:

$$\left.\begin{aligned} \frac{d}{dx}\left[\frac{V(x,s)}{V(\ell,s)}\right] &= -Z(x,s)\left[\frac{I(x,s)}{V(\ell,s)}\right] \\ \frac{d}{dx}\left[\frac{I(x,s)}{V(\ell,s)}\right] &= -Y(x,s)\left[\frac{V(x,s)}{V(\ell,s)}\right] \end{aligned}\right\} 0 < x < \ell$$

$$\left[\frac{V(x,s)}{V(\ell,s)}\right]_{x=\ell} = 1$$

$$\left[\frac{I(x,s)}{V(\ell,s)}\right]_{x=\ell} = 0.$$

Since $Z(x, s)$ and $Y(x, s)$ are entire functions, $\left[V(x, s)/V(\ell, s)\right]$ and $\left[I(x, s)/V(\ell, s)\right]$ are entire functions too, by the theorems of Horn and Goursat stated earlier. The input impedance of the line is

$$Z_I(s) = \frac{V(0,s)}{I(0,s)} = \frac{\left[\frac{V(0,s)}{V(\ell,s)}\right]}{\left[\frac{I(0,s)}{V(\ell,s)}\right]},$$

and it is certainly a meromorphic function, possibly having poles at the (isolated) zeros of $\left[I(0, s)/V(\ell, s)\right]$.

The consideration of the short-circuited line is quite similar, and the details will not be given here.

So much for the input impedance of the line. Other transfer characteristics may be studied; if we are interested in voltage transformation, then

$$A(s) = \frac{V(x,s)}{V(0,s)} = \frac{\left[\dfrac{V(x,s)}{V(\ell,s)}\right]}{\left[\dfrac{V(0,s)}{V(\ell,s)}\right]} = \frac{\left[\dfrac{V(x,s)}{I(\ell,s)}\right]}{\left[\dfrac{V(0,s)}{I(\ell,s)}\right]}$$

is a meromorphic function (the last part of the expression is to be employed in the case of a short-circuited line).

APPENDIX #10

A Supplement to Appendix #9

Since this is an Appendix to an Appendix, it should be permissible to restate the restated problem. We are investigating the properties of the solutions of

$$\left.\begin{aligned} Z_T(s)\frac{dM(x,s)}{dx} &= -\left[R(x)+s\,L(x)\right]N(x,s) \\ \frac{dN(x,s)}{dx} &= -\left[G(x)+s\,C(x)\right]Z_T(s)\,M(x,s) \end{aligned}\right\} 0<x<\ell$$

subject to the boundary conditions

$$M(\ell,s) = 1$$

$$N(\ell,s) = 1.$$

It has been shown in (A-9) that $M(x, s)$ and $N(x, s)$ cannot possibly have singularities in the finite s-plane at other than the poles and zeros of the rational function $Z_T(s)$. We must now determine the exact nature of these singularities, knowing as yet only that they cannot be branch points.

Suppose that $Z_T(s)$ has a pth-order pole ($p > 0$) at $s = s_o$, a finite point in the s-plane. We shall now show that $M(x, s)$ is analytic at $s = s_o$, and that $N(x, s)$ has a pth-order pole there.

The Laurent's-series expansion of $Z_T(s)$ about $s = s_o$ can be assumed known;

$$Z_T(s) = \sum_{n=-p}^{\infty} c_n (s-s_o)^n$$

throughout some annular neighborhood of $s = s_o$. Let us try to determine the functions $a_n(x)$ and $b_n(x)$ in the expansions

$$M(x,s) = \sum_{n=0}^{\infty} a_n(x)(s-s_o)^n$$

$$N(x,s) = \sum_{n=-p}^{\infty} b_n(x)(s-s_o)^n .$$

In order that the boundary conditions of the restated restated problem be satisfied, these coefficient functions must satisfy

$$a_o(\ell) = 1$$

$$a_n(\ell) = 0, \quad n = 1, 2, \cdots$$

and

$$b_0(\ell) = 1$$

$$b_n(\ell) = 0, \quad n = -p, -p+1, \cdots, -1, 1, 2, \cdots .$$

Substituting the expansions into the equations of the problem, we get

$$\left[\sum_{n=-p}^{\infty} c_n (s-s_0)^n\right]\left[\sum_{n=0}^{\infty} a_n'(x)\,(s-s_0)^n\right]$$

$$= -\left[R(x) + s_0 L(x) + (s-s_0) L(x)\right]\left[\sum_{n=-p}^{\infty} b_n(x)(s-s_0)^n\right]$$

and

$$\left[\sum_{n=-p}^{\infty} b_n'(x)(s-s_0)^n\right] = -\left[G(x) + s_0 C(x) + (s-s_0) C(x)\right]$$

$$\times \left[\sum_{n=-p}^{\infty} c_n (s-s_0)^n\right]\left[\sum_{n=0}^{\infty} a_n(x)(s-s_0)^n\right].$$

Equating the coefficients of $(s-s_0)^{-p}$ in the two equations, we get the subsidiary differential-equations problem

$$\left.\begin{aligned} c_{-p}a_0'(x) &= -\left[R(x)+s_0L(x)\right]b_{-p}(x) \\ b_{-p}'(x) &= -\left[G(x)+s_0C(x)\right]c_{-p}a_0(x) \end{aligned}\right\} 0<x<\ell$$

$$a_0(\ell) = 1$$

$$b_{-p}(\ell) = 0 .$$

The solution of this problem yields functions $a_0(x)$ and $b_{-p}(x)$ which are not identically zero.*

Equating the coefficients of $(s-s_0)^{-p+1}$, we get the subsidiary differential-equations problem

$$\left.\begin{aligned} & c_{-p}a_1'(x) + c_{-p+1}a_0'(x) \\ &\qquad = -\left[R(x)+s_0L(x)\right]b_{-p+1}(x) - L(x)\,b_{-p}(x) \\ & b_{-p+1}'(x) = -\left[G(x)+s_0C(x)\right]\left[c_{-p}a_1(x)+c_{-p+1}a_0(x)\right] \\ &\qquad - C(x)\,c_{-p}\,a_0(x) \end{aligned}\right\} 0<x<\ell$$

$$a_1(\ell) = 0$$

$$b_{-p+1}(\ell) = 1 \quad \text{for } p = 1$$

$$b_{-p+1}(\ell) = 0 \quad \text{for } p = 2, 3, \cdots .$$

*Had we assumed higher-order singularities of $M(x, s)$ and $N(x, s)$ at $s = s_0$, we should have come up with differential equations and boundary conditions satisfied by solutions which vanish identically, showing that the higher-order singularities were not really present.

The solution of this problem yields the functions $a_1(x)$ and $b_{-p+1}(x)$. It is clear that the process can be continued step-wise, allowing each of the functions $a_n(x)$ and $b_n(x)$ to be determined. Thus the hypothesis as to the behavior of $M(x, s)$ and $N(x, s)$ in the vicinity of $s = s_0$ is verified.

The analysis of the behavior of $M(x, s)$ and $N(x, s)$ near a zero of $Z_T(s)$ is, by the symmetry of the problem, quite similar to the foregoing, so that if $Z_T(s)$ has a qth-order zero $(q > 0)$ at $s = s_a$, a finite point in the s-plane, then $M(x, s)$ has a qth-order pole at $s = s_a$, and $N(x, s)$ is analytic there.

It is of interest to check this analysis by comparison with the known results for the ordinary transmission line, in which R, L, G, and C are uniform in $0 \leqq x \leqq \ell$. It can be shown [after a little manipulation of some results in (R-38)] that

$$\left[\frac{V(x,s)}{V(\ell,s)}\right] = \cosh\left[\sqrt{Z(s)Y(s)}(\ell - x)\right] + \left[\frac{Z(s)}{Z_T(s)}\right]\left[\frac{\sinh\left[\sqrt{Z(s)Y(s)}(\ell-x)\right]}{\sqrt{Z(s)Y(s)}}\right],$$

where $Z(s) = R + Ls$ and $Y(s) = G + Cs$. Thus the poles of $[V(x, s)/V(\ell, s)]$ agree in position and order with the zeros of $Z_T(s)$. Similarly,

$$\left[\frac{I(x,s)}{I(\ell,s)}\right] = \cosh\left[\sqrt{Z(s)Y(s)}\,(\ell-x)\right] + Z_T(s)\,Y(s)\left[\frac{\sinh\left(\sqrt{Z(s)Y(s)}\,(\ell-x)\right)}{\sqrt{Z(s)Y(s)}}\right]$$

has poles which agree with those of $Z_T(s)$. This agrees exactly with the general analysis above.

APPENDIX #11

Explicit Continuation by Borel Summation

Borel's method of summation is discussed in (R-53, R-54, R-55). The procedure is as follows: Suppose that we are given a power series

$$f(s) = \sum_{n=0}^{\infty} a_n s^n$$

whose radius of convergence is unity. With this functional element we associate an entire function

$$\phi(s) = \sum_{n=0}^{\infty} \frac{a_n}{\underline{\lfloor n}} s^n.$$

Then the function defined by the integral

$$f_1(s) = \int_0^{\infty} e^{-t} \phi(st) dt$$

is exactly equal to f(s) within the circle $|s| = 1$. What is more, the integral converges and represents an analytic function in the interior of a certain convex polygon which is often a more extensive region than the interior of the unit circle. Thus $f_1(s)$ represents the analytic continuation of f(s) outside the unit circle within the polygon. The integral diverges outside this polygon, by the way.

As an example of the method, suppose that

$$f(s) = \sum_{n=0}^{\infty} s^n .$$

The circle of convergence of this series is certainly $|s| = 1$, and $a_n = 1$ for all n. The associated entire function is

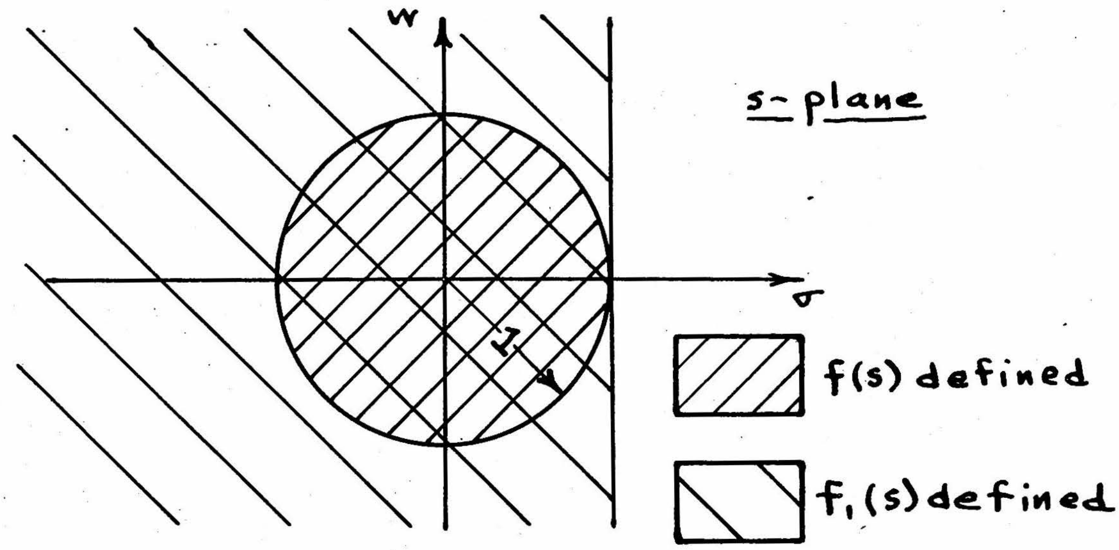
$$\phi(s) = \sum_{n=0}^{\infty} \frac{s^n}{\underline{|n}} = e^s$$

by comparison with a well-known series expansion for exp(s). Borel's integral is, then,

$$f_1(s) = \int_0^{\infty} e^{-t} e^{st} dt = \int_0^{\infty} e^{-(1-s)t} dt .$$

This integral converges for $\sigma \equiv \mathrm{Re}(s) < 1$, so that

$$f_1(s) = \frac{1}{1-s} , \quad \sigma < 1 .$$



In this example the (degenerate) polygon of summability is the entire half-plane $\sigma < 1$. If we expand $f_1(s)$ in a Taylor's series about $s = 0$, we discover that $f_1(s) \equiv f(s)$ for $|s| < 1$, since the corresponding coefficients of powers of s are equal. So, $f_1(s)$ is the unique analytic continuation of f(s) outside the unit circle in the region $\sigma < 1$.

Of course, we could have summed the series for f(s) by inspection to begin with; one suspects that continuation by Borel's method is not of great utility in most practical cases. However, the existence of the integral expression for $f_1(s)$ outside the unit circle is a comfort and a reassurance. Most authors merely prove the classical uniqueness theorems of analytic continuation, never even trying to carry out the process.

Nobody seems to have taken the trouble to continue a Laurent's-series functional element* by Borel summation. Since

*Such a series is just as good a functional element as a power series.

the Taylor's-series portion of a Laurent's series offers no difficulties, let us omit all but the negative-exponent terms. Suppose that

$$g(s) = \sum_{n=1}^{\infty} \frac{b_n}{s^n}$$

converges outside the unit circle, diverging for $|s| < 1$. With this functional element we associate an entire function

$$\theta(s) = \sum_{n=1}^{\infty} \frac{b_n}{\lfloor n} s^n .$$

Then

$$g_1(s) = \int_0^{\infty} e^{-t} \theta\left(\frac{t}{s}\right) dt$$

is the integral of interest.

As an example of the method, suppose that we are given

$$g(s) = \sum_{n=1}^{\infty} \frac{1}{s^n} .$$

This series converges for $|s| > 1$ and diverges for $|s| < 1$; $b_n = 1$ for all n. The associated entire function is

$$\theta(s) = \sum_{n=1}^{\infty} \frac{s^n}{\lfloor n} = e^s - 1$$

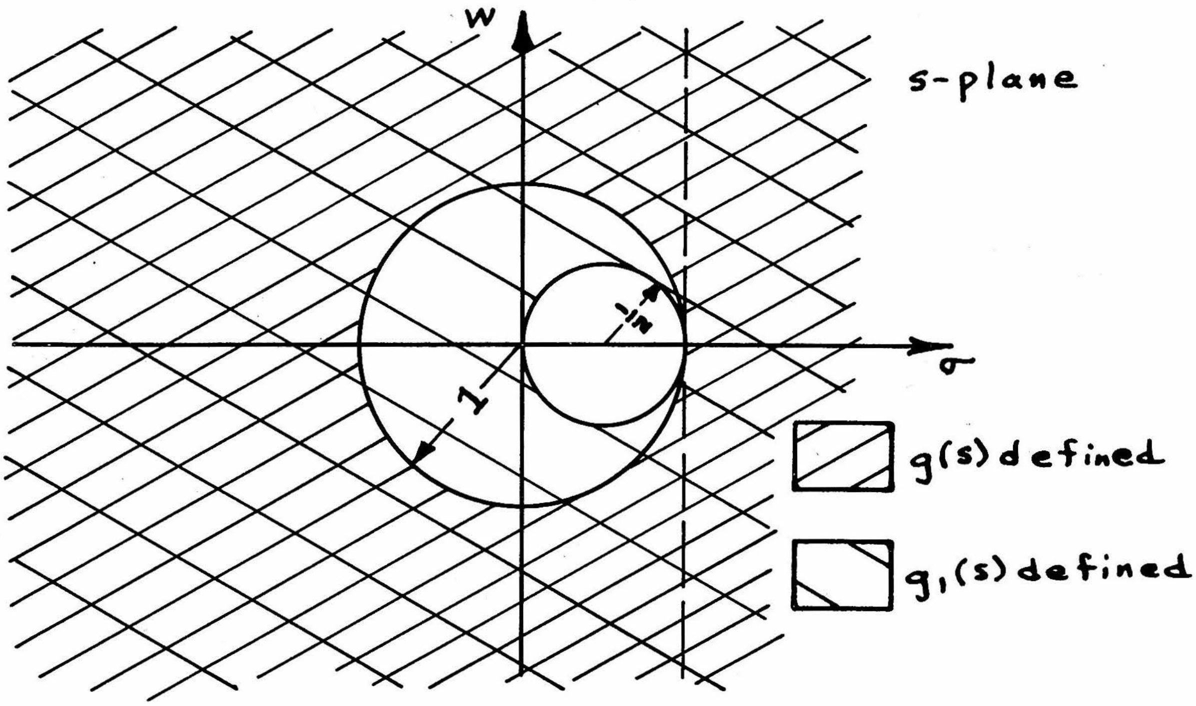
by inspection. The integral of interest is

$$g_1(s) = \int_0^\infty e^{-t}\left[e^{\frac{t}{s}} - 1\right] dt = \int_0^\infty e^{-(1-\frac{1}{s})t} dt - 1.$$

This integral converges for all points s for which $\mathrm{Re}(1-1/s) > 0$. This is the region $(\sigma - \frac{1}{2})^2 + w^2 > (\frac{1}{2})^2$, the exterior of a certain circle. In this region, then,

$$g_1(s) = \frac{1}{1-\frac{1}{s}} - 1 = \frac{s}{s-1} - 1 = \frac{1}{s-1}.$$

The Laurent's-series expansion of $g_1(s)$ about $s = 0$ for $|s| > 1$ agrees with the original expression for $g(s)$, so that $g_1(s)$ is the analytic continuation of $g(s)$ into the portion of the unit circle outside the circle $(\sigma - \frac{1}{2})^2 + w^2 = (\frac{1}{2})^2$.



$g_1(s)$ and $f_1(s)$ differ only in sign, and their boundaries of existence are an inverse pair with respect to the unit circle.

APPENDIX #12

The Significance of a Certain Restriction on T(s)

It was shown in Section (E), Part II, that T(s) cannot increase in magnitude faster than s^k as $s \to \infty$ in any wedge-region described by

$$|\arg(s-\sigma_W)| \leqq \alpha < \pi/2,$$

where $\sigma_W > \sigma_H\{T(s)\}$.

Consider $T(s) = \exp(st_a)$, where t_a is positive. This proposed transfer characteristic is an entire function (thus possessing no natural boundaries) and is real for s real. It does not satisfy the requirement stated in the first paragraph above, since $|T(s)| \to \infty$ "too fast" as $s \to \infty$ in any wedge-region described above. More explicitly (taking $\sigma_W = 0$), since

$$s = |s|\{\cos[\arg(s)] + j\sin[\arg(s)]\}$$

$$|\arg(s)| \leqq \alpha < \pi/2$$

$$\cos[\arg(s)] \geqq \cos\alpha > 0,$$

then

$$|T(s)| = \exp(t_a \sigma) = \exp\{t_a |s| \cos[\arg(s)]\}$$

$$|T(s)| \geqq \exp\{t_a \cos\alpha\ |s|\},$$

and since $t_a \cos\alpha > 0$, $|T(s)| \to \infty$ faster than any $|s^K| = |s|^K$, by a well-known property of the real exponential function.

This negative conclusion is quite understandable when we consider to just what sort of hypothetical device this outlawed T(s) corresponds. Recalling that a transfer characteristic equal to $\exp(-t_d s)$ ($t_d > 0$) produces a <u>delay</u> of t_d units of time of the output signal g(t) behind the input signal f(t), we see that $T(s) = \exp(t_a s)$ ($t_a > 0$) would produce an output signal g(t) which <u>anticipates</u> the input signal f(t) by t_a units of time. Such a prediction device for arbitrary input functions f(t) is certainly nonsensical.

This cause-and-effect conclusion is exactly the essential feature of the Paley-Wiener-Wallman criterion discussed in Section (E), Part IV. Though that criterion and the tests developed in this paper are by no means equivalent, there seems to be a close connection between them. Since both come from Fourier-/Laplace-transformation analysis, this is not surprising.

APPENDIX #13

The Semi-Idealized Low-Pass Filter

For this device, with $M(w) = |T(jw)|$,

$$M(w) = 1, \quad |w| < 1$$

$$M(w) = \epsilon, \quad |w| > 1.$$

The filter is low-pass for $\epsilon < 1$; there is, however, nothing to prevent one's taking $\epsilon > 1$ and having essentially a high-pass filter.

In studying this problem by the methods developed in Part II and Part III, it is easiest to start with the expression for T(s) suggested by the work in (R-63) and then to show that it has the proper magnitude for $s = jw$ and fulfills other requirements on transfer characteristics in general. Consider (R-64)

$$T(s) = \exp\left[\frac{2 \operatorname{Log} \epsilon}{\pi} \arctan(s)\right]$$

$$= \exp\left[\frac{2 \operatorname{Log} \epsilon}{\pi} \cdot \frac{\operatorname{Log}\left(\frac{1+js}{1-js}\right)}{2j}\right]$$

$$= \exp\left[j\,\frac{\operatorname{Log} \epsilon}{\pi} \cdot \operatorname{Log}\left(\frac{1-js}{1+js}\right)\right],$$

where the principal branch of the logarithmic function is to be taken. T(s) is real for s real, and if we introduce branch cuts running in the negative σ-direction from the points s = $\pm$j, T(s) is holomorphic for $\sigma > \sigma_H\{T(s)\} = 0$. T(s) has no natural boundaries. Letting $s \to \infty$ along a ray s = r exp(jθ), $|\theta| < \pi/2$, θ constant,

$$\frac{1-js}{1+js} = \frac{1-jre^{j\theta}}{1+jre^{j\theta}} = \frac{(1-r^2)-j2r\cos\theta}{1-2r\sin\theta+r^2}$$

approaches -1-j0 as $r \to \infty$, since cos $\theta > 0$. Then

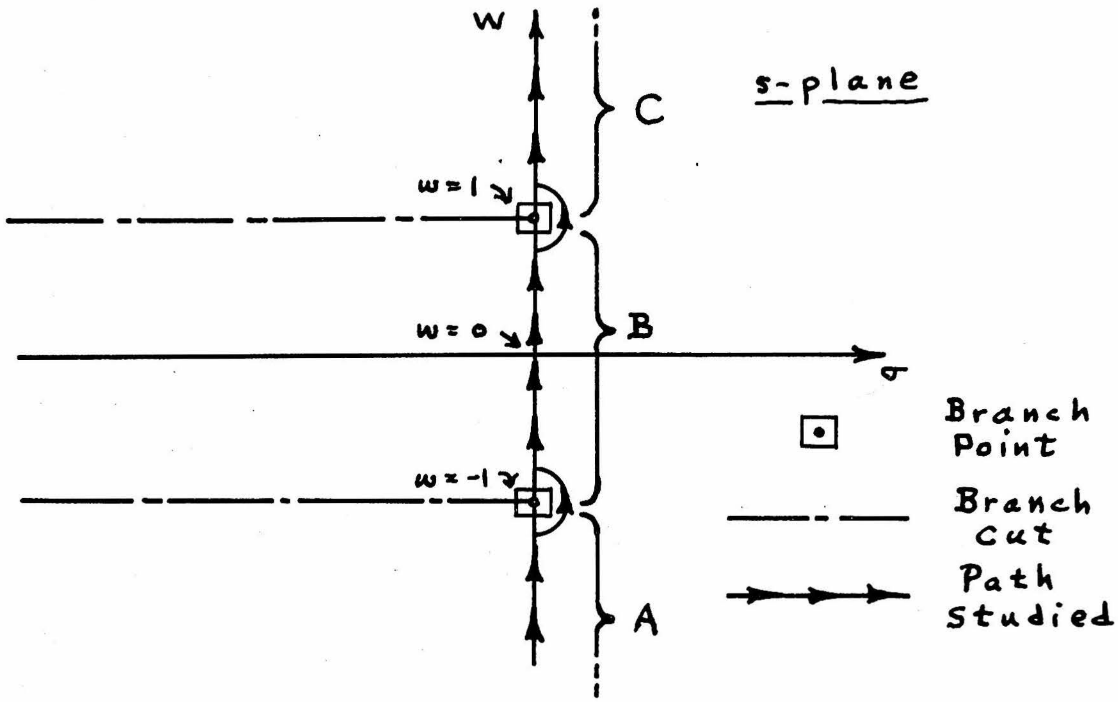
$$\text{Log}\left(\frac{1-js}{1+js}\right) \to \text{Log}(-1-j0) = -j\pi$$

as $r \to \infty$, and

$$T(s) = T[re^{j\theta}] \to \exp\left[j\frac{\text{Log}\,\epsilon}{\pi}(-j\pi)\right] = \epsilon$$

as $r \to \infty$. Thus T(s) possesses all the properties of a transfer characteristic derived in Section (E), Part II.

Let us now determine T(jw). The path of observation in the s-plane detours to the right around the branch points at s = $\pm$ j, in the manner shown below.



The crucial matter here is the proper determination of

$$\mathrm{Log}\left(\frac{1-js}{1+js}\right) = \mathrm{Log}\left|\frac{1-js}{1+js}\right| + j\left[\arg(1-js) - \arg(1+js)\right]$$

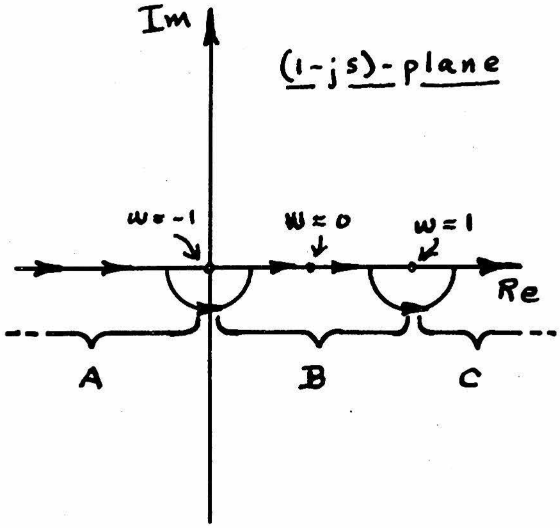
as the representative point s moves along the path shown in the figure above through the ranges A, B, and C.

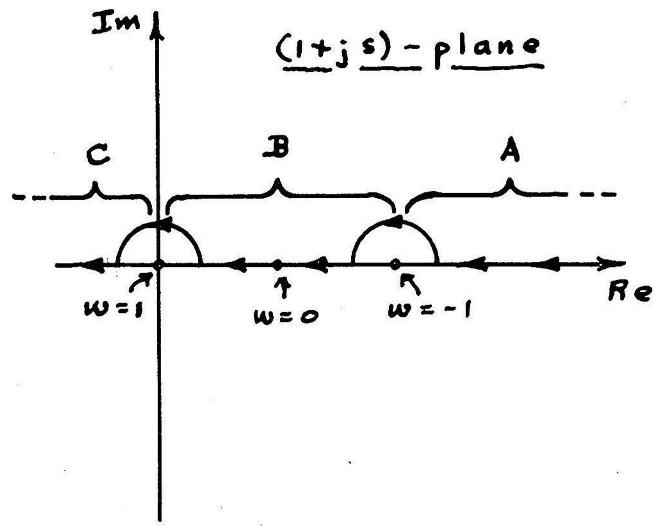
For $s = jw$, $\mathrm{Log}\left|\frac{1-js}{1+js}\right| = \mathrm{Log}\left|\frac{1+w}{1-w}\right|$ is a real-valued function of w with odd-function symmetry properties. Indeed,

$$-\frac{\mathrm{Log}\,\epsilon}{\pi}\,\mathrm{Log}\left|\frac{1+w}{1-w}\right| = \frac{|\mathrm{Log}\,\epsilon|}{\pi}\,\mathrm{Log}\left|\frac{1+w}{1-w}\right|$$

($0 < \epsilon < 1$) is effectively the phase-lag function which Wallman presents (R-63). So this part checks out very well.

The investigation of the imaginary part of the logarithm requires a little care, however. In the s-plane, the path of the representative point s is that shown in the figure above. In the (1-js)-plane and the (1 + js)-plane, the path becomes





The values of arg(1-js) and arg(1 + js) can now be found for the ranges A, B, and C of the path.

Range	Arg(1-js)	Arg(1+js)	$Im\left[Log\left(\frac{1-js}{1+js}\right)\right]$
A	$-\pi$	0	$-\pi$
B	0	0	0
C	0	π	$-\pi$

Thus in range B ($|w| < 1$), we find that $j\frac{Log\,\epsilon}{\pi}(j0) = 0$ and $|T(jw)| = 1$, as it should. In the ranges A and C ($|w| > 1$), we find that $j\frac{Log\,\epsilon}{\pi}(-j\pi) = Log\,\epsilon$ and $|T(jw)| = \epsilon$, as it should.

The given $M(w) = |T(jw)|$ has been checked exactly by the

T(s) considered, so that we agree completely with Wallman in this example. This T(s) is not unique, since an additional factor $\exp(-st_d)$ ($t_d > 0$), for example, might be added without materially affecting the situation.

No exact physical realization of this proposed transfer characteristic has as yet been found. If the hypothesis of meromorphicity advanced in Section (E), Part II, is correct, none ever will be.

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