LUMPED PARAMETER ANALOGIES FOR CONTINUOUS MECHANICAL SYSTEMS

Thesis by

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ABSTRACT

This thesis is an investigation of the analogies between lumped or continuous mechanical systems and electric circuits. Methods are developed for obtaining electric circuit analogs of linearized mechanical systems. Essential features of the development are the use of coordinate transformations and the transformation properties of certain fundamental matrices. These make possible a general treatment of the problem of obtaining efficient analogs.

The general theory is developed in part II and it is shown that an electric circuit using linear passive, bi-lateral elements and ideal transformers may be constructed for any of the linear mechanical systems considered.

In part III a new approach to the problem of circuit analogies for beams is developed and the methods of part II are applied to obtain new and more accurate analogies for the dynamic behavior of beams with up to six degrees of freedom. A discussion of analogies for frames is given and the effects of the so-called shear deflection and of combined lateral and axial loads in beams is investigated.

In part IV the errors due to lumping of distributed mass and distributed external force are investigated for some simple systems.

A discussion and summary of the thesis is given on pages 11 to 14.

ACKNOWLEDGEMENTS

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PART I

INTRODUCTION

1.1 A Survey of the Computational Problem

The dynamic behavior of elastic systems undergoing small displacements from a position of equilibrium form an important class of problems in engineering. Vibrations of machinery, flutter of airplane wings and earthquake response of buildings are typical examples. Such problems are solved in various ways most of which may be classified under three headings.

One approach makes simplifying assumptions, then writes differential equations and solves these by the methods of mathematical analysis. In practice this method is limited to systems involving relatively few simultaneous, linear, differential equations with constant coefficients. An example is the free-vibration solution for beams with various end conditions. Many treatises and papers have been written on such problems and some of the important ones are listed in the appended bibliography.

Another method of solution solves the pertinent equations, whether or not they are explicitly written, by numerical methods. These numerical methods can be used on a wider class of problems than can be solved analytically, but the answers obtained are less general in nature. For example, they can be used to obtain numerical solutions to linear equations with non-constant coefficients. Among the best known examples are the Holzer method for finding the normal modes and frequencies of torsional vibration of a shaft and the corresponding method for beams developed by Myklestad (1), Prohl (2), and others. Southwell (3), and Hardy Cross (4), have developed well known iterative numerical methods to calculate deflections of elastic systems under static loads.

The third type of approach and the one used in this thesis constructs a physical model, and the problem is solved not by digital calculation but by making physical measurements on the model. The model may or may not be of the same physical form as the prototype. Examples in which it is the same are the study of stresses by the photo-elastic technique and the study of suspension bridge behavior by the testing of a dynamically similar model. Models of a physical form different from the prototype are called analogs. They are models because the mathematical equations which describe their behavior are the same as, or approximate those, which describe the prototype behavior. The analog method of solution is most often used on problems which are too complex to be solved by the other methods described. An example of such an analog is the use of a two-dimensional, electric conductor to solve Laplace's equation. $\nabla^2 v = 0$

Electric circuit analogies for mechanical systems form a particularly important class of such methods of solution, and it is with such analogies for the small displacement behavior of elastic systems that this thesis is primarily concerned. Steady state and transient behavior of complex elastic systems have been studied by this method. An example, which will be discussed in this thesis, is the determination of the first seventeen normal modes and frequencies of a complete airplane. Many types of linear and non-linear control systems have also been investigated. Electric analogs for lumped mechanical elements and for simple mechanical systems have been described by many authors, among which Gardner and Barnes (7) give an excellent treatment. The literature on electric circuit analogies for continuous mechanical systems and for lumped systems with more than one degree of freedom is not extensive. It will be discussed throughout this thesis. Of particular importance is the work of McCann and MacNeal (8), MacNeal (9) and Kron (10).

In the present part of the thesis some general discussion of methods of formation of analogies and of the errors involved is first discussed. Restrictions on the mechanical systems to be studied are then developed and some notation is introduced. In section 1.7 a discussion and summary of the material presented in parts II, III and IV is given.

1.2 Two Methods of Obtaining Lumped Equations

The analogies which will be considered are formed by using a finite number of discrete electrical elements. If they are to represent continuous mechanical systems with an infinite number of degrees of freedom, an approximation or lumping must be made in obtaining the analog from the prototype. There are two general methods by which this is done. In one method the differential equations of motion for the continuous system are written in terms of some particular coordinate system. The lumping is then accomplished by converting the differential equations to difference equations. This method was used by McCann and MacNeal (8) and by Kron (11) in his analogy for the elastic field. The other method lumps the distributed mass of segments of the system into rigid body equivalents and the distributed external forces into statically equivalent concentrated forces and applies these to a one-dimensional continuous elastic network. The resulting lumped, or finite degree of freedom system is then mathematically described in terms of a set of coordinate systems, and the circuit analog for these equations is constructed. This was the method used by Kron (10) in his analogy for beams and by Myklestad (1) and Prohl (2) in their numerical methods. One of the main contributions of this thesis stems from application of this method of lumping. [The two methods, which will hereafter be called the finite difference method and the lumped method, yield the same result when applied to systems, such as a shaft in torsion, which are described by the wave equation.] They give different results and the lumped method more closely approximates the continuous system when they are applied to higher order systems such as beams. The two methods will be compared and the analogs obtained by McCann and MacNeal and by Kron will be described in part III.

1.3 Steps in Solving a Problem by Electric Circuit Analogy

The process of solving a problem by electric circuit analogy may be outlined in four steps. These are:

1. The system must be defined and any necessary assumptions and approximations such as that of linearity made. It must then be replaced, if necessary, by an equivalent lumped system.

2. The lumped system must be described mathematically in terms of a coordinate system or set of coordinate systems.

3. An electric circuit is devised whose behavior corresponds to the mathematical description of step 2.

4. The physical electric circuit is constructed and the solution of the problem is obtained by making measurements upon it.

These steps are interrelated, for any one of the four depends upon what is required of the other three. The lumping used, coordinate systems chosen, and circuits devised depend upon the accuracy desired, the cost involved, and flexibility required in the resulting circuit.

1.4 Nature of the Errors Involved

The value of a solution to any problem concerning the physical world depends upon how closely the solution approximates the corresponding real physical behavior. The difference between the solution of a problem and the corresponding real behavior will be called the total error. The total error involved in the solution of physical problems by lumped analogies is the sum of partial errors which arise in the following way:

1. Error is caused by initial definition and simplification of the system. This error is common to all methods of solution of problems and it will be discussed no further in this thesis. Examples are the assumptions of linearity, pure mass, concentrated loads, pure impulse, conservative systems, and so forth.

2. Error is caused by lumping of continuous systems. This is illustrated by the question: how closely does the dynamic behavior of a massless beam with n equally spaced masses correspond to that of the same beam with the same total mass distributed over its length?

3. Some circuits may be considerably simplified by omitting elements which have a small effect, and this introduces error. For instance, the finite difference beam analogy represents a simplification of one form of the lumped analogy in which one negative inductance per cell is omitted.

4. Error is caused by imperfections in the circuit elements. This error might be classified under two headings: First, as biased error, that due to known effects such as resistance of inductances and leakage and magnetizing inductance of transformers; second, as random error such as that due to imperfect calibration of elements.

All of these errors are important and unfortunately all are rather difficult to analyze. More will be said about them and some quantitative information on lumping error will be given in part IV.

1.5 Nature of the Lumped System Equations

The behavior of a lumped elastic system may be specified by a finite set of generalized coordinates (x_1--- x_n). For the small displacements of the lumped system about a position of equilibrium, a set of Lagrange's equations in the form of eq.(1) may be written. The theory of such small oscillations is presented by Karman and Biot (12), chapters III, V and VI, in the form used in this thesis.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) + \frac{\partial U}{\partial x_i} + \frac{\partial D}{\partial \dot{x}_i} = F_i \qquad (1)$$

(i = 1 --- n) where n is the number of degrees of freedom of the system

In this equation T, U, D, are positive definite quadratic forms which are respectively the first non-zero or quadratic terms in Taylor's series expansions of the kinetic energy, the potential energy and a dissipation function about the position of equilibrium. The x_i are the generalized coordinates and the F_i are the corresponding external generalized forces. The meaning of the technical term, small-displacement, is that the higher order terms of the Taylor's series are small compared with the quadratic terms.

Equations (1) may be written in the form:

$$m_{ij}\ddot{x}_j + b_{ij}\dot{x}_j + k_{ij}x_j = F_i \qquad j,i = (1 --- n) \qquad (2)$$

where the repeated index summation convention is used and where the mass coefficients m_{ij}, the viscous damping coefficients b_{ij}, and the spring coefficients k_{ij} are real constants.

Equations (1) may also be written in the matrix form (3).

$$[m]\,[\ddot{x}] + [b]\,[\dot{x}] + [k]\,[x] = [F] \qquad (3)$$

The matrices $[m]$, $[b]$, and $[k]$ are matrices of positive definite quadratic forms and as such they are symmetric, have positive diagonal terms, and positive discriminants.* Furthermore they are non-singular. In what follows these matrices with the time derivative operator included will be referred to as a group by the term physical coefficient matrix with the symbol $[d]$ or d_{ij}. Thus $[m]\frac{d^2}{dt^2}$, $[b]\frac{d}{dt}$ and $[k]$ are $[d]$'s. The reason for using matrix notation and some discussion of eq.(3) will be given in section 1.7. Circuit analogies for these physical coefficient matrices are the main subject of part II.

Equations (2) and (3) may be interpreted as equations for an n loop or n node-pair electric network with linear, passive, bi-lateral elements. For the loop network the F_i are voltage sources and the $\dot{x}_i$ are loop currents while for the node network the F_i are current sources and the $\dot{x}_i$ are node-pair voltages. If D'Alembert's principle is used, eq.(3) states that the sum of the forces acting at a point is zero. In the loop analogy this corresponds to Kirchhoff's voltage law and in the node analogy it corresponds to Kirchhoff's current law. This correspondence forms the basis for electric circuit analogy solution of mechanical systems. In practice the electric elements available are linear and it is this fact which restricts consideration to small displacements and to systems with viscous damping.

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* For good treatments of matrix algebra in a form pertinent to this thesis see:
Le Corbeiller, P. (13), Matrix Analysis of Electric Networks
Frazer, Duncan and Collar (14), Elementary Matrices

1.6 Coordinate Transformations

The form of the electric circuit analogous to a physical coefficient matrix $[d]$ depends upon the values of the components d_{ij}. These in turn depend upon the coordinates chosen to describe the mechanical system. For a given problem it may be that Newton's law, eq.(3), is not expressed in a coordinate system suitable for the construction of an electric circuit, or it may be that two distinct coordinate systems have been used and these must be related electrically. In either case a transformation of coordinates must be made. Such transformations will now be studied.

Suppose a set of generalized coordinates $\bar{x}_i$ is given as functions of another set x_i by:

(4) $$x_i = x_i(\bar{x}_1 \cdots\cdots \bar{x}_n) \qquad i = (1 \cdots\cdots m)$$

Displacements are then related by:

(5) $$dx_i = \frac{\partial x_i}{\partial \bar{x}_j} d\bar{x}_j$$

If displacements of points are measured from the origin of coordinates then they are the same quantities as the coordinates of the points, and the same symbol, x_i, may be used for both. The partial derivatives may be expanded in a Taylor's series about the origin. If displacements are small only the constant term need be retained for a first order approximation and the differential notation in eq.(5) may be dropped. Thus eq.(5) may be written:

(6) $$x_i = a_{ij}\bar{x}_j \qquad \text{or} \qquad [x] = [a][\bar{x}]$$

where the a_{ij} are real constants which may be positive, negative or zero.

The technical term, generalized coordinate, includes the specification of independent coordinates and this implies that $[a]$ is non-singular and has an inverse, $[a]^{-1}$. The terms on the diagonal of $[a]$ represent physically a change of scale or unit of measurement along the same axes. The off-diagonal terms represent a change of coordinate axes. A set of n quantities which transform in the same manner as do the coordinates of a point in an n space is defined as a contravariant vector (see Le Corbeiller (13)). It follows that the x_i form a contravariant vector, $[x]$. Since the transformation is independent of time, the generalized velocities $\dot{x}_i$, accelerations $\ddot{x}_i$, and in fact all time derivatives of $[x]$, form contravariant vectors.

The generalized forces F_i are defined as quantities which, when multiplied by the corresponding generalized displacements and then summed, give the invariant, work. This relation is used to find the manner of transformation of generalized forces as follows:

(7)
$$
\begin{aligned}
[F'][\delta x] &= \delta W \\
[F'][a][\delta \bar{x}] &= \delta W = [\bar{F}'][\delta \bar{x}] \\
[F'][a] &= [\bar{F}'] \\
[F] &= [a']^{-1}[\bar{F}]
\end{aligned}
$$

A set of quantities which transform as the inverse transposed matrix of the coordinate transformation is defined as a covariant vector (see Le Corbeiller (13)). Hence the set of generalized forces and the sets of all time derivatives of these forces form covariant vectors.

Time, which is the independent variable, may be transformed by a scale change or change in the unit of time. Such a transformation will be specified by:

(8) $$t = N\,\bar{t}$$

The manner in which the physical coefficient matrices, introduced on page 7, transform when the coordinate system is transformed by $[x] = [a][\bar{x}]$ and $t = N\,\bar{t}$, will now be determined. It will be recalled that a physical coefficient matrix relates a set of forces to a set of displacements or time derivatives of displacements. In the notation introduced on page 7 the derivative operator was included in d but in all but the last equation below, the derivative operator will be explicitly written as $p^n = \frac{d^n}{dt^n}$ or $\bar{p}^n = \frac{d^n}{d\bar{t}^n}$. The behavior equation for a system is:

$$[d\,p^n][x] = [F]$$

When the coordinate transformations, $[x] = [a][\bar{x}]$, $[F] = [a']^{-1}[\bar{F}]$ and $t = N\,\bar{t}$, are made, this equation becomes:

$$\left[d\,\frac{\bar{p}^n}{N^n}\right][a][\bar{x}] = [a']^{-1}[\bar{F}]$$

Upon premultiplying both sides by $[a']$ this becomes:

$$[a']\left[d\,\frac{\bar{p}^n}{N^n}\right][a][\bar{x}] = [\bar{F}]$$

Now $[\bar{d}]$ relates $[\bar{F}]$ and $[\bar{x}]$ by $[\bar{d}][\bar{x}] = [\bar{F}]$ and therefore by comparison with the equation above,

(9) $$[\bar{d}] = [a']\left[\frac{d}{N^n}\right][a]$$

which is the required transformation law.

1.7 Discussion of the Thesis

The quadratic forms which appear in eq.(1) are invariant to coordinate system change; they have a definite value no matter what coordinate system is chosen to specify the behavior of the system. The set of equations (2) are derived from eq.(1) and the values of the physical coefficients m_{ij}, b_{ij}, and k_{ij} in this set do depend on the particular coordinate system. The form of the electric circuit analog and the type of electrical elements used also depend on the particular set of values of the physical coefficients. Therefore there are at least as many circuit analogs for a physical system as there are coordinate systems in which the physical system may be described. The usefulness, flexibility and cost of construction of the circuit analogs for one physical system may vary widely, and usually there will be an optimum circuit for the given computational problem. One of the contributions of this thesis is the development of methods for choosing a coordinate system for a given physical problem which will yield an optimum electric circuit analog. An important tool which is used both in choosing coordinate systems and in devising circuit analogies for a given set of behavior equations is the transformation of coordinates. Depending upon the conditions in a specific problem, this transformation may be made mathematically, before the circuit is devised, or electrically and in the circuit.

The set of equations (2) taken as a group have a meaning apart from any particular coordinate system. This is conveniently expressed mathematically by writing the set in matrix form as eq.(3), and by providing a rule for the transformation of the matrices when the coordinates are transformed. The physical coefficient matrices of eq.(3) may be called respectively a generalized mass, a generalized viscous damper and a generalized spring. These generalized quantities relate the set of generalized forces, called the vector force, acting on a system to the corresponding set of generalized displacements which as a group are called the vector displacement. While an analogous electric circuit is in principle always constructed for a set of behavior equations, still, with these concepts in mind, one may say that the passive circuit analog itself is constructed not for the behavior equations (2), but for the physical coefficient matrices of eq.(3). Currents and voltages are applied to the circuit analog in the same way that forces and displacements are applied to the corresponding physical system.

In part II some required nomenclature is developed and the analogy for a coordinate transformation is given. A general investigation is then made of circuit analogies for physical coefficient matrices, that is, for the generalized masses, viscous dampers and springs which make up a physical system. This investigation uses coordinate transformations and has as an object the discovery of useful circuits for certain classes of matrices and the development of methods by which useful circuits may be devised. One meaning of useful circuit in this sense is a circuit which requires no negative elements, and in order to eliminate such elements, ideal transformers are usually required. The object of part II may thus be stated as the investigation of analogies for physical systems which require a minimum number of ideal transformers, no negative elements and which give maximum accuracy.

In part III analogies for a restricted class of generalized beams are developed by using the concepts and analogies of part II and one new concept. This concept is that the displacement of one end of a one-dimensional elastic system is the sum of a rigid body displacement caused by the displacement of the other end and an elastic displacement caused by the applied force across the elastic system. The nomenclature and pertinent equations are first worked out for a one-coordinate system such as a shaft in torsion. The two-coordinate case of a beam bending in a principal plane is then discussed and this is followed by a three-coordinate example of the general six-coordinate case. Part III also discusses some important subsidiary topics.

In part IV the error caused by lumping of distributed mass is discussed for some particular systems. The error caused by lumping of a distributed external force on a cantilever beam is investigated and some results of errors due to circuit imperfections are given.

The thesis may be summarized as follows:

A. The essential developments and new contributions to electric analog computation are:

1. An electric circuit analog for a generalized coordinate system transformation is given.

2. Coordinate system transformations and the circuit analogs for them are used to obtain analogs for any linear mechanical system whose behavior is described by a finite set of generalized coordinates. These analogs use only ideal transformers and passive, bilateral, electric elements.

3. Methods are developed for obtaining efficient circuit analogs; that is, analogs which use a minimum number of transformers.

4. A circuit analog for the small displacements of a massless rigid body and for the forces acting on the body is given.

5. The concept that the displacement of one end of a one-

dimensional elastic system is the sum of a rigid body part and an elastic deformation is used to obtain an efficient circuit analog for such a system. This analog is constructed using circuits obtained under items 2 and 4 above.

B. The concepts and analogs outlined in A permit the construction of analogous circuits for beams which have the following important features.

1. They are significantly more accurate and not appreciably more complex than the finite difference analogs hitherto used. Numerical comparisons of the two analogs with corresponding continuous systems are given in part IV.

2. They are much more flexible and efficient than the analogs using negative electric elements which are given by Kron (10) and others.

3. The differential equation for the beams considered need not be explicitly stated; only the specification of strain energy is needed. This fact permits the construction of analogs for generalized six-coordinate systems such as curved beams.

C. Examples of the theory and analogs developed are given throughout the thesis and various subsidiary topics relating to beams and frames are discussed.

PART II

ANALOGIES FOR ELEMENTARY SYSTEMS

The purpose of part II is to investigate and develop useful circuit analogies for physical coefficient matrices and their inverses. This is done with the aid of coordinate system transformations. Except for some examples, the matrices considered are general in nature. In part III the matrices for some particular systems will be given.

In section 2.1 some electro-mechanical analogous quantities are discussed and the system used in this thesis is explained. Section 2.2 considers the analogy for a coordinate system transformation and section 2.3 that for a massless rigid body. Two-coordinate physical coefficient matrices are discussed in section 2.4a and their inverses in section 2.4b. In 2.4c some examples are given. The general multi-coordinate physical coefficient matrix is considered in section 2.5a. Its inverse is the subject of 2.5b and examples are given in 2.5c.

2.1 Electro-mechanical Analogous Quantities

In section 1.5 mention was made of the fact that Newton's law could be made analogous to either Kirchhoff's current law or to his voltage law. Each gives rise to a whole family of analogous quantities, only a few of which are usually useful. For the sake of simplicity in exposition only the current law analogy will be developed in this thesis; the voltage law analogy can be developed in precisely the same manner. Both are used in actual computational work, although the current law analogy is the more common for continuous mechanical systems.*

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* The circuits obtained from the current law analogy are topologically similar to the corresponding mechanical systems. (Gardner and Barnes chap. II). For example, if a spring connects two points mechanically then correspondingly an inductance connects two nodes electrically. This fact is a great aid in visualizing and in constructing circuit analogs.

Kirchhoff's current law states that the sum of currents flowing into a circuit node or junction must be zero. The same is true for all time derivatives and time integrals of current. Newton's law with D'Alembert's principle makes an equivalent statement for forces acting on a rigid body. Hence force may be made analogous to current or any of its time derivatives or time integrals and, depending upon the physical coefficients used, displacement will be analogous to voltage or to some time derivative or integral of voltage. Three of the most useful sets of analogous quantities are given in table I. A complete table of both families is given in Gardner and Barnes (7) page 64.

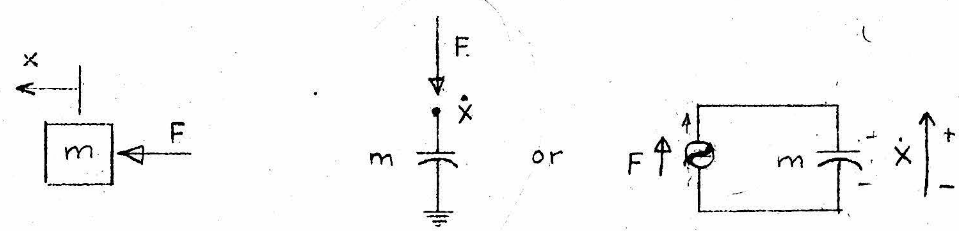
Table I

Electro-mechanical Analogous Quantities

Mechanical quantity	Symbol	Analogy 1	Analogy 2	Analogy 3 (for static probs.)
force	F	current	$\frac{d}{dt}$(current)	current
displacement	x	$\int$(voltage) dt	voltage	voltage
velocity	$\dot{x}$	voltage	$\frac{d}{dt}$(voltage)	$\frac{d}{dt}$(voltage)
mass	m	capacity	same as analogy 1	none
spring	k	inverse induct.		conductance
compliance	g	inductance		resistance
viscous damper	b	conductance		none
inverse damper	b^{-1}	resistance		none

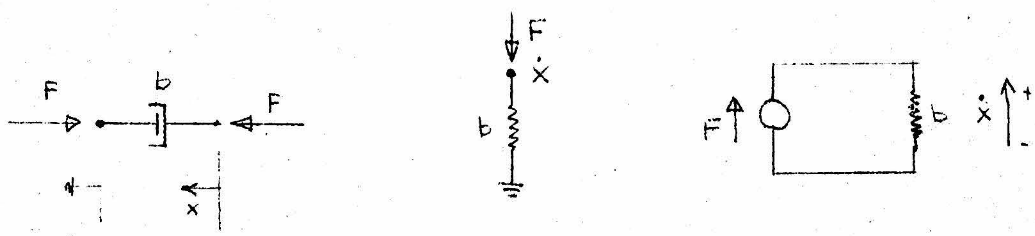
For convenience and simplicity only analogy 1 will be used in the remainder of the thesis.

The elementary circuit analogs in Fig. 1 specify the sign conventions and illustrate the method which will be used. The external force acting on a point is positive when in the same direction as the corresponding displacement. Such a force is represented by a current flowing into the corresponding node. Both currents and forces are indicated by closed arrows —▷. Voltage positive above a reference ground is indicated by $\dot{x}$ placed at the node. Other potential differences are indicated by + and − and by open arrows —>. With each figure the behavior equation is given in the notation used in this thesis and in conventional electrical notation.



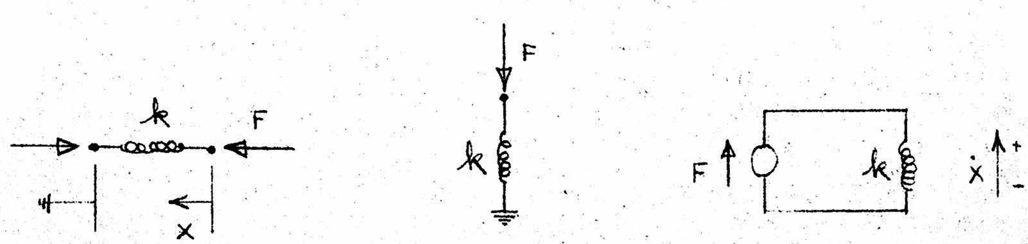
$F = m\ddot{x}$ $\qquad\qquad$ $i = C\frac{de}{dt}$

Fig. 1a



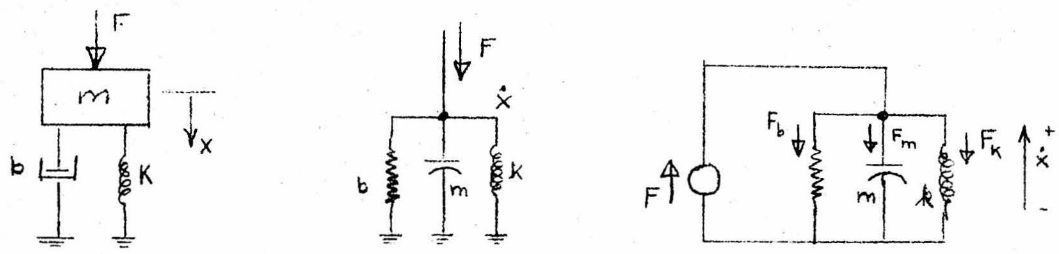
$F = b\dot{x}$ $\qquad\qquad$ $i = Ge$ or $e = iR$

Fig. 1b



$F = k\int \dot{x}\,dt$ $\qquad\qquad$ $i = \frac{1}{L}\int e\,dt$ or $e = L\frac{di}{dt}$

Fig. 1c



$F = F_m + F_b + F_K$ This is Newton's law and Kirchhoff's current law where:

$F_m = m\ddot{x}$ $F_b = b\dot{x}$ $F_K = k\int \dot{x}\, dt$

Fig. 1d.

2.2 Circuit Analogy for Coordinate Transformation

The electric circuit analog for a coordinate transformation is an ideal transformer network. To show this, consider the transformation specified by eqs.(10a) which are given in expanded form as eqs.(10b).

(10a)
$$[x] = [a][\bar{x}] \qquad [\bar{x}] = [a]^{-1}[x]$$
$$[F] = [a']^{-1}[\bar{F}] \qquad [\bar{F}] = [a'][F]$$

(10b)
$$\begin{aligned} x_1 &= a_{11}\bar{x}_1 + a_{12}\bar{x}_2 - - - + a_{1n}\bar{x}_n & \bar{F}_1 &= a_{11}F_1 + a_{21}F_2 - - - + a_{n1}F_n \\ x_2 &= a_{21}\bar{x}_1 + a_{22}\bar{x}_2 - - - + a_{2n}\bar{x}_n & \bar{F}_2 &= a_{12}F_1 + a_{22}F_2 - - - + a_{n2}F_n \\ &\vdots & &\vdots \\ x_n &= a_{n1}\bar{x}_1 + a_{n2}\bar{x}_2 - - - + a_{nn}\bar{x}_n & \bar{F}_n &= a_{1n}F_1 + a_{2n}F_2 - - - + a_{nn}F_n \end{aligned}$$

These equations can represent relations between voltages and between currents by making the coefficients the turns ratios of ideal transformers. The windings of the transformers are connected so that the voltages and currents add in the proper manner. Either n multiple-winding or n^2 two-winding transformers may be used. The network using multiple-winding transformers is given in Fig. 2. The numbers, a_{ij}, indicated are the ratios of turns on the coils representing the unbarred coordinate system to the turns on the coils representing the barred coordinate system.

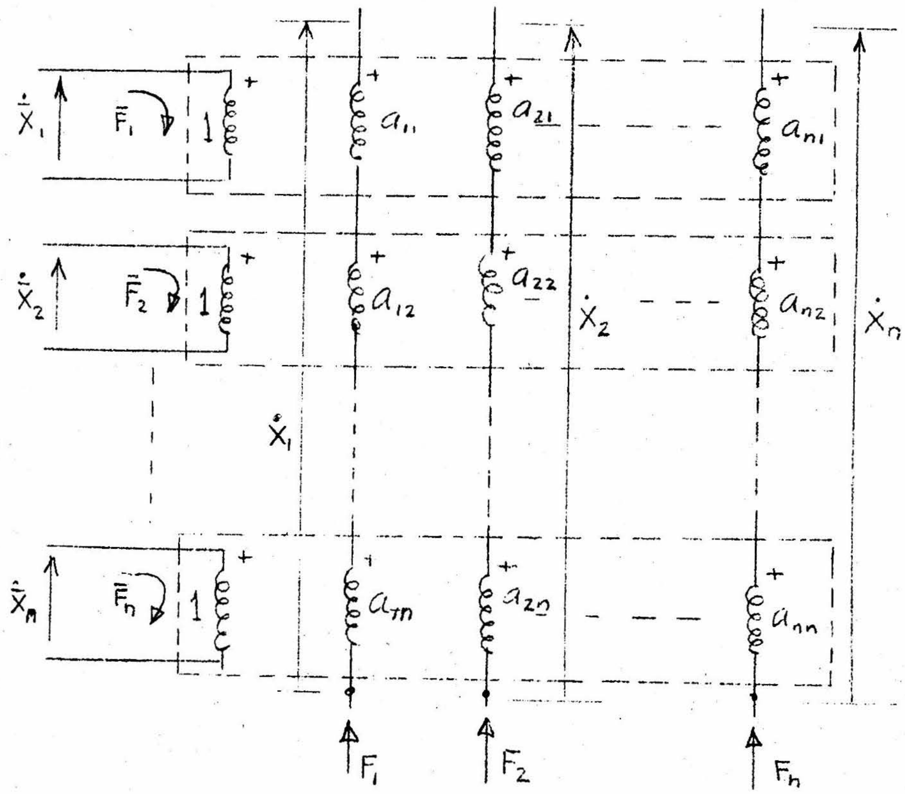


Fig. 2.

When isolation is not required, that is, when all voltages are measured from a common node, one line of each circuit may be grounded, and the circuit may be indicated as shown in Fig. 3. The set of currents flowing into one side of the coordinate system transformation (C.S.T.) are analogous to a generalized vector force. The set flowing out of the other side represent the same generalized force but in a different

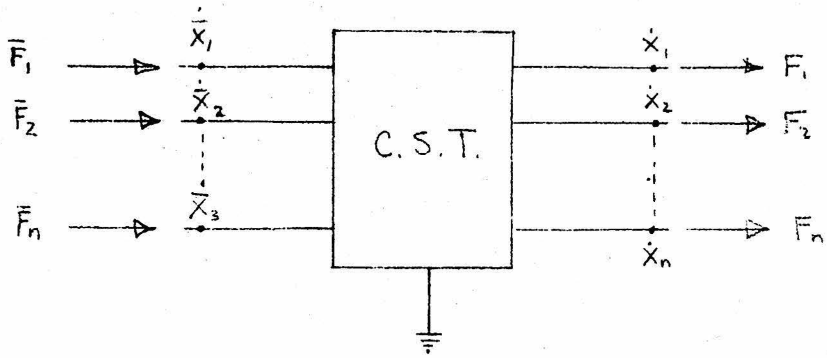


Fig. 3.

coordinate system. The same statement is true for the set of voltages or generalized velocity vector.

A particularly important circuit for the applications is that for two coordinates with no scale change and with a common reference node. This is illustrated in Fig. 4. Eq.(11) is the corresponding transformation.

(11) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +1 & +a_{12} \\ 0 & +1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

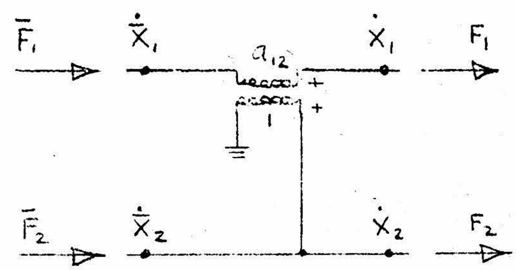
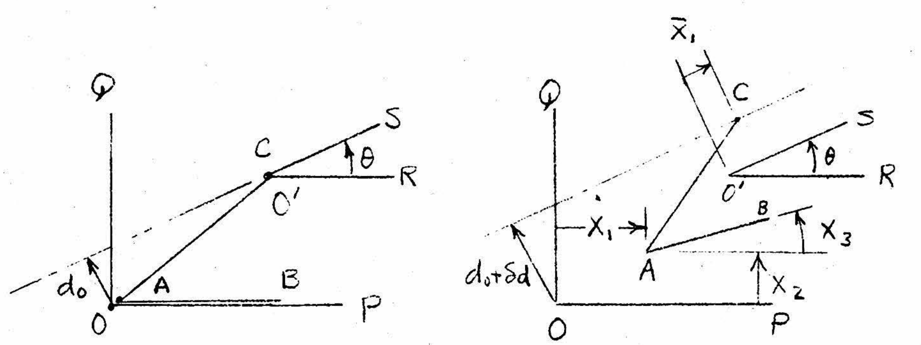


Fig. 4.

2.3 Circuit Analogy for a Massless Rigid Body

An important mechanical element whose electric circuit analogy for small displacements will now be developed is the massless rigid body. To specify the position of a rigid body six independent coordinates are required. The specification of a general displacement requires nine quantities but if displacements are small a set of six independent coordinates suffices. By the same arguments used in section 1.6, one can say that if the origin of coordinates is taken at the initial position from which the small displacements take place, then the position coordinates and the displacement components are identical. The small displacement components transform as the coordinates do and therefore they form a contravariant vector.

As an example, consider the displacements of a rigid plane body in its own plane (Fig. 5). The number of degrees of freedom in this case is three. Let the displacements in the x_λ coordinate system be the cartesian displacements, x_1, x_2, of point A measured from the fixed origin, O, and the small rotation, x_3, of the body about O.



before displacement after displacement

Fig. 5.

In Fig. 5 the lines OP, OQ, O′R, O′S, are coordinate axes fixed in space, while the lines AB and AC are reference lines fixed in the body. d is the normal distance from O to the line through C parallel to O′S. Consider another coordinate system in which the vector displacement is $[\bar{x}]$.

This coordinate system is non-orthogonal and may have separate origins and separate length scales for each coordinate. Let $\bar{x}_1$ be one component of this vector defined as the displacement of point C in the direction O′S. Eq.(12) relates the displacement components in the two coordinate systems.

$$\bar{x}_1 = x_1 \cos\theta + x_2 \sin\theta - d\,x_3 \tag{12}$$

Two similar independent relations can be written for two other points and the three equations as a group would transform $[x]$ into $[\bar{x}]$. In eq.(12), d is a constant, d_0, plus a function, δd, of x_1, x_2, x_3. If displacements are small enough so that in the term, $d_0 + \delta d$, $\delta d \ll d_0$, then the transformations of the type of eq.(12) are linear and the coordinate transformation can be written:

$$[\bar{x}] = [\alpha][x] \tag{13}$$

([α] will be used hereafter to designate a massless rigid body)

This relation can be looked upon as a change of coordinate system used to express the small displacement $[x]$, or it can be looked upon as a rigid body relation which gives the displacements of a set of points $\bar{x}_i$ when the body is displaced by $[x]$. From the latter viewpoint there is no reason to restrict the $\bar{x}_i$ to an independent set, and accordingly $[\alpha]$ may be a rectangular matrix with an infinite number of rows. Unless the coordinates are independent, $[\alpha]$ is singular and the equations cannot be inverted.

The example was given for a planar system because it was simple to illustrate. For the general case the relations are similar but six coordinates are involved.

Forces acting on the rigid body can be treated in the following manner. Assume that the forces act along the various coordinate axes. Since the rigid body can store no energy, the total work done in a small displacement must be zero. Hence:

$$[F'][x] = [\bar{F}'][\bar{x}]$$

(14) $$[F'][x] = [\bar{F}'][\alpha][x]$$

$$[F] = [\alpha'][\bar{F}] \qquad \text{([}\alpha\text{] may be rectangular)}$$

Physically this means that if the set of n forces and couples, $\bar{F}_\lambda$, is given, then the six independent components, F_i, needed to specify the vector force are obtained by eq.(14).

Except for the fact that $[\alpha]$ may be singular, that is, have no inverse, eqs.(13) and (14) are identical to eqs.(6) and (7). Accordingly, the analog for a massless rigid body is the transformer network of Fig. 2 with any necessary extra transformer windings. It can be concluded that a massless rigid body, which is a generalized lever, is a mechanical analog of a mathematical coordinate transformation or of an electric circuit transformer network. One of the most useful ways to use the concepts of this section is to consider eqs. (13) and (14) as relating the forces and displacements at two ends of a rigid bar. From this point of view and with $[\alpha]$ non-singular, if $[F]$ and $[x]$ are the vector external force acting upon one end, and the vector displacement of the same end of the bar, then $[\bar{x}] = [\alpha][x]$ and $[\bar{F}] = -[\alpha']^{-1}[F]$ are the external force acting upon, and the displacement of, the other end of the bar. The minus sign arises because both $[F]$ and $[\bar{F}]$ represent the total vector force acting on the body, and for the body to be in equilibrium, the total vector force must be zero.

As an example of this concept consider the analog for a spring-mounted engine at the end of a stiff nacelle on an airplane wing. The system is idealized as shown in Fig. 6.

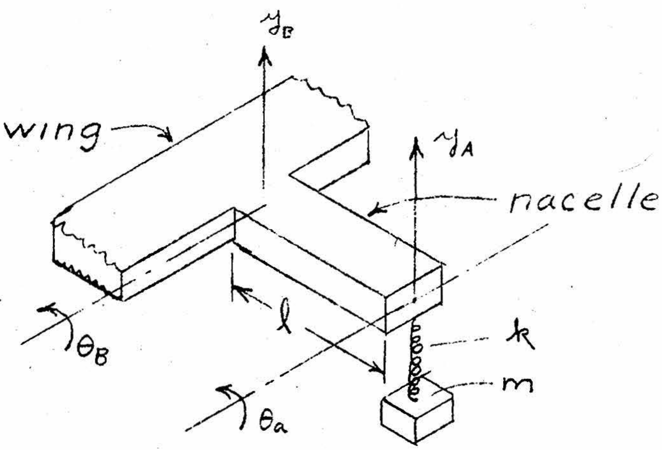


Fig. 6.

The relation between the displacements at ends A and B of the bar is:

$$\begin{bmatrix} y_B \\ \theta_B \end{bmatrix} = \begin{bmatrix} +1 & -l \\ 0 & +1 \end{bmatrix} \begin{bmatrix} y_A \\ \theta_A \end{bmatrix}$$

The analog for this equation is given in Fig. 4. The analog for the spring-mass system is given in Fig. 1c. The two analogs are connected to form the complete analog which is given in Fig. 7.

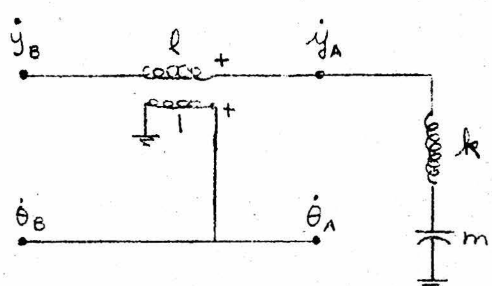


Fig. 7.

2.4 Circuit Analogies for Two-coordinate Physical Coefficient Matrices and their Inverses

In section 2.1 some elementary circuit analogies for one-coordinate systems were given. Analogies for two-coordinate generalized masses, viscous dampers and springs and their inverses will now be investigated. In section 2.5 a similar investigation for the multiple-coordinate case will be made. Section 2.4a discusses the equation $[F] = [d][X]$ while in 2.4b the inverse, $[x] = [d]^{-1}[F]$, is considered. Two simple examples of the theory are given in 2.4c.

2.4a. Consider equations (15) which mechanically relate a vector force to a vector displacement while electrically they are equations for a four terminal network with two self-admittances and one mutual admittance.

$$
\begin{aligned}
F_1 &= d_{11} x_1 + d_{12} x_2 \\
F_2 &= d_{12} x_1 + d_{22} x_2
\end{aligned} \qquad (15)
$$

A four-terminal, mutual, inverse inductance is not a practical single element to use in electric analog computation. Practical considerations thus require the mutual term to be a two terminal element common to two nodes. In order to construct a circuit for eqs.(15) using only two terminal elements, quantities are added and subtracted to the right sides of the equations so that they become eqs.(16). If the reference nodes for x_1 and x_2 are the same, Fig. 8 may be constructed from these equations by inspection. This circuit is a simple π network.

$$
\begin{aligned}
F_1 &= x_1(d_{11}+d_{12}) - d_{12}(x_1 - x_2) \\
F_2 &= -d_{12}(x_2 - x_1) + x_2(d_{22}+d_{12})
\end{aligned} \qquad (16)
$$

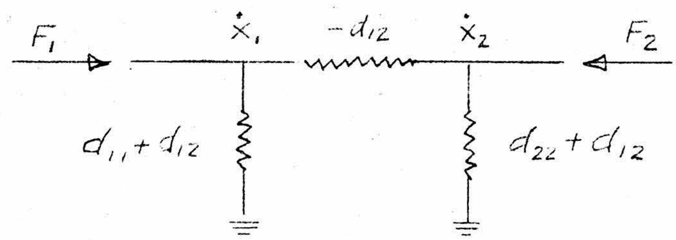


Fig. 8.

There are three situations in which the π network of Fig. 8 loses its simplicity. These are:

1. If the reference nodes for eqs.(16) are not the same, isolation is required. To accomplish this either a transformer or the circuit of Fig. 9 may be used. This circuit always requires two negative admittances.

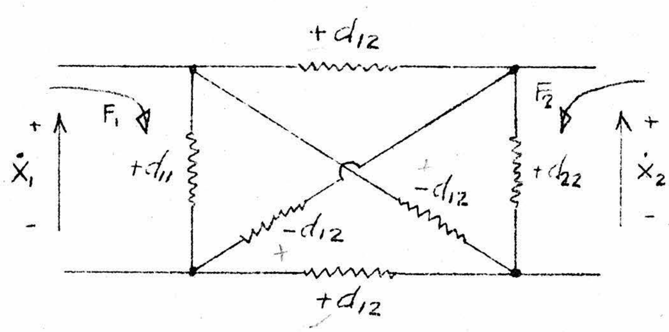


Fig. 9.

2. If d_{12} in eqs.(15) is a positive quantity then Fig. 8 requires a negative admittance. For steady state, fixed frequency, generalized spring or mass problems, where a negative inverse inductance is a capacity and vice versa, a negative admittance may easily be used. In all other types of problems, negative admittances require active elements such as feedback amplifiers.

3. If d_{12} is negative, it is possible that $d_{11} + d_{12} < 0$ or $d_{22} + d_{12} < 0$, a condition which would require negative admittances.

It will now be shown that no matter what the values of the d_{ij}, circuits for $[d]$ may be drawn which use no more than one transformer, have no negative admittances, and which give isolation (permit two reference nodes). These circuits are devised by using concepts of coordinate system transformation. A scale change transformation will first be investigated and then a coordinate axes change will be considered. Note that if the three conditions below are met, no transformer or negative admittance is required.

1. d_{12} is negative
2. $d_{11} > |d_{12}|$, $d_{22} > |d_{12}|$
3. The reference nodes are common (no isolation needed).

The scale change transformation, eq.(17), is introduced and used to transform $[d]$ of eq.(15) so that in the new coordinate system eqs.(18) are obtained. a_1 or a_2 may be positive or negative, and each a_i with value other than +1 will require one transformer.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \tag{17}$$

$$\begin{aligned} \bar{F}_1 &= d_{11} a_1^2 \bar{x}_1 + d_{12} a_1 a_2 \bar{x}_2 \\ \bar{F}_2 &= d_{12} a_1 a_2 \bar{x}_1 + d_{22} a_2^2 \bar{x}_2 \end{aligned} \tag{18}$$

Quantities are added and subtracted from the right sides of eqs(18) and eqs.(19) are obtained.

$$\bar{F}_1 = \left(d_{11}a_1^2 - |\beta a_1 a_2 d_{12}|\right)\bar{X}_1 + |\beta a_1 a_2 d_{12}|\left(\bar{X}_1 + \frac{\bar{X}_2}{\beta}\right) \tag{19}$$

$$\bar{F}_2 = |d_{12}a_1a_2|\left(\frac{\bar{X}_2}{|\beta|} + \frac{|\beta|}{\beta}\bar{X}_1\right) + \left(d_{22}a_2^2 - \left|\frac{d_{12}a_1a_2}{\beta}\right|\right)\bar{X}_2$$

where β is a pure number whose sign is the same as that of $a_1a_2d_{12}$.

Eqs.(19) are equivalent to the circuit of Fig. 10.

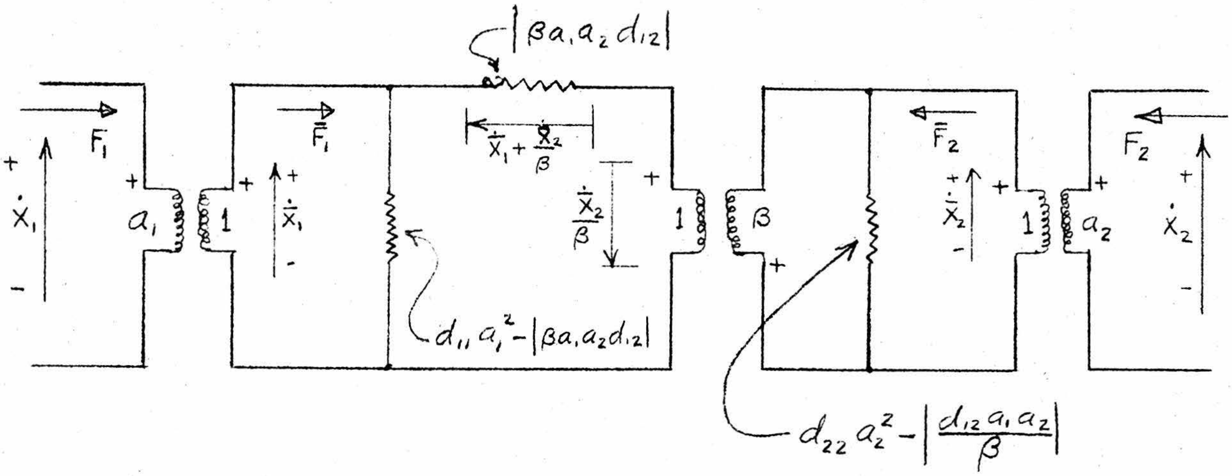


Fig. 10.

In eqs.(19) and the analogous circuit, Fig. 10, the quantity β is a turns ratio and polarity on an internal transformer in the circuit which accomplishes isolation, inverts voltage and current and effectively changes scale. Transformers such as this, in what hereafter will be called the β position, are very important and are used repeatedly in the rest of the thesis.

An inspection of the circuit shows that any one of the three transformers serves to isolate the currents and to remove the restriction on the sign of d_{12}. The requirement that no negative admittance be used imposes the inequalities (20).

$$(20) \qquad d_{11} \geqslant \frac{\beta a_2}{a_1} d_{12} \qquad d_{22} \geqslant \frac{a_1}{\beta a_2} d_{12}$$

If the second inequality is inverted and substituted into the first, the condition is obtained for an $\frac{\beta a_2}{a_1}$ to exist which simultaneously satisfies the inequalities. This condition is that $d_{11} d_{22} \geqslant d_{12}^2$. The discriminant of $[d]$ is $d_{11} d_{22} - d_{12}^2$ which is greater than zero because $[d]$ is the matrix of a quadratic form. Therefore, inequalities (20) are always satisfied and an $\frac{\beta a_2}{a_1}$ can always be chosen which will yield only positive admittances. Furthermore any one of the three transformers can be used alone to obtain a suitable value of $\frac{\beta a_2}{a_1}$. The term $\frac{\beta a_2}{a_1}$ may be chosen so that either inequality, but not both together, is an equality and hence that the corresponding admittance is zero.

If the transformers were perfect any one of the three positions would be equally good. However they have, among other imperfections, leakage and magnetizing inductance. These can easily be combined with the circuit inductors when $[d]$ is a generalized spring. For this particular case the β position would be best.

The elementary coordinate axes transformation (Fig. 4 page 20) applied to eq.(15) is important enough to deserve investigation. Consider the transformations:

$$(21) \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +1 & +a_{12} \\ 0 & +1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

$$(22) \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +1 & 0 \\ +a_{21} & +1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

These transformations applied to eqs.(15) yield eqs.(23) and (24) respectively.

$$(23)\qquad \begin{aligned} \bar{F}_1 &= d_{11}\bar{x}_1 + (d_{11}a_{12}+d_{12})\bar{x}_2 \\ \bar{F}_2 &= (d_{11}a_{12}+d_{12})\bar{x}_1 + (d_{11}a_{12}^2+2a_{12}d_{12}+d_{22})\bar{x}_2 \end{aligned}$$

$$(24)\qquad \begin{aligned} \bar{F}_1 &= (d_{11}+2a_{21}d_{12}+d_{22}a_{21}^2)\bar{x}_1 + (d_{22}a_{21}+d_{12})\bar{x}_2 \\ \bar{F}_2 &= (d_{22}a_{21}+d_{12})\bar{x}_1 + d_{22}\bar{x}_2 \end{aligned}$$

The corresponding circuit does not isolate and is not particularly important unless $a_{12} = -\frac{d_{12}}{d_{11}}$ or $a_{21} = -\frac{d_{12}}{d_{22}}$. These particular values reduce $[d]$ to a diagonal matrix. Eqs.(23) and (24) then become eqs.(25) and (26) with corresponding circuits, Figs. 11 and 12.

$$(25)\qquad \begin{aligned} \bar{F}_1 &= d_{11}\bar{x}_1 \\ \bar{F}_2 &= \frac{d_{11}d_{22}-d_{12}^2}{d_{11}}\bar{x}_2 \end{aligned}$$

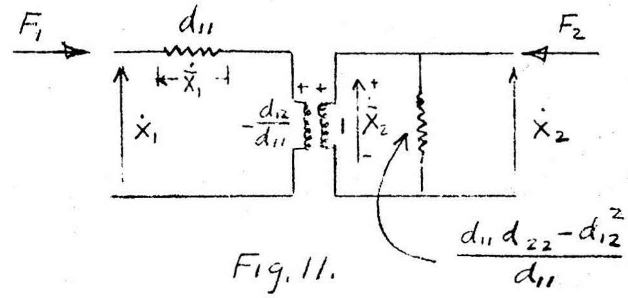


Fig. 11.

$$(26)\qquad \begin{aligned} \bar{F}_1 &= \frac{d_{11}d_{22}-d_{12}^2}{d_{22}}\bar{x}_1 \\ \bar{F}_2 &= d_{22}\bar{x}_2 \end{aligned}$$

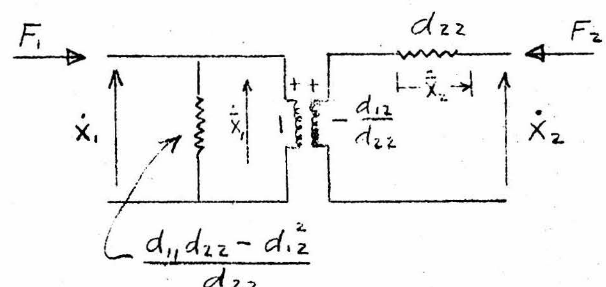


Fig. 12.

The circuits of Figs, 11 and 12 are identical to that of Fig. 10 when $a_1 = a_2 = +1$ and $\beta = \frac{d_{11}}{d_{12}}$, $\beta = \frac{d_{12}}{d_{22}}$ respectively.

2.4b Circuit analogies for two-coordinate physical coefficient matrices were discussed in section 2.4a. Analogies for their inverses will now be considered.

The behavior of a system may be specified by an equation $[F] = [d][x]$ or, since $[d]$ is non-singular, by an inverse equation, $[x] = [\hat{d}][F]$, where $[\hat{d}] = [d]^{-1}$, and time derivative operators are replaced by time integral operators. The inverse of eqs.(15) can then be written as eqs.(27)

$$x_1 = \hat{d}_{11} F_1 + \hat{d}_{12} F_2 \tag{27}$$
$$x_2 = \hat{d}_{12} F_1 + \hat{d}_{22} F_2$$

The corresponding circuit for a common reference node is a simple T. It is drawn in Fig. 13.

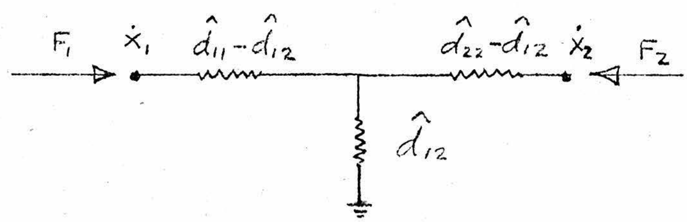


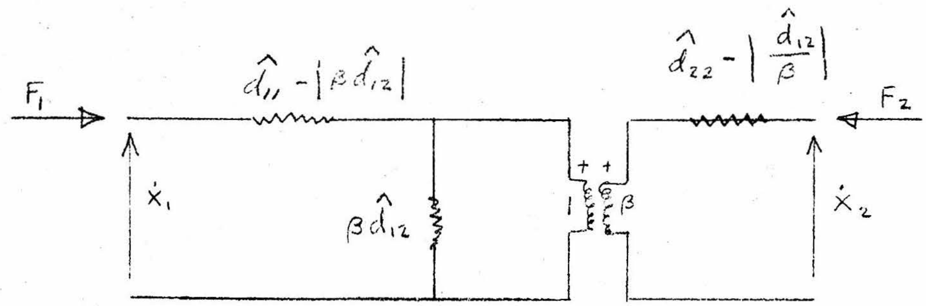
Fig. 13.

The $\hat{d}_{ij}$ are now impedances and the condition for no negative impedances is that $\hat{d}_{12}$ be positive and that $\hat{d}_{11} > \hat{d}_{12}$ and $\hat{d}_{22} > \hat{d}_{12}$. The relation between $\hat{d}_{ij}$ and d_{ij} is:

$$\hat{d}_{11} = \frac{d_{22}}{\Delta} \qquad \hat{d}_{22} = \frac{d_{11}}{\Delta} \qquad \hat{d}_{12} = -\frac{d_{12}}{\Delta}$$

where: $d_{11} d_{22} - d_{12}^2 = \Delta$

These relations and Fig. 13 show that if a transformer is or is not required for the circuit for $[d]$ then correspondingly it is or is not required in the circuit for $[\hat{d}]$. The same transformations and development can be made for eqs.(27) as was done for eqs.(15). The resulting circuit for a transformer in the β position is shown in Fig. 14.



β is a pure number whose sign is that of $\hat{d}_{12}$

Fig. 14.

Either the physical coefficient equation, eq.(15), or its inverse, eq.(27), can be used with the same coordinate system to relate the vector force to the vector displacement and correspondingly either the circuit of Fig. 10 or of Fig. 13 may be used as the electric analog for the physical system.

2.4c Two simple examples of circuit analogs for a two-coordinate physical coefficient matrices will now be given.

Consider a rigid body mass constrained by forces applied at a point a distance ℓ from the center of mass. In Fig. 15 these forces are shown as spring forces. A rigid body mass such as this is a useful idealization of a lumped mass in a non-symmetrical beam such as an airplane wing.

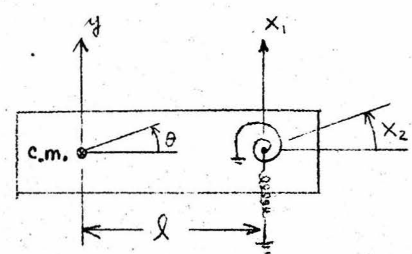


Fig. 15

The inertia properties of the rigid body mass expressed in the y,θ coordinate system are given by eq.(28).

$$(28)\qquad \begin{bmatrix} F_y \\ F_\theta \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix}$$

If a transformation to the x_1, x_2 coordinate system is made by eq.(29), then the inertia properties are expressed by eq.(30).

$$(29)\qquad \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} +1 & -\ell \\ 0 & +1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$$

$$(30)\qquad \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} +m & -m\ell \\ -m\ell & m\ell^2 + I \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$$

The analogous circuit is that of Fig. 8 where the admittances are now capacities. The circuit is drawn in Fig. 16.

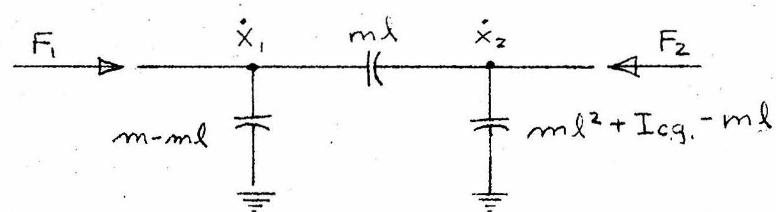


Fig. 16.

If $\ell > 1$ then one capacity would be negative unless one of the methods discussed in this section were used to eliminate it.

As a second example consider a narrow cantilever beam loaded with a force and moment applied at the end (Fig. 17). If the clamped end is restrained from warping, then Castigliano's theorem may be used to obtain the spring matrix (eq.31).*

* See Timoshenko, Theory of Elasticity, (18) page 149.

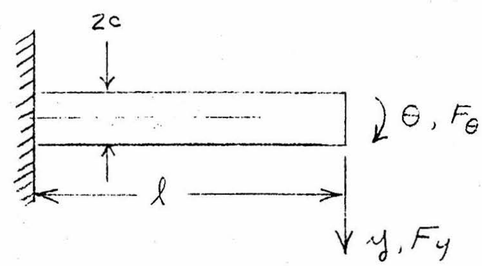


Fig. 17.

$$(31) \qquad \begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{\ell^3}{3EI} + \frac{2}{5}\frac{\ell c^2}{GI}, & +\frac{\ell^2}{2EI} \\ +\frac{\ell^2}{2EI} & \frac{\ell}{EI} \end{bmatrix} \begin{bmatrix} F_y \\ F_\theta \end{bmatrix}$$

Equation (31) is in the inverse physical coefficient matrix form, for which the general circuit analog is drawn in Fig. 13. The impedances are now inductances and the circuit is given in Fig. 18.

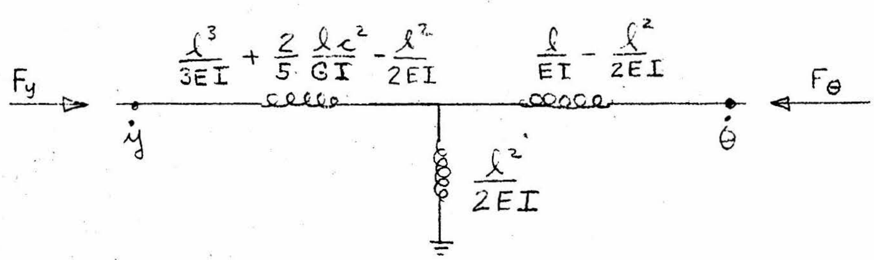


Fig. 18.

Any negative inductances may be eliminated by the methods developed in this section.

2.5 Electric Circuit Analogies for Multi-coordinate Physical Coefficient Matrices.

Multi-coordinate physical coefficient matrices will now be analyzed by methods similar to those used for the two-coordinate case. Circuit analogies for $[d]$ and $[\hat{d}] = [d]^{-1}$ will be obtained in sections 2.5a and 2.5b, and some general remarks about the number of transformers required for an arbitrary $[d]$ will be made. As examples, the analogs for a generalized mass and a generalized spring are given in section 2.5c.

2.5a Consider eq.(32) which corresponds to eqs.(15) for the two-coordinate case.

$$(32) \qquad \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{12} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1n} & d_{2n} & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

If quantities are added and subtracted from the right sides of the constituent equations of eq.(32), then the set of eqs.(33) are obtained.

$$(33) \qquad F_i = (d_{i1} + d_{i2} \cdots + d_{in})x_i - d_{i1}(x_i - x_1) \cdots - d_{in}(x_i - x_n) \qquad i = (1 \cdots n)$$

If the reference nodes for the n displacements, x_i, are common, the equivalent circuit for eqs.(33) is the generalized π. It is illustrated for n = 4 in Fig. 19.

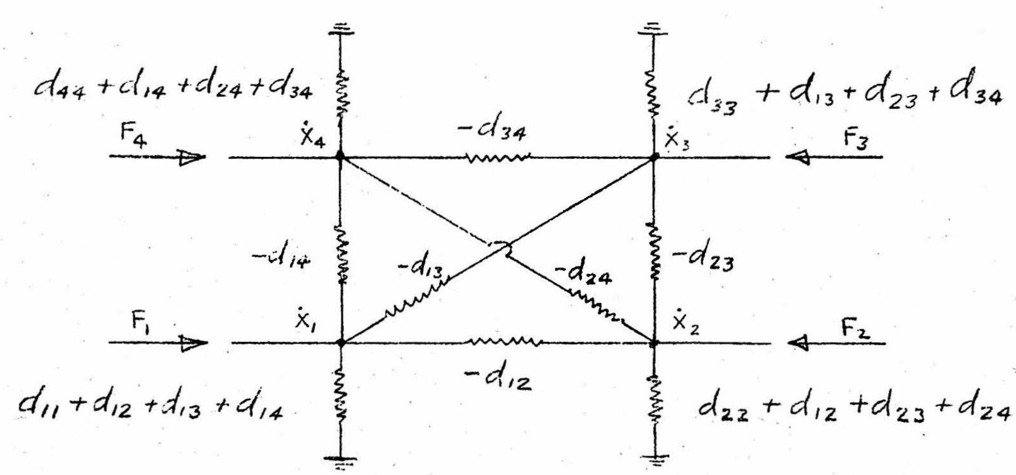


Fig. 19.

If all the off-diagonal d_{ij} are negative, and if every diagonal d_{ij} is greater than the sum of the absolute values of the off-diagonal d_{ij} in the same row, then no negative admittances are required. If negative admittances are to be eliminated, or if isolation is necessary, then transformers must be used. These transformers may be used in any of three positions which correspond to a scale change, an internal change in the β position, or a coordinate axes change. We proceed in the same manner as in the two-dimensional case and study the first two positions simultaneously and after that the coordinate axes change.

The scale change transformation, eq.(34), is used to transform $[d]$ so that in the new coordinate system eq.(35) is obtained. Quantities are added and subtracted from the right sides of the constituent equations of eq.(35) and eqs.(36) are obtained.

$$(34) \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

$$(35) \qquad \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \vdots \\ \bar{F}_n \end{bmatrix} = \begin{bmatrix} d_{11}a_1^2 & d_{12}a_1a_2 & \cdots & d_{1n}a_1a_n \\ d_{12}a_1a_2 & d_{22}a_2^2 & \cdots & d_{2n}a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ d_{1n}a_1a_n & d_{2n}a_2a_n & \cdots & d_{nn}a_n^2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

$$(36)\qquad \bar F_i = +\sum_{j=1}^{j=i-1}\left(|a_i a_j d_{ij}|\right)\left(\frac{\bar X_i}{|\beta_{ij}|} + \frac{|\beta|}{\beta}\bar X_j\right) + \left(d_{ii}a_i^2 - \sum_{j=1}^{j=i-1}\left|\frac{a_i a_j d_{ij}}{\beta_{ij}}\right| - \sum_{j=i+1}^{j=n}|\beta_{ij}a_i a_j d_{ij}|\right)\bar X_i$$
$$+\sum_{j=i+1}^{j=n}\left(|\beta_{ij} a_i a_j d_{ij}|\right)\left(\bar X_i + \frac{\bar X_j}{\beta_{ij}}\right)$$

where β_{ij} is a pure number whose sign is the same as that of $a_i a_j d_{ij}$. (The summation convention is not used). Because the terms of eq.(36) are awkward to write, a shorthand notation will be introduced. This notation will be used throughout the remainder of the thesis. Define:

$$\delta_{ai} = d_{ii}a_i^2 - \sum_{j=1}^{j=i-1}\left|\frac{a_i a_j d_{ij}}{\beta_{ij}}\right| - \sum_{j=i+1}^{j=n}|\beta_{ij}a_i a_j d_{ij}|$$

$$\delta_{Mij} = |a_i a_j \beta_{ij} d_{ij}|$$

When this notation is used for the case of three coordinates, eqs.(36) become eqs.(37).

$$\bar F_1 = \delta_{a1}\bar X_1 + \delta_{M12}\left(\bar X_1 + \frac{\bar X_2}{\beta_{12}}\right) + \delta_{M13}\left(\bar X_1 + \frac{\bar X_3}{\beta_{13}}\right)$$

$$(37)\qquad \bar F_2 = \frac{\delta_{M12}}{|\beta_{12}|}\left(\frac{\bar X_2}{|\beta_{12}|} + \frac{|\beta_{12}|}{\beta_{12}}\bar X_1\right) + \delta_{a2}\bar X_2 + \delta_{M23}\left(\bar X_2 + \frac{\bar X_3}{\beta_{23}}\right)$$

$$\bar F_3 = \frac{\delta_{M13}}{|\beta_{13}|}\left(\frac{\bar X_3}{|\beta_{13}|} + \frac{|\beta_{13}|}{\beta_{13}}\bar X_1\right) + \frac{\delta_{M23}}{|\beta_{23}|}\left(\frac{\bar X_3}{|\beta_{23}|} + \frac{|\beta_{23}|}{\beta_{23}}\bar X_2\right) + \delta_{a3}\bar X_3$$

The equivalent circuit for eqs.(37) is drawn in Fig. 20.

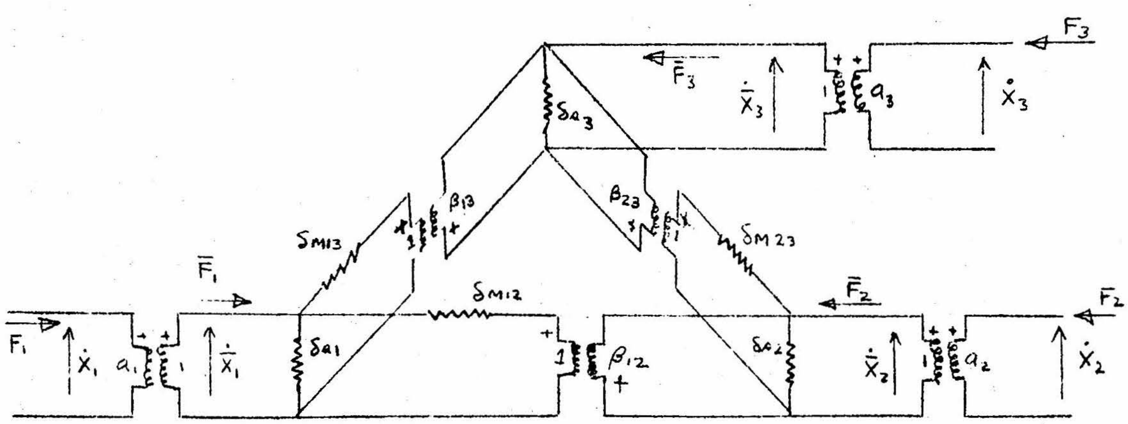


Fig. 20.

Transformers may be needed to do any combination of the following three things.

1. Establish isolation
2. Make the δ_{ei} positive
3. Invert voltage and current if d_{ij} is a positive term

The effect of transformer position on accomplishing these three things will now be discussed.

Any node-pair may be isolated either by its a_i transformer or by the β_{ij} transformers which couple it to the other nodes. Since there are n node-pairs, complete isolation may be accomplished by n-1 transformers in the a_i position or by $\frac{n(n-1)}{2}$ transformers in the β_{ij} position.

If an a_i is negative it has the effect of changing the sign of all the off-diagonal d_{ij} of the corresponding row and column. Transformers in the β_{ij} position change the sign of the corresponding d_{ij} only. These two methods of inversion can be used together. So used, they mean that no more transformers will be required for sign inversion than the number of positive d_{ij} on one side of the diagonal. This number is required only in the extreme case when the signs alternate in going along the rows; in the usual case fewer will be required.

The condition that all the δ_{ei} be positive imposes the ineqs. (38).

$$
(38)\qquad
\begin{array}{lllll}
d_{11} \geq & 0 & + \left|\frac{\beta_{12} a_2}{a_1} d_{12}\right| & - - - - - - & + \left|\frac{\beta_{1n} a_n}{a_1} d_{1n}\right| \\
d_{22} \geq & + \left|\frac{a_1}{\beta_{12} a_2} d_{12}\right| & 0 & - - - - - & + \left|\frac{\beta_{2n} a_n}{a_2} d_{2n}\right| \\
\vdots & \vdots & \vdots & & \vdots \\
d_{nn} \geq & + \left|\frac{a_1}{\beta_{1n} a_n} d_{1n}\right| & + \left|\frac{a_2}{\beta_{2n} a_n} d_{2n}\right| & - - - - - & 0
\end{array}
$$

Inspection of inequalities (38) shows that the scale change transformers

a_i change all the terms in the corresponding row and column while the β_{ij} transformers change only individual terms. The two positions are not equivalent as they are in the two dimensional case.

The existence of a set of $\frac{\beta_{ij} a_j}{a_i}$ which satisfies inequalities (38) will now be investigated. For these inequalities to be satisfied the left hand terms must certainly be larger than the individual terms on the right hand sides. This condition gives a set of inequalities of the type:

$$\left|\frac{a_1}{\beta_{12} a_2}\right| \geq \left|\frac{d_{12}}{d_{11}}\right| \quad - - - - - - \quad \left|\frac{a_1}{\beta_{1n} a_n}\right| \geq \left|\frac{d_{1n}}{d_{11}}\right|$$

$$\left|\frac{\beta_{12} a_2}{a_1}\right| \geq \left|\frac{d_{12}}{d_{22}}\right| \quad - - - \quad \left|\frac{a_2}{\beta_{2n} a_n}\right| \geq \left|\frac{d_{2n}}{d_{22}}\right|$$

$$\left|\frac{\beta_{1n} a_n}{a_1}\right| \geq \left|\frac{d_{1n}}{d_{nn}}\right| , \quad \left|\frac{\beta_{2n} a_n}{a_2}\right| \geq \left|\frac{d_{2n}}{d_{nn}}\right|$$

If these inequalities are substituted into inequalities (38) the inequalities (39) are obtained.

$$\text{(39)}\qquad \begin{aligned} d_{11} &\geq 0 + \frac{d_{12}^2}{d_{22}} + \frac{d_{13}^2}{d_{33}} + - - - - - - - + \frac{d_{1n}^2}{d_{nn}} \\ d_{22} &\geq + \frac{d_{12}^2}{d_{11}} + 0 + \frac{d_{23}^2}{d_{33}} + - - - - - - + \frac{d_{2n}^2}{d_{nn}} \\ &\;\vdots \\ d_{nn} &\geq + \frac{d_{1n}^2}{d_{11}} + \frac{d_{2n}^2}{d_{22}} + \frac{d_{3n}^2}{d_{33}} + - - - + \frac{d_{(n-1)n}^2}{d_{(n-1)(n-1)}} + 0 \end{aligned}$$

Inequalities (39) are necessary but not sufficient conditions for the existence of a set of $\frac{\beta_{ij} a_j}{a_i}$ which will make all the $\delta_{\lambda i}$ positive. This means that unless they are satisfied no set of $\frac{\beta_{ij} a_j}{a_i}$ exists and even if they are satisfied a set of $\frac{\beta_{ij} a_j}{a_i}$ may not exist.

The fact that $[d]$ is the matrix of a positive definite quadratic form means that all the discriminants of $[d]$ are positive. This fact yields the inequalities (40) which will always be satisfied by physical coefficient matrices.

$$(40) \qquad d_{11} \geqslant \frac{d_{12}^2}{d_{22}} \; ; \quad d_{11} \geqslant \frac{d_{13}^2}{d_{33}} \qquad \text{or in general for any } j \neq i, \; d_{ii} \geqslant \frac{d_{ij}^2}{d_{jj}}$$

A comparison of inequalities (39) and (40) shows that physical coefficient matrices may exist for which no electric circuit can be constructed with positive admittances and using transformers in the a_i or β_{ij} positions. For the two-coordinate case inequalities (39) and (40) are identical and inequalities (39) are both necessary and sufficient.

The coordinate axes transformation will now be investigated. An important theorem pertaining to this transformation states that an analogous circuit for any $[d]$ can always be constructed using at most $\frac{n(n-1)}{2}$ transformers. The proof is that successive elementary transformations of the type, eq.(41), can be performed where each transformation makes the corresponding off-diagonal term zero. Since there are $\frac{n(n-1)}{2}$ off-diagonal terms, this is the number of transformers required to reduce $[d]$ to a diagonal matrix.

$$(41) \qquad \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} +1 & +a_{ij} & 0 & 0 \\ 0 & +1 & 0 & 0 \\ \\ 0 & 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \\ \bar{x}_n \end{bmatrix}$$

A three coordinate $[d]$ transformed by eq.(41) is given in eq.(42).

$$(42) \qquad \begin{bmatrix} d_{11} & d_{11}a_{12} + d_{12} & d_{13} \\ d_{11}a_{12} + d_{12} & d_{11}a_1^2 + 2d_{12}a_{12} + d_{22} & d_{13}a_{12} + d_{23} \\ d_{13} & d_{13}a_{12} + d_{23} & d_{33} \end{bmatrix}$$

For isolation to be obtained with a coordinate axes transformation the off-diagonal terms must be made zero. This requires a definite turns ratio on the a_{ij} transformers, for instance, in the example $a_{12} = -\frac{d_{11}}{d_{12}}$. If isolation is not required the values of a_{ij} may be arbitrary within certain limits and still yield positive δ_{oi} and sign inversion for positive d_{ij}. The minimum number of elementary coordinate axes transformations required for any $[d]$ depends on the relative numerical values of the components of $[d]$.

In practical computation with an electric analog computer, transformers are the most expensive and the most troublesome elements used. Therefore it is always desirable to choose coordinate systems so that a minimum number are required. Unfortunately no general formula can be given which will explicitly determine this optimum coordinate system. In general, a coordinate axes, a scale change and an internal β_{ij} position transformation will be required simultaneously.

2.5b. Circuit analogies for the inverse physical coefficient matrix, $[d]^{-1} = [\hat{d}]$, will now be investigated. They can be treated in much the same manner as that used for the matrix $[d]$. Consider eq.(42)

$$(42) \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} \hat{d}_{11} & \hat{d}_{12} & \cdots & \hat{d}_{1n} \\ \hat{d}_{12} & \hat{d}_{22} & \cdots & \hat{d}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{d}_{1n} & \hat{d}_{2n} & \cdots & \hat{d}_{nn} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$$

This equation may be transformed by a scale change and by adding and subtracting quantities in exactly the same way as its inverse was transformed. This means that eqs.(34), (35), (36) and (37) can be applied to eq.(42) if in these equations d_{ij} is replaced by $\hat{d}_{ij}$, x_i by F_i and F_i

by x_i. The circuit for a three coordinate $[d]^{-1}$ is shown in Fig. 21 where, for simplicity, the scale change transformers are not shown.

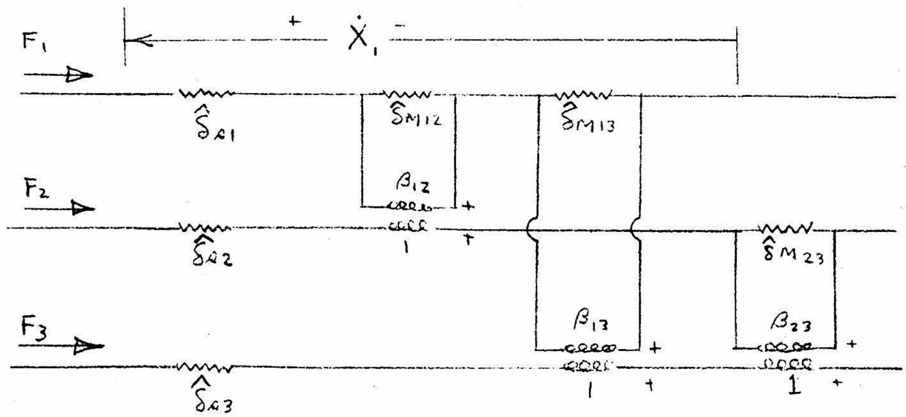


Fig. 21.

The condition that the $\hat{\delta}_{\alpha i}$ be positive imposes the inequalities (38), and a necessary condition that any combination of a_i and β_{ij} transformers exists which satisfy these inequalities is given by inequalities (39). (In these inequalities d_{ij} is to be replaced by $\hat{d}_{ij}$). If no set of a_i and β_{ij} exist, then a coordinate axes change must be made. In the same manner as in the circuit for $[d]$, transformers may be used in the a_i or β_{ij} positions to invert voltage and current if any one of the $\hat{d}_{ij}$ is negative.

If the three conditions on page 38 are satisfied, then the circuit for $[d]$ requires no transformers. This is not true in the circuit for $[d]^{-1}$. For $n>3$, transformers in at least some of the β_{ij} positions will always be required.* No β_{ij} transformers are required for $n = 3$ but for all physical problems likely to be encountered one scale change transformer is needed for isolation if the reference nodes are common and two are needed if they are not. The circuit for the three-coordinate case

* This can be proved by topological arguments. See Gardner and Barnes (7) page 49.

when the reference nodes are common is drawn in Fig. 22.

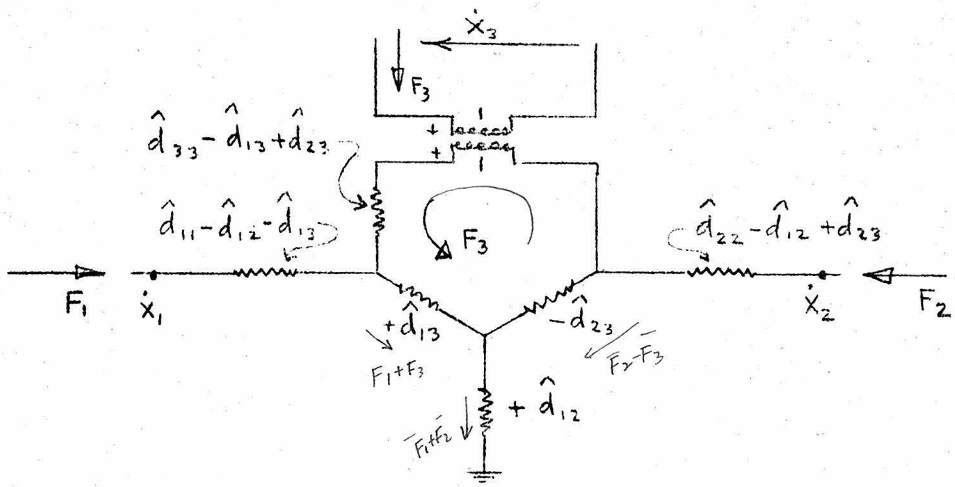


Fig. 22.

Another coordinate circuit may be coupled to that of Fig. 22 to yield the circuit for $n = 4$, only by using three β_{ij} transformers. Furthermore, three is a minimum number obtained only when the δ_{ii} are all positive. The minimum number of transformers for any circuit can be obtained by combining circuits such as Fig. 21 and Fig. 22 with those obtained by coordinate axes change.

For two coordinates it was found that if a transformer was or was not required for $[d]$ then correspondingly one was or was not required for $[\hat{d}]$. This is no longer true in the multi-coordinate case. $[d]$ may have no zero terms and require many transformers while $[\hat{d}]$ may have many zero terms and require few transformers, or the converse may be true. Because of this fact the search for a circuit with minimum number of transformers which is analogous to any multi-coordinate physical system should include investigation of both $[d]$ and $[\hat{d}]$.

2.5c. Two simple examples of multi-coordinate physical coefficient matrices will now be given.

An airplane engine is suspended by springs so that it can yaw, roll and move parallel to the axis of the wing. These motions correspond to the antisymmetric vibrations of the airplane as a whole. The circuit analog for the three-degree-of-freedom rigid mass mounted on the springs is required. To describe the mass a coordinate system will be used whose three components are the small displacements along the spring axes of the three points in the mass at which the springs are attached. Fig. 23 illustrates the situation. The behavior of the rigid body mass expressed in a cartesian coordinate system with origin at the center of mass is given by eq.(43).

$$\begin{bmatrix} F_y \\ M_x \\ M_z \end{bmatrix} = \begin{bmatrix} m & 0 & 0 \\ 0 & I_{xx} & I_{xz} \\ 0 & I_{xz} & I_{zz} \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta}_x \\ \ddot{\theta}_z \end{bmatrix} \tag{43}$$

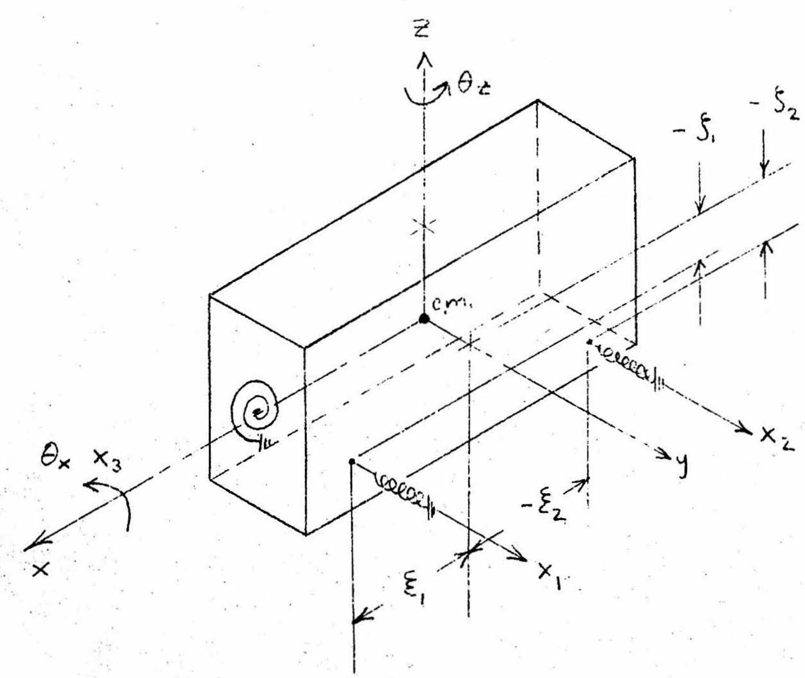


Fig. 23.

The linear transformation from the y, θ_x, θ_z to the generalized coordinates x_1, x_2, x_3 is given by:

$$x_1 = y - \zeta_1 \theta_x + \xi_1 \theta_z$$
$$x_2 = y - \zeta_2 \theta_x + \xi_2 \theta_z$$
$$x_3 = \theta_x$$

When these equations are inverted they become eq.(44).

$$\text{(44)} \qquad \begin{bmatrix} y \\ \theta_x \\ \theta_z \end{bmatrix} = \frac{1}{\xi_2 - \xi_1} \begin{bmatrix} \xi_2 & -\xi_1 & \xi_2\zeta_1 - \xi_1\zeta_2 \\ 0 & 0 & +1 \\ -1 & +1 & \zeta_2 - \zeta_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The components of the mass matrix obtained when eq.(43) is transformed by eq.(44) are:

$$m_{11} = + \frac{m\xi_2^2 + I_{zz}}{(\xi_2 - \xi_1)^2}$$

$$m_{12} = - \frac{m\xi_1\xi_2 + I_{zz}}{(\xi_2 - \xi_1)^2}$$

$$m_{13} = + \frac{m\xi_2(\zeta_1\xi_2 - \zeta_2\xi_1) + I_{zz}(\zeta_1 - \zeta_2)}{(\xi_2 - \xi_1)^2} - \frac{I_{xz}}{\xi_2 - \xi_1}$$

(45)

$$m_{22} = + \frac{m\xi_1^2 + I_{zz}}{(\xi_2 - \xi_1)^2}$$

$$m_{23} = + \frac{m\xi_1(\zeta_2\xi_1 - \zeta_1\xi_2) + I_{zz}(\zeta_2 - \zeta_1)}{(\xi_2 - \xi_1)^2} - \frac{I_{xz}}{\xi_2 - \xi_1}$$

$$m_{33} = + \frac{m(\xi_1\zeta_2 - \xi_2\zeta_1)^2 + I_{zz}(\zeta_1 - \zeta_2)^2}{(\xi_2 - \xi_1)^2} + I_{xx} + \frac{2I_{xz}(\zeta_2 - \zeta_1)}{\xi_2 - \xi_1}$$

For the specific problem considered the numerical values are:

$\mathcal{E}_1 = -14.2$ in. $\qquad \mathcal{E}_2 = +55.8$ in.

$\zeta_1 = +10.8$ in. $\qquad \zeta_2 = -25.8$ in.

$$m = 11.1 \ \frac{\text{lb-sec}^2}{\text{in}}$$

$$I_{xx} = 3560 \ \text{lb-in-sec}^2$$

$$I_{zz} = 34000 \ \text{lb-in-sec}^2$$

$$I_{xz} = 0$$

These numerical values substituted into eqs.(45) together with a scale change of $x_3 = \frac{1}{30}\bar{x}_3$ yield eq.(46). No transformer is required for this scale change because the spring to which the mass is attached is also transformed. The problem is now to find a circuit analog for eq.(46) which has no negative admittances. Isolation is not required because the reference nodes, which correspond to the equilibrium position of the mass, are common.

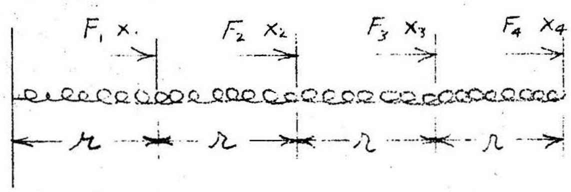
$$(46) \qquad \begin{bmatrix} F_1 \\ F_2 \\ \bar{F}_3 \end{bmatrix} = \begin{bmatrix} +14.0 & -5.14 & +9.47 \\ -5.14 & +7.40 & -8.23 \\ +9.47 & -8.23 & +14.4 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\bar{x}}_3 \end{bmatrix}$$

Since for each row the sum of the absolute values of the off-diagonal terms is greater than the diagonal term and since d_{13} is positive, it is apparent that some sort of transformation is required. Neither the mass matrix of eq.(46) nor its inverse satisfy the inequalities (39), therefore a coordinate axes transformation must be made. An inspection of the three possible elementary transformations shows that the a_{13} position is preferable. This transformation gives the matrix (47)

$$(47) \qquad \begin{bmatrix} m_{11} & m_{12} & a_{13}m_{11} + m_{13} \\ m_{12} & m_{22} & a_{13}m_{12} + m_{23} \\ a_{13}m_{11} + m_{13} & a_{13}m_{12} + m_{23} & a_{13}^2 m_{11} + 2a_{13}m_{13} + m_{33} \end{bmatrix}$$

When numerical values are substituted into matrix (47) the requirement that all the δ_{oi} be positive and that the transformed d_{13} be negative imposes the condition that $+1.31 > a_{13} > +1.16$. As a convenient value take $a_{13} = 1.25$ which gives a transformer turns ratio of .8 to 1. The resulting circuit is drawn in Fig. 26. The single transformer has not only made all the δ_{oi} positive but has also inverted the sign of the d_{13}.

As a second example consider a uniform cantilever shaft twisted by couples equally spaced along its length. This is the same problem as that of a bar loaded with pure tensile forces, Fig. 24, or that of a beam loaded with bending moments.



k = spring constant

$= \frac{JG}{r}$ for the shaft

$= \frac{E}{r}$ for the bar

$= \frac{EI}{R}$ for the beam

Fig. 24.

In the form $[x] = [\hat{d}][F]$, which yields the impedance analogy, the behavior of the system is described by the influence coefficient matrix eq.(48). This is the form in which many complicated mechanical systems are described and is that which is obtained from Castigliano's theorem.

$$(48) \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{k} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

It is apparent that the analog for this matrix requires many transformers. If the equation is inverted to the admittance analogy form, $[F] = [d][x]$, eq.(49) is obtained.

$$(49) \qquad \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = k \begin{bmatrix} +2 & -1 & 0 & 0 \\ -1 & +2 & -1 & 0 \\ 0 & -1 & +2 & -1 \\ 0 & 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

In this form all the δ_{oi} except δ_{o1} are zero, all the δ_{Mij} are negative and no isolation is required. The circuit, Fig. 25, is a generalized π with many zero admittances. The admittances are inverse inductances.

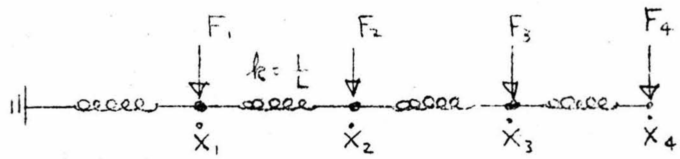


Fig. 25.

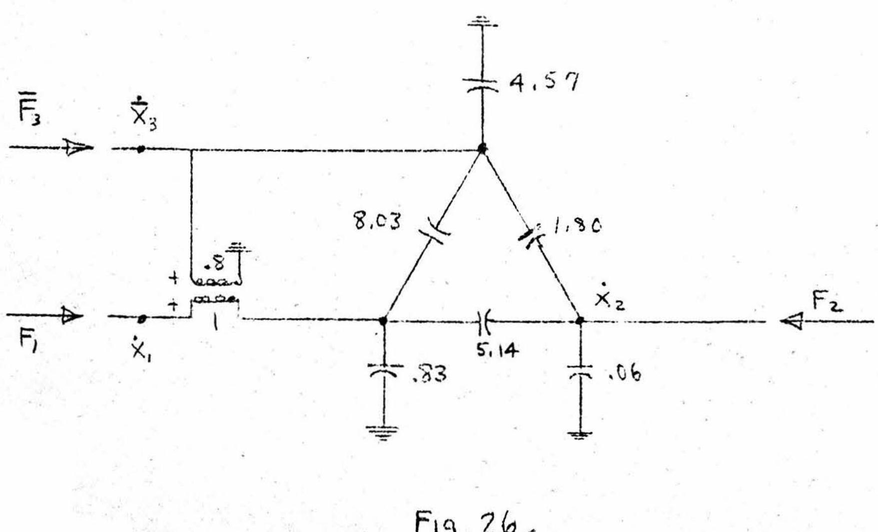


Fig. 26.

PART III

APPLICATIONS TO CONTINUOUS SYSTEMS

The analogies for elementary systems developed in part II may be considered as tools which will be applied in part III to continuous systems. Section 3.1 considers the shaft loaded by concentrated torques from a different viewpoint than was done in section 2.5. Nomenclature and ideas which normally would not be used on so simple a system are introduced in order that a parallelism may be made with the case of the simple beam, section 3.2, and with the generalized case. The methods and nomenclature developed in sections 3.1 and 3.2 are generalized and applied to a more complex system in section 3.3. In the remaining sections of part III various topics pertaining to beams and frames are discussed.

3.1 Notation and Concepts used in the Analysis of Beams. The Shaft in Torsion

In section 2.5c the behavior of a uniform massless shaft loaded by equally spaced concentrated couples was specified by an influence coefficient matrix, or by its inverse, the spring matrix. The electric circuit analogies for both matrices are given in part II. The influence matrix method of approach given in section 2.5c is not convenient for more complicated systems nor for boundary conditions which are a function of time. We now consider a much more powerful method of describing the mechanical system and of obtaining circuit analogies.

Imagine that the shaft (or bar in tension) is divided into n sections which have no mass. Between any two sections is a point upon which the external forces act and which has a lumped point mass equal to that of the shaft in the neighborhood of the point. For the present D'Alembert's principle will be used and this mass will be accounted for by an externally applied inertia force. Each of the sections has a spring constant, k, equal to that of the corresponding section of the continuous shaft. The sections need not be equal in length nor the shaft uniform, but for convenience in the discussion that follows the k's are assumed equal. The situation and the nomenclature used is illustrated in Fig. 27.

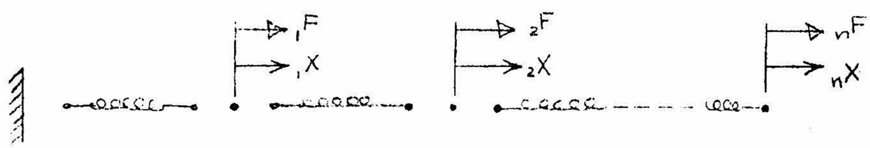


Fig. 27.

Each section may be considered as an elementary system whose behavior may be described by $F = k\,x$ or $x = g\,F$. The nomenclature used to describe this elementary system and more complicated ones to follow will now be developed. For a reason which will be apparent later this will be done using matrix notation, although in the case of the shaft the matrices have only one component and the square brackets may be ignored. The two ends of the spring are denoted by a and b. $[F_{ai}]$, $[F_{bi}]$ are the external forces acting on the ends, and $[x_{ai}]$ and $[x_{bi}]$ are the displacements of the ends. These quantities are positive in the direction shown in Fig. 28. An i post-subscript denotes the ith component of the matrix while an i pre-subscript indicates the ith section in the complete system.

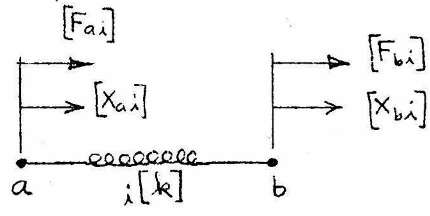


Fig. 28.

A very important concept which forms the basis for much that follows and which will now be explained is that $[x_b]$ can be considered as the sum of a rigid body displacement, $[x_\lambda]$, and an elastic displacement, $[x_e]$. Suppose that the spring is rigidly clamped and that $[F_{a\lambda}]$ is then applied to end a and that it is displaced by $[x_a]$. The theory of rigid body displacements developed in section 2.3 is applied to obtain:

$$[x_\lambda] = [\alpha][x_a] \qquad \text{(for the shaft section, } \alpha = +1\text{)}$$
$$[F_b] = -[\alpha']^{-1}[F_a]$$

The clamp on the spring is then removed, end a is not moved and end b displaces elastically, $[x_e]$, under the load $[F_b]$. Using these concepts the behavior of the system may be specified by eqs.(50). This section of the thesis and the two that follow are concerned with the investigation of these very important equations.

(50.1) $$[F_b] = [k][x_e]$$

(50.2) $$[x_b] = [x_\lambda] + [x_e]$$

(50.3) $$[x_\lambda] = [\alpha][x_a]$$

(50.4) $$[F_b] = -[\alpha']^{-1}[F_a]$$

(50.5) $$[x_e] = [\delta][F_b]$$

For the case of the shaft section these equations become eqs.(51)

(51)
$$F_b = k\, x_e$$
$$x_b = x_a + x_e$$
$$F_b = -F_a$$

If the a end of the spring were fixed or grounded, $[x_a]$ would be zero, $[x_e]$ would equal $[x_b]$ and eqs.(51) would reduce to the form $[F] = [k][x]$

to which the theory of part II could be applied directly. In general, both ends of the spring move and both displacements must be represented by node pair voltages in the electric circuit. There are two general methods of obtaining a circuit analogy for eq.(50) and these correspond to the analogies for $[d]$ and $[d]^{-1}$ studied in part II.

The admittance method proceeds by reducing and rearranging eqs.(50) in the following manner. Eq.(50.3) is substituted into eq.(50.2). The result is then substituted into eq.(50.1) to give eq.(52.1). Eq.(50.4) is then substituted into eq.(52.1) to give eq.(52.2). These two equations are combined into one matrix equation, eq.(52.3) and this equation is then another statement of the relations contained in eqs.(50).

(52.1) $$[F_b] = [k]\Big[[x_b]-[\alpha][x_a]\Big]$$

(52.2) $$[F_a] = -[\alpha]^{t}[k]\Big[[x_b]-[\alpha][x_a]\Big]$$

(52.3) $$\begin{bmatrix}[F_a]\\ [F_b]\end{bmatrix} = \begin{bmatrix}h_{aa} & h_{ab}\\ h_{ab} & h_{bb}\end{bmatrix}\begin{bmatrix}[x_a]\\ [x_b]\end{bmatrix}$$

For the shaft these equations become:

$$F_b = +k(x_b - x_a)$$

$$F_a = -k(x_b - x_a)$$

The matrix $[h]$ in eq.(52.3) is symmetrical and singular. That it is so follows from the reciprocity theorem and the fact that only the relative displacement of the two end points depends on the force applied across the spring. The circuit analogy for eq.(52.3) is the generalized π. Isolation is not required because $[x_a]$ and $[x_b]$ are measured from the at rest or zero position. In the circuit for the shaft section, Fig. 29, both the $\delta\alpha_\lambda$ are zero.

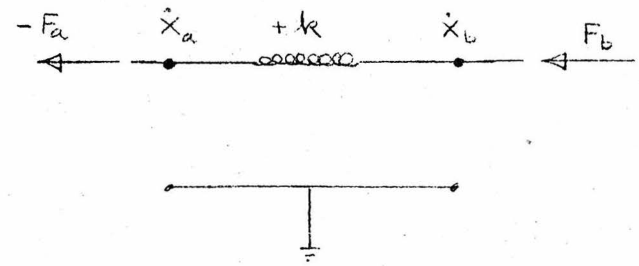


Fig. 29.

The impedance method proceeds by constructing circuits for the rigid body transformation eq.(50.3) and either the spring or inverse spring matrices, eq.(50.1) or eq.(50.5). Then in order to add $[\mathbf{x}_e]$ to $[\mathbf{x}_\lambda]$ the two circuits are placed in series. The resulting circuit for the shaft section, eqs.(51) is given in Fig. 30.

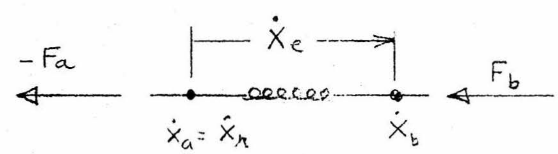


Fig. 30.

A comparison of Fig. 29 and Fig. 30 shows that for the shaft section the two circuits are the same. Note that positive F_b corresponds to tension in the spring and that both displacements (voltages) and tensions (currents) may be measured in the analog.

Circuit analogies for the individual sections of the shaft have now been obtained. The next step is to combine these circuits to obtain the circuit for the complete shaft. To do this Newton's law is applied to the points between the sections. As a notation convention let each section assume the number of the junction point on its right and let the forces, displacements or parameters of the ith section be denoted by a pre-subscript. The external forces applied to the junction points and the displacements of the points have no letter subscripts. A typical junction point with all the forces and displacements labelled is shown in Fig. 31.

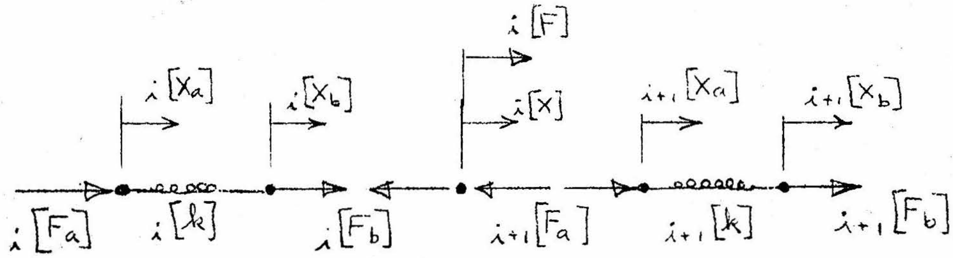


Fig. 31.

In all such systems the ends of the sections have the same displacements as the points to which they are attached. Hence:

(53) $${}_i[x_b] = {}_i[x] = {}_{i+1}[x_a]$$

Newton's law applied to the point gives:

(54) $${}_i[F] = {}_i[F_b] + {}_{i+1}[F_a] \qquad i = (1 \cdots n)$$

For the case of the shaft with $n = 3$ eqs.(55) are obtained.

(55) $$\begin{aligned} F_1 &= 2\,k\,x_1 - k\,x_2 \\ F_2 &= -\,k\,x_1 + 2\,k\,x_2 - k\,x_3 \\ F_3 &= \qquad\quad -k\,x_2 + k\,x_3 \end{aligned}$$

Equations (54) and (55) are also Kirchhoff's law for currents flowing into a node. Together with eq.(53) they show that the individual circuits representing the separate sections should be placed in series to form the complete analog, Fig. 32. Eqs.(55) and Fig. 32 are identical to those obtained in section 2.5c by inverting the influence coefficient matrix, (eq.(49) and Fig. 26).

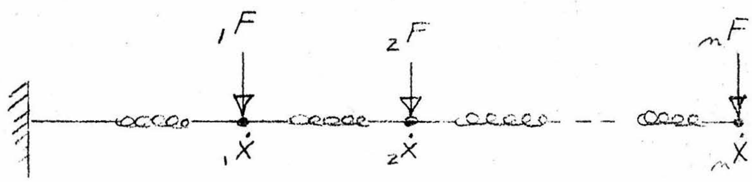


Fig. 32.

So far in the discussion the $_iF$ have been considered as applied external forces of any nature whatsoever. If the points each have a point mass $_i[m]$ and no external applied forces then the $_i[F]$ are given by eq.(56).

(56) $$_i[F] = -_i[m]\,_i[\ddot{x}]$$

The analog for a one-coordinate mass is a condenser, therefore when the circuit analogs for the masses are substituted for the external forces the circuit becomes that of Fig. 33.

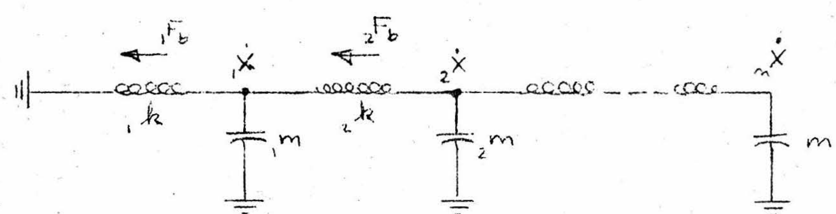


Fig. 33.

Fig. 33 is the well known ladder network which is the finite difference analog for continuous systems described by the one dimensional wave equation.

3.2 Analogies for a Beam Bending in a Principal Plane

In this section analogies will be obtained for a beam bending in a principal plane by precisely the same method as was used for the shaft in section 3.1. The matrix equations will be the same except that here they will have two coordinates instead of one. It will be assumed for the time being that the beam behavior follows the technical theory of beams, that is, that cross sections plane in the unloaded beam remain plane in the loaded beam. The beam is loaded with concentrated transverse forces ${}_iP$ and bending couples ${}_iM$ whose positive directions are shown in Fig. 34.

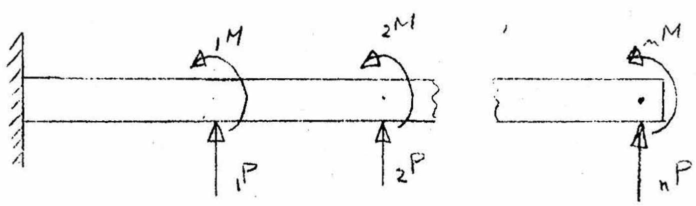


Fig. 34.

Imagine that the beam is divided into n massless sections and that between any two sections is a rigid plane upon which the external force and couple act. The nomenclature used and the positive directions of quantities are shown for a typical beam section in Fig. 35.

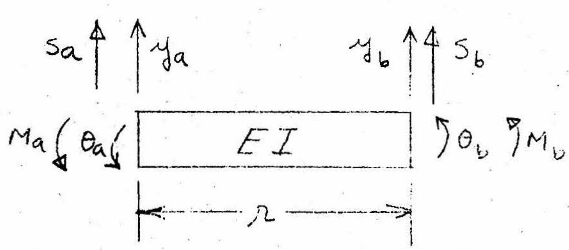


Fig. 35.

The concepts which were developed in section 3.1 and which are embodied in eqs.(50) will now be applied to the beam section of Fig. 35. Eqs.(50) for the beam section become eqs.(57). (Eqs.(50) are written at the side for reference)

(57.1) $$\begin{bmatrix} S_b \\ M_b \end{bmatrix} = EI \begin{bmatrix} \frac{12}{r^3} & -\frac{6}{r^2} \\ -\frac{6}{r^2} & +\frac{4}{r} \end{bmatrix} \begin{bmatrix} y_e \\ \theta_e \end{bmatrix} \qquad [F_b] = [\mathbf{k}][\mathbf{x}_e]$$

(57.2) $$\begin{bmatrix} y_b \\ \theta_b \end{bmatrix} = \begin{bmatrix} y_r \\ \theta_r \end{bmatrix} + \begin{bmatrix} y_e \\ \theta_e \end{bmatrix} \qquad [\mathbf{x}_b] = [\mathbf{x}_r] + [x_e]$$

(57.3) $$\begin{bmatrix} y_r \\ \theta_r \end{bmatrix} = \begin{bmatrix} +1 & +r \\ 0 & +1 \end{bmatrix} \begin{bmatrix} y_a \\ \theta_a \end{bmatrix} \qquad [\mathbf{x}_r] = [\boldsymbol{\alpha}][\mathbf{x}_a]$$

(57.4) $$\begin{bmatrix} S_b \\ M_b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ +r & -1 \end{bmatrix} \begin{bmatrix} S_a \\ M_a \end{bmatrix} \qquad [F_b] = -[\alpha']^{-1}[F_a]$$

(57.5) $$\begin{bmatrix} y_e \\ \theta_e \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} +\frac{r^3}{3} & +\frac{r^2}{2} \\ +\frac{r^2}{2} & +r \end{bmatrix} \begin{bmatrix} S_b \\ M_b \end{bmatrix} \qquad [\mathbf{x}_e] = [g][F_b]$$

The components of $[\alpha]$ are obtained from the geometry of the beam section. The components of $[g]$ may be obtained by using Castigliano's theorem in the following manner. From the theory of strength of materials we obtain the expression for strain energy in the beam:

$$U = \frac{1}{2}\int_0^r \frac{M^2}{EI}\, d\xi \qquad (\xi = \text{distance measured from end b toward end a})$$

In the case considered here $M = S_b\xi + M_b$. If EI is a function of r it should be taken into account in the integration. Here, for definiteness in the example, EI has been assumed constant. Thus:

$$U = \frac{1}{2EI}\left(\frac{S_b^2 r^3}{3} + M_b^2 r + S_b M_b r^2\right)$$

Eq.(57.5) is obtained by:

$$\frac{\partial U}{\partial S_b} = y_e = +\frac{\lambda^3}{3EI} S_b + \frac{\lambda^2}{2EI} M_b$$

$$\frac{\partial U}{\partial M_b} = \theta_e = +\frac{\lambda^2}{2EI} S_b + \frac{\lambda}{EI} M_b$$

The two general methods of approach discussed in section 3.1 will now be applied to obtain analogies for eqs.(57). It will be found that the circuits finally obtained by the two methods will be identical although the circuits obtained in intermediate steps are quite different.

The admittance analogy is obtained by the same method that was used in section 3.1. Eqs.(57) are substituted into eqs.(52) and eqs.(59) are obtained as a result.

(59.1)
$$\begin{bmatrix} S_b \\ M_b \end{bmatrix} = EI \begin{bmatrix} +\frac{12}{\lambda^3} & -\frac{6}{\lambda^2} \\ -\frac{6}{\lambda^2} & +\frac{4}{\lambda} \end{bmatrix} \begin{bmatrix} y_b - (y_a + \lambda\theta_a) \\ \theta_b - \theta_a \end{bmatrix}$$

(59.2)
$$\begin{bmatrix} S_a \\ M_a \end{bmatrix} = EI \begin{bmatrix} -1 & 0 \\ -\lambda & -1 \end{bmatrix} \begin{bmatrix} +\frac{12}{\lambda^3} & -\frac{6}{\lambda^2} \\ -\frac{6}{\lambda^2} & +\frac{4}{\lambda} \end{bmatrix} \begin{bmatrix} y_b - (y_a + \lambda\theta_a) \\ \theta_b - \theta_a \end{bmatrix}$$

(59.3)
$$\begin{bmatrix} S_a \\ M_a \\ S_b \\ M_b \end{bmatrix} = EI \begin{bmatrix} +\frac{12}{\lambda^3} & +\frac{6}{\lambda^2} & -\frac{12}{\lambda^3} & +\frac{6}{\lambda^2} \\ +\frac{6}{\lambda^2} & +\frac{4}{\lambda} & -\frac{6}{\lambda^2} & +\frac{2}{\lambda} \\ -\frac{12}{\lambda^3} & -\frac{6}{\lambda^2} & +\frac{12}{\lambda^3} & -\frac{6}{\lambda^2} \\ +\frac{6}{\lambda^2} & +\frac{2}{\lambda} & -\frac{6}{\lambda^2} & +\frac{4}{\lambda} \end{bmatrix} \begin{bmatrix} y_a \\ \theta_a \\ y_b \\ \theta_b \end{bmatrix}$$

The circuit for eq.(59.3) is the generalized π. It is shown with negative admittances in Fig. 36.

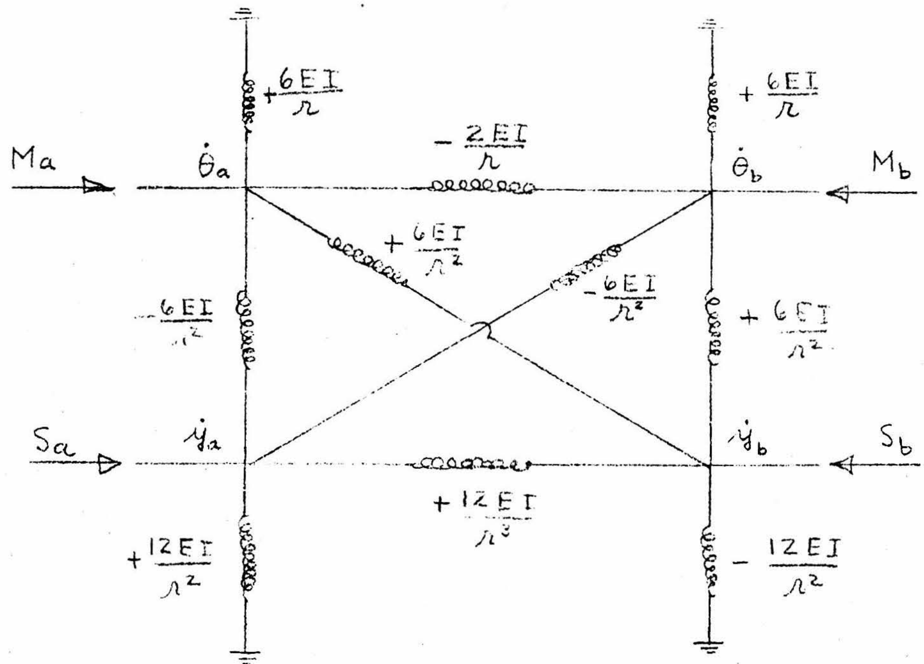


Fig. 36. (the indicated values are of inverse inductance)

Eq.(59.3) and Fig. 36 are the same as those given by Kron (10).*

To obtain the circuit for the complete beam, Newton's law is applied to the rigid planes between the beam sections. The convention for designating the sections and rigid planes between the sections which was explained at the bottom of page 53 will be used. Eqs.(53) and (54) for the beam become:

$$(60) \qquad \begin{bmatrix} {}_iy \\ {}_i\theta \end{bmatrix} = \begin{bmatrix} {}_iy_b \\ {}_i\theta_b \end{bmatrix} = \begin{bmatrix} {}_{i+1}y_a \\ {}_{i+1}\theta_a \end{bmatrix}$$

$$(61) \qquad \begin{bmatrix} {}_iP \\ {}_iM \end{bmatrix} = \begin{bmatrix} {}_iS_b \\ {}_iM_b \end{bmatrix} + \begin{bmatrix} {}_{i+1}S_a \\ {}_{i+1}M_a \end{bmatrix}$$

Upon substituting the values for $[F_b]$ and $[F_a]$ obtained from eqs.(59.1)

---

* Kron obtains Fig. 36 by a different approach than is used here and he uses his own "tensorial" methods. See his book (15).

and (59.2) in eq.(61), and by making use of eqs.(60), eq.(62) is obtained. The terms of eq.(62) are added to give eq.(63). When eq.(63) is written for a beam with a particular number of cells and the boundary conditions are considered, it is an equation of the form $[F] = [d][x]$ for which analogies were studied in part II. The inverse of eq.(63) for any given beam becomes the influence coefficient equation $[x] = [g][F]$. In part IV this equation is given for a cantilever beam.

$$(62)\quad \begin{bmatrix} {}_iP \\ {}_iM \end{bmatrix} = EI \begin{bmatrix} +\frac{12}{\lambda^3} & -\frac{6}{\lambda^2} \\ -\frac{6}{\lambda^2} & +\frac{4}{\lambda} \end{bmatrix} \begin{bmatrix} {}_iy - {}_{i+1}y - {}_{i-1}\theta_\lambda \\ {}_i\theta - {}_{i-1}\theta \end{bmatrix} + EI \begin{bmatrix} -\frac{12}{\lambda^3} & +\frac{6}{\lambda^2} \\ -\frac{6}{\lambda^2} & +\frac{2}{\lambda} \end{bmatrix} \begin{bmatrix} {}_{i+1}y - {}_iy - {}_i\theta_\lambda \\ {}_{i+1}\theta - {}_i\theta \end{bmatrix}$$

$$(63)\quad \begin{bmatrix} {}_iP \\ {}_iM \end{bmatrix} = EI \begin{bmatrix} -\frac{12}{\lambda^3} & -\frac{6}{\lambda^2} & +\frac{24}{\lambda^3} & 0 & -\frac{12}{\lambda^3} & +\frac{6}{\lambda^2} \\ +\frac{6}{\lambda^2} & +\frac{2}{\lambda} & 0 & +\frac{8}{\lambda} & -\frac{6}{\lambda^2} & +\frac{2}{\lambda} \end{bmatrix} \begin{bmatrix} {}_{i-1}y \\ {}_{i-1}\theta \\ {}_iy \\ {}_i\theta \\ {}_{i+1}y \\ {}_{i+1}\theta \end{bmatrix}$$

Eqs.(60) to (64) show that the circuit for a complete beam is obtained by placing the section circuits of Fig. 36 in series. When this is done some of the admittances which are in parallel cancel. The circuit for a cantilever beam with $n = 3$ is shown in Fig. 37. The boundary condition at the clamped end is imposed by grounding θ_a and y_a. This shorts some of the admittances and places others in parallel making possible some simplifications not shown in the figure. The circuit is the same as that obtained from $[k]$ of eq.(63).

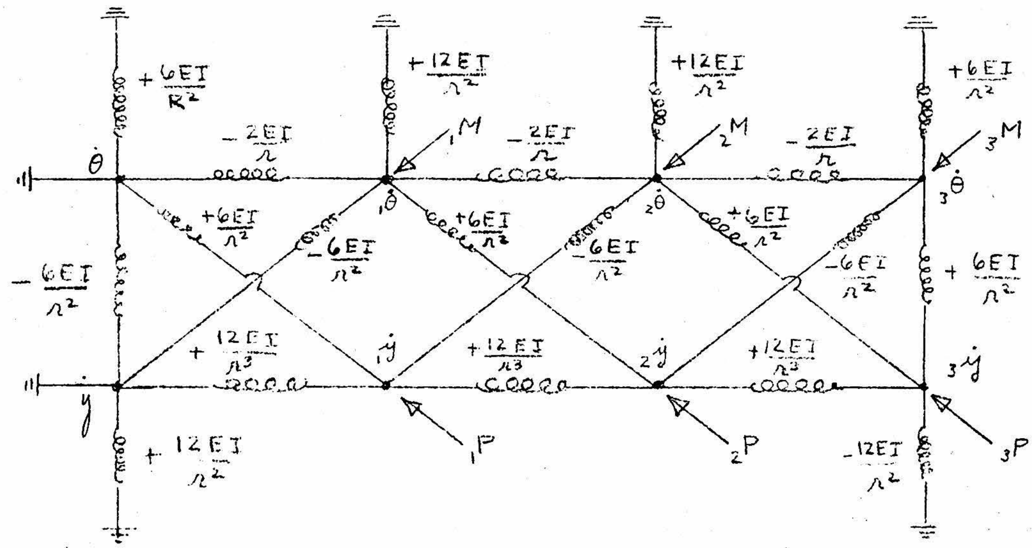


Fig. 37.

Eq.(59.3), the equation which describes the behavior of the beam section in the admittance method, yields negative admittances in its equivalent circuit and provides for no scale change in passing from the mechanical to the electrical system. To eliminate these effects the methods of part II will be used and in particular the scale change, eq.(64), and coordinate axes transformation, eq.(65), will be applied.

$$
\begin{aligned}
y_a &= a\,\bar{y}_a & S_a &= \frac{1}{a}\bar{S}_a \\
\theta_a &= b\,a\,\bar{\theta}_a & M_a &= \frac{1}{ba}\bar{M}_a \\
y_b &= a\,\bar{y}_b & S_b &= \frac{1}{a}\bar{S}_b \\
\theta_b &= b\,a\,\bar{\theta}_b & M_b &= \frac{1}{ba}\bar{M}_b
\end{aligned}
\tag{64}
$$

$$
\begin{bmatrix} \bar{y}_a \\ \bar{\theta}_a \\ \bar{y}_b \\ \bar{\theta}_b \end{bmatrix} =
\begin{bmatrix} +1 & +a_{12} & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & +a_{24} \\ 0 & 0 & 0 & +1 \end{bmatrix}
\begin{bmatrix} x_{a1} \\ x_{a2} \\ x_{b1} \\ x_{b2} \end{bmatrix}
\tag{65}
$$

If a_{12} and a_{24} are taken as $a_{12} = -\frac{br}{2}$, $a_{24} = +\frac{br}{2}$ and eqs.(64) and (65) are used to transform eq.(59.3), then the very much simplified eq.(66)

results. Eq.(65) with the values of a_{12} and a_{24} substituted are given as eqs.(67).

$$(66)\qquad \begin{vmatrix} F_{a1} \\ F_{a2} \\ F_{b1} \\ F_{b2} \end{vmatrix} = EI \begin{bmatrix} +\frac{12a^2}{r^3} & 0 & -\frac{12a^2}{r^3} & 0 \\ 0 & +\frac{b^2a^2}{r} & 0 & -\frac{b^2a^2}{r} \\ -\frac{12a^2}{r^3} & 0 & +\frac{12a^2}{r^3} & 0 \\ 0 & -\frac{b^2a^2}{r} & 0 & +\frac{b^2a^2}{r} \end{bmatrix} \begin{bmatrix} x_{a1} \\ x_{a2} \\ x_{b1} \\ x_{b2} \end{bmatrix}$$

$$(67)\qquad \begin{aligned} \bar{y}_a &= x_{a1} - \frac{br}{2} x_{a2} & \bar{S}_a &= F_{a1} \\ \bar{\theta}_a &= x_{a2} & \bar{M}_a &= +\frac{br}{2} F_{a1} + F_{a2} \\ \bar{y}_b &= x_{b1} + \frac{br}{2} x_{b2} & \bar{S}_b &= F_{b1} \\ \bar{\theta}_b &= x_{b2} & \bar{M}_b &= -\frac{br}{2} F_{b1} + F_{b2} \end{aligned}$$

The analogous circuit for eq.(66) is given if Fig. 38. In this figure the currents and voltages are given by eqs.(67).

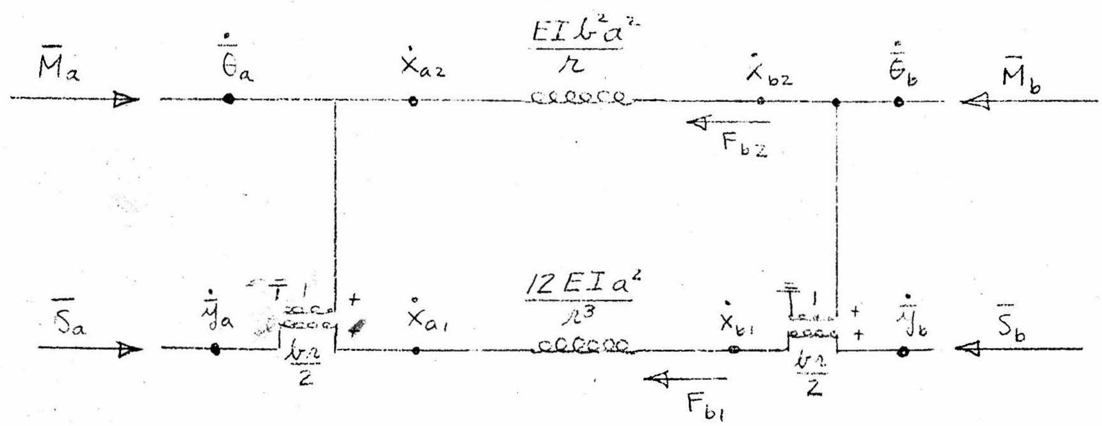


Fig. 38.

The circuit analog for a beam of more than one section is obtained by placing n circuits like Fig. 38 in series. This circuit will be discussed after the impedance method of approach has been investigated.

Analogies obtained by the impedance method will now be studied. A circuit which can take several forms will be obtained which, upon a coordinate axes transformation, becomes the circuit of Fig. 38. In this method the circuit for either the spring equation eq.(57.1), or the inverse spring equation eq.(57.5), is placed in series with the circuit for the rigid body transformation, eq.(57.3). All of these circuits were studied in detail in part II. The reference nodes for $\bar{y}_e$ and $\bar{\theta}_e$ are $\bar{y}_r$ and $\bar{\theta}_r$, and since these are not the same, isolation is required. Furthermore, $|k_{12}| > k_{22}$ so that a transformer is required to eliminate a negative δ_{22}. Since we are dealing with two-dimensional, physical coefficient matrices either the T or the π circuit (see section 2.4) may be used and the single transformer may be placed in the β, the a_1, or the a_2 position. In order to construct a practical electric circuit the scale changes used in the admittance analogy are introduced and used to transform all the eqs.(57). When this is done eqs.(57.1) and (57.3) become eqs.(68.1) and (68.3).

$$(68.1) \qquad \begin{bmatrix} \bar{S}_b \\ \bar{M}_b \end{bmatrix} = EI \begin{bmatrix} +a^2\frac{12}{r^3} & -a^2 b\frac{6}{r^2} \\ -a^2 b\frac{6}{r^2} & +a^2 b^2\frac{4}{r} \end{bmatrix} \begin{bmatrix} \bar{y}_e \\ \bar{\theta}_e \end{bmatrix}$$

$$(68.3) \qquad \begin{bmatrix} \bar{y}_r \\ \bar{\theta}_r \end{bmatrix} = \begin{bmatrix} +1 & +br \\ 0 & +1 \end{bmatrix} \begin{bmatrix} \bar{y}_a \\ \bar{\theta}_a \end{bmatrix}$$

The combined circuit using the π network (see Fig. 10) with the transformer in the β position is given in Fig. 39. This circuit has the advantage that the bending moments, M_b, and shear forces, S_b, are currents which may be measured in the circuit.

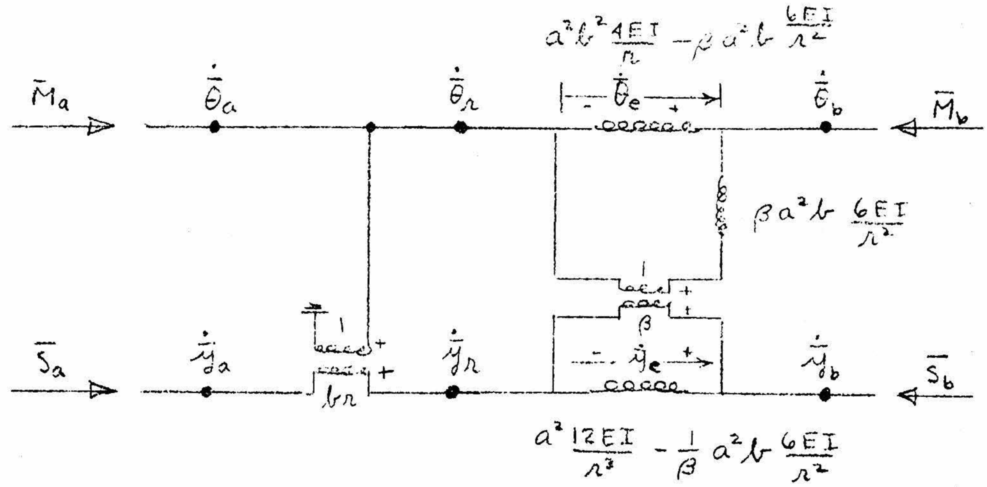


Fig. 39.

Eqs.(68) and the corresponding circuit of Fig. 39 will now be simplified by means of a coordinate axes transformation. The elementary coordinate axes transformation of eq.(69) is applied to the beam section spring equation, eq.(68.1). If $a_{12} = -\frac{k_{12}}{k_{11}} = +\frac{rb}{2}$, the spring equation becomes eq.(70).

$$(69) \qquad \begin{bmatrix} \bar{y}_e \\ \bar{\theta}_e \end{bmatrix} = \begin{bmatrix} +1 & +a_{12} \\ 0 & +1 \end{bmatrix} \begin{bmatrix} x_{e1} \\ x_{e2} \end{bmatrix}$$

$$(70) \qquad \begin{bmatrix} F_{b1} \\ F_{b2} \end{bmatrix} = \begin{bmatrix} +a^2\frac{12EI}{r^3} & 0 \\ 0 & +\frac{a^2b^2EI}{r} \end{bmatrix} \begin{vmatrix} x_{e1} \\ x_{e2} \end{vmatrix}$$

If the rigid body displacements are transformed in the same manner as the elastic displacements, that is, by the matrix of eq.(69), and if the right side of eq.(68.3) is substituted for the rigid body displacements then eq.(71) is obtained.

(71) $$\begin{bmatrix} +1 & +br \\ 0 & +1 \end{bmatrix} \begin{bmatrix} \bar{y}_a \\ \bar{\theta}_a \end{bmatrix} = \begin{bmatrix} \bar{y}_r \\ \bar{\theta}_r \end{bmatrix} = \begin{bmatrix} +1 & +\frac{rb}{2} \\ 0 & +1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{r1} \\ \mathbf{x}_{r2} \end{bmatrix}$$

This equation is rearranged to obtain:

(71b) $$\begin{bmatrix} \bar{y}_a \\ \bar{\theta}_a \end{bmatrix} = \begin{bmatrix} +1 & -\frac{br}{2} \\ 0 & +1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{a1} \\ \mathbf{x}_{a2} \end{bmatrix}$$ where, by definition: $\mathbf{x}_{a1} = \mathbf{x}_{r1}$, $\mathbf{x}_{a2} = \mathbf{x}_{r2}$

If the scale change of eq.(64) and the transformation matrix of eq.(69) are applied to eq.(57.2), eqs.(72) are obtained.

(72) $$\begin{bmatrix} \bar{y}_b \\ \bar{\theta}_b \end{bmatrix} = \begin{bmatrix} +1 & +\frac{rb}{2} \\ 0 & +1 \end{bmatrix} \begin{bmatrix} \vec{\mathbf{x}}_{b1} \\ \vec{\mathbf{x}}_{b2} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{x}_{b1} \\ \mathbf{x}_{b2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{a1} \\ \mathbf{x}_{a2} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{e1} \\ \mathbf{x}_{e2} \end{bmatrix}$$

The equivalent circuit for eqs.(70), (71) and (72) is the same as that obtained by the admittance method. The circuit is given in Fig. 38 and the equations which define the voltages and currents in this circuit are eqs.(67).

Figs. 38 and 39 are two useful circuits for a single beam section. A circuit for a beam composed of several sections is obtained by placing several of these section circuits in series. When this is done, at each section junction two transformers are in series with the rigid-plane displacement node $_i y$ between them. The two transformers may be replaced by one with the node point $_i y$ at the center tap. The resulting circuit is shown in Fig. 40. Note that the values of r and EI may differ from section to section.

The center-tapped transformers may be eliminated if the external force vector $\begin{bmatrix} {}_iP \\ {}_iM \end{bmatrix}$ is transformed into the coordinate system used in the i + 1 beam section. This transformation of the external vector force will in general require a transformer which for some important special

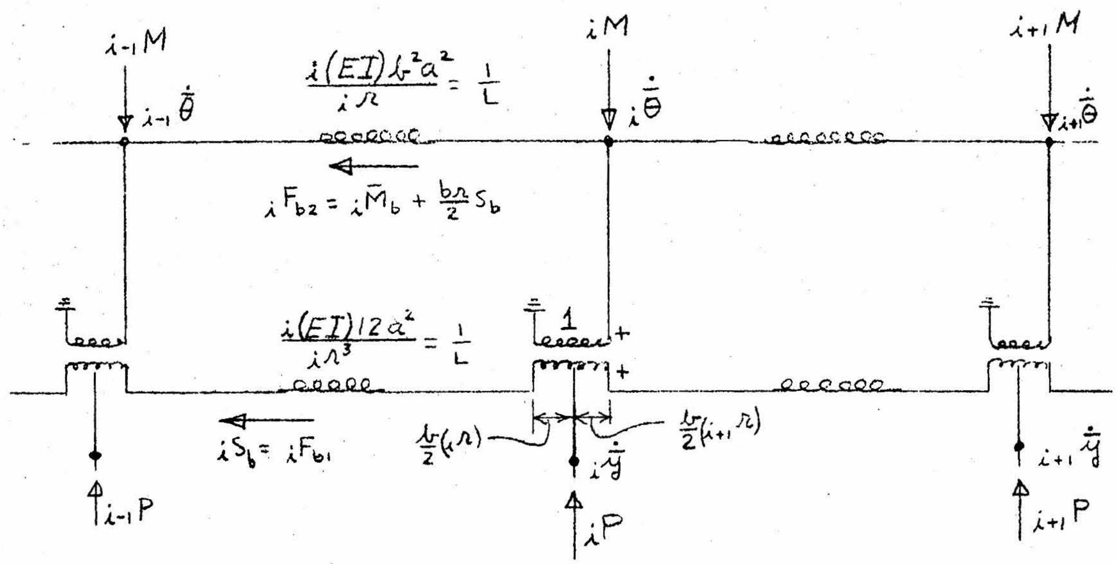


Fig. 40.

cases may be eliminated. The appropriate transformation equations are eqs.(67), and these applied to $\begin{bmatrix} {}_iP \\ {}_iM \end{bmatrix}$ give eqs.(73).

$$(73) \qquad {}_iF_1 = {}_iP \qquad\qquad {}_iy = {}_ix_1 - \frac{b({}_{i+1}r)}{2}$$

$$\qquad {}_iF_2 = {}_iM - \frac{b({}_{i+1}r)}{2} \qquad\qquad {}_i\theta = {}_ix_2$$

The circuit for the transformation is given in Fig. 41a.

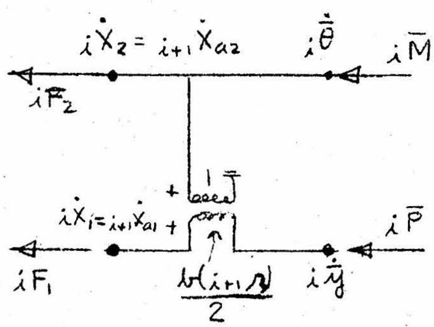


Fig. 41a.

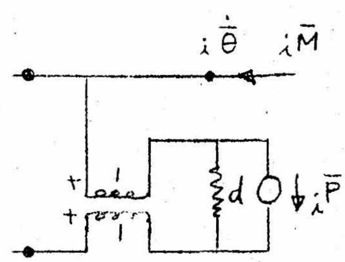


Fig. 41b.

If ${}_iP$ is an external force generator or if it is proportional to ${}_iy$ or to a time derivative or integral of ${}_iy$, and if $b = \frac{2}{{}_{i+1}\ell}$, then the circuit becomes that of Fig. 41b and it is apparent that the transformer may be omitted.

As an example suppose that ${}_iP$ and ${}_iM$ are inertia forces given by eq.(74).

$$(74) \qquad -\begin{bmatrix} {}_iP \\ {}_iM \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} {}_i\ddot{y} \\ {}_i\ddot{\theta} \end{bmatrix}$$

When the scale change transformation (75) is made this equation becomes eq.(76).

$$(75) \qquad t = N\bar{t} \qquad\qquad y = a\bar{y} \qquad\qquad \theta = b\,a\,\bar{\theta}$$

$$(76) \qquad -\begin{bmatrix} {}_i\bar{P} \\ {}_i\bar{M} \end{bmatrix} = \begin{bmatrix} \frac{m a^2}{N^2} & 0 \\ 0 & \frac{I \ell^2 a^2}{N^2} \end{bmatrix} \begin{bmatrix} {}_i\ddot{\bar{y}} \\ {}_i\ddot{\bar{\theta}} \end{bmatrix} \qquad \text{where} \quad \ddot{\bar{y}} = \frac{d^2\bar{y}}{d\bar{t}^2}$$

The circuit analog for each of the mass coefficients is a condenser. In Fig. 42 these condensers are shown in the circuit for a cantilever beam which has beam sections of equal length. The boundary conditions at the clamped end are imposed by grounding the displacement nodes while those at the free end are imposed by applying no shear force or bending couple currents.

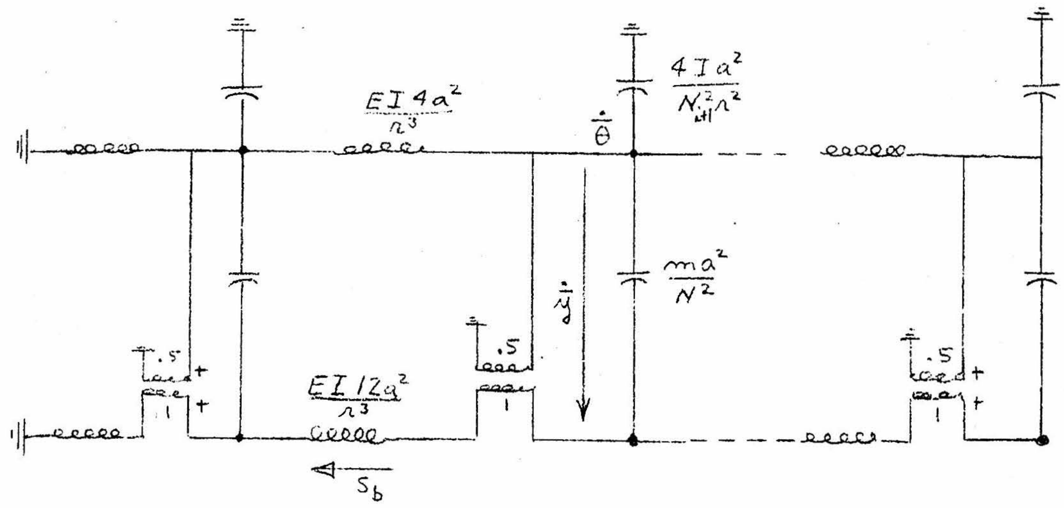


Fig. 42 (values shown are for $\frac{1}{L}$ and C)

3.3 The General One-dimensional Elastic Structure

The methods and matrix equations developed in sections 3.1 and 3.2 may be applied to a beam with six degrees of freedom. In the same manner as before, the beam is divided into massless sections which are separated by rigid planes upon which the external forces act. In the general case the displacement vectors have six components which specify translation and rotation and the force vectors correspondingly have six components which specify force and couple. The elastic behavior of a beam section is usually specified by an influence matrix which is obtained from the strain energy by using Castigliano's theorem. For the systems considered in this and previous sections no non-linear forces such as those involved in buckling are allowed. Such forces will be discussed in section 3.6. As before, the principal idea used is that the displacement of end b of a beam section is the sum of an elastic part caused by the applied forces and a rigid body part caused by the displacement of end a. The circuit for the section is obtained by placing circuits for the two parts in series.

As an example of a more general beam, the circuit analogy for the bending of a curved beam out of its plane will be obtained in this section. Other systems such as the bending of a beam with a product of inertia or the bending and twisting of a beam about an axis other than the center of twist can be obtained in a similar manner. It will be assumed that the beam has a circular cross section, that its center line is a circular arc and that the radius of the cross section is small compared to the radius of the center line. These assumptions are made for simplicity in exposition and in order that a comparison of natural frequencies of vibration may be made with a known solution of the continuous case; they are not a restriction on the method used.

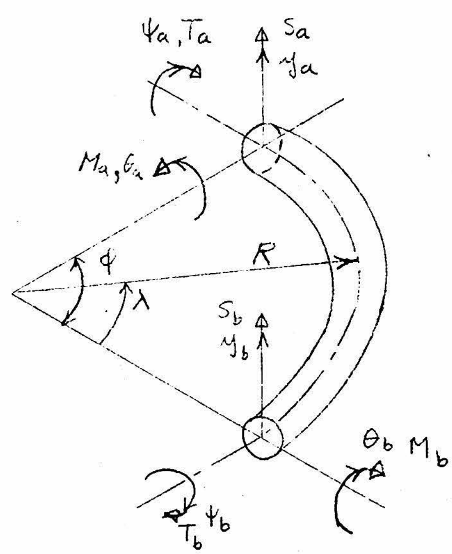


Fig. 43.

In terms of the notation for a typical beam section shown in Fig. 43, the strain energy in the beam section is given by:

$$U = \int_0^{\phi} \left(\frac{M_\lambda^2}{2EI} + \frac{T_\lambda^2}{2JG} \right) R\, d\lambda \tag{77}$$

where: $M_\lambda = F_b R \sin\lambda + M_b \cos\lambda - T_b \sin\lambda$

$T_\lambda = F_b R(1-\cos\lambda) + M_b \sin\lambda + T_b \cos\lambda$

The elastic behavior of the beam can be specified by:

$$\begin{bmatrix} y_e \\ \theta_e \\ \psi_e \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{bmatrix} \begin{bmatrix} S_b \\ M_b \\ T_b \end{bmatrix} \quad (50.5) \quad [x_e] = [g][F_b] \tag{78}$$

The influence coefficients in equation (78) may be calculated from eq.(77) by using Castigliano's theorem. The result of this calculation is:

(79)

$$g_{11} = \frac{R^3}{4EI}\left[2\phi - \sin 2\phi + \frac{EI}{JG}\left(6\phi + \sin 2\phi - 8\sin\phi\right)\right]$$

$$g_{12} = \frac{R^2}{4EI}\left[1 - \cos 2\phi + \frac{EI}{JG}\left(3 - 4\cos\phi + 2\sin\phi\right)\right]$$

$$g_{13} = \frac{R^2}{4EI}\left[\sin 2\phi - 2\phi + \frac{EI}{JG}\left(4\sin\phi - 2\phi - \sin 2\phi\right)\right]$$

$$g_{22} = \frac{R}{4EI}\left[2\phi + \sin 2\phi + \frac{EI}{JG}\left(2\phi - \sin 2\phi\right)\right]$$

$$g_{23} = \frac{R}{4EI}\left[-(1 - \cos 2\phi) + \frac{EI}{JG}\left(1 - \cos 2\phi\right)\right]$$

$$g_{33} = \frac{R}{4EI}\left[2\phi - \sin 2\phi + \frac{EI}{JG}\left(2\phi + \sin 2\phi\right)\right]$$

The rigid body relation, eq.(50.2), is given by:

$$\begin{bmatrix} y_n \\ \theta_n \\ \psi_n \end{bmatrix} = \begin{bmatrix} +1 & +R\sin\phi & +R(1-\cos\phi) \\ 0 & +\cos\phi & +\sin\phi \\ 0 & -\sin\phi & +\cos\phi \end{bmatrix} \begin{bmatrix} y_a \\ \theta_a \\ \psi_a \end{bmatrix} \quad (50.2) \quad [x_n] = [\alpha][x_a] \tag{80}$$

When $\phi \to 0$ and $R \to \infty$ in such a manner that $R\phi$ = constant = r, these equations become those for a simple beam in bending, section 3.2, and a shaft in torsion, section 3.1.

In accordance with eqs.(50.3) and (50.4), the analog for the beam section is obtained by placing the inverse spring circuit from eq.(78), in series with the rigid body circuit from eq.(80). Both of these circuits were studied in part II although the specific values of the electrical elements depend upon the particular problem considered. To yield a practical electric circuit, the scale changes, eqs.(81) are introduced.

$$(81) \qquad \begin{aligned} y_j &= a\,\bar{y}_j \\ \theta_j &= ab_1\,\bar{\theta}_j \quad (j = a,b,e,r) \\ \psi_j &= ab_2\,\bar{\psi}_j \end{aligned} \qquad \begin{aligned} S_i &= \frac{1}{a}\bar{S}_i \\ M_i &= \frac{1}{ab_1}\bar{M}_i \quad (i = a,b) \\ T_i &= \frac{1}{ab_2}\bar{T}_i \end{aligned}$$

When these scale changes are introduced into the inverse spring equation (78) and the rigid body relation (80), they become eqs.(82) and (83) respectively.

$$(82) \qquad \begin{bmatrix} \bar{y}_e \\ \bar{\theta}_e \\ \bar{\psi}_e \end{bmatrix} = \begin{bmatrix} g_{11} & \frac{g_{12}}{b_1} & \frac{g_{13}}{b_2} \\ \frac{g_{12}}{b_1} & \frac{g_{22}}{b_1^2} & \frac{g_{23}}{b_1 b_2} \\ \frac{g_{13}}{b_2} & \frac{g_{23}}{b_1 b_2} & \frac{g_{33}}{b_2^2} \end{bmatrix} \begin{bmatrix} \bar{S}_b \\ \bar{M}_b \\ \bar{T}_b \end{bmatrix} \qquad [\bar{x}_e] = [\bar{g}][\bar{F}_b]$$

$$(83) \qquad \begin{bmatrix} \bar{y}_r \\ \bar{\theta}_r \\ \bar{\psi}_r \end{bmatrix} = \begin{bmatrix} +1 & +b_1 R \sin\phi & +b_2 R(1-\cos\phi) \\ 0 & +\cos\phi & +\frac{b_2}{b_1}\sin\phi \\ 0 & -\frac{b_1}{b_2}\sin\phi & +\cos\phi \end{bmatrix} \begin{bmatrix} \bar{y}_a \\ \bar{\theta}_a \\ \bar{\psi}_a \end{bmatrix} \qquad [\bar{x}_r] = [\bar{\alpha}][\bar{x}_a]$$

A circuit for the beam section when $\phi = 22.5^\circ$ and $\frac{EI}{JG} = (1+\nu) = 1.3$ will now be developed. For this value of ϕ and $\frac{EI}{JG}$ the values of g_{ij} are:

$$g_{11} = \frac{R^3}{4EI}(+0.080681) \qquad g_{22} = \frac{R}{4EI}(+1.594284)$$

(84) $$g_{12} = \frac{R^2}{4EI}(+0.307958) \qquad g_{23} = \frac{R}{4EI}(+0.087878)$$

$$g_{13} = \frac{R^2}{4EI}(-0.040063) \qquad g_{33} = \frac{R}{4EI}(+2.018548)$$

To simplify the rigid body relation, take:

(85) $$b_1 = \frac{1}{R \sin\phi} \qquad b_2 = \frac{1}{R \sin^2\phi}$$

The rigid body relation then becomes:

(86) $$\begin{bmatrix} \bar{y}_r \\ \bar{\theta}_r \\ \bar{\psi}_r \end{bmatrix} = \begin{bmatrix} +1 & +1 & +.520 \\ 0 & +.924 & +1 \\ 0 & -.146 & +.924 \end{bmatrix} \begin{bmatrix} \bar{y}_a \\ \bar{\theta}_a \\ \bar{\psi}_a \end{bmatrix}$$

The inverse spring equation becomes

(87) $$\begin{bmatrix} \bar{y}_e \\ \bar{\theta}_e \\ \bar{\psi}_e \end{bmatrix} = \frac{R^3}{4EI\,a^2} \begin{bmatrix} +0.08068 & +0.11785 & -0.00586 \\ +0.11785 & +0.23348 & +0.00492 \\ -0.00586 & +0.00492 & +0.04329 \end{bmatrix} \begin{bmatrix} \bar{S}_b \\ \bar{M}_b \\ \bar{T}_b \end{bmatrix}$$

To accomplish isolation at least two transformers are required in the analog for eq.(87). In additon the negative impedance resulting from $\bar{g}_{12} > \bar{g}_{11}$ must be removed. A practical way to do this is to make a local scale change in the $\bar{y}_e$ component and to isolate the $\bar{\theta}_e$ component. A circuit for a beam section in which this is done is given in Fig. 44. Although the general values of eqs.(82) and (83) are given on the circuit diagram, it is valid as drawn only for a $[g]$ which has components with the same signs and nearly the same values as that of eq.(87). Other values of components might require transformers in the β_{ij} positions or a coordinate axes change.

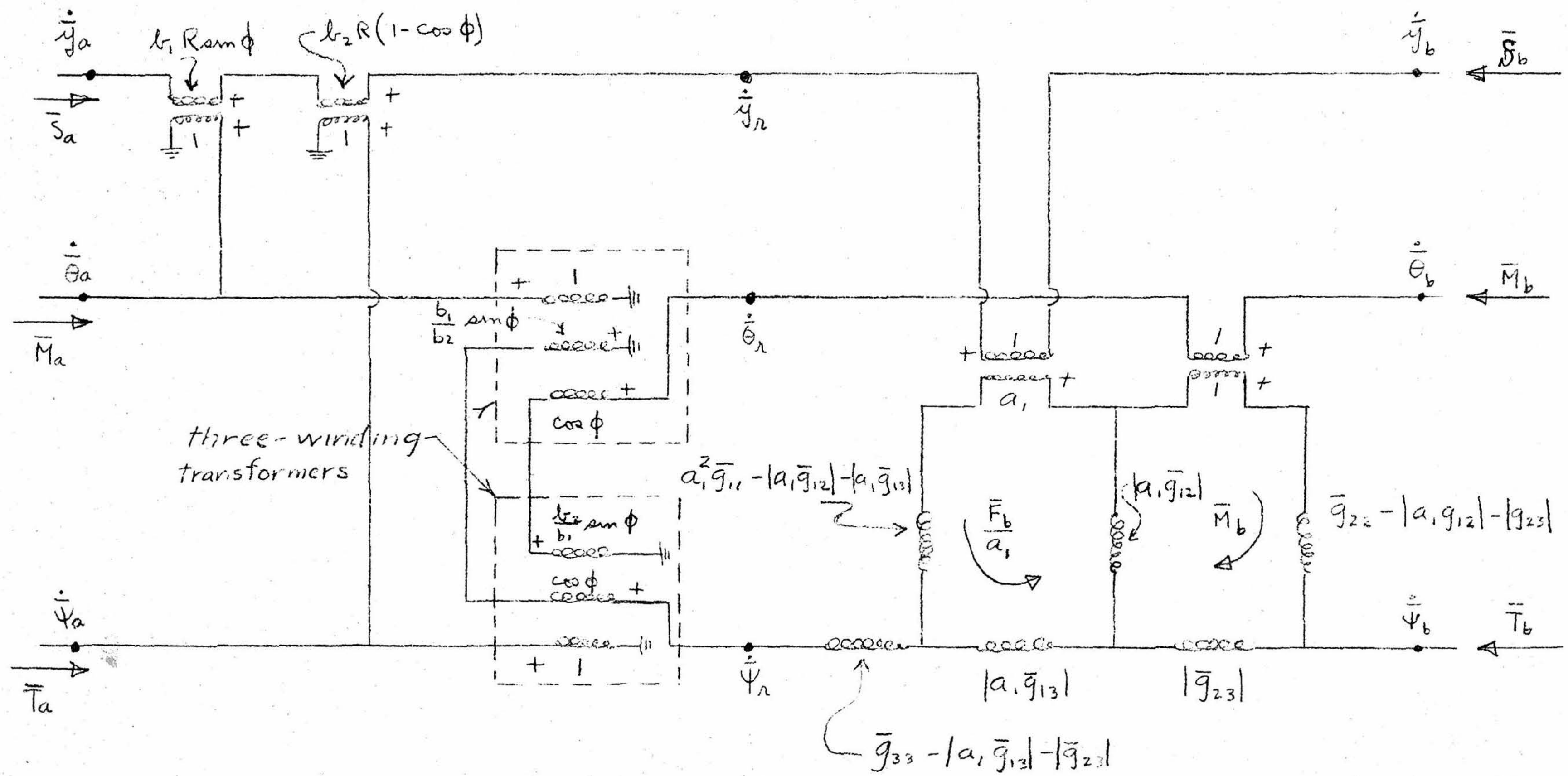

$\dot{y}_a$
$\bar{S}_a$
$b_1 R \sin\phi$
$b_2 R(1-\cos\phi)$
$\dot{y}_r$
$\dot{y}_b$
$\bar{S}_b$
$\dot{\theta}_a$
$\bar{M}_a$
$\frac{b_1}{b_2}\sin\phi$
$\cos\phi$
$\dot{\theta}_r$
a_1
$\dot{\theta}_b$
$\bar{M}_b$
three-winding transformers
$a_1^2\bar{g}_{11} - |a_1\bar{g}_{12}| - |a_1\bar{g}_{13}|$
$\frac{\bar{F}_b}{a_1}$
$|a_1\bar{g}_{12}|$
$\bar{g}_{22} - |a_1 g_{12}| - |g_{23}|$
$\dot{\psi}_a$
$\bar{T}_a$
$\dot{\psi}_r$
$|a_1\bar{g}_{13}|$
$|\bar{g}_{23}|$
$\dot{\psi}_b$
$\bar{T}_b$
$\bar{g}_{33} - |a_1\bar{g}_{13}| - |\bar{g}_{23}|$

Fig. 44

A lumped circuit analogy for the out of plane free vibrations of a uniform full ring will now be given. The shape of the fundamental mode of such a ring is indicated in Fig. 45. The + and − indicate movements out of and into the plane of the figure. The boundary conditions on the octant which are shown in the figure are obtained by symmetry.

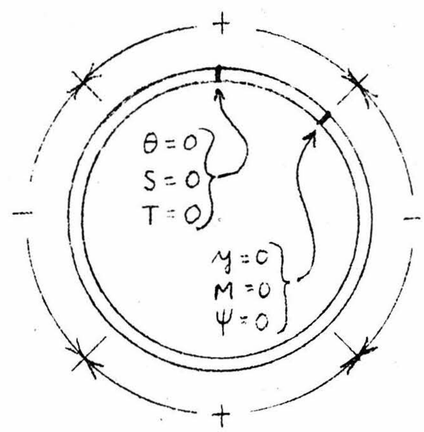
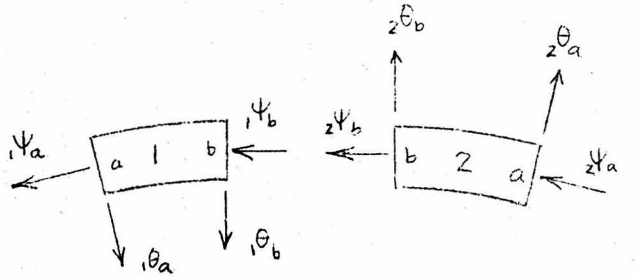


Fig. 45.

Because of symmetry, to model the full ring when it vibrates in the fundamental or higher modes with an integral number of quarter wave lengths in an octant, it is only necessary to model the octant. To model the octant in the example considered here, two of the beam sections studied above will be placed in series. The boundary conditions make it possible to eliminate some of the rigid body transformers, and to take advantage of this, the b ends of the sections are joined together instead of end a to ~~end~~ a b as has been done previously. The coordinate systems used on the beam sections are shown in Fig. 46. An inspection of the figure shows that the same equations describe both beam sections and therefore the analogs for the two sections are identical. However at the section junction a transformer is needed to relate

${}_2\theta_b$ to ${}_1\theta_b$ because their positive directions are opposite to each other.



(all y's are positive out of the page)

Fig. 46.

A lumped mass equal to the mass of the center half of the octant is placed at the section junction and a lumped mass equal to one-quarter of the mass of the octant is placed at the a end of section 1. The mass which would normally be at end a of section 2 is grounded. In the analog the rotary inertias, which are condensers connected to the slope and twist nodes, have been neglected. If μ = mass of the beam per unit length, then the mass of one-half an octant is:

$$m = \int_0^{\frac{\pi}{8}} \mu R \, d\phi = \mu R \frac{\pi}{8}$$

When the scale changes, $y = a\bar{y}$, and $t = N\bar{t}$, are made the mass becomes:

(88) $$m = \mu R \frac{\pi}{8} \frac{a^2}{N^2}$$

The complete circuit with the parameter values of eqs.(86), (87) and (88) is given if Fig. 47.

The expression for the natural frequencies for the distributed mass ring, for which Fig. 47 is the lumped mass analog, is given by Timoshenko (22), page 410. This expression is:

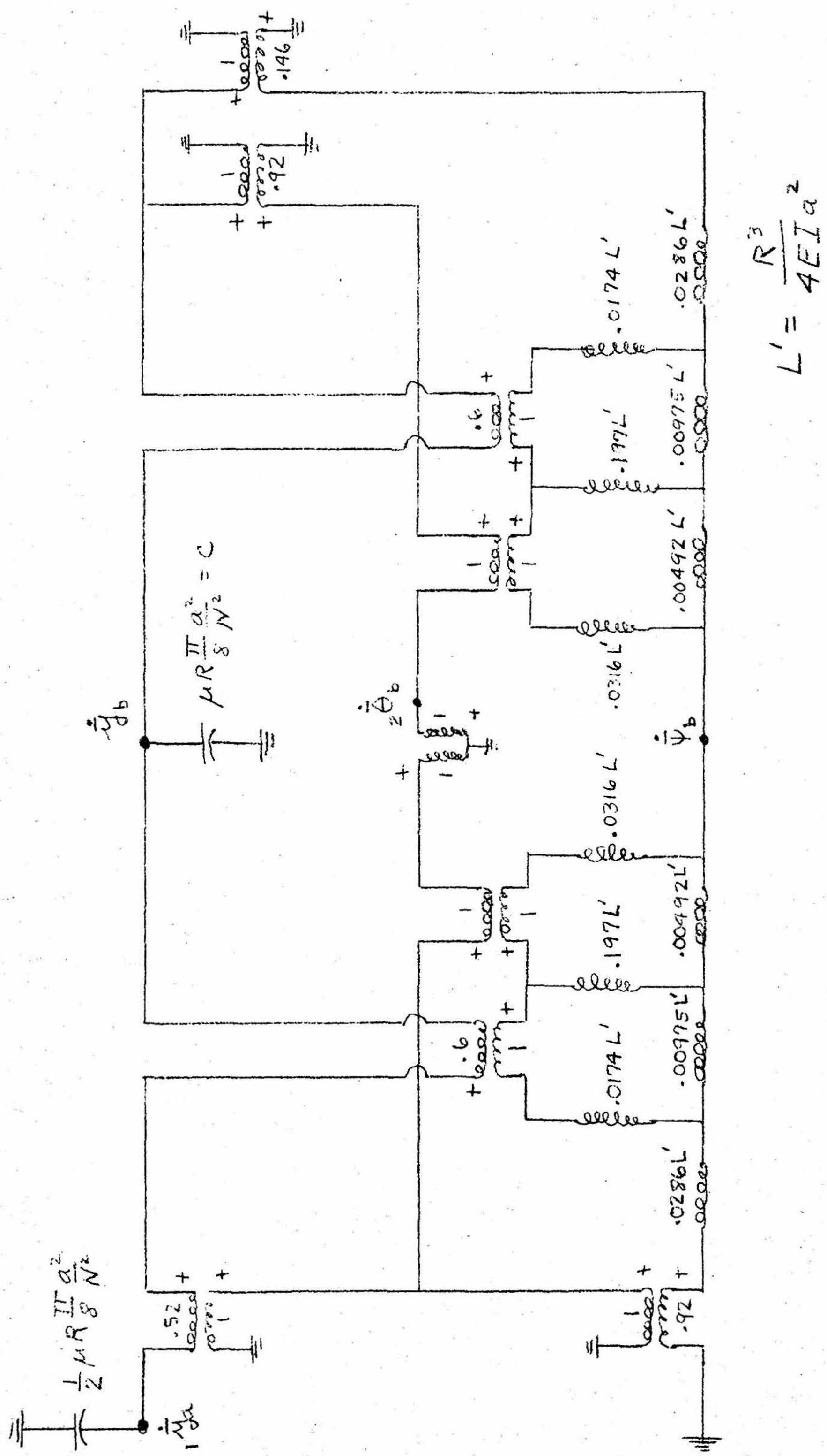


Fig. 47.

(89) $$\omega_{ci} = \sqrt{\frac{EI}{\mu R^4}\,\frac{i^2(i^2-1)^2}{i^2+1+\nu}} \qquad i = (2 \cdots \infty)$$

For the fundamental mode and for $\nu = 0.3$ this expression becomes:

(90) $$\omega_c = \sqrt{\frac{EI}{\mu R^4}}\,(2.606) \qquad \text{or} \qquad \frac{\omega_c}{\sqrt{\frac{EI}{\mu R^4}}\,(2.606)} = 1$$

The lumped mass frequency obtained from the circuit frequency, $\bar{f} = 2\pi\bar{\omega}_{Le}$, can be put into this dimensionless form by substituting the values of the circuit elements obtained from Fig. 47 into the term $\sqrt{\frac{EI}{\mu R^4}}$. The circuit element values are:

(91) $$C = \mu R \frac{\pi}{8}\frac{a^2}{N^2} \qquad L' = \frac{R^3}{4EIa^2}$$

Making the substitution one obtains:

(92) $$\sqrt{\frac{EI}{\mu R^4}} = \frac{1}{N}\sqrt{\frac{\pi}{8}}\,\frac{1}{2\sqrt{L'C}}$$

Thus since $\omega_{Le} = \frac{\bar{\omega}_{Le}}{N} = \frac{2\pi\bar{f}}{N}$ the frequency parameter becomes:

(93) $$\frac{\omega_{Le}}{\sqrt{\frac{EI}{\mu R^4}}\,(2.606)} = \frac{2\pi\bar{f}}{N\frac{1}{N}\sqrt{\frac{\pi}{8}}\,\frac{1}{2\sqrt{L'C}}\,(2.606)} = 7.70\sqrt{L'C}\;\bar{f}$$

(the Le subscript indicates the lumped beam frequency calculated by the electric circuit). The ratio of the continuous beam frequency to the lumped-mass beam frequency as calculated by the analogous electric circuit is obtained by dividing eq.(90) by eq.(93). The result is:

(94) $$\frac{\omega_c}{\omega_{Le}} = \frac{1}{\bar{f}\,(7.70)\sqrt{L'C}}$$

3.4 The Finite Difference Beam Analogy

In the first part of this section the finite difference analogy developed by McCann and MacNeal for the 4th order differential equation describing the behavior of a straight beam will be given. This analogy will then be compared with the lumped analogy developed in section 3.2.

To describe the beam the coordinate system of Fig. 48 will be used.

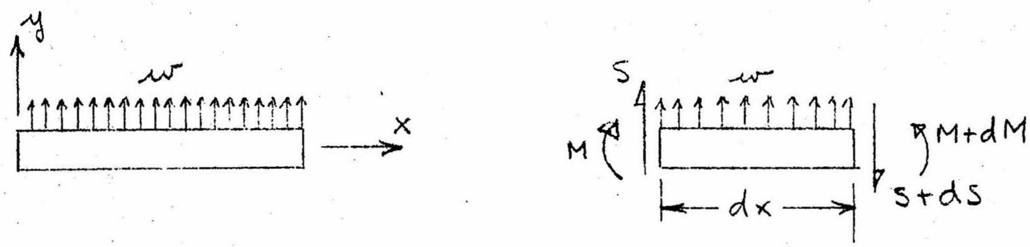


Fig. 48.

From the theory of the strength of materials the differential equation for the deflection of the neutral axis is:

$$\frac{\partial}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = w \qquad (95)$$

This equation could be converted to a finite difference equation immediately, but it is more instructive to consider the four first order equations used to derive it.

The equilibrium conditions on the beam element are:

$$\sum \text{forces} = 0 \qquad S + w\,dx = S + dS$$

or

$$\frac{dS}{dx} = w \qquad (96.1)$$

$\sum$ moment = 0 $\qquad M + S\,dx + P\,dx\frac{dx}{2} = M + dM$

retaining only first order terms, the following is obtained:

(96.2) $$\frac{dM}{dx} = + S$$

Hooke's law gives:

(96.3) $$EI\frac{d\theta}{dx} = + M$$

where:

(96.4) $$\frac{dy}{dx} = + \theta$$

Differentials are replaced by differences and eqs.(97) are obtained. In these equations the post-subscript indicates the section or cell number in the beam.

(97.1) $$S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}} = + (w\ \Delta x)_i$$

(97.2) $$M_{i+\frac{1}{2}} - M_i = + S_{i+\frac{1}{2}}\ \Delta x_{i+\frac{1}{2}}$$

(97.3) $$\theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}} = + \frac{M_i\ \Delta x_i}{EI}$$

(97.4) $$y_{i+1} - y_i = \theta_{i+\frac{1}{2}}\ \Delta x_{i+\frac{1}{2}}$$

Electrically these equations have the following meaning:

Eq(97.1) is Kirchhoff's current law.

Eq(97.2) is a ratio of currents.

Eq(97.3) is a relation between voltage and current. (ohms law)

Eq(97.4) is a ratio of voltages.

The analogous circuit for eqs.(97), which is the finite difference analog for a simple beam, is given in Fig. 49.

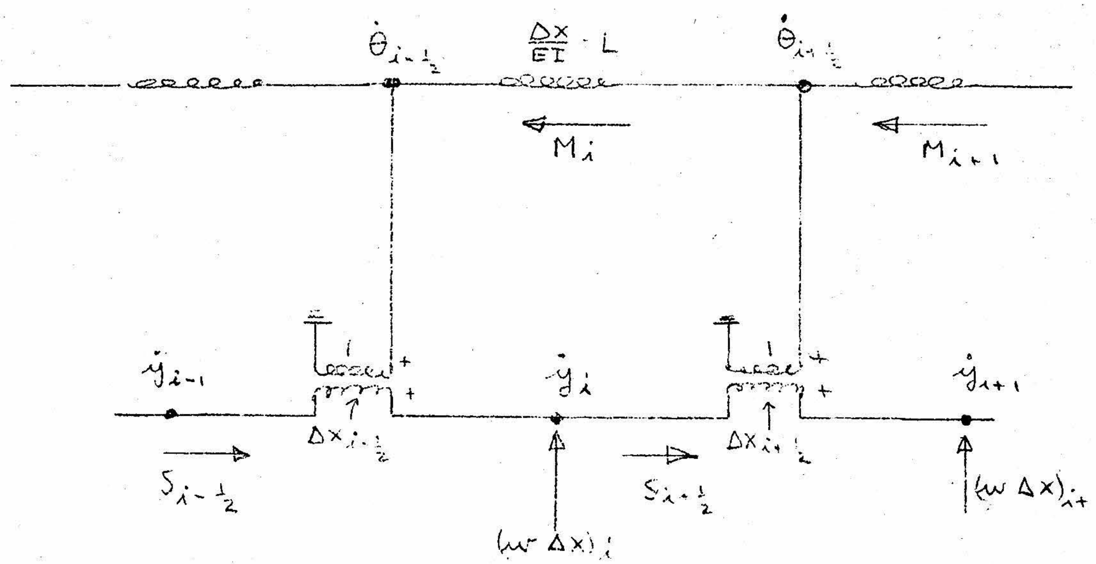


Fig. 49.

For dynamic problems the external lumped loads, $(w\ \Delta x)_i$ will include inertia forces and these will be obtained by connecting condensers to the displacement nodes. The analog has the advantage that displacements, slopes, bending moments and shear forces are easily measured quantities.

The mechanical analog for Fig. 49 is shown in Fig. 50. The analogy is true for small displacements only. The masses and the fulcrums of the T shaped levers are constrained to move vertically. Any external forces can be applied to the junction points. The boundary conditions shown in the figure are for a cantilever beam.

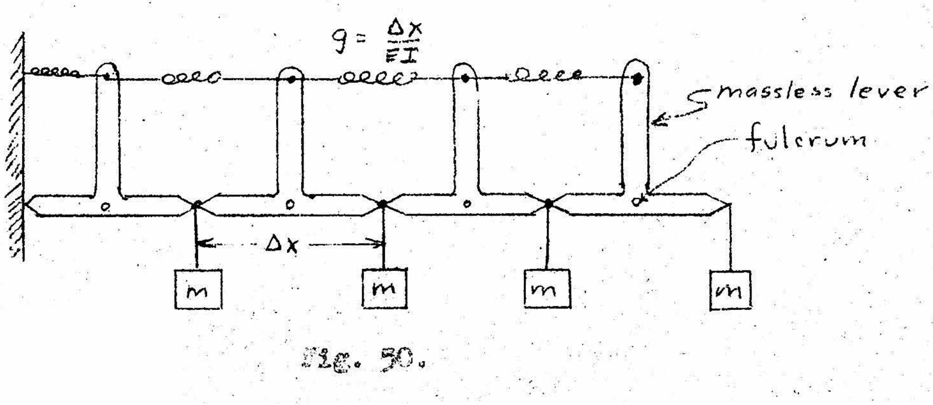


Fig. 50.

To compare the finite difference analogy of Fig. 49 with the lumped analogy developed in section 3.2 we transform the latter into a form similar to the former. To do this the inverse spring equation, (57.5), for the simple beam is written with $[x_e] = [x_b] - [\alpha][x_a]$ substituted for $[x_e]$. Eqs.(98) are obtained as a result.

$$y_b - (y_a + r\,\theta_a) = \frac{r^3}{3EI} S_b + \frac{r^2}{2EI} M_b \qquad (98.1)$$

$$\theta_b - \theta_a = \frac{r^2}{2EI} S_b + \frac{r}{EI} M_b \qquad (98.2)$$

Both sides of eq.(98.1) are divided by r and both S_b terms are divided and multiplied by r to give eqs.(99).

$$\begin{aligned} \frac{y_b - y_a}{r} - \theta_a &= \frac{r}{3EI}(r\,S_b) + \frac{r}{2EI} M_b \\ \theta_b - \theta_a &= \frac{r}{2EI}(r\,S_b) + \frac{r}{EI} M_b \end{aligned} \qquad (99)$$

The rigid body relation of forces is also required. This is obtained from eq.(57.4).

$$\begin{aligned} M_a &= -M_b + r\,S_a \\ S_a &= -S_b \end{aligned} \qquad (100)$$

In eqs.(99) the reference nodes for the two relative displacements on the left hand sides are common, namely θ_a. The analogous circuit for these equations is the simple T network but since $\frac{r}{2EI} > \frac{r}{3EI}$, one inductance will be negative. The circuit for one section is given in Fig. 51. This circuit is the same as that for one cell of the finite difference analogy, Fig. 49, except for the negative inductance which is not present in the latter. The effect on errors of omitting this inductance will be investigated in part IV.

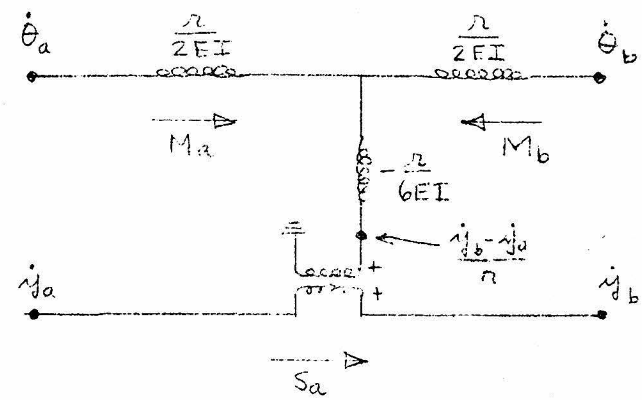


Fig. 51.

3.5 Shear Deflection in Beams

In sections 3.1 to 3.3 the technical theory of beams was assumed in which plane sections remain plane. This assumption makes the problem truly one-dimensional, that is, the relative displacements of two ends of a beam section can be specified by not more than six quantities which are proportional to the same number of generalized force components. In actuality a beam section is a three-dimensional elastic region and the displacements of the ends can only be specified by three displacement functions of two space variables each. Plane sections remain plane only for the case of pure bending; for lateral loads deflections are greater than that predicted by the technical theory. In texts on strength of materials in which the technical theory is used to obtain the bending stiffness of beams, it is shown that the shear stress is a function of the lateral shear force and the shape of the beam cross section. The shear stress causes a shear strain, which means a distortion of cross sections, and the shear strain produces a deflection additional to that obtained from bending. There are various approximate ways in which this additional deflection may be calculated. The most common* is to assume a slope caused by shear alone of magnitude:

(101) $$\frac{dy_1}{dx} = \frac{(\tau_{xy})_{y=0}}{G} = \frac{\alpha S}{AG} = \frac{\alpha c^2}{3IG} S.$$

y_1 = shear deflection

α = factor which multiplies the average shear stress to obtain the shear stress at the centroid of a cross section, = 3/2 for rectangular cross sections

c = one half the depth of a symmetrical beam

In the beam sections considered in this thesis the shear force is constant over the section length, and the shear deflection of a beam section is thus:

(102) $$(y_b - y_a)_{shear} = \frac{\alpha S}{AG} r = \frac{\alpha r c^2}{3GI} S$$

This additional deflection is a one-dimensional effect, that is, it depends only on the shear force in the beam section. This simple shear deflection term can be included in the beam equations of section 3.2 by simply adding $\frac{\alpha R}{AG}$ to the g_{11} term of eq.(57.5). This term then becomes $\frac{r^3}{3EI} + \frac{\alpha r}{AG}$ which in the electric circuit means a correspondingly larger inductance. In Fig. 51, which is the circuit for one section of the lumped analogy in the finite difference form, the negative inductance arises from $\frac{r}{3EI} - \frac{r}{2EI} = -\frac{r}{6EI}$. When shear effects are included, the corresponding inductance is:

(103) $$\frac{r}{3EI} + \frac{\alpha}{AGr} - \frac{r}{2EI} = -\frac{r}{6EI} + \frac{\alpha}{AGr}$$

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* For example see Timoshenko (17), Strength of Materials, 2nd ed. page 170

Hence when the shear term is large enough, no negative inductance is required and the circuit of Fig. 51 becomes a very useful and practical circuit. For a solid rectangular beam with Poisson's ratio of 0.3 the ratio of beam depth, h, to length, r, at which the inductance is either zero or positive can be found by:

$$\alpha = 3/2 \qquad\qquad A = b\,h$$

$$\frac{E}{G} = 2(1+\nu) = 2.6 \qquad\qquad I = \frac{bh^3}{12}$$

substitute these values in $\frac{r}{6EI} \leq \frac{\alpha}{AGr}$ and obtain the condition:

(104) $$\frac{r}{h} \leq 1.4$$

The shear deflection term of eq.(102) does not depend on the boundary conditions on the beam section as it should to be correct for all situations, and it is for this reason that its validity is open to question. In the plane stress or two-dimensional approximation to a beam* the shear term of eq.(102) for $\alpha = 3/2$ is obtained only for the case of a rectangular cantilever beam in which the built in end is free to warp. For all other boundary conditions the term of eq.(102) is not correct although the correct term is known for only a few cases. For instance,** the strain energy in shear may be calculated and Castigliano's theorem used to obtain the shear term for an end-loaded cantilever with the built in end restrained from warping. In this case:

(105) $$y_{shear} = \frac{2}{5}\,\frac{r c^2}{GI}\,S$$

In summary we may say that if the magnitude of the shear term can

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* Timoshenko, Theory of Elasticity (18), page 37

** " " page 149

be obtained, it can be incorporated into the circuit analogy easily. However, in all but the most elementary cases a correct shear term is not known.

3.6 Combined Axial and Transverse Loads.

When an axial force acts on a beam that undergoes lateral deflection, the effect of the force depends on the lateral deflection and in this sense the system is non-linear. Consider a beam in bending with an axial load T. (Fig. 52a).

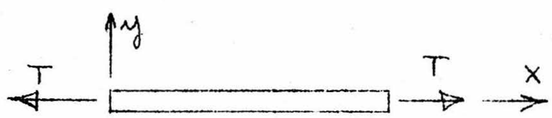


Fig. 52a.

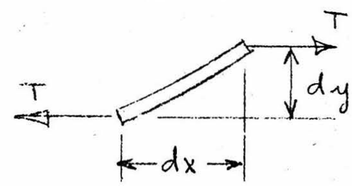


Fig. 52b.

The axial load may be thought of as applying a distributed external bending couple, $-T\frac{dy}{dx}$, to the beam. This is shown in Fig. 52b where T dy is the moment applied to the beam element of length dx. In the electric circuit analogy for the beam this distributed moment must be lumped into equivalent concentrated moments which are applied at the ${}_i\theta$ nodes of the beam. (Fig. 38 or 40). If T is a variable, electronic multipliers are required in which the inputs are the voltages ${}_i\theta$ and T and the output is the current ${}_iM$. When T is a constant, the current ${}_iM = (-T\ {}_i\theta\ q)$ may be obtained by connecting inductances between ground and the ${}_i\theta$ nodes. In this equation q is the length of beam over which the distributed moments are lumped; in most cases it will equal the beam section length, r. If T is negative, that is, if the beam is in compression, either electronic negative inductances must be used or, in the steady state case at a known frequency, condensers may be used as negative inductances. In the static case where the beam masses are zero, buckling loads correspond to resonant frequencies of the circuit. When the scale changes of eq.(64) are made, the value of the T inverse inductors become $\frac{1}{L} = T q b^2 a^2$. When condensers are used for negative inductances their value is:

$$C = \frac{1}{\bar{\omega}^2 L} = \frac{T q b^2 a^2}{\bar{\omega}^2} \tag{106}$$

In Fig. 42 these condensers would be in parallel with the rotary inertia condensers. In the figure, $b = \frac{2}{\rho}$ and for $q = r$, the axial load condenser values would be $\frac{T a^2}{\rho \bar{\omega}^2}$.

3.7 Lumping of Distributed Forces.

Within the stated assumptions of one-dimensional theory, the electric circuit analogies developed in sections 3.1 to 3.3 are exact for the case of concentrated loads. When distributed loads act on a continuous system they must be replaced by equivalent lumped loads in the analogy. The word, equivalent, in this case can have many meanings. For instance, the lumped loads may be statically equivalent to the distributed loads which they replace, or they may be concentrated loads which produce deflections at the points of application equal to the deflections of the same points under the distributed loads.* The method used in this thesis of replacing distributed loads by statical equivalents does not require any knowledge whatsoever of the solution in order to make the lumping, and it is thus probably the most general and best to use on complicated

* Another method of treating systems with distributed loads is to change the effective spring constants so that upon application of lumped loads, deflections nearly equal to those caused by equivalent distributed loads are obtained. This method is discussed by Horvay and Ormondroyd (19).

dynamical systems.

Distributed loads caused by the motion of the system itself form an important class of such loads. Examples are inertia forces due to distributed mass, transverse forces caused by an elastic foundation or a viscous fluid, and distributed moments caused by axial forces. In this case the distributed forces are effectively combined by lumping the distributed mass, spring or viscous damper and applying the resultant lumped element at the node points of the elastic system or circuit analogy. Thus the total inertia force on a section of a simple beam with mass per unit length μ is $\int_0^q \mu \ddot{y}\, dx$. In the lumped analogy this force is replaced by ${}_i\ddot{y} \int_0^q \mu\, dx$ where ${}_i\ddot{y}$ should be at or near the center of mass of the mass, $\int_0^q \mu\, dx$. In the general case, $\ddot{y}$ and μ are functions of x, and the integral of the distributed inertia forces is not exactly equivalent to the lumped inertia force. That is:

$$\text{(in general)} \qquad \int_0^q \mu \ddot{y}\, dx \neq {}_i y \int_0^q \mu\, dx$$

If $\ddot{y}$ and μ are expanded in a Taylor's series about their value at one end of the beam section it can be shown that for y and μ constant or linear functions of x the equality is true.

The errors involved in some particular types of lumping on some specific systems will be investigated in part IV. An analysis of the problem in general would be interesting but difficult. It should be pointed out that these lumping errors are not peculiar to electric analogy methods but occur in most numerical methods of solving continuous mechanical systems.

3.8 Frames

The manner in which beam sections are placed in series to form beams has been discussed. In the same manner, beams may be combined to form frames. If a frame is a structure such as a building in which the dynamic behavior of the building as a whole is of interest, then only one or two sections are required for each beam and the masses of the beams may be lumped at the beam junctions. On the other hand, if the frame is a structure such as an airplane in which the mode shape of each wing is important, then enough sections must be used in the beam to adequately represent the number of modes desired.

In a frame each member can be described in a coordinate system which yields the best circuit analog for that member. The various members, each in an optimum coordinate system, are then connected by transformer networks. In a frame with only mutually perpendicular members the coordinate systems will differ normally by scale change factors only, but in a frame with oblique members, coordinate axes changes will usually be required.

An example of a simple plane frame with oblique members is given in Fig. 52a. The coordinate axes chosen to represent each beam are shown in Fig. 52b and the resulting circuit diagram in Fig. 53. The methods of the preceeding section may be used to take account of the combined axial and lateral loads. If the external force had been other than along one of the six coordinate directions at the beam junction, it would have to be transformed into components along these directions. The coordinate axes transformation at the junction may be eliminated if the same coordinate system is used for both beams. This usually complicates one of the beam circuits so that more transformers are required in it. The best method to use depends upon the specific problem.

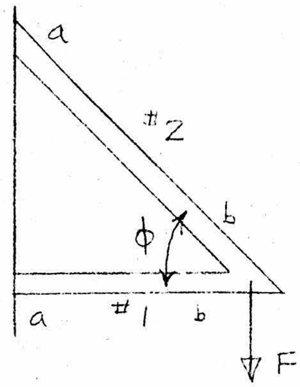


Fig. 52a.

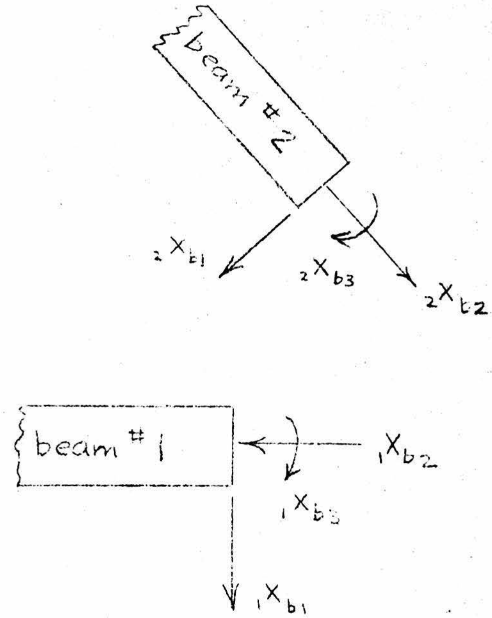


Fig. 52b.

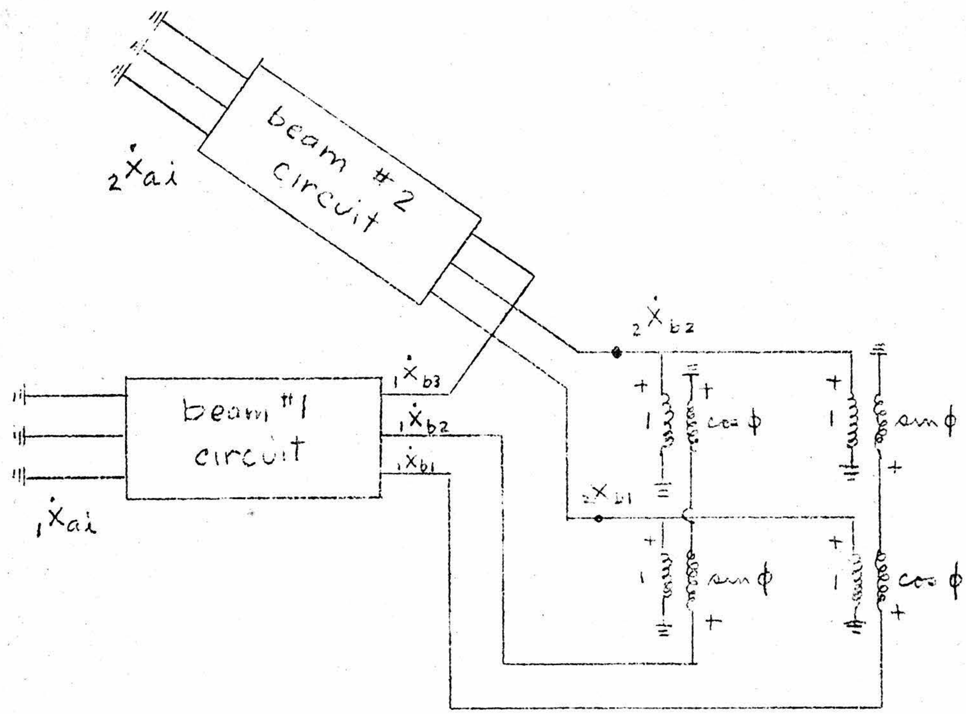
The coordinate system transformation is:

$$ {}_1x_{b_1} = {}_2x_{b_1}\cos\phi + {}_2x_{b_2}\sin\phi $$

$$ {}_1x_{b_2} = {}_2x_{b_1}\sin\phi - {}_2x_{b_2}\cos\phi $$

$$ {}_1x_{b_3} = {}_2x_{b_3} $$

As another example of a frame, Figs. 54 and 55 show analogous circuits for an airplane whose wing, fuselage and tail surfaces are beams which bend in a principal plane and twist. The problem represents an actual airplane and is one that was solved on the Electric Analog Computer of the Analysis Laboratory at the California Institute of Technology. The airplane is symmetrical, and therefore symmetrical and antisymmetrical modes of vibration exist and may be studied separately. In the symmetrical modes the wing and horizontal stabilizer tips move together and the fuselage bends in a vertical plane but does



(The four two-winding transformers in the circuit may be replaced by two three-winding transformers)

Fig. 53.

not twist. The vertical fin is considered rigid in these modes. In the antisymmetrical modes the wing and horizontal stabilizer tips move oppositely, the fuselage bends sidewise and twists and the vertical fin bends and twists. The wings and horizontal stabilizer bend and twist in both cases. Because of the symmetry of the airplane it is only necessary to model the half airplane lying on one side of the longitudinal axis. The finite difference beam analog is shown in the figures.

The centers of mass of the beam sections are in general not on the center of twist of the beams, yet the coordinate system used to describe the behavior of the masses is the deflection of, and rotation about, the center of twist. Therefore, the situation is exactly that of the example in section 2.4 and the rigid body mass analog is the circuit of Fig. 16. The centers of mass are not all on the same side of the center of twist and therefore β position transformers are used in some of the mass networks to change the sign of the m_{12}.

The masses and stiffnesses of the fuselage are much larger than those of the wing and tail members, and to keep the electrical elements within an optimum size for computer use, different scales were used for the different beams. The scale change transformers shown in the figure were therefore required. Note, for example, that twist in the wing is coupled to slope in the fuselage.

In the analogy for the symmetrical modes the engine can move vertically and can pitch. The coordinate system used to describe its behavior has as components the displacements of the points of attachment of the mounting springs. The circuit analog is the π network shown. The engine in the antisymmetrical case is the same as that given as an example in section 2.5c. The circuit analog is the generalized π with a coordinate axes transformation. In Fig. 55, the engine analog has three transformers instead of the one required in Fig. 24. The reason for the difference is that at the time the airplane problem was solved the methods developed in this thesis were not known and the coordinate system used to describe the engine behavior was not optimum.

Rotary inertia of the beam sections was neglected in all cases but those such as the fuselage sections carrying the wing in yaw or the vertical fin in pitch.

The various sub-circuits mentioned above are labelled on the circuit diagrams.

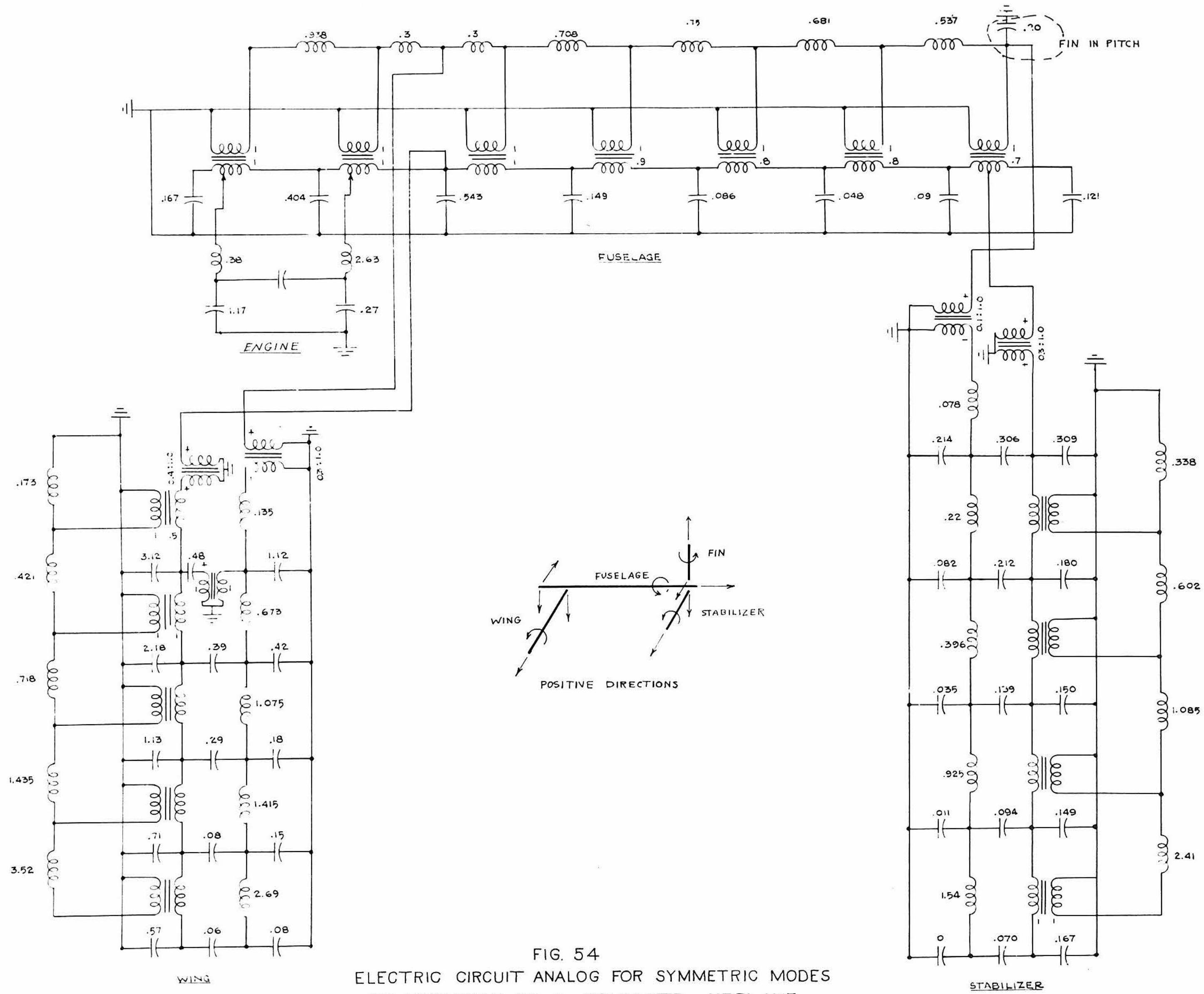


FIG. 54
ELECTRIC CIRCUIT ANALOG FOR SYMMETRIC MODES
OF VIBRATION OF A COMPLETE AIRPLANE

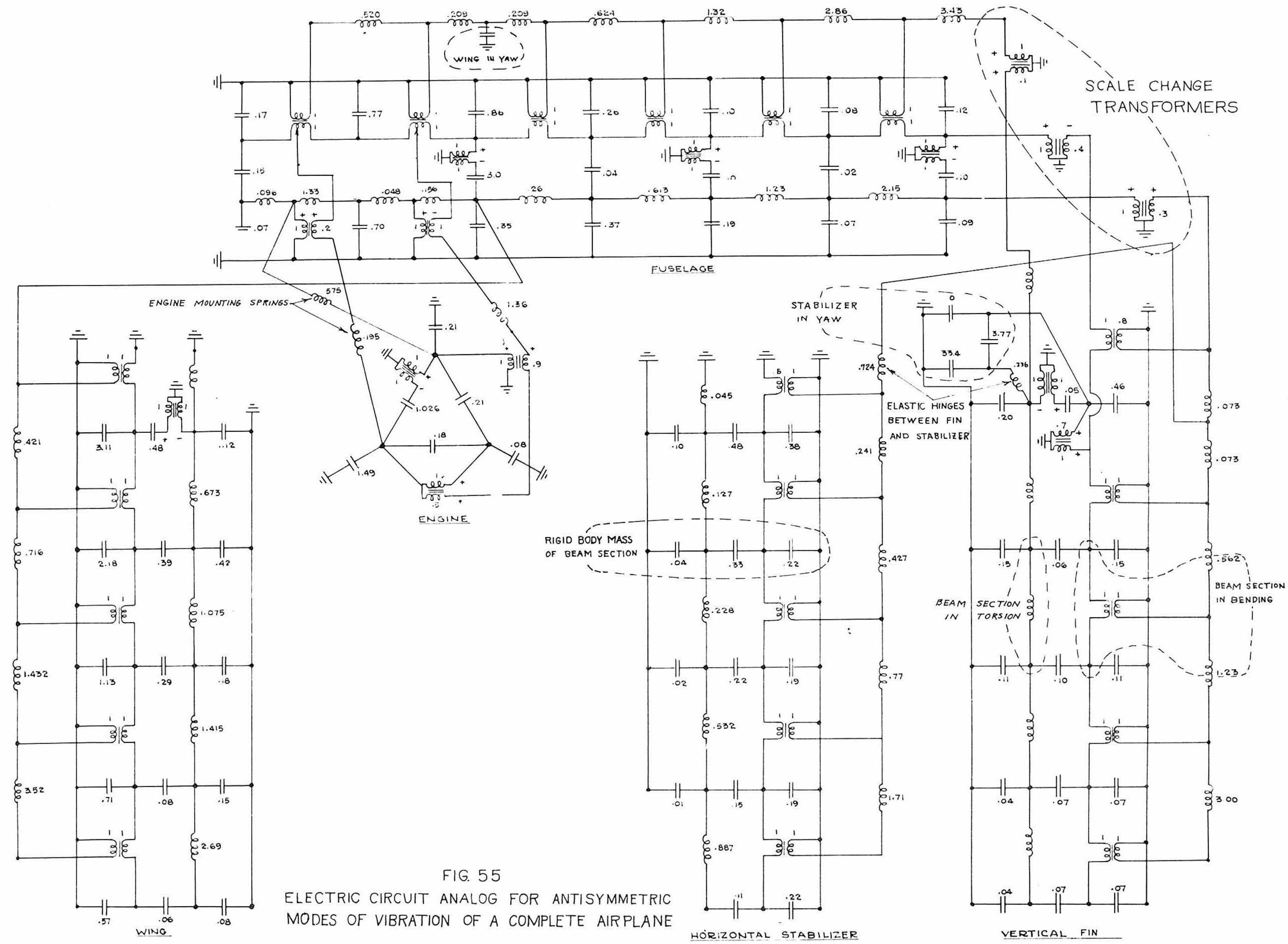


FIG. 55
ELECTRIC CIRCUIT ANALOG FOR ANTISYMMETRIC MODES OF VIBRATION OF A COMPLETE AIRPLANE

3.9 Analogies for Two and Three-Dimensional Systems

An electric circuit is topologically a system of lines and this means that to obtain analogies for two or three-dimensional continuous systems they must be replaced by a grid or mesh of one-dimensional systems. Another way to state the situation is that, excluding the time variable, two and three-dimensional continuous systems are described by partial differential equations while electric networks are always described by sets of ordinary differential equations. After the two or three-dimensional system is replaced by an equivalent one-dimensional mesh, the methods developed in this thesis can be used, with no particular complication, to obtain a circuit analogy.

The equivalence of one-dimensional meshes to two-dimensional problems has been quite widely studied although much remains to be done. The mesh analogy for a membrane is well known.* The electric circuit analog is a network of inductors topologically the same as the mesh of springs. A mesh of bars which have axial and shear springs can be used to approximate the problem of plane stress and plane strain. Coupling between bars in perpendicular directions is caused by Poisson's ratio. Kron (11) has given an analogy for this mesh and MacNeal in an unpublished note has used a finite difference approximation to the differential equations to obtain the same result in a somewhat more general fashion. The analogy can also be constructed using the impedance method discussed in section 3.1 and 3.2. Grinter (5), page 13, has given an equivalent mesh for plane stress and plane strain problems in which beams in axial strain and bending replace the shear members of the other mesh analogy. He shows that the effect of Poisson's ratio may

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* Marcus (20) has written a book on the subject of mesh approximations for two dimensional elastic systems.

often be neglected in this mesh and when this can be done the circuit for the mesh is simpler than that for the finite difference equations of elasticity.

An elastic plate may also be approximated by an orthogonal mesh of beams which bend and twist. MacNeal (9) has obtained a finite difference analogy for a plate by converting the differential equations for a plate (see Timoshenko (21)) into finite difference form. Newmark (5), page 138, has briefly discussed the corresponding mechanical model. The analogy consists of an orthogonal net of the finite difference beams discussed in section 3.4 which twist and bend and are interconnected by springs which correspond to Poisson's ratio. If the beams developed in section 3.2 of this thesis were used in place of the finite difference beams, considerable improvement in accuracy should result.

Three dimensional problems can be handled in the same way as two dimensional ones but an almost prohibitively large number of electric elements would be required for all but the simplest problems.

PART IV

ERRORS

The nature of the errors involved in the solution of problems of continuous mechanical systems by electric circuit analogies was outlined in section 1.4. A complete study of these errors would include an investigation of the effect of circuit element imperfections and of lumping of parameters on the transient and steady state response of systems. Unfortunately these error studies are difficult and require long and tedious numerical calculations. In the present part of this thesis an error study of some simple uniform systems will be made. It seems reasonable that the results may be applied qualitatively to systems which are not too greatly non-uniform.

In section 4.1 the deflections of a uniform cantilever beam under a uniform distributed load will be obtained and compared with the deflection of the same beam under equivalent lumped loads and with the displacement of the equivalent finite difference beam. Explicit expressions for deflection error for the general case of n concentrated (lumped) loads will be obtained.

In section 4.2 the natural frequencies and mode shapes of a uniform cantilever beam will be compared with those for the lumped mass analogy, the finite difference beam, and with the results obtained from an analogous electric circuit for the lumped mass beam. The comparison is made for $n = 1$ to $n = 4$ sections in the beam. Since the finite difference beam may be considered as a simplification of the lumped mass beam, this comparison evaluates for this particular system the three types of error mentioned in section 1.4.

In section 4.3 the continuous beam, the lumped mass beam and the finite difference beam are compared for the case of pinned-pinned ends with $n = 3$.

The methods used in part IV may be applied to beams with other boundary conditions and with a greater number of sections than are studied here. When and if the errors due to circuit elements are known and understood, an electric analog computer could be used to study the errors due to lumping and this would obviate the discouraging amount of numerical calculation otherwise necessary.

4.1 Comparison of the Deflection of a Cantilever Beam under a Uniform Distributed Load with that of the same Beam under a Set of Equivalent Concentrated Loads and with the Deflection of the Equivalent Finite Difference Beam.

In this section the deflection of a uniform cantilever beam under a uniform distributed transeverse load will be obtained by the theory of strength of materials. The distributed forces will then be replaced by a set of statically equivalent concentrated forces and the influence coefficient matrix for this system obtained. The terms of this matrix will then be summed in such a manner that the deflections of the points at which the forces are applied are obtained. These deflections will then be compared with those of the same points under the distributed load. Finally the finite difference equation for a beam under uniform distributed load is written and the solution given. This solution is then compared with the other two. The results are expressed as deflection errors which are functions of position in the beam and number of sections into which the beam is divided. This error study of a static system is important because it is one of the few that can be solved for a general number of beam sections. Qualitatively, the results can probably be applied to systems with some non-uniformity.

Consider the beam of Fig. 56a.

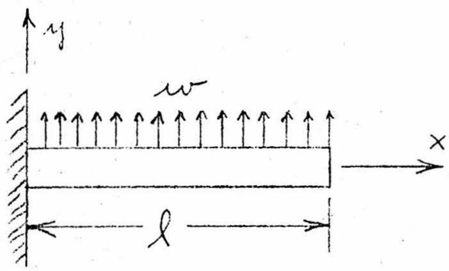

Fig. 56a.

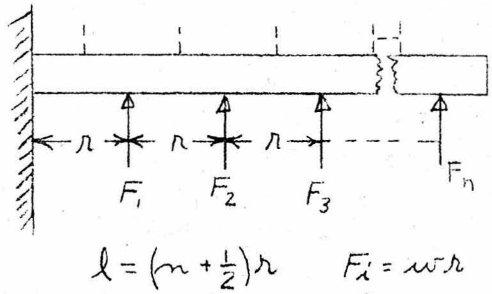

Fig. 56b.

The differential equation for this beam is $\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = w$ and its solution is:

(107) $$y = \frac{wx^2}{24EI}\left(6l^2 - 4lx + x^2\right)$$

The lumped load approximation to Fig. 56a is given in Fig. 56b. To develop the deflection equation for this lumped load beam we next consider the end loaded cantilever of Fig. 57.

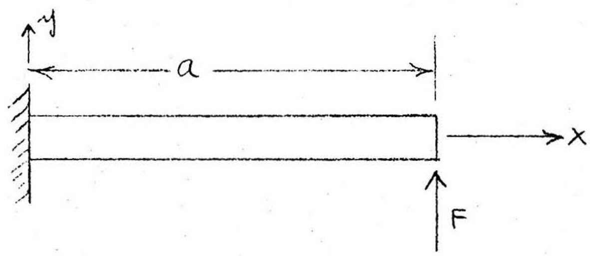

Fig. 57.

From strength of materials theory one obtains for this beam,

(108) $$y = \frac{Fx^2}{6EI}(3a - x)$$

and if the substitutions $a = jr$ and $x = ir$ are made, eq.(109) is obtained. (i and j are positive integers which indicate position).

(109) $$y_i = \frac{F_j \ell^3}{6EI}\left[i^2(3j - i)\right] \qquad j, i = (1 - - - n)$$

This equation gives the beam deflection at the point i due to a force at point j when $j \geq i$. From it an influence coefficient matrix can be constructed. That is:

(110) $$y_i = \sum_{j=1}^{j=n} g_{ij} F_j$$

where $$g_{ij} = \frac{\ell^3}{6EI}\left[i^2(3j - i)\right] \quad \text{for} \quad j \geq i \ (i, j = 1 - - - n)$$

and $g_{ij} = g_{ji}$ by the reciprocity theorem.

A few values of this matrix are given in eq.(111). Note that the g_{ij} terms are unique and do not depend on n. This means that the matrix for n sections is obtained by taking only the first n rows and columns from the general matrix.

(111) $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \frac{\ell^3}{6EI} \begin{bmatrix} 2 & 5 & 8 & 11 & 14 \\ 5 & 16 & 28 & 40 & 52 \\ 8 & 28 & 54 & 81 & 108 \\ 11 & 40 & 81 & 128 & 176 \\ 14 & 52 & 108 & 176 & 250 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix}$$

The summation of eq.(110) will now be performed explicitly. Since the value of g_{ij} given in eq.(110) is true only for $j \geq i$, the symmetry properties of the matrix will be used and the summation will be made in two parts. The path of summation is shown by arrows in eq.(111). First the terms in the ith column are summed down to, but not including, the diagonal term. This sum is:

$$\sum_{k=1}^{k=i-1} k^2(3i - k)$$

The second sum is formed by starting at the ith column and summing along the ith row. This sum is:

$$\sum_{k=i}^{k=n} i^2 (3k-i)$$

(The reader should remember that i, j, and k are dummy indexes standing for integers. Whether they specify a row or column in the matrix depends on their position in the various terms). Since all the F_i are equal, eq.(110) with the method of summation indicated is given by:

(112) $$y_i = \frac{Fr^3}{6EI}\left[\sum_{k=1}^{k=i-1} k^2(3i-k) + \sum_{k=i}^{k=n} i^2(3k-i)\right]$$

The first sum is the sum of two finite power series while the second is the sum of an arithmetic progression. The result of the summation is:

(113) $$y_{Li} = \frac{Fr^3}{48EI} i^2\left[2i^2 - (4+8n)i + (2+12n+12n^2)\right]$$

where the L subscript indicates the lumped load system.

The deflections of the equally spaced points, $x = ir$, $(i = 1 \cdots n)$, will now be obtained for the continuously loaded cantilever, eq.(107). When the substitutions:

$$l = (n + \tfrac{1}{2})r \quad , \quad x = i\,r \quad , \quad r\,w = F$$

are made in eq.(107) it becomes:

(114) $$y_{ci} = \frac{Fr^3}{48EI} i^2\left[2i^2 - (4+8n)i + (3+12n+12n^2)\right]$$

where the c subscript indicates the distributed load or continuous system.

The difference in deflection of the point $x = i\,r$ under the continuous and the lumped loads is given by the difference of eq.(114) and eq.(113).

This difference is the remarkably simple relation:

$$(115) \qquad (y_c - y_L)_i = \frac{F r^3}{48EI} i^2$$

(Note that y_{ci} is always greater than y_{Li})

In terms of w, x and ℓ the error becomes:

$$(116) \qquad (y_c - y_L)_x = \frac{F w \ell^2 x^2}{48EI\,(n+\frac{1}{2})^2} \quad \text{(x is a discrete variable)}$$

The error ratio in dimensionless form is:

$$(117) \qquad \left(\frac{y_c - y_L}{y_c}\right)_x = \frac{1}{2\left(n+\frac{1}{2}\right)^2\left[6 - 4\frac{x}{\ell} + \left(\frac{x}{\ell}\right)^2\right]}$$

The two equations above show that the maximum error magnitude and the maximum percent error occur at $\frac{x}{\ell} = \frac{n}{n+\frac{1}{2}}$.

The deflection of the equivalent finite difference beam will now be obtained. The differential equation for the beam with distributed load can be converted to a finite difference equation by replacing differentials with differences. When this is done the finite difference equation that is obtained is:

$$(118) \qquad y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = \frac{w(\Delta x)^4}{EI} \qquad \Delta x = r, \quad \ell = (n+\tfrac{1}{2})r$$

The general solution to this inhomogenous difference equation is:

$$(119) \qquad y_i = A_0 + A_1 i + A_2 i^2 + A_3 i^3 + \frac{w r^4}{24EI} i^4$$

The boundary conditions on the beam are expressed by:

$$y_0 = 0 \qquad y_{n+1} - y_n = y_n - y_{n-1}$$

$$y_{-1} = y_{+1} \qquad y_{n+2} - y_{n+1} = y_{n+1} - y_n$$

These boundary conditions are substituted into eq.(119) and after considerable algebraic work the arbitrary constants are evaluated and the solution, eq.(120) is obtained.

$$(120) \qquad y_{fi} = \frac{Fr^3}{48EI}\left[2i^4 - (8n+4)i^3 + i^2(12n^2 + 12n - 2) + i(8n+4)\right]$$

where the f subscript indicates the finite difference beam.

The difference in deflection at the point i between the continuous and finite difference beams is given by eq.(114) minus eq.(120). This is:

$$(121) \qquad (y_c - y_f)_i = \frac{Fr^3}{48EI}\left[5i^2 - 8ni - 4i\right]$$

Since $n > 1$, eq.(121) states that $y_{ci} < y_{fi}$. The error ratio in dimensionless form for the finite difference beam is:

$$(122) \qquad \left(\frac{y_c - y_f}{y_c}\right)_x = \frac{1}{2\left(\frac{x}{l}\right)^2\left(n+\frac{1}{2}\right)^2}\left(\frac{5\left(\frac{x}{l}\right)^2 - \frac{1}{n+\frac{1}{2}}(8n+4)\frac{x}{l}}{6 - 4\frac{x}{l} + \frac{x^2}{l^2}}\right)$$

Eq.(121) shows that the maximum error occurs at $i = \frac{4n+2}{5}$ and eq.(122) shows that the maximum percent error occurs at x = minimum = r.

In table II are given values of the percent error for the lumped beam, eq.(117), and the finite difference beam eq.(122), for values of n from 1 to 6.

Table II

Values of error ratio in percent for the lumped and finite difference approximations to a uniformly loaded cantilever beam.

	$n=1$		$n=2$		$n=3$	
position	$\frac{y_c-y_L}{y_c}\cdot 100$	$\frac{y_c-y_f}{y_c}\cdot 100$	$\frac{y_c-y_L}{y_c}\cdot 100$	$\frac{y_c-y_f}{y_c}\cdot 100$	$\frac{y_c-y_L}{y_c}\cdot 100$	$\frac{y_c-y_f}{y_c}\cdot 100$
x_1	5.89	-41.1	1.75	-26.3	.826	-19.0
x_2			2.32	-11.6	1.01	- 9.09
x_3					1.23	- 5.35
	$n=4$		$n=5$		$n=6$	
x_1	.478	-14.8	.312	-12.1	.218	-10.3
x_2	.559	- 7.83	.353	- 6.00	.248	- 5.11
x_3	.654	- 4.58	.401	- 3.88	.271	- 3.34
x_4	.762	- 3.05	.456	- 2.74	.302	- 2.42
x_5			.518	- 1.97	.337	- 1.82
x_6					.375	- 1.39

4.2 Comparison of Normal Modes and Frequencies of a Continuous Cantilever Beam with those of the Lumped Mass Beam, the Finite Difference Beam, and an Analogous Electric Circuit.

In this section the natural frequencies and normal modes of a cantilever beam carrying lumped masses will be obtained and compared with those of a continuous cantilever. The natural frequencies will also be calculated for the finite difference beam and in addition the results of an electric circuit solution on the CIT Electric Analog Computer will be given. The results are summarized in the error information given in tables IV and V.

4.2a The first four normal modes and frequencies for the continuous beam will first be obtained. Consider the beam of Fig. 56a in which the distributed load is an inertia loading. The differential equation in this case is:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = \omega_c^2 \mu y \tag{123}$$

where μ = mass per unit length. The solution to this equation for clamped-free boundaries is: (Timoshenko (22), page 344, or Rayleigh (23), page 276.)

$$y = A\left[\cos ax - \cosh ax + \frac{\cos al + \cosh al}{\sin al + \sinh al}(\sinh ax - \sin ax)\right] \tag{124}$$

where:

$$a^2 = \omega_c\sqrt{\frac{\mu}{EI}} \tag{125}$$

(the c subscript indicates the continuous mass beam)

and the first four values of a are given by:

$$a_1 l = 1.875104$$

$$a_2 l = 4.694091$$

$$a_3 l = 7.854757$$

$$a_4 l = 10.995541$$

The corresponding values of the frequency parameter, $\omega_c l^2\sqrt{\frac{\mu}{EI}} = a_i^2 l^2$, are given in table IV, column 3.

The mode shapes of the lumped mass analogy are expressed by deflections of points along the beam. To make a comparison of mode shapes, the deflections of the same points in the continuous mass beam must be calculated. In both cases the deflection at any point x_i is made dimensionless by dividing by the deflection at point x_1. The calculation is made using the function of eq.(124). This function and its derivative have been calculated by Fung in Air Force Technical Report No. 5761, part II, (24). Using these tables the deflection ratios given in table V, column 3 are obtained.

4.2b The natural frequencies and mode shapes of the continuous beam having been obtained, the next step is a similar calculation for the lumped mass beam. The lumped approximation to Fig. 56a is Fig. 56b where the concentrated forces are now inertia forces of magnitude:

(126) $$F_i = \omega^2 \mu r\, y_i \qquad \text{where } \mu r = \text{lumped mass}, \quad l = (n+\tfrac{1}{2})r$$

These values of the forces substituted into the inverse spring equation for the beam, eq.(111), give the frequency equation (128). In this equation:

(127) $$\lambda = \frac{6EI}{\mu r^4 \omega_L^2} = \frac{6EI(n+\frac{1}{2})^4}{\mu l^4 \omega_L^2}$$ (the L subscript indicates the lumped mass beam)

(128) $$\begin{vmatrix} 2-\lambda & 5 & 8 & 11 \\ 5 & 16-\lambda & 28 & 40 \\ 8 & 28 & 54-\lambda & 81 \\ 11 & 40 & 81 & 128-\lambda \end{vmatrix} = 0$$

Expansion of the determinant of eq.(128) for n = 1 to 4 gives the following frequency equations

$$
(129)\qquad
\begin{aligned}
n &= 1 \qquad & -\lambda + 2 &= 0\\
n &= 2 \qquad & +\lambda^2 - 18\lambda + 7 &= 0\\
n &= 3 \qquad & -\lambda^3 + 72\lambda^2 - 131\lambda + 26 &= 0\\
n &= 4 \qquad & \lambda^4 - 200\lambda^3 + 1065\lambda^2 - 722\lambda + 97 &= 0
\end{aligned}
$$

The roots of these frequency equations are given below, and in table IV, column 4, are given the values of the frequency parameter, $\omega_L l^2\sqrt{\frac{\mu}{EI}} = \frac{\sqrt{6}\,(n+\frac{1}{2})^2}{\sqrt{\lambda}}$.

$$
(130)\qquad
\begin{aligned}
n &= 1 \qquad & \lambda &= 2\\
n &= 2 \qquad & \lambda &= 17.602325\ ,\ 0.397675\\
n &= 3 \qquad & \lambda &= 70.1375\ ,\ 1.6357\ ,\ 0.2267\\
n &= 4 \qquad & \lambda &= 194.54474\ ,\ 4.68662\ ,\ 0.58757\ ,\ 0.18106
\end{aligned}
$$

The normal mode shapes for the lumped mass beam are calculated from the equations of motion. (The elements of the determinant of eq.(128) are the coefficients of these equations.) Thus, for n = 2 the equations of motion give the displacement ratio by:

$$
(131a)\qquad
\begin{aligned}
\lambda y_1 &= 2y_1 + 5y_2\\
\lambda y_2 &= 5y_1 + 16y_2
\end{aligned}
\qquad \text{and} \qquad \frac{y_2}{y_1} = \frac{\lambda - 2}{5} \quad \binom{\text{for}}{n = 2}
$$

The displacement ratios for other values of n are obtained in the same way. The results are:

$$
(131b)\qquad
\frac{y_2}{y_1} = \frac{7\lambda - 4}{2\lambda + 3}
\qquad
\frac{y_3}{y_1} = \frac{4(7\lambda - 4)}{5\lambda - 46}
\qquad \binom{\text{for}}{n = 3}
$$

$$\frac{y_2}{y_1} = \frac{(40\lambda-25)(+\lambda^2-182\lambda+351)+(12)(133-8\lambda)}{(12)(28\lambda-344)+(24+11\lambda)(+\lambda^2-182\lambda+351)}$$

(131c) $$\frac{y_3}{y_1} = \frac{(40\lambda-25)(28\lambda-344)-(133-8\lambda)(11\lambda+24)}{(+\lambda^2-182\lambda+351)(11\lambda+24)+(12)(28\lambda-344)} \qquad \begin{pmatrix}\text{for}\\ n=4\end{pmatrix}$$

$$\frac{y_4}{y_1} = \frac{(40\lambda+108)(28\lambda-16)+(8\lambda+12)(11\lambda+54)}{(8\lambda+12)(+\lambda^2-182\lambda+351)-(40\lambda+188)(12)}$$

The values of these ratios for the values of λ_1 and λ_2 given in eqs.(130) are tabulated in table V, column 4.

4.2c The next system considered is the finite difference beam. The finite difference equation for a vibrating beam can be solved without difficulty but for clamped-free boundary conditions the frequency equation which is obtained is a very complicated transcendental equation whose roots are very difficult to determine. Instead of using this general solution for the finite difference cantilever beam, the same method which was used for the lumped mass beam will be used.

The deflection at a point x = ir on the finite difference beam due to a load at a point x = jr is given by:

(132) $$y_i = \sum_{j=1}^{j=n} g_{ij} F_j \qquad i = (1 - - n)$$

The value of g_{ij} can conveniently be determined from the circuit analogy for the finite difference beam. Its value is:

(133) $$g_{ij} = \frac{r^3}{2EI}\left[\frac{i}{3}\left(3ji - i^2 + 1\right)\right] \qquad \text{for } j \geq i$$

$$g_{ij} = g_{ji}$$

The inverse spring equation, (132), when written in matrix form with numerical values of g_{ij} obtained from eq.(133) is:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \frac{\lambda^3}{2EI} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 10 & 14 \\ 3 & 10 & 19 & 28 \\ 4 & 14 & 28 & 44 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \tag{134}$$

The frequency equation for the system is obtained from eq.(134) when the F_i are inertia forces given by eq.(126). The frequency equation is:

$$\begin{vmatrix} 1-\lambda_f & 2 & 3 & 4 \\ 2 & 6-\lambda_f & 10 & 14 \\ 3 & 10 & 19-\lambda_f & 28 \\ 4 & 14 & 28 & 44-\lambda_f \end{vmatrix} = 0 \tag{135}$$

where

$$\lambda_f = \frac{2EI}{\mu \lambda^4 \omega_f^2} \tag{136}$$

(the f subscript indicates the finite difference beam)

Expansion of the determinant of eq.(135) for n = 1 to 4 gives the following frequency equations.

(137)

n = 1	$-\lambda_f + 1 = 0$
n = 2	$+\lambda_f^2 - 7\lambda_f + 2 = 0$
n = 3	$-\lambda_f^3 + 26\lambda_f^2 - 26\lambda_f + 4 = 0$
n = 4	$+\lambda_f^4 - 70\lambda_f^3 + 174\lambda_f^2 - 76\lambda_f + 8 = 0$

The roots of these equations are given below, and the values of the frequency parameter $\omega_f \ell^2 \sqrt{\frac{\mu}{EI}} = \frac{\sqrt{2}(n+\frac{1}{2})^2}{\sqrt{\lambda_f}}$ are given in table IV, col. 5.

(138)

n = 1	λ =	1			
n = 2	λ =	6.70156	0.29844		
n = 3	λ =	24.96496	0.84555	0.18949	
n = 4	λ =	67.4364	2.0406	0.36261	0.16033

The mode shape for the finite difference beam may be calculated in exactly the same manner as that for the lumped mass beam, although it will not be done here.

4.2d   To make a limited investigation of errors due to computer elements, the lumped-mass beam analog was tested on the CIT Electric Analog Computer. The results when compared with the lumped-mass beam calculated values give the error due to circuit imperfections.

The electric circuit calculations were made on the circuit of Fig. 58. This is the same circuit as that of Fig. 42 but with the rotary inertia neglected. In the physical circuit used, the leakage inductance of the transformers was included in the values of the circuit inductances and the magnetizing inductance was made negligibly small by operating at high enough frequencies.

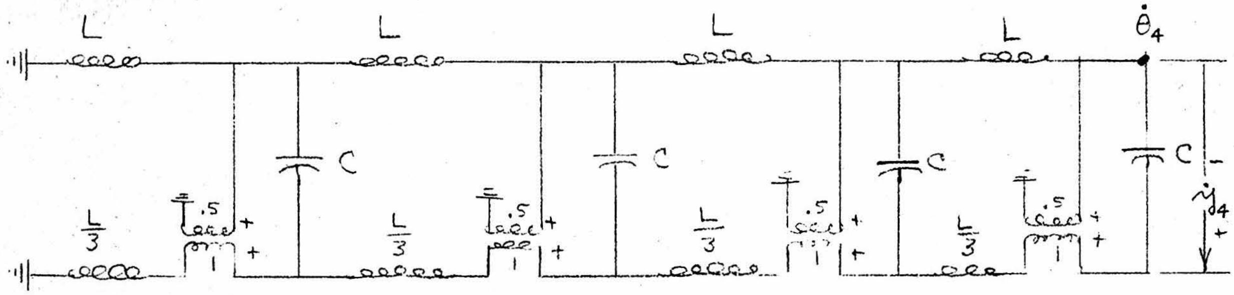

Fig. 58.

The relationship between the circuit element values, the circuit frequencies, and the corresponding beam frequencies is obtained in the following manner.

The lumped beam frequency is given by eqs.(127) which are:

$$(127) \qquad \omega_L = \sqrt{\frac{6EI}{m r^3}}\,\frac{1}{\sqrt{\lambda}} \;, \quad \sqrt{\lambda} = \frac{\sqrt{6}\,(n+\frac{1}{2})^2}{\omega_L \ell^2 \sqrt{\frac{\mu}{EI}}} \;, \quad m = \mu r \;, \quad \ell = (n+\tfrac{1}{2}) r$$

From Fig. 42 the values of the circuit elements in Fig. 58 are found to be:

$$L = \frac{\ell^3}{4EIa^2} \qquad C = \frac{ma^2}{N^2}$$

These values substituted into eq.(127) give:

(139) $$\frac{N\omega_{Le}}{2\pi} = \bar{f}_e = \frac{1}{2\pi}\frac{\sqrt{1.5}}{\sqrt{\lambda}}\frac{1}{\sqrt{LC}}$$ (the e subscript indicates the electric circuit)

Where $\bar{f} = 2\pi\,\bar{\omega}_{Le} = 2\pi N \omega_{Le}$ is the computer frequency.

When the value of $\sqrt{\lambda}$ from the second of eqs.(127) is substituted into (139), the desired relation is obtained.

(140) $$\omega_{Le}\,\ell^2\sqrt{\frac{\mu}{EI}} = \bar{f}_e\sqrt{LC}\,(m+\tfrac{1}{2})^2(4\pi)$$

In the circuit the exciting voltage was adjusted so that in each case y_1 was equal to 1. The values of the circuit elements used and the circuit frequencies obtained are given in table III. The corresponding values of the frequency parameter, $\omega_{Le}\,\ell^2\sqrt{\frac{\mu}{EI}}$, are given in table IV, column 6. The mode shape data from the circuit are given in table V, column 5. A comparison of the data obtained from the electric circuit with the calculated values for the lumped-mass beam yields the errors due to circuit imperfections alone. These data for natural frequencies are given in table IV, column 9.

Table III

Circuit Values for Fig. 58

number of sections n	mode number	L henries	C mfd	f_{LE} c.p.s
1	1	1	1	141
2	1	.1	.1	457
2	2	1	1	310
3	1	.1	.1	231
3	2	1	1	155.5
3	3	1	1	409.5
4	1	.1	.1	140.5
4	2	.3	.3	304
4	3	1	1	256
4	4	1	1	458

Table IV

Frequency Parameters and Frequency Errors for Free Vibrations of a Cantilever Beam

1	2	3	4	5	6	7	8	9
		frequency parameters				Percent error		
n	mode	$\omega_c l^2\sqrt{\frac{\mu}{EI}}$	$\omega_L l^2\sqrt{\frac{\mu}{EI}}$	$\omega_f l^2\sqrt{\frac{\mu}{EI}}$	$\omega_{Le} l^2\sqrt{\frac{\mu}{EI}}$	$\frac{\omega_c-\omega_L}{\omega_c}\cdot 100$	$\frac{\omega_c-\omega_f}{\omega_c}\cdot 100$	$\frac{\omega_L-\omega_{Le}}{\omega_L}\cdot 100$
1	1	3.516015	3.8977	3.18197	3.26	-10.86	+ 9.50	-2
2	1	3.516015	3.6490	3.41438	3.36	- 3.783	+ 2.89	+ 2
	2	22.034490	24.277	16.1797	16.2	- 1.018	+26.57	- .3
3	1	3.516015	3.5829	3.4673	3.47	- 1.902	+ 1.385	0
	2	22.034490	23.463	18.8399	19.2	- 6.485	+14.50	-2
	3	61.697208	63.025	39.7980	39.8	- 2.152	+35.49	0
4	1	3.516015	3.55624	3.4873	3.50	- 1.143	+ .816	- .7
	2	22.034490	22.9124	20.0474	20.3	- 3.985	+ 9.018	-1
	3	61.697208	64.708	47.5575	47.9	- 4.88	+22.91	- .4
	4	120.90192	116.571	71.5211	71.5	+ 3.582	+40.84	0

Table V

Mode Shape Data for Vibrating Cantilever Beam

n	mode	$\left(\frac{y_2}{y_1}\right)_C$	$\left(\frac{y_2}{y_1}\right)_L$	$\left(\frac{y_2}{y_1}\right)_{LE}$	positions of $\frac{y_2}{y_1}$
2	1	+ 3.1558	+ 3.1205	+ 3.07	$y_{\frac{x}{l}=\frac{4}{5}}$ / $y_{\frac{x}{l}=\frac{2}{5}}$
	2	- .1025	- .3205	- .33	
3	1	+ 3.4108	+ 3.3987	+ 3.36	$y_{\frac{x}{l}=\frac{4}{7}}$ / $y_{\frac{x}{l}=\frac{2}{7}}$
	2	+ 1.2915	+ 1.1879	+ 1.19	
	3	- .4670	- .7008	- .71	
4	1	+ 3.5177	+ 3.5457	+ 3.50	$y_{\frac{x}{l}=\frac{4}{9}}$ / $y_{\frac{x}{l}=\frac{2}{9}}$
	2	+ 2.0252	+ 1.9794	+ 1.98	
	3	+ .4803		+ .240	
	4	- .7790		- .96	
		$\left(\frac{y_3}{y_1}\right)_C$	$\left(\frac{y_3}{y_1}\right)_L$	$\left(\frac{y_3}{y_1}\right)_{LE}$	positions of $\frac{y_3}{y_1}$
3	1	+ 6.4427	+ 6.3928	+ 6.30	$y_{\frac{x}{l}=\frac{6}{7}}$ / $y_{\frac{x}{l}=\frac{2}{7}}$
	2	- .6530	- .7880	- .82	
	3	- .0927	+ .2151	+ .232	
4	1	+ 6.9495	+ 6.9919	+ 6.92	$y_{\frac{x}{l}=\frac{6}{9}}$ / $y_{\frac{x}{l}=\frac{2}{9}}$
	2	+ 1.1977	+ 1.0764	+ 1.12	
	3	- .9660		- .74	
	4	+ .2311		+ .60	
		$\left(\frac{y_4}{y_1}\right)_C$	$\left(\frac{y_4}{y_1}\right)_L$	$\left(\frac{y_4}{y_1}\right)_{LE}$	positions of $\frac{y_4}{y_1}$
4	1	+10.7639	+10.8074	+10.70	$y_{\frac{x}{l}=\frac{8}{9}}$ / $y_{\frac{x}{l}=\frac{2}{9}}$
	2	- 1.3345	- 1.4383	- 1.42	
	3	+ .2204		+ .32	
	4	+ .2061		- 1.62	

4.3 Comparison of Normal Modes and Frequencies of a Distributed Mass Pinned-pinned Beam with those of the Lumped Mass and Finite Difference Equivalents.

In this section the normal modes and frequencies of a distributed mass, uniform, pinned-pinned beam will be given. The frequencies and mode shapes for the equivalent lumped mass beam with n = 3 will then be found. The general solution of the finite difference equation for pinned-pinned ends will be given. Finally the frequencies and mode shapes for the three cases will be compared.

The solution of the beam equation, eq.(123), with distributed mass loading for pinned-pinned boundaries is: (Timoshenko (22), page 338)

(141) $$y = A \sin \frac{s\pi x}{l} \sin \omega_c t$$

where:

(142) $$\frac{l^2}{\pi^2} \omega_c \sqrt{\frac{\mu}{EI}} = s^2 \qquad s = 1 \cdots \infty$$

The natural frequencies and mode shapes for a lumped mass beam may be determined by the same methods as were used for the cantilever beam. For the pinned-pinned beam with n = 3 the frequency equation is:

(143) $$\begin{vmatrix} 18-\lambda & 22 & 14 \\ 22 & 32-\lambda & 22 \\ 14 & 22 & 18-\lambda \end{vmatrix} = 0$$

where:

(144) $$\lambda = \frac{24\,EI}{\mu\, r^4\, \omega_L^2} = \frac{24\,EI\,(n+1)^4}{\mu\, l^4\, \omega_L^2}$$

Eq.(143) reduces to:

(145) $$(4-\lambda)(\lambda^2 - 64\lambda + 56) = 0$$

The roots of this equation are:

(146) $$\lambda = 32 \pm \sqrt{968}\,,\ 4 \quad ; \text{ or } \quad \lambda = 63.1127\ ,\ 4\ ,\ 0.8873$$

The corresponding values of the frequency parameter are:

(147) $$\frac{\ell^2}{\pi^2}\omega_L\sqrt{\frac{\mu}{EI}} = 0.999695 \quad , \quad 3.97096 \quad , \quad 8.430919$$

The mode shapes of the beam are obtained from the equations of motion. They are specified by:

(148)
$$y_1 = \pm y_3$$
$$\frac{y_1}{y_2} = \frac{7\lambda + 18}{11\lambda - 44}$$

The value, $\lambda = 4$, substituted into this last equation gives a singularity which indicates the second mode, in which by symmetry, the point, $\frac{x}{\ell} = \frac{1}{2}$, does not move. When the quadratic roots are substituted into the second of eqs.(144) and the expression is simplified, the following value is obtained.

(149) $$\frac{y_1}{y_2} = \frac{y_{\frac{x}{\ell}=\frac{1}{4}}}{y_{\frac{x}{\ell}=\frac{1}{2}}} = \pm\frac{\sqrt{2}}{2}$$

A comparison of these values with the corresponding values from eq.(141) shows that there is no mode shape error for the lumped mass beam for n =3.

The finite difference equation for inertia loading is:

(150) $$y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = \frac{\omega_f^2 \mu r^4}{EI} y_i$$

The general solution for this homogeneous equation is obtained by assuming that $y_i = e^{ai}$. This value is substituted into eq.(150) and the solution found to be:

(151) $$y_i = A\cos ai + A_2 \sin ai + A_3 \cosh ai + A_4 \sinh ai$$

Where:

(152) $$\cos a = 1 - \frac{\sqrt{\frac{\omega_f^2 \mu r^4}{EI}}}{2} \quad \text{or} \quad \sqrt[4]{\frac{\omega_f^2 \mu r^4}{EI}} = 2\sin\frac{a}{2}$$

The boundary conditions are:

(153) $$y_{-1} = -y_{+1} \qquad y_n = -y_{n+2}$$
$$y_0 = 0 \qquad y_{n+1} = 0$$

When these boundary conditions are substituted into eq.(151) the solution below is obtained:

(154) $$y_i = A(\sin ai)\sin\omega_f t = A\sin\frac{s\pi x}{l}\sin\omega_f t$$
$$a = \frac{s\pi}{n+1} = \frac{s\pi h}{l} \qquad s = (1 \cdots n)$$

When this value of a is substituted into eq.(152), the frequency parameter for the finite difference beam is found to be:

(155) $$\frac{l^2}{\pi^2}\omega_f\sqrt{\frac{\mu}{EI}} = \left[\frac{n+1}{\pi}\,2\sin\frac{s\pi}{2(n+1)}\right]^2$$

The numerical values of this parameter for n = 3 are:

(156) $$\frac{l^2}{\pi^2}\omega_f\sqrt{\frac{\mu}{EI}} = 0.97449 \;,\quad 1.8006 \;,\quad 2.3526$$

A comparison of eqs.(154) and eq.(141) shows that there is no mode shape error for the finite difference pinned-pinned beam for any value of n.

The frequencies for the three beams may be compared by giving the frequency errors. These are given in table VI.

Table VI

mode	$\frac{\omega_c - \omega_L}{\omega_c}\cdot 100$	$\frac{\omega_c - \omega_f}{\omega_c}\cdot 100$
1	.031 %	5.04 %
2	.726 %	18.94 %
3	6.32 %	38.5 %

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