Dominant Strategy Implementation on Private Goods Domains with Indivisibilities

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Dedication

To Grandmother

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Abstract

We consider the allocation of indivisible goods to agents who may have private information about their preferences. Standard allocation rules such as Walrasian equilibria or administrative processes fail to perform satisfactorily in this setting. In particular, they are not compatible with individual incentives. Thus, the planner faces an implementation problem, a problem of designing an institution (or mechanism) that induces appropriate incentives for the agents. We examine allocation rules, called social choice functions, for which this implementation problem is solvable, using the dominant strategy solution concept, which requires the implementing mechanism to provide a best action for each agent which does not depend on the other agents' actions. Social choice functions that satisfy this requirement are called strategyproof. We investigate primarily two domains of preferences, the universal private goods domain (Chapter 3), which is only restricted by the assumption that the agents are selfish, and the strict private goods domain (Chapters 1 and 2), which rules out, in addition, indifference between any two distinct allocations to any agent.

In Chapter 1, we consider the allocation of a single indivisible object. Necessary and sufficient conditions for strategyproofness are established, and the relationship between strategyproofness, efficiency, and Pareto-optimality is examined. It is shown that if an indirect form of manipulation, bossiness, is also ruled out, then we obtain a Gibbard-Satterthwaite-type impossibility result. We also prove that all strategyproof, nonbossy, and Pareto-optimal social choice functions are serial dictatorships.

We investigate the allocation of heterogeneous and indivisible objects in Chapters 2 and 3. The objects are heterogeneous in the sense that they typically have different values to

an agent. A most important characteristic of our model is that the valuation of the objects depend on what other objects they are obtained with. In Chapter 2, we establish that all strategyproof, strongly nonbossy, and Pareto-optimal social choice functions are serial dictatorships, where strong nonbossiness is a slightly stricter condition than bossiness. We also characterize the set of strategyproof, nonbossy, and Pareto-optimal social choice functions. Namely, we show that they are dictatorial sequential choice functions, which indicates that the consequences of the Gibbard-Satterthwaite theorem can only be escaped on the strict private goods domain by choosing bossy social choice functions. We also explore two restricted domains, which express complementarity, and, respectively, substitutibility of the objects. Finally, we briefly examine full implementation and social choice correspondences, allocation rules that may prescribe multiple outcomes to preference profiles.

In Chapter 3, we explore the allocation of heterogeneous indivisible objects when monetary transfers can be used to induce the right incentives for the agents. When the utility functions are additively separable and linear in the currency in which the transfers are paid, a mechanism is strategyproof and value maximizing if, and only if, it is a Groves mechanism. We impose further criteria, namely, envyfreeness and individual rationality, to choose among the Groves mechanisms. We show that none of the Groves mechanisms is envyfree on the universal private goods domain. However, we characterize the sets of envyfree, and the sets of both envyfree and individually rational Groves mechanisms on the two examined restricted domains. Some revenue related criteria are also examined.

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Introduction

We consider the allocation of indivisible goods to agents who may have private information about their preferences. Standard allocation rules such as Walrasian equilibria or administrative processes fail to perform satisfactorily in this setting. In particular, they are not compatible with individual incentives. Thus, the planner faces an *implementation* problem, a problem of designing an institution (or a *mechanism*) that induces appropriate incentives for the agents. In this study, we examine allocation rules that can be implemented, that is, for which the implementation problem is solvable. The allocation rules are called *social choice functions* or *social choice correspondences*, which embody the desired rules of the allocation, such as efficiency and fairness. Social choice functions assign one particular allocation of the private goods, an *outcome*, to each preference profile of the agents, while social choice correspondences may prescribe multiple outcomes to preference profiles. Given a set of possible preferences for each agent, the planner wishes to design a mechanism such that, following their individual incentives, the agents choose strategies that lead to the outcome(s) prescribed in the social choice function or correspondence.

We wish to find social choice functions that are incentive compatible, or in other words, *strategyproof.* When strategyproofness is required, attention is restricted to *direct* mechanisms, mechanisms that ask the agents to report their own preferences. This is due to the well-known revelation principle, which says that for any mechanism that has *dominant strategy* equilibria, there exists a strategyproof direct mechanism. A dominant strategy is a strategy which an agent finds the best, regardless of what the other agents' strategies are. Dominant strategy equilibria are desirable, because they eliminate any strategic interaction among the agents. Admittedly, the existence of dominant strategy equilibria is a very strong requirement. Other common solution concepts, such as Nash-equilibrium and Bayesian-Nash-equilibrium are less demanding. The Nash-equilibrium concept, however, requires the agents to have full information about each other's preferences, while dominant strategy equilibria do not require any such information. The Bayesian-Nash solution concept, which also tends to produce better results, is based only on the knowledge of the agents' prior distributions. Nonetheless, the exact knowledge of these prior distributions is typically crucial to these results. Since dominant strategy mechanisms are robust in the sense that they do not use the information structure in the economy, it is essential to explore them, however gloomy the results may be.

The dominant strategy solution concept, which is in the focus of our study, is indeed very demanding, which is illustrated by the famous Gibbard-Satterthwaite theorem. It states that in the context of voting the only social choice functions which induce truthful reporting of the preferences designate some favored voter who dictates the outcome. Underlying this impossibility theorem is the assumption that all conceivable preferences of the agents are admissible.[1] When the allocation of private goods is considered, the outcomes have as many components as agents, each component representing the allotted bundle of private goods for some agent. If we assume that the agents are *selfish*, i.e., that they only care about their own bundle of goods, then not all conceivable preferences are admissable. In

[1] We need to remark, however, that a similar impossibility result has been established for various restricted preferences. For example, Barbera and Peleg (1990) proved this negative result for continuous preferences, and Zhou (1991a) for continuous and convex preferences.

particular, preferences other than indifference are ruled out between any two outcomes for any agent when the agent's component is the same in the two outcomes. This set of preferences, which we call the *universal private goods domain* (of preferences) is the topic of our study.[2] It is the largest private goods domain, as no restriction other than selfishness is imposed on the preferences. Studies that examine strategyproofness in the context of allocating private goods focus on divisible goods, so that a further a priori structure (e.g., continuity, quasi-concavity, etc.) is imposed on the preferences (see, for example, Zhou (1991b) and Barbera and Jackson (1995)[3]).[4] We examine the allocation of a single indivisible object and the allocation of heterogeneous indivisible objects, so that further a priori restrictions need not be imposed on the domain. In the case of heterogeneous objects, this is due to the fact that the valuation of an object may not be independent of the other objects that it is obtained with, when the indivisible objects are heterogeneous. We also consider two restricted domains, where preferences are assumed to express some degree of complementarity, and, respectively, substitutibility, among the objects.

In sum, the problem we study differs from the problems examined in the relevant literature in two essential ways. Firstly, some of the related results can be found in the social choice theory literature, which are different in that they apply to voting problems (or public goods allocation problems), as opposed to private goods allocation problems. Secondly, the literature on private goods allocation problems tend to focus on divisible private goods, so

[2]A model with an n-dimensional outcome space was first formulated by Sen (1970) in a different context. It is commonly used to model private goods allocation problems, as described here.

[3]Satterthwaite and Sonnenschein (1981) is an exception, who study the allocation of private goods and do not impose further restrictions on the domain beyond selfishness and a condition, called broad applicability, which requires that the set of admissible utility functions is open, a condition that amounts to certain "richness" of the domain. They impose, however, several differentiability conditions on the mechanism, called regularity, which ensures that the mechanism is "smooth."

[4]Production economies were considered by Moulin and Shenker (1992).

that these results don't apply to our problem either.[5] Given that we consider indivisible private goods, the outcome space is assumed to be finite in this study, and if the indivisible goods are also heterogeneous, then no further assumption beyond selfishness needs to be imposed on the preferences. Therefore, the domain of preferences we are interested in is in between the universal domain that was investigated by Gibbard (1973) and Satterthwaite (1975), and the domains that are usually examined in the context of private goods allocation problems, where the private goods are divisible and the outcome space is infinite. There are several studies on decomposable outcome spaces that are relevant to our study. Moreno and Walker (1991) investigated strategyproofness when the agents' interests are partially decomposable, and proved a Gibbard-Satterthwaite-type theorem. Other related work includes Le Breton and Sen (1995a, 1995b), who characterized strategyproofness on product domains for strict and weak orderings, and Sprumont (1994) who considered separable domains. Strategyproofness for a multidimensional outcome space was also investigated by Border and Jordan (1983) and Barbera et al. (1993).

Another line of research that is related to our work is the investigation of the existence of Arrow social welfare functions on the so called private alternatives domains, started by Kalai and Ritz (1980). They considered, as the largest private alternatives domain, a domain that is only restricted by the selfishness asumption, and, in addition, they ruled out indifferences between any two alternatives for any agent, such that the agent's component in the two alternatives is not identical. We call this domain the *strict private goods domain*, which is the domain we examine in some parts of this sudy, mainly for simplicity. Continuing this line of research, Ritz (1983) established a a reciprocity result for private alterna-

[5]For a brief survey of the literature on strategyproofness in private goods economies, see Sprumont (1994).

tives domains between Arrow-type social welfare functions and Gibbard-Satterthwaite-type social choice functions (in fact, he allows for social choice correspondences). Given Example 1 in Kalai and Ritz (1980), Theorem 3 of Ritz (1983) implies that the universal private goods domain does not admit any rational, strategyproof, nonbossy, and nondictatorial social choice correspondence. It is important to point out that requiring rationality is not reasonable in the context of allocating private goods, as it requires that for every set of outcomes the social choice function has to select the best element of an Arrow social welfare function. In the voting or public goods context, it is a reasonable requirement, since it takes into account that some outcomes (alternatives) may not be feasible. In the context of private goods allocation problems, however, an outcome not being "feasible" in this sense means that some particular *distribution* of the fixed amounts of private goods is not feasible, which does not make much sense. When private goods are being allocated, feasibility problems arise if the *amount* of some private goods available for distribution is reduced. Then the outcome space "shrinks" accordingly, but no particular distribution of the reduced amounts of private goods should be treated as infeasible. For example, in our context, if there are two agents and two objects, a and b, then the outcome space is the following: $\{(a,b),(b,a),(ab,0),(0,ab),(a,0),(0,a),(b,0),(0,b),(0,0)\}$, where ab indicates the set of objects containing both a and b. Then a "feasible" set of outcomes, one that the rationality condition applies to, is, for example, $\{(a,b),(a,0),(0,b)\}$. Clearly, it is very unusual to restrict agent 1 to obtaining only object a, and agent 2 to obtaining only object b, if both objects are available, not to mention that it violates Pareto-optimality. In other words, it is an imposition on the outcomes without appealing to feasibility. If, on the other hand, say, object a is not available any more, then the set of feasible outcomes becomes $\{(0,b),(b,0),(0,0)\}$, which is a reasonable restriction. Therefore, this literature is not directly relevant to our investigation, although some results are closely related, as we will see.

Some more comments on our model are in order. Firstly, we would like to point out that, although the set of admissible preferences over their own components of the outcome (which we call the agent's allocation) is symmetric for the agents, they are not symmetric over the outcomes, on any private goods domain. Secondly, we would like to emphasize the importance of the feasibility constraints. If the same set of allocations were available to each agent, regardless of what the others get, there would be no conflict to solve. As opposed to voting, or a public goods economy, in this case each agent would get her favorite allocation. That is, for private goods allocation problems, the conflict stems from the fact that the amount of private goods available for allocation is fixed, i.e., from the scarcity of the resources expressed in the feasibility constraints. Thirdly, we assume that the goods don't have to be allocated, which is a reasonable assumption if negative valuations are allowed and efficiency is to be achieved. This ensures individual rationality when there are no monetary compensations.

In Chapter 1, we study the allocation of a single indivisible object, while in Chapter 2, we investigate the allocation of heterogeneous indivisible objects, both for the strict private goods domain. In these chapters, we are concerned with pure division, that is we do not allow the use of monetary transfers. In Chapter 3, we examine the allocation of heterogeneous indivisible objects for the universal private goods domain, when monetary transfers are allowed, and the agents' utilities are additively separable and linear in the currency in which they pay the compensation. We don't examine the allocation of a single indivisible object in the context of transfer mechanisms, since it is well-known that, for example, the use of a second-price auction provides satisfactory results for this problem. It is not known, however, whether standard auction mechanisms are satisfactory when several objects are for sale, objects whose valuations are not independent of each other.

Chapter 1

Nontransfer Mechanisms for Allocating a Single Indivisible Object

We examine the problem of allocating a single indivisible object to one of several selfish agents who may or may not desire the object, using a strategyproof mechanism. The objective is to give away the object without receiving any monetary payments, according to criteria such as efficiency, using a "nice" (e.g. nondictatorial) mechanism. It is assumed that the object is not necessarily awarded to any agent, which is a reasonable assumption when the planner's first priority is efficiency.

In the context of trading, Myerson and Satterthwaite (1983) have studied the problem of selling an indivisible object when there is a single buyer, and Makowski and Mezzetti (1993) examined the same problem with many buyers, both in the Bayesian framework.

The problem of allocating a single object without any monetary transfers was considered by Kim and Ledyard (1994) also in the Bayesian framework, and by Glazer and Ma (1989) in the complete information framework. Kim and Ledyard (1994) found that it is impossible to design an ex post efficient Bayesian incentive compatible mechanism for allocating the object, where the agents only know the distribution of other agents' valuations of the object. Glazer and Ma (1989) constructed multistage mechanisms with a unique subgame perfect equilibrium outcome. These outcomes are efficient in the sense that the agent with the highest valuation gets the object, without any monetary transfers being made at equilibrium. In this study, we don't allow the consideration of any monetary transfers, even if payments are only made out of equilibrium. Our planner does not consider balanced transfers either. If only Pareto-optimality is required where the welfare function depends on the allocation of the object and on the payments made between the potential recipients and the supplier of the object, balanced transfers would be acceptable. However, in contexts that are not marketlike (e.g. within a company) or where it is not politically viable to require any compensations (e.g. where traditionally the object is allocated without any compensations and the potential recipients cannot be coerced to pay), no transfers of any form are accepted. This is the case we examine in this study. The above mentioned two papers are also different in that they require that the agents have some information (complete information in the case of Glazer and Ma (1989)) about other agents' preferences. Since in this study mechanisms are required to be strategyproof, that is, the mechanism has to ensure that honest behavior is a dominant strategy for every agent and every preference profile, it is not necessary for the agents to have any information about the others' preferences. Of course, it is assumed that the agents know their own valuation of the object, and both the agents and the planner know the set of admissible preferences. Some related results can also be found in Dasgupta et al. (1978) and Satterthwaite and Sonnenschein (1981), which will be discussed in the course of the exposition.

The criteria regarding the desired rules of the outcome are embodied in social choice functions, functions that assign exactly one outcome to any preference profile of the agents. When strategyproofness is required, attention is restricted to direct mechanisms, mechanisms that ask the agents to report their own preferences, due to the well-known revelation principle. Therefore, a direct mechanism that implements a social choice function will mirror the social choice function, in the sense that the outcome of the mechanism will coincide with the outcome prescribed by the social choice function for each preference profile. Thus, the criteria applied to the mechanisms apply to the social choice functions as well.

Although the results in this paper are of interest on their own, the elementary and intuitive proofs also offer some insight into more general aspects of the problem of allocating indivisible private goods by using strategyproof mechanisms. Throughout the paper, special care is taken to emphasize which results are specific to the single object allocation problem due to its simple structure, which is helpful in identifying others that are potential candidates for generalization.

This chapter is organized as follows. The notation and definitions are introduced in Section 1.1. In Section 1.2, necessary and sufficient conditions for a social choice function to be strategyproof are derived, which are specific to the single object allocation problem. Efficiency and Pareto-optimality are analyzed in Section 1.3, and the relationship between strategyproofness and Pareto-optimality of a social choice function is established. In Section 1.4, we examine the implications of strategyproofness on the desirability of the mechanism.

In particular, it is shown that a Gibbard-Satterthwaite-type impossibility is escaped on the private goods domain for our problem. However, if the indirect form of manipulation is also ruled out, which typically arises in the context of private goods allocation problems, namely, bossiness, then the analog of the Gibbard-Satterthwaite theorem holds. The special case where there are only two agents is also considered in this section. In Section 1.5, we characterize the set of strategyproof, nonbossy, and Pareto-optimal social choice functions. We conclude in Section 1.6.

1.1 Notation and Definitions

There are $n \geq 2$ agents and one object to be allocated among the agents. Let N denote the set of n agents. An *outcome* $x = (x^1, \ldots, x^n)$ is such that $x^i \in \{0, 1\}$, where

$$x^i = \begin{cases} 1 & \text{if the object is given to agent } i \\ 0 & \text{otherwise,} \end{cases}$$

$\forall i \in N$. Clearly, an outcome x is *feasible* if at most one agent gets the object, i.e., if $\sum_{i \in N} x^i \leq 1$. Denote the set of feasible outcomes by X.

Let θ^i denote the value that agent i places on the object. We assume that $\theta^i \in \Re \setminus \{0\}, \forall i \in N$, that is, the agents cannot be indifferent between obtaining and not obtaining the object. Let Θ^i be the set of admissible values for each agent i, i.e., $\Theta^i = \Re \setminus \{0\}$. Denote the set of preferences for all agents by Θ. Let $\theta \in \Theta$ be a profile of the agents, and $\theta^{-i} \in \Theta^{-i}$ be the profile of all the agents except for agent i.

Each agent i is assumed to be *selfish*, that is, each i only cares about the ith component of x, which implies that she is indifferent between any two outcomes that have the same ith component. Formally, $U(x, \theta^i) = U(x^i, \theta^i), \forall x \in X, \forall i \in N$.

Definition 1 A *social choice function* is a function $f : \Theta \mapsto X$.

Let $f^i(\theta)$ denote the assignment prescribed to agent i by f at θ.

Definition 2 An SCF f is *strategyproof* if $\forall \theta \in \Theta$, $\forall i \in N$, $\forall \tilde{\theta}^i \in \Theta^i$, $U(f^i(\theta), \theta^i) \geq U(f^i(\tilde{\theta}^i, \theta^{-i}), \theta^i)$. *If* $\exists i \in N$ such that $U(f^i(\theta), \theta^i) < U(f^i(\tilde{\theta}^i, \theta^{-i}), \theta^i)$ for some $\theta \in \Theta, \tilde{\theta}^i \in \Theta^i$, then we say that f *is manipulable* and agent i can *manipulate* it.

1.2 Necessary and Sufficient Conditions for Strategyproofness

First we define two characterisitics of an SCF f, *ordinality* and *positive responsiveness*, which together are necessary and sufficient conditions for f to be strategyproof in our context.

Definition 3 An SCF f is *ordinal* if $\forall \theta \in \Theta, \forall i \in N, \forall \tilde{\theta}^i \in \Theta^i$ such that $\theta^i, \tilde{\theta}^i > 0$ or $\theta^i, \tilde{\theta}^i < 0, f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i})$.

Definition 4 An SCF f satisfies *positive responsiveness (PR)* if $\forall \theta \in \Theta, \forall i \in N$, $\forall \tilde{\theta}^i \in \Theta^i$ (a) $\theta^i > 0, \tilde{\theta}^i < 0$ and $f^i(\theta) = 0$ imply that $f^i(\tilde{\theta}^i, \theta^{-i}) = 0$, and (b) $\theta^i < 0, \tilde{\theta}^i > 0$ and $f^i(\theta) = 1$ imply that $f^i(\tilde{\theta}^i, \theta^{-i}) = 1$.

Proposition 1 *An SCF is strategyproof if, and only if, it is ordinal and PR.*

Proof:

Strategyproofness $\Rightarrow$ ordinality

Let f be strategyproof and not ordinal. Then $\exists\theta \in \Theta, i \in N$, and $\tilde{\theta}^i \in \Theta^i$ such that we have either $\theta^i, \tilde{\theta}^i > 0$ or $\theta^i, \tilde{\theta}^i < 0$ and $f^i(\theta) \neq f^i(\tilde{\theta}^i, \theta^{-i})$. Since $f^i(\bar{\theta}) \in \{0,1\}, \forall i \in N, \forall \bar{\theta} \in \Theta$, assume, without loss of generality, that $f^i(\theta) = 1$ and $f^i(\tilde{\theta}^i, \theta^{-i}) = 0$. Given that f is strategyproof, $\theta^i > 0$, otherwise agent i would report $\tilde{\theta}^i$ and get 0 which she would prefer to 1 if $\theta^i < 0$. Similarly, f's strategyproofness implies that $\tilde{\theta}^i < 0$. We have reached a contradiction.

Strategyproofness $\Rightarrow$ PR

Let f be strategyproof and not PR. Then $\exists\theta \in \Theta, i \in N$ and $\tilde{\theta}^i \in \Theta^i$ such that $\theta^i > 0, \tilde{\theta}^i < 0, f^i(\theta) = 0$, and $f^i(\tilde{\theta}^i, \theta^{-i}) = 1$. Since f is strategyproof, we must have $U(0, \theta^i) \geq U(1, \theta^i)$, which implies that $\theta^i < 0$, a contradiction.

Ordinality and PR $\Rightarrow$ strategyproofness

Let f be ordinal, PR, and manipulable. Then $\exists\theta \in \Theta, i \in N$, and $\tilde{\theta}^i \in \Theta^i$ such that $U(f^i(\theta), \theta^i) < U(f^i(\tilde{\theta}^i, \theta^{-i}), \theta^i)$. Then either (a) $f^i(\theta) = 0, f^i(\tilde{\theta}^i, \theta^{-i}) = 1$, and $\theta^i > 0$, or (b) $f^i(\theta) = 1, f^i(\tilde{\theta}^i, \theta^{-i}) = 0$, and $\theta^i < 0$. However, PR implies that $\tilde{\theta}^i > 0$ for (a) and $\tilde{\theta}^i < 0$ for (b). Therefore, f is not ordinal in either case, which is a contradiction.□

We would like to remark here that ordinality and PR together are equivalent to the well-known IPM condition for strategyproofness (see, for example, Laffont and Maskin (1982)). The proof of this is straightforward and is left to the reader. We will work with the ordinality and PR properties, since they better facilitate the following analysis, which will be clear throughout this paper. For now, let us say that, for the allocation of a single object, the condition for strategyproofness has been split into an independence property (ordinality) and a monotonicity property (PR). This is useful, because the independence property, ordinality, is very intuitive and is easily checked, and therefore it helps in ruling out manipulable SCF's. Although in the voting context, where the outcomes are of a political nature, cardinal valuations may not make sense, for resource allocation problems they are of importance. It is usually implicitly assumed in the implementation literature that only ordinal preferences can be elicited when monetary payments are not used.[1] For private goods allocation problems, this informational constraint has considerable consequences. While a somewhat trivial condition, ordinality has important implications for the efficiency of strategyproof mechanisms. This will be discussed in the next section.

1.3 Efficiency

Given the necessity of ordinality for strategyproofness, it follows immediately that it is not possible to design an *efficient* strategyproof mechanism, a mechanism which assigns the object to the agent who values it most. The same is shown in the Bayesian framework by Kim and Ledyard (1994, Theorem 1). However, ordinality implies more than that. It rules out any interpersonal utility level comparisons, and therefore, less stringent efficiency criteria, such as assigning the object to an agent whose value for it is within the $k (k \leq n)$ highest positive values, or even assigning it to an agent whose value is not the lowest among the positive valuations, cannot be implemented. Therefore, we resort to Pareto-optimality as a criterion of efficiency, given that Pareto-optimal SCF's may satisfy ordinality.

Definition 5 An SCF f is *Pareto-optimal* if $\forall \theta \in \Theta$, there does not exist $y \in X$ such that $U(y^i, \theta^i) \geq U(f^i(\theta), \theta^i), \forall i \in N,$ and for some $j \in N, U(y^j, \theta^j) > U(f^i(\theta), \theta^j)$.

[1] An explicit discussion of this issue with regard to Nash-implementation can be found in Maskin (1986).

Note that the stronger notion of Pareto-optimality is used, which is more appropriate in this context than the other, weaker version.[2] However, this Pareto-optimality condition is still very weak, in the sense that it typically allows for several different outcomes.

If an SCF f is Pareto-optimal then $\forall i \in N, \forall \theta \in \Theta, f^i(\theta) = 1$ implies that $\theta^i > 0$. Therefore, a Pareto-optimal SCF also satisfies *individual rationality*, where f is individually rational if $\forall i \in N, \forall \theta \in \Theta, \theta < 0$ implies that $f^i(\theta) = 0$. Individual rationality alone, however, is satisfied by an imposed mechanism, for example, in which the object is never awarded. Pareto-optimality, on the other hand, implies *citizen sovereignty.*

Definition 6 An SCF f satisfies *citizen sovereignty (CS)* if $\forall x \in X, \exists \theta \in \Theta$ such that $f(\theta) = x$.

Next, we are able to prove a positive result, which states that any Pareto-optimal and ordinal SCF is strategyproof.

Proposition 2 *If an SCF is ordinal and Pareto-optimal then it is strategyproof.*

Proof: Notice that if an SCF violates PR then $\exists i \in N$ and $\theta \in \Theta$ such that $\theta^i < 0$ and $f^i(\theta) = 1$. This, however, implies that f is not Pareto-optimal. Therefore, Pareto-optimality implies PR, which, together with Proposition 1, yields the required result.□

From the above proof it is also clear that if PR is violated, then individual rationality does not hold either. Thus, any individually rational ordinal SCF is strategyproof.

We would like to point out that the above result only holds in the context of the single object allocation problem, and it does not generalize to more complex problems, for example,

[2]In fact, the weak version of Pareto-optimality, which only requires that the outcome not be strictly preferred to another feasible outcome by all agents at any profile, is automatically satisfied, as long as there are at least three agents.

where there is more than one object to allocate. This is illustrated in Example 1. Given the proof of Proposition 2, this should not come as a surprise. The proof is based on a relationship between Pareto-optimality and PR, namely, that any Pareto-optimal SCF also satisfies PR. Since Pareto-optimality is an intraprofile property (i.e., it can be determined whether the outcome is Pareto-optimal for a given profile), while PR is an interprofile property, this relationship is clearly due to the simple structure of our problem, and cannot hold in general. Before the example is provided, we need an appropriate generalization of the ordinality property for the case where there is more than one object to allocate. Note that in this case each $x \in X$ is a matrix, and each $\theta^i \in \Theta^i$ is a vector.

Definition 7 An SCF f is *ordinal* if $\forall \theta \in \Theta, \forall i \in N, \forall \tilde{\theta}^i \in \Theta^i$ such that $\forall x, y \in X, U(x^i, \theta^i) > U(y^i, \theta^i) \Leftrightarrow U(x^i, \tilde{\theta}^i) > U(y^i, \tilde{\theta}^i)$, $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i})$.

Example 1 *An ordinal, Pareto-optimal, and manipulable SCF for allocating more than one object.*

Let there be two agents and two objects to be allocated among them. Then the two agents have strict preferences over the elements of the set $\{a, b, ab, 0\}$, where a and b are the two objects, ab indicates the allocation to an agent when the agent gets both objects, and 0 denotes the allocation to an agent when she doesn't get anything. Consider the following SCF. If both agents prefer a to b or if both prefer b to a, give agent 1 her first choice, and then give agent 2 her first choice from the remaining object(s). Otherwise, if the two agents' preference orderings are not the same over a and b, then give agent 2 her first choice, and then agent 1 her first choice from the remaining object(s). This SCF is Pareto-optimal and ordinal. However, it is not strategyproof. Consider the following reported preferences.

$(ab, b, a, 0)$ for agent 1, and $(ab, a, b, 0)$ for agent 2. Since 1 prefers b to a and 2 prefers a to b, agent 2 gets her first choice, ab, and agent 1 gets 0. However, agent 1 can manipulate the outcome by reporting $(ab, a, b, 0)$ and obtaining ab, her first choice, instead of 0, her last choice.□

Our next question is, which strategyproof mechanisms satisfy Pareto-optimality? In order to answer this question, we need the concept of *bossiness*, which was introduced by Satterthwaite and Sonnenschein (1981). An SCF is bossy if there exists at least one agent whose preferences can change in a way that the prescribed allocation is different for some other agent(s), but not for herself, while everyone else's preferences are unchanged. Intuitively, this is an undesirable property, given that the mechanism mirrors the SCF that it implements. This means that the agent who can change some other agent's allocation without changing her own may use her "power" by accepting a bribe or blackmailing. Thus, in the presence of bossiness, the predictibility of the outcomes becomes questionable, which is what we wanted to avoid in the first place by requiring strategyproofness. Note also that bossiness is only a concern when indifference over outcomes is allowed. In particular, when private goods are being allocated and the agents are selfish, indifferences cannot be ruled out, so that a mechanism may allow agents to change the allocations for others without changing their own allocation. For a further discussion of bossiness see Ritz (1983),[3] and Section 2.2.

Definition 8 An SCF f is *bossy* if $\exists \theta \in \Theta, i \in N$, and $\tilde{\theta}^i \in \Theta^i$ such that $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i})$ and $f^j(\theta) \neq f^j(\tilde{\theta}^i, \theta^{-i})$ for some $j \in N$. An SCF f is *nonbossy* if it is not bossy. If

[3]Ritz (1983) calls bossy social choice functions *corruptible*, and also defines corruptibility for social choice correspondences.

$\exists i \in N$ such that $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i})$ and $\exists j \in N$ such that $f^j(\theta) \neq f^j(\tilde{\theta}^i, \theta^{-i})$ for some $\theta \in \Theta, \tilde{\theta}^i \in \Theta$, then we say that agent i is *bossy.*

In order to answer the earlier question, we prove Proposition 3, which says that a strategyproof, nonbossy, and CS SCF is Pareto-optimal.

Proposition 3 *If an SCF is strategyproof, nonbossy, and CS then it is Pareto-optimal.*

Proof: Let an SCF f be strategyproof, nonbossy, CS, and not Pareto-optimal. By Proposition 1, f is ordinal and PR. Since f is not Pareto-optimal, we have one of the following two cases:

(a) $\exists \theta \in \Theta$ such that $\theta^i > 0$ for some $i \in N$ and $f(\theta) = \mathbf{0}$.

(b) $\exists \tilde{\theta} \in \Theta$ such that $\tilde{\theta}^i < 0$ for some $i \in N$ and $f^i(\tilde{\theta}) = 1$.

Let's look at the two cases in turn.

(a) Since f satisfies CS, $\exists \tilde{\theta} \in \Theta$ such that $f^i(\tilde{\theta}) = 1$. Consider the sequence of profiles

$$(\theta^1, \ldots, \theta^n)$$
$$\vdots$$
$$(\theta^1, \ldots, \theta^{j-1}, \tilde{\theta}^j, \ldots, \tilde{\theta}^n)$$
$$\vdots$$
$$(\tilde{\theta}^1, \ldots, \tilde{\theta}^n).$$

Let $i = 1$. Since f is nonbossy, the outcome either does not change when θ^j is replaced by $\tilde{\theta}^j$ in the above sequence of profiles, or $f^j(\theta^1, \ldots, \theta^{j-1}, \tilde{\theta}^j, \ldots, \tilde{\theta}^n) = 1$ for $j = 2, \ldots, n$. Thus, $f^1(\theta^1, \tilde{\theta}^{-1}) = f^i(\theta^i, \tilde{\theta}^{-i}) = 0$. Since $f^i(\tilde{\theta}) = 1, \tilde{\theta}^i < 0$, by ordinality. However, this violates PR.

(b) Since f satisfies CS, $\exists \theta \in \Theta$ such that $f(\theta) = \mathbf{0}$. Now we can repeat the first part of the argument in case (a) to get that $f^i(\theta^i, \tilde{\theta}^{-i}) = 0$. Since $f^i(\tilde{\theta}) = 1, \theta^i > 0$, by ordinality. However, this violates PR.□

We would like to note here that there is a similar result in Dasgupta, Hammond and Maskin (1978, Theorem 3.3.1) for the domain where all strict orderings are admissible, although in a much more general framework. Their result does not require nonbossiness, since indifference between outcomes is ruled out.

1.4 A Gibbard-Satterthwaite-type Impossibility Result for Nonbossy Mechanisms

The focus of this section is whether a Gibbard-Satterthwaite-type result holds for the private goods domain when there is a single object to allocate. That is, one would like to see whether the strategyproof mechanisms used to implement the chosen SCF can be nondictatorial. First, we demonstrate with an example that a Gibbard-Satterthwaite-type impossibility is escaped in our context. The example is given for the case where there are at least three agents, since the two-agent case will be treated later.

Definition 9 An SCF f is *dictatorial* if $\exists i \in N$ such that $\forall \theta \in \Theta, \forall x \in X, f(\theta) = x$ only if $\forall y \in X, U(x, \theta^i) \geq U(y, \theta^i)$. Then i is called a *dictator* for f. An SCF f is *nondictatorial* if it is not dictatorial.

Example 2 *A strategyproof, CS, and nondictatorial SCF for* $n \geq 3$.

Let $n = 3$. Let $(1, 2, 3)$ be a fixed ordering of the three agents. Consider the following SCF f. If there is an odd number of agents whose values are positive for the object, give the object to the first agent in the above fixed ordering whose value is positive. If there is an even number of agents whose values are positive for the object, give the object to the second agent in the above ordering whose value is positive. If all agents have negative values for the object then don't give the object to any one of them. Clearly, this SCF satisfies citizen sovereignty. On the basis of the following table, containing all the different profiles and outcomes, it is easy to verify that f is also strategyproof and nondictatorial.

1	$\boxed{+}$	$+$	$+$	$-$	$\boxed{+}$	$-$	$-$	$-$
2	$+$	$\boxed{+}$	$-$	$+$	$-$	$\boxed{+}$	$-$	$-$
3	$+$	$-$	$\boxed{+}$	$\boxed{+}$	$-$	$-$	$\boxed{+}$	$-$

(In the table $+$ means a positive value and $-$ means a negative value for the object. The columns represent profiles and the rows represent agents. The outcome for each profile is indicated by the boxes.)

The example generalizes to $n > 3$. The SCF f, as defined above, clearly satisfies CS for any number of agents. Since no agent with a negative value will obtain the object, only agents with a positive value have any reason to manipulate. However, if an agent reports a negative value instead of a true positive one, which is the only way for her to change her allocation, given f, then she will not obtain the object. Thus, f is strategyproof for $n > 3$ agents. To see that f is also nondictatorial for $n > 3$, consider the profile $(+, \ldots, +)$ for the n agents, i.e., where each agent's value is positive for the object. Then, if n is odd, agent 1 gets the object according to f, and if n is even, then agent 2 gets the object. Thus, agents $3, \ldots, n$ are not dictators. Notice, however, that if the profile $(+, \ldots, +, -)$ is reported then agent 2 gets the object if n is odd, and agent 1 gets the object if n is even. Therefore, agents 1 and 2 are not dictators for f either.□

Notice, first, that the SCF does not only satisfy CS in the above example, but it is also Pareto-optimal. Secondly, note that f is bossy. For example, 2 is bossy when agents 1 and 3 both report $+$, and agent 3 is bossy when agents 1 and 2 both report $+$. The next proposition verifies that this observation is true in general, that is, any Pareto-optimal, strategyproof, and nonbossy mechanism is dictatorial.

Proposition 4 *If an SCF is strategyproof, nonbossy, and Pareto-optimal then it is dictatorial.*

Proof: First note that $j \in N$ is a dictator with respect to an SCF f if $f^j(\theta) = 1$ whenever $\theta^j > 0$, and if $f^j(\theta) = 0$ whenever $\theta^j < 0$ for $\theta \in \Theta$. Fix a strategyproof, nonbossy, and Pareto-optimal SCF f. Let $\theta \in \Theta$ be such that $\theta^i > 0, \forall i \in N$. Then $\exists j \in N$ such that $f^j(\theta) = 1$, by Pareto-optimality. Then $\forall i \neq j$ the ordinality of f (using Proposition 1) implies that $\forall \tilde{\theta}^i > 0, f^i(\tilde{\theta}^i, \theta^{-i}) = 0$, and PR implies (again, using Proposition 1) that $\forall \tilde{\theta}^i < 0, f^i(\tilde{\theta}^i, \theta^{-i}) = 0$. Therefore, $\forall i \neq j$, agent i cannot change the outcome for herself, as long as agent j's reported value is θ^j. Since f is nonbossy, $f^j(\theta^j, \tilde{\theta}^{-j}) = 1, \forall \tilde{\theta}^{-j} \in \Theta^{-j}$, including $\tilde{\theta}^i = \theta^i, \forall i \neq j$. Then $f^j(\tilde{\theta}) = 1, \forall \tilde{\theta}^j > 0, \forall \tilde{\theta}^{-j} \in \Theta^{-j}$, by ordinality. However, we also have $f^j(\tilde{\theta}) = 0, \forall \tilde{\theta}^j < 0, \forall \tilde{\theta}^{-j} \in \Theta^{-j}$, since f is Pareto-optimal. Therefore, j is a dictator for f, and f is dictatorial.□

Given Proposition 3, Pareto-optimality can be replaced by citizen sovereignty in Proposition 5, which gives an analog to the Gibbard-Satterthwaite theorem for nonbossy mechanisms.

Corollary 1 *If an SCF is strategyproof, nonbossy, and CS then it is dictatorial.*

Corollary 1 shows that if no manipulations in the form of bossiness are allowed then the Gibbard-Satterthwaite theorem carries over to the private goods domain when a single object is being allocated. Another immediate implication of Proposition 4, combined with Proposition 2, is the following.

Corollary 2 *If an SCF is ordinal, nonbossy, and Pareto-optimal then it is dictatorial.*

Now we turn to the case where there are only two agents, which is somewhat different from the general case. First, it is shown that for this special case any Pareto-optimal SCF is nonbossy.

Proposition 5 *If* $n = 2$ *and an SCF is Pareto-optimal then it is nonbossy.*

Proof: For any Pareto-optimal SCF f, we have $f^1(+,-) = 1, f^2(-,+) = 1$, and $f(-,-) = \mathbf{0}$ for possible profiles of the agents, based only on preference orderings. Furthermore, Pareto-optimality also requires that either $f^1(+,+) = 1$ or $f^2(+,+) = 1$. Now it is easy to check that f is nonbossy.□

If f is also strategyproof in the above proof then the ordinality of f implies that if $f^i(+,+) = 1$ for some profile $(+,+)$, where $i \in \{1,2\}$, then $f^i(+,+) = 1$ for any profile $(+,+)$. Thus, either 1 or 2 is a dictator for f. Therefore, we can obtain the following two corollaries, both of which are also implied by earlier results.

Corollary 3 *If* $n = 2$ *and an SCF is strategyproof and Pareto-optimal then it is dictatorial.*

Corollary 3 follows from Propositions 4 and 5.

Corollary 4 *If* $n = 2$ *and an SCF is ordinal and Pareto-optimal then it is dictatorial.*

This corollary is implied by Corollary 3 and Proposition 2.

It follows from Corollary 1 that if an SCF is strategyproof, nondictatorial, and CS then it is bossy. Then Proposition 5 implies that the SCF is not Pareto-optimal when $n = 2$. An example of such an SCF f is given by $f(+,+) = (1,0), f(-,+) = (0,1)$, and $f(+,-) = f(-,-) = (0,0)$. Finally, it should be remarked that Corollary 4 does not generalize to the case where there is more than one object to allocate. This is illustrated by the SCF in Example 1, which is ordinal, Pareto-optimal, and nondictatorial. Example 1 is also a counterexample to the generalization of Corollary 2.

1.5 Serial Dictatorship

Remark that, unlike in the voting context, dictatorship alone does not characterize the set of strategyproof, nonbossy, and Pareto-optimal SCF's. When there are only two agents, a Pareto-optimal and dictatorial SCF is strategyproof and nonbossy. Therefore, we have a complete characterization for the two-agent case. This, however, is not true for more than two agents. Take, for example, a Pareto-optimal and dictatorial SCF such that agent 1 is the dictator with respect to f, and $f(0,8,7) = (0,1,0), f(0,7,7) = (0,0,1), f(-2,7,7) = (0,1,0)$. Clearly, f is not strategyproof, since agent 2 can manipulate it, and agent 1 is bossy, so it is not nonbossy either. We show next that the set of strategyproof, nonbossy, and Pareto-optimal SCF's is characterized by *serial dictatorships*. In our context,[4] a serial dictatorship is a mechanism in which the agents have priorities for the object in a predetermined order. That is, in a serial dictatorship, the object is awarded to the first agent

[4]For more on serial dictatorships, see Satterthwaite and Sonnenschein (1981) and Section 2.3.

in a fixed ordering of the agents who reports a positive value. Satterthwaite and Sonnenschein established a similar result in a lot more general framework. However, they require the mechanisms to satisfy numerous differentiability conditions, and, although they don't require Pareto-optimality, conditions in addition to strategyproofness and nonbossiness are imposed on the mechanism, conditions that are not yet well understood, in order to get serial dictatorships.

In the following, let σ denote a permutation of N.

Definition 10 An SCF f is a *serial dictatorship* if $\exists \sigma = (\sigma^1, \dots, \sigma^n)$ such that $\forall \theta \in \Theta, \forall i \in N, f^{\sigma^i}(\theta) = 1$ if $\theta^{\sigma^i} > 0$, and $\forall j \in N, j < i, \theta^{\sigma^j} < 0$, otherwise $f^{\sigma^i}(\theta) = 0$. We then call σ the *hierarchy* associated with f.

Proposition 6 *An SCF is strategyproof, nonbossy, and Pareto-optimal if, and only if, it is a serial dictatorship.*

Proof: Suppose f is a serial dictatorship. It is Pareto-optimal, since if $f(\theta) = \mathbf{0}$, then $\forall i \in N, \theta^i < 0$, and if $f^i(\theta) = 1$ then $\theta^i > 0$. To see that f is nonbossy, let the hierarchy associated with f be $\sigma = (1, \dots, n)$. Suppose $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i}) = 0$ for some $i \in N, \theta \in \Theta, \tilde{\theta}^i \in \Theta^i$. If $f(\theta) = \mathbf{0}$ then $\theta^j < 0, \forall j \in N$, by Pareto-optimality. This implies that $f^j(\tilde{\theta}^i, \theta^{-i}) = 0, \forall j \in N, j \neq i$, since f is Pareto-optimal. If $f(\theta) \neq \mathbf{0}$ then $\exists j \in N, j < i$, such that $f^j(\theta) = 1$. Then $\forall t \in N, t < j, \theta^t < 0$, so that $f^j(\tilde{\theta}^i, \theta^{-i}) = 1$. This proves that f is nonbossy. It is straightforward to verify that f is ordinal. Therefore, given Proposition 2, it follows that f is strategyproof.

Now we prove the converse. Let f be strategyproof, nonbossy, and Pareto-optimal. Suppose $\exists \theta, \tilde{\theta} \in \Theta, i, j \in N, i \neq j$, such that $f^i(\theta) = 1, f^j(\tilde{\theta}) = 1, \tilde{\theta}^i > 0$, and $\theta^j > 0$. Let

$\bar{\theta}^t < 0, \forall t \in N, t \neq i,j$. Since f is ordinal, $f^t(\bar{\theta}^t, \theta^{-t}) = 0, \forall t \in N, t \neq i,j$ such that $\theta^t < 0$. Since f is PR, $f^t(\bar{\theta}^t, \theta^{-t}) = 0, \forall t \in N, t \neq i,j$ such that $\theta^t > 0$. Then by nonbossiness, $f^i(\theta^i, \theta^j, \bar{\theta}^{-i,j}) = 1$. A similar argument shows that $f^j(\tilde{\theta}^i, \tilde{\theta}^j, \bar{\theta}^{-i,j}) = 1$. Given that f is Pareto-optimal, $\theta^i > 0$ and thus $f^i(\tilde{\theta}^i, \theta^j, \bar{\theta}^{-i,j}) = 1$, by ordinality. Similarly, $\tilde{\theta}^j > 0$ by Pareto-optimality, and so $f^j(\tilde{\theta}^i, \theta^j, \bar{\theta}^{-i,j}) = 1$, by ordinality. This is a contradiction. Therefore, $\forall \theta, \tilde{\theta} \in \Theta, \forall i,j \in N$, if $f^i(\theta) = 1, \theta^j > 0$, and $\tilde{\theta}^i > 0$ then $f^j(\tilde{\theta}) = 0$. This determines an ordering of the agents, $\sigma = (\sigma^1, \ldots, \sigma^n)$, such that $\forall \theta \in \Theta$, if $f^{\sigma^i}(\theta) = 1$ for some $i \in N$ then $\forall j \in N, j \neq i$, such that $\theta^{\sigma^j} > 0$, we have $j > i$. Then $\forall \theta \in \Theta, \forall j \in N$ such that $j < i$, we have $\theta^{\sigma^j} < 0$. Since this is true for each $i \in N$, f is a serial dictatorship.□

Given Propositions 3 and 2, we have the following corollaries.

Corollary 5 *An SCF is strategyproof, nonbossy, and CS if, and only if, it is a serial dictatorship.*

Corollary 6 *An SCF is ordinal, nonbossy, and Pareto-optimal if, and only if, it is a serial dictatorship.*

Clearly, Corollary 6 does not hold for more complex allocation problems.

A natural and convenient decentralization of a serial dictatorship is to ask the first agent in the hierarchy associated with the mechanism whether she wants the object. If she turns it down then we ask the second agent, etc., until one of the agents takes the object, or until we have asked each agent. This decentralized mechanism greatly reduces the informational requirements, while retaining the properties of a serial dictatorship. Notice that the ordering of the agents is exogeneously given in a serial dictatorship. Thus, the supplier of the object (or society) may determine the order. It can be set up as a priority ranking, incorporating some criteria of justice or other known characteristics of the individuals.

1.6 Discussion

In sum, we have showed in this chapter that if there is one indivisible object to be allocated among several selfish agents, the best the planner can do without monetary transfers is to give the object to an agent who desires it, but whose value may not be the highest among the agents, using a mechanism that is either dictatorial or bossy. It has also been verified that all strategyproof, nonbossy, and Pareto-optimal mechanisms are serial dictatorships. The results have been shown for the strict private goods domain. However, they can be extended to the universal private goods domain with minor modifications. We consider only the strict private goods domain here to preserve symmetry between Chapters 1 and 2.

Finally, we would like to discuss the possibility of generalizing these results to allocation problems with more than one indivisible object. As it is demonstrated by Example 1, Proposition 2, and thus, Corollaries 2, 4, and 6, cannot be generalized, as they follow from the simple structure of the single object allocation problem. Proposition 1 is explicitly written for our problem, so that deriving an independence and a monotonicity condition for strategyproofness might be difficult in general. However, it is clear that the necessity of the generalized ordinality condition holds in general. This leaves Propositions 3, 4, 5, and 6, together with Corollaries 1, 3, and 5. We consider more complex allocation problems in Chapter 2, where we verify which results hold when heterogeneous indivisible objects are being allocated.

Chapter 2

Nontransfer Mechanisms for Allocating Heterogeneous Indivisible Objects

In this chapter we examine strategyproof mechanisms for allocating heterogeneous indivisible objects. Similarly to Chapter 1, we exclude the possibility of transfers, and focus on the strict private goods domain.

This problem is an extension of the much studied assignment problem, where objects are assigned to agents on a one-to-one basis. The extension from the assignment problem to the problem of allocating heterogeneous indivisible objects is motivated by its potential applications: the FCC auction of PCS spectrum rights, the allocation of tracking time on NASA's worldwide Deep Space Network of antennas, the selling of tract lease rights to oil companies, the allocation of takeoff and landing rights to airlines, and the coordination of the use of various other shared facilities. A common feature of these resource allocation problems is that the value of an object to an agent is typically not independent of the other objects assigned to her. That is, the objects may decrease or increase each other's value when obtained together, which is a most important characteristic of our model.

The assignment model was first formulated by Shapley and Shubik (1972). A number of papers studied the properties of different mechanisms that can be applied to the assignment problem. Among the papers that examine nontransfer mechanisms, Gardenfors (1973) studied a positional voting system, and a chit mechanism was considered by Hylland and Zeckhauser (1979). Olson (1991) took a more systematic approach to finding mechanisms with desired properties. Experimental results on some of the proposed mechanisms were provided by Olson and Porter (1991, 1994).

In our model each agent may obtain any set of objects, which we call a *package*.[1] The objects are *heterogeneous* in the sense that they typically have different values to the individual. It is assumed that the value of an object to an agent may not be independent of the other objects assigned to her. Therefore, any valuation of the packages is admissable. However, we assume, for technical reasons, that the agents are not indifferent among the packages, including the *null package*, the package that does not contain any object. We also assume that the objects need not be assigned. If the objects in a package are not assigned to any agent, we say that the package is *unassigned*. Of course, we assume that the agents are *selfish*, i.e., that they only care about their own component of the *outcome*, which we call their *allocation*.

The performance criteria are expressed in *social choice functions (SCF)* that prescribe

[1]This terminology follows that of Rassenti et al. (1982).

a single outcome to each preference profile of the agents. We briefly examine *social choice correspondences (SCC's)* as well, which may prescribe multiple outcomes to each profile.

We distinguish between strategyproofness (truthful implementation) and *full implementation*, as is usual in the literature. Strategyproofness is only concerned with direct mechanisms, i.e., with mechanisms in which the strategy space is the set of different preferences for each agent, given the revelation principle. An SCF is strategyproof if there exists a direct mechanism for which honest announcement of the preferences is an equilibrium strategy for each agent. Full implementation is defined for mechanisms with general strategy spaces. An SCF (SCC) is fully implementable in dominant strategies if there exists a mechanism for which the set of dominant strategy equilibria coincides with the set of assignments prescribed by the SCF (SCC) for each preference profile.

We introduce the notation and definitions in Section 2.1. In Section 2.2, the criteria imposed on the SCF's are discussed. In particular, we require the SCF's to be *nonbossy.* Nonbossiness means, just as in Chapter 1, that an agent can change her preferences in a way that changes the prescribed allocations to others without changing her own. In Section 2.3, we show that all strategyproof, strongly nonbossy, and, Pareto-optimal SCF's are serial dictatorships, where strong nonbossiness is a slightly stricter condition than nonbossiness. A Gibbard-Satterthwaite-type impossibility result is established for nonbossy SCF's in Section 2.4, and the set of strategyproof, nonbossy, and Pareto-optimal SCF's is characterized. We examine two restricted domains in Section 2.5, the strict superadditive and strict substitute domains, and show that there exist strategyproof, nonbossy, Pareto-optimal, and nondictatorial SCF's on both of these domains. In Section 2.6, we demonstrate that if an SCF is nonbossy, the requirement for full implementation in dominant strategies is not stricter than for strategyproofness. Finally, we show that SCC's can only be fully implemented in dominant strategies by bossy mechanisms. We conclude in Section 2.7.

2.1 Notation and Definitions

There are $n \geq 2$ agents and $k \geq 2$ objects to be allocated among the agents. Let N denote the set of n agents, and K be the set of k objects. Let $\mathcal{K}$ denote the union of the power set of K and the null package. That is, $\mathcal{K}$ is the set of packages, including the null package.

An *outcome* x from N to K is an $n \times (2^k - 1)$ matrix, in which each element x_a^i is defined by

$$x_a^i = \begin{cases} 1 & \text{if package } a \text{ is assigned to agent } i \\ 0 & \text{otherwise,} \end{cases}$$

$\forall i \in N$, $\forall a \in \mathcal{K}$. To make the notation simple, we will write that $x^i = a$ when $x_a^i = 1$ and $x_b^i = 0$, $\forall b \in \mathcal{K}$, $b \neq a$. If agent i is not assigned any package as part of outcome x, then $x^i = 0$. Let $M_b = \{a \in \mathcal{K} | \ a \bigcap b \neq \emptyset\}$, $\forall b \in \mathcal{K} \setminus \{0\}$.

An outcome x is *feasible* if each agent gets at most one package, i. e., $\sum_{a \in \mathcal{K}} x_a^i \leq 1$, $\forall i \in N$, and no object is assigned more than once as an element of some package, i. e., $\sum_{i \in N} \sum_{a \in M_b} x_a^i \leq 1$, $\forall b \in \mathcal{K} \setminus \{0\}$. Denote the set of feasible outcomes by $\mathcal{X}$.

Let θ_a^i denote the value that agent i places on package a. Then $\theta^i = (\theta_1^i, \ldots, \theta_{2^k-1}^i)$ is a list of the values placed by agent i on the set of packages, which we will refer to as *preferences*. The value of the null package is zero to each agent i with any preferences θ^i. We assume that each agent i is selfish, that is, that each agent i only cares about the ith element of x, which implies that she is indifferent between any two allocations that have the same ith element. Formally, $U(x,\theta^i) = U(x^i,\theta^i), \forall x \in \mathcal{X}, \forall i \in N$. We also assume that each agent has strict preferences over the allocations. That is, $\forall \theta^i \in \Theta^i$, $\theta^i_a \neq \theta^i_b$ whenever $a \neq b, \forall a,b \in \mathcal{K}$. Let Θ^i be the set of admissable preferences for agent i, so that $\theta^i \in \Theta^i$, $\forall i \in N$. Denote the set of admissable preferences for all agents by $\Theta = \times_{i\in N}\Theta^i$. Thus, Θ represents the strict private goods domain, given N and K. Let $\theta \in \Theta$ denote a profile of the agents. Similarly, let θ^{-i} denote the profile of all the agents except for agent i.

Definition 11 A *social choice function* is a function $f : \Theta \mapsto \mathcal{X}$.

Definition 12 A *mechanism* (g, S) is a set of strategy spaces S_i, $\forall i \in N$, where $S = \times_{i\in N}S_i$, and a function $g : S \mapsto \mathcal{X}$.

Definition 13 A *direct mechanism* g is a mechanism for which agent *i's* strategy space is $S_i = \Theta^i$, $\forall i \in N$, so that $S = \Theta$.

Let $f^i(\theta)$ denote the allocation prescribed to agent i by f at θ, and let $g^i(\theta)$ denote i's allocation resulting from mechanism g, when the reported profile is θ.

Definition 14 An SCF f is *strategyproof* if $\forall\theta \in \Theta$, $\forall i \in N$, $\forall\tilde{\theta^i} \in \Theta^i$, $U(f^i(\theta),\theta^i) \geq U(f^i(\tilde{\theta^i},\theta^{-i}),\theta^i)$. If f is not strategyproof then it is *manipulable.* Then $\theta \in \Theta, i \in N$ and $\tilde{\theta}^i \in \Theta^i$ such that $U(f^i(\theta),\theta^i) < U(f^i(\tilde{\theta^i},\theta^{-i}),\theta^i)$. We then say that agent i can *manipulate* at θ via $\tilde{\theta^i}$.

Definition 15 An SCF f is *nonbossy* if $\forall\theta \in \Theta$, $\forall i,j \in N$, $\forall\tilde{\theta^i} \in \Theta^i$, if $f^i(\theta) = f^i(\theta^i,\tilde{\theta}^{-i}$, then $f^j(\theta) = f^j(\theta^i,\tilde{\theta}^{-i}$. If f is not nonbossy then it is *bossy.* Then $\exists\theta \in \Theta, i,j \in N$ *and* $\tilde{\theta}^i \in \Theta^i$ *such that* $f^i(\theta) = f^i(\theta^i,\tilde{\theta}^{-i}$, and $f^j(\theta) \neq f^j(\theta^i,\tilde{\theta}^{-i}$. We then say that i is *bossy* at θ versus $(\tilde{\theta}^i,\theta^{-i})$.

Definition 16 An SCF f is *Pareto-optimal* if $\forall \theta \in \Theta$, there does not exist $y \in \mathcal{X}$ such that $\forall i \in N, U(y^i, \theta^i) \geq U(f^i(\theta), \theta^i)$, and, for some $j \in N, U(y^j, \theta^j) > U(f^j(\theta), \theta^j)$.

Let top(θ^i) denote the top-ranked package according to θ^i. That is, $\forall i \in N, \forall \theta^i \in \Theta^i, \forall p \in \mathcal{K}, U(\text{top}(\theta^i), \theta^i) \geq U(p, \theta^i)$.

Definition 17 An SCF f is *nondictatorial* if there does not exist $i \in N$ such that $\forall \theta \in \Theta, f^i(\theta) = \text{top}(\theta^i)$. If f is not nondictatorial then it is *dictatorial.* Then $\exists i \in N$ such that $\forall \theta \in \Theta, f^i(\theta) = \text{top}(\theta^i)$. We then say that i is a *dictator* for f.

2.2 Nonbossiness, Efficiency, and Information Constraints

In this section, we discuss the requirements that are imposed on the SCF's besides strategyproofness. First of all, we require the SCF's to be nonbossy. This criterion was introduced by Satterthwaite and Sonnenschein (1981), and used subsequently by Ritz (1983, 1985), Olson (1991), and Barbera and Jackson (1995). Since it rules out an indirect form of manipulation, namely, where an agent can change other agents' allocations by changing her messages even though her own allocation does not change, nonbossiness is quite desirable intuitively. Indeed, strategyproofness is required in order to be able to predict outcomes in a reliable way, and this reliability is at risk when bossy behavior is allowed, since the bossy agent is completely indifferent among the allocations to the others that she "controls" with her strategy. Admittedly, nonbossiness and strategyproofness together amount to more than the impossibility of deviating alone, as Barbera and Jackson (1995, Lemma 4) showed that they imply a weak form of coalitional strategyproofness. However, while coalitional strategyproofness may not be required in any form for voting problems or for public goods economies, it is also true that bossy behavior does not typically arise in these contexts. In fact, when indifferences are not admissable, bossiness obviously cannot occur, and ruling out indifferences might be quite agreeable in the above mentioned contexts. In contrast, when private goods are being allocated to selfish agents, indifferences cannot be ruled out.

A useful result is that strategyproofness and nonbossiness together imply monotonicity.

Definition 18 An SCF f satisfies *monotonicity* if $\forall \theta, \tilde{\theta} \in \Theta$ such that $f(\theta) = x$, if $\forall y \in \mathcal{X}, \forall i \in N, U(x^i, \theta^i) \geq U(y^i, \theta^i) \Rightarrow U(x^i, \tilde{\theta}^i) \geq U(y^i, \tilde{\theta}^i)$ then $f(\tilde{\theta}) = x$.

Lemma 1 [2] *A strategyproof and nonbossy SCF is monotonic.*

Proof: Suppose f is strategyproof, nonbossy, and $\exists \theta, \tilde{\theta} \in \Theta$ such that $f(\theta) = x$ and $\forall y \in \mathcal{X}, \forall i \in N, U(x^i, \theta^i) \geq U(y^i, \theta^i) \Rightarrow U(x^i, \tilde{\theta}^i) \geq U(y^i, \tilde{\theta}^i)$. Let $f(\tilde{\theta}^1, \theta^{-1}) = z$. Then either $z = x$ or $z^1 \neq x^1$, by f's nonbossiness. If $z^1 \neq x^1$ then strategyproofness implies that $U(x^1, \theta^1) > U(z^1, \theta^1)$. Then by assumption, $U(x^1, \tilde{\theta}^1) > U(z^1, \tilde{\theta}^1)$, given that $z^1 \neq x^1$. However, this contradicts f's strategyproofness. Therefore, $z = x$, and $f(\tilde{\theta}^1, \theta^{-1}) = x$. Repeating the same argument for $i = 2, \ldots, n$, we get that $f(\tilde{\theta}) = x$, as required.□

Remark that strategyproofness alone is equivalent to the IPM property, as was shown by Dasgupta et al. (1978).[3] However, it is not equivalent to *strong positive association,* (SPA) on the private goods domain, in contrast with the domain that consists of all strict preferences (in the following, *strict domain*), for which the equivalence was shown by Muller and Satterthwaite (1977). In fact, SPA is equivalent to monotonicity[4] so that on the private

[2]Essentially the same result is shown in Olson (1991, Lemma 8.11) and Barbera and Jackson (1995, Lemma 2), although both in a somewhat different setting, and using completely different terminology. The proof is given here for self-containment.

[3]For the correct version of IPM, see, for example, Maskin (1982).

[4]Yet another name for this property appears in Moulin (1988) who calls it strong monotonicity. We follow the majority, and call this property monotonicity.

goods domain, SPA implies strategyproofness, but strategyproofness alone does not imply SPA. Thus, Lemma 1 underlines that on the private goods domain strategyproofness and nonbossines together rule out the same sources of strategic behavior as strategyproofness alone on the strict domain.

Note that monotonicity has serious implications for effeciency, if one wants to maximize the sum of the utilities for the agents. Clearly, these utilitarian-type SCF's are not strategyproof and nonbossy.[5] In fact, just as in the case of a single object, information on cardinal utilities cannot be used when strategyproofness and nonbossiness are required. The question is, how much of the information can be retained if bossiness is allowed. It is clear that strategyproofness alone implies ordinality (see Definition 7). Therefore, for an efficient SCF to be strategyproof, the following condition needs to hold. For all $i \in N, \theta \in \Theta$, and $\tilde{\theta}^i \in \Theta^i$ such that $U(x^i, \theta^i) > U(y^i, \theta^i) \Leftrightarrow U(x^i, \tilde{\theta}^i) > U(y^i, \tilde{\theta}^i), \forall x, y \in \mathcal{X}$, if $f(\theta) \in \text{argmax}_{x \in \mathcal{X}} \sum_{i \in N} U(x^i, \theta^i)$ then $f(\tilde{\theta}^i, \theta^{-i}) \in \text{argmax}_{x \in \mathcal{X}} \sum_{i \in N} U(x^i, \theta^i)$. Since this condition does not hold, efficient SCF's are not strategyproof. Therefore, we use Pareto-optimality as a criterion of efficiency, as for the single object problem, since Pareto-optimality can be determined using ordinally interpreted utility functions (preference orderings). This shows that nonbossiness does not relevantly restrict the information that can be used in determining outcomes, since intrapersonal comparisons of utility levels[6] are meaningful under monotonicity, and that's all the information Pareto-optimality requires.

Satterthwaite and Sonnenschein (1981) suggest that one interpretation of their negative results is that bossy mechanisms should be used, as efficiency and strategyproofness

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[5]See also Le Breton and Sen (1995a).

[6]See Bossert (1991). This information requirement is also often referred to as ordinal noncomparibility.

seem to be in conflict when bossiness is ruled out. For our problem, it is efficiency and strategyproofness that are in conflict, when the sum of the agents' welfare is desired to be maximal, whether or not the mechanism is bossy. Furthermore, for achieving Pareto-optimality, nonbossiness does not seem to be detrimental in our context. In fact, just as in the case where there is only a single object to be allocated, Pareto-optimality and bossiness are incompatible when there are only two agents, which is demonstrated below. Accordingly, the results that require Pareto-optimality in the following sections can be restated without the nonbossiness assumption for the two-agent case.

Lemma 2 *If there are only two agents then a Pareto-optimal SCF is nonbossy.*

Proof: Let $n = 2$ and let f be Pareto-optimal and bossy. Suppose agent 1 is bossy. Then $\exists \theta \in \Theta$ and $\tilde{\theta}^1 \in \Theta^1$ such that $f^1(\theta^1, \theta^2) = f^1(\tilde{\theta}^1, \theta^2)$ and $f^2(\theta^1, \theta^2) \neq f^2(\tilde{\theta}^1, \theta^2)$. Let $f(\theta^1, \theta^2) = x$ and $f(\tilde{\theta}^1, \theta^2) = y$, so that $x^1 = y^1$ and $x^2 \neq y^2$. Then either $U(x^2, \theta^2) > U(y^2, \theta^2)$ or $U(x^2, \theta^2) < U(y^2, \theta^2)$, and $x^1 \bigcap x^2 = \emptyset, x^1 \bigcap y^2 = \emptyset$, by feasibility. This implies that either y or x is not Pareto-optimal.□

In sum, we will require an SCF to be strategyproof, nonbossy, and Pareto-optimal. An example of an SCF that satisfies all three requirements is a Pareto-optimal serial dictatorship with a single hierarchy. In the next section, we provide a characterization of these special SCF's.

2.3 Characterization of Strategyproof, Strongly Nonbossy, and Pareto-optimal Social Choice Functions

A Pareto-optimal serial dictatorship with a single hierarchy, which we call simply a *serial dictatorship*, is a mechanism in which the agents get their favorite allocation from a feasible set (the remaining objects), according to a predetermined order. That is, the outcomes of a serial dictatorship correspond to a decentralized mechanism in which the agent who is ranked first chooses her favorite allocation from the fixed set of objects K, then the second agent chooses her favorite allocation from the remaining objects, etc, until all the objects are taken, or until we get to the last agent, whichever happens first. Note that since the first agent gets to "choose" from the set of all the objects, and all the subsequent agents "choose" from all the objects available after the higher ranked agents made their choices, these SCF's are Pareto-optimal. This contrasts with the observation of Satterthwaite and Sonnenschein (1981) that serial dictatorships violate Pareto-optimality, which they demonstrate with an example of a production economy. Since we do not consider production, a serial dictatorship is Pareto-optimal in our framework. It is also easy to verify that a serial dictatorship is strategyproof and nonbossy.

Similarly to Satterthwaite and Sonnenschein (1981), we examine which additional requirements, if imposed on an SCF, would imply that it is a serial dictatorship. It turns out that a mild strengthening of nonbossiness, which we call *strong nonbossiness*, is enough to constrain the choice of appropriate SCF's to serial dictatorships, when required in addition to strategyproofness and Pareto-optimality. Strong nonbossiness means that if an agent deviates at some profile, with the result that the extra objects that she obtains (if any) are unassigned at the given profile, and the objects that she loses (if any) remain unassigned at the new profile, then the other agents' allocations remain unchanged. In other words, strong nonbossiness requires that if an agent's action does not affect the others through the feasibility constraints then it should not affect the other agents at all. Clearly, strong nonbossiness implies nonbossiness, but bossiness does not imply strong nonbossiness, even if strategyproofness is also required. It can also be shown (analogously to Lemma 2) that a Pareto-optimal SCF is strongly nonbossy if there are only two agents. It is also interesting to point out that if there is only a single object to allocate then nonbossiness implies strong nonbossiness, which explains why only nonbossiness is required to get serial dictatorships in Proposition 6.

Definition 19 An SCF f is *strongly nonbossy* if $\forall i \in N, \forall \theta \in \Theta$, and $\forall \tilde{\theta}^i \in \Theta^i$such that $f^i(\theta) \bigcap f^j(\tilde{\theta}^i, \theta^{-i}) = \emptyset$ and $f^i(\tilde{\theta}^i, \theta^{-i}) \bigcap f^j(\theta) = \emptyset, \forall j \in N, j \neq i$, we have $f^j(\theta) = f^j(\tilde{\theta}^i, \theta^{-i})$, $\forall j \in N, j \neq i$.

For $\mathcal{Y} \subseteq \mathcal{X}, i \in N$, and $\theta^i \in \Theta^i$, let $c(\mathcal{Y}, \theta^i) = \{x \in \mathcal{Y} \mid \forall y \in \mathcal{Y}, U(x, \theta^i) \geq U(y, \theta^i)\}$ be the set of the best outcomes in $\mathcal{Y}$ for agent i with preferences θ^i. For $\mathcal{Y} \subseteq \mathcal{X}, i \in N$, and $\theta^i \in \Theta^i$, let $c^i(\mathcal{Y}, \theta^i) = x^i$ such that $U(x^i, \theta^i) \geq U(y^i, \theta^i), \forall y \in \mathcal{Y}$, where $x \in \mathcal{Y}$. Given that only strict preferences over allocations are admissable, $c^i(\mathcal{Y}, \theta^i)$ is a singleton for each agent i and θ^i. Since it will be clear in the following which SCF we refer to, $c(\mathcal{Y}, \theta^i)$ and $c^i(\mathcal{Y}, \theta^i)$ are not indexed for f, just as in other definitions to follow.

Let $\Sigma(N)$ denote the set of permutations of N. Then $\sigma \in \Sigma(N)$ is an ordered list of the agents, i.e., $\sigma = (\sigma^1, \ldots, \sigma^n)$. For the following definition, let the null package be defined as the empty set, i.e., let $0 = \emptyset$.

Definition 20 An SCF f is a *serial dictatorship* if $\exists \sigma \in \Sigma(N)$ such that $\forall \theta \in \Theta, f^{\sigma^1}(\theta) = c^{\sigma^1}(\mathcal{K}, \theta^{\sigma^1}) = \text{top}(\theta^{\sigma^1})$, and for $j \in N \setminus \{1\}, f^{\sigma^j}(\theta)$ are defined recursively by $f^{\sigma^j}(\theta) = c^{\sigma^j}(\mathcal{K} \setminus \bigcup_{i=1}^{j-1}\{f^{\sigma^i}(\theta)\}, \theta^{\sigma^j})$. We then call σ the *d-hierarchy* associated with f.

Now we are ready to prove the characterization theorem.

Proposition 7 *An SCF f is strategyproof, strongly nonbossy, and Pareto-optimal if, and only if, it is a serial dictatorship.*

Proof: It is easy to check that a serial dictatorship is strategyproof, strongly nonbossy, and Pareto-optimal. In order to prove that a strategyproof, strongly nonbossy, and Pareto-optimal SCF is a serial dictatorship, we need to introduce some definitions. The proof will proceed by several lemmas.

Let $\sigma : \Theta \mapsto \Sigma(N)$ be a function that assigns an ordered list of the agents to each profile. With a slight abuse of notation, we denote $\sigma(\theta)$ by σ_θ so that $\sigma_\theta = (\sigma_\theta^1, \ldots, \sigma_\theta^n), \forall \theta \in \Theta$. Then, if $\sigma_\theta^i = j$, we write that $\sigma_\theta(j) = i$.

Definition 21 An SCF f is *multihierarchical* if $\exists \sigma : \Theta \mapsto \Sigma(N)$ such that $\forall i, j \in N$, if $\sigma_\theta(i) < \sigma_\theta(j)$ then $U(f^i(\theta), \theta^i) > U(f^j(\theta), \theta^i)$, unless $f^i(\theta) = f^j(\theta) = 0$. We then call σ_θ an *m-hierarchy* associated with f *at* θ.

Thus, if f is multihierarchical then there exists a "hierarchy" of the agents for each profile, not necessarily the same for each profile, such that each agent prefers her allocation at that profile to the allocation of all the agents at the same profile who rank lower than she in the hierarchy for that profile. Thus, loosely speaking, if there is a "conflict" among agents at some profile then it is resolved according to the hierarchy at that profile.

Definition 22 The *top set* $T(j,\theta)$ for each agent j and profile θ contains the packages that j prefers to her allocation at that profile, given some SCF f. That is, $T(j,\theta) = \{p \in \mathcal{K} \mid U(p,\theta^j) > U(f^j(\theta),\theta^j)\}, \forall j \in N, \forall \theta \in \Theta$.

Clearly, $\forall \theta \in \Theta, \forall j \in N, 0 \notin T(j,\theta)$, since the objects need not be assigned.

Definition 23 Given an SCF f, agent i *beats* agent j at θ, if $f^i(\theta) \in T(j,\theta)$. This relationship is denoted by $B(\theta)$. That is, if i beats j at θ, then we write $iB(\theta)j$.

Lemma 3 *A Pareto-optimal SCF is multihierarchical.*

Proof: Let f be a Pareto-optimal SCF. Then $\forall \theta \in \Theta, B(\theta)$ is acyclic for f. That is, $\forall \theta \in \Theta$, if $i_1 B(\theta) i_2 B(\theta) \cdots B(\theta) i_t$ for $i_l \in N, l = 1,\ldots,t$, $2 \leq t \leq n$, then $\neg(i_t B(\theta) i_1)$.[7] This implies that $\forall \theta \in \Theta, \exists \sigma \in \Sigma(N)$ such that $\forall i,j \in N$ if $iB(\theta)j$ then $\sigma_\theta(j) > \sigma_\theta(i)$. Then $\forall i,j \in N, \forall \theta \in \Theta, \sigma_\theta(j) > \sigma_\theta(i)$ implies that $\neg(jB(\theta)i)$, which in turn implies that $U(f^i(\theta),\theta^i) \geq U(f^j(\theta),\theta^i)$. But $f^i(\theta) \neq f^j(\theta)$, unless $f^i(\theta) = f^j(\theta) = 0$, since $f(\theta)$ is not feasible otherwise. Thus, $U(f^i(\theta),\theta^i) > U(f^j(\theta),\theta^i), \forall i,j \in N, \forall \theta \in \Theta$ if $\sigma_\theta(j) > \sigma_\theta(i)$, unless $f^i(\theta) = f^j(\theta) = 0$. Therefore, $\forall \theta \in \Theta, \sigma_\theta$ is an m-hierarchy associated with f at θ, and thus f is multihierarchical.□

Let $\theta^i \in \begin{pmatrix} a \\ b \end{pmatrix}$ denote some preferences of agent i such that a is ranked first, y is ranked second, and the rest of the preferences is arbitrary. We use a similar notation for profiles. For example, $\theta \in \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ if $\theta^1 \in \begin{pmatrix} a \\ b \end{pmatrix}$ and $\theta^2 \in \begin{pmatrix} c \\ d \end{pmatrix}$. Furthermore, we

[7] The logic symbol $\neg$ means 'not' in this study.

write $f\begin{pmatrix} a & c \\ b & d \end{pmatrix} = x$ to indicate that f assigns outcome x to all profiles in $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Lemma 4 *For every strategyproof, strongly nonbossy, and Pareto-optimal SCF there exists a single m-hierarchy that is associated with it at each profile.*

Proof: Let f be strategyproof, strongly nonbossy, and Pareto-optimal. For Steps 1–3, fix $i, j \in N$ and $\theta, \tilde{\theta} \in \Theta$ such that $f^i(\theta) = K, iB(\theta)j$, and $jB(\tilde{\theta})i$. By Pareto-optimality, $\forall i \in N, \exists \theta \in \Theta$ such that $f^i(\theta) = K$. If there do not exist j and $\tilde{\theta}$ such that $iB(\theta)j$, and $jB(\tilde{\theta})i$, where $f^i(\theta) = K$ then the lemma holds. Let $f(\theta) = x$ and $f(\tilde{\theta}) = y$.

Step 1: If $iB(\tilde{\theta})j$ such that $f^i(\theta) = K$ and $jB(\tilde{\theta})i$ for some $j \in N$ and $\tilde{\theta} \in \Theta$ then $f^j(\tilde{\theta}) \neq K$.

Suppose $f^j(\theta) = K$. Let $\bar{\theta}^i, \bar{\theta}^j \in \begin{pmatrix} K \\ 0 \end{pmatrix}$. Let $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i, j\}$. Given that $iB(\theta)j$, f's monotonicity implies that $f(\bar{\theta}) = x$. However, since $jB(\tilde{\theta})i$, monotonicity also implies that $f(\bar{\theta}) = x$. Since $x = y$ contradicts feasibility, $f^j(\tilde{\theta}) \neq K$.

Step 2: If $jB(\tilde{\theta})i$ for some $i, j \in N, \tilde{\theta} \in \Theta$ then $\exists \bar{\theta} \in \Theta$ with $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i, j\}$ such that $jB(\bar{\theta})i$.

Suppose that $\forall \bar{\theta} \in \Theta$ such that $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i, j\}$, $\neg(jB(\bar{\theta})i)$. Let $\bar{\bar{\theta}}^i \in \begin{pmatrix} y^j \\ y^i \\ 0 \end{pmatrix}$ and $\bar{\bar{\theta}}^l \in \begin{pmatrix} y^l \\ 0 \end{pmatrix}, \forall l \in N \setminus \{i\}$. Then $f(\bar{\bar{\theta}}) = y$ by monotonicity, given that $jB(\tilde{\theta})i$ and $y^j \in T(i, \tilde{\theta}^{-i})$. If $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i, j\}$ then Pareto-optimality and feasibility imply that $f^i(\bar{\bar{\theta}}^i, \bar{\bar{\theta}}^j, \bar{\theta}^{-i,j}) = y^j$, given that $\neg\left(jB(\bar{\bar{\theta}}^i, \bar{\bar{\theta}}^j, \bar{\theta}^{-i,j})i\right)$. However, since f is strongly nonbossy, Pareto-optimality implies that $f^i(\bar{\bar{\theta}}^i, \bar{\bar{\theta}}^j, \bar{\bar{\theta}}^{\bar{L}}, \bar{\theta}^L) = y^i, \forall L \subseteq N \setminus \{i,j\}$, where $\bar{L} = N \setminus (\{i,j\} \bigcup L)$. For $L = N \setminus \{i,j\}$ we get that $f^i(\bar{\bar{\theta}}^i, \bar{\bar{\theta}}^j, \bar{\theta}^{-i,j}) = y^i$. This implies that $y^i = y^j$, so that $y^i = y^j = 0$, given the feasibility constraints. However, $y^j \in T(i, \tilde{\theta}^{-i})$ implies that $y^j \neq 0$, which is a contradiction. Therefore, if $jB(\tilde{\theta})i$ for some $i, j \in N$, and $\tilde{\theta} \in \Theta$ then $\exists \bar{\theta} \in \Theta$ with $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i,j\}$ such that $jB(\bar{\theta})i$.

Step 3: If $jB(\bar{\theta})i$ such that $f^j(\bar{\theta}) \neq K$ and $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i,j\}$ then $\exists \bar{\bar{\theta}} \in \Theta$ such that $jB(\bar{\bar{\theta}})i$ and $f^j(\bar{\theta}) = K$.

Let $f(\bar{\theta}) = z$ and let $i = 1, j = 2$. By assumption, $z^2 \neq K$, and since $2B(\bar{\theta})1, z^2 \neq 0$. By Pareto-optimality, $z^l = 0, \forall l \in N \setminus \{1,2\}$. By monotonicity, $f \begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ z^1 & 0 & & & \end{pmatrix} = z = (z^1, z^2, 0, \ldots, 0)$. Now consider some profile in $\begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ 0 & 0 & & & \end{pmatrix}$. If $z^1 \neq 0$ then Pareto-optimality implies that either agent 1 or agent 2 gets z^2 at this profile. If agent 1 gets z^2 then she can manipulate at $\begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ z^1 & 0 & & & \end{pmatrix}$ via $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}$. Therefore, Pareto-optimality yields $f \begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ 0 & 0 & & & \end{pmatrix} = (0, z^2, 0, \ldots, 0)$. Then $f \begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ K & 0 & & & \\ 0 & & & & \end{pmatrix} = (0, z^2, 0, \ldots, 0)$, since the other Pareto-optimal outcome, $(z^2, 0, \ldots, 0)$, would enable agent 1 to manipulate at $\begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ 0 & 0 & & & \end{pmatrix}$ via $\begin{pmatrix} z^2 \\ K \\ 0 \end{pmatrix}$. Now consider some profile in $\begin{pmatrix} z^2 & K & 0 & \cdots & 0 \\ K & z^2 & & & \\ 0 & 0 & & & \end{pmatrix}$. There are two Pareto-optimal outcomes at these profiles, $(z^2, 0, \ldots, 0)$ and $(0, K, 0, \ldots, 0)$, given the feasibility constraints. If the outcome is the former then agent 2 can manipulate at these profiles via $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}$. Therefore, $f^2\begin{pmatrix} z^2 & K & 0 & \cdots & 0 \\ K & z^2 & & & \\ 0 & 0 & & & \end{pmatrix} = K$. Letting one of these profiles be $\bar{\bar{\theta}}$, we get that $f^j(\bar{\bar{\theta}}) = f^2(\bar{\bar{\theta}}) = K$, and $K \in T(i, \bar{\bar{\theta}})$ implies that $jB(\bar{\bar{\theta}})i$, as desired.

Step 4: If $iB(\theta)j$ for some $i, j \in N$ and $\theta \in \Theta$ such that $f^i(\theta) = K$ then $\forall \tilde{\theta} \in \Theta, \neg(jB(\tilde{\theta})i)$.

This step follows from Steps 1-3. Suppose $iB(\theta)j$ for some $i, j \in N$ and $\theta \in \Theta$ such that $f^i(\theta) = K$ and $jB(\tilde{\theta})i$ for some $\tilde{\theta} \in \Theta$. Since $jB(\tilde{\theta})i, \exists \bar{\theta}$ with $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i, j\}$ such that $jB(\bar{\theta})i$, by Step 2. Then $\exists \bar{\bar{\theta}} \in \Theta$ such that $f^j(\bar{\bar{\theta}}) = K$ and $jB(\bar{\bar{\theta}})i$, by Step 3. However, this contradicts the assumption that $iB(\theta)j$ where $f^i(\theta) = K$, by Step 1. Therefore, $\forall i, j \in N$ if $iB(\theta)j$ for some $\theta \in \theta$ such that $f^i(\theta) = K$ then $\forall \tilde{\theta} \in \Theta, \neg(jB(\tilde{\theta})i)$.

Step 5: There exists $\sigma \in \Sigma(N)$ such that σ is an m-hierarchy associated with f at each profile.

Let $\theta_{[1]} \in \begin{pmatrix} K & \cdots & K \\ 0 & \cdots & 0 \end{pmatrix}$. Then $\exists i \in N$ such that $f^i(\theta_{[1]}) = K$, by Pareto-optimality, and then $f^j(\theta_{[1]}) = 0, \forall j \in N \setminus \{i\}$, given the feasibility constraints. Let $i = 1$. Now let $\theta_{[2]} \in \begin{pmatrix} 0 & K & \cdots & K \\ & 0 & \cdots & 0 \end{pmatrix}$. Then $\exists i' \in N \setminus \{1\}$ such that $f^{i'}(\theta_{[2]}) = K$, by Pareto-optimality, and so $f^j(\theta_{[2]}) = 0, \forall j \in N \setminus \{i'\}$, given the feasibility constraints. Let $i' = 2$. Continuing in the same manner, we get an ordering of the agents, $\sigma = (1, \ldots, n)$. Now fix $i, j \in N$. If $i < j$ then $iB(\theta_{[i]})j$ and $f^i(\theta_{[i]}) = K$. Then Step 4 implies that $\forall \theta \in \Theta, \neg(jB(\theta)i)$. Thus, $\forall \theta \in \Theta, U(f^i(\theta), \theta^i) > U(f^j(\theta), \theta^i)$, unless $f^i(\theta) = f^j(\theta) = 0$. Therefore, $\sigma = (1, \ldots, n)$ is an m-hierarchy associated with f at each profile θ.□

Let $o(i, \theta^{-i}) = \{x \in \mathcal{X} \mid \exists \theta^i \in \Theta^i \text{such that} f(\theta) = x\}$ denote agent *i's option set* (for f) at profile θ. Let $o^i(i, \theta^{-i}) = \{p \in \mathcal{K} \mid \exists \theta^i \in \Theta^i \text{such that} f^i(\theta) = p\}$. That is, $o^i(i, \theta^{-i})$ is the set of allocations that agent i can get by deviating her messages when the other agents' report is θ^{-i}. Clearly, $0 \in o^i(i, \theta^{-i}), \forall i \in N, \forall \theta \in \Theta$.

Lemma 5 *If an SCF is strategyproof, strongly nonbossy, and Pareto-optimal such that there exists a single m-hierarchy, σ, associated with it at each profile then it is a serial dictatorship with d-hierarchy σ.*

Proof: Let f be strategyproof, strongly nonbossy, and Pareto-optimal such that $\sigma = (1, \ldots, n)$ is an m-hierarchy associated with it at each profile θ.

Step 1: If $p \in T(i, \theta)$ then $\exists j \in N, j < i$, such that $p \bigcap f^j(\theta) \neq \emptyset$.

Fix $i \in N, \theta \in \Theta$, and $p \in \mathcal{K}, p \neq 0$. Suppose that $p \in T(i, \theta)$ for some $i \in N$ and $\forall j \in N \backslash \{i\}, j < i, p \bigcap f^j(\theta) = \emptyset$. Then Pareto-optimality implies that $\exists j \in N \backslash \{i\}$ such that $p \bigcap f^j(\theta) \neq \emptyset$. Suppose there are $t \geq 1$ such agents, $j_1, \ldots, j_t$, i.e., for $l = 1, \ldots, t, p \bigcap f^{j_l} \neq \emptyset$ such that $j_l > i$. Let $f(\theta) = x$. Let $\bar{\theta}^j \in \begin{pmatrix} x^j \\ 0 \end{pmatrix}, \forall j \in N \setminus \{i\}$ and let $\bar{\theta}^i \in \begin{pmatrix} p \\ x^i \\ 0 \end{pmatrix}$.

Then monotonicity implies that $f(\bar{\theta}) = x$. By strong nonbossiness and Pareto-optimality, $f(\bar{\theta}^i, \bar{\theta}^J, 0, \ldots, 0) = (x^i, x^J, 0, \ldots, 0)$, where $J = \{j_1, \ldots, j_t\}, x^J = (x^{j_1}, \ldots, x^{j_t})$ and 0 denotes a strategy in (0). For simplicity, let us ignore all $j \notin J, j \neq i$ for the rest of this proof, since their strategies will be kept the same (a strategy in (0)) and therefore they won't play any role. Thus, we have $f\begin{pmatrix} p & x^{j_1} & \cdots & x^{j_t} \\ x^i & 0 & \cdots & 0 \\ 0 & & & \end{pmatrix} = (x^i, x^{j_1}, \ldots, x^{j_t})$. Now consider a profile in $\begin{pmatrix} p & x^{j_1} & \cdots & x^{j_1} \\ x^{j_1} & 0 & \cdots & 0 \\ 0 & & & \end{pmatrix}$. Since $\neg(j_1 B(\theta) i), \forall \theta \in \Theta, j_1$ cannot get x_{j_1} at this profile, given that $p \bigcap x^{j_1} \neq 0$. Then, by Pareto-optimality, agent i gets either p or x^{j_1}. If i gets p then she can manipulate at $\begin{pmatrix} p & x^{j_1} & \cdots & x^{j_t} \\ x^i & 0 & \cdots & 0 \\ 0 & & & \end{pmatrix}$ via $\begin{pmatrix} p \\ x^{j_1} \\ 0 \end{pmatrix}$. Therefore, agent i gets x^{j_1}, and thus Pareto-optimality implies that $f\begin{pmatrix} p & x^{j_1} & \cdots & x^{j_t} \\ x^{j_1} & 0 & \cdots & 0 \\ 0 & & & \end{pmatrix} = (x^{j_1}, 0, x^{j_2}, \ldots, x^{j_t})$. By monotonicity, (or Pareto-optimality and nonbossiness), we get $f\begin{pmatrix} p & 0 & x^{j_2} & \cdots & x^{j_t} \\ x^{j_1} & & 0 & \cdots & 0 \\ 0 & & & & \end{pmatrix} = (x^{j_1}, 0, x^{j_2}, \ldots, x^{j_t})$. Now we can continue by replacing iteratively i's strategy with $\begin{pmatrix} p \\ x^{j_2} \\ 0 \end{pmatrix}$, $\begin{pmatrix} p \\ x^{j_3} \\ 0 \end{pmatrix}$, etc. When we get to $\begin{pmatrix} p \\ x^{j_{t-1}} \\ 0 \end{pmatrix}$, we get

$f\begin{pmatrix} p & 0 & \cdots & 0 & x^{j_t} \\ x^{j_{t-1}} & & & & 0 \\ 0 & & & & \end{pmatrix} = (x^{j_{t-1}}, 0, \dots, 0, x^{j_t})$. Then $f\begin{pmatrix} p & 0 & \cdots & 0 & x^{j_t} \\ x^{j_t} & & & & 0 \\ 0 & & & & \end{pmatrix} =$ $(p, 0, \dots, 0)$, since $\neg(j_t B(\theta) i), \forall \theta \in \Theta$, and so j_t cannot get x^{j_t}. Then, agent i gets p, by Pareto-optimality. However, in this case, agent i can manipulate at $\begin{pmatrix} p & 0 & \cdots & 0 & x^{j_t} \\ x^{j_{t-1}} & & & & 0 \\ 0 & & & & \end{pmatrix}$ via $\begin{pmatrix} p \\ x^{j_t} \\ 0 \end{pmatrix}$, which contradicts f's strategyproofness. Therefore, $\forall i \in N, \forall p \in \mathcal{K}, p \neq 0, \forall \theta \in \Theta$ if $p \in T(i, \theta)$ then $\exists j \in n, j < i$ such that $p \bigcap f^j(\theta) \neq \emptyset$.

Step 2: f is a serial dictatorship where σ is the d-hierarchy associated with f.

For this step, set $0 = \emptyset$. Fix $i \in N$ and $p \in \mathcal{K}$. Let θ^i be such that $\text{top}(\theta^i) = p$, and suppose that $\theta^{-i} \in \Theta^{-i}$ is such that $\forall j \in N \setminus \{i\}, j < i, p \bigcap f^j(\theta) = \emptyset$. Then $p \notin T(i, \theta)$, by Step 1, and so $f^i(\theta) = p$. This proves that

$$\forall i \in N, \forall p \in \mathcal{K}, \forall \theta \in \Theta, \text{ if } p \bigcap f^j(\theta) = \emptyset, \forall j \in N \setminus \{i\}, j < i, \text{ then } p \in o^i(i, \theta^{-i}). \quad (2.1)$$

Since $\forall j \in N \setminus \{1\}, j > 1$, we have $p \in o^1(1, \theta^{-1})$, $\forall p \in \mathcal{K}, \forall \theta^{-1} \in \Theta^{-1}$, which implies that $o^1(1, \theta^{-1}) = \mathcal{K}$. Now fix $i \in N \setminus \{1\}$. Suppose $p \bigcap f^1(\theta) \neq \emptyset$ and $p \in o^i(i, \theta^{-i})$, for some $\theta \in \Theta$ and $p \in \mathcal{K}$. Then $\exists \tilde{\theta}^i \in \Theta^i$ such that $f^i(\tilde{\theta}^i, \theta^{-i}) = p$. Then $f^1(\tilde{\theta}^i, \theta^{-i}) \neq f^1(\theta)$, given the feasibility constraints. Clearly, if f is strategyproof and nonbossy, then the outcome at every profile is the best option at that profile for each agent. That is, $\forall \theta \in \Theta, \forall i \in N, f^i(\theta) = c^i(o(i, \theta^{-i}), \theta^i)$. Since $o^1(1, (\tilde{\theta}^i, \theta^{-i,1})) = \mathcal{K}$, this implies that $f^1(\tilde{\theta}^i, \theta^{-i}) = f^1(\theta) = c^1(\mathcal{K}, \theta^1)$, which is a contradiction. Then $\forall i \in N, \forall p \in \mathcal{K}, \forall \theta \in \Theta$, if $p \bigcap f^1(\theta) \neq \emptyset, p \notin o^2(2, \theta^{-2})$. This implies, together with (2.1), that $o^i(i, \theta^{-i}) = \mathcal{K} \setminus \{f^1(\theta)\}, \forall \theta \in \Theta$. Now fix $i \in N \setminus \{1, 2\}$. Suppose $p \bigcap f^2(\theta) \neq \emptyset$ and $p \in o^i(i, \theta^{-i})$, for some $\theta \in \Theta, p \in \mathcal{K}$. Then a similar argument to the one applied to agent 1 above shows that this is a contradiction, and we can imply that $o^3(3, \theta^{-3}) = \mathcal{K} \setminus \{(f^1(\theta)\} \bigcup \{f^2(\theta)\}$, using (2.1). Continuing iteratively, we get that $\forall \theta \in \Theta, \forall i \in N \setminus \{1\}, o^i(i, \theta^{-i}) = \mathcal{K} \setminus \bigcup_{l=1}^{i-1} \{f^l(\theta)\}$, where $o^1(1, \theta^{-1}) = \mathcal{K}$. Thus, we have $f^1(\theta) = c^1(\mathcal{K}, \theta^1)$ and for $i \in N \setminus \{1\}, f^i(\theta) = c^i(\mathcal{K} \setminus \bigcup_{l=1}^{i-1} \{f^l(\theta)\}, \theta^{-i}), \forall \theta \in \Theta$. Therefore, f is a serial dictatorship such that the d-hierarchy associated with f is $(1, \ldots, n)$.□

Proposition 7 follows immediately from the three lemmas.

It should be remarked that Satterthwaite and Sonnenschein's result (1981, Theorem 2) does not imply ours. Although they do not require Pareto-optimality, they impose a number of differentiabilty assumptions on the social choice function (which they call regularity) and assume that each agents' consumption set is convex. These assumptions clearly do not apply to economies with indivisibilities.

It is interesting, however, to compare their sufficiency condition for serial dictatorships to ours. Satterthwaite and Sonnenschein use a binary relation defined on the set of agents for each profile, namely the *affect* relation, throughout their analysis. An agent affects another agent at a given profile, if she can change the allocation for the other agent by deviating alone at that profile.[8] Satterthwaite and Sonnenschein's sufficiency condition is that the affect relation is *everywhere total* (in the following, ET), i.e., at each profile for any two agents at least one of them affects the other. Our result, therefore, looks surprising, since the strong nonbossiness condition rules out some affect relations under certain circumstances.

[8]For a formal definition, see Definition 24.

Furthermore, a serial dictatorship in our context does not satisfy ET. To see this, take three agents, say agents 1, 2, and 3, such that $f^1(\theta) = p$ at some profile θ, where $p \in \mathcal{K}, p \neq 0$, and $\theta^2, \theta^3 \in \begin{pmatrix} p \\ 0 \end{pmatrix}$. Since agent 1 beats both agents 2 and 3 at θ, the d-hierarchy σ associated with the serial dictatorship f is such that $\sigma(1) < \sigma(2)$ and $\sigma(1) < \sigma(3)$. Now assume that $\sigma(2) < \sigma(3)$, so that $f^3(\theta) = 0$, by Pareto-optimality. Then agent 1 cannot affect agent 3 at this profile, since for each $\tilde{\theta}^1 \in \Theta^1$, if $f^1(\tilde{\theta}^1, \theta^{-1}) \bigcap p \neq \emptyset$ then, given the feasibility constraints, Pareto-optimality requires that $f^3(\tilde{\theta}^1, \theta^{-1}) = 0$. In addition, if $f^1(\tilde{\theta}^1, \theta^{-1}) \bigcap p = \emptyset$ then $f^3(\tilde{\theta}^1, \theta^{-1}) = 0$ again, since $\sigma(2) < \sigma(3)$, and $\theta^2 \in \begin{pmatrix} p \\ 0 \end{pmatrix}$. Notice, however, that it is Pareto-optimality that seems to be in conflict with ET. Indeed, if an SCF satisfies ET in our context, then it cannot be Pareto-optimal. (To check this, take a profile in which two agents' first choice is the null package.) Thus, the ET condition is too restrictive in our context. Pareto-optimality, however, is too restrictive in the Satterthwaite-Sonnenschein model, as they remark that for some standard convex and compact allocation possibility sets, the set of Pareto-optimal SCF's is empty. Although the two conditions are not necessarily compatible, they are essentially similar in their effects. To see this, note that Satterthwaite and Sonnenschein don't require any form of citizen sovereignty, that is, variation in the outcomes. Therefore, their Theorem 1 is consistent with an imposed mechanism,[9] which says that for a strategyproof, nonbossy, and regular mechanism, the affect relationship is acyclic at any profile, if the domain is some open set of utility functions. That is, an imposed mechanism which yields the same outcome at any profile, a mechanism for which no agent affects any other agent at any profile, would satisfy the theorem. Therefore, in order to get

[9]This is pointed out in Muller and Satterthwaite (1986).

a serial dictatorship, they need to require some variation in the outcomes, and ET implies just that. In light of Proposition 9, our Pareto-optimality requirement has essentially a similar effect.

This still does not explain the sufficiency of strong nonbossiness. Remark that in our model, since the objects need not be allocated, and the value of any package may be negative to an agent, Pareto-optimality requires that at some profiles not all the objects are allocated. Apparently, serial dictatorship can be avoided using this type of lack of "conflict," so that when some variation in the outcomes are ruled out in these "no conflict" situations, and that's what strong nonbossiness amounts to, the ordering of the agents induced by Pareto-optimality must be the same for all profiles, causing the mechanism to be a serial dictatorship.

2.4 Characterization of Strategyproof, Nonbossy, and Pareto-optimal Social Choice Functions

In this section we would like to characterize the set of strategyproof, nonbossy, and Pareto-optimal SCF's. First we prove that, just as for the single object case, any strategyproof, nonbossy, and Pareto-optimal SCF is dictatorial. Notice that in the definition of a dictatorial mechanism (Definition 17), a dictator is not a dictator in the strong sense that, given any profile of the other agents, the dictator can "determine" the outcome, i.e., the allocations to the other agents as well.[10] Our definition is more apropriate in the context of private goods

[10]This is not to be confused with the distinction between weak and strong dictatorship in Muller and Satterthwaite (1986), which has to do with the feasible sets of alternatives, i.e., whether the agent is a dictator over a single feasible set or every feasible set. Satterthwaite (1975) distinguishes between fully and partially dictatorial voting procedures, which depends on whether it is in the dictator's power to impose any

allocation problems because the nonexistence of the conventional dictatorship is a very weak requirement. It would be ruled out by Pareto-optimality (or nonbossiness) alone.[11] This weaker definition, however, is in the spirit of the original definition of dictatorship (see Gibbard (1973), for example), in that a dictator can get her first choice regardless of the others' will. Thus, given the feasibility constraints, our dictator affects the outcomes of the other agents, which makes the distribution of power lopsided. However, since the dictator may be indifferent among outcomes that give her top choice to her, a dictatorial mechanism, as defined in this study, may take into account other agents' preferences as well.

Proposition 8 *A strategyproof, nonbossy, and Pareto-optimal SCF is dictatorial.*

First we provide two lemmas, which will be used in the proof of the proposition. Both lemmas and the definitions to follow are based on Barbera (1983), who proves the Gibbard-Satterthwaite theorem using the concept of pivotal voters.

A *reshuffling* of a preference ordering around an outcome x is another preference ordering under which x preserves the same relative position to all the other outcomes. Formally, for $\theta^i \in \Theta^i$ and $x \in \mathcal{X}, \tilde{\theta}^i \in \Theta^i$ is a reshuffling of θ^i around x if $\forall y \in \mathcal{X}, U(x,\theta^i) \geq U(y,\theta^i) \Leftrightarrow U(x,\tilde{\theta}^i) \geq U(y,\tilde{\theta}^i)$. Let $r(x,\theta^i)$ denote the set of reshufflings of θ^i around x. Clearly, no agent can change the outcome of a strategyproof and nonbossy SCF f at any profile by changing her reported preferences to a reshuffling around that outcome. This follows immediately from monotonicity, or can be verified directly by checking that if

outcome or whether she is constrained to some subset of the possible outcomes. This is also different from the distinction discussed here.

[11]Zhou (1991) proves for the two-agent case, where private goods are divisible and the admissable utility functions are continuous, strictly quasi-concave, and increasing, that any Pareto-optimal and strategyproof mechanism is *inversely dictatorial.* A mechanism is inversely dictatorial if one agent gets 0 at each profile. This is also a very weak requirement in our context, since the agents may have a negative evaluation for any package, and thus Pareto-optimality alone ensures that an SCF is not inversely dictatorial.

$f(\theta) \neq f(\tilde{\theta}^i, \theta^{-i})$ for some agent i, profile θ, and $\tilde{\theta}^i \in r(f(\theta), \theta^i)$ then, since f is nonbossy, agent i can manipulate f either at θ via $\tilde{\theta}^i$, or at $(\tilde{\theta}^i, \theta^{-i})$ via θ^i.

Let $(\theta^i)^x$ denote the preferences obtained from θ^i when x is ranked first, preserving the ordering of all the other outcomes in θ^i. Similarly, let $(\theta^i)_x$ denote the preference ordering when x is ranked last, $(\theta^i)^{x,y}$ when x is ranked first and y is ranked second, and $(\theta^i)^x_y$ when x is ranked first and y is ranked last.

The lemma to follow states that no agent can change the option set of any other agent at any profile, by changing her preferences to a reshuffling around the outcome of f at that profile, provided f is strategyproof and nonbossy.

Lemma 6 *If an SCF f is strategyproof and nonbossy, then $\forall \theta \in \Theta, \forall i, j \in N, \forall \tilde{\theta}^i \in r(f(\theta), \theta^i), o(j, \theta^{-j}) = o(j, (\tilde{\theta}^i, \theta^{-i,j}))$.*

Proof: Let f be strategyproof and nonbossy. Let $\theta \in \Theta, i, j \in N$, and $\tilde{\theta}^i \in r(f(\theta), \theta^i)$. We will show that $o(j, (\tilde{\theta}^i, \theta^{-i,j})) \subseteq o(j, \theta^{-j})$. Since $f(\tilde{\theta}^i, \theta^{-i}) = f(\theta)$, and $\theta^i \in r(f(\theta), \tilde{\theta}^i)$, a similar argument will prove that $o(j, \theta^{-j}) \subseteq o(j, (\tilde{\theta}^i, \theta^{-i,j}))$, which establishes the desired result.

Suppose $o(j, (\tilde{\theta}^i, \theta^{-i,j})) \not\subseteq o(j, \theta^{-j})$. Then $\exists y \in \mathcal{X}$ such that $y \in o(j, (\tilde{\theta}^i, \theta^{-i,j}))$ and $y \notin o(j, \theta^{-j})$. Let $f(\theta) = f(\tilde{\theta}^i, \theta^{-i}) = x$. Then $x \neq y$, since $x \in o(j, \theta^{-j})$. Since $x = c(o(j, \theta^{-j}), \theta^j)$ and y $\notin o(j, \theta^{-j})$, we have $x = c(o(j, \theta^{-j}), (\theta^j)^y) = f((\theta^j)^y, \theta^{-j})$. However, $y \in o(j, (\tilde{\theta}^i, \theta^{-i,j}))$ implies that $c(o(j, (\tilde{\theta}^{-i}, \theta^{-i,j}), (\theta^j)^y) = y$, so $f(\tilde{\theta}^i, (\theta^j)^y, \theta^{-i,j}) = y$. In sum, we have $f(\theta^i, (\theta^j)^y, \theta^{-i,j}) = x$, and $f(\tilde{\theta}^i, (\theta^j)^y, \theta^{-i,j}) = y$. If $x^i \neq y^i$, then $\theta^i \in r(x, \tilde{\theta}^i)$ implies that agent i can manipulate either at $(\theta^i, (\theta^j)^y, \theta^{-i,j})$ via $\tilde{\theta}^i$ or at $f(\tilde{\theta}^i, (\theta^j)^y, \theta^{-i,j})$ via θ^i. This contradicts f's strategyproofness, hence $x^i = y^i$. However, in this case nonbossiness implies that $x = y$, which is a contradiction.□

The next lemma is about the agents' ability to affect each other's allocation. An agent *affects* another agent at a given profile if she can change the other agent's allocation by deviating her messages.

Definition 24 For an SCF f, agent i *affects* agent j at $\theta \in \Theta$, if $\exists \tilde{\theta^i} \in \Theta^i$ such that $f^j(\theta) \neq f^j(\tilde{\theta^i}, \theta^{-i})$. We then write that $iA(\theta)j$.

The following lemma states that if two agents can affect one another at some profile, then at least one of them is able to "get" the allocation the other one "imposes" on her, by deviating her message at that profile, or the "imposed" allocation is the null package for at least one of them, given a strategyproof and nonbossy SCF.

Lemma 7 *If an SCF f is strategyproof and nonbossy then $\forall \theta \in \Theta, \forall i,j \in N, i \neq j$, such that $iA(\theta)j$ and $jA(\theta)i$, and $\forall \tilde{\theta^i} \in \Theta^i, \forall \tilde{\theta^j} \in \Theta^j$ such that $f^j(\tilde{\theta^i}, \theta^{-i}) \neq f^j(\theta)$ and $f^i(\tilde{\theta^j}, \theta^{-j}) \neq f^i(\theta)$, we have one of four cases: (a) $f^j(\tilde{\theta^i}, \theta^{-i}) = f^j(\tilde{\theta^j}, \theta^{-j})$, (b) $f^i(\tilde{\theta^j}, \theta^{-j}) = f^i(\tilde{\theta^i}, \theta^{-i})$, (c) $f^j(\tilde{\theta^i}, \theta^{-i}) = 0$, or (d) $f^i(\tilde{\theta^j}, \theta^{-j}) = 0$.*

Proof: Let f be strategyproof and nonbossy. Let $\theta \in \Theta, i,j \in N, i \neq j$, such that $iA(\theta)j$ and $jA(\theta)i$. Fix $\tilde{\theta^i} \in \Theta^i$ such that $f^j(\tilde{\theta^i}, \theta^{-i}) \neq f^j(\theta)$, and fix $\tilde{\theta^j} \in \Theta^j$ such that $f^i(\tilde{\theta^j}, \theta^{-j}) \neq f^i(\theta)$. Let $f(\theta) = x, f(\tilde{\theta^i}, \theta^{-i}) = y$, and $f(\tilde{\theta^j}, \theta^{-j}) = z$. Then $x, y \in o(i, \theta^{-i})$, $x, z, \in o(j, \theta^{-j})$, $z^i \neq x^i$, and $y^j \neq x^j$. Suppose $y^j \neq z^j, y^i \neq z^i, y^j \neq 0$, and $z^i \neq 0$. Since $z^i \neq 0$, it is possible that $U(x^i, \theta^i) < U(z^i, \theta^i)$, and, similarly, since $y^j \neq 0$ it is possible that

$U(x^j, \theta^j) < U(y^j, \theta^j)$. Then we can define $\bar{\theta}^i, \bar{\theta}^j, \hat{\theta}^i$, and $\hat{\theta}^j$ as follows. Let

$$\bar{\theta}^i = \begin{cases} (\theta^i)_{z^i} & \text{if} \quad U(x^i, \theta^i) > U(z^i, \theta^i) \\ (\theta^i)^{z^i} & \text{if} \quad U(x^i, \theta^i) < U(z^i, \theta^i), \end{cases}$$

$$\bar{\theta}^j = \begin{cases} (\theta^j)_{y^j} & \text{if} \quad U(x^j, \theta^i) > U(y^j, \theta^j) \\ (\theta^j)^{y^j} & \text{if} \quad U(x^j, \theta^j) < U(y^j, \theta^j), \end{cases}$$

$$\hat{\theta}^i = \begin{cases} (\theta^i)^{y^i}_{z^i} & \text{if} \quad U(x^i, \theta^i) > U(z^i, \theta^i) \\ (\theta^i)^{z^i, y^i} & \text{if} \quad U(x^i, \theta^i) < U(z^i, \theta^i), \end{cases}$$

and

$$\hat{\theta}^j = \begin{cases} (\theta^j)^{z^j}_{y^j} & \text{if} \quad U(x^j, \theta^i) > U(y^j, \theta^j) \\ (\theta^j)^{y^j, z^j} & \text{if} \quad U(x^j, \theta^j) < U(y^j, \theta^j). \end{cases}$$

Note that $\bar{\theta}^i \in r(x^i, \theta^i), \bar{\theta}^j \in r(x^j, \theta^j), \hat{\theta}^i \in r(z^i, \bar{\theta}^i)$, and $\hat{\theta}^j \in r(y^j, \bar{\theta}^j)$.

Since $\bar{\theta}^j \in r(x^j, \theta^j), o(i, (\bar{\theta}^j, \theta^{-i,j})) = o(i, \theta^{-i})$, by Lemma 6. Then $y \in o(i, (\bar{\theta}^j, \theta^{-i,j}))$, so $c(o(i, (\bar{\theta}^j, \theta^{-i,j}), \hat{\theta}^i) = y$ if $U(x^i, \theta^i) > U(z^i, \theta^i)$. If $U(x^i, \theta^i) < U(z^i, \theta^i)$ then $f(\theta) \neq z$ indicates that $z \notin o(i, \theta^{-i})$, and so $z \notin o(i, (\bar{\theta}^j, \theta^{-i,j}))$. Thus, if $U(x^i, \theta^i) < U(z^i, \theta^i)$, we also have $c(o(i, (\bar{\theta}^j, \theta^{-i,j}), \hat{\theta}^i) = y$. Therefore, $f(\hat{\theta}^i, \bar{\theta}^j, \theta^{-i,j}) = y$. Using a similar argument for agent j, we can show that $f(\bar{\theta}^i, \hat{\theta}^j, \theta^{-i,j}) = y$. But then, given that $\hat{\theta}^i \in r(z^i, \bar{\theta}^i)$ and $\hat{\theta}^j \in r(y^j, \bar{\theta}^j)$, we get that $f(\hat{\theta}^i, \hat{\theta}^j, \theta^{-i,j}) = y = z$, which is a contradiction.□

Proof of Proposition 8:

Let f be strategyproof, nonbossy, and Pareto-optimal.

Step 1: Identification of the dictator.

Let $\theta^i \in \begin{pmatrix} K \\ 0 \end{pmatrix}, \forall i \in N$. Then Pareto-optimality implies that there exists an agent, say agent 1, who gets package K at θ. That is, given the feasibility constraints, $f(\theta) = (K, 0, \ldots, 0)$.

Step 2: No agent can affect the dictator at a profile where each agent's first choice is K and second choice is 0.

Let $\bar{\theta}^1 \in (0)$. Then $\exists i \in N \backslash \{1\}$ such that $f^i(\bar{\theta}^1, \theta^{-1}) = K$, by Pareto-optimality. Let this agent be agent 2, so that $f(\bar{\theta}^1, \theta^{-1}) = (0, K, 0, \ldots, 0)$, by feasibility. Then $1A(\theta)2$. Suppose $2A(\theta)1$. Then, by Lemma 7, we have one of three cases: (a) $\exists \tilde{\theta}^2 \in \Theta^2$ such that $f^2(\tilde{\theta}^2, \theta^{-2}) = K$, or (b) $\exists \tilde{\theta}^2 \in \Theta^2$ such that $f^1(\tilde{\theta}^2, \theta^{-2}) = f^1(\bar{\theta}^1, \theta^{-1}) = 0$, or (c)$f^2(\bar{\theta}^1, \theta^{-1}) = 0$. Clearly, (c) doesn't hold. If (a) holds then agent 2 can manipulate at θ via $\tilde{\theta}^2$. If (b) holds then Pareto-optimality implies that either agent 2 gets package K at $(\tilde{\theta}^2, \theta^{-2})$, which leads to the same contradiction as in case (a), or some agent other than 1 or 2 gets package K at $(\tilde{\theta}^2, \theta^{-2})$, which implies that agent 2 is bossy. Therefore, $\neg(2A(\theta)1)$.

Next, we show that $\forall i \in N \setminus \{1, 2\}, \neg(iA(\theta)1)$. Fix $i \in N \setminus \{1, 2\}$. Suppose $iA(\theta)1$. Then $\exists \tilde{\theta}^i \in \Theta^i$ such that $f^1(\tilde{\theta}^i, \theta^{-i}) \neq f^1(\theta)$. By nonbossiness, $f^i(\tilde{\theta}^i, \theta^{-i}) \neq f^i(\theta) = 0$. We know that $f^i(\tilde{\theta}^i, \theta^{-i}) \neq K$, otherwise agent i can manipulate at θ via $\tilde{\theta}^i$. Therefore, $f^i(\tilde{\theta}^i, \theta^{-i}) = p$, where $p \in \mathcal{K}, p \neq K, p \neq 0$. Then feasibility and Pareto-optimality imply that $f^j(\tilde{\theta}^i, \theta^{-i}) = 0, \forall j \in N \setminus \{i\}$. Now let $\hat{\theta}^i \in \begin{pmatrix} K \\ p \end{pmatrix}$. Then Pareto-optimality implies that $\exists j \in N$ such that $f^j(\hat{\theta}^i, \theta^{-i}) = K$. If $f^i(\hat{\theta}^i, \theta^{-i}) = K$, then i can manipulate at θ via $\hat{\theta}^i$. If $f^j(\hat{\theta}^i, \theta^{-i}) = K$ for some $j \in N \setminus \{1, i\}$ then i is bossy at $(\hat{\theta}^i, \theta^{-i})$ versus θ. Therefore, $f^1(\hat{\theta}^i, \theta^{-i}) = K$, and the feasibility constraints imply that $f^i(\hat{\theta}^i, \theta^{-i}) = 0$. However, in this case, agent i can manipulate at $(\hat{\theta}^i, \theta^{-i})$ via $\tilde{\theta}^i$, which contradicts f's strategyproofness. This completes the proof that $\forall i \in N \setminus \{1\}, \neg(iA(\theta)1$.

Step 3: No coalition of the $n-1$ non-dictators can change the outcome, as long as the dictator's first choice is K.

Given Step 2, no agent other than 1 can change the outcome at $\theta = \begin{pmatrix} K & \cdots & K \\ 0 & \cdots & 0 \end{pmatrix}$, by changing her strategy alone. Now we want to show that no coalition of the $n-1$ agents, excluding agent 1, can change the outcome at θ by jointly deviating. Assume the contrary. Then $\exists \tilde{\theta}^{-1} \in \Theta^{-1}$ such that $f(\theta^1, \tilde{\theta}^{-1}) \neq (K, 0, \ldots, 0)$. If $n = 2$, then Step 3 holds by Step 2, so let $n \geq 3$. Let $\tilde{\Theta}^{-1} \subset \Theta^{-1}$ be a subset of the set of preference profiles for the $n-1$ agents, such that $\forall \tilde{\theta}^{-1} \in \tilde{\Theta}^{-1}, f(\theta^1, \tilde{\theta}^{-1}) \neq (K, 0, \ldots, 0)$. For all $\tilde{\theta}^{-1} \in \tilde{\Theta}^{-1}$, let $L(\tilde{\theta}^{-1}) = \left\{ i \in N \setminus \{1\} \mid \tilde{\theta}^i \notin \begin{pmatrix} K \\ 0 \end{pmatrix} \right\}$. Let $l = \min_{(\tilde{\theta}^{-1}) \in \tilde{\Theta}^{-1}} \{ \mid L(\tilde{\theta}^{-1}) \mid \}$, i.e., l is the minimum number of the agents contained in any coalition in $N \setminus \{1\}$ that can jointly change the outcome at θ by deviating their strategies. Note that $l \geq 2$, by Step 2.

Now fix $\tilde{\theta}^{-1} \in \tilde{\Theta}^{-1}$ such that $L(\tilde{\theta^{-1}}) = l$. Let $L = \left\{ i \in N \setminus \{1\} \mid \tilde{\theta}^i \notin \begin{pmatrix} K \\ 0 \end{pmatrix} \right\}$, and let $f(\theta^1, \tilde{\theta}^{-1}) = f \begin{pmatrix} K, & \tilde{\theta}^L, & K & \cdots & K \\ 0 & & 0 & \cdots & 0 \end{pmatrix} = \mathrm{x}$, assuming, without loss of generality, that $\forall i \in L, i \leq l+1$. Then monotonicity implies that $f \begin{pmatrix} K & x^L & K & \cdots & K \\ 0 & & 0 & \cdots & 0 \end{pmatrix} = x$, where $x^L = (x^2, \ldots, x^{l+1})$. Given that $\mid L \mid = l$, $f \begin{pmatrix} K, & \theta^i, & x^{L \setminus \{i\}} & K & \cdots & K \\ 0 & & & 0 & \cdots & 0 \end{pmatrix} =$

$(K,0,\ldots,0), \forall i \in L$, where $\theta^i \in \begin{pmatrix} K \\ \\ 0 \end{pmatrix}$. Since $x \neq (K,0,\ldots,0)$, $x^1 = 0$, by Pareto-optimality and feasibility. Now let $\bar{L} = N \setminus (L \bigcup \{1\})$, so that N can be partitioned into $\{1\}, L$, and $\bar{L}$. (Note that $\bar{L} = \emptyset$ if $l = n-1$.) We know that $x^{\bar{L}} = (x^{l+2}, \ldots, x^n) = (0, \ldots, 0)$, otherwise some $j \in \bar{L}$ gets package K at $(\theta^1, \tilde{\theta}^{-1})$, and thus each $i \in L$ is bossy at $(\theta^1, \tilde{\theta}^{-1})$ versus $(\theta^1, \theta^i, \tilde{\theta}^{-1,i})$, given the feasibility constraints. We also know that $\exists i^* \in L$ such that $x^{i^*} \neq 0$, otherwise Pareto-optimality requires that either $x^1 = K$ or $x^j = K$ for some $j \in \bar{L}$, which is a contradiction. But then $\forall i \in L \setminus \{i^*\}, x^i \neq 0$, otherwise i is bossy at $(\theta^1, \tilde{\theta}^{-1})$ versus $(\theta^1, \theta^i, \tilde{\theta}^{-1,i})$. Therefore, $\forall i \in L, x^i \neq 0$. Given that $\mid L \mid \geq 2$ and $\forall i \in L, x^i \neq 0$, the feasibility constraints imply that $x^i \neq K, \forall i \in L$. Therefore, $\mid L \mid \leq k$.

Now we will show that

$$
\begin{aligned}
&f\begin{pmatrix} K & K & x^3 & \cdots & x^{l+1} & K & \cdots & K \\ \\ 0 & x^2 & & & & 0 & \cdots & 0 \end{pmatrix} = \\
&f\begin{pmatrix} K & K & \cdots & K & x^{i+1} & \cdots & x^{l+1} & K & \cdots & K \\ \\ 0 & x^2 & \cdots & x^i & & & & 0 & \cdots & 0 \end{pmatrix} = \qquad (2.2) \\
&f\begin{pmatrix} K & K & \cdots & K & K & \cdots & K \\ \\ 0 & x^2 & \cdots & x^{l+1} & 0 & \cdots & 0 \end{pmatrix} = x.
\end{aligned}
$$

First notice that no agent other than 1 can get K, as long as agent 1 and each agent $j \in \bar{L}$ report $\begin{pmatrix} K \\ \\ 0 \end{pmatrix}$, since otherwise some agent $i \in L$ is bossy, given the feasibility constraints. (If $\mid L \mid = 2$ and one agent in L gets K then the other agent in L is the bossy agent.) If the outcome were $(K, 0, \ldots, 0)$ for any of the above preference profiles then the appropriate agent i ($i \in L$) can manipulate via (x^i). Therefore, Pareto-optimality implies that (2.2) holds.

Using monotonicity, we get that $f\begin{pmatrix} 0 & K & \cdots & K & K & \cdots & K \\ & x^2 & \cdots & x^{l+1} & 0 & \cdots & 0 \end{pmatrix} = x$. Now take $i^* \in L$ such that

$$f^{i^*}\begin{pmatrix} 0 & K & \cdots & K & 0 & \cdots & 0 \\ & 0 & \cdots & 0 & & & \end{pmatrix} = K, \tag{2.3}$$

where $\forall i \in L$, i's strategy is $\begin{pmatrix} K \\ 0 \end{pmatrix}$, and $\forall i \notin L$, i's strategy is (0). Note that i^* satisfying (2.3) exists by Pareto-optimality. Let $i^* = 2$, without loss of generality. If agent 2 gets K at

$$\begin{pmatrix} 0 & K & K & \cdots & K & 0 & \cdots & 0 \\ & 0 & x^3 & \cdots & x^{l+1} & & & \end{pmatrix}$$

then agent 2 can manipulate at

$$\begin{pmatrix} 0 & K & \cdots & K & 0 & \cdots & 0 \\ & x^2 & \cdots & x^{l+1} & 0 & \cdots & 0 \end{pmatrix}$$

via $\begin{pmatrix} K \\ 0 \end{pmatrix}$. If some other $i \in L, i \neq 2$ gets K at that profile, then monotonicity implies that (2.3) is contradicted. Therefore,

$$f\begin{pmatrix} 0 & K & K & \cdots & K & 0 & \cdots & 0 \\ & 0 & x^3 & \cdots & x^{l+1} & & & \end{pmatrix} = (0, 0, x^3, \ldots, x^{l+1}, 0, \ldots, 0),$$

by Pareto-optimality. Then monotonicity implies that

$$f\begin{pmatrix} 0 & 0 & K & \cdots & K & 0 & \cdots & 0 \\ & & x^3 & \cdots & x^{l+1} & & & \end{pmatrix} = (0,0,x^3,\ldots,x^{l+1},0,\ldots,0).$$

Now let $L^2 = L \setminus \{2\}$, and apply the same argument to L^2 as the one applied to L above. Letting $i^* = 3$, where $i^* \in L^2$ satisfies

$$f^{i^*}\begin{pmatrix} 0 & 0 & K & \cdots & K & 0 & \cdots & 0 \\ & & 0 & \cdots & 0 & & & \end{pmatrix} = K,$$

we get that

$$f\begin{pmatrix} 0 & 0 & K & K & \cdots & K & 0 & \cdots & 0 \\ & & 0 & x^4 & \cdots & x^{l+1} & & & \end{pmatrix} =$$

$$f\begin{pmatrix} 0 & 0 & 0 & K & \cdots & K & 0 & \cdots & 0 \\ & & & x^4 & \cdots & x^{l+1} & & & \end{pmatrix} = (0,0,0,x^4,\ldots,x^{l+1},0,\ldots,0).$$

Continuing iteratively until we get to L^{l-1}, we find that

$$f^{l+1}\begin{pmatrix} 0 & \cdots & 0 & K & K & 0 & \cdots & 0 \\ & & & 0 & x^{l+1} & & & \end{pmatrix} = x^{l+1},$$

which violates Pareto-optimality. Note that we can get this contradiction for any number of agent in L, as long as $\mid L \mid \geq 2$, and regardless of the size of $\bar{L}$, which might be the empty set. Furthermore, since $2 \leq\mid L \mid\leq k$, we need at least two objects. Therefore, this proof applies to any number of agents such that $n \geq 3$ and any number of objects such that $k \geq 2$.

Therefore, $\forall\tilde{\theta}^{-1} \in \Theta^{-1}, f^1(\theta^1,\tilde{\theta}^{-1}) = K$, where $\theta^1 \in (K)$. But then $\forall\tilde{\theta}^1 \in \Theta^1$ such that $\tilde{\theta}^1 \in \begin{pmatrix} K \\ 0 \end{pmatrix}, \forall\tilde{\theta}^{-1} \in \Theta^{-1}, f^1(\tilde{\theta}) = K$, by monotonicity, which is what we wanted to show.□

Step 4: No coalition of the $n-1$ non-dictators can change the outcome for the dictator at any profile.

Let $\tilde{\theta}^1 \in \begin{pmatrix} p \\ K \\ 0 \end{pmatrix}$, where $p \in \mathcal{K}, p \neq K, p \neq 0$. Suppose $f^1(\tilde{\theta}) \neq p$ for some $\tilde{\theta}^{-1} \in \Theta^{-1}$. Then $f^1(\tilde{\theta}) = K$, otherwise Step 3 implies that agent 1 can manipulate at $\tilde{\theta}$ via $\theta^1 \in (K)$. However, in this case $f(\tilde{\theta}) = (K, 0, \ldots, 0)$, given the feasibility constraints, and thus the outcome $(p, 0, \ldots, 0)$ Pareto-dominates $(k, \ldots, 0)$ at $\tilde{\theta}$. Therefore, Pareto-optimality implies that $f^1(\tilde{\theta}) = p$. Then, by monotonicity, $\forall\tilde{\theta}^1 \in (p), \forall\tilde{\theta}^{-1} \in \Theta^{-1}, f^1(\tilde{\theta}) = p$. Finally, if $\tilde{\theta}^1 \in (0)$ then $f^1(\tilde{\theta}) = 0, \forall\tilde{\theta}^{-1} \in \Theta^{-1}$, by Pareto-optimality. Thus, together with Step 3, we have $\forall\tilde{\theta} \in \Theta, f^1(\tilde{\theta}) = \text{top}(\tilde{\theta}^1)$. Therefore, agent 1 is a dictator, and f is dictatorial.□

In order to get an analog of the Gibbard-Satterthwaite theorem for nonbossy mechanisms on the private goods domain, we show that a strategyproof and nonbossy SCF that satisfies *citizen sovereignty* is Pareto-optimal.

Definition 25 An SCF f satisfies *citizen sovereignty (CS)* if $\forall x \in \mathcal{X}, \exists\theta \in \Theta$ such that *f*$(\theta) = x$.

Proposition 9 *A strategyproof, nonbossy, and CS SCF is Pareto-optimal.*

Proof: Let f be strategyproof, nonbossy, CS, and not Pareto-optimal. Then $\exists x, y \in \mathcal{X}$ with $f(\theta) = x$ for some $\theta \in \Theta$, such that $\forall i \in N, U(y^i, \theta^i) \geq U(x^i, \theta^i)$, and for some

$j \in N, U(y^j, \theta^j) > U(x^j, \theta^j)$. Define $\tilde{\theta} \in \Theta$ as follows. For each $i \in N$ such that $x^i \neq y^i$, let $\tilde{\theta}^i \in \begin{pmatrix} y^i \\ \\ x^i \end{pmatrix}$, and for each $i \in N$ such that $x^i = y^i$, let $\tilde{\theta}^i \in (y^i)$. Then $f(\tilde{\theta}) = x$, by monotonicity. Since f is CS, $\exists \bar{\theta} \in \Theta$ such that $f(\bar{\theta}) = y$. Now let $\hat{\theta}^i \in (y^i)$. Then $f(\hat{\theta}) = y$, by monotonicity. However, $\tilde{\theta}^i \in r(y, \hat{\theta}^i), \forall i \in N$ so that $x = y$. This is a contradiction, since $U(y^j, \theta^j) > U(x^j, \theta^j)$, for some $j \in N$.□

Corollary 7 *A strategyproof, nonbossy, and CS SCF is dictatorial.*

The corollary follows directly from Propositions 8 and 9.

Notice that not all dictatorial mechanisms are strategyproof, nonbossy, and Pareto-optimal, unlike on the strict domain. In our context indifferences cannot be ruled out entirely, and we defined dictatorship accordingly. Therefore, if the dictator is indifferent among outcomes that give her top allocation to her, which implies that some objects are available for allocation among the rest of the agents (at least one), then there is still room for manipulation and bossiness, and it is possible to get a Pareto-dominated outcome. In the next proposition, we characterize the set of strategyproof, nonbossy, and Pareto-optimal SCF's.

For the following definition, let the null package be defined as the empty set, i.e., let $0 = \emptyset$.

Definition 26 An SCF f is a *sequential choice function* if $\exists \sigma : \Theta \mapsto \Sigma(N)$ such that $\forall \theta \in \Theta, f^{\sigma_\theta^1}(\theta) = c^{\sigma_\theta^1}(\mathcal{K}, \theta^{\sigma_\theta^1}) =$ top $(\theta^{\sigma_\theta^1})$, and, for $j \in N \setminus \{1\}, f^{\sigma_\theta^j}(\theta)$ are defined recursively by $f^{\sigma_\theta^j}(\theta) = c^{\sigma_\theta^j}(\mathcal{K} \setminus \bigcup_{i=1}^{j-1}\{f^{\sigma_\theta^i}(\theta)\}, \theta^{\sigma_\theta^j})$. We then call σ_θ an *s-hierarchy* associated with f at θ.

Definition 27 An SCF f is a *dictatorial sequential choice function* if it is a sequential choice function such that $\forall \theta, \tilde{\theta} \in \Theta, \sigma^1_\theta = \sigma^1_{\tilde{\theta}}$, and, $\forall j \in N \setminus \{1\}$, if $f^{\sigma^i_\theta}(\theta) = f^{\sigma^i_{\tilde{\theta}}}(\tilde{\theta})$ for $i = 1, \ldots, j-1$, then $\sigma^j_\theta = \sigma^j_{\tilde{\theta}}$.

A dictatorial sequential choice mechanism is a mechanism in which for each profile there exists an ordering of the agents such that the first agent in the ordering gets her favorite allocation, then, from the remaining objects, the second agent in the ordering gets her favorite allocation, etc., until we run out of either the objects or the agents. However, the ordering of the agents at the different profiles is not arbitrary. For each profile, the first agent in the ordering must be the same, hence the name *dictatorial.* Moreover, the ordering of the rest of the agents may only vary at the different profiles as a function of the allocations of the preceeding agents.

Proposition 10 *An SCF is strategyproof, nonbossy, and Pareto-optimal if, and only if, it is a dictatorial sequential choice function.*

Proof:

(a) First we prove that a dictatorial sequential choice function is strategyproof, nonbossy, and Pareto-optimal. It is easy to verify that a sequential choice function is Pareto-optimal, hence we will only show i) strategyproofness and ii) nonbossiness.

i) Let f be a dictatorial sequential choice function. First we show that an agent cannot change her rank in the appropriate orderings by deviating alone. Fix $\theta \in \Theta, j \in N$, and $\tilde{\theta}^j \in \Theta^j$. Let $\sigma_\theta(j) = t$ and $\sigma_{(\tilde{\theta}^j, \theta^{-j})}(j) = l$, where $t, l \in N$. Suppose $t \neq l$. If $t = 1$ then $l = 1$, so $t \neq l$ implies that $t \neq 1$. By symmetry, $l \neq 1$. Suppose $t = 2$. Then, since $t \neq 1$, and $l \neq 1$, $f^{\sigma^1_\theta}(\theta) = f^{\sigma^1_{(\tilde{\theta}^j, \theta^{-j})}}(\tilde{\theta}^j, \theta^{-j}) = \text{top}(\theta^{\sigma^1_\theta})$, which implies that $t \neq 2$, and by symmetry, $l \neq 2$. Continuing iteratively, we get that $t \notin N$, which is a contradiction. Therefore, $\forall \theta \in \Theta, \forall j \in N, \forall \tilde{\theta}^j \in \Theta^j, \sigma_\theta(j) = \sigma_{(\tilde{\theta}^j,\theta^{-j})}(j)$.

Now keep $\theta \in \Theta, j \in N$, and $\tilde{\theta}^j \in \Theta^j$ fixed and let $\sigma_\theta(j) = \sigma_{(\tilde{\theta}^j,\theta^{-j})}(j) = t$, where $t \in N$. Clearly, j cannot manipulate if $t = 1$. If $t = 2$ then $\theta^{\sigma^1_\theta} = \theta^{\sigma^1_{(\tilde{\theta}^j,\theta^{-j})}}$ implies that $f^{\sigma^1_\theta}(\theta) = f^{\sigma^1_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j,\theta^{-j}) = \text{top}(\theta^{\sigma^1_\theta})$. Then $f^j(\theta) = c^j(\mathcal{K} \setminus \{\text{top}(\theta^{\sigma^1_\theta}\},\theta^j)$ and $f^j(\tilde{\theta}^j,\theta^{-j}) = c^j(\mathcal{K} \setminus \{\text{top}(\theta^{\sigma^1_\theta}\},\tilde{\theta}^j)$, so that j cannot manipulate. Similarly, if $t > 2$, then $\theta^{\sigma^1_\theta} = \theta^{\sigma^1_{(\tilde{\theta}^j,\theta^{-j})}}$ implies that $f^{\sigma^1_\theta}(\theta) = f^{\sigma^1_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j,\theta^{-j}) = \text{top}(\theta^{\sigma^1_\theta})$, which in turn implies that $\sigma^2_\theta = \sigma^2_{(\tilde{\theta}^j,\theta^{-j})}$. Then $\theta^{\sigma^2_\theta} = \theta^{\sigma^2_{(\tilde{\theta}^j,\theta^{-j})}}$, which implies that $f^{\sigma^2_\theta}(\theta) = f^{\sigma^2_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j,\theta^{-j})$, etc, till we get to $t-1$. In sum, $f^{\sigma^i_\theta}(\theta) = f^{\sigma^i_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j,\theta^{-j})$, for $i = 1,\ldots,t-1$, and so $\mathcal{K} \setminus \bigcup_{i=1}^{t-1}\{f^{\sigma^i_\theta}(\theta)\} = \mathcal{K} \setminus \bigcup_{i=1}^{t-1}\{f^{\sigma^i_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j,\theta^{-j})\}$. Therefore, agent j cannot manipulate for any $t \in N$. Since θ, j, and $\tilde{\theta}^j$ were chosen arbitrarily, this proves that f is strategyproof.

ii) Fix $\theta \in \Theta, j \in N$, and $\tilde{\theta}^j \in \Theta^j$. Then $\sigma_\theta(j) = \sigma_{(\tilde{\theta}^j,\theta^{-j})}(j)$ and $f^{\sigma^i_\theta}(\theta) = f^{\sigma^i_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j,\theta^{-j})$ for $i = 1,\ldots,t-1$, where $\sigma_\theta(j) = t$, by i). Suppose $f^j(\theta) = f^j(\tilde{\theta}^j,\theta^{-j})$. Then $\sigma^{t+1}_\theta = \sigma^{t+1}_{(\tilde{\theta}^j,\theta^{-j})}$, and $\mathcal{K} \setminus \bigcup_{i=1}^{t}\{f^{\sigma^i_\theta}(\theta)\} = \mathcal{K} \setminus \bigcup_{i=1}^{t}\{f^{\sigma^i_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j,\theta^{-j})\}$, which implies that $f^{\sigma^{t+1}_\theta}(\theta) = f^{\sigma^{t+1}_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j,\theta^{-j})$. This, in turn, implies that $\sigma^{t+2}_\theta = \sigma^{t+2}_{(\tilde{\theta}^j,\theta^{-j})}$. Continuing iteratively, we get that $f^{\sigma^l_\theta}(\theta) = f^{\sigma^l_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j,\theta^{-j})$ for $l = t+1,\ldots,n$, which proves that f is nonbossy.

(b) Conversely, we prove that a strategyproof, nonbossy, and Pareto-optimal SCF is a dictatorial sequential choice function. Suppose f is strategyproof, nonbossy, and Pareto-optimal. By Proposition 8, f is dictatorial. Let agent 1 be the dictator. Fix $\theta^1 \in \Theta^1$, and let $\mathcal{K}_2 = \mathcal{K}\setminus\{\text{top}(\theta^1)\}$. Now let f_2 be an SCF which is defined for the set of agents $N_2 = N\setminus\{1\}$ and the set of packages $\mathcal{K}_2$ such that $\forall \tilde{\theta}^{-1} \in \Theta^{-1}, \forall i \in N_2, f^i_2(\tilde{\theta}^{-1}) = f^i(\theta^1,\tilde{\theta}^{-1})$. Since f is Pareto-optimal, and $f^1(\theta^1,\tilde{\theta}^{-1}) = \text{top}(\theta^1), \forall \tilde{\theta}^{-1} \in \Theta^{-1}, f_2$ is also Pareto-optimal. Since f

is strategyproof and nonbossy, no agent $i \in N_2$ can manipulate or be bossy at $(\theta^1, \tilde{\theta}^{-1})$ for any $\tilde{\theta}^{-1} \in \Theta^{-1}$. Therefore, f_2 is strategyproof and nonbossy. Thus, by Proposition 8, f_2 is dictatorial. (If $\mathcal{K}_2$ is a singleton, use Proposition 4, instead of Proposition 8.) Let agent 2 be the dictator for f_2. Note that the identity of the dictator for f_2 may only depend on θ^1. Now fix $\theta^2 \in \Theta^2$, etc. Repeating the same argument for $n = 2, \ldots, n-1$, this proves that f is a sequential choice function such that $\forall \theta, \tilde{\theta} \in \Theta, \sigma^1_\theta = \sigma^1_{\tilde{\theta}}$, and, for $j = 2, \ldots, n-1, \sigma^j(\theta)$ depends only on $\theta^{\sigma^i_\theta}, i = 1, \ldots, j-1, \forall \theta \in \Theta$, where σ_θ is an s-hierarchy associated with f at θ.

Now fix $\sigma_\theta \in \Sigma(N), \forall \theta \in \Theta$, such that σ_θ is an s-hierarchy associated with f at θ. Let $\sigma^1_\theta = 1, \forall \theta \in \Theta$, and fix $\theta^1 \in \Theta^1$. Let $\tilde{\theta}^1 \in \Theta^1$ be such that, $f^1(\theta) = f^1(\tilde{\theta}^1, \theta^{-1})$, where $\theta^1 \neq \tilde{\theta}^1$. (For example, let $\tilde{\theta}^1 \in r(f(\theta), \theta^1)$.) Note that $\theta^{-1} \in \Theta^{-1}$ is arbitrary. Now let $\sigma^2_\theta = i$ and $\sigma^2_{(\tilde{\theta}^1, \theta^{-1})} = j$. Suppose $i \neq j$. Since f is a sequential choice function, $f^i(\theta) = c^i(\mathcal{K} \setminus \{\text{top}(\theta^1)\}, \theta^i)$ and $f^j(\tilde{\theta}^1, \theta^{-1}) = c^j(\mathcal{K} \setminus \{\text{top}(\tilde{\theta}^1)\}, \theta^j)$. However, $\text{top}(\theta^1) = \text{top}(\tilde{\theta}^1)$. Therefore, $f^j(\tilde{\theta}^1, \theta^{-1}) = c^j(\mathcal{K} \setminus \{\text{top}(\theta^1)\}, \theta^j)$. Now suppose, without loss of generality, that θ^i and θ^j satisfy $c^i(\mathcal{K} \setminus \{\text{top}(\theta^1)\}, \theta^i) = c^j(\mathcal{K} \setminus \{\text{top}(\theta^1\}), \theta^j)$. Then $f^i(\theta) = f^i(\tilde{\theta}^1, \theta^{-1})$ is not feasible, which violates f's nonbossiness. Therefore, $i = j$, and thus $\forall \theta^1, \tilde{\theta}^1 \in \Theta^1$ such that $f^1(\theta) = f^1(\tilde{\theta}^1, \theta^{-1})$ implies that $\sigma^2_\theta = \sigma^2_{\tilde{\theta}}$, if $f^1(\theta) = f^1(\tilde{\theta})$, since σ^2_θ depends only on θ^1 and $\sigma^2_{\tilde{\theta}}$ depends only on $\tilde{\theta}^1$. Repeating the same argument for $j = 3, \ldots, n$, we get that f is a dictatorial sequential choice function.□

Finally, we would like to remark that if bossiness is allowed, then a strategyproof and Pareto-optimal SCF need not be dictatorial, as long as there are at least three agents. (If there are only two agents then Lemma 2 implies, together with Proposition 8, that any strategyproof and Pareto-optimal SCF is dictatorial.) We give an example of a nondictatorial, strategyproof, and Pareto-optimal SCF below.

Example 3 [12] *A nondictatorial, strategyproof, and Pareto-optimal SCF where $n = 3$.*

Let f be a sequential choice function. Define $\tilde{\Theta} = \{\theta \in \Theta \mid \text{ if } \sigma_\theta = (1,2,3), f^3(\theta) = 0 \text{ and if } \sigma_\theta = (2,1,3), f^3(\theta) = 0\}$, where σ_θ is an s-hierarchy associated with f at θ. Now fix $p \in \mathcal{K}, p \neq K, p \neq 0$. Let $\sigma_\theta = (1,2,3), \forall\theta \notin \tilde{\Theta}$ and $\forall\theta \in \tilde{\Theta}$ if $\theta^3 \in (p)$. Otherwise, let $\sigma_\theta = (2,1,3)$. Clearly, f is Pareto-optimal, since it is a sequential choice function. It is nondictatorial, since, for example,

$$f\begin{pmatrix} p & p & p \\ \mathcal{K}\setminus\{p\} & \mathcal{K}\setminus\{p\} & \end{pmatrix} = (p, \mathcal{K}\setminus\{p\}, 0),$$

and

$$f\begin{pmatrix} p & p & \mathcal{K}\setminus\{p\} \\ \mathcal{K}\setminus\{p\} & \mathcal{K}\setminus\{p\} & \end{pmatrix} = (\mathcal{K}\setminus\{p\}, p, 0).$$

To see that f is strategyproof, note that agents 1 and 2 cannot affect the ordering at any profile, and that agent 3 can only affect the ordering when she is indifferent. This example works with any number of objects such that $k \geq 2$, and can easily be generalized to more than three agents.□

Of course, the above defined SCF is bossy. In particular, agent 3 is bossy at some profiles where she does not get any object, for example, at the above displayed two profiles.

In sum, when designing Pareto-optimal and strategyproof mechanisms for allocating heterogeneous objects, one may chose between dictatorial and bossy mechanisms.

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[12] A similar example is provided in Satterthwaite and Sonnenschein (1981, Endnote 2), in the context of divisible goods. This is a very natural example of a nondictatorial and bossy mechanism, where an agent, who is a "loser" at certain profiles, gets to alternate the dictators at those profiles.

2.5 Restricted Domains

In this section we examine two subdomains of the strict private goods domain, the *strict superadditive* and the *strict substitute* domains. Although we are unable to characterize the set of strategyproof, nonbossy, and Pareto-optimal social choice functions on these domains, we explore some of the possibilities and give illustrative examples. The most important finding is that both of these domain restrictions are sufficient to escape the consequences of the Gibbard-Satterthwaite theorem, even if nonbossiness is required. That is, the planner can design nondictatorial, strategyproof, nonbossy, and Pareto-optimal mechanisms if the agents are known to have only strict superadditive or strict substitute preferences. The next step towards fairness, *anonymity*, however, cannot be achieved. As usual, we call an SCF anonymous if a permutation of the agents does not change the outcome. This contrasts with the results of Barbera and Jackson (1995) who characterized the set of strategyproof, nonbossy, anonymous, and tie-free SCF's in the context of a pure exchange economy with divisible goods. The impossibility of finding anonymous SCF's for our model is due to the fact that the goods are indivisible and heterogeneous. To see this, consider, for example, a profile where each agent's preferences are identical. Clearly, ties have to be broken in a non-anonymous way.[13]

First we examine the strict superadditive domain, which expresses complementary effects among the objects. An agent's preferences are superadditive if each object has a nonnegative value to the agent and the packages do not reduce each other's value when obtained together.

[13]Note that in Chapter 3, where transfers are allowed, there are no anonymous mechanisms in this sense either. However, a transfer mechanism can be anonymous for our model with regard to the agents' utilities. In Chapter 3, we use anonymity in that sense.

In other words, each package is worth to the agent at least as much as the sum of the packages contained in it. Formally, agent i's preferences, θ^i, are superadditive if $\forall p \in \mathcal{K}, U(p, \theta^i) \geq 0$, and $\forall p, p' \in \mathcal{K}$ such that $p \bigcap p' = \emptyset, U(p \bigcup p', \theta^i) \geq U(p, \theta^i) + U(p', \theta^i)$. If we rule out indifference over packages, just as for the strict private goods domain, we get the strict superadditive domain.

Definition 28 Agent i has *strict superadditive preferences* if $\forall p \in \mathcal{K} \setminus \{0\}, U(p, \theta^i) > 0$, *and* $\forall p, p' \in \mathcal{K} \setminus \{0\}$ such that $p \bigcap p' = \emptyset, U(p \bigcup p', \theta^i) > U(p, \theta^i) + U(p', \theta^i)$. A *strict superadditive profile* is a profile in which each agent's preferences are strict superadditive preferences. The *strict superadditive domain,* denoted by Θ^+ consists of the set of strict superadditive profiles.

Note that Lemma 1 holds for the strict superadditive domain as well, so that only preference orderings can be used when an SCF is strategyproof and nonbossy. Given a strict superadditive preference profile, each agent's first choice is K and last choice is 0. Moreover, $p \subset p'$ indicates that $U(p', \theta^i) > U(p, \theta^i)$, and if $U(p', \theta^i) > U(p, \theta^i)$ then $p' \not\subseteq p, \forall i \in N, \forall \theta^i \in (\Theta^i)^+$.[14] Let f^+ denote an SCF which is defined for the strict superadditive domain.

First we redefine citizen sovereignty (CS), since for superadditive preferences some outcomes will never be Pareto-optimal, for example, in the extreme case when each agent gets the null package.

Definition 29 An SCF f *satisfies* CS* if $\forall x \in X$ such that x is Pareto-optimal for some

[14]The superadditive preferences for packages are similar to the seperable preferences for outcomes in Barbera et al. (1991). The only difference is that we require each package (or object) to be preferred to the null package.

$\theta \in \Theta, \exists \tilde{\theta} \in \Theta$ such that $f(\tilde{\theta}) = x$.

Note that CS* is consistent with CS for the strict private goods domain, so that the earlier propositions hold when CS* is substituted for CS.

First we show that if f^+ is strategyproof and nonbossy then it does not satisfy CS*. Since Lemma 2 holds on the strict superadditive domain as well, this also proves that if there are only two agents, then strategyproofness alone violates CS*.

Proposition 11 *A strategyproof and nonbossy SCF on the strict superadditive domain is not CS*.*

Proof: Let f^+ be strategyproof, nonbossy, and CS*. Then $\exists \theta \in \Theta^+$ such that $f(\theta) = (K, 0, \ldots, 0)$. Keeping agent 1's strategy, θ^1, fixed, and replacing the other agents' strategies, one at a time, with new strategies, we either find that $f(\theta^1, \tilde{\theta}^{-1}) = (K, 0, \ldots, 0), \forall \tilde{\theta}^{-1} \in (\Theta^{-1})^+$, or we find two strategy profiles, $(\theta^1, \tilde{\theta}^{-1})$ and $(\theta^1, \bar{\theta}^i, \tilde{\theta}^{-1,i})$ such that $f(\theta^1, \tilde{\theta}^{-1}) = (K, 0, \ldots, 0)$ and $(\theta^1, \bar{\theta}^i, \tilde{\theta}^{-1,i}) \neq (K, 0, \ldots, 0)$, for some $i \in N \setminus \{1\}$, where $\tilde{\theta}^j$ may or may not be the same strategy as θ^j, for $j = 2, \ldots, n$. In the latter case, if $f^i(\theta^1, \bar{\theta}^i, \tilde{\theta}^{-i,1}) \neq 0$ then i can manipulate at $(\theta^1, \tilde{\theta}^{-1})$ via $\bar{\theta}^i$, and if $f^i(\theta^1, \bar{\theta}^i, \tilde{\theta}^{-i,1}) = 0$ then i is bossy at $(\theta^1, \tilde{\theta}^{-1})$ versus $(\theta^1, \bar{\theta}^i, \tilde{\theta}^{-i,1})$. Therefore, strategyproofness and nonbossiness imply that $f(\theta^1, \tilde{\theta}^{-1}) = (K, 0, \ldots, 0), \forall \tilde{\theta}^{-1} \in (\Theta^{-1})^+$. But then $f(\tilde{\theta}) = (K, 0, \ldots, 0), \forall \tilde{\theta} \in \Theta^+$, otherwise agent i could manipulate via θ^1. This implies that f^+ is not CS*, which is a contradiction.□

On the strict superadditive domain, Pareto-optimality does not imply CS*, unlike on the strict private goods domain, for which any Pareto-optimal SCF satisfies CS. Thus, we can still find strategyproof, nonbossy, and Pareto-optimal SCF's on the strict superadditive domain. One such example is a dictatorial mechanism, which is a unique mechanism on this domain, once the identity of the dictator is determined. This is because the dictator gets package K for any strict superadditive profile. Therefore, the dictator completely determines the outcome for all the agents in this case, and the outcome does not vary with the reported preferences. This implies that a dictatorial sequential choice function and a serial dictatorship are identical on this domain. Not surprisingly, nonbossiness implies strong nobossiness, which is due to the fact that on this domain the contention for the objects is high at any profile, or, in other words, any Pareto-optimal outcome requires that all the objects are allocated. Moreover, any sequential choice function is dictatorial, which can be inferred from the proof of Proposition 11.

Clearly, a dictatorial mechanism is strategyproof, nonbossy, and Pareto-optimal on the strict superadditive domain. However, it is most unappealing. Thus, we need to search for nondictatorial mechanisms with more desirable features. It can be seen from the proof of Proposition 11 that if we want to avoid dictatorship, then no agent can get package K at any profile. We present next, as an example, a set of SCF's that are strategyproof, nonbossy, Pareto-optimal, and nondictatorial. We call these SCF's quasi-dictatorial quota choice functions, since one agent almost always gets more than one object according to these SCF's, while the other agents get at most one object.

Let P_m indicate the set of packages in K that contain exactly m objects, where the null object is excluded. Let $P_m : \mathcal{K}'$ indicate the set of packages in $\mathcal{K}' \subseteq \mathcal{K}$ that contain exactly m objects, excluding the null object.

Example 4 *Quasi-dictatorial quota choice functions: strategyproof, nonbossy, Pareto-optimal, and nondictatorial SCF's on the strict superadditive domain.*

An SCF f is a quasi-dictatorial quota choice function if $\exists\sigma \in \Sigma(N)$ such that $\forall\theta \in \Theta, f^{\sigma^1}(\theta) = c^{\sigma^1}(P_m, \theta^{\sigma^1})$ such that $\max\{k-n+1, 1\} \leq m \leq k-1$, for $j = 2, \ldots, k-m+1$, f^{σ^j} are defined recursively by $f^{\sigma^j}(\theta) = c^{\sigma^j}(P_1 : \mathcal{K} \setminus \bigcup_{i=1}^{j-1}\{f^i(\theta)\}, \theta^{\sigma^j})$, and $f^{\sigma^j}(\theta) = 0$ for $j = k-m+2, \ldots, n$.

Thus, the first agent in the ordering σ gets her favorite package containing m objects, where m is minimum 1, or the number of the objects left if each of the other agents obtains an object, whichever is bigger. Furthermore, m is maximum $k-1$, so as to avoid dictatorship. The rest of the agents obtain at most one object according to the ordering σ, just as in a serial dictatorship where each agent can get at most one object. The number of the agents who obtain an object depends on the number of the objects and on m, the number of the objects in agent σ^1's package. These SCF's are obviously nondictatorial. It is also easy to verify that a quasi-dictatorial quota choice function is strategyproof, nonbossy, and Pareto-optimal on the strict superadditive domain.□

Note that a quasi-dictatorial quota choice function is not Pareto-optimal on the strict private goods domain. Moreover, if any agent other than σ^1 was allowed to get more than one object in a mechanism that is otherwise similar to the above examples then the mechanism wouldn't be Pareto-optimal on the strict superadditive domain either. For example, if $\sigma^1 = 1, \sigma^2 = 2$, agent 1 gets $p_1 = \{a, b, c\}$, and agent 2 gets $p_2 = \{d, e\}$ then the outcome $f(\theta)$ at $\theta \in \Theta^+$ such that $U(\{b, c, d, e\}), \theta^1) > U(\{a, b, c\}, \theta^1)$ and $U(\{a\}, \theta^2) > U(\{d, e\}, \theta^2)$ is not Pareto-optimal. Notice also that the ordering of the agents does not have to be the same at each profile, as we defined for the quasi-dictatorial quota choice functions. The orderings may vary as a function of the allocations of the preceeding agents, just as for the dictatorial sequential choice functions. Nonetheless, we have seen that if the ordering of the agents is the same at each profile, we still don't need to have a serial dictatorship. Since Lemma 3 holds for the strict superadditive domain as well, this implies that Lemma 5 does not hold for this domain.

Next, we illustrate with an example that not all strategyproof, nonbossy, Pareto-optimal, and nondictatorial mecanisms are quasi-dictatorial quota choice mechanisms, even if the orderings of the agents vary as indicated above.

Example 5 *A strategyproof, nonbossy, Pareto-optimal, and nondictatorial SCF on the strict superadditive domain for three agents and two objects.*

Let there be three agents, 1, 2, and 3, and two objects, a and b. The SCF is given below.

3

a

2

		a	b
1	a	(a,b,0)	(a,b,0)
	b	(b,a,0)	(a,b,0)

b

3

		a	b
1	a	(a,b,0)	(a,b,0)
	b	(b,a,0)	(a,b,0)

Since there are only two objects, each agent has only two strict superadditive preferences: $(\{a,b\},\{a\},\{b\},\{0\})$ or $(\{a,b\},\{b\},\{a\},\{0\})$. The former is indicated by a, the latter is indicated by b. In the above tables, agent 1 chooses row, agent 2 chooses column, and agent 3 chooses table. Clearly, this SCF is strategyproof, nonbossy, Pareto-optimal, and nondictatorial. It is not a quasi-dictatorial quota choice function, since both agents 1 and 2 get their favorite object at only 3/4 of the profiles.□

The possibilities of designing strategyproof, nonbossy, Pareto-optimal, and nondictatorial SCF's seems to depend on the number of the agents and the objects. The more agents and objects there are, the more variation we can get regarding the fairness of the SCF's, where we mean by fairness the extent to which the agents are favored relatively equally. However, as we discussed earlier, none of the SCF's is anonymous, so fairness is a priori very limited in this context.

The restriction to substitute preferences takes us back to the assignment problem, since an agent's preferences are called substitute if any package is worth to the agent at most as much as the object that has the highest value to her among the objects that are contained in the given package. If we also require that preferences over packages are strict then we get the following definition.

Definition 30 Agent i has *strict substitute preferences* if $\forall p, p' \in \mathcal{K}\backslash\{0\}$ such that $p \bigcap p' = \emptyset, U(p \bigcup p', \theta^i) < max\{U(p, \theta^i), U(p', \theta^i)\}$. A *strict substitute profile* is a profile in which each agent's preferences are strict substitute preferences. The *strict substitute domain,* denoted by Θ^- consists of the set of strict substitute profiles.

Since the objects may remain unassigned, it is clear that on the strict substitute domain any Pareto-optimal outcome can only involve the allocation of singleton packages. Thus, we can use our a priori knowledge about the preferences to restrict the outcome space to outcomes in which no agent receives more than one object. Therefore, on the strict substitute domain, an assignment x from N to K is an $n \times k - 1$ matrix, in which each element x_a^i is defined by

$$x_a^i = \begin{cases} 1 & \text{if agent } i \text{ obtains object } a \\ 0 & \text{otherwise,} \end{cases}$$

$\forall i \in N, \forall a \in K$. Let X denote the set of feasible outcomes where each allocation is a singleton or the null package. That is, $\forall x \in X, \sum_{i \in N} x_a^i \leq 1, \forall a \in K$, and $\sum_{a \in K} x_a^i \leq 1, \forall i \in N$. Let f^- denote an SCF that is defined for the strict substitute domain, i.e., $f^- : \Theta^- \mapsto X$.

Since preferences are strict, we have $\forall a, b \in K, a \neq b$, either $U(a, \theta^i) > U(b, \theta^i)$ or $U(a, \theta^i) < U(b, \theta^i), \forall i \in N, \forall \theta^i \in \Theta^-$. However, no further restrictions apply, since now only preferences over the objects are relevant. Therefore, the admissible preferences are the same on the strict substitute and on the strict private goods domain. The only difference is in the feasibility constraints. While all the objects can be assigned simultaneously, this is not true for the packages. As we will illustrate that a strategyproof, nonbossy, and Pareto-optimal SCF need not be dictatorial on the strict substitute domain, this also underlines the importance of the feasibility constraints.

Remark that there is also less contention for the objects on the strict substitute domain than on the strict superadditive domain, given that the objects are substitutes of each other on the strict substitute domain, rather than complements. Note also that Lemmas 6 and 7 hold on the strict substitute domain, together with Proposition 9.

First we illustrate with an example that requiring nonbossiness is important on this domain as well. We give an example of a strategyproof and bossy SCF f^- where $n = k$, and show how a bossy agent can change the allocations for others at a particular profile such that a Pareto-optimal outcome is turned into a Pareto-dominated outcome.

Example 6 *A strategyproof and bossy SCF on the strict substitute domain where the number of agents and objects are equal.*

Let there be five agents and five objects, i.e., let $n = k = 5$. Call the objects a, b, c, d, and e. Define an SCF as follows. If there is no conflict between any two agents' first choices, give everyone her first choice. If there is a conflict, break the ties according to the following table.

1	2	3	4	5
a	b	c	d	e
b	c	d	e	a
c	d	e	a	b
d	e	a	b	c
e	a	b	c	d

That is, for example, if agents 1 and 2 both have a as their first choice then the tie is broken in favor of agent 1, since she has a above agent 2's in the tie-breaking table. (Note that any two agents have different ranks for any object in the table, so that ties can always be broken this way.) Then eliminate the first choices of those agents who "lost" when in conflict with others, and check whether the remaining first choices are in conflict for any two agents. Keep iterating this way until all conflicts are eliminated, and award the current "first choice" to each agent, which is now feasible. This defines a single-valued, nonempty SCF, since ties are always broken, and each agent is "unbeatable" when she holds as a current first choice her top ranked object in the tie-breaking table. The SCF is strategyproof, since no gain can be made from dishonest reporting when the other agents' reports are fixed, however, losses may occur due to dishonesty. To see this, note that if an agent beats another one when they "compete" for a particular object, this agent will beat the other one any time when the object in question comes up as a current first choice for the other agent. However, an agent may get stuck with an object less desirable to her than another one lower in the reported ordering. Now consider the following profile, where only preference orderings are indicated.

1	2	3	4	5
[c]	[a]	[d]	[e]	[b]
e	c	c	b	d
b	d	e	d	c
d	b	a	c	e
a	e	b	a	a

Since there is no conflict among the agents, everyone gets her original first choice at this profile, which is indicated by the boxes. In the next profile, only agent 2's strategy is changed, and the outcome (indicated by the boxes) is (b, a, c, d, e).

1	2	3	4	5
~~c~~	~~c~~	~~d~~	~~e~~	~~b~~
~~e~~	$\boxed{a}$	$\boxed{c}$	~~b~~	~~d~~
$\boxed{b}$	d	e	$\boxed{d}$	~~c~~
d	b	a	c	$\boxed{e}$
a	e	b	a	a

Note that agent 2's allocation is a at both profiles. Therefore, agent 2 is bossy. She alone changed the allocation for everyone else, and, without exception, from better to worse, so that at the latter profile the outcome is not Pareto-optimal.□

For the case where $n = k$ and each agent owns an object,[15] Ma (1994) shows that a strategyproof, Pareto-optimal, and individually rational mechanism is the strict core mechanism. This model is more structured than ours, so that, given that the strict core of such an economy consists of exactly one outcome,[16] the criteria imposed on a mechanism define it uniquely.

Svensson (1994) proposes serial dictatorships as weakly fair, strategyproof, and Pareto-optimal mechanisms to allocate indivisible goods to agents on a one-to-one basis, where the number of the agents and the number of the goods are equal. He calls an outcome weakly fair if $\forall i \in N, U(a, \theta^i) \geq U(b, \theta^i)$ such that agent i receives object a and there exists some agent j who receives object b, where $\sigma(i) < \sigma(j)$, according to some ordering $\sigma \in \Sigma(N)$. Admittedly, a serial dictatorship is a much more appealing mechanism on the strict substitute domain, when agents are restricted to get at most one object, than on

[15]This model is known as the Shapley-Scarf housing market. See Shapley and Scarf (1974).
[16]See Postlewaite and Roth (1977).

the strict private goods domain. However, if $n > k$ then some agents (namely, the last $n - k$ in the ordering σ) will only be able to obtain an object if sufficient number of agents rank the null object high enough in their preference orderings who are above them in the "pecking order." That is, these agents will typically not receive an object, and thus a serial dictatorship can hardly be called even weakly fair in this case. Nonetheless, it is strategyproof, nonbossy, Pareto-optimal, and also weakly fair in the sense defined above.

A strategyproof, nonbossy, and Pareto-optimal mechanism need not be dictatorial on the strict substitute domain. Given that Lemma 3 holds for the strict substitute domain, we know that any SCF satisfying the above criteria is multihierarchical. Then it is easy to verify that if each agent can get at most one object, a strategyproof, nonbossy, and Pareto-optimal SCF is a sequential choice function on this domain, given that Pareto-optimality requires that each agent prefers her allocation to each of the unassigned objects, if there are any. This finding is stated in the next proposition.

Proposition 12 *A strategyproof, nonbossy, and Pareto-optimal SCF on the strict substitute domain is a sequential choice function.*

As the proposition indicates, dictatorship can be avoided if the s-hierarchies associated with a sequential choice function vary, that is, if there is no single hierarchy. Clearly, not every system of hierarchies yields a strategyproof and nonbossy SCF. However, it is not clear which patterns of varying hierarchies are appropriate in order to make the SCF strategyproof and nonbossy. Now we provide an example of a strategyproof, nonbossy, Pareto-optimal, and nondictatorial SCF for two agents and two objects. The SCF is a sequential choice function with varying hierarchies, as implied by Proposition 12. Hence, since there are only two agents, it is nondictatorial.

Example 7 [17] *A strategyproof, nonbossy, Pareto-optimal, and nondictatorial SCF on the strict substitute domain.*

There are two agents, 1 and 2, and two objects, a and b.

1 \ 2	(a b 0)	(a 0 b)	(b a 0)	(b 0 a)	(0 a b)	(0 b a)
(a b 0)	(a,b)	(a,0)	(a,b)	(a,b)	(a,0)	(a,0)
(a 0 b)	(a,b)	(a,0)	(a,b)	(a,b)	(a,0)	(a,0)
(b a 0)	(b,a)	(b,a)	(a,b)	(a,b)	(b,0)	(b,0)
(b 0 a)	(b,a)	(b,a)	(0,b)	(0,b)	(b,0)	(b,0)
(0 a b)	(0,a)	(0,a)	(0,b)	(0,b)	(0,0)	(0,0)
(0 b a)	(0,a)	(0,a)	(0,b)	(0,b)	(0,0)	(0,0)

□

Proposition 12 also indicates that if there exists a single m-hierarchy associated with a strateyproof, nonbossy, and Pareto-optimal SCF at each profile $\theta \in \Theta^-$ then the SCF is a serial dictatorship. This is a similar finding to Lemma 5, except that strong nonbossiness

[17] This example is also a counterexample to Theorem 1 in Olson (1991), which states that for any strategyproof and nonbossy SCF defined on the substitute domain, $A(\theta)$ is acyclic for each $\theta \in \Theta^-$. Since he does not allow negative valuations for the objects, it is enough to look at the preferences $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} b \\ a \end{pmatrix}$. Thus, we have

1 \ 2	$(a\ b)$	$(b\ a)$
$(a\ b)$	(a,b)	(a,b)
$(b\ a)$	(b,a)	(a,b)

At the profile $\theta = \begin{pmatrix} b & a \\ a & b \end{pmatrix}$, we have $1A(\theta)2$ and $2A(\theta)1$, which indicates that $A(\theta)$ is not acyclic.

is not required on the strict substitute domain. It is interesting, since bossiness does not imply strong nonbossiness on this domain.

2.6 Full Implementation and Social Choice Correspondences

In this section first we examine the connection between strategyproofness (truthful implementation) and full implementation of SCF's. Interestingly, strategyproofness of a nonbossy SCF in dominant strategies implies that it is also fully implementable in dominant strategies.[18] Thus, when looking at nonbossy SCF's, we can restrict our attention to strategyproofness. This is not a surprising result, given that strategyproofness and full implementation of SCF's in dominants strategies are identical on the strict domain.[19]

Given a mechanism (g, S), a profile $s \in S$ is a *dominant strategy profile* of g at $\theta \in \Theta$ if $\forall i \in N, \forall \tilde{s}^i \in S^i, U(g^i(s), \theta^i) \geq U(g^i, (\tilde{s}^i, s^{-i}), \theta^i)$.

Definition 31 An SCF f is *fully implementable in dominant strategies* if there exists a mechanism (g, S) such that $g(s) = f(\theta), \forall s \in S$ such that s is a dominant strategy profile of f at θ. We then say that (g, S) *fully implements* f.

Proposition 13 *A strategyproof and nonbossy SCF is fully implementable in dominant strategies.*

Proof: Let f be strategyproof and nonbossy. Let $g = f$, so that $g(\theta) = f(\theta), \forall \theta \in \Theta$. Since f is strategyproof, θ is a dominant strategy profile of g at $\theta, \forall \theta \in \Theta$. Fix $\theta \in \Theta$ such that $\exists \tilde{\theta} \in \Theta, \tilde{\theta} \neq \theta$, such that $\tilde{\theta}$ is another dominant strategy profile of g at θ. (If there do

[18]This finding is stated in Olson (1991) but the proof is not correct.
[19]See Dasgupta et al. (1978, Corollary 4.1.4).

not exist such θ and $\tilde{\theta}$ then g fully implements f.) Then $g^1(\tilde{\theta}^1, \theta^{-1}) = g^1(\theta)$. Moreover, f's nonbossiness implies that $g^i(\tilde{\theta}^1, \theta^{-1}) = g^i(\theta)$, $\forall i \in N \setminus \{1\}$. Thus, $g(\tilde{\theta}^1, \theta^{-1}) = g(\theta)$. Similarly, $g^2(\tilde{\theta}^1, \tilde{\theta}^2, \theta^{-1,2}) = g^2(\tilde{\theta}^1, \theta^{-1})$, and $g^i(\tilde{\theta}^1, \tilde{\theta}^2, \theta^{-1,2}) = g^i(\tilde{\theta}^1, \theta^{-1})$, $\forall i \in N \setminus \{2\}$, by f's nonbosiness, so that $g(\tilde{\theta}^1, \tilde{\theta}^2, \theta^{-1,2}) = g(\tilde{\theta}^1, \theta^{-1})$. Repeating the same argument for agents $3, 4, \ldots, n$, we get that $g(\theta) = g(\tilde{\theta})$. Since this holds for any two dominant strategy profiles, we have $g(\tilde{\theta}) = f(\theta)$ for each dominant strategy profile $\tilde{\theta}$ at θ.□

Our next finding is that if a mechanism fully implements a *social choice correspondence* in dominant strategies then it must be bossy.

Definition 32 A *social choice correspondence (SCC)* is a correspondence $f : \Theta \mapsto P(\mathcal{X})$, where $P(\mathcal{X})$ is the power set of $\mathcal{X}$.

Let $\bar{S}_g(\theta)$ denote the set of dominant strategy profiles of g at $\theta, \forall \theta \in \Theta$, and let $g(\bar{S}_g(\theta))$ denote the set of outcomes assigned to the profiles that are dominant strategy profiles at θ.

Definition 33 An SCC f is *fully implementable in dominant strategies* if there exists a mechanism (g, S) such that $g(\bar{S}_g(\theta)) = f(\theta), \forall \theta \in \Theta$. We then say that (g, S) *fully implements* f.

Definition 34 *A mechanism (g, S) is* bossy *if $\exists s \in S$, $i, j \in N$, and $\tilde{s}^i \in S^i$ such that $g^i(s) = g^i(\tilde{s}^i, s^{-i})$ and $g^j(s) \neq g^j(\tilde{s}^i, s^{-i})$.*

Proposition 14 *If an SCC is fully implementable in dominant strategies, then the mechanism that fully implements it is bossy.*

Proof: Let an SCC f be fully implemented by (g, S). Since f is an SCC, $\exists \theta \in \Theta$ such that $|f(\theta)| > 1$. Let s be a dominant strategy profile of g at $\theta \in \Theta$, where $|f(\theta)| > 1$.

Let $g(s) = x$. Since $x \in f(\theta)$, but $x \neq f(\theta)$, we can replace the strategies in s by other dominant strategies of g at θ, for one agent at a time, until we get a profile $\tilde{s} \in S$ such that $g(\tilde{s}) = y \neq x$. Thus, $\exists j \in N$ with $s^j \in S^j$ such that $g(s^j, \tilde{s}^{-j}) = x$. Since both s^j and $\tilde{s}^j$ are dominant strategies for agent j with respect to θ^j, we must have $x^j = y^j$. Thus, (g, S) is bossy.□

Finally, we present an example of a bossy mechanism that fully implements an SCC. For simplicity, the SCC in the example does not satisfy CS.

Example 8 *A bossy mechanism that fully implements an SCC in dominant strategies.*

Let $n = 3$, $k = 4$, and call the objects a, b, c, and d. Let $\tilde{\Theta}^1$ denote the set of preferences such that $\forall \theta^1 \in \tilde{\Theta}^1$, agent 1 prefers object a to object c, and $\forall \theta^1 \notin \tilde{\Theta}^1$, agent 1 prefers object c to object a. Let $\tilde{\Theta}^2$ denote the set of preferences such that $\forall \theta^2 \in \tilde{\Theta}^2$, agent 2 prefers object a to object b, and $\forall \theta^2 \notin \tilde{\Theta}^2$, agent 2 prefers object b to object a.

Allocations:

$x = (a, b, d)$

$y = (c, a, d)$

$z = (c, b, d)$

SCC f:

$$f(\theta) = \begin{cases} \{x, y\} & \text{if } \theta^1 \in \tilde{\Theta}^1 \text{ and } \theta^2 \in \tilde{\Theta}^2 \\ \{x\} & \text{if } \theta^1 \in \tilde{\Theta}^1 \text{ and } \theta^2 \notin \tilde{\Theta}^2 \\ \{y\} & \text{if } \theta^1 \notin \tilde{\Theta}^1 \text{ and } \theta^2 \in \tilde{\Theta}^2 \\ \{z\} & \text{if } \theta^1 \notin \tilde{\Theta}^1 \text{ and } \theta^2 \notin \tilde{\Theta}^2 \end{cases}$$

Mechanism (g, S) that fully implements f : Fix $\tilde{s}^1 \in S^1$, $\tilde{s}^2 \in S^2$, and $\tilde{s}^3 \in S^3$. Below, s^i denotes any $s^i \in S^i$ such that $s^i \neq \tilde{s}^i, \forall i \in N$.

$$g(\tilde{s}^1, \tilde{s}^2, \tilde{s}^3) = x, \quad g(\tilde{s}^1, \tilde{s}^2, s^3) = y, \quad g(\tilde{s}^1, s^2, \tilde{s}^3) = x,$$
$$g(\tilde{s}^1, s^2, s^3) = x, \quad g(s^1, \tilde{s}^2, \tilde{s}^3) = y, \quad g(s^1, \tilde{s}^2, s^3) = y,$$
$$g(s^1, s^2, \tilde{s}^3) = z, \quad g(s^1, s^2, s^3) = z.$$

Dominant strategies: For agent 3, $\forall s^3 \in S^3$ is a dominant strategy, since she is indifferent among outcomes x, y, and z. Agent 1 has one dominant strategy: $\tilde{s}^1$, if $\theta^1 \in \tilde{\Theta}^1$. If $\theta^1 \notin \tilde{\Theta}^1$, each $s^1 \neq \tilde{s}^1$ is a dominant strategy. Similarly to agent 1, agent 2 has one dominant strategy: $\tilde{s}^2$, if $\theta^2 \in \tilde{\Theta}^2$. If $\theta^2 \notin \tilde{\Theta}^2$, each $s^2 \neq \tilde{s}^2$ is a dominant strategy.

The mechanism is bossy, since agent 3 can change the allocations to agents 1 and 2, without changing her own allocation.□

Note that our last result contrasts with the finding in Dasgupta et al. (1978, Corollary 4.1.3) that on the strict domain any fully implementable social choice correspondence is single-valued, i.e., a social choice function. On the strict private goods domain, this will only hold if we require the mechanism to be nonbossy.

2.7 Discussion

We presented two main results in this chapter. Firstly, we showed that all strategyproof, strongly nonbossy, and Pareto-optimal SCF's are serial dictatorships. Secondly, we proved that all strategyproof, nonbossy, and Pareto-optimal SCF's are dictatorial sequential choice functions. It is interesting to note that the concepts of strong nonbossiness and bossiness are identical on domains of high conflict (e.g., on the strict superadditive domain) if Pareto-optimality is also required, or when the contention for the object(s) is high due to the feasibility constraints (e.g., when a single object is being allocated). Thus, using a serial dictatorship may be necessary if the potential conflict of interests is severe. We remark that the two results are the same if there is only a single object, given that serial dictatorships and dictatorial social choice functions are also identical in this case.

Since a Gibbard-Satterthwaite-type impossibility result holds for nonbossy mechanisms, we can draw the usual conclusion. If nonbossiness is desired, the planner needs to have a priori information about the agent's preferences in order to avoid dictatorships. Accordingly, we demonstrated that for strict superadditive preferences, or, alternatively, for strict substitute preferences there exist strategyproof, nonbossy, Pareto-optimal, and nondictatorial SCF's.

The results in this chapter were established for strict preferences over allocations. We conjecture that they would also hold on the universal private goods domain, just as the results in Chapter 1 can be extended to this larger domain. Since an agent's choice from a given choice set may not be uniquely defined when weak preferences are allowed, this may lead to difficulties in defining Pareto-optimal SCF's.[20]

[20]For an illustration of this problem see, for example, Svensson (1994).

Chapter 3

Transfer Mechanisms for Allocating Heterogeneous Indivisible Objects

In this chapter we examine mechanisms that use monetary transfers for allocating heterogeneous, indivisible objects. As in Chapter 2, the valuations of the objects are interdependent, and each agent may obtain more than one object. The focus of this chapter is the class of Groves mechanisms, which are value maximizing and strategyproof for the multi-object allocation problem we investigate. Since the class of Groves mechanisms allows for a wide variety of transfer schemes, the planner may impose further criteria regarding the revenue distribution, in order to choose among these mechanisms. We examine the fairness of Groves mechanisms when heterogeneuos, indivisible objects are being allocated, using the well-known *envyfreeness* as a criterion of fairness. Individual rationality and other revenue related criteria are also investigated.

The results we present here are the only ones so far on the fairness of Groves mechanisms. There exists a relevant literature, however, on the existence of fair allocations for the assignment problem, (see, for example, Svensson (1983), Alkan et al. (1991), and Tadenuma and Thomson (1991)). Alkan (1991) extends the multi-item auction of Demange et al. (1986) to a more general domain of preferences where income effects are present. Bikchandani and Mamer (1994) consider prices for a multi-object allocation problem with interdependent values and provide conditions under which market clearing prices for the objects exist. The mostly negative results in their paper illustrate why we take a package assignment approach in our study, that is, why packages instead of objects are auctioned off in the Groves-type sealed bid auctions that we examine. Namely, since an agent does not attach a value to an object itself when obtained together with other objects, the efficiency of any mechanism that assigns objects independently of one another would be severely reduced. Several experimental papers explore the properties of mechanisms proposed for use in various applications (e.g., Grether et al. (1981), Rassenti et al. (1982), Ledyard et al. (1994)).[1] Our results build partly on the literature on Groves mechanisms, most importantly, on Groves (1973), Green and Laffont (1977, 1979), and Moulin (1986), and partly on the extensive literature on the assignment problem.[2]

Our model involves a fixed number of agents, a fixed number of heterogeneous and indivisible objects, and a perfectly divisible currency in which the agents can be charged for the packages. As earlier, the agents are taken to be *selfish*, that is, it is assumed

[1]There is also a substantial literature on practical issues related to the FCC auction. See, for example, Bykowsky et al. (1995).

[2]For a review of the literature on the assignment problem see, for example, Roth and Sotomayor (1990).

that they are indifferent among the assignments to other agents, as long as their own individual assignments are unchanged. Since the values of the objects are interdependent, any valuation of the packages is admissable. Moreover, in this chapter indifference between packages is admissable, that is, we examine the universal private goods domain. Since an outcome in this model consists of a particular distribution of the packages and a set of transfers, we will refer to any particular distribution of the packages as an assignment, and the allocations will also be referred to as individual assignments.

We make some assumptions that are required to study Groves mechanisms, along with other assumptions that are particular to the package allocation problem. Since the Groves mechanisms are strategyproof, the agents need not know anything about each other's valuations of the packages. An assignment is called *optimal* if the total value obtained by the agents is maximized. Thus, Groves mechanisms choose an optimal assignment for any preference profile of the agents. In this sense, if we ignore the transfers, Groves mechanisms are efficient. It is natural to assume that the objects need not be assigned, since negative valuations are allowed. Therefore, in order to determine the optimal assignment(s), the zero value (the value that an agent gets if no package is assigned to her) plays a role. This implies that the values cannot be shifted by a constant for all the agents, which contrasts with the public goods case. In fact, the values in our case are unique,[3] as we also have to be able to measure the difference between the values of any package for any two agents. This also implies that there is a unique dominant strategy for each agent, and thus the Groves mechanisms that we examine are normalized.[4] We assume that the agents' utility functions

[3]Although each value for each agent may be multiplied by the same positive constant, this transformation would only mean that we changed the unit of measurement.

[4]See Green and Laffont (1977).

are additively separable and linear in the currency, i. e., when an agent is assigned a package in a given outcome (a feasible assignment of packages and a set of charges), her utility is the difference between the value of her assignment to her and the transfer paid for it.

The chapter is organized as follows. The notation and the definitions are given in Section 3.1. In Section 3.2, we prove that the Groves mechanisms are the only strategyproof and efficient direct mechanisms on the universal private goods domain and on the superadditive and substitute domains. In Section 3.3, we present first an impossibility result, which says that all Groves mechanisms fail to be envyfree on the universal private goods domain. However, there exist envyfree Groves mechanisms on both examined restricted domains. We characterize the class of envyfree Groves mechanisms on both the superadditive and substitute domains. In Section 3.4, we impose a further restriction on the Groves mechanisms: individual rationality. We derive a condition for an envyfree Groves mechanism to be individually rational on both restricted domains, after characterizing the set of individually rational Groves mechanisms. Finally, we characterize the pivotal mechanism in Section 3.5, and illustrate that the universal private goods domain is not as restrictive for some revenue related criteria as the universal private goods domain when Groves mechanisms are used. We conclude in Section 3.6.

3.1 Notation and Definitions

There are $n \geq 2$ agents and $k-1$ objects to be allocated among the agents.[5] We require that $k \geq 3$, since this means that there are at least two objects to allocate. If $k = 2$ then the

[5]Note that our notation is inconsistent with that of Chapter 2. We use this notation in order to present the results and the proofs in this chapter in a significantly simpler form than the usual notation would allow us.

interdependency of the values does not play a role, and we have a much simpler problem. Let N denote the set of n agents. As a notational convention, it will be assumed that the set of objects, K, contains an artificial "null object". Thus, $|K| = k$. Both N and K are assumed to be finite and nonempty. We will refer to any set of objects as a package. Let $\mathcal{K}$ be the set of packages, which includes the null package, the package that consists of the null object.

An *assignment* x from N to K is an $n \times (2^{k-1} - 1)$ matrix, in which each element x_p^i is defined by

$$x_p^i = \begin{cases} 1 & \text{if package } p \text{ is assigned to agent } i \\ 0 & \text{otherwise,} \end{cases}$$

$\forall i \in N$, $\forall p \in \mathcal{K} \setminus \{0\}$. As earlier, we write that $x^i = p$ when $x_p^i = 1$ and $x_t^i = 0$, $\forall t \in \mathcal{K}$, $t \neq p$. If an agent is not assigned any other package, it will be assumed that she is assigned package 0, which may be assigned to more than one agent. If agent i is assigned package 0 as part of assignment x, then we write that $x^i = 0$. Feasibility is defined as in the previous chapter. Let $M_r = \{p \in \mathcal{K} | \ p \bigcap r \neq \emptyset\}$, $\forall r \in \mathcal{K}$. An assignment x is *feasible* if each agent gets at most one package, i. e., $\sum_{p \in \mathcal{K} \setminus \{0\}} x_p^i \leq 1$, $\forall i \in N$, and no object is assigned more than once as an element of some package, i. e., $\sum_{i \in N} \sum_{p \in M_r} x_p^i \leq 1$, $\forall r \in \mathcal{K} \setminus \{0\}$. Denote the set of feasible assignments by $\mathcal{X}$. If a package is assigned to an agent as part of an assignment, x, it will be said that the package is assigned to the agent *under* x. If a package is not assigned to any agent as part of an assignment x, it will be said that it is *unassigned* under x. Agent is a *winner* under if $x^i \neq 0$. Agent is a *loser* under if $x^i = 0$. Let $V(x) = \{i \in N \mid x^i \neq 0\}$ denote the set of winners under assignment x. An *outcome* $a = (x, t)$ consists of a feasible assignment $x \in \mathcal{X}$ and a set of transfers from the agents $t = (t_1, \ldots, t_n)$.

Let θ_p^i denote the value that agent i places on package p; that is, agent i's willingness to pay for package p, where $\theta_p^i \in \Re$, $\forall i \in N, \forall p \in \mathcal{K}$. Let $\theta_0^i = 0, \forall i \in N$. Then $\theta_K^i = (\theta_1^i, \ldots, \theta_{2^{k-1}-1}^i)$ is a set of the values placed by agent i on the set of packages. Let Θ_K^i be the set of preferences for agent i, so that $\theta_K^i \in \Theta_K^i$, $\forall i \in N$. Denote the set of admissable preferences for all agents by $\Theta_K^N = \times_{i \in N} \Theta_K^i$.[6] Θ_K^N, thus, represents the universal private goods domain. Let $\bar{\Theta}_K^N$ denote an arbitrary subset of Θ_K^N. We call $\theta_K^N \in \Theta_K^N$ a *profile* of the agents. Each *environment* is characterized by (N, K, θ_K^N). θ_K^{N-i}denotes the profile of all the agents except for i, and θ_{K-p}^N is the profile of the n agents for all the packages except for the packages in M_p. Assuming that each agent cares only about her own payoff, agent i with preferences θ_K^i has the additively separable utility function $U((x,t), \theta_K^i) = U((x^i, t_i), \theta_K^i) = \sum_{p \in \mathcal{K}} x_p^i \theta_p^i - t_i$.

An *optimal* assignment $x^* \in \mathcal{X}$ with respect to θ_K^N is such that $\sum_{i \in N} \sum_{p \in \mathcal{K}} x_p^{*i} \theta_p^i = max_{x \in \mathcal{X}} \left\{ \sum_{i \in N} \sum_{p \in \mathcal{K} x_p^i \theta_p^i} \right\}$. Thus, it is clear that an optimal assignment exists for any profile θ_K^N. However, an optimal assignment is not necessarily unique for a given profile. Therefore, we denote the set of optimal assignments with respect to θ_K^N by $X(\theta_K^N)$. The *value* of an assignment $x \in \mathcal{X}$ is given by the sum of the values that the agents place on the packages that are assigned to them. Let $W(\theta_K^N)$ denote the value of an optimal assignment in the environment (N, K, θ_K^N), i. e., let $W(\theta_K^N) = \max_{x \in \mathcal{X}} \left\{ \sum_{i \in N} \sum_{p \in \mathcal{K}} x_p^i \theta_p^i \right\}$. Similarly, let $W(\theta_K^{N-l}) = \max_{x \in \mathcal{X}} \left\{ \sum_{i \in N, i \neq l} \sum_{p \in \mathcal{K}} x_p^i \theta_p^i \right\}$, and $W(\theta_{K-r}^N) = \max_{x \in \mathcal{X}} \left\{ \sum_{i \in N} \sum_{p \in \mathcal{K}, p \notin M_r} x_p^i \theta_p^i \right\}$, etc.

Definition 35 A *mechanism* (g, S) is a set of strategy spaces S_i, $\forall i \in N$, where $S =$

[6]This notation is also different from that of Chapter 2. It is necessary, since, for example, we will use the notation θ_K^{N+i}, which refers to a profile of $n+1$ agents, which could be easily confused with θ^i if it was denoted by θ^{+i}.

$\times_{i \in N} S_i$, and a function $g : S \mapsto \mathcal{X} \times \Re^{\,n}$. Thus, for a strategy profile $s \in S$, the outcome is $g(s) = (x(s), (t_1(s), \ldots, t_n(s))$, where $x(s) \in \mathcal{X}$, and $t_i(s) \in \Re$, $\forall i \in N$.

Definition 36 A *direct mechanism* $(g, \bar{\Theta}_K^N)$is a mechanism for which agent i's strategy space is $S_i = \bar{\Theta}_K^i$, $\forall i \in N$, so that $S = \bar{\Theta}_K^N$.

Definition 37 A direct mechanism $(g, \bar{\Theta}_K^N)$is *strategyproof* if truthful revelation is a dominant strategy for each agent and profile. That is, if $U(g(\theta_K^N), \theta_K^i) \geq U(g(\tilde{\theta}_K^i, \theta_K^{N-i}), \theta_K^i)$, $\forall \theta_K^N \in \bar{\Theta}_K^N$, $\forall \tilde{\theta}_K^i \in \bar{\Theta}_K^i$, $\forall i \in N$.

Definition 38 A direct mechanism $(g, \bar{\Theta}_K^N)$is *efficient* if it provides an optimal assignment for any profile $\theta_K^N \in \bar{\Theta}_K^N$.

We now define the class of Groves mechanisms in our framework.

Definition 39 A direct mechanism $(g, \bar{\Theta}_K^N)$ is a *Groves mechanism* if $g = (x, t)$ is such that $\forall \theta_K^N \in \bar{\Theta}_K^N$, $x \in X(\theta_K^N)$, and $\forall \theta_K^N \in \bar{\Theta}_K^N$, $\forall i \in N$, $t_i(\theta_K^N) = f_i(\theta_K^{N-i}) - W(\theta_{K-(i)}^{N-i})$, where $x^i = (i)$ and f_i is an arbitrary deterministic function of θ_K^{N-i}, $\forall i \in N$.

If a Groves mechanism $(g, \bar{\Theta}_K^N)$ uses $\mathbf{f} = (f_1, \ldots, f_n)$ in its transfer rule t, it will be denoted by $(G(\mathbf{f}), \bar{\Theta}_K^N)$. It is straightforward to verify that the Groves mechanisms are indeed strategyproof for the package assignment problem. Given that $X(\theta_K^N)$is not necessarily a singleton for each profile θ_K^N, we need to assume that the Groves mechanisms involve a tie-breaking method. Note, that there exist some tie breaking methods that select an optimal assignment arbitrarily within $X(\theta_K^N)$for θ_K^N; for example, the ones that depend only on the labeling of the agents and the packages. Thus, for profiles that yield multiple optimal assignments, a Groves mechanism may result in any of the optimal assignments, depending on the labeling.[7] This feature of the Groves mechanisms will be used throughout, rather than dealing with *extended* Groves mechanisms (Green and Laffont (1977)) that are not single-valued.

Recall that an agent's preferences are superadditive if each object has a nonnegative value to the agent and the packages do not reduce each other's values when obtained together.

Definition 40 Agent i has *superadditive preferences* if $\forall t \in \mathcal{K}, \theta_t^i \geq 0$, and $\forall t, t' \in \mathcal{K}$ such that $t \bigcap t' = \emptyset$, $\theta_{t \bigcup t'}^i \geq \theta_t^i + \theta_{t'}^i$. A *superadditive profile* is a profile in which each agent's preferences are superadditive. The *superadditive domain,* denoted by $(\Theta_K^N)^+$, consist of the set of superadditive profiles.

Now recall that an agent's preferences are called substitute if any package is worth to the agent at most as much as the object that has the highest value to her among the objects that are contained in the given package.

Definition 41 Agent i has *substitute preferences* if $\forall t, t' \in \mathcal{K}$ such that $t \bigcap t' = \emptyset$, $\theta_{t \bigcup t'}^i \leq max \left\{\theta_t^i, \theta_{t'}^i\right\}$. A *substitute profile* is a profile in which each agent's preferences are substitute. The *substitute domain,* denoted by $(\Theta_K^N)^-$, consists of the set of substitute profiles.

In any substitute environment $\left(N, K, \theta_K^N \in (\Theta_K^N)^-\right)$, there exists an optimal assignment that only involves singleton packages. Then, examining any substitute environment, we can restrict our attention to assignments in which each agent may obtain at most one object,

[7] It may seem that a Groves mechanism using such a tie-breaking rule cannot be anonymous or neutral, since ties are broken in a non-anonymous and non-neutral way. However, these Groves mechanisms are neutral in the sense that any agent gets the same utility under any optimal assignment. Furthermore, if the f_i's are chosen to be identical for each agent i, these Groves mechanisms will be anonymous in the sense that the labeling of the agents will not affect their final utility levels.

just as in the previous chapter. Accordingly, an assignment is an $n \times (k-1)$ matrix in an environment (N, K, θ_K^N) for $\theta_K^N \in (\Theta_K^N)^-$. Note that this modification only helps to simplify the proofs but does not restrict the generality of the results.

3.2 Uniqueness of Groves Mechanisms

First we would like to verify that the Groves mechanisms are the only strategyproof and effecient mechanisms on the universal private goods domain and on the two restricted domains, the superadditive and substitute domains. This can be verified using Holmstrom's result (Holmstrom (1975)) that on a convex domain of preferences any efficient and strategyproof mechanism is a Groves mechanism. The universal private goods domain and the superadditive domain are convex, which is easily checked. The substitute domain is not convex. However, it contains a convex subset, namely, when each profile consists only of preferences according to $\theta^i_{t \bigcup t'} \leq \min\{\theta^i_t, \theta^i_{t'}\}$, $\forall t, t' \in \mathcal{K}$ such that $t \bigcap t' = \emptyset$. Since this domain is convex, Holmstrom's theorem applies to it. This means that for any domain containing this domain, in particular, for the substitute domain, the uniqueness of Groves mechanisms still holds.

However, it is interesting to prove this directly, by using a method similar to that of Theorem 1 in Green and Laffont (1977), which shows that any strategyproof and efficient mechanism is a Groves mechanism on the universal domain. To do this, we use an auxiliary description of Groves mechanisms, just as in Green and Laffont (1977).

Lemma 8 *An efficient direct mechanism is a Groves mechanism if, and only if, it satisfies the following two properties.*

i) $\forall\theta^N_K \in \Theta^N_K, \forall i \in N, \forall\tilde{\theta}^i_K \in \Theta^i_K$, *if* $x \in X(^N_K)$ *and* $x \in X(\tilde{\theta}^i_K, \theta^{N-i}_K)$ *then* $t_i(\theta^N_K) = t_i(^{\theta^i}_K, \tilde{\theta}^{N-i}_K)$.

ii) $\forall\theta^N_K \in \Theta^N_K, \forall i \in N, \forall\tilde{\theta}^i_K \in \Theta^i_K$, *if* $x \in X(^N_K)$ *and* $x \in X(\tilde{\theta}^i_K, \theta^{N-i}_K)$ *then* $t_i(\theta^N_K) - t_i(^{\theta^i}_K, \theta^{N-i}_K) = W(\theta^{\tilde{N}-i}_{K-\tilde{x}^i}) - W(\theta^{N-i}_{K-x^i})$.

Proof: Obvious.□

Proposition 15 *A strategyproof and efficient direct mechanism on the universal private goods domain is a Groves mechanism.*

Proof:

i) If a strategyproof and efficient direct mechanism (g, Θ^N_K) violates i) in Lemma 8 then $\exists\theta^N_K \in \Theta^N_K, i \in N$, and $\tilde{\theta}^i_K \in \Theta^i_K$ such that $x \in X(^N_K), x \in X(\tilde{\theta}^i_K, \theta^{N-i}_K)$ and $t_i(\theta^N_K) > t_i(^{\theta^i}_K, \tilde{\theta}^{N-i}_K)$. Then $\theta^i_{x^i} - t_i(\tilde{\theta}^i_K, \theta^{N-i}_K) > \theta^i_{x^i} - t_i(\theta^N_K)$, so g is not strategyproof.

ii) (a) If a direct mechanism (g, θ^N_K) violates ii) in Lemma 8 then $\exists\theta^N_K \in \Theta^N_K, i \in N$, and $\tilde{\theta}^i_K \in \Theta^i_K$ such that $x \in X(^N_K), x \in X(\tilde{\theta}^i_K, \theta^{N-i}_K)$ and $t_i(\theta^N_K) - t_i(^{\theta^i}_K, \theta^{N-i}_K) = W(\theta^{\tilde{N}-i}_{K-\tilde{x}^i}) - W(\theta^{N-i}_{K-x^i})$ $+\epsilon$, where $\epsilon > 0$.

(b) If $x^i = \tilde{x}^i$ then $W(\theta^{N-i}_{K-\tilde{x}^i}) = W(\theta^{N-i}_{K-x^i})$ and $\theta^i_{x^i} = \theta^i_{\tilde{x}^i}$ so that $\theta^i_{\tilde{x}^i} - t_i(\tilde{\theta}^i_K, \theta^{N-i}_K) = \theta^i_{x^i} - t_i(\theta^N_K) + \epsilon$. This, however, contradicts strategyproofness, since i can manipulate at θ^{N_K} via $\tilde{\theta}^i_K$.

(c) If $x^i \neq \tilde{x}^i$ and $\tilde{x}^i \neq 0$ then let $\bar{\theta}^i_{x^i} = \theta^i_{x^i}, \bar{\theta}^i_{\tilde{x}^i} = W(\theta^N_K) - W(\theta^{N-i}_{K-\tilde{x}^i})$, and $\bar{\theta}^i_p = 0$ for $p \in \mathcal{K}, p \neq x^i, p \neq \tilde{x}^i$. (Note that if $x^i = 0$ then $\bar{\theta}^i_{x^i} = 0$.) Then $\bar{\theta}^i_{x^i} + W(\theta^{N-i}_{K-x^i}) = W(\theta^N_K) = \bar{\theta}^i_{\tilde{x}^i} + W(\theta^{N-i}_{K-\tilde{x}^i})$, and $\bar{\theta}^i_p + W(\theta^{N-i}_{K-p}) = W(\theta^{N-i}_{K-p}) \leq W(\theta^N_K), \forall p \in \mathcal{K}, p \neq x^i, p \neq \tilde{x}^i$. Thus, $x \in X(\bar{\theta}^i_K, \theta^{N-i}_K)$ and so $t_i(\theta^N_K) = t_i(\bar{\theta}^i_K, \theta^{N-i}_K)$, by i). Then we have $\bar{\theta}^i_{\tilde{x}^i} - t_i(\tilde{\theta}^i_K, \theta^{N-i}_K) = \bar{\theta}^i_{x^i} - t_i(\bar{\theta}^i_K, \theta^{N-i}_K) + W(\theta^{N-i}_{K-\tilde{x}^i}) - W(\theta^{N-i}_{K-x^i}) - \bar{\theta}^i_{x^i} + \bar{\theta}^i_{\tilde{x}^i}) + \epsilon$. Since $W(\theta^{N-i}_{K-\tilde{x}^i}) + \bar{\theta}^i_{\tilde{x}^i} =$

$W(\theta^{N-i}_{K-x^i}) + \bar{\theta}^i_{x^i}$, this implies that $\bar{\theta}^i_{\tilde{x}^i} - t_i(\tilde{\theta}^i_K, \theta^{N-i}_K) = \bar{\theta}^i_{x^i} - t_i(\bar{\theta}^i_{x^i}, \theta^{N-i}_K) + \epsilon$. This, however, contradicts strategyproofness, since agent i can manipulate at $(\bar{\theta}^i_K, \theta^{N-i}_K)$ via $\tilde{\theta}^i_K$.

(d) If $x^i \neq \tilde{x}^i$ and $\tilde{x}^i \neq 0$ then let $\bar{\theta}^i_{x^i} = W(\theta^{N-i}_K) - W(\theta^{N-i}_{K-x^i})$, and let $\bar{\theta}^i_p = 0, \forall p \in \mathcal{K}, p \neq x^i$. Then $\bar{\theta}^i_{x^i} + W(\theta^{N-i}_{K-x^i}) = W(\theta^{N-i}_K)$, and $\bar{\theta}^i_p + W(\theta^{N-i}_{K-p}) = W(\theta^{N-i}_{K-p}) \leq W(\theta^{N-i}_K), \forall p \in \mathcal{K}, p \neq x^i$. Thus, $x \in X(\bar{\theta}^i_K, \theta^{N-i}_K)$ and so $t_i(\theta^N_K) = t_i(\bar{\theta}^i_K, \theta^{N-i}_K)$, by i). Then we have $\bar{\theta}^i_{\tilde{x}^i} - t_i(\theta^i_K, \theta^{N-i}_K) = \bar{\theta}^i_{x^i} - t_i(\bar{\theta}^i_{x^i}, \theta^{N-i}_K) + W(\theta^{N-i}_{K-\tilde{x}^i}) - W(\theta^{N-i}_{K-x^i}) - \bar{\theta}^i_{x^i} + \bar{\theta}^i_{\tilde{x}^i}) + \epsilon$. Since $W(\theta^{N-i}_{K-\tilde{x}^i}) = W(\theta^{N-i}_K)$ and $\bar{\theta}^i_{\tilde{x}^i} = 0$, this implies that $\bar{\theta}^i_{\tilde{x}^i} - t_i(\tilde{\theta}^i_K, \theta^{N-i}_K) = \bar{\theta}^i_{x^i} - t_i(\bar{\theta}^i_K, \theta^{N-i}_K) + \epsilon$. This, however, contradicts strategyproofness, since agent i can manipulate at $(\bar{\theta}^i_K, \theta^{N-i}_K)$ via $\tilde{\theta}^i_K$.□

Note that Proposition 15 also proves uniqueness for the substitute domain. To prove a similar result for the superadditive domain, we need to modify the above proposition. The modifications are given in the Appendix in Proposition 15'.

3.3 Envyfreeness

In this section we examine the envyfreeness of Groves mechanisms. An outcome is envyfree if no agent prefers any other agent's outcome to her own. A direct mechanism is envyfree if it provides an envyfree outcome for each possible profile of the agents.

Definition 42 An *outcome* $a = (x, t)$ is *envyfree* for θ^N_K if $U\left((x^i, t_i), \theta^i\right) \geq U\left((x^j, t_j), \theta^i\right)$, $\forall i, j \in N$.

Definition 43 A *direct mechanism* $(g, \bar{\Theta}^N_K)$ is *envyfree* if $g(\theta^N_K)$ is envyfree for all $\theta^N_K \in \bar{\Theta}^N_K$.

One remark is in order about the interpretation of envyfree outcomes in Groves mechanisms, given that the Groves-price of a package depends on the agent who obtains the package. In a Groves mechanism, the utility of agent j's outcome to agent i is the value of j's assigned package to i, minus the Groves tax agent j is charged for her package, i.e., $U\left((x^j, t_j), \theta^i\right) = \theta^i_{(j)} - f_j(\theta_K^{N-j}) + W(\theta^{N-j}_{K-(j)})$, where j's assigned package is (j). Thus, regardless that agent i would not have been charged $f_j(\theta_K^{N-j}) - W(\theta^{N-j}_{K-(j)})$, were she assigned package (j), envyfreeness requires that we take into account the *realized* outcome, rather than a *hypothetical* one that would have occurred under a different assignment.

We show first that there are no envyfree Groves mechanisms on the universal private goods domain. Prior to that, we provide three lemmas that will frequently be used in our analysis. The first one provides an equivalent condition for the envyfreeness of Groves mechanims.

Lemma 9 (General Condition for Envyfreeness (GC)) *A Groves mechanism* $(G(\mathbf{f}), \bar{\Theta}^N_K)$*is envyfree if and only if* $\theta^j_{(j)} - \theta^i_{(j)} \geq f_i(\theta_K^{N-i}) - f_j(\theta_K^{N-j})$, $\forall i, j \in N$, $\forall \theta^N_K \in \bar{\Theta}^N_K$, $\forall x \in X(\theta^N_K)$, *where* $x^j = (j)$.

Proof: A Groves mechanism $(G(\mathbf{f}), \bar{\Theta}^N_K)$is envyfree if and only if

$$\theta^i_{(i)} - f_i(\theta_K^{N-i}) + W(\theta^{N-i}_{K-(i)}) \geq \theta^i_{(j)} - f_j(\theta_K^{N-j}) + W(\theta^{N-j}_{K-(j)}), \tag{3.1}$$

$\forall i, j \in N$, $\forall \theta^N_K \in \bar{\Theta}^N_K$, $\forall x \in X(\theta^N_K)$, where $x^j = (j)$ and $x^i = (i)$. (3.1) is equivalent to $W(\theta^N_K) - f_i(\theta_K^{N-i}) \geq \theta^i_{(j)} - f_j(\theta_K^{N-j}) + W(\theta^N_K) - \theta^j_{(j)}$, which is just $\theta^j_{(j)} - \theta^i_{(j)} \geq f_i(\theta_K^{N-i}) - f_j(\theta_K^{N-j})$.□

The result of this lemma will be referred to in the following as the GC. The next lemma states that in an envyfree Groves mechanism each agent who is a loser under at least one optimal assignment will get the same utility.

Lemma 10 (Loser's Equality (LE)) *If* $(G(\mathbf{f}), \bar{\Theta}_K^N)$ *is an envyfree Groves mechanism then* $\forall j, j' \in N$ *and* $\forall \theta_K^N \in \bar{\Theta}_K^N$ *such that* $x^j = 0$ *for some* $x \in X(\theta_K^N)$, *and* $\bar{x}^{j'} = 0$ *for some* $\bar{x} \in X(\theta_K^N)$ *(including* $x = \bar{x}$*),* $f_j(\theta_K^{N-j}) = f_{j'}(\theta_K^{N-j'})$.

Proof: Fix $\theta_K^N \in \Theta_K^N, x \in X(\theta_K^N)$, and $j \in N$. If $j \notin V(x)$, then $\theta_{xj}^j = 0$ and $\theta_{xj}^i = 0$, $\forall i \in N$. Thus, the GC can be written as $f_j(\theta_K^{N-j}) \geq f_i(\theta_K^{N-i})$, $\forall i \in N$, which implies the result.□

The result of Lemma 10 will be referred to as LE. Next, we show that for an envyfree Groves mechanism $(G(\mathbf{f}), \bar{\Theta}_K^N)$, f_i's are identical for each agent i. That is, an envyfree Groves mechanism is anonymous in the sense that it does not discriminate among agents regarding their final utilities. Since anonymity is a basic equity property, this is not a surprising result.

Lemma 11 (Anonymity) *If a Groves mechanism* $(G(\mathbf{f}), \bar{\Theta}_K^N)$ *is envyfree, then* $\theta_K^{N-i} = \theta_K^{N-j}$ *implies that* $f_i(\theta_K^{N-i}) = f_j(\theta_K^{N-j})$, $\forall i, j \in N$.

Proof: If $\theta_K^{N-i} = \theta_K^{N-j}$, then $\theta_K^i = \theta_K^j$. Thus, $\theta_{xj}^i = \theta_{xj}^j$ and $\theta_{xi}^i = \theta_{xi}^j$, $\forall x \in X(\theta_K^N)$, and therefore the GC implies that $f_i(\theta_K^{N-i}) = f_j(\theta_K^{N-j})$.□

In the following we will say, as shorthand, that f is envyfree on some domain of preferences, rather than the Groves mechanism that uses f. Given Lemma 11, we will use f without the subscript that refers to an agent whenever f is envyfree.

Proposition 16 *There exist no envyfree Groves mechanisms on the universal private goods domain.*

Proof: Consider θ_K^N, $\bar{\theta}_K^N$, and $\tilde{\theta}_K^N \in \Theta_K^N$:

θ_K^N	{a}	{b}	{a,b}
1	[3]	4	11
2	6	[9]	10

$\tilde{\theta}_K^N$	{a}	{b}	{a,b}
1	[6]	7	11
2	6	[9]	10

$\bar{\theta}_K^N$	{a}	{b}	{a,b}
1	3	4	[11]
2	6	7	[11]

(The rows correspond to agents and the columns correspond to packages; the boxes indicate the optimal assignments.)

For $n > 2$, $k > 3$, let all the additional values be zeros in all three preference profiles.

If f is envyfree then the GC requires that $3-6 \geq f(\theta_K^{N-2}) - f(\theta_K^{N-1})$, i. e., $3 \leq f(\theta_K^{N-1}) - f(\theta_K^{N-2})$, and $9 - 7 \geq f(\tilde{\theta}_K^{N-1}) - f(\tilde{\theta}_K^{N-2})$, i. e., $2 \geq f(\tilde{\theta}_K^{N-1}) - f(\tilde{\theta}_K^{N-2})$. The anonymity of f implies that $f(\theta_K^{N-1}) = f(\tilde{\theta}_K^{N-1})$, $f(\bar{\theta}_K^{N-1}) = f(\tilde{\theta}_K^{N-2})$, and $f(\bar{\theta}_K^{N-2}) = f(\theta_K^{N-2})$. From the GC or LE we also have $f(\bar{\theta}_K^{N-1}) = f(\bar{\theta}_K^{N-2})$, so $f(\theta_K^{N-2}) = f(\tilde{\theta}_K^{N-2})$. Therefore, f is envyfree if $3 \leq f(\theta_K^{N-1}) - f(\theta_K^{N-2}) \leq 2$, a contradiction. Thus, no Groves mechanism is envyfree on the domain $\{\theta_K^N, \tilde{\theta}_K^N, \bar{\theta}_K^N\}$. This implies that no Groves mechanism is envyfree on any domain that contains $\{\theta_K^N, \tilde{\theta}_K^N, \bar{\theta}_K^N\}$, and, in particular, on the universal private goods domain Θ_K^N.□

Since there exists a set of transfers for any optimal assignment that make the outcome envyfree,[8] we can conclude that envyfreeness and efficiency are not incompatible, in general, for the package allocation problem. Thus, the next question to ask is whether there exist envyfree Groves mechanisms on the two restricted domains. The answer is positive. First we identify the envyfree Groves mechanisms on the superadditive domain.

[8]The existence of these transfers follows directly from an application of the linear programming duality theorem to the assignment problem (see Shapley and Shubik (1972)). This is because the prices of envyfree outcomes correspond to price equilibria that clear the buyers' markets, where only the partitioning of K that results in the optimal assignment is taken into account, since unassigned packages (whether or not unassigned due to the feasibility constraints) need not be priced.

Proposition 17 *A Groves mechanism* $(G(\mathbf{f}),(\Theta_K^N)^+)$ *is envyfree if and only if* $f(\theta_K^{N-j}) = h\left(W(\theta_K^{N-j})\right)$ *for* $\theta_K^{N-j} \in (\Theta_K^{N-1})^+$, *where h is an arbitrary function satisfying* $0 \le h(\varphi + d) - h(\varphi) \le d,\ d \ge 0,\ \varphi \ge 0.$

Proof: See Appendix.

The sufficiency proof of the proposition is relatively straightforward, and is based on the fact that the Groves mechanism $(G(\mathbf{f}),(\theta_K^N)^+)$ such that $f(\theta_K^{N-j}) = W(\theta_K^{N-j})$, known as the *pivotal mechanism*[9] is envyfree, which constitutes the first part of the proof. The necessity proof consists of three parts. First we show that f can only depend on the values that each agent has for the package she is assigned as part of an optimal assignment when agent j is excluded. We call these values the *optimal values.* To get an intuitive idea about this proof, consider, for example, the following two superadditive profiles.

θ_K^N	{a}	{b}	{a,b}
1	2	[6]	11
2	[7]	4	12
3	5	3	8
j	0	[6]	6

$\tilde{\theta}_K^N$	{a}	{b}	{a,b}
1	0	6	6
2	7	4	12
3	5	3	8
j	0	6	6

There are two optimal assignments with respect to θ_K^N. In one of them 1 gets $\{b\}$ and j gets nothing, while in the other one j gets $\{b\}$ and 1 gets nothing. Thus, by LE, $f(\theta_K^{N-1}) = f(\theta_K^{N-j})$. We also have $f(\theta_K^{N-1}) = f(\tilde{\theta}_K^{N-1})$ and $f(\tilde{\theta}_K^{N-1}) = f(\tilde{\theta}_K^{N-j})$ by anonymity, so that we get $f(\theta_K^{N-j}) = f(\tilde{\theta}_K^{N-j})$. If similar arguments are applied to agents 2 and 3, it can be seen that f can only depend on the optimal values 6 and 7. In the second

[9] See for example Moulin (1986).

part of the necessity proof it is shown that, in fact, f only depends on the sum of the optimal values, $W(\theta_K^{N-j})$. The main idea behind this proof is that, since multiple optimal assignments are possible, f has to depend on the only common aspect of the optimal values in different optimal assignments, namely, their sum. Finally, a simple argument shows that the restriction on the function h, given in the statement of the proposition, has to hold, where $f(\theta_K^{N-j}) = h(W(\theta_K^{N-j}))$. Notice that given this restriction, f can be any constant $c \in \Re$, since $h(\varphi + d) - h(\varphi) = c - c = 0$, for every $d \geq 0$, and $\varphi \geq 0$. If f takes the linear form $f(\varphi) = a\varphi + c$ where a and c are constants, then the restriction on f implies that $0 \leq a \leq 1$. However, envyfreeness does not restrict the value of c.

We now characterize the class of envyfree Groves mechanisms on the substitute domain.

Proposition 18 *A Groves mechanism* $\left(G(\mathbf{f}), (\Theta_K^N)^-\right)$ *is envyfree if and only if* $f(\theta_K^{N-j}) = h\left(\left\{W(\theta_{K-p}^{N-j}) | p \in K\right\}\right)$ *for* $\theta_K^{N-j} \in (\Theta_K^{N-1})^-$, *where* h *is an arbitrary function satisfying* $h\left(\left\{W(\theta_{K-p}^{N-i}) | p \in K\right\}\right) - h\left(\left\{W(\theta_{K-p}^{N-j}) | p \in K\right\}\right) \leq max_{p \in K}\left\{W(\theta_{K-p}^{N-i}) - W(\theta_{K-p}^{N-j})\right\}$, $\forall \theta_K^N \in (\Theta_K^N)^-$, $\forall i, j \in N$, *if* $j \in V(x)$ *for some* $x \in X(\theta_K^N)$, *otherwise* $h\left(\left\{W(\theta_{K-p}^{N-i}) | p \in K\right\}\right) - h\left(\left\{W(\theta_{K-p}^{N-j}) | p \in K\right\}\right) \leq 0$.

Proof: See Appendix.

Instead of going into details of the proof, which has a similar structure to the proof of Proposition 2, we would like to provide some intuition to highlight the differences between the two results. On the substitute domain, the envyfreeness of a Groves mechanism requires that f depends on the values of the optimal assignments where the agent in question and each object in turn is excluded from the assignment, rather than on the single value, $W(\theta_K^{N-j})$, as for the superadditive case. Consider the following preference profiles.

θ_K^{N-j}	{a}	{b}
1	5	[6]
2	[4]	3

θ_K^N	{a}	{b}
1	[5]	6
2	4	3
j	0	[6]

$\tilde{\theta}_K^N$	{a}	{b}
1	0	[6]
2	[4]	3
j	0	6

Notice that agent 1 gets $\{a\}$ as part of the optimal assignment with respect to θ_K^N, whereas 1 is assigned $\{b\}$ optimally with respect to θ_K^{N-j}. Thus, the argument used for the superadditive domain to prove that $f(\theta_K^{N-j}) = f(\tilde{\theta}_K^{N-j})$ cannot be used here. In this case, the function f will depend not only on the optimal values, but also on the *substitute values*, where they exist. The substitute value for agent i is the value of the package to i which she gets optimally with respect to θ_K^N if $\theta_{x^i}^j = \theta_{x^i}^i, \theta_p^j = 0, \forall p \neq x^i$, for $x \in X(\theta_K^{N-j})$. In the above example, agent i's substitute value is 5. In fact, it turns out that $f(\theta_K^{N-j})$ only depends on the set $\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right\}$. Therefore, as we illustrated, the differences in the two characterization results are due to the fact that $\forall \theta_K^{N-j} \in (\Theta_K^N)^+$, $\forall i \in N \setminus \{j\}$, $\forall x \in X(\theta_K^{N-j})$, $\exists \tilde{x} \in X(\theta_{K-x^i}^{N-j})$ such that $i \notin V(\tilde{x})$. That is, if the preferences are superadditive and a package, assigned originally to agent i under some optimal assignment, is excluded in a new environment, then there exists an optimal assignment in this new environment under which i is a loser. The substitute domain, however, does not have this consistency property, precisely because of the substitute nature of the objects, which accounts for the more complicated characterization of the envyfree mechanisms on this domain.

Proposition 3 implies that, for example, $f(\theta_K^{N-j}) = \sum_{p\in K} \alpha_p W(\theta_{K-p}^{N-j}) + c$ is envyfree on $(\Theta_K^N)^-$, where $0 \leq \alpha_p \leq 1$, $\forall p \in K$, $\sum_{p\in K} \alpha_p = 1$, and $c \in \Re$. Notice that here f cannot be a constant, which contrasts with the superadditive case.

3.4 Individual Rationality

While envyfreeness ensures a fair distribution of the packages and revenues among the agents, it does not say anything about the extent of the extracted revenue. In this section, we impose an individual rationality constraint in order to ensure that the agents are willing to participate in the auction. A mechanism is individually rational if it gives a nonnegative utility to each agent for any profile.

Definition 44 A direct mechanism $(g, \bar{\Theta}_K^N)$is *individually rational* if $g = (x, t)$ is such that $\sum_{p \in \mathcal{K}} x_p^i \theta_p^i - t_i(\theta_K^N) \geq 0$, $\forall i \in N$, $\forall \theta_K^N \in \bar{\Theta}_K^N$.

Since the Groves mechanisms are efficient, for these mechanisms the individual rationality constraint places an upper bound on the Groves taxes that can be extracted from the agents, given that the assignment is optimal. First we need to determine what this upper bound is. A Groves mechanism $(G(\mathbf{f}), \bar{\Theta}_K^N)$ is individually rational if, and only if, $\theta_{x^i}^i - t_i(\theta_K^N) = \theta_{x^i}^i - f_i(\theta_K^{N-i}) + W(\theta_{K-x^i}^{N-i}) = W(\theta_K^N) - f_i(\theta_K^{N-i}) \geq 0, \forall i \in N, \forall \theta_K^N \in \bar{\Theta}_K^N$, where $x \in X(\theta_K^N)$.[10] Now fix arbitrary $\theta_K^{N-i} \in \bar{\Theta}_K^{N-1}$. Let $\theta_K^i = \mathbf{0}$.[11] Then $W(\theta_K^N) = W(\theta_K^{N-i})$. Since f_i cannot depend on θ_K^i, this implies that $f_i(\theta_K^{N-i}) \leq W(\theta_K^{N-i})$, $\forall i \in N$, $\forall \theta_K^N \in \bar{\Theta}_K^N$ is required for a Groves mechanism $(G(\mathbf{f}), \bar{\Theta}_K^N)$ to be individually rational. This is also a sufficient condition for individual rationality, since $W(\theta_K^N) \geq W(\theta_K^{N-i}), \forall i \in N, \forall \theta_K^N \in \Theta_K^N$.

[10] In fact, a Groves mechanism is individually rational if, and only if, $t_i(\theta_K^N) \leq 0$ whenever $\theta_K^i = \mathbf{0}$, since $t_i(\theta_K^{N-i}, \mathbf{0}) = f_i(\theta_K^{N-i}) - W(\theta_{K-x^i}^{N-i}) = f_i(\theta_K^{N-i}) - W(\theta_K^N) \leq 0$ implies that $f_i(\theta_K^{N-i}) \leq W(\theta_K^N)$, and vice versa.

[11] We use the notation $\mathbf{d} = (d, \ldots, d)$ or $\mathbf{d} = \begin{pmatrix} d & d & d & \cdots \\ d & d & d & \cdots \\ d & d & d & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$, depending on the dimensions of the vector or matrix $\mathbf{d}$ stands for.

Thus, an individually rational Groves mechanism cannot charge the agents more than the pivotal mechanism does, and, conversely, any Groves mechanism with this upper bound on the taxes is individually rational.

Given Propositons 17 and 18, it can be seen that the pivotal mechanism is envyfree on both the superadditive and substitute domains. Thus, the pivotal mechanism is an example of an individually rational and envyfree Groves mechanism on both examined domains.

Now we would like to characterize the entire menus of Groves mechanisms that are individually rational and envyfree on both restricted domains. Interestingly, the condition for individual rationality, namely, that for $\theta_K^{N-j} = \mathbf{0}$ the value of f is nonpositive, is the same for both domains, although it cannot directly be deduced from the GC. We prove the condition for the superadditive domain in Proposition 19, and for the substitute domain in Proposition 20. Notice that it is a special case of the general condition for individual rationality, since $W(\mathbf{0}) = 0$, and thus the necessity proofs are omitted in the following two propositions. This also indicates that the envyfree requirement places a structure on the Groves transfers that is congruous with individual rationality.

Proposition 19 *An envyfree Groves mechanism* $\left(G(\mathbf{f}), (\Theta_K^N)^+\right)$ *is individually rational if and only if* $f(\mathbf{0}) \leq 0$.

Proof: From Proposition 17, if h is envyfree on $(\Theta_K^N)^+$, then we have $h\left(W(\theta_K^{N-j})\right) - h(0) \leq W(\theta_K^{N-j})$, $\forall \theta_K^{N-j} \in (\theta_K^{N-j})^+$. We also have $f(\mathbf{0}) = h(0) \leq 0$. Thus, $f(\theta_K^{N-j}) = h\left(W(\theta_K^{N-j})\right) \leq W(\theta_K^{N-j})$, $\forall \theta_K^{N-j} \in (\theta_K^{N-j})^+$, which implies that the Groves mechanism $\left(G(\mathbf{f}), (\Theta_K^N)^+\right)$ is individually rational.□

If an envyfree f on the superadditive domain has the linear form $f(W(\theta_K^{N-j})) = aW(\theta_K^{N-j}) + c$ for some constants a and $c, 0 \leq a \leq 1$, the individual rationality constraint is satisfied if $c \leq 0$. Since it is also a requirement for individual rationality, a positive amount that is constant across agents and profiles cannot be added to the charges in any individually rational envyfree Groves mechanism for which f has the above form.

Proposition 20 *An envyfree Groves mechanism* $\big(G(\mathbf{f}), (\Theta_K^N)^-\big)$ *is individually rational if and only if* $f(\mathbf{0}) \leq 0$.

Proof: See Appendix.

Notice that a similar argument to the proof of Proposition 19 would only imply for this domain that $f(\theta_K^{N-j}) \leq W(\theta_K^{N-j})$, where $\theta_K^{N-j-i} = \mathbf{0}$ for some $i \in N \setminus \{j\}$. Thus, we need a different argument for the substitute domain. The proof is given in the Appendix, since it requires definitions and lemmas from the proof of Proposition 18, which is also given in the Appendix.

Suppose $f(\theta_K^{N-j}) = \sum_{p \in K} \alpha_p W(\theta_{K-p}^{N-j}) + c$ for some envyfree f on $(\Theta_K^N)^-$, where $0 \leq \alpha_p \leq 1$, $\forall p \in K$, $\sum_{p \in K} \alpha_p = 1$, and c is some constant. Then, given that $W(\theta_K^{N-j}) = 0$ implies that $W(\theta_{K-p}^{N-j}) = 0$, $\forall p \in K$, the individual rationality constraint is satisfied if $c \leq 0$. Therefore, given that it is also a necessary condition, these envyfree Groves mechanisms will not be individually rational if an additional positive constant amount is charged to the agents, similarly to the superadditive case.

3.5 The Pivotal Mechanism and Revenue Extraction

It is well-known that the pivotal mechanism plays a prominent role in the class of Groves mechanisms. The pivotal mechanisms are also important in our context. As we have seen in the last two sections, they are envyfree on both the superadditive and substitute domains, and provide the upper bound for the revenue that can be extracted when the individual rationality constraint is satisfied.[12] The pivotal mechanism can be characterized in terms of other requirements regarding revenue extraction. In the next proposition we show that if the planner wants to maximize the agents' utility (i.e., minimize the extracted revenue) such that a) she does not subsidize any agent, or b) no agent gets higher utility than by obtaining the assignment of her choice while paying nothing, then the planner will have to choose a pivotal mechanism in either case.

Proposition 21 *a) A Groves mechanism minimizes the extracted revenue and does not subsidize any agent at any profile if, and only if, it is a pivotal mechanism.*

b) A Groves mechanism minimizes the extracted revenue and does not allow any agent at any profile to get higher utility than by getting her first choice without paying if, and only if, it is a pivotal mechanism.

Proof: a) A pivotal mechanism satisfies the no subsidy requirement since $t_i(\theta_K^N) = W(\theta_K^{N-i}) - W(\theta_{K-x^i}^{N-i}) \geq 0, \forall \theta_K^N \in \Theta_K^N, \forall x \in X(\theta_K^N)$, and $\forall i \in N$. Now fix arbitrary $\theta_K^{N-i} \in \Theta_K^{N-i}$. Let $\theta_K^i = \mathbf{0}$. Then $\exists x \in X(\Theta_K^N)$ such that $x^i = 0$, so that $W(\theta_K^{N-i}) = W(\theta_{K-x^i}^{N-i})$.

[12] Each pivotal mechanism yields the same budget surplus, given that $\forall \theta_K^N \in \Theta_K^N, \sum_{j \in N} W(\theta_K^{N-j}) - \sum_{j \in N} W(\theta_{K-x^j}^{N-j}) = \sum_{j \in N} W(\theta_K^{N-j}) - (n-1)W(\theta_K^N), \forall x \in X(\theta_K^N)$. Thus, the extracted revenue is uniquely defined, regardless of which pivotal mechanism is used. This is also true for any Groves mechanism (see the remark on neutrality in Footnote 7 in this chapter).

This proves that only a pivotal mechanism minimizes the extracted revenue subject to $t_i(\theta_K^N) \geq 0, \forall \theta_K^N \in \Theta_K^N$.

b) A pivotal mechanism satisfies $U(\theta_K^N, \theta_K^i) \leq \max_{p\in\mathcal{K}}\theta_p^i, \forall \theta_K^N \in \Theta_K^N, \forall i \in N$, since $W(\theta_K^{N-i})$ $+\max_{p\in\mathcal{K}}\theta_p^i \geq W(\theta_{K-x^i}^{N-i}) + \theta_{x^i}^i = W(\theta_K^N)$, and thus $U(\theta_K^N, \theta_K^i) = W(\theta_K^N) - W(\theta_K^{N-i}) \leq$ $\max_{p\in K}\theta_p^i, \forall \theta_K^N \in \Theta_K^N, \forall x \in X(\theta_K^N), \forall i \in N$. Now fix arbitrary $\theta_K^{N-i} \in \Theta_K^{N-i}$. Let $\theta_K^i = \mathbf{0}$. Then $W(\theta_K^N) = W(\theta_K^{N-i})$ and $\max_{p\in\mathcal{K}}\theta_p^i = 0$. Then $U(\theta_K^N, \theta_K^i) \leq \max_{p\in\mathcal{K}}, \forall \theta_K^N \in \Theta_K^N, \forall i \in$ N requires that $f(\theta_K^{N-i}) \geq W(\theta_K^N) - \max_{p\in\mathcal{K}}\theta_p^i = W(\theta_K^{N-i}), \forall \theta_K^N \in \Theta_K^N, \forall i \in N$. Thus, only the pivotal mechanisms minimize the extracted revenue subject to $U(\theta_K^N, \theta_K^i) \leq \max_{p\in\mathcal{K}}\theta_p^i$.□

Note that the above characterization results hold for both the superadditive and substitute domains, since $\theta_K^i = \mathbf{0}$ is included in both restricted domains.

The above characterization results are negative results. They state that a revenue minimizing planner must choose a pivotal mechanism if one of the above requirements is desirable. We remark that the stated requirements are very reasonable ones. The first one says that no agent should be subsidized, which is natural to ask for if commercial objects are auctioned off and the efficiency of the assignment of the objects is first priority. The second one says that no one should get more benefit from the assignment than by obtaining her first choice for free, which is very reasonable for a planner who merely wishes to achieve an efficient assignment. It follows from the proposition that whether the planner is a revenue maximizer when the agents are individually rational, or a revenue minimizer subject to one of the above natural upper bounds on the agents' final utilities, she has no other choice but a pivotal mechanism. Thus, Proposition 21 illustrates the restrictive nature of the Groves transfers concerning the revenue choices and reveals that strategyproofness is bought at a considerable expense.

Proposition 21 also holds on the universal domain.[13] Our next result demonstrates that the universal private goods domain is not as restrictive as the universal domain with respect to other revenue related criteria. Moulin (1986) proves that any anonymous and *feasible* Groves mechanism (i.e., a Groves mechanism which does not generate a budget deficit at any profile) satisfying the criterion of *no free ride* is a pivotal mechanism. For a public goods problem, free riding means manipulating the mechanism by abstaining. For a private goods allocation problem, it is not possible to manipulate by not participating, since an abstaining agent stays at her initial (zero) utility. However, one may want to insure that the agents participate to achieve greater efficiency. Thus, a natural analog of the no free ride criterion is individual rationality (it is also the formal analog for our problem). In the next proposition we illustrate that the pivotal mechanisms are not the only mechanisms that avoid budget deficit and satisfy individual rationality. We give an example of a feasible Groves mechanism that does not collect more revenue (and for some profiles it collects less) than the pivotal mechanisms. Thus, we are able to demonstrate that the natural structure of the universal private goods domain allows for more choices than the universal domain in this respect. That is, the contrast between our result and Moulin's (1986, Theorem 1) is due to the restrictions of the universal private goods domain, i.e., that the conflict among agents is somewhat reduced when agents only care about one component of the outcome. In particular, on the universal domain one can specify, for any profile of a coalition $N \setminus \{i\}$ of $n-1$ agents, a valuation of the public projects for agent i such that no agent is pivotal, i.e., such that the efficient public project is unchanged for any $N \setminus \{j\}$ coalition for each $j \in N$. This is not always possible for our private goods allocation problem. In fact, on

[13]For details see Moulin (1986).

the universal private goods domain an agent is not pivotal if she is a loser under some optimal assignment. It is clear, however, that one cannot specify for all $N \setminus \{i\}$ profile θ_K^{N-i} a valuation θ_K^i for agent i such that each agent is a loser under some optimal assignment.

Definition 45 A direct mechanism $(g, \bar{\Theta}_K^N)$ is *feasible* if $\sum_{j \in N} t_j(\theta_K^N) \geq 0, \forall \theta_K^N \in \bar{\Theta}_K^N$.

Proposition 22 *There exist anonymous, feasible, and individually rational Groves mechanisms on the universal private goods domain that don't yield a higher budget surplus at any profile than the pivotal mechanisms, and yield a lower budget surplus at some profiles.*

Proof: Define $(G(\mathbf{f}^m), \Theta_K^N)$ by $f_i^m(\bar{\theta}_K^{N-i}) = W(\bar{\theta}_K^{N-i}) - h(\bar{\theta}_K^{N-i})$, where $h(\bar{\theta}_K^{N-i}) = 1/n$ $\min_{\theta_K^i \in \Theta_K^i} \left[\sum_{j \in N} W(\theta_K^i, \bar{\theta}_K^{N-i-j}) - \sum_{j \in N} W\left(\theta^i_{K-x^j_{[\theta_K^i]}}, \bar{\theta}^{N-i-j}_{K-x^j_{[\Theta_K^i]}}\right)\right], \forall i \in N, \forall \bar{\theta}_K^{N-i} \in$ Θ_K^{N-i} such that $x_{[\theta_K^i]} \in X(\theta_K^i, \bar{\theta}_K^{N-i}), \forall \theta_K^i \in \Theta_K^i$. The Groves mechanisms $(G(\mathbf{f}^m), \Theta_K^N)$, as defined above, are anonymous. We will show that they are also individually rational and feasible. Since $W(\theta_K^i, \bar{\theta}_K^{N-i-j}) - W\left(\theta^i_{K-x^j_{[\theta_K^i]}}, \bar{\theta}^{N-i-j}_{K-x^j_{[\theta_K^i]}}\right) \geq 0, \forall (\theta_K^i, \bar{\theta}_K^{N-i}) \in \Theta_K^N, \forall x_{[\theta_K^i]} \in$ $X(\theta_K^i, \bar{\theta}_K^{N-i}), \forall j \in N$, we have $h(\bar{\theta}_K^{N-i}) \geq 0, \forall \bar{\theta}_K^{N-i} \in \Theta_K^{N-i}$, so that these Groves mechanisms are individually rational. A Groves mechanism $(G(\mathbf{f}), \Theta_K^N)$ is feasible if $\sum_{j \in N} W(\theta_{K-x^j}^{N-j}) \leq \sum_{j \in N} f_j(\theta_K^{N-j}), \forall \theta_K^N \in \Theta_K^N, \forall x \in X(\theta_K^N)$. Thus, feasibility can be verified by checking that $\min_{\theta_K^i \in \Theta_K^i} \left[\sum_{j \in N} W(\theta_K^i, \bar{\theta}_K^{N-i-j}) - \sum_{j \in N} W\left(\theta^i_{K-x^j_{[\theta_K^i]}}, \bar{\theta}^{N-i-j}_{K-x^j_{[\Theta_K^i]}}\right)\right]$ $\leq \sum_{j \in N} W(\bar{\theta}_K^{N-j}) - \sum_{j \in N} W(\bar{\theta}_{K-\bar{x}^j}^{N-j}), \forall \bar{\theta}_K^N \in \Theta_K^N, \forall \bar{x} \in X(\bar{\theta}_K^N)$, which holds.

Finally, we need to show that these mechanisms don't yield a higher budget surplus than the pivotal mechanisms for any profile, which is guaranteed by the individual rationality constraint, and that they yield a lower budget surplus for some profiles. We illustrate the latter by an example of such a profile for $n = 3, k = 3$. Define $\bar{\theta}_K^N$ as follows.

$\bar{\theta}_K^N$	{a}	{b}	{a,b}
1	7	0	0
2	8	0	0
3	9	0	0

Take $\bar{\theta}_K^{N-3}$ and specify $\theta_K^3 \in \Theta_K^3$. Then $W(\theta_K^3, \bar{\theta}_K^{N-3}) = \theta_{x^3}^3$, where $x^3 = \{a\}$, or $x^3 = \{a,b\}$, or $W(\theta_K^3, \bar{\theta}_K^{N-3}) = \theta_{x^3}^3 + 8$, where $x^3 = \{b\}$. If $W(\theta_K^3, \bar{\theta}_K^{N-3}) = \theta_{x^3}^3$, where $x^3 = \{a\}$ or $x^3 = \{a,b\}$ then $W(\theta_K^3, \bar{\theta}_K^{N-3-1}) = W(\theta_K^3, \bar{\theta}_K^{N-3-2}) = \theta_{x^3}^3$. Since $W(\bar{\theta}_K^{N-3}) = 8$, this gives $\sum_{j\in N} W(\theta_K^3, \bar{\theta}_K^{N-3-j}) - \sum_{j\in N} W(\theta_{K-x^j}^3, \bar{\theta}_{K-x^j}^{N-3-j}) = \sum_{j\in N} W(\theta_K^3, \bar{\theta}_K^{N-3-j}) - 2W(\theta_K^3, \bar{\theta}_K^{N-3}) = 2\theta_{x^3}^3 + 8 - 2\theta_{x^3}^3 = 8$, where $x \in X(\theta_K^3, \bar{\theta}_K^{N-3})$. If $W(\theta_K^3, \bar{\theta}_K^{N-3}) = \theta_{x^3}^3 + 8$, where $x^3 = \{b\}$ then $\sum_{j\in N} W(\theta_K^3, \bar{\theta}_K^{N-3-j}) - 2W(\theta_K^3, \bar{\theta}_K^{N-3}) = 2\theta_{x^3}^3 + 2\cdot 8 + 7 - 2\theta_{x^3}^3 - 2\cdot 8 = 7$, since in this case $W(\theta_K^3, \bar{\theta}_K^{N-3-1}) = \theta_{x^3}^3 + 8$, and $W(\theta_K^3, \bar{\theta}_K^{N-3-2}) = \theta_{x^3}^3 + 7$. Thus, $h(\bar{\theta}_K^{N-3}) = 1/3\cdot 7$. We can calculate, similarly, that $h(\bar{\theta}_K^{N-2}) = 1/3\cdot 7$, and $h(\bar{\theta}_K^{N-1}) = 1/3\cdot 8$. Therefore, $\sum_{j\in N} f^m(\bar{\theta}_K^{N-j}) = \sum_{j\in N} W(\bar{\theta}_K^{N-j}) - \sum_{j\in N} h(\bar{\theta}_K^{N-i}) = 9+9+8-1/3(7+7+8) = 18\ 2/3 < 26 = \sum_{j\in N} W(\bar{\theta}_K^{N-j})$. Given this example, it is easy to find similar examples for any $n \geq 2, k \geq 3$ for which $\sum_{j\in N} h(\bar{\theta}_K^{N-j}) > 0$.□

Note that the above proposition can easily be modified to hold for restricted domains as well.

3.6 Discussion

When it is possible to use monetary transfers for allocating heterogeneous indivisible objects, one can design more desirable strategyproof mechanisms than without compensations. In particular, not only efficient and anonymous mechanisms exist in this case, but we were also able to identify envyfree Groves mechanisms. We also investigated individual rationality, which is an issue when compensations are possible. However, the positive results were obtained at some expense. Firstly, although we allowed indifferences, the utility functions are assumed to be additively separable. Secondly, the Groves mechanisms are not generically budget balancing.[14]Thus, unless we assume that the balance is absorbed by the planner, these mechanisms are not Pareto-optimal. Nonetheless, we were able to demonstrate that the universal private goods domain is less restrictive than the universal domain with respect to some revenue related criteria. Thirdly, it is straightforward to verify that the Groves mechanisms are bossy, if we redefine bossiness in terms of utilities. However, bossy Groves mechaninsms are not as undesirable as bossy nontransfer mechanisms are, given that the bossiness of a Groves mechanism only affects the extracted revenue, while the optimal assignment is preserved.

[14]See Hurwicz and Walker (1990).

Appendix A

Proofs

Proposition 15 for the superadditive domain

Proposition 15' *A strategyproof and efficient direct mechanism on the superadditive domain is a Groves mechanism.*

Proof: Note that parts i), ii)(a), and ii) (b) in the proof of Proposition 15 also hold for the superadditive domain. Thus, assume that a strategyproof and efficient direct mechanism $(g, (\Theta_K^N)^+)$ violates ii) in Lemma 8 and that $x^i \neq \tilde{x}^i$. We need to consider four cases.

1) $x^i \not\subset \tilde{x}^i, \tilde{x}^i \not\subset x^i, \tilde{x}^i \neq 0$.

Part ii) (c) in the proof of Proposition 15 applies to this case with the following modifications:

$\bar{\theta}_p^i = 0$ if $x^i \not\subset p, \tilde{x}^i \not\subset p$,

$\bar{\theta}_p^i = \bar{\theta}_{x^i}^i$ if $x^i \subset p, \tilde{x}^i \not\subset p$,

$\bar{\theta}_p^i = \bar{\theta}_{\tilde{x}^i}^i$ if $x^i \not\subset p, \tilde{x}^i \subset p$,

$\bar{\theta}_p^i = \bar{\theta}_{x^i}^i + \bar{\theta}_{\tilde{x}^i}^i$ if $x^i \subset p, \tilde{x}^i \subset p$.

2) $\tilde{x}^i = 0$. (Note that in this case $x^i \not\subseteq \tilde{x}^i, \tilde{x}^i \not\subseteq x^i$, and $x^i \neq 0$, since $x^i \neq \tilde{x}^i$.)

Part ii) (d) in the proof of Proposition 15 applies to this case with the following modifications:

$\bar{\theta}^i_p = 0$ if $x^i \not\subset p$,

$\bar{\theta}^i_p = \bar{\theta}^i_{x^i}$ if $x^i \subset p$.

3) $x^i \subset \tilde{x}^i$. (Note that in this case $x^i \neq 0$ and $\tilde{x}^i \neq 0$.)

Part ii) (c) in the proof of Proposition 15 applies to this case with the following modifications:

$\bar{\theta}^i_p = 0$ if $x^i \not\subset p$,

$\bar{\theta}^i_p = \bar{\theta}^i_{x^i}$ if $x^i \subset p, \tilde{x}^i \not\subset p$,

$\bar{\theta}^i_p = \bar{\theta}^i_{\tilde{x}^i}$ if $\tilde{x}^i \subset p$.

Remark: $W(\theta^{N-i}_{K^i_{\tilde{x}}}) \geq W(\theta^{N-i}_{K-\tilde{x}^i})$, since $x^i \subset \tilde{x}^i$. Then $\bar{\theta}^i_{\tilde{x}^i} = W(\theta^N_K) - W(\theta^{N-i}_{K-\tilde{x}^i}) \geq \theta^i_{x^i} = \bar{\theta}^i_{x^i}$, so $\bar{\theta}^i_K$ is superadditive.

4) $\tilde{x}^i \subset x^i$. (Note that in this case $x^i \neq 0$ and $\tilde{x}^i \neq 0$.)

Part ii) (c) in the proof of Proposition 15 applies to this case with the following modifications:

$\bar{\theta}^i_p = 0$ if $\tilde{x}^i \not\subset p$,

$\bar{\theta}^i_p = \bar{\theta}^i_{\tilde{x}^i}$ if $x^i \not\subset p, \tilde{x}^i \subset p$,

$\bar{\theta}^i_p = \bar{\theta}^i_{x^i}$ if $x^i \subset p$.

Remark: $W(\theta^{N-i}_{K^i_{\tilde{x}}}) \leq W(\theta^{N-i}_{K-\tilde{x}^i})$, since $\tilde{x}^i \subset x^i$. Then $\bar{\theta}^i_{\tilde{x}^i} = W(\theta^N_K) - W(\theta^{N-i}_{K-\tilde{x}^i}) \leq \theta^i_{x^i} = \bar{\theta}^i_{x^i}$, so $\bar{\theta}^i_K$ is superadditive.□

Proof of Proposition 17

Sufficiency

First we show that if $f(\theta_K^{N-j}) = W(\theta_K^{N-j}), \forall \theta_K^{N-j} \in (\Theta_K^{N-j})^+$ then f is envyfree on $(\Theta_K^N)^+$.

Fix $\theta_K^N \in (\Theta_K^N)^+$ and $i, j \in N$. Let $x \in X(\theta_K^N)$, $x^j = (j)$, and $x^i = (i)$.

1. If $j \notin V(x)$, then $\theta_{(j)}^j = 0$, $\theta_{(j)}^i = 0$, and $W(\theta_K^{N-j}) = W(\theta_K^N)$. Given that $W(\theta_K^N) \geq W(\theta_K^{N-i})$, $\forall \theta_K^N \in (\Theta_K^N)^+$, $\forall i \in N$, the GC holds for this case.

2. If $j \in V(x)$, then $(i) \bigcap (j) = \emptyset$, given that package (i) and package (j) are assigned simultaneously. Since $W(\theta_{K-(j)}^{N-j}) + \theta_{(j)}^j = W(\theta_K^N)$,

$$W(\theta_{K-(j)}^{N-j}) + \theta_{(j)}^j \geq W(\theta_K^{N-i}). \tag{A.1}$$

Now consider

$$W(\theta_K^{N-j}) \geq W(\theta_{K-(j)-(i)}^{N-j-i}) + \theta_{(i)\cup(j)}^i, \tag{A.2}$$

which holds, since agent i may not get the package consisting of packages (i) and (j), as part of an optimal assignment, when agent j is excluded from the assignment. Given that packages (i) and (j) do not contain any common object, $W(\theta_K^N) \in (\Theta_K^N)^+$ implies that $\theta_{(i)\cup(j)}^i \geq \theta_{(i)}^i + \theta_{(j)}^i$. Thus, from (A.2) we get $W(\theta_K^{N-j}) \geq W(\theta_{K-(j)-(i)}^{N-j-i}) + \theta_{(i)}^i + \theta_{(j)}^i$. Furthermore, since $W(\theta_{K-(j)-(i)}^{N-j-i}) + \theta_{(i)}^i = W(\theta_{K-(j)}^{N-j})$, we have

$$W(\theta_K^{N-j}) \geq W(\theta_{K-(j)}^{N-j}) + \theta_{(j)}^i. \tag{A.3}$$

Adding (A.1) and (A.3), we get $W(\theta_K^{N-j}) + \theta_{(j)}^j \geq W(\theta_K^{N-i}) + \theta_{(j)}^i$, which is equivalent to the GC for i, j, and θ_K^N.

This completes the proof that $f(\theta_K^{N-j}) = W(\theta_K^{N-j})$ is envyfree on $(\Theta_K^N)^+$.

Now let $W(\theta_K^{N-i}) - W(\theta_K^{N-j}) = d$.

(a) $d \geq 0$

Given our assumption on h, we have

$h\left(W(\theta_K^{N-j}) + d\right) - h\left(W(\theta_K^{N-j})\right) \leq d = W(\theta_K^{N-i}) - W(\theta_K^{N-j}) \leq \theta_{(j)}^j - \theta_{(j)}^i$. Thus, $\theta_{(j)}^j - \theta_{(j)}^i \geq h\left(W(\theta_K^{N-i})\right) - h\left(W(\theta_K^{N-j})\right)$ holds, as required.

(b) $d < 0$

First we need to show that $\theta_{(j)}^j \geq \theta_{(j)}^i$. Suppose that $\theta_{(j)}^i > \theta_{(j)}^j$. Then $\theta_K^N \in (\Theta_K^N)^+$ implies that $\theta_{(i)\cup(j)}^i > \theta_{(i)}^i + \theta_{(j)}^j$, which implies that $x \notin X(\theta_K^N)$. A contradiction. Therefore,

$$\theta_{(j)}^j - \theta_{(j)}^i \geq 0. \tag{A.4}$$

Since $d < 0$, we have $h\left(W(\theta_K^{N-j})\right) - h\left(W(\theta_K^{N-j}) + d\right) \geq 0$, i. e., $h\left(W(\theta_K^{N-j}) + d\right) - h\left(W(\theta_K^{N-j})\right) \leq 0$. Together with (A.4), this yields $\theta_{(j)}^j - \theta_{(j)}^i \geq h\left(W(\theta_K^{N-i})\right) - h\left(W(\theta_K^{N-j})\right)$, as required.

Necessity

The necessity proof consists of three parts, which are summarized below.

I. If f is envyfree on $(\Theta_K^N)^+$ then there exists a function $\hbar$ such that $f(\theta_K^{N-j}) = \hbar\left(\{\theta_{x^i}^i \mid i \in V(x)\}\right)$, $\forall \theta_K^{N-j} \in (\Theta_K^{N-j})^+$, $\forall x \in X(\theta_K^{N-j})$.

II. If $\hbar$ is envyfree on $(\Theta_K^N)^+$ then $\hbar$ can be written as a function of the sum of the optimal values in its argument, i. e., there exists a function h such that for any profile $\theta_K^{N-j} \in (\Theta_K^{N-j})^+$ with a set of optimal values Φ, $\hbar(\Phi) = h\left(\sum_{\phi\in\Phi}\phi\right)$.

III. If f is envyfree on $(\Theta_K^N)^+$and $f(\theta_K^{N-j}) = h(W(\theta_K^{N-j})) = h(\varphi)$ then $0 \leq h(\varphi + d) - h(\varphi) \leq d$, $\forall\varphi \geq 0$, $\forall d \geq 0$.

I.

Claim 1 *If f is envyfree on $(\Theta_K^N)^+$, $\forall\theta_K^{N-j}, \tilde{\theta}_K^{N-j} \in (\Theta_K^{N-j})^+$ such that*

i) $V(x) = V(\tilde{x})$,

ii) $x^i = \tilde{x}^i$, $\forall i \in V(x)$, and

iii) $\theta^i_{x^i} = \tilde{\theta}^i_{x^i}$, $\forall i \in V(x)$,

where $x \in X(\theta_K^{N-j})$, and $\tilde{x} \in X(\tilde{\theta}_K^{N-j})$, we have $f(\theta_K^{N-j}) = f(\tilde{\theta}_K^{N-j})$.

Proof: Fix $\theta_K^{N-j} \in (\Theta_K^{N-j})^+$ and $\tilde{\Theta}_K^{N-j} \in (\Theta_K^{N-j})^+$ such that conditions i)–iii) hold for them, $x \in X(\theta_K^{N-j})$, and $\tilde{x} \in X(\tilde{\Theta}_K^{N-j})$. Let $j = n$ and $Q_{[0]K}^{N-j} = \theta_K^{N-j}$. Define $\left\{Q_{[i]K}^{N-j} \mid i = 1, \ldots, n-1\right\}$ as follows. Let $Q^t_{[i]p} = \theta^t_{x^t}, \forall p \in K$ such that $\mathrm{x}^t \subseteq p$ and $Q^t_{[i]p} = 0, \forall p \in K$ such that $x^t \not\subseteq p$ for $i = 1, \ldots, n-1$, $t = 1, \ldots, i$. Let $Q^t_{[i]K} = \theta^t_{[i]K}$ for $i = 1, \ldots, n-1$, $t = i+1, \ldots, n-1$.

Now we want to show that if $Q^j_{[i-1]K} = Q^j_{[i]K} = Q^i_{[i]K}$ then $f(Q^{N-i}_{[i-1]K}) = f(Q^{N-j}_{[i-1]K})$, and $f(Q^{N-i}_{[i]K}) = f(Q^{N-j}_{[i]K})$. There are two cases to consider.

(a) $i \notin V(x)$.

If $i \notin V(x)$ then $\exists\ \bar{x} \in X(\theta_K^N)$, where $\theta_K^j = \theta_K^i$, such that $i \notin V(\bar{x})$. Since $i \notin V(x)$, $\exists\ x_{[i]} \in X(Q_{[i]K}^{N-j})$ such that $i \notin V(x_{[i]})$. Then $\exists\ \bar{x}_{[i]} \in X(Q_{[i]K}^N)$, such that $i \notin V(\bar{x}_{[i]})$. Similarly, if $i \notin V(x)$, then $\exists\ x_{[i-1]} \in X(Q_{[i-1]K}^{N-j})$ such that $i \notin V(x_{[i-1]})$. Then, given the definition of $\left\{Q_{[i]K}^{N-j} \mid i=1,\ldots,n-1\right\}$, $\exists\ \bar{x}_{[i-1]} \in X(Q_{[i-1]K}^{N})$ such that $i \notin V(\bar{x}_{[i-1]})$. Also, $\exists\ \bar{\bar{x}}_{[i]} \in X(Q_{[i]K}^{N})$ such that $j \notin V(\bar{\bar{x}}_{[i]})$, and $\bar{\bar{x}}_{[i-1]} \in X(Q_{[i-1]K}^{N})$ such that $j \notin V(\bar{\bar{x}}_{[i-1]})$. Thus, LE implies that $f(Q_{[i]K}^{N-i}) = f(Q_{[i]K}^{N-j})$ and $f(Q_{[i-1]K}^{N-i}) = f(Q_{[i-1]K}^{N-j})$.

(b) $i \in V(x)$.

Let $x_{[i-1]} \in X(Q_{[i-1]K}^{N})$, where $Q_{[i-1]K}^{j} = Q_{[i]K}^{i}$. We have the following cases.

1. $i \notin V(x_{[i-1]})$, $j \notin V(x_{[i-1]})$.

2. $i \notin V(x_{[i-1]})$, $j \in V(x_{[i-1]})$.

   Then $\exists\ x'_{[i-1]} \in X(Q_{[i-1]K}^{N})$ such that $i \in V(x'_{[i-1]})$ with $x'^{i}_{[i-1]} = x^{j}_{[i-1]}$, and $j \notin V(x'_{i-1]})$.

3. $i \in V(x_{[i-1]})$, $j \notin V(x_{[i-1]})$, $x^{i}_{[i-1]} = x^{i}$.

   Then $\exists\ x'_{[i-1]} \in X(Q_{[i-1]K}^{N})$ such that $i \notin V(x'_{[i-1]})$, $j \in V(x'_{[i-1]})$, and $x'^{j}_{[i-1]} = x^{i}$.

4. $i \in V(x_{[i-1]})$, $j \notin V(x_{[i-1]})$, $x^{i}_{[i-1]} \neq x^{i}$.

   Then given that $i \in V(x)$, $\exists\ \bar{x}_{[i-1]} \in X(Q_{[i-1]K}^{N})$ such that $i \in V(\bar{x}_{[i-1]})$ and $\bar{x}^{i}_{[i-1]} = x^{i}$. This implies that $\exists\ x'_{[i-1]} \in X(Q_{[i-1]K}^{N})$ such that $i \notin V(x'_{[i-1]})$, $j \in V(x'_{[i-1]})$, and $x'^{j}_{[i-1]} = x^{i}$.

5. $i \in V(x_{[i-1]})$, $j \in V(x_{[i-1]})$.

   Then, since $x^{i}_{[i-1]} \bigcap x^{j}_{[i-1]} = \emptyset$, and $\theta_K^{N-j} \in (\Theta_K^{N-j})^{+}$, we have

$$\theta^{i}_{x^{j}_{[i-1]}} + \theta^{i}_{x^{i}_{[i-1]}} \leq \theta^{i}_{x^{j}_{[i-1]} \bigcup x^{i}_{[i-1]}}. \tag{A.5}$$

   However, (A.5) must be an equality, since $x_{[i-1]} \in Q_{[i-1]K}^{N}$. Then $\exists\ \bar{\bar{x}}_{[i-1]} \in$

$X(Q^N_{[i-1]K})$ such that $\bar{\bar{x}}^i_{[i-1]} = x^j_{[i-1]} \bigcup x^i_{[i-1]}$ and $\bar{\bar{x}}^j_{[i-1]} = 0$. If $\bar{\bar{x}}^i_{[i-1]} = x^i$ then case 2. applies, and if $\bar{\bar{x}}^i_{[i-1]} \neq x^i$ then case 3. applies.

In sum, if $i \in V(x)$, there exist two optimal assignments (or possibly just one) with respect to $Q^N_{[i-1]K}$, $\bar{x}_{[i-1]}$and $\bar{\bar{x}}_{[i-1]}$, ($\bar{x}_{[i-1]} = \bar{\bar{x}}_{[i-1]}$ is not excluded), such that $j \notin V(\bar{x}_{[i-1]})$ and $i \notin V(\bar{\bar{x}}_{[i-1]})$. Then LE implies that $f(Q^{N-i}_{[i-1]K}) = f(Q^{N-j}_{[i-1]K})$.

Now consider $Q^N_{[i]K}$. We know that $\exists\ x_{[i]} \in X(Q^N_{[i]K})$ such that $x^i_{[i]} = x^i$ and $x^j_{[i]} = 0$. Then $\exists x'_{[i]} \in X(Q^N_{[i]K})$ such that $x'^j_{[i]} = x^i$ and $x'^i_{[i]} = 0$. Thus, LE implies that $f(Q^{N-i}_{[i]K}) = f(Q^{N-j}_{[i]K})$.

If $Q^j_{[i-1]K} = Q^j_{[i]K}$ then $Q^N_{[i-1]K}$ and $Q^N_{[i]K}$ only differ in agent i's values, i. e., $Q^{N-i}_{[i-1]K} = Q^{N-i}_{[i]K}$. Then the anonymity of f implies that $f(Q^{N-i}_{[i-1]K}) = f(Q^{N-i}_{[i]K})$. Therefore, we get

$$f(Q^{N-j}_{[i-1]K}) = f(Q^{N-i}_{[i-1]K}) = f(Q^{N-i}_{[i]K}) = f(Q^{N-j}_{[i]K}).$$

Since this holds for $i = 1, \ldots, n-1$, we have $f(Q^{N-j}_{[0]K}) = f(Q^{N-j}_{[n-1]K})$, where $Q^{N-j}_{[0]K} = \theta^{N-j}_K$. Using a similar method for $\tilde{\theta}^{N-j}_K \in (\Theta^{N-j}_K)^+$, we get $f(\tilde{\theta}^{N-j}_K) = f(\tilde{Q}^{N-j}_{[0]K}) = f(\tilde{Q}^{N-j}_{[n-1]K})$, where $\tilde{Q}^N_{[i]K}(i = 1, \ldots, n-1)$ are defined similarly to $Q^N_{[i]K}(i = 1, \ldots, n-1)$, using $\tilde{\theta}^{N-j}_K$ instead of θ^{N-j}_K. Notice that $Q^{N-j}_{[n-1]K} = \tilde{Q}^{N-j}_{[n-1]K}$. Then $f(\theta^{N-j}_K) = f(\tilde{\theta}^{N-j}_K)$ follows, as required.□

Corollary 8 *If f is envyfree on $(\Theta^N_K)^+$, $\forall \theta^{N-j}_K$, $\tilde{\theta}^{N-j}_K \in (\Theta^{N-j}_K)^+$ such that $|V(x)| = |V(\tilde{x})|$, and $\forall i \in V(x)$, $\exists \tilde{i} \in V(\tilde{x})$ such that $x^i = \tilde{x}^{\tilde{i}}$ and $\theta^i_{x^i} = \tilde{\theta}^{\tilde{i}}_{\tilde{x}^{\tilde{i}}}$, where $x \in X(\theta^{N-j}_K)$, and $\tilde{x} \in X(\tilde{\theta}^{N-j}_K)$, we have $f(\theta^{N-j}_K) = f(\tilde{\theta}^{N-j}_K)$.*

Corollary 8 directly follows from Claim 1, since the agents can be relabeled.

Claim 2 *If f is envyfree on $(\Theta_K^N)^+$, $\forall \theta_K^{N-j}, \tilde{\theta}_K^{N-j} \in (\Theta_K^{N-j})^+$ such that $V(x) = V(\tilde{x})$, and $\theta_{x^i}^i = \tilde{\theta}_{\tilde{x}^i}^i$, $\forall i \in V(x)$, where $x \in X(\theta_K^{N-j})$, and $\tilde{x} \in X(\tilde{\theta}_K^{N-j})$, we have $f(\theta_K^{N-j}) = f(\tilde{\theta}_K^{N-j})$.*

Proof: Label the objects so that $x^i \bigcup \tilde{x}^i \neq \emptyset$. This can be done, for example, by the following method. Let the first winner have object 1 in both of her winning packages, the second winner have object 2 in both of her winning packages, etc. The labeling of the rest of the objects in the packages is arbitrary. Now define $\bar{\theta}_K^{N-j}$ as follows. Let $\bar{\theta}_{x^i}^i = \theta_{x^i}^i$ and $\bar{\theta}_{\tilde{x}^i}^i = \tilde{\theta}_{\tilde{x}^i}^i$, $\forall i \in V(x)$. If $x^i \subseteq p$ or $\tilde{x}^i \subseteq p$, let $\bar{\theta}_p^i = \theta_{x^i}^i = \tilde{\theta}_{\tilde{x}^i}^i$, otherwise let $\bar{\theta}_p^i = 0$, $\forall i \in V(x)$. Let $\bar{\theta}_p^i = 0$, $\forall i \notin V(x)$, $\forall p \in \mathcal{K}$. Then $\exists$ x and $\tilde{x} \in X(\bar{\theta}_K^{N-j})$. Since $\bar{\theta}_K^{N-j} \in (\Theta_K^{N-j})^+$, Claim 1 implies that for an envyfree f on $(\Theta_K^N)^+$, we have $f(\theta_K^{N-j}) = f(\bar{\theta}_K^{N-j}) = f(\tilde{\theta}_K^{N-j})$.□

Corollary 9 *If f is envyfree on $(\Theta_K^N)^+$ then $\forall \theta_K^{N-j}, \tilde{\theta}_K^{N-j} \in (\Theta_K^{N-j})^+$ such that $\{\theta_{x^i}^i \mid i \in V(x)\} = \left\{\tilde{\theta}_{\tilde{x}^i}^i \mid i \in V(\tilde{x})\right\}$ we have $f(\theta_K^{N-j}) = f(\tilde{\theta}_K^{N-j})$, where $x \in X(\theta_K^{N-j})$ and $\tilde{x} \in X(\tilde{\theta}_K^{N-j})$. That is, there exists a function $\hbar$ such that $f(\theta_K^{N-j}) = \hbar\left(\{\theta_{x^i}^i \mid i \in V(x)\}\right)$, $\forall \theta_K^{N-j} \in (\Theta_K^{N-j})^+$, $\forall x \in X(\theta_K^{N-j})$.*

Corollary 9 follows from Claim 1, Corollary 8 and Claim 2.

II.

Claim 3 *If $\hbar$ is envyfree on $(\Theta_K^N)^+$ then $\hbar$ can be written as a function of the sum of the optimal values in its argument, i. e., there exists a function h such that for any profile $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$ with a set of optimal values Φ, $\hbar(\Phi) = h\left(\sum_{\phi \in \Phi} \phi\right)$.*

Proof: Let Φ and Φ' be two finite sets of nonnegative numbers such that $\sum_{\phi\in\Phi}\phi = \sum_{\phi'\in\Phi'}\phi'$, where Φ and Φ' are sets of optimal values for $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$ and $\theta_K^{'N-j} \in (\Theta_K^{N-j})^-$, respectively. Let $|\Phi| = T$, $|\Phi'| = T'$ and $m = \max\{T, T'\}$. If $T > T'$ let $\phi'_{T'+1} = \phi'_{T'+2} = \cdots = \phi'_m = 0$. If $T' > T$ let $\phi_{T+1} = \phi_{T+2} = \cdots = \phi_m = 0$. Denote $K' \setminus \{0\}$ by $\bar{K}$. Let there be m^2 objects, i.e., $K' = \{0, 1, \ldots, m^2\}$. Let $Y = \{y_1, \ldots, y_m\}$ and $Y' = \{y'_1, \ldots, y'_m\}$ be two sets of packages such that

(1a) $|y_i| = m$, $i = 1, \ldots, m$,

(1b) $|y'_i| = m$, $i = 1, \ldots, m$,

(2a) $y_i \bigcap y_l = \emptyset$, $\forall\{i, l\} \in \{1, \ldots, m\} \times \{1, \ldots, m\}$,

(2b) $y'_i \bigcap y'_l = \emptyset$, $\forall\{i, l\} \in \{1, \ldots, m\} \times \{1, \ldots, m\}$,

(3a) $\bigcup_{i=1}^{m} y_i = \bar{K}$,

(3b) $\bigcup_{i=1}^{m} y'_i = \bar{K}$, and

(4) $\mid y_i \bigcap y'_l \mid = 1$, $\forall\{i, l\} \in \{1, \ldots, m\} \times \{1, \ldots, m\}$.

To see that Y and Y' satisfying conditions (1a)–(4) exist, we show how to construct them. Let $y_i = \{(i-1)m + 1, (i-1)m + 2, \ldots, im\}$, $i = 1, \ldots, m$. Then $|y_i| = m$, $i = 1, \ldots, m$, satisfying (1a), $y_i \bigcap y_l = \emptyset$, $\forall\{i, l\} \in \{1, \ldots, m\} \times \{1, \ldots, m\}$, satisfying (2a), and $\bigcup_{i=1}^{m} y_i = \bar{K}$, satisfying (3a). Let $y'_i = \{km + i \mid k = 0, \ldots, m-1\}$, $i = 1, \ldots, m$. Then $|y'_i| = m$, $i = 1, \ldots, m$, satisfying (1b), $y'_i \bigcap y'_l = \emptyset$, $\forall\{i, l\} \in \{1, \ldots, m\} \times \{1, \ldots, m\}$, satisfying (2b), and $\bigcup_{i=1}^{m} y'_i = \bar{K}$, satisfying (3b). Condition (4) is also satisfied, since $y_i \bigcap y'_l = \{(i-1)m + l\}$, so $|y_i \bigcup y'_l| = 1$, $\forall\{i, l\} \in \{1, \ldots, m\} \times \{1, \ldots, m\}$.

Now define $\theta_{K'}^{N'-j}$, where $|N'| = m+1$, as follows. Let $\theta_{y_i}^i = \phi_i$ and $\theta_{y'_i}^i = \phi'_i$, $i = 1, \ldots, m$. For $i = 1, \ldots, m$, *i)* if $y_i \subseteq p$ and $\left(y_i \bigcup y_i'\right) \not\subseteq p$, let $\theta_p^i = \phi_i$,

ii) if $y_i' \subseteq p$ and $\left(y_i \bigcup y_i'\right) \not\subseteq p$, let $\theta_p^i = \phi_i'$,

iii) if $\left(y_i \bigcup y_i'\right) \subseteq p$, let $\theta_p^i = \max\{\phi_i, \phi_i'\}$, and

iv) if $y_i \not\subseteq p$ and $y_i' \not\subseteq p$, let $\theta_p^i = 0$.

Given conditions (1a)–(4), $\exists\ x, x' \in X(\theta_{K'}^{N'-j}$ such that $x^i = y_i$ and $x'^i = y_i'$ for $i = 1, \ldots, m$. Since f and thus $\hbar$ are single-valued, this implies that for an envyfree $\hbar$ on $(\Theta_K^N)^+$, $\hbar(\Phi) = \hbar(\Phi')$ must hold. Thus, there exists a function h such that $\hbar(\Phi) = h\left(\sum_{\phi\in\Phi} \phi\right)$.□

Remark: The proof of Claim 3 may seem to require that if a Groves mechanism $(G(\mathbf{f}), \theta_K^N)$ is envyfree then (G(**f**), $\theta_{K'}^{N'}$) is also envyfree, for arbitrary $n' \geq 2$ and $k' \geq 3$, given that we defined in the proof $\theta_{K'}^{N'-j}$, where $|N'| = m + 1$ and $|K'| = m^2 + 1$. However, this is not the case. In part I. of the necessity proof we showed that if f is envyfree on $(\Theta_K^N)^+$, then f will only depend on $\{\theta_{x^i}^i \mid i \in V(x)\}, \forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-, \forall x \in X(\theta_K^{N-j})$. Thus, we need m to be no less than the number of the winners, T or T', which holds, since $m = \max\{T, T'\}$. Since an envyfree f does not depend on the values of the losers, we can have an arbitrary number of losers. In the proof it is $n - T - 1$ for θ_K^{N-j} and $n - T' - 1$ for $\theta_K'^{N-j}$. As for the number of the objects, part I. of the necessity proof also impies that $k - 1 \geq |V(x)|$. However, it need not matter how many objects the winning packages consist of. In the proof of Claim 3, $|K' - 1| = m^2 \geq m = \max\{T, T'\}$, so this condition is also met.

III.

Claim 4 *If* f *is envyfree on* $(\Theta_K^N)^+$ *and* $f(\theta_K^{N-j}) = h(W(\theta_K^{N-j}))$ *then* $0 \leq h(\varphi + d) - h(\varphi) \leq d$, $\forall \varphi \geq 0$, $\forall d \geq 0$.

Proof: Fix $\varphi \geq 0$ and $d \geq 0$. Define $\theta_K^N \in (\Theta_K^N)^+$ as follows. Let $\theta_{pK}^j = \varphi + d$ and

$\theta^i_{p_K} = \varphi$, where p_K is the package that contains all $k-1$ objects. Let all the other values be zeros for i and $j \in N$. Let all the valus be zeros for all the other agents in N. Then $\exists\ x \in X(\theta^N_K)$ such that $x^j = p_K$ and $x^i = 0$, and $\exists\ \bar{x} \in X(\theta^{N-j}_K)$ such that $\bar{x}^i = p_K$. Also, $W(\theta^{N-j}_K) = \varphi$, and $W(\theta^{N-i}_K) = \varphi + d$. Thus, using the results in I. and II., the GC requires that $\theta^j_{x^j} - \theta^i_{x^j} \geq h\left\{W(\theta^{N-i}_K)\right\} - h\left\{W(\theta^{N-j}_K)\right\}$, or, equivalently, that $\varphi + d - \varphi \geq h(\varphi + d) - h(\varphi)$. Furthermore, LE implies that $h(\varphi + d) \geq h(\varphi)$, since $i \notin V(x)$. In sum, $0 \leq h(\varphi + d) - h(\varphi) \leq d$, as required.□

Proof of Proposition 18

Definitions and lemmas for the sufficiency proof

Definition 46 A *loop* with respect to $\theta_K^N \in (\Theta_K^N)^-$ is a sequence of agents, $(v_1, \ldots, v_t), t \leq n$, such that $\exists\ x$ and $\bar{x} \in X(\theta_K^N)$ with $x^{v_j} = (v_j) \neq 0$, $j = 1, \ldots, t$, and $\bar{x}^{v_j} = (v_{j+1})$, $j = 1, \ldots, t-1$, $\bar{x}^{v_t} = (v_1)$.

Definition 47 Let $n > k$. An $x-$ *loop* with respect to $\theta_K^N \in X(\theta_K^N)$ is a sequence of agents, $(v_1, \ldots, v_t)$, $t \leq n$, such that $\exists\ x$ and $\bar{x} \in X(\theta_K^N)$ with $x^{v_j} = (v_j) \neq 0$, $j = 1, \ldots, t$, and $\bar{x}^{v_j} = (v_{j+1})$, $j = 1, \ldots, t-1$. Furthermore, agent v_t is a loser under $\bar{x}$, or gets an object that is unassigned under x, and object (v_1) is unassigned under $\bar{x}$ or assigned to an agent under $\bar{x}$ who is a loser under x.

Lemma 12 *Take* $\theta_K^N \in (\Theta_K^N)^-$ *and* $j \in N$. *If* $\exists\ x \in X(\theta_K^N)$ *such that* $x^j = 0$, *then* $\exists\ \bar{x} \in X(\theta_{K-p}^N)$ *such that* $\bar{x}^j = 0$, $\forall p \in K$.

Proof: Fix $\theta_K^N \in (\Theta_K^N)^-$ and $j \in N$ such that $x^j = 0$ for $x \in X(\theta_K^N)$. Fix $p \in K$. Let $\bar{x} \in X(\theta_{K-p}^N)$. If $\bar{x}^j = 0$, we are done. If $\bar{x}^j \neq 0$, then agent j gets a object, say $(v_1) \neq 0$ under $\bar{x}$. Then $\theta_{(v_1)}^j = 0$. If $\theta_{(v_1)}^j = 0, \exists \bar{\bar{x}} \in X(\theta_{K-p}^N)$ such that $\bar{\bar{x}}^j = 0$. If $\theta_{(v_1)}^j > 0$ then (v_1) is assigned under x, otherwise j would have got it. So object (v_1) is assigned to agent $v_1 \neq j$ under x. Agent v_1 is either a loser under $\bar{x}$ or gets an object, $(v_2) \neq 0$. Object (v_2) is either unassigned under x or assigned to an agent, say $v_2 \notin \{v_1, j\}$. Agent v_2 is either a loser under $\bar{x}$ or gets an object, $(v_3) \neq 0$, etc. Repeating the same argument, we find an agent, v_t, $(1 \leq t \leq n-1)$, who is a loser under $\bar{x}$, or assigned an object that is unassigned under x. However, given that $x \in X(\theta_K^N)$, $(v_1, \ldots, v_t)$ is an x-loop with respect to θ_{K-p}^N. Thus, $\exists\ \bar{\bar{x}} \in X(\theta_{K-p}^N)$ such that $\bar{\bar{x}}^j = 0.\square$

Lemma 13 *Let* $\theta_K^N \in (\Theta_K^N)^-$ *and* $j \in N$. *If* $\exists\; x \in X(\theta_K^N)$ *such that* $x^j = (j)$, *then* $\exists\; \bar{x} \in X(\theta_{K+(j)}^N)$ *such that* $\bar{x}^j = (j)$.

Proof: Let $\theta_K^N \in (\Theta_K^N)^-$, and $x \in X(\theta_K^N)$ such that $x^i = (i)$, $\forall i \in N$. Fix $j \in N$. Let $\tilde{x} \in X(\theta_{K+(j)}^N)$. Let *original* refer to x in this proof. If $(j) = 0$ then the lemma holds trivially. So assume that $(j) \neq 0$.

We know that either one object (j) is unassigned, assigned to an original loser, or both object (j)'s are assigned to original winners under $\tilde{x}$.

(a) If one object (j) is unassigned under $\tilde{x}$ then $W(\theta_{K+(j)}^N) = W(\theta_K^N)$. Thus, given that $x \in X(\theta_K^N)$, $\exists\; \bar{x} \in X(\theta_{K+(j)}^N)$ with $\bar{x}^j = (j)$.

(b) If one object (j) is assigned to an original loser, agent l, under $\tilde{x}$ then $W(\theta_{K+(j)}^N) = W(\theta_K^{N-l}) + \theta_{(j)}^l = W(\theta_K^N) + \theta_{(j)}^l$. Thus, given that $x \in X(\theta_K^N)$, $\exists\; \bar{x} \in X(\theta_{K+(j)}^N)$ with $\bar{x}^j = (j)$.

(c) If both object (j)'s are assigned to original winners under $\tilde{x}$, then suppose one of them is assigned to agent v_1 who is an original winner.

Then object (v_1) may be unassigned, assigned to an original loser, or assigned to an original winner, agent v_2, under $\tilde{x}$. Since there are n agents and k objects (excluding the null object), repeating this argument we find either $t \leq n$ such that object (v_t) is unassigned under $\tilde{x}$, or $t \leq k$ such that agent v_t is an original loser. Thus, there exists a sequence of agents, $(v_1, \ldots, v_t)$, $t \leq \min\{n, k\}$, such that $\tilde{x}^{v_{i+1}} = (v_i)$, $i = 1, \ldots, t-1$, and $\tilde{x}^{v_1} = (j)$. Moreover, $\tilde{x}^i = (i)$, $\forall i \notin \{v_1, \ldots, v_t\}$. Therefore, there exists $l \in \{1, \ldots, t\}$ such that $v_l = j$, otherwise the originally assigned object (j) is j's assignment under $\tilde{x}$. In fact, $j \neq v_t$, since j is an original winner, and (j) is assigned under $\tilde{x}$. However, it implies that $(j, v_l, v_{l-1}, \ldots, v_1)$ is a loop wit respect to $\theta^N_{K+(j)}$. Therefore, $\exists\ \bar{x} \in X(\theta^N_{K+(j)})$ with $\bar{x}^j = (j)$.□

Lemma 14 *Let $\theta^N_K \in (\Theta^N_K)^-$, $x \in X(\theta^N_K)$, and $(p) \in K$ such that $(p) \neq 0$ and (p) is unassigned under x. Let* original *refer to x. Then $\exists \bar{x} \in X(\theta^N_{K-(p)})$ such that all but one original winners are assigned some originally assigned object under $\bar{x}$, and one original winner is either a loser under $\bar{x}$ or gets assigned an originally unassigned object.*

Proof: Let $\theta^N_K \in (\Theta^N_K)^-, x \in X(\theta^N_K)$. Let $x^i = (i), \forall i \in N$, and $(p) \in K$ such that $(p) \neq 0$ and is assigned under x, where *original* refers to x. If p gets an originally unassigned object, or p is a loser under some optimal assignment $\bar{x} \in X(\theta^N_{K-p})$, we are done, since then $W(\theta^N_{K-(p)}) = W(\theta^N_K) - \theta^p_{(p)} + \theta^p_u$, where u is either an originally assigned object or $u = 0$. If p gets an originally assigned object, (v_1), then v_1 is either a loser under some optimal assignment $\bar{x} \in X(\theta^N_{K-(p)})$, or v_1 gets assigned an originally unassigned object, or gets an originally assigned object. Repeating the same argument, we find a sequence of original winners, $(v_1, \ldots, v_t), 1 \leq t \leq n-1$, such that for some $\bar{x} \in X(\Theta^N_{K-(p)}), \bar{x}^{v_i} = (v_{i+1})$ for $i = 1, \ldots, t-1, \bar{x}^{v_t} = u$, where $u = 0$ or u is an originally unassigned object, and $\forall i \notin (v_1, \ldots, v_t), \bar{x}^i = (i)$.□

Lemma 15 *Let $\theta^N_K \in (\Theta^N_K)^-$, $x \in X(\theta^N_K)$, and $(j) \in K$ such that $(j) \neq 0$ and (p) is unassigned under x. Let* original *refer to x. Then $\exists \bar{x} \in X(\theta^N_{K+(j)})$ such that one of the following two cases holds for it.*

a) The winners are the original winners and the assigned objects are either the originally assigned objects, or the originally assigned objects with one exception and the extra object

(j).

b) The winners are the original winners plus one original loser, the assigned objects are the originally assigned objects and the extra object (j).

Proof: Follows from Lemma 14.

Sufficiency

Let $f(\theta_K^{N-j}) = h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right\}\right)$, $\forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-$, such that $h\left(\left\{W(\theta_{K-p}^{N-i}) \mid p \in K\right\}\right) - h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right\}\right) \leq \max_{p \in K}\left\{W(\theta_{K-p}^{N-i}) - W(\theta_{K-p}^{N-j})\right\}$, $\forall \theta_K^N \in (\Theta_K^N)^-, \forall i, j \in N$, if $j \in V(x_{\theta_K^N})$, where $x_{\theta_K^N} \in X(\theta_K^N)$, and $h\left(\left\{W(\theta_{K-p}^{N-i}) \mid p \in K\right\}\right) - h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right\}\right) \leq 0$ otherwise. According to the GC, f is envyfree on $(\Theta_K^N)^-$ if $\theta_{(j)}^j - \theta_{(j)}^i \geq f(\theta_K^{N-i}) - f(\theta_K^{N-j})$, where $x_{\theta_K^N}^j = (j)$. If j is a loser under $x_{\theta_K^N}$ then $\theta_{(j)}^j = \theta_{(j)}^i = 0$, thus, the GC holds in this case. Given the assumptions on f and h, it is enough to show that $\theta_{(j)}^j - \theta_{(j)}^i \geq \max_{p \in K}\left\{W(\theta_{K-p}^{N-i}) - W(\theta_{K-p}^{N-j})\right\}$, i. e., that

$$W(\theta_{K-p}^{N-j}) + \theta_{(j)}^j \geq W(\theta_{K-p}^{N-i}) + \theta_{(j)}^i, \tag{A.6}$$

$\forall \theta_K^N \in (\Theta_K^N)^-, \forall i, j \in N, \forall p \in K$ if $\exists x_{\theta_K^N} \in X(\theta_K^N)$ such that $j \in V(x_{\theta_K^N})$. Let $\theta_K^N \in (\Theta_K^N)^-$ and let $x_{\theta_K^N} \in X(\theta_K^N)$ such that $x_{\theta_K^N}^j = (j)$, $\forall j \in N$. Fix $i, j \in N$, and $p \in K$ such that $j \in V(x_{\theta_K^N})$.

1. If j is a winner under $x_{\theta_K^N}$ and $p = (j)$, we have $W(\theta_{K-(j)}^{N-j}) + \theta_{(j)}^j = W(\theta_K^N) \geq W(\theta_{K-(j)}^{N-i}) + \theta_{(j)}^i$, thus, (A.6) holds.

2. If j is a winner under $x_{\theta_K^N}$ and $p = 0$, then $W(\theta_{K-p}^{N-j}) = W(\theta_K^{N-j})$ and Lemma 13 implies that $W(\theta_K^{N-j}) + \theta_{(j)}^j = W(\theta_{K+(j)}^N)$. Since $W(\theta_{K+(j)}^N) \geq W(\theta_K^{N-i}) + \theta_{(j)}^i =$

$W(\theta^{N-i}_{K-p}) + \theta^i_{(j)}$, (A.6) holds in this case, too.

3. It remains to show that (A.6) holds if j is a winner under $x_{\theta^N_K}$, and $p \notin \{(j), 0\}$. Inequality (A.6) holds, in general, if $\exists\, x_{\theta^N_{K-p+(j)}} \in X(\theta^N_{K-p+(j)})$ such that $x^j_{\theta^N_{K-p+(j)}} = (j)$, since it implies that $W(\theta^{N-j}_{K-p}) + \theta^j_{(j)} = W(\theta^N_{K-p+(j)})$, and $W(\theta^N_{K-p+(j)}) \geq W(\theta^{N-i}_{K-p}) + \theta^i_{(j)}$.

   Fix $x_{\theta^N_{K-p}} \in X(\theta^N_{K-p})$. If $x^j_{\theta^N_{K-p}} = (j)$ then $\exists\, x_{\theta^N_{K-p+(j)}} \in X(\theta^N_{K-p+(j)})$ such that $x^j_{\theta^N_{K-p+(j)}} = (j)$, by Lemma 13. So assume that $x^j_{\theta^N_{K-p}} \neq (j)$. If object p is unassigned under $x_{\theta^N_K}$ then $W(\theta^N_{K-p}) = W(\theta^N_K)$ and $\exists\, x'_{\theta^N_{K-p}} \in X(\theta^N_{K-p})$ such that $x'^j_{\theta^N_{K-P}} = (j)$, which, in turn, implies that $\exists\, x'_{\theta^N_{K-p+(j)}} \in X(\theta^N_{K-p+(j)})$ with $x'^j_{\theta^N_{K-p+(j)}} = (j)$, given Lemma 13. If p is assigned to some agent under $x_{\theta^N_K}$, let $(p) = p$, so that object (p) is assigned to agent p under $x_{\theta^N_K}$. If $x^p_{\theta^N_{K-(p)}} = 0$, then $\exists\, x'_{\theta^N_{K-(p)}} \in X(\theta^N_{K-(p)})$ such that $x'^i_{\theta^N_{K-(p)}} = (i)$, $\forall i \in N$, $i \neq p$, and so Lemma 13 implies that $\exists\, x'_{\theta^N_{K-(p)+(j)}} \in X(\theta^N_{K-(p)+(j)})$ such that $x'^j_{\theta^N_{K-(p)+(j)}} = (j)$. If p gets a object, say (v_1), under $x_{\theta^N_{K-(p)}}$, such that $\theta^p_{(v_1)} > 0$, and (v_1) is unassigned under $x_{\theta^N_K}$ then $\exists\, x'_{\theta^N_{K-(p)}} \in X(\theta^N_{K-(p)})$ such that $x'^j_{\theta^N_{K-(p)}} = (j)$, and thus Lemma 13 implies the required result again. If agent v_1 is assigned (v_1) under $x_{\theta^N_K}$, v_1 may be a loser under $x_{\theta^N_{K-(p)}}$, v_1 may get a object, (v_2) under $x_{\theta^N_{K-(p)}}$ that is unassigned under $x_{\theta^N_K}$, or v_1 may get a object, (v_2), that is assigned to agent v_2 under $x_{\theta^N_K}$, etc. Therefore, we can find a sequence of agents, $(p, v_1, \ldots, v_t)$, $0 \leq t \leq n-1$, with $x^p_{\theta^N_{K-(p)}} = (v_1)$, $x^{v_i}_{\theta^N_{K-(p)}} = (v_{i+1})$, $i = 1, \ldots, t-1$, such that $\forall i \notin \{p, v_1, \ldots, v_t\}$, $x^i_{\theta^N_{K-(p)}} = (i)$. Since $x^j_{\theta^N_{K-(p)}} \neq (j)$, $\exists l$, $1 \leq l \leq t$ with $v_l = j$. We also know that in this case either v_t gets a object under $x_{\theta^N_{K-(p)}}$ that is unassigned under $x_{\theta^N_K}$, or v_t is a loser under $x_{\theta^N_{K-(p)}}$.

Fix $x_{\theta^N_{K-(p)+(j)}} \in X(\theta^N_{K-(p)+(j)})$. Consider the following cases regarding $x_{\theta^N_{K-(p)+(j)}}$.

(a) If one object (j) is unassigned under $x_{\theta^N_{K-(p)+(j)}}$, or it is assigned to an agent who is a loser under $x_{\theta^N_K}$, then $(j, v_{l+1}, v_{l+2}, \ldots, v_t)$ is an x-loop with respect to $\theta^N_{K-(p)+(j)}$, and so $\exists\, x'_{\theta^N_{K-(p)+(j)}} \in X(\theta^N_{K-(p)+(j)})$ such that $x'^j_{\theta^N_{K-(p)+(j)}} = (j)$.

(b) If both object (j)'s are assigned under $x_{\theta^N_{K-(p)+(j)}}$ to agents who are winners under $x_{\theta^N_K}$ then Lemma 14 and Lemma 15 imply that $\exists x'_{\theta^N_{K-(p)+(j)}} \in X(\theta^N_{K-(p)-(j)})$ such that the set of winners and the set of assigned objects under it are described by one of the five cases indicated in the following table. (*Original* refers here to $x_{\theta^N_K}$.)

In the table, V denotes the set of original winners with the exception of agent p. Let $v \in V$, such that $v = j$ is possible. (V) denotes the set of originally assigned objects, except for object (p). Let $(w) \in (V)$, such that $v = w$ is possible. Agent l is an original loser, and object (u) is an originally unassigned object. Package 0 is denoted by $(\emptyset)$ in the table, and a dummy agent who is "assigned" any unassigned object is denoted by $\emptyset$. The set of winners under $x'_{\theta^N_{K-(p)+(j)}} \in X(\theta^N_{K-(p)+(j)})$ is described in the first column, and the set of assigned objects is described in the second column for the five possible cases. The table shows that, for each of the five cases, $x'_{\theta^N_{K-(p)+(j)}}$ can be described by a permutation of the set of slots, $(V) \bigcup \{(j), (\emptyset), (u)\}$, for a fixed order of the set of agents, $V \bigcup \{j, l, \emptyset\}$. The fixed assignments are indicated in the third column, and the fourth column contains a set of agents and a set of objects for each case, such that any assignment resulting from a permutation is possible. Note that since we already ruled out the case where (j) is assigned to an original loser, in cases

4 and 5 l cannot get (j). However, all the other permutations are possible in these cases. Notice also that $\theta^l_u = 0$, so that l is essentially a loser in case 1. Also note that v is not necessarily the same agent in different cases, that is, it's a generic element of V. Similarly, (w) is a generic element of (V).

Case	Winners	Assigned objects	Fixed assignments	Permutations
1	$V - v + p$	$(V) - (w) + (j)$	$\begin{bmatrix} v \\ (\emptyset) \end{bmatrix} \begin{bmatrix} l \\ (u) \end{bmatrix} \begin{bmatrix} \emptyset \\ (w) \end{bmatrix}$	$V \setminus v, p$ (V) $\setminus(w), (j)$
2	$V + p$	$(V) - (w) + (j)$	$\begin{bmatrix} l \\ (\emptyset) \end{bmatrix} \begin{bmatrix} \emptyset \\ (w) \end{bmatrix}$	V, p (V) $\setminus(w), (j), (u)$
3	$V + p$	$(V) + (j)$	$\begin{bmatrix} l \\ (\emptyset) \end{bmatrix} \begin{bmatrix} \emptyset \\ (u) \end{bmatrix}$	V, p (V), (j)
4	$V - v + p + l$	$(V) + (j)$	$\begin{bmatrix} v \\ (\emptyset) \end{bmatrix} \begin{bmatrix} \emptyset \\ (u) \end{bmatrix}$	$V \setminus v, p, l$ (V), (j)
5	$V + l + p$	$(V) + (j) + (u)$	$\begin{bmatrix} \emptyset \\ (\emptyset) \end{bmatrix}$	V, p, l (V), (j), (u)

Let $\mathcal{V} = V \bigcup \{p, l, \emptyset\}$. It is clear that the optimal assignment $x_{\theta^N_K}$ is given by $x^v_{\theta^N_K} = (v)$, $\forall v \in V, x^p_{\theta^N_K} = (p), x^l_{\theta^N_K} = (\emptyset)$, and $x^{\emptyset}_{\theta^N_K} = (u)$. Take the sequence of agents $(p, a_1, \ldots, a_t)$, where $a_i \in \mathcal{V}$ for $i = 1, \ldots, t$ such that $x'^{p}_{\theta^N_{K-(p)+(j)}} = x^{a_i}_{\theta^N_K}, x'^{a_i}_{\theta^N_{K-(p)+(j)}} = x^{a_{i+1}}_{\theta^N_K}$ for $i = 1, \ldots, t-1$, and $x'^{a_t}_{\theta^N_{K-(p)+(j)}} = (j)$.

Now take the complement set $\mathcal{V} \setminus \{p, a_1, \ldots, a_t\}$. Clearly, this set includes j. Given the way we choose $\{p, a_1, \ldots, a_t\}$, there exists a permutation of the complement set such that the objects that the agents in this complement set are assigned under $x'_{\theta^N_{K-(p)+(j)}}$ are assigned as originally. Therefore, this assignment must be optimal, too, which implies that $\exists x''_{\theta^N_{K-(p)+(j)}} \in X(\theta^N_{K-(p)+(j)})$ such that $x''_{\theta^N_{K-(p)+(j)}} = (j)$ for all five cases.□

Definitions and lemmas for the necessity proof

Lemma 14 implies that, $\forall \theta^{N-i}_K \in (\Theta^{N-j}_K)^-$, $\forall x \in X(\theta^{N-i}_K)$, where $x^i = (i)$, $\forall i \in N \backslash \{j\}$, and $\forall (p_1) \in K \backslash \{0\}$ such that (p_1) is assigned under x, there exists a sequence of agents, $(p_1, \ldots, p_t)$, $1 \leq t \leq n-1$, such that $\bar{x}^{p_l} = (p_{l+1})$ for $l = 1, \ldots, t-1$, where $\bar{x} \in X(\theta^{N-j}_{K-(p_1)})$, p_{l+1} is assigned under x for $l = 1, \ldots, t-1$, p_t is either a loser under $\bar{x}$, or p_t is assigned a object under $\bar{x}$ that is unassigned originally, and $\forall i \notin \{p_1, \ldots, p_t\}$, $\bar{x}^i = (i)$. Call such a sequence of agents for $(p_1) \in K \setminus \{0\}$ and $x \in X(\theta^{N-j}_K)$ a *(p_1)-chain under x*, denoted by $(p_1) \sim x$ if no agent is included more than once in the sequence, and if $\sum_{l=1}^{t-1} \left(\theta^{p_l}_{(p_{l+1})} + \theta^{p_{l+1}}_{(p_{l+1})}\right) + \theta^{p_t}_{(p_{t+1})} > 0$ for $t > 1$. Call $\bar{x}$ *compatible* with $(p_1) \sim x$.

Lemma 16 *Let $\theta^{N-j}_K \in (\Theta^{N-j}_K)^-$, $x \in X(\theta^{N-j}_K)$, and let a (p_1)-chain under x be $(p_1) \sim x$ such that $(p_1) \sim x = (p_1, \ldots, p_t)$, where $(p_1) \in K \setminus \{0\}$ and (p_1) is assigned under x. Let $x^i = (i)$, $\forall i \in N \setminus \{j\}$. Then $\forall p_l \in (p_1, \ldots, p_t), \exists \bar{x} \in X(\theta^{N-j}_{K-(p_l)})$ such that $\bar{x}^{p_l} = (p_{l+1})$. $(p_{l+1}) \in \mathcal{T}_{p_l}(P_{p_l}(x), \theta^{N-j}_K)$.*

Proof: Let the assumptions of the lemma hold. Take p_l such that $1 \leq l \leq t$. Let a (p$_l$)-chain under x be $(q_1, q_2, \ldots, q_{t'})$, where $t' \leq n$, and $q_1 = p_l$. If $\{p_1, p_2, \ldots, p_{l-1}\} \bigcap \{q_1, q_2, \ldots, q_{t'}\} = \emptyset$, then $\exists\, \bar{x} \in X(\theta^{N-j}_{K-(p_l)})$ such that $\bar{x}^{p_l} = (p_{l+1})$. If $\{p_1, p_2, \ldots, p_{l-1}\} \bigcap \{q_1, q_2, \ldots, q_{t'}\} \neq \emptyset$ then let $p_{l'}$ be the first agent in $(p_1) \sim x$ such that $\exists\, i$, $2 \leq i \leq t'$, with $p_{l'} = q_i$, where $1 \leq l' \leq l-1$. Let p denote the object that is assigned to p_t under

$\tilde{x} \in X(\theta^{N-j}_{K-(p_1)})$ such that $\tilde{x}$ is compatible with $(p_1) \sim x$. Let q' denote the object that is assigned to $q_{t'}$ under $\tilde{x}' \in X(\theta^{N-j}_{K-q_1})$ such that $\tilde{x}'$ is compatible with $(q_1) \sim x$. Given that $\tilde{x} \in X(\theta^{N-j}_{K-(p_1)})$, we have $\left(\theta^{p_{l'}}_{(p_{l'+1})} - \theta^{p_{l'}}_{(p_{l'})}\right) + \left(\theta^{p_{l'+1}}_{(p_{l'+2})} - \theta^{p_{l'+1}}_{(p_{l'+1})}\right) + \cdots + \left(\theta^{p_t}_{p} - \theta^{p_t}_{(p_t)}\right) \geq \left(\theta^{q_i}_{(q_{i+1})} - \theta^{q_i}_{(q_i)}\right) + \left(\theta^{q_{i+1}}_{(q_{i+2})} - \theta^{q_{i+1}}_{(q_{i+1})}\right) + \cdots + \left(\theta^{q_{t'}}_{q'} - \theta^{q_{t'}}_{(q_{t'})}\right)$. Given that $x \in X(\theta^{N-j}_{K})$, we also have $\left(\theta^{p_{l'}}_{(p_{l'+1})} - \theta^{p_{l'}}_{(p_{l'})}\right) + \left(\theta^{p_{l'+1}}_{(p_{l'+2})} - \theta^{p_{l'+1}}_{(p_{l'+1})}\right) + \cdots + \left(\theta^{p_l}_{(q_2)} - \theta^{p_l}_{(q_1)}\right) + \left(\theta^{q_2}_{(q_3)} - \theta^{q_2}_{(q_2)}\right) + \cdots + \left(\theta^{q_{i-1}}_{(q_i)} - \theta^{q_{i-1}}_{(q_{i-1})}\right) \leq 0$.

If we add the above two inequalities, we get $\left(\theta^{p_{l'}}_{(p_{l'+1})} - \theta^{p_{l'}}_{(p_{l'})}\right) + \left(\theta^{p_{l'+1}}_{(p_{l'+2})} - \theta^{p_{l'+1}}_{(p_{l'+1})}\right) + \cdots + \left(\theta^{p_t}_{p} - \theta^{p_t}_{(p_t)}\right) \geq \left(\theta^{p_{l'}}_{(p_{l'+1})} - \theta^{p_{l'}}_{(p_{l'})}\right) + \left(\theta^{p_{l'+1}}_{(p_{l'+2})} - \theta^{p_{l'+1}}_{(p_{l'+1})}\right) + \cdots + \left(\theta^{p_l}_{(q_2)} - \theta^{p_l}_{(q_1)}\right) + \left(\theta^{q_2}_{(q_3)} - \theta^{q_2}_{(q_2)}\right) + \cdots + \left(\theta^{q_{i-1}}_{(q_i)} - \theta^{q_{i-1}}_{(q_{i-1})}\right) + \left(\theta^{q_i}_{(q_{i+1})} - \theta^{q_i}_{(q_i)}\right) + \cdots + \left(\theta^{q_{t'}}_{q'} - \theta^{q_{t'}}_{(q_{t'})}\right)$, i.e., $\left(\theta^{p_l}_{(p_{l+1})} - \theta^{p_l}_{(p_l)}\right) + \cdots + \left(\theta^{p_t}_{(p)} - \theta^{p_t}_{(p_t)}\right) \geq \left(\theta^{p_l}_{(q_2)} - \theta^{p_l}_{(q_1)}\right) + \cdots + \left(\theta^{q_i}_{(q_{i+1})} - \theta^{q_i}_{(q_i)}\right) + \cdots + \left(\theta^{q_{t'}}_{p'} - \theta^{q_{t'}}_{(q_{t'})}\right)$. This implies that $\exists\ \bar{x} \in X(\theta^{N-j}_{K-(p_l)})$ such that $\bar{x}^{p_l} = (p_{l+1})$.□

Lemma 17 *Let* $\tilde{x} \in X(\theta^{N-j}_{K-x^i})$, *where* $x \in X(\theta^{N-j}_{K}), i \in N \setminus \{j\}$, *and* $\theta^{N-j}_{K} \in (\Theta^{N-j}_{K})^{-}$. *Then* $\exists x \in X(\theta^{N}_{K})$, *where* $\theta^j = \theta^i$ *such that* $\bar{x}^i = \tilde{x}^i$ *and* $\bar{x}^j = x^i$.

Proof: Let the assumptions of the lemma hold. We have $W(\theta^{N-j-i}_{K-x^i-\tilde{x}^i}) + \theta^i_{\tilde{x}^i} = W(\theta^{N-j}_{K-x^i})$. Since no agent is included in an x^i-chain more than once, it follows from Lemma 16 that $W(\theta^{N-j-i}_{K-x^i-\tilde{x}^i}) + \theta^i_{x^i} = W(\theta^{N-j}_{K-\tilde{x}^i})$. But then $W(\theta^{N-j-i}_{K-x^i-\tilde{x}^i}) + \theta^j_{x^i} = W(\theta^{N-i}_{K-\tilde{x}^i})$. This implies that $W(\theta^{N-j-i}_{K-x^i-\tilde{x}^i}) + \theta^i_{\tilde{x}^i}\theta^j_{x^i} = W(\theta^{N}_{K})$ as required.□

Definition 48 Agent i is an L_0-agent with respect to $\theta^{N-j}_{K} \in (\Theta^{N-j}_{K})^{-}$ if $W(\theta^{N}_{K}) = W(\theta^{N-j}_{K})$, where $\theta^j_K = \theta^i_K$. Agent i is an L_*-agent with respect to $\theta^{N-j}_{K} \in (\Theta^{N-j}_{K})^{-}$ if $W(\theta^{N}_{K}) > W(\theta^{N-j}_{K})$, where $\theta^j_K = \theta^i_K$.

Definition 49 Let $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$, $x \in X(\theta_K^{N-j})$, and $i \in N \setminus \{j\}$. Then agent i's *substitute set* with respect to (x^i, θ_K^{N-j}) is $\mathcal{T}_i(x^i, \theta_K^{N-j}) = \left\{x^i \in K \mid \bar{x} \in X(\theta_K^N), \theta_K^j = \theta_K^i, \text{ and} \bar{x}^j = x^i\right\}$.

Lemma 18 *Let $\theta_K^{N-j} \in (\Theta_K^{N-j})^-, x \in X(\Theta_K^{N-j})$, and let a (p_1)-chain under x be $(p_1) \sim x$ such that $(p_1) \sim x = (p_1, \dots, p_t)$, where $(p_1) \in K \setminus \{0\}$ and (p_1) is assigned under x. Let $x^i = (i), \forall i \in N \setminus \{j\}$. Then $\forall p_l \in \{p_1, \dots, p_t\}, (p_{l+1}) \in \mathcal{T}_{p_l}(x^{p_l}, \theta_K^{N-j})$.*

Proof: Follows from Lemma 16 and Lemma 17.

Definition 50 Let $x \in X(\theta_K^{N-j})$ for $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$. Let $\Gamma(x, \theta_K^{N-j}) = \{\gamma_{x,\theta_K^{N-j}}\}$, where $\gamma_{x,\theta_K^{N-j}} = \{T_i \mid T_i \in \mathcal{T}_i(x^i, i \in N \setminus \{j\}$.Let $T_i(\gamma_{x,\theta_K^{N-j}})$ be the object that is specified for agent i by $\gamma_{x,\theta_K^{N-j}}$.

Definition 51 Let $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$ and $x \in X(\theta_K^{N-j})$. The *substitute-reduced form* of θ_K^{N-j} with respect to $\gamma_{x,\theta_K^{N-j}} \in \Gamma(x, \theta_K^{N-j})$, $\bar{\theta}_K^{N-j}$, is defined as follows. For all $i \in N \setminus \{j\}$, $\bar{\theta}^i_{x^i} = \theta^i_{x^i}$, $\bar{\theta}^i_{T_i(\gamma_{x,\theta_K^{N-j}})} = \theta^i_{T_i(\gamma_{x,\theta_K^{N-j}})}$, and all the other values are zeros in $\bar{\theta}_K^{N-j}$.

Denote the substitute-reduced form of θ_K^{N-j} with respect to $\gamma_{x,\theta_K^{N-j}}$ by $R_{\gamma_{x,\theta_K^{N-j}}}(\theta_K^{N-j})$.

Lemma 19 *If f is envyfree on $(\Theta_K^N)^-$ then $\forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-$, $\forall x \in X(\theta_K^{N-j})$, $\forall \gamma_{x,\theta_K^{N-j}} \in \Gamma(x, \theta_K^{N-j})$, $f(\theta_K^{N-j}) = f\left(R_{\gamma_{x,\theta_K^{N-j}}}(\theta_K^{N-j})\right)$.*

Proof: Let $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$, $x \in X(\theta_K^{N-j})$, and $\gamma_{x,\theta_K^{N-j}} \in \Gamma(x, \theta_K^{N-j})$. Let $Q_{[0]K}^{N-j} = \theta_K^{N-j}$. Let $j = n$ and let $\bar{\theta}_K^{N-j} = R_{\gamma_{x,\theta_K^{N-j}}}(\theta_K^{N-j})$. Define $\{Q_{[i]K}^{N-j} \mid i = 1, \dots, n-1\}$

as follows. Let $Q^t_{[i]K} = \bar{\theta}^t_K$ for $i = 1, \ldots, n-1$, $t = 1, \ldots, i$. Let $Q^t_{[i]K} = \theta^t_K$ for $i = 1, \ldots, n-1$, $t = i+1, \ldots, n-1$. Fix $i \in N \setminus \{j\}$. Let $Q^j_{[i-1]K} = Q^j_{[i]K} = Q^i_{[i]K}$. Then $\exists\ x_{[i-1]} \in X(Q^N_{[i-1]K})$ and $x_{[i]} \in X(Q^N_{[i]K})$ such that $x^i_{[i-1]} = x^i$, $x^i_{[i]} = x^i$, $x^j_{[i-1]} = T_i(\gamma_{x,\theta^{N-j}_K})$, and $x^j_{[i]} = T_i(\gamma_{x,\theta^{N-j}_K})$. Then the GC implies that for an envyfree f on $(\Theta^N_K)^-$ we have

$$Q^j_{[i-1]T_i(\gamma_{x,\theta^{N-j}_K})} - Q^i_{[i-1]T_i(\gamma_{x,\theta^{N-j}_K})} \geq f(Q^{N-i}_{[i-1]K}) - f(Q^{N-j}_{[i-1]K}),$$

and

$$Q^i_{[i-1]x^i} - Q^j_{[i-1]x^i} \geq f(Q^{N-j}_{[i-1]K}) - f(Q^{N-i}_{[i-1]K}).$$

Since $Q^j_{[i-1]T_i(\gamma_{x,\theta^{N-j}_K})} = Q^i_{[i-1]T_i(\gamma_{x,\theta^{N-j}_K})}$, and $Q^j_{[i-1]x^i} = Q^i_{[i-1]x^i}$, we get

$$f(Q^{N-i}_{[i-1]K}) = f(Q^{N-j}_{[i-1]K}). \tag{A.7}$$

A similar argument applies to $Q^N_{[i]K}$, so that

$$f(Q^{N-i}_{[i]K}) = f(Q^{N-j}_{[i]K}). \tag{A.8}$$

Since $Q^N_{[i-1]K}$ and $Q^N_{[i]K}$ only differ in agent i's values, the anonymity of f implies that $f(Q^{N-i}_{[i-1]K}) = f(Q^{N-i}_{[i]K})$. Therefore, together with (A.7) and (A.8) we get $f(Q^{N-j}_{[i-1]K}) = f(Q^{N-j}_{[i]K})$. Since this is true for arbitrary $i \in N \setminus \{j\}$, $f(Q^{N-j}_{[0]K}) = f(Q^{N-j}_{[n-1]K})$. Given that $Q^{N-j}_{[0]K} = \theta^{N-j}_K$ and $Q^{N-j}_{[n-1]K}) = R_{\gamma_{x,\theta^{N-j}_K}}(\theta^{N-j}_K)$, this means that $f(\theta^{N-j}_K) = f\left(R_{\gamma_{x,\theta^{N-j}_K}}(\theta^{N-j}_K)\right)$.□

Definition 52 Let $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$, $x \in X(\theta_K^{N-j})$, $\gamma_{x,\theta_K^{N-j}} \in \Gamma(x, \theta_K^{N-j})$, and $i \in N \setminus \{j\}$. Agent i is an L_t*-agent* with respect to $\gamma_{x,\theta_K^{N-j}}$ if $\left| x^i \sim \gamma_{x,\theta_K^{N-j}} \right| = t$.

Definition 53 Let $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$, $x \in X(\theta_K^{N-j})$, $(p_1) \in K \setminus \{0\}$, and $\gamma_{x,\theta_K^{N-j}} \in \Gamma(x, \theta_K^{N-j})$. If (p_1) is assigned under x, a (p_1)*-chain* with respect to $\gamma_{x,\theta_K^{N-j}}$, denoted by $(p_1) \sim \gamma_{x,\theta_K^{N-j}}$, is the sequence of agents $\{p_1, \ldots, p_t\}$, $t \leq n$, such that $x^{p_l} = (p_l)$ *and* $(p_{l+1}) = T_l(\gamma_{x,\theta_K^{N-j}})$ *for* $l = 1, \ldots, t$. If (p_1) is unassigned under x, $(p_1) \sim \gamma_{x,\theta_K^{N-j}} = \emptyset$.

Given Lemma 16, $\forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-$, $\forall (p_1) \in K$, $\exists\ \bar{x} \in X(\theta_{K-(p_1)}^{N-j})$ such that $(p_1) \sim \gamma_{x,\theta_K^{N-j}} = \{p_1, \ldots, p_t\}$, $\gamma_{x,\theta_K^{N-j}} \in \Gamma(x, \theta_K^{N-j})$, $\bar{x}^{p_l} = (p_{l+1})$. Furthermore, $\forall i \notin (p_1) \sim \gamma_{x,\theta_K^{N-j}}$, $\bar{x}^i = x^i$, where $x \in X(\theta_K^{N-j})$. If $\bar{x} \in X(\theta_{K-(p_1)}^{N-j})$ satisfies the above conditions, $\bar{x}$ is called *compatible* with $(p_1) \sim \gamma_{x,\theta_K^{N-j}}$. Notice that if $i \in \{p_1, \ldots, p_t\}$ is an L_0-agent with respect to θ_K^{N-j}, then $i = p_t$, and i is a loser under $\tilde{x}$.

<u>Necessity</u>

The necessity proof consists of three parts, which are summarized below.

I. If f is envyfree on $(\Theta_K^N)^-$ then there exists a function $\hbar$ such that $f(\theta_K^{N-j}) = \hbar\left(\left\{\left(\theta_{x^i}^i, W(\theta_{K-x^i}^{N-j})\right) \mid i \in N \setminus \{j\}\right\}, W(\theta_K^{N-j})\right), \forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-, \forall x \in X(\theta_K^{N-j})$.

II. If f is envyfree on $(\Theta_K^N)^-$ then there exists a function h such that $f(\theta_K^{N-j}) = h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right)\right\}, \forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-$.

III. If $f(\theta_K^{N-j})$ is envyfree on $(\Theta_K^N)^-$ and $f(\theta_K^{N-j}) = h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right)\right\}$ for some function h then $h\left(\left\{W(\theta_{K-p}^{N-i}) \mid p \in K\right)\right\} - h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right)\right\} \leq$

$\max_{p\in K}\left\{W(\theta^{N-i}_{K-p})-W(\theta^{N-j}_{K-p})\right\}, \forall\theta^{N-j}_{K}\in(\Theta^{N-j}_{K})^{-}$, $\forall i,j\in N$ for $j\in V(x)$, where $x\in X(\theta^{N}_{K})$, and $h\left(\left\{W(\theta^{N-i}_{K-p})\mid p\in K\right)\right\}-h\left(\left\{W(\theta^{N-j}_{K-p})\mid p\in K\right)\right\}\leq 0$ otherwise.

I.

Claim 5 *If f is envyfree on $(\Theta^{N}_{K})^{-}$ then there exists a function $\hbar$ such that* $f(\theta^{N-j}_{K})=\hbar\left(\left\{\left(\theta^{i}_{x^i},W(\theta^{N-j}_{K-x^i})\right)\mid i\in N\setminus\{j\}\right\},W(\theta^{N-j}_{K})\right)$, $\forall\theta^{N-j}_{K}\in(\Theta^{N-j}_{K})^{-}$, $\forall x\in X(\theta^{N-j}_{K})$.

Proof: Let $\theta^{N-j}_{K}\in(\Theta^{N-j}_{K})^{-}$ and $\tilde{\theta}^{N-j}_{K}\in(\Theta^{N-j}_{K})^{-}$ such that $W(\theta^{N-j}_{K})=W(\tilde{\theta}^{N-j}_{K})$, and $\exists\ x\in X(\theta^{N-j}_{K})$ and $\tilde{x}\in X(\tilde{\theta}^{N-j}_{K})$ with $\left\{\left(\theta^{i}_{x^i},W(\theta^{N-j}_{K-x^i})\right)\mid i\in N\setminus\{j\}\right\}=\left\{\left(\tilde{\theta}^{i}_{\tilde{x}^i},W(\tilde{\theta}^{N-j}_{K-\tilde{x}^i})\right)\mid i\in N\setminus\{j\}\right\}$. Label the agents and the objects so that $\theta^{i}_{x^i}=\tilde{\theta}^{i}_{\tilde{x}^i}$ and $x^i=\tilde{x}^i$, $\forall i\in N\setminus\{j\}$. Let $\gamma_{x,\theta^{N-j}_{K}}\in\Gamma(x,\theta^{N-j}_{K})$ and $\gamma_{\tilde{x},\tilde{\theta}^{N-j}_{K}}\in\Gamma(\tilde{x},\tilde{\theta}^{N-j}_{K})$. Let $\bar{\theta}^{N-j}_{K}=R_{\gamma_{x,\theta^{N-j}_{K}}}(\theta^{N-j}_{K})$ and $\bar{\bar{\theta}}^{N-j}_{K}=R_{\gamma_{\tilde{x},\tilde{\theta}^{N-j}_{K}}}(\tilde{\theta}^{N-j}_{K})$. Then, given Lemma 19, we have $f(\bar{\theta}^{N-j}_{K})=f(\theta^{N-j}_{K})$ and $f(\bar{\bar{\theta}}^{N-j}_{K})=f(\tilde{\theta}^{N-j}_{K})$. We know that $x=\tilde{x}$, $x\in X(\bar{\theta}^{N-j}_{K})$, and $x\in X(\bar{\bar{\theta}}^{N-j}_{K})$. Furthermore, $\bar{\theta}^{i}_{x^i}=\bar{\bar{\theta}}^{i}_{x^i}, W(\bar{\theta}^{N-j}_{K-x^i})=W(\bar{\bar{\theta}}^{N-j}_{K-\tilde{x}^i}), \forall i\in N\setminus\{j\}$, and $W(\bar{\theta}^{N-j}_{K})=W(\bar{\bar{\theta}}^{N-j}_{K})$. Then, $\forall i\in N\setminus\{j\}$, if i is an L_0-agent with respect to $\bar{\theta}^{N-j}_{K}$, then i is an L_0-agent with respect to $\bar{\bar{\theta}}^{N-j}_{K}$, and if i is an L_*-agent with respect to $\bar{\theta}^{N-j}_{K}$, then i is an L_*-agent with respect to $\bar{\bar{\theta}}^{N-j}_{K}$. Thus, $\bar{\theta}^{N-j}_{K}$ and $\bar{\bar{\theta}}^{N-j}_{K}$ may only differ in the values of L_*-agents.

Suppose there are n' ($n'\leq k-1$) L_*-agents with respect to $\bar{\theta}^{N-j}_{K}$ and $\bar{\bar{\theta}}^{N-j}_{K}$. Label the agents so that agents $1,2,\ldots,n'$ are L_*-agents, and if l is an L_t-agent with respect to $\gamma_{x,\tilde{\theta}^{N-j}_{K}}$, while l' is an $L_{t'}$-agent with respect to $\gamma_{x,\tilde{\theta}^{N-j}_{K}}$, then $l<l'$ implies that $t\leq t'$.

Define $\{Q^{N-j}_{[i]K} \mid i = 0, \ldots, n'\}$ as follows. Let $Q^{N-j}_{[0]K} = \bar{\theta}^{N-j}_{K}$. Let $Q^{t}_{[i]K} = \bar{\bar{\theta}}^{t}_{K}$ for $i = 1, \ldots, n'$, $t = 1, \ldots, i$, and let $Q^{t}_{[i]K} = \bar{\theta}^{t}_{K}$ for $i = 1, \ldots, n'$, $t = i+1, \ldots, n-1$.

First we will show that $W(Q^{N-j}_{[i]K}) = W(\bar{\theta}^{N-j}_{K}) = W(\bar{\bar{\theta}}^{N-j}_{K})$ for $i \leq n'$. We know that $W(Q^{N-j}_{[i]K}) \geq W(\bar{\theta}^{N-j}_{K})$, since the value of assignment x with respect to $Q^{N-j}_{[i]K}$ is $W(\bar{\theta}^{N-j}_{K})$. If $x_{[i]} = x$ then $W(Q^{N-j}_{[i]K}) = W(\bar{\theta}^{N-j}_{K})$. Take a sequence of agents, $(v_1, \ldots, v_t)$, $t \leq n'+1$, such that $x^{v_l} = (v_l)$, $x^{v_l}_{[i]} = (v_{l+1})$. If $x_{[i]} \neq x$ then $\forall v \in n \setminus \{j\}$ such that $x^{v}_{[i]} \neq x^{v}, x^{v}_{[i]} = T_v$, where

$$T_v = \begin{cases} T_v(\gamma_{x,\theta^{N-j}_{K}}) & \text{if} \quad v > i \\ T_v(\gamma_{\tilde{x},\bar{\theta}^{N-j}_{K}}) & \text{if} \quad v \leq i, \end{cases}$$

otherwise $x_{[i]} \notin X((Q^{N-j}_{[i]K})$, given that $W(Q^{N-j}_{[i]K}) \geq W(\theta^{N-j}_{K})$. for $l = 1, \ldots, t-1$, and, for $l = 1, \ldots, t$, where $T_{v_t} = 0$ or T_{v_t} is unassigned under x.

If $v_l > i$, $l = 1, \ldots, t$ for any such sequence then the value of assignment $x_{[i]}$ is the same with respect to $Q^{N-j}_{[i]K}$ and $\bar{\theta}^{N-j}_{K}$. Therefore, in this case $W(Q^{N-j}_{[i]K}) \leq W(\bar{\theta}^{N-j}_{K})$. If $\exists\, l$, $1 \leq l \leq t$, such that $v_l \leq i$, let the first such agent in $v_1, \ldots, v_t$ be $v_{l'}$. Then , given the ordering of the agents, $\forall v_l \in (v_1, \ldots, v_t)$ such that $l > l'$, $v_l \leq i$, so that $(v_{l+1}) = T_{v_l}(\gamma_{\tilde{x},\bar{\theta}^{N-j}_{K}})$ for $l > l'$. Since $x^{v_{l'}} = \tilde{x}^{v_{l'}}, W(\bar{\theta}^{N-j}_{K-x^{v_{l'}}}(x)) = W(\bar{\bar{\theta}}^{N-j}_{K-\tilde{x}^{v_{l'}}})$, $\sum_{l=1}^{l'-1}\left(\bar{\theta}^{v_l}_{T_{v_l}} - \bar{\theta}^{v_l}_{(v_l)}\right) + \sum_{l=l'}^{t}\left(\bar{\bar{\theta}}^{v_l}_{T_{v_l}} - \bar{\bar{\theta}}^{v_l}_{(v_l)}\right) = \sum_{l=1}^{t}\left(\bar{\theta}^{v_l}_{T_{v_l}} - \bar{\theta}^{v_l}_{(v_l)}\right) \leq 0$. Thus, $W(Q^{N-j}_{[i]K}) \leq W(\bar{\theta}^{N-j}_{K})$. Therefore, we get

$$W(Q^{N-j}_{[i]K}) = W(\bar{\bar{\theta}}^{N-j}_{K}), \; \forall i \leq n'. \tag{A.9}$$

Notice that the above argument also implies that

$$W(Q^{N-j}_{[i]K-(p)}) = W(\bar{\theta}^{N-j}_{K-(p)}), \forall i \leq n', \forall p \in K \setminus \{0\}. \tag{A.10}$$

Fix $i \in \{1, \ldots, n'\}$. Let $Q^j_{[i-1]K} = Q^j_{[i]K} = Q^i_{[i]K}$. Let $\bar{\theta}^j_K = \bar{\theta}^i_K$ and $\bar{\bar{\theta}}^j_K = \bar{\bar{\theta}}^i_K$. Given (A.9) and Lemma 18, we have $W(Q^N_{[i]K}) = W(Q^{N-j}_{[i]K-x^i}) + Q^j_{[i]x^i}$. Thus, $W(Q^N_{[i]K}) = W(\bar{\theta}^{N-j}_{[i]K-x^i}) + Q^j_{[i]x^i}) = W(\bar{\theta}^N_K)$, by (A.10). Given that $Q^j_{[i]K} = Q^i_{[i]K}$, there exist $\bar{x}$ and $\bar{\bar{x}} \in X(Q^N_{[i]K})$ such that $\bar{x}^i = x^i$ and $\bar{\bar{x}}^j = x^i$. Since $Q^i_{[i]x^i} = Q^j_{[i]x^i}$, the GC implies that

$$f(Q^{N-j}_{[i]K}) = f(Q^{N-i}_{[i]K}), \tag{A.11}$$

if f is envyfree on $(\Theta^N_K)^-$.

Now consider $Q^N_{[i-1]K}$. For $i = 1$, $Q^N_{[0]K} = \bar{\theta}^N_K$, so that $W(Q^N_{[0]K}) = W(\bar{\bar{\theta}}^N_K)$. Let $2 \leq i \leq n'$, let $\bar{x} \in X(Q^N_{[i-1]K})$. Consider the following three different assignments of agent j under $\bar{x}$.

(a) $\bar{x}^j = \tilde{x}^i$.

(b) $\bar{x}^j = T_i(\gamma_{\tilde{x},\tilde{\theta}^{N-j}_K})$.

(c) $\bar{x}^j \notin \{\tilde{x}^i, T_i(\gamma_{\tilde{x},\tilde{\theta}^{N-j}_K})\}$.

(a) The value of $\bar{x}$, with the restriction that $\bar{x}^j = \tilde{x}^i$, is $W(Q^{N-j}_{[i-1]K-\tilde{x}^i})+ Q^j_{[i-1]\tilde{x}^i} = W(\bar{\bar{\theta}}^{N-j}_{K-\tilde{x}^i}) + \bar{\bar{\theta}}^j_{\tilde{x}^i} = W(\bar{\bar{\theta}}^N_K)$, by (A.10).

(b) The value of $\bar{x}$, with the restriction that $\bar{x}^j = T_i(\gamma_{\tilde{x},\tilde{\theta}^{N-j}_K})$, is $W(Q^{N-j}_{[i-1]K-T_i(\gamma_{\tilde{x},\tilde{\theta}^{N-j}_K})})+ Q^j_{[i-1]T_i(\gamma_{\tilde{x},\tilde{\theta}^{N-j}_K})} = W(\bar{\bar{\theta}}^{N-j}_{K-T_i(\gamma_{\tilde{x},\tilde{\theta}^{N-j}_K})}) + \bar{\bar{\theta}}^j_{T_i(\gamma_{\tilde{x},\tilde{\theta}^{N-j}_K})} \leq W(\bar{\bar{\theta}}^N_K)$, by (A.10).

(c) The value of $\bar{x}$, with the restriction that $\bar{x}^j \notin \{\tilde{x}^i, T_i(\gamma_{\bar{x},\bar{\theta}_K^{N-j}})\}$, is $W(Q_{[i-1]K}^{N-j}) = W(\bar{\bar{\theta}}_K^{N-j}) \leq W(\bar{\bar{\theta}}_K^N)$, by (A.10).

In sum, $W(Q_{[i-1]K}^N) = W(\bar{\bar{\theta}}_K^N)$ for $i \in \{1, \ldots, n'\}$. Then $\exists\ \bar{x} \in X(Q_{[i-1]K}^N)$ such that $\bar{x}^j = x^i$. Furthermore, since $Q_{[i-1]K}^{N-i} = Q_{[i]K}^{N-j}$, we have $W(Q_{[i-1]K-\bar{x}^i}^{N-i}) = W(Q_{[i]K-\bar{x}^i}^{N-j})$. Thus, by (A.10), $W(Q_{[i-1]K-\bar{x}^i}^{N-i}) = W(Q_{[i-1]K-\bar{x}^i}^{N-j})$. Therefore, $W(Q_{[i-1]K}^N) = W(Q_{[i-1]K-P_i(\bar{x})}^{N-j}) + Q_{[i-1]\bar{x}^i}^j = W(Q_{[i-1]K-P_i(\bar{x})}^{N-i}) + Q_{[i-1]\bar{x}^i}^i$, which implies that $\exists \bar{\bar{x}} \in X(Q_{[i-1]K}^N)$ such that $\bar{\bar{x}}^i = x^i$. Since $Q_{[i-1]x^i}^i = Q_{[i-1]x^i}^j$, the GC implies that

$$f(Q_{[i-1]K}^{N-j}) = f(Q_{[i-1]K}^{N-i}), \tag{A.12}$$

if f is envyfree on $(\Theta_K^N)^-$. Since $Q_{[i-1]K}^N$ and $Q_{[i]K}^N$ only differ in agent i's values, $Q_{[i-1]K}^{N-i} = Q_{[i]K}^{N-i}$. Then

$$f(Q_{[i-1]K}^{N-i}) = f(Q_{[i]K}^{N-i}), \tag{A.13}$$

by the anonymity of f. Furthermore, (A.11), (A.12), and (A.13) imply that $f(Q_{[i-1]K}^{N-j}) = f(Q_{[i]K}^{N-j})$. Since the same argument holds for each $i \in \{1, \ldots, n'\}$, $f(Q_{[0]K}^{N-j}) = f(Q_{[n']K}^{N-j})$. However, $Q_{[0]K}^{N-j} = \bar{\theta}_K^{N-j}$ and $Q_{[n']K}^{N-j} = \bar{\bar{\theta}}_K^{N-j}$. Therefore, $f(\bar{\theta}_K^{N-j}) = f(\bar{\bar{\theta}}_K^{N-j})$, as required.□

II.

Claim 6 *If f is envyfree on $(\Theta_K^N)^-$ then there exists a function h such that* $f(\theta_K^{N-j}) = h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right\}\right)$, $\forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-$.

Proof: Take $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$ and $\tilde{\theta}_K^{N-j} \in (\Theta_K^{N-j})^-$ such that $\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right\} =$

$\left\{W(\tilde{\theta}_{K-p}^{N-j}) \mid p \in K\right\}$.

Label the agents and the objects so that $x \in X(\theta_K^{N-j})$ and $x \in X(\tilde{\theta}_K^{N-j})$, with $x^i = (i)$, $\forall i \in N \setminus \{j\}$, $W(\theta_{K-p}^{N-j}) = W(\tilde{\theta}_{K-p}^{N-j})$, $\forall p \in K$, and $\theta_{(i)}^i > \tilde{\theta}_{(i)}^i$ for $i = 1, \ldots, t$, $\theta_{(i)}^i \leq \tilde{\theta}_{(i)}^i$ for $i = t{+}1, \ldots, n{-}1$, where $t < n-1$ is the number of agents for whom $\theta_{(i)}^i > \tilde{\theta}_{(i)}^i$. In addition, let the L_0-agents with respect to θ_K^{N-j} be the last ones. This is consistent with the ordering, since if l is an L_0-agent with respect to θ_K^{N-j} then $\tilde{\theta}_{(l)}^l \geq \theta_{(l)}^l$. For suppose not. Then $\tilde{\theta}_{(l)}^l < \theta_{(l)}^l$, and, thus, $W(\theta_K^{N-j}) - \theta_{(l)}^l < W(\tilde{\theta}_K^{N-j}) - \tilde{\theta}_{(l)}^l$. However, since l is an L_0-agent with respect to $W(\theta_K^{N-j})$, Lemma 17 implies that $W(\theta_{K-(l)}^{N-j}) = W(\theta_K^{N-j+l}) - \theta_{(l)}^l = W(\theta_K^{N-j}) - \theta_{(l)}^l$. Therefore, we get $W(\theta_{K-l}^{N-j}) < W(\tilde{\theta}_K^{N-j}) - \tilde{\theta}_{(l)}^l = W(\tilde{\theta}_{K-(l)}^{N-j-l}) \leq W(\tilde{\theta}_{K-(l)}^{N-j})$, which is a contradiction.

Let $\gamma_{x,\theta_K^{N-j}} \in \Gamma(x, \theta_K^{N-j})$ and $\gamma_{x,\tilde{\theta}_K^{N-j}} \in \Gamma(x, \tilde{\theta}_K^{N-j})$. Let $\bar{\theta}_K^{N-j} = R_{\gamma_{x,\theta_K^{N-j}}}(\theta_K^{N-j})$ and $\bar{\bar{\theta}}_K^{N-j} = R_{\gamma_{x,\tilde{\theta}_K^{N-j}}}(\tilde{\theta}_K^{N-j})$. Then $f(\theta_K^{N-j}) = f(\bar{\theta}_K^{N-j})$ and $f(\tilde{\theta}_K^{N-j}) = f(\bar{\bar{\theta}}_K^{N-j})$, by Lemma 19, if f is envyfree on $(\Theta_K^N)^-$. In the rest of the proof, we will ignore agents who are losers under x, since they always have 0 value for any object according to both $\bar{\theta}_K^{N-j}$ and $\bar{\bar{\theta}}_K^{N-j}$ and thus they don't play any role. Let there be $n'-1$ winners with respect to $\bar{\theta}_K^{N-j}$ and $\bar{\bar{\theta}}_K^{N-j}$, i.e., let $|V(x)| = n'-1$. Let $j = n'$.

Define $\{\theta_{[i]K}^N \mid i = 0, \ldots, n'-1\}$ as follows. Let $\theta_{[0]K}^{N-j} = \bar{\theta}_K^{N-j}$. Let $\theta_{[i]K}^l = \bar{\theta}_K^l$ for $i = 1, \ldots, n'-1$, $l = i{+}1, \ldots, n'-1$. Let $\theta_{[i](l)}^l = \bar{\bar{\theta}}_{(l)}^l$, for $i = 1, \ldots, n'-1, l = 1, \ldots, i$. Define

$$T_l = \begin{cases} T_l(\gamma_x, \theta_K^{N_j}) & \text{if } l \text{ is an } L_*\text{-agent with respect to } \bar{\theta}_K^{N-j} \\ u & \text{if } l \text{ is an } L_0\text{-agent with respect to } \bar{\theta}_K^{N-j}, \text{ and } n' < k \\ 0 & \text{otherwise,} \end{cases}$$

where u is unassigned under x. Now let $\theta^l_{[i]T_l(\gamma_{x,\theta_K^{N-j}})} = \bar{\theta}^l_{T_l(\gamma_{x,\theta_K^{N-j}})} + \bar{\bar{\theta}}^l_{(l)} - \bar{\theta}^l_{(l)}$, and $\theta^l_{[i]p} = 0$, $\forall p \in K$, $p \notin \{(l), T_l(\gamma_{x,\theta_K^{N-j}})\}$ for $i = 1,\ldots,n-1$, $l = 1,\ldots,i$. If $T_l = 0$, let $\theta^l_{[i](t)} = \theta^t_{[i-1](t)} + \bar{\bar{\theta}}^l_{(l)} - \bar{\theta}^l_{(l)}$, where $t \neq l$ is an L_0-agent with respect to $\theta^{N-j}_{[l-1]K}$, and $\theta^l_{[i]p} = 0, i = 1,\ldots,n'-1, l = 1,\ldots,i$. Let $\theta^j_{[i]K} = \theta^i_{[i+1]K}$ for $i = 0,\ldots,n'-1$. ($\theta^j_{[n']K}$ is not defined because it does not have a role in the proof.) To see that $\left\{\theta^N_{[i]K} \mid i = 0,\ldots,n'-1\right\}$ exists, we need to show that if $n' = k, \forall l \leq n'-1$ such that l is an L_0-agent with respect to $\bar{\theta}^{N-j}_K$, $\exists\, t \leq n'-1$ such that t is an L_0-agent with respect to $\bar{\theta}^{N-j}_{[l-1]K}$, and $t \neq l$. Since an L_0-agent with respect to θ^{N-j}_K, t, is also an L_0-agent with respect to $\bar{\theta}^{N-j}_{[l-1]K}$ if $t > l-1$, there exists an L_0-agent with respect to $\theta^{N-j}_{[l-1]K}$ other than l, unless l is the last L_0-agent with respect to θ^{N-j}_K, given the ordering of the agents. So we need to show that if l is the last L_0-agent with respect to $\bar{\theta}^{N-j}_K$, i. e., if $l = n'-1$, then $\exists\, t \leq n'-2$ such that t is an L_0-agent with respect to $\theta^{N-j}_{[l-1]K} = \theta^{N-j}_{[n'-2]K}$. We will show this later.

First we prove that $W(\theta^{N-j}_{[i]K-p}) = W(\theta^{N-j}_{[i-1]K-p}) + \bar{\bar{\theta}}^i_{(i)} - \bar{\theta}^i_{(i)}$, for $p \in K, i \leq n'-1$, if $T_i \neq 0$. $i = 1,\ldots,n-1$, $p \in K$. We show this in the following order.

(a) $i = 1$, $p = (1)$.

(b) $i = 1$, $p = T_1(\gamma_{x,\theta_K^{N-j}})$.

(c) $i = 1$, $p = 0$.

(d) $i = 1$, $p \in K \setminus \{(1), T_1(\gamma_{x,\theta_K^{N-j}}), 0\}$.

(e) $i = 2,\ldots,n-1$, $p \in K$.

Let $T_1 = T_1(\gamma_{x,\theta_K^{N-j}})$.

(a) We have $W(\bar{\bar{\theta}}_K^{N-j}) = W(\bar{\bar{\theta}}_{K-(1)}^{N-j-1}) + \bar{\bar{\theta}}_{(1)}^1 \leq W(\bar{\bar{\theta}}_{K-(1)}^{N-j}) + \bar{\bar{\theta}}_{(1)}^1$. Thus, $W(\theta^{N-j}[0]K) \leq W(\theta^{N-j}[0]K - (1)) + \bar{\bar{\theta}}_{(1)}^1$, and $W(\theta^{N-j-1}[0]K - (1)) + \bar{\theta}_{(1)}^1 \leq W(\theta^{N-j}[0]K - (1)) + \bar{\bar{\theta}}_{(1)}^1$. Then, for $p \neq T_1$, we have $W(\theta_{[1]K-(1)-p}^{N-j-1}) + \theta_p^1 = W(\theta_{[0]K-(1)-p}^{N-j-1}) + \theta_p^1 \leq W(\theta_{[0]K-1}^{N-j-1}) \leq W(\theta_{[0]K-(1)}^{N-j}) + \bar{\bar{\theta}}_{(1)}^1 - \bar{\theta}_{(1)}^1$. Using Lemma 16, we also have $W(\theta_{[1]K-(1)-T_1}^{N-j-1}) + \theta_{[1]T_1}^1 = W(\theta_{[1]K-(1)-T_1}^{N-j-1}) + \theta_{T_1}^1 + \bar{\bar{\theta}}_{(1)}^1 - \bar{\theta}_{(1)}^1 = W(\theta_{[0]K-(1)-T_1}^{N-j-1}) + \theta_{T_1}^1 + \bar{\bar{\theta}}_{(1)}^1 - \bar{\theta}_{(1)}^1 = W(\theta_{[0]K-(1)}^{N-j}) + \bar{\bar{\theta}}_{(1)}^1 - \bar{\theta}_{(1)}^1$. Therefore, $W(\theta_{[1]K-(1)}^{N-j}) = W(\theta_{[0]K-(1)}^{N-j}) + \bar{\bar{\theta}}_{(1)}^1 - \bar{\theta}_{(1)}^1$.

(b) $W(\theta_{[0]K}^{N-j}) \leq W(\theta_{[0]K-(1)}^{N-j}) + \bar{\bar{\theta}}_{(1)}^1$, as was shown above. Then, given that $W(\theta_{[0]K-T_1}^{N-j-1}) + \bar{\theta}_{T_1}^1 \leq W(\theta_{[0]K}^{N-j})$, we get $W(\theta_{[0]K-T_1}^{N-j-1}) + \bar{\theta}_{T_1}^1 \leq W(\theta_{[0]K-(1)}^{N-j}) + \bar{\bar{\theta}}_{(1)}^1$. Then $W(\theta_{[0]K-T_1}^{N-j-1}) \leq W(\theta_{[0]K-(1)-T_1}^{N-j-1}) + \bar{\bar{\theta}}_{(1)}^1$, and, using Lemma 18 $W(\theta_{[1]K-T_1}^{N-j-1}) = W(\theta_{[0]K-T_1}^{N-j-1}) \leq W(\theta_{[0]K-T_1}^{N-j}) - \bar{\theta}_{(1)}^1 + \bar{\bar{\theta}}_{(1)}^1$, since $i \notin T_1 \sim \gamma_{x,\theta_K^{N-j}}$, and, thus, agent 1 gets (1) under $\bar{\bar{x}} \in X(\theta_{[0]K-T_1}^{N-j})$. This shows that the value of an optimal assignment with respect to $\theta_{[1]K-T_1}^{N-j}$, with the restriction that agent 1 gets a object other than (1), is no more than $W(\theta_{[0]K-T_1}^{N-j}) - \bar{\theta}_{(1)}^1 + \bar{\bar{\theta}}_{(1)}^1$. The value of an optimal assignment with respect to $\theta_{[1]K-T_1}^{N-j}$, with the restriction that agent 1 gets (1), is $W(\theta_{[1]K-T_1-(1)}^{N-j-1}) + \bar{\bar{\theta}}_{(1)}^1 = W(\theta_{[0]K-T_1-(1)}^{N-j-1}) + \bar{\bar{\theta}}_{(1)}^1 = W(\theta_{[0]K-T_1}^{N-j}) - \bar{\theta}_{(1)}^1 + \bar{\bar{\theta}}_{(1)}^1$. Thus, $W(\theta_{[1]K-T_1}^{N-j}) = W(\theta_{[0]K-T_1}^{N-j}) - \bar{\theta}_{(1)}^1 + \bar{\bar{\theta}}_{(1)}^1$.

(c) The result in **(a)** implies that $\exists\, \bar{x} \in X(\theta_{[1]K-(1)}^{N-j})$ such that $\bar{x}^1 = T_1$, and the result in **(b)** implies that $\exists\, \bar{\bar{x}} \in X(\theta_{[1]K-T_1}^{N-j})$ such that $\bar{\bar{x}}^1 = (1)$. Thus, $\exists\, x' \in X(\theta_{[1]K}^{N-j})$ such that either $x'^1 = (1)$ or $x'^1 = T_1$, by Lemma 16. Since $W(\theta_{[1]K-(1)}^{N-j-1}) + \theta_{[1](1)}^1 = W(\theta_{[0]K-(1)}^{N-j-1}) + \bar{\bar{\theta}}_{(1)}^1 = W(\theta_{[0]K}^{N-j}) - \bar{\theta}_{(1)}^1 + \bar{\bar{\theta}}_{(1)}^1 \geq W(\theta_{[0]K-T_1}^{N-j-1}) + \bar{\theta}_{T_1}^1 - \bar{\theta}_{(1)}^1 + \bar{\bar{\theta}}_{(1)}^1 = W(\theta_{[1]K-T_1}^{N-j-1}) + \theta_{[1]T_1}^1$, $\exists\, x' \in X(\theta_{[1]K}^{N-j})$ such that $x'^1 = (1)$, and, therefore,

$$W(\theta^{N-j}_{[1]K}) = W(\theta^{N-j}_{[0]K}) - \bar{\theta}^1_{(1)} + \bar{\bar{\theta}}^1_{(1)}.$$

(d) From **(c)** we know that $x \in X(\theta^{N-j}_{[1]K})$. From **(a)**, **(b)**, and Lemma 18, $\exists\, \gamma_{x,\theta^{N-j}_{[1]K}} \in \Gamma(x, \theta^{N-j}_{[1]K})$ such that $T_1(\gamma_{x,\theta^{N-j}_{[1]K}}) = T_1$. Then Lemma 18 implies that agent 1 gets either (1) or T_1 optimally with respect to $\theta^{N-j}_{[1]K-p}$, for $p \in K$. Now fix $p \in K$ and $\tilde{x} \in X(\theta^{N-j}_{[1]K-p})$. If $\tilde{x}^1 = (1)$ then $W(\theta^{N-j-1}_{[1]K-p-(1)}) + \bar{\bar{\theta}}^1_{(1)} \geq W(\theta^{N-j-1}_{[1]K-p-T_1}) + \bar{\theta}^1_{T_1} - \bar{\theta}^1_{(1)} + \bar{\bar{\theta}}^1_{(1)}$. Thus, $W(\theta^{N-j-1}_{[0]K-p-(1)}) + \bar{\theta}^1_{(1)} \geq W(\theta^{N-j-1}_{[0]K-p-(1)}) + \bar{\theta}^1_{T_1}$, so that $\exists\, \tilde{x}' \in X(\theta^{N-j}_{[0]K-p})$ such that $\tilde{x}'^1 = (1)$. Therefore, in this case, $W(\theta^{N-j}_{[1]K-p}) = W(\theta^{N-j-1}_{[1]K-p-(1)}) + \theta^1_{[1](1)} = W(\theta^{N-j-1}_{[0]K-p-(1)}) + \bar{\bar{\theta}}^1_{(1)} = W(\theta^{N-j}_{[0]K-p}) - \bar{\theta}^1_{(1)} + \bar{\bar{\theta}}^1_{(1)}$. Similarly, if $\tilde{x}^1 = T_1$ then $\exists\, \tilde{x}' \in X(\theta^{N-j}_{[0]K-p})$ such that $\tilde{x}'^1 = T_1$. Thus, in this case, $W(\theta^{N-j}_{[1]K-p}) = W(\theta^{N-j-1}_{[1]K-p-T_1}) + \theta^1_{[1]T_1} = W(\theta^{N-j-1}_{[0]K-p-T_1}) + \bar{\theta}^1_{T_1} - \bar{\theta}^1_{(1)} + \bar{\bar{\theta}}^1_{(1)} = W(\theta^{N-j}_{[0]K-p}) - \bar{\theta}^1_{(1)} + \bar{\bar{\theta}}^1_{(1)}$.

(e) For $i = 2, \ldots, t$, the arguments used in **(a)**–**(d)** applied to $\theta^{N-j}_{[i]K}$ and $\theta^{N-j}_{[i-1]K}$ instead of $\theta^{N-j}_{[1]K}$ and $\theta^{N-j}_{[0]K}$, using the fact that the optimal assignments with respect to $\theta^{N-j}_{[i-1]K-p}$ are the same as with respect to $\theta^{N-j}_{[i-2]K-p}$, show that $W(\theta^{N-j}_{[i]K-p}) = W(\theta^{N-j}_{[i-1]K-p}) + \bar{\theta}^i_{(i)} - \bar{\bar{\theta}}^i_{(i)}$. For $i = t+1, \ldots, n-1$, $\bar{\bar{\theta}}^i_{(i)} \geq \bar{\theta}^i_{(i)}$, so it is clear that the optimal assignments with respect to $\theta^{N-j}_{[i]K-p}$ are the same as with respect to $\theta^{N-j}_{[i-1]K-p}$, $\forall p \in K$. Thus, we have $W(\theta^{N-j}_{[i]K-p}) = W(\theta^{N-j}_{[i-1]K-p}) + \bar{\bar{\theta}}^i_{(i)} - \bar{\theta}^i_{(i)}$, for $i = t+1, \ldots, n-1$, $p \in K$.

In sum,

$$W(\theta^{N-j}_{[i]K-p}) = W(\theta^{N-j}_{[i-1]K-p}) + \bar{\bar{\theta}}^i_{(i)} - \bar{\theta}^i_{(i)}, \forall p \in K, \text{ for } i \leq n' - 1 \text{ such that } T_i \neq 0. \tag{A.14}$$

Next, we prove (A.14) for $i \leq n' - 2$, where $T_i = 0$. If $T_i = 0$ then $\bar{\bar{\theta}}^i_{(i)} \geq \bar{\theta}^i_{(i)}$, as we showed earlier. Since the case where $\bar{\bar{\theta}}^i_{(i)} = \bar{\theta}^i_{(i)}$ is trivial, we assume that $\bar{\bar{\theta}}^i_{(i)} > \bar{\theta}^i_{(i)}$. Let t be an L_0-agent with respect to $\bar{\theta}^{N-j}_{[i-1]K}$ such that $t \neq i$, $t \leq n' - 1$, and let $\theta^i_{[i](t)} = \theta^t_{[i-1](t)} + \bar{\bar{\theta}}^i_{(i)} - \bar{\theta}^i_{(i)}$.

(a) $p = 0$

We have $X(\theta^{N-j}_{[i-1]K}) = X(\theta^{N-j}_{[i]K})$, unless

$$\theta^t_{[i](i)} + \theta^i_{[i](t)} > \theta^t_{[i](t)} + \theta^i_{[i](i)} \qquad \text{(A.15)}$$

holds, since $\theta^i_{[i](i)} > \theta^i_{[i-1](i)}$, $\theta^i_{[i](t)} > \theta^i_{[i-1](t)}$, and each agent's values are the same in $\theta^{N-j}_{[i-1]K}$ and $\theta^{N-j}_{[i]K}$, except for i's. However, (A.15) doesn't hold, given that $\theta^t_{[i](t)} + \bar{\bar{\theta}}^i_{(i)} \geq \theta^t_{[i](t)} + \bar{\bar{\theta}}^i_{(i)} - \bar{\theta}^i_{(i)}$, and $\theta^t_{[i](i)} = 0$, which implies that $\theta^t_{[i](t)} + \theta^i_{[i](i)} \geq \theta^i_{[i]t} + \theta^t_{[i](i)}$. Thus,

$$W(\theta^{N-j}_{[i]K}) = W(\theta^{N-j}_{[i-1K}) + \bar{\bar{\theta}}^i_{(i)} - \bar{\theta}^i_{(i)}. \qquad \text{(A.16)}$$

(b) $p \neq 0$

$\mathcal{T}_i((i), \theta^{N-j}_{[i]K}) = (t)$, so Lemma 18 and (A.16) imply that $\forall p \in K \setminus \{0\}$, $\forall \bar{x} \in X(\theta^{N-j}_{[i]K-p})$, we have either $\bar{x}^i = (i)$ or $\bar{x}^i = (t)$.

Let $\bar{\bar{x}} \in X(\theta^{N-j}_{[i-1]K-p})$ for some $p \in K$. Since i is an L_0-agent with respect to $\theta^{N-j}_{[i-1]K}$, we have either $\bar{\bar{x}}^i = (i)$ or $\bar{\bar{x}}^i = 0$, by Lemma 18.

i) If $\bar{\bar{x}}^i = (i)$, then $\exists\, \bar{x} \in X(\theta^{N-j}_{[i]K-p})$ such that $\bar{x}^i = (i)$. Suppose $\bar{x}^i \neq (i)$. Then $\bar{x}^i = (t)$, and thus $\bar{x}^t = 0$, given that t is an L_0-agent with respect to $\theta^{N-j}_{[i]K}$.

Let $p \sim \gamma_{x,\theta^{N-j}_{[i]K}} = \{\ldots, v, i, t\}$ for $\gamma_{x,\theta^{N-j}_{[i]K}} \in \Gamma(x, \theta^{N-j}_{[i]K})$, where $\bar{x}$ is compatible with $p \sim \gamma_{x,\theta^{N-j}_{[i]K}}$. Then $\theta^v_{[i](i)} - \theta^i_{[i](i)} + \theta^i_{[i](t)} - \theta^t_{[i](t)} > 0$, which implies that $\theta^v_{[i](i)} - \theta^i_{[i](i)} + \theta^i_{[i](t)} - \theta^t_{[i](t)} = \theta^v_{[i-1](i)} - \theta^i_{[i](i)} + \bar{\bar{\theta}}^i_{(i)} - \bar{\theta}^i_{(i)} = \theta^v_{[i-1](i)} - \theta^i_{[i-1](i)} > 0.$

However, this means that $\bar{\bar{x}}^i = 0$, a contradiction. Therefore, we have $W(\theta^{N-j}_{[i-1]K-p}) = W(\theta^{N-j-i}_{[i-1]K-p-(i)}) + \bar{\theta}^i_{(i)} = W(\theta^{N-j-i}_{[i]K-p-(i)}) + \bar{\theta}^i_{(i)} = W(\theta^{N-j}_{[i]K-p}) - \bar{\bar{\theta}}^i_{(i)} + \bar{\theta}^i_{(i)}$, where $\bar{\bar{x}}^i = (i)$.

ii) If $\bar{\bar{x}}^i = 0$, then $\bar{\bar{x}}^t = (t)$, since t is an L_0-agent with respect to $\theta^{N-j}_{[i-1]K}$, and so $t \notin p \sim \gamma_{x,\theta^{N-j}_{[i-1]K}}$, where $\gamma_{x,\theta^{N-j}_{[i-1]K}} \in \Gamma(x, \theta^{N-j}_{[i-1]K})$, and x is compatible with $p \sim \gamma_{x,\theta^{N-j}_{[i-1]K}}$. Then, since $\bar{\bar{\theta}}^i_{(i)} - \bar{\theta}^i_{(i)} = \theta^i_{[i](t)} - \theta^t_{[i](t)} > 0$, there exists $\bar{x} \in X(\theta^{N-j}_{[i]K-p})$ such that $\bar{x}^i = (t)$ and $\bar{x}^t = 0$. Therefore, we have $W(\theta^{N-j}_{[i-1]K-p}) = W(\theta^{N-j-i}_{[i-1]K-p}) = W(\theta^{N-j-i-t}_{[i-1]K-p-(t)}) + \theta^t_{[i-1](t)} = W(\theta^{N-j-i-t}_{[i]K-p-(t)}) + \theta^t_{[i-1](t)} = W(\theta^{N-j-t}_{[i]K-p}) - \theta^i_{[i](t)} + \theta^t_{[i-1](t)} = W(\theta^{N-j}_{[i]K-p}) - \bar{\bar{\theta}}^i_{(i)} + \bar{\theta}^i_{(i)}$.

In sum, together with (A.14), we have

$$W(\theta^{N-j}_{[i]K-p}) = W(\theta^{N-j}_{[i-1]K-p}) + \bar{\bar{\theta}}^i_{[i](i)} - \bar{\theta}^i_{(i)}, \forall p \in K, \text{ for } i \le n'-2, \text{ where } n = k. \quad \text{(A.17)}$$

Now we are ready to prove that there exists an agent $t \le n' - 2$ such that t is an L_0-agent with respect to $\theta^{N-j}_{[n-2]K}$, if $n' = k$. Let $l = n' - 1$, i.e., the last L_0-agent with respect to $\bar{\theta}^{N-j}_K$. Since $n' = k$, each object is assigned under x, and then Lemma 14 and Lemma 17 imply that there exists at least one L_0-agent with respect to $\bar{\bar{\theta}}^{N-j}_K$, say t, such that $t \le n' - 1$. Since l is an L_0-agent with respect to $\bar{\theta}^{N-j}_K$, $\bar{\theta}^l_{(l)} \le \bar{\bar{\theta}}^l_{(l)}$. The case where $\bar{\theta}^l_{(l)} = \bar{\bar{\theta}}^l_{(l)}$ is trivial, so let $\bar{\theta}^l_{(l)} < \bar{\bar{\theta}}^l_{(l)}$. Then

l is not an L_0-agent with respect to $\bar{\bar{\theta}}_K^{N-j}$, and thus $t \neq l$. Since t is an L_0-agent with respect to $\bar{\bar{\theta}}_K^{N-j}$, we have $W(\bar{\bar{\theta}}_{K-(t)}^{N-j}) = W(\bar{\bar{\theta}}_K^{N-j}) - \bar{\bar{\theta}}_{(t)}^t$. Then using Lemma 17, $W(\theta_{[n'-2]K-(t)}^{N-j}) = W(\theta_{[l-1]K-(t)}^{N-j}) = \mathrm{W}(\bar{\bar{\theta}}_{K-(t)}^{N-j}) - \bar{\bar{\theta}}_{(l)}^l + \bar{\theta}_{(l)}^l = \mathrm{W}(\bar{\bar{\theta}}_K^{N-j}) - \bar{\bar{\theta}}_{(t)}^t - \bar{\bar{\theta}}_{(l)}^l + \bar{\theta}_{(l)}^l =$ $W(\theta_{[n'-2]K}^{N-j}) - \theta_{[n'-2](t)}^t$, which means that t is an L_0-agent with respect to $\theta_{[n'-2]K}^{N-j}$. Therefore, there exists an L_0-agent with respect to $\theta_{[n'-2]K}^{N-j}$, other than l.

Since t is an L_0-agent with respect to $\theta_{[n'-2]K}^{N-j}$, we can let $\theta_{[n'-2]T_t}^t = 0$ without loss of generality. Thus, the same argument applies to $\theta_{[n'-2]K}^{N-j}$ and $\theta_{[n'-1]K}^{N-j}$ as to $\theta_{[i-1]K}^{N-j}$ and $\theta_{[i]K}^{N-j}$ for $i \leq n' - 2$, when $n' = k$, and we have $W(\theta_{[n'-1]K-p}^{N-j}) = W\theta_{[n'-2]K-p}^{N-j}) + \bar{\bar{\theta}}_{(n'-1)}^{n'-1} - \bar{\theta}_{(n'-1)}^{n'-1}$. Together with (A.14) and (A.17), we get

$$W(\theta_{[i]K-p}^{N-j}) = W\theta_{[i-1]K-p}^{N-j}) + \bar{\bar{\theta}}_{(i)}^i - \bar{\theta}_{(i)}^i, \forall p \in K, \forall i \leq n' - 1.$$

Now take $i \in \{1, \dots, n'-1\}$, and suppose $n' < k$. The value of an optimal assignment with respect to $\theta_{[i-1]K}^N$, with the restriction that j gets $p \in K$ such that $p \notin \{(i), T_i\}$, is $W(\theta_{[i-1]K-p}^{N-j}) \leq W(\theta_{[i-1]K}^{N-j})$. The value of an optimal assignment with respect to $\theta_{[i-1]K}^N$, with the restriction that j gets (i), is $W(\theta_{[i-1]K-(i)}^{N-j}) + \theta_{[i-1](i)}^j = W(\theta_{[0]K-(i)}^{N-j}) + \sum_{l=1}^{i-1} \left(\bar{\bar{\theta}}_{(l)}^l - \bar{\theta}_{(l)}^l\right) + \bar{\bar{\theta}}_{(i)}^i = W(\bar{\bar{\theta}}_{K-(i)}^{N-j})$ $+ \sum_{l=1}^{i-1} \left(\bar{\bar{\theta}}_{(l)}^l - \bar{\theta}_{(l)}^l\right) + \bar{\bar{\theta}}_{(i)}^i = W(\bar{\bar{\theta}}_K^{N-j+i}) + \sum_{l=1}^{i-1} \left(\bar{\bar{\theta}}_{(l)}^l - \bar{\theta}_{(l)}^l\right) \geq W(\bar{\bar{\theta}}_K^{N-j}) + \sum_{l=1}^{i-1} \left(\bar{\bar{\theta}}_{(l)}^l - \bar{\theta}_{(l)}^l\right)$ $= W(\theta_{[i-1]K}^{N-j})$, using (A.18) and Lemma 17. The value of an optimal assignment with respect to $\theta_{[i-1]K}^N$, with the restriction that j gets T_i, is $W(\theta_{[i-1]K-T_i}^{N-j}) + \theta_{[i-1]T_i}^j =$ $W(\theta_{[0]K-T_i}^{N-j}) + \sum_{l=1}^{i-1} \left(\bar{\bar{\theta}}_{(l)}^l - \bar{\theta}_{(l)}^l\right) + \bar{\theta}_{T_i}^i + \bar{\bar{\theta}}_{(i)}^i - \bar{\theta}_{(i)}^i = W(\theta_{[0]K-T_i-(i)}^{N-j-i})$ $+ \sum_{l=1}^{i-1} \left(\bar{\bar{\theta}}_{(l)}^l - \bar{\theta}_{(l)}^l\right) + \bar{\theta}_{T_i}^i + \bar{\bar{\theta}}_{(i)}^i = W(\theta_{[0]K-(i)}^{N-j}) - \bar{\theta}_{T_i}^i + \sum_{l=1}^{i-1} \left(\bar{\bar{\theta}}_{(l)}^l - \bar{\theta}_{(l)}^l\right) + \bar{\theta}_{T_i}^i + \bar{\bar{\theta}}_{(i)}^i =$ $W(\theta_{[i-1]K-(i)}^{N-j}) + \theta_{[i-1](i)}^j$. Therefore, $\exists\ x^* \in X(\theta_{[i-1]K}^N)$ such that $x^{*j} = (i), x^{*i} = T_i$, and

$\exists x^{**} \in X(\theta^N_{[i-1]K})$ such that $x^{**j} = T_i$ and $x^{**i} = (i)$.

When $n' = k$ and i is an L_0-agent with respect to $\bar{\theta}^{N-j}_K$, we only need to check two cases: $p = (i)$, and $p \neq (i)$. Thus, the above arguments show that in this case $\exists x^* \in X(\theta^N_{[i-1]K})$ such that $x^{*j} = (i)$. Since i is an L_0-agent with respect to $\bar{\theta}^N_{[i-1]K}$, we know that $x^{*t} = (t)$, where $\theta^i_{[i]t} = \theta^t_{[i-1](t)} + \bar{\bar{\theta}}^i_{(i)} = \bar{\theta}^i_{(i)}$. Now redefine $\theta^i_{[i-1](t)=\theta^t_{[i-1](t)}}$, for $i \leq n'-1$, such that $T_i = 0$. Notice that this modification does not change any of the earlier results, since t is an L_0-agent with respect to $\theta^{N-j}_{[i-1]K}$. Then $\exists x^{**} \in X(\theta^{N-j}_{[i-1]K})$ such that $x^{**j} = (i)$ and $x^{**i} = (t)$. Since $\theta^i_{[i-1](i)} + \theta^j_{[i-1](t)} = \bar{\theta}^i_{(i)} + \theta^i_{[i](t)} = \bar{\theta}^i_{(i)} + \bar{\bar{\theta}}^t_{(t)} + \bar{\bar{\theta}}^i_{(i)} - \bar{\theta}^i_{(i)} = \bar{\theta}^i_{(i)} + \bar{\bar{\theta}}^t_{(t)} = \theta^i_{[i-1](t)} + \theta^j_{[i-1](i)}, \exists x^{***} \in X(\theta^N_{[i-1]K})$ such that $x^{***j} = (t)$ and $x^{***i} = (i)$. Now redefine $T_i = (t)$ for $i \leq n'-1$ if i is an L_0-agent with respect to $\bar{\theta}^{N-j}_K$ and $n' = k$, and let T_i be defined as before otherwise. Then, if f is envyfree on $(\Theta^N_K)^-$, the GC requires that $\theta^j_{[i-1](i)} - \theta^i_{[i-1](i)} \geq f(\theta^{N-i}_{[i-1]K}) - f(\theta^{N-j}_{[i-1]K})$, i. e, that $\bar{\theta}^i_{(i)} - \bar{\bar{\theta}}^i_{(i)} \leq f(\theta^{N-j}_{[i-1]K}) - f(\theta^{N-i}_{[i-1]K})$. The GC also requires that $\theta^i_{[i-1]T_i} - \theta^j_{[i-1]T_i} \geq f(\theta^{N-j}_{[i-1]K}) - f(\theta^{N-i}_{[i-1]K})$, i.e, that $\bar{\theta}^i_{(i)} - \bar{\bar{\theta}}^i_{(i)} \geq f(\theta^{N-j}_{[i-1]K}) - f(\theta^{N-i}_{[i-1]K})$. Thus, $f(\theta^{N-j}_{[i-1]K}) - f(\theta^{N-i}_{[i-1]K}) = \bar{\theta}^i_{(i)} - \bar{\bar{\theta}}^i_{(i)}$, $i = \ldots, n-1$. Since $\theta^{N-i}_{[i-1]K} = \theta^{N-j}_{[i]K}$ for $i = 1, \ldots, n-1$, we get

$$
\begin{aligned}
f(\theta^{N-j}_{[0]K}) &= f(\theta^{N-j}_{[0]K}) + \bar{\theta}^1_{(1)} - \bar{\bar{\theta}}^1_{(1)} = f(\theta^{N-j}_{[1]K}) + \bar{\theta}^1_{(1)} - \bar{\bar{\theta}}^1_{(1)} \\
&= f(\theta^{N-2}_{[1]K}) + \bar{\theta}^2_{(2)} - \bar{\bar{\theta}}^2_{(2)} + \bar{\theta}^1_{(1)} - \bar{\bar{\theta}}^1_{(1)} = \cdots \\
&= f(\theta^{N-j}_{[n-1]K}) + \textstyle\sum_{l=1}^{n-1} \left(\bar{\theta}^l_{(l)} - \bar{\bar{\theta}}^l_{(l)}\right).
\end{aligned}
$$

Given that $\sum_{l=1}^{n-1} \left(\bar{\theta}^l_{(l)} - \bar{\bar{\theta}}^l_{(l)}\right) = \sum_{l=1}^{n-1} \bar{\theta}^l_{(l)} - \sum_{l=1}^{n-1} \bar{\bar{\theta}}^l_{(l)} = W(\bar{\theta}^{N-j}_K) - W(\bar{\bar{\theta}}^{N-j}_K) = 0$, we have $f(\theta^{N-j}_{[0]K}) = f(\theta^{N-j}_{[n-1]K})$, and, therefore, $f(\bar{\theta}^{N-j}_K) = f(\theta^{N-j}_{[n-1]K})$. Notice that, given (A.14), $\left\{\left(\theta^i_{[n-1](i)}, W(\theta^{N-j}_{[n-1]K-(i)})\right) \mid i \in \setminus\{j\}\right\} = \left\{\left(\bar{\bar{\theta}}^i_{(i)}, W(\bar{\bar{\theta}}^{N-j}_{K-(i)})\right) \mid i \in N \setminus \{j\}\right\}$, and $W(\theta^{N-j}_{[n-1]K}) = W(\bar{\bar{\theta}}^{N-j}_K)$. Then Claim 6 implies that $f(\theta^{N-j}_{[n-1]K}) = f(\bar{\bar{\theta}}^{N-j}_K)$. Therefore, $f(\bar{\theta}^{N-j}_K) = f(\bar{\bar{\theta}}^{N-j}_K)$, as required.□

III.

Claim 7 *If* $f(\theta_K^{N-j})$ *is envyfree on* $(\Theta_K^N)^-$ *and* $f(\theta_K^{N-j}) = h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right)\right\}$ *for some function* h *then* $h\left(\left\{W(\theta_{K-p}^{N-i}) \mid p \in K\right)\right\} - h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right)\right\} \leq max_{p\in K}\left\{W(\theta_{K-p}^{N-i}) - W(\theta_{K-p}^{N-j})\right\}, \forall\theta_K^{N-j} \in (\Theta_K^{N-j})^-$, $\forall i,j \in N$ *for* $j \in V(x)$, *where* $x \in X(\theta_K^N)$, *and* $h\left(\left\{W(\theta_{K-p}^{N-i}) \mid p \in K\right)\right\} - h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right)\right\} \leq 0$ *otherwise.*

Proof: Take $\theta_K^N \in (\Theta_K^N)^-$ and let $x \in X(\theta_K^N)$. Let $x^j = (j), \forall j \in N$. Fix $i,j \in N$. If $j \notin V(x)$ then $\theta_{(j)}^j = \theta_{(j)}^i = 0$. Therefore, $f(\theta_K^{N-i}) - f(\theta_K^{N-j}) = h(\{W(\theta_{K-p}^{N-i}) \mid p \in K\} - h(\{W(\theta_{K-p}^{N-j}) \mid p \in K\} \leq 0$ if f is envyfree on $(\Theta_K^N)^-$ and $f(_K^{N-j}) = h(\{W(\theta_{K-p}^{N-j}) \mid p \in K\}$. If $j \in V(x)$, let $\max_{p\in K}\{W(\theta_{K-p}^{N-i}) - W(\theta_{K-p}^{N-j})\} = d$, where $d \in \Re$. Define $\bar{\theta}_K^N \in (\Theta_K^N)^-$ as follows. Let $\bar{\theta}_{(j)}^i = \theta_{(j)}^j - d, \bar{\theta}_p^i = \theta_p^i, \forall p \neq (j), \forall K$, and let $\bar{\theta}_K^t = \theta_K^t, \forall t \neq i, t \in N$. Now take $\bar{x} \in X(\bar{\theta}_K^N)$. If $\bar{x}^i \neq (j)$ then $W(\bar{\theta}_K^N) = W(\theta_K^N)$. We also have $W(\theta_{K-(j)}^{N-i}) - W(\theta_{K-(j)}^{N-j}) \leq d$, i.e., $W(\theta_{K-(j)}^{N-i}) + \theta_{(j)}^j - d \leq W(\theta_{K-(j)}^{N-i}) + \theta_{(j)}^j$, and thus $W(\theta_{K(j)}^{N-i}) + \bar{\theta}_{(j)}^i \leq W(\theta_K^N)$. This implies that $x \in X(\bar{\theta}_K^N)$ and $W(\bar{\theta}_K^N) = W(\theta_K^N$. Now let $\tilde{x} \in X(\bar{\theta}_{K-p}^{N-j})$ for $p \in K$. If $\tilde{x} \neq (j)$ then $W(\bar{\theta}_{K-p}^{N-j}) = W(\theta_{K-p}^{N-j})$. We also have $W(\bar{\theta}_{K-p-(j)}^{N-j-i}) + \bar{\theta}_{(j)}^i = W(\theta_{K-p-(j)}^{N-j-i}) + \theta_{(j)}^j - d \leq W(\theta_{K-p}^{N-i}) - d \leq W(\theta_{K-p}^{N-j})$. Therefore, $W(\bar{\theta}_{K-p}^{N-j}) = W(\theta_{K-p}^{N-j}), \forall p \in K$. We also know that $W(\bar{\theta}_{K-p}^{N-i}) = W(\theta_{K-p}^{N-i}), \forall p \in K$, since θ_K^N and $\bar{\theta}_K^N$ only differ in agent i's values. Thus, using the GC, $\bar{\theta}_{(j)}^j - \bar{\theta}_{(i)}^i = \max_{p\in K}\{W(\theta_{K-p}^{N-i}) - W(\theta_{K-p}^{N-j})\}$ implies that if f is envyfree on $(\Theta_K^N)^-$ and $f(\theta_K^{N-j}) = h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right)\right\}$ then $\max_{p\in K}\left\{W(\theta_{K-p}^{N-i}) - W(\theta_{K-p}^{N-j})\right\} \geq h\left(\left\{W(\theta_{K-p}^{N-i}) \mid p \in K\right)\right\} - h\left(\left\{W(\theta_{K-p}^{N-j}) \mid p \in K\right)\right\}$.□

Proof of Proposition 20

Fix $\theta_K^{N-j} \in (\Theta_K^{N-j})^-$. Let $\theta_K^j = \mathbf{0}$. Then $W(\theta_{K-(p)}^{N-j}) = W(\theta_{K-(p)}^N)$, $\forall (p) \in K$. We will now show that $\max_{(p)\in K} \left\{W(\theta_{K-(p)}^N) - W(\theta_{K-(p)}^{N-i})\right\} \leq \theta_{(i)}^i, \forall \imath \in V(x)$, where $x^i = (i)$ and $x \in X(\theta_K^N)$. Fix $(p) \in K$ and $i \in V(x)$.

1. If $i \notin (p) \sim \gamma_{x,\theta_K^N}$, where $\gamma_{x,\theta_K^N} \in \Gamma(x, \theta_K^N)$, then $\exists\ \bar{x} \in X(\theta_{K-(p)}^N)$ such that $\bar{x} = (i)$, by Lemma 14. Then $W(\theta_{K-(p)}^N) - \theta_{(i)}^i = W(\theta_{K-(p)-(i)}^{N-i}) \leq W(\theta_{K-(p)}^{N-i})$, so that $W(\theta_{K-(p)}^N) - W(\theta_{K-(p)}^{N-i}) \leq \theta_{(i)}^i$.

2. If $i \in (p) \sim \gamma_{x,\theta_K^N}$, let $(p) \sim \gamma_{x,\theta_K^N} = \{p, v_1, \ldots, v_l, i, \ldots\}$. We have

$$W(\theta_{K-(p)}^{N-i}) \geq W(\theta_K^N) - \theta_{(p)}^p - \textstyle\sum_{t=1}^l \theta_{(v_t)}^{v_t} - \theta_{(i)}^i + \theta_{(v_1)}^p + \textstyle\sum_{t=1}^{l-1} \theta_{(v_{t+1})}^{v_t} + \theta_{(i)}^{v_l},$$

   and thus,

$$W(\theta_{K-(p)}^{N-i}) + \theta_{(i)}^i \geq W(\theta_{K-(i)}^N) - \theta_{(p)}^p - \textstyle\sum_{t=1}^l \theta_{(v_t)}^{v_t} + \theta_{(v_1)}^p + \textstyle\sum_{t=1}^{l-1} \theta_{(v_{t+1})}^{v_t} + \theta_{(i)}^{v_l} =$$

$$W\left(\theta_{K-(i)-(p)-\sum_{t=1}^l (v_t)}^{N-p-\sum_{t=1}^l v_t}\right) + \theta_{(v_1)}^p + \textstyle\sum_{t=1}^{l-1} \theta_{(v_{t+1})}^{v_t} + \theta_{(i)}^{v_l} = W(\theta_{K-(p)}^N).$$

   This implies that $W(\theta_{K-(p)}^N) - W(\theta_{K-(p)}^{N-i}) \leq \theta_{(i)}^i$.

We have covered both possible cases. Therefore,

$$\max_{(p)\in K} \left\{W(\theta_{K-(p)}^N) - W(\theta_{K-(p)}^{N-i})\right\} \leq \theta_{(i)}^i,$$

$\forall i \in V(x)$. We also know that if $i \notin V(x)$ then $\theta_{(i)}^i = 0$. Therefore, if f is envyfree on $(\Theta_K^N)^-$, then Proposition 18 implies that $f(\theta_K^{N-j}) - f(\theta_K^{N-i}) \leq \theta_{(i)}^i$, $\forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-$, $\forall i \in N \setminus \{j\}$ $\theta_K^j = \mathbf{0}$. Repeating the same step for each $i \in N \setminus \{j\}$ with $= \mathbf{0}$ and summing up the inequalities for $i \in N \setminus \{j\}$, we get $f(\theta_K^{N-j}) - f(\mathbf{0}) \leq \sum_{i \in N\setminus\{j\}} \theta_{(i)}^i = W(\theta_K^{N-j})$,

$\forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-$. Thus, $f(\theta_K^{N-j}) \leq W(\theta_K^{N-j})$, $\forall \theta_K^{N-j} \in (\Theta_K^{N-j})^-$ if $f(\mathbf{0}) \leq 0$ and f is envyfree on $(\Theta_K^N)^-$. Therefore, an envyfree Groves mechanism is individually rational if $f(\mathbf{0}) \leq 0$.□

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