SINGULAR PERTURBATION PROBLEMS

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Gordon Eric Latta

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Abstract

The equations considered in this paper are linear differential equations in one and two independent variables. The problem at hand is to study solutions of boundary value problems for these equations in their dependence on a small parameter ϵ. Specifically, the equations are of the form

$$\text{(A)} \qquad \epsilon N\phi + M\phi = 0$$

where M, N are linear differential expressions, and $\epsilon > 0$ is a small parameter; the order n of N is greater than the order m of M.

It is found, in certain cases, that the solution of a boundary value problem for (A), say $\phi(P,\epsilon)$ tends non uniformly to a function $u(P)$ satisfying the "reduced equation" $M\phi = 0$, and even assumes the original boundary values on certain portions of the boundary of the region in question.

When the regions of non uniform convergence are located, an asymptotic expansion in terms of specific functions of ϵ, for ϵ small, is obtained.

Section two deals with a class of ordinary differential equations, while sections three and four deal with partial differential equations. In particular, it appears from the results of section four, that methods used in this paper should carry over to the non-linear Navier-Stokes equations of which the Oseen equations of the last section are a linearized approximation. This is being investigated at present.

SECTION 1

Certain of the boundary value problems for differential equations occurring in mathematical physics are customarily "simplified" by neglecting some of the higher order derivatives appearing in the equations. (One of the foremost examples is Prandtl's "boundary layer" theory.) In those cases in which any justification of this procedure is attempted, physical arguments are frequently used. One of the main difficulties involved in supplying any rigorous proofs of convergence, or uniformity of approximation of the solutions of these simplified problems to those of the full problem, lies in the present lack of a representation of these solutions, even granting their existence. We shall endeavor to throw a little more light on certain of these problems by using firstly a heuristic argument, and then proving the validity of the results of these arguments in certain cases.

Thus we consider a special class of boundary value problems for equations of the form

$$\epsilon N\phi + M\phi = 0, \tag{1.1}$$

with suitable boundary conditions, where $N\phi$, $M\phi$, are linear differential expressions involving one or more independent variables. N is in general taken to be of a higher order than M. We shall limit ourselves to the case of constant coefficients, although this restriction is for convenience only, the method being general. In this way we try to minimize the difficulties involved in dealing with the actual equation

1.1) and attempt to isolate those properties of the solution to 1.1), namely $\phi(P, \epsilon)$, which cause difficulty as $\epsilon \to 0$. Several of these difficulties are apparent immediately.

a.) M is of lower order than N, and hence the solution of the equation $M\phi = 0$ cannot, in general, satisfy as many boundary conditions as can that of the full equation. Thus we expect to obtain a region of non-uniform convergence of $\phi(P, \epsilon)$ toa limit as $\epsilon \to 0, P \to$ the boundary, provided such a limit exists. Such regions have been called "boundary layers" and we adopt this terminology.

b.) Let us call the equation $M\phi = 0$, the reduced equation. In the case of partial differential equations it may happen that the reduced equation has no properly posed boundary value problem for the region given for the full equation. In other words, "what is the proper boundary value problem to pose for $M\phi = 0$ in order to expect convergence of $\phi_{\epsilon \to 0}(P, \epsilon)$ to the solution of this problem?"

We could add considerably to this list, but can only give an answer to such questions in certain simple cases. As a first step, we consider the boundary layer terms. By using existence theorems, or physical arguments, we must first decide which boundary conditions we must drop to pose a proper problem for the reducedequation. This is fairly simple for equations of lower orders. Then following Prandtl's ideas, we "blow up" the region R in which the solution to 1.1) is desired, by making a change of variable such as $x = \epsilon^{\lambda} t$, let $\epsilon \to 0$, and by suitably choosing λ , we obtain a new equation, in general simpler than the original. Again for low order equations, this "simple" equation suggests the form of the non-uniformly converging terms in the solution of 1.1). In the cases considered, this procedure gives rise to an asymptotic expansion for $\phi(P, \epsilon)$, for sufficiently small ϵ .

With no claim to originality, we call problems of the afore mentioned type "singular perturbation problems" for differential equations. It is to be noted, however, that the singular nature of the problem depends not only on the loss of certain highest derivatives, but on the given region in which the solution is to be obtained. Thus, for example, the solution of the first boundary value problem for $\phi_{xx} + \epsilon\phi_{yy} - 2a\phi_y = 0$ in the upper half plane tends uniformly, as $\epsilon \to 0$, to the solution of the first boundary value problem for $\phi_{xx} - 2a\phi_y = 0$ with the same boundary values and region.

Another term which has appeared in the literature relating to singular perturbation problems is "sub-characteristics", or the characteristics of the reduced equation. As will be shown, these sub-characteristics play a major role in obtaining an asymptotic development of the solutions of our singular equations.

In conclusion, we refer to [1] , [2] , [3] , for other results in singular perturbation problems.

In Section Two, we shall consider a problem in ordinary differential equations, but, for simplicity, shall restrict ourselves to illustrating the method by means of an example. The object of this section will be to develop a technique which can be generalized to linear partial differential equations of the same form. The results obtained, for the most part, have already been obtained by Wasow [1] by other methods.

Section Three deals with the first boundary value problem for the equation

1.2) $$\phi_{xx} + \epsilon \phi_{yy} - 2a\phi_{y} = 0$$

ϵ, and $a > 0$, and are constant. This changes form from elliptic to parabolic as $\epsilon \to 0$. The methods used in Section Two enable us to obtain an asymptotic expansion for the solution of 1.2), exhibiting the boundary layer terms.

Finally, in Section Four, we consider a system of partial differential equations, and find that in this case also, the heuristic approach gives the essential terms of the solution with no difficulty.

Section 2. The equation $\epsilon\, y^{(n)} + \alpha_1\, y^{(n-1)} + \dots + \alpha_n\, y = 0$

We consider now the equation

2.1) $\epsilon\, y^{(n)} + \alpha_1\, y^{(n-1)} + \dots + \alpha_n\, y = 0$

in the interval (0,1) with boundary conditions of the form

2.2) $y^{(\lambda_i)}(0) = l_i \qquad i = 1, \dots, r, \quad n > \lambda_1 > \lambda_2 > \dots$

$y^{(\tau_i)}(1) = m_i \qquad i = r+1, \dots n, \quad n > \tau_1 > \dots$

$y^{(k)}$ stands for $\frac{d^k y}{dx^k}$.

$\epsilon > 0$ is a small constant, for simplicity we assume the α_i to be constant, $\alpha_1 \neq 0$. Assume also that the solution to $\alpha_1\, y^{(n-1)} + \dots + \alpha_n\, y = 0 \qquad y^{(\lambda_i)}(0) = l_i$

$i = 2, \dots, r \qquad y^{(\tau_i)}(1) = m_i \quad i = r+1, \dots n$

is unique.

The reduced equation, $\alpha_1 y^{(n-1)} + \dots + \alpha_n\, y = 0$ will be referred to as $My = 0$ in the sequel.

This, and similar problems have been solved by Wasow [1] under fairly general conditions, and he gives sufficient conditions for the existence of a limit as $\epsilon \to 0$, and at the same time derives a representation formula which exhibits the form of the boundary layer terms. Unfortunately, his formulae are not quite suitable for calculation purposes, and do not suggest any method for obtaining a complete asymptotic expansion of the solution to such equations as 2.1).

As a preliminary to the equation discussed in Sec. 3, we investigate the solution to 2.1) by a heuristic argument, and then establish the validity of the results of this argument. In this way, we arrive at the complete asymptotic expansion, and also obtain a method which lends itself to a natural generalization to partial differential equations.

In the following, the heuristic results will be proven by an example, and the general results stated.

2.3) $\epsilon y''' + a y'' + b y' + c y = 0$ in $(0,1)$

2.4) $y(0) = \alpha$, $y'(0) = \beta$, $y(1) = \gamma$

We may as well assume $a > 0$ since $x_1 = 1 - x$ changes the sign of the second term of 2.3). In lemma 2.1) we shall prove that the solution to 2.1) satisfying 2.2) tends to a finite limit as $\epsilon \to 0$ provided $r \geq 1$ if $\alpha_1 > 0$, or $r \leq n - 1$ if $\alpha_1 < 0$. Assume that the solution of 2.3), 2.4), tends to a bounded solution as $\epsilon \to 0$. Then if $y(x, \epsilon)$ is the desired solution, we would like to know where to expect the region of non-uniform convergence, since the reduced equation can only satisfy two arbitrary boundary conditions. Let us put $x = \epsilon^{\lambda} t$ in 2.3). We get

2.5) $\epsilon \dddot{y} + a \epsilon^{\lambda} \ddot{y} + b \epsilon^{2\lambda} \dot{y} + c \epsilon^{3\lambda} y = 0$

where the dot stands for differentiation with respect to t, and $\lambda > 0$.

If now we let $\epsilon \to 0$, we must distinguish several cases.

a) $0 < \lambda < 1$ in which case 2.5) tends formally

2.5') to $a \ddot{y} = 0$

b) $\lambda > 1$, and we get $\dddot{y} = 0$

c) $\lambda = 1$, giving $\dddot{y} + a \ddot{y} = 0$

This gives us a certain amount of information about $y(x, \epsilon)$;

putting $x = \epsilon^{\lambda} t$ we expect

2.5'') a') $\lim_{\epsilon \to 0} y(\epsilon^{\lambda} t, \epsilon) = A_1 + B_1 t$ if $0 < \lambda < 1$

b') $\lim_{\epsilon \to 0} y(\epsilon^{\lambda} t, \epsilon) = A_2 + B_2 t + C_2 t^2$ if $\lambda > 1$

c') $\lim_{\epsilon \to 0} y(\epsilon t, \epsilon) = A_3 + B_3 t + C_3 e^{-at}$

Alternately, let us consider the indicial equation relative to 2.3). This is

2.6) $$\epsilon z^3 + a z^2 + b z + c = 0$$

Whence* we infer that for ϵ sufficiently small, there are no multiple roots , and that one of the roots, say z_3, tends to ∞ as $\epsilon \to 0$ in such a way that $\epsilon z_3 \to -a$ as $\epsilon \to 0$, while z_1, z_2 tend to finite limits.

Comparing this with c') above we find that substituting $x = \epsilon t$ in 2.3), and letting $\epsilon \to 0$ gives us the exponential term which must be associated with the non-uniform convergence. In fact this substitution is directly related to the Puiseaux polygon construction for determining a system of linearly independent solutions of such an equation as 2.3). [5]

Again, from c'), if the limit on the left exists, then we expect $B_3 = 0$, and $y(x, \epsilon) = A_3(x, \epsilon) + C_3(x, \epsilon) e^{\frac{-ax}{\epsilon}}$

for ϵ small,

where $$\lim_{\epsilon \to 0} A_3(\epsilon t, \epsilon) = A_3$$
$$\lim_{\epsilon \to 0} C_3(\epsilon t, \epsilon) = C_3.$$

If we further assume $A_3(x, \epsilon)$, $C_3(x, \epsilon)$ to have a series development in powers of ϵ , then we are led to consider

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* See Appendix 1.

an expression such as

2.7) $$y(x,\epsilon) = u_0(x) + \epsilon u_1(x) + \dots + e^{\frac{-ax}{\epsilon}}\{h_0(x) + \epsilon h_1(x) + \cdots\}$$

Moreover, if 2.7) is correct, (and we do not make any precise statements about the equality), we see that the non-uniformity occurs near $x = 0$. This, in fact, is the case. If, on the other hand $a<0$, we would find the boundary layer near $x = 1$.

Since we are not certain that our series 2.7) is convergent, let us take a finite series, and add a correction term

2.8) $$y(x,\epsilon) = u_0(x) + \epsilon u_1(x) + \dots + \epsilon^p u_p(x) + e^{\frac{-ax}{\epsilon}}\{h_0(x) + \epsilon h_1(x) + \dots + \epsilon^q h_q(x)\} + v(x,\epsilon).$$

Substitute this into 2.3) and collect terms. Formally equating each coefficient of $\epsilon^\nu e^{\frac{-ax}{\epsilon}}$, and of ϵ^ν, to zero, we get

2.9)
$$\begin{aligned}
&\text{a)} \quad Lh_0 = ah_0' - bh_0 = 0\\
&\text{b)} \quad Lh_1 = 2h_0'' - \frac{b}{a}h_0' - \frac{c}{a}h_0\\
&\text{c)} \quad Lh_2 = 2h_1'' - \frac{b}{a}h_1' - \frac{c}{a}h_1 - h_0'''\\
&\text{d)} \quad Lh_q = 2h_{q-1}'' - \frac{b}{a}h_{q-1}' - \frac{c}{a}h_{q-1} - h_{q-2}'''\\
&\text{e)} \quad Mu_0 = au_0' + bu_0' + cu_0 = 0\\
&\text{f)} \quad Mu_k = -u_{k-1}''' \qquad k = 1, 2, \dots, p\\
&\text{g)} \quad \epsilon v''' + Mv = -\epsilon^{p+1}u_p''' - \epsilon^q e^{\frac{-ax}{\epsilon}}(bh_q' + ch_q - 2ah_q'' + h_{q-1}''' + \epsilon h_q''')
\end{aligned}$$

The boundary conditions become

$$\text{a)}\quad y(0) = \alpha = u_0(0) + \epsilon u_1(0) + \dots + \epsilon^p u_p(0) + h_0(0) + \epsilon h_1(0) + \dots + \epsilon^q h_q(0) + v(0,\epsilon)$$

2.10) b) $y'(0) = \beta = u_o'(0) + \epsilon u_1'(0) + \dots + \epsilon^p u_p'(0)$
$+ h_o'(0) + \epsilon h_1'(0) + \dots + \epsilon^q h_q'(0)$
$+ v'(0, \epsilon) - \frac{a}{\epsilon}(h_o(0) + \epsilon h_1(0)$
$+ \dots + \epsilon^q h_q(0))$

c) $y(1) = \gamma = u_o(1) + \epsilon u_1(1) + \dots + \epsilon^p u_p(1)$
$+ e^{-\frac{1}{\epsilon}}(h_o(1) + \epsilon h_1(1) + \dots + \epsilon^q h_q(1))$
$+ v(1, \epsilon)$

Now in order that 2.8) be an asymptotic expansion, we require $|v(x, \epsilon)| = 0(\epsilon^{p+1})$ and $q \geq p + 1$. Then for a bounded solution as $\epsilon \to 0$, from 2.10b) we infer $h_o(0) = 0$. But then 2.9a) gives $h_o(x) \equiv 0$. In the general case of equation 2.1), by the same procedure, we find that for p, q > n, we must have $h_o = h_1 = \dots = h_{\lambda_1 - 1}(x) \equiv 0$.

Next the terms in 2.10) which are independent of ϵ give

2.11) $u_o(0) = \alpha$
$u_o'(0) - a\, h_1(0) = \beta$
$u_o(1) = \gamma$

Using 2.9) we see that 2.11) determine $u_o(x)$ and $h_1(x)$ uniquely.(1) Namely

2.12) $Mu_o = 0$
$u_o(0) = \alpha\ , \ u_o(1) = \gamma$

which is the reduced equation, and we have had to drop the highest order derivative at x = 0 in order to get the boundary condition for 2.12). If a < 0, the opposite end point relays the same role. This is general; for equation 2.1), we obtain the boundary conditions for the reduced

(1) We assume the solution to 2.12) to be unique.

equation by dropping $y^{(\lambda_1)}(0) = 1$, from the set 2.2) . (hence the condition $r \geq 1$ for $\alpha_1 > 0$) .

$$2.13) \qquad \begin{aligned} &Lh_1 = 0 \\ &ah_1(0) = u_o'(0) - \beta \end{aligned}$$

The coefficients of ϵ^{κ} in 2.10) yield

$$2.14) \qquad \begin{aligned} &u_k(0) + h_k(0) = 0 \\ &u_k'(0) + h_k'(0) = ah_{k+1}(0) \\ &k = 1, 2, \ldots p \\ &u_k(1) = 0 \end{aligned}$$

since we require $u_i(x)$, $h_i(x)$ to be independent of ϵ . Furthermore, to have a solvable system of equations 2.9), we must take $q = p+1$. Then from 2.14) and 2.9) using 2.12) and 2.13) we see that u_o, h_1 are determined; this determines u_1, which in turn gives h_2 and so on.

$$u_o, h_1 \rightarrow u_1 \rightarrow h_2 \rightarrow \ldots \rightarrow u_p \rightarrow h_{p+1} = q .$$

The correction term satisfies

$$2.15) \quad \epsilon v''' + Mv = -\epsilon^{p+1}\left\{ u_p''' + e^{\frac{-ax}{\epsilon}} (b h_{p+1}' + c h_{p+1} - 2ah_{p+1}'' - h_p''' + h_{p+1}''')\right\}$$

$$2.16) \qquad \begin{aligned} &v(0,\epsilon) = -\epsilon^{p+1} h_{p+1}(0) \\ &v'(0,\epsilon) = -\epsilon^{p+1} h_{p+1}'(0) \\ &v(1,\epsilon) = -\epsilon^{p+1}\left\{ \exp(-\tfrac{1}{\epsilon})\epsilon^{-(p+1)} \left[(h_o(1) + \epsilon h_1(1) + \ldots + \epsilon^{p+1} h_{p+1}(1)\right]\right\} \end{aligned}$$

Thus we have $v(x,\epsilon) = \epsilon^{p+1} v_1(x,\epsilon)$ and we shall show, in general, that $|v_1(x,\epsilon)| \leq M_1$ independent of ϵ , where M_1 is a constant. Thus $y(x,\epsilon) \rightarrow u_o(x)$ uniformly in $0 < \delta \leq x \leq 1$, and we may take $\delta = 0$ if $\lambda_1 \neq 0$.

This gives us the desired asymptotic expansion since p is arbitrary, although, depending on the orders of the derivatives given in the boundary conditions, there is an integer p_0, $0 \leq p_0 \leq n$, so that the expansion 2.8) remains finite only for $p \geq p_0$.

Summarizing, we take the indicial equation relative to 2.5'c) which is $z^3 + az^2 = 0$; the non trivial root is $z = -a$. If $a > 0$, the boundary layer term is at $x = 0$, $a < 0$, the boundary layer term is at $x = 1$.

Taking $a > 0$, for example, we then let

$$2.17) \qquad \begin{aligned} y(x,\epsilon) &= u_0(x) + \epsilon u_1(x) + \dots + \epsilon^p u_p(x) \\ &+ \epsilon^{\lambda_1} e^{\frac{-ax}{\epsilon}} (h_0(x) + \epsilon h_1(x) + \dots + \epsilon^p h_p(x)) \\ &+ \epsilon^{p+1} v(x,\epsilon) \end{aligned}$$

where λ_1 is the highest order derivative appearing in the boundary conditions at $x = 0$. (We have relabelled the h_i's for convenience). Then we must show $|v(x,\epsilon)| \leq M_1$, independent of ϵ, and take $p \geq n$ to ensure the validity of 2.17) ($n = 3$ in our example, and we need only $p \geq 1$). The same statements hold for equation 2.1).

For the equation

$$2.18) \qquad \begin{aligned} &\epsilon y^{(n)} + \alpha_1 y^{(m)} + \dots + \alpha_n y = 0 \qquad \text{in } (0,1) \\ &y^{(\lambda i)}(0) = l_i \\ &y^{(\tau i)}(1) = m_i \end{aligned}$$

the indicial equation becomes $z^n + \alpha_1 z^m = 0$ and the exponents carrying the boundary layer terms are the $n - m$

values of $(-\alpha_1)^{\frac{1}{n-m}}$. For those with negative real part, there is a boundary layer term at the origin, and for those with real part positive the boundary layer terms are at $x = 1$. For convergence as $\epsilon \to 0$, we must have at least as many boundary conditions at each end point as there are values of $(-\alpha_1)^{\frac{1}{n-m}}$ with positive, (negative) real parts. Then we use equation 2.17) with extra terms added, one set for each boundary layer term, and powers of $\epsilon^{\frac{1}{n-m}}$ instead of ϵ. A special case arises when $n - m = 2k$. It is then possible to have two values of $(-\alpha_1)^{\frac{1}{n-m}}$ which are pure imaginary. An investigation of sufficient conditions for the existence of a limit as $\epsilon \to 0$ yields the same results as given by Wasow and hence we refer to [1] .(1)

Note that no where in the above have we required the α_i to be constant. In fact we need only the condition $\alpha_1(x) \neq 0$ in $(0,1)$, and $\alpha_i(x) \in C^\infty$ (x) in $(0,1)$ to obtain the complete asymptotic expansion for 2.1). In those cases in which $\alpha_1(x)$ has a zero in $(0,1)$, we have a so-called turning point, and very little work seems to have been done in general for this type of equation. Wasow [4] and others have treated special cases in which the coefficients are analytic functions. At present, we leave this problem aside, although in special cases the above method

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(1) It is not our purpose to reproduce Wasow's results but merely to give the method, illustrating by examples. For completeness, we give the independent proof of the boundedness of the correction term is 2.8).

yields the correct expansions.

Finally, we turn to our Lemma, and prove the existence of a bounded solution to 2.1) as $\epsilon \to 0$, which, in conjunction with 2.8) - 2.16) establishes the existence of a limit.

Theorem 2.2

Let $y(x, \epsilon)$ denote the solution to

$$\epsilon y^{(n)} + \alpha_1 y^{(n-1)} + \dots + \alpha_n y = f(x, \epsilon)$$

2.19) $y^{(\lambda_i)}(0) = l_i(\epsilon) \qquad i = 1, \dots r \qquad n > \lambda_1 > \dots\dots$

$y^{(\tau_i)}(1) = m_i(\epsilon) \qquad i = r+1, \dots n \qquad n > \tau_1 > \dots\dots$

Where a) $My = \alpha_1 y^{(n-1)} + \dots + \alpha_n y = 0$

$y^{(\lambda_i)}(0) = l_i(\epsilon) \qquad i = \ , 2 \dots r$

$y^{(\tau_i)}(1) = m_i(\epsilon) \qquad i = r+1, \dots n$

$\Rightarrow y(x, \epsilon)$ unique

b) $f(x, \epsilon) \in C^{(0)}(x)$ in $(0,1)$, and bounded independent of ϵ.

c) $|l_i(\epsilon)|$, $|m_i(\epsilon)|$ are bounded independent of ϵ

d) $r \geq 1$, $\alpha_1 > 0$

Then $|y(x, \epsilon)|$ is bounded, independent of ϵ, for all ϵ sufficiently small, and the solution to 2.19) with $f = l_i = m_i = 0$ is zero uniquely.

Proof:

Let us form a system of equations from 2.19)

$$y = \xi_1, \ y' = \xi_2, \dots y^{(n-1)} = \xi_n$$

Then 2.19) becomes

2.20) $$\frac{d\xi}{dx} = \left(\frac{A}{\epsilon} + B\right)\xi + \tau(x, \epsilon)$$

where ξ, τ are column vectors, and $\|\tau\| \leq m_i$, a constant, independent of ϵ. In fact, the following results are valid for an arbitrary matrix A whose characteristic roots have negative real parts, or zero real part with no elementary divisors.

Now define $P \equiv \lim_{t \to \infty} e^{At}$. We can interpret A, P geometrically in the following way.

Let $\mathcal{L}$ be the vector space spanned by $(\xi_1, \ldots \xi_n)$ and consider A as a transformation of L into L.

Then if
$$AL \equiv \{\xi \mid \xi = A\eta\}$$
$$N \equiv \{\xi \mid A\xi = 0\}$$
we have: P is the projection of L onto N through AL, and it is a simple matter to verify that N and AL are complementary.

By direct calculation we find

2.21) $$P = I + \frac{A}{\alpha_1}$$

and the reduced equation is

2.22) $$\frac{d\eta}{dx} = PB\eta \qquad \text{where } \eta \in L.$$

Then we shall prove

2.23) $$\lim_{\epsilon \to 0} e^{(\frac{A}{\epsilon}+B)x} = e^{PBx} P \quad \text{for } 0 < x \leq 1.$$

The characteristic roots of $\frac{A}{\epsilon} + B$ are given by

$$\left| \lambda I - (\tfrac{A}{\epsilon} + B) \right| = 0$$

or

$$\epsilon \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_n = 0$$

All the roots of this equation are distinct, unless $\alpha_n = 0$, in which case $\frac{A}{\epsilon} + B$ has no elementary divisors. Hence $\frac{A}{\epsilon} + B$ is equivalent to a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ 0 & \ddots & \lambda_n \end{pmatrix} \qquad \text{where } \lambda_i = \lambda_i(\epsilon).$$

For definiteness, let $\lambda_n = -\frac{\alpha_1}{\epsilon} + \sigma_n(\epsilon)$

$\lambda_i = \sigma_i(\epsilon) \qquad i = 1, \ldots n-1$

Then $\sigma_i(\epsilon)$ tend to finite limits as $\epsilon \to 0$.

Thus $\quad e^{(\frac{A}{\epsilon} + B)x}$ is equivalent to

$$\begin{pmatrix} e^{\lambda_1 x} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n x} \end{pmatrix}$$ which tends to a finite

limit as $\epsilon \to 0$,* and so the solution to the initial value problem

$$\frac{d\xi}{dx} = (\frac{A}{\epsilon} + B)\xi \quad , \quad \xi(0) = \xi_0 \text{ is given by}$$

$$\xi(x) = e^{(\frac{A}{\epsilon} + B)x} \xi_0, \text{ where}$$

$$\|\xi(x)\| \leq \| e^{(\frac{A}{\epsilon} + B)x} \| \, \| \xi_0 \| \leq M_2 \text{ if } \|\xi_0\| \text{ is}$$

bounded.

On the other hand, let us solve this same initial value problem by obtaining its asymptotic expansion up to terms of order ϵ^n. Then the correction term $v(x, \epsilon)$ satisfies

2.24) $\quad \epsilon v^{(n)} + Mv = \epsilon^{n+1} k(u_i^{(\mu)}, h_i^{(\nu)}, e^{-\frac{\alpha_1 x}{\epsilon}}, \epsilon)$

$\qquad v^{(\nu)}(0) = \epsilon^{n+1} m_\nu$

Where k is a linear function of its arguments which is bounded in x and ϵ, and m_ν are bounded as $\epsilon \to 0$. Forming a system from 2.24) we get

$$\frac{d\sigma}{dx} = (\frac{A}{\epsilon} + B)\sigma + \epsilon J(x, \epsilon)$$

$$\sigma(0) = \epsilon\sigma_0 \qquad \text{where } \|\sigma_0\| \leq M_3$$

$$\|J\| \leq M_4$$

* Appendix Two

The solution is given by

$$\sigma(x) = \epsilon\, e^{(\frac{A}{\epsilon} + B)x} \sigma_o + \int_o^x e^{(\frac{A}{\epsilon} + B)(x - u)} J(u, \epsilon)\,du$$

and hence

$$\| \sigma(x) \| \le \epsilon M_5 \quad \text{since } \| e^{(\frac{A}{\epsilon} + B)x} \| \text{ is bounded}$$

independent of ϵ and x for bounded $x \geq 0$.

Thus from the fact that the initial values are $P \xi_o$, we have established

$$\lim_{\epsilon \to 0} e^{(\frac{A}{\epsilon} + B)x} \xi_o = e^{PBx} P \xi_o \quad \text{for } x > 0.$$

Since this is true for every bounded ξ_o, we have

$$2.25) \qquad \lim_{\epsilon \to 0} e^{(\frac{A}{\epsilon} + B)x} = e^{PBx} P$$

Returning to 2.19), we note that

$y_1(x, \epsilon) = \int_o^x e^{(\frac{A}{\epsilon} + B)(x - u)} \tau(u, \epsilon)\,du$ solves the inhomogeneous equation with zero initial conditions.

Then let $y(x, \epsilon) = y_1(x, \epsilon) + y_2(x, \epsilon)$

and we have $\epsilon y_2^{(n)} + My_2 = 0$

$$y_2^{(\lambda_i)}(0) = l_i(\epsilon)$$

$$y_2^{(\tau_i)}(1) = m_i(\epsilon), \quad m_i(\epsilon) \text{ bounded.}$$

Also $\| y_1(x, \epsilon) \|$ is bounded in $(0,1)$ for all ϵ, and we show $| y_2(x, \epsilon) |$ bounded as $\epsilon \to 0$.

The problem for $y_2(x, \epsilon)$ can be stated as follows:

$\frac{d\xi}{dx} = (\frac{A}{\epsilon} + B)\xi$; where at $x = 0$, ξ lies in a linear manifold S_o of dimension ρ_o, and at $x = 1$ lies in S_1 of dimension ρ_1, so that $\rho_o + \rho_1 = n$.

By hypothesis, $(e^{PB} PS_o) \cap S_1$ consists of one point, and hence, using 2.25), for all $0 < \epsilon < \epsilon_o$ we have

$(e^{\frac{(A+B)}{\epsilon}} S_0) \cap S_1$ consists of one point i.e., it tends to a unique limit as $\epsilon \to 0$. Thus for $\epsilon < \epsilon_0$, $\| \xi(x,\epsilon) \| \leq M_0$ for $x > 0$, and since $r \geq 1$ there is at least one component ξ_k , $k \geq 0$, which is given at $x = 0$. Then ξ_k is bounded uniformly in $(0,1)$ and hence so is $\xi_1 = y(x,\epsilon)$. Thus $y(x,\epsilon)$ tends uniformly to a limit in $(0 < \delta \leq x \leq 1)$, and we may let $\delta = 0$ if $\lambda_1 \neq 0$.

Sec. 3 The First Boundary Value Problem for the Equation $\phi_{xx} + \epsilon\phi_{yy} - 2a\phi_y = 0$.

In the following section, R will denote a finitely connected (1), bounded open region in the x, y plane, whose boundary S has a parametric representation of class $C^{(2)}(s)$. The term "characteristics" will be reserved for the characteristics of the reduced equation $\phi_{xx} - 2a\phi_y = 0$, namely, the lines y = constant. $\bar{S}$ is that portion of S which has a characteristic tangent, S^* is the subset of $\bar{S}$ from which we enter R as y decreases. We assume, further, that any straight line passing through R cuts S in a finite number of points only, or coincides with a portion of S with a finite number of extra, isolated crossings. Thus $\bar{S}_i$ will denote that portion of S which coincides with the characteristic $y = y_i$, while S_k will denote a segment of S between two successive arcs (points) $\bar{S}$.

We wish to study the solution to the equation

3.1) $\phi_{xx} + \epsilon\phi_{yy} - 2a\phi_y = 0$ in R

3.2) $\phi(P) = \phi_0$ on S

where $\phi_0(s)$, s = arc length on S, is of class $C^4(s)$ on each of the arcs of S_i, $\bar{S}_i$. ϵ, and "a" are positive constants.

We shall prove

<u>Theorem 3.1</u>:

The solution to 3.1) with boundary conditions 3.2) has an asymptotic expansion of the form

3.3) $\phi(P, \epsilon) = u(P) + \sum_i \exp\left[-\frac{2a}{\epsilon}(y_i - y)\right] h_i(P) + \epsilon^{1-\delta} v(P, \epsilon)$,

$\delta > 0$ arbitrary, for ϵ sufficiently small, where $|\epsilon^{\delta} v(P, \epsilon)|$ is bounded uniformly in R + S independent of ϵ for every $\delta > 0$, and u(P) satisfies the reduced equation in R and tends to ϕ_o on $S - S^*$. The terms $\exp\left[\frac{-2a}{\epsilon}(y_i - y)\right] h_i(P)$ are the boundary layer terms, which tend to zero in R as $\epsilon \to 0$, but remain finite on certain arcs of $\bar{S}$.

Before giving the proof of this theorem, let us apply our heuristic argument, and show how the boundary layer terms are obtained. As a first step, we consider the reduced equation

3.4) $\phi_{xx} - 2a\phi_y = 0$ in R

3.5) $\phi(P) = \phi_o$ on $S - S^*$.

Let us denote this solution by u(P). This is a well determined function in R, but is determined on S^*, and hence it is near S^* that we expect to locate the boundary layer terms. Consider a neighborhood of one of the arcs of S^*, say $\bar{S}_\kappa$ given by $y = y_\kappa$.

Let $y_\kappa - y = \epsilon\eta$, substitute in 3.1) and we get

3.6) $$\epsilon\phi_{xx} + \phi_{\eta\eta} + 2a\phi_\eta = 0$$

Defining

$$\lim_{\epsilon \to 0} \phi(x, \epsilon\eta, \epsilon) \equiv \phi^*(x, \eta), \text{ we have}$$

$$\phi^*_{\eta\eta} + 2a\phi^*_\eta = 0, \text{ or}$$

$$\phi^*(x, \eta) = A(x) + B(x)\exp(-2a\eta)$$

Hence we expect, for small ϵ,

3.7) $\phi(P, \epsilon) = A(x,y,\epsilon) + B(x,y,\epsilon) \exp\left(-\frac{g(P)}{\epsilon}\right)$ near $\overline{S}_K$ where $\lim\limits_{\epsilon \to 0} \frac{g(x, \epsilon\eta)}{\epsilon} = 2a\eta$.

In fact, to obtain the first terms in an asymptotic expansion for small ϵ , we are led to try an expression of the form

3.8) $$\phi(P, \epsilon) = u(P) + e^{\frac{-g(P)}{\epsilon}} h(P) + \epsilon \sigma(P, \epsilon).$$

It appears that this procedure should give the boundary layer terms, and hence the asymptotic expansion for the solutions of linear differential equations of the form $\epsilon N\phi + M\phi = 0$, provided the order of N is at most one higher than that of M in any one variable.

The procedure to be followed is very simple. We obtain $\phi(P,\epsilon) = u(P) + \psi(P, \epsilon) + \epsilon\sigma(P,\epsilon)$ where u(P) is the solution of the reduced equation, $\psi(P,\epsilon)$ embodies the boundary layer terms, and $\epsilon\sigma$ is the correction term. If we can obtain a bound for $|\sigma|$, independent of ϵ for ϵ small, then we have the desired expansion. Note that if $N\phi$, $M\phi$, have constant coefficients, then under the above conditions, the boundary layer terms turn out to be exponentials in general.

<u>Lemma 3.1</u> (Principle of the Maximum)

Let u(P) be of class $C^2(P)$ in R, and $u_{xx} + u_{yy} + Au_x + Bu_y + Cu = D$ in R, where A, B, C are continuous in R.

Then if $C \le -\delta < 0$, $|D| \le m_1$ in R, we have $|u| \le m_2$ in R + S provided u is continuous in R + S, and $|u|$ is bounded on S [3.]

(Note: Here, and in the sequel, m_i will denote a positive constant, independent of ϵ. If a bound is a function of ϵ, we denote it by $m_i(\epsilon)$.)

In particular, if $D \equiv 0$, then $u(P)$ cannot have a positive maximum or a negative minimum in R.

Thus for 3.1), setting $\phi = \sigma(K - e^{\mu x})$ where K, μ are constants, we have

$$\sigma_{xx} + \epsilon\sigma_{yy} - \frac{2\mu\sigma_x e^{\mu x}}{K - e^{\mu x}} - \frac{\mu^2 e^{\mu x} v}{K - e^{\mu x}} = 0$$

and since R is bounded, we can choose K, μ so that

$$K - e^{\mu x} \geq \delta > 0 \text{ in } R + S$$

$$\mu^2 e^{\mu x} \geq \delta \; (K - e^{\mu x})$$

Hence we infer that for 3.1) we have

$$|\phi(P)| \leq m_1$$

Lemma 3.2

Let $G_R(P,Q)$ denote the classical Green's function for 3.1) in R. Then we have

a) $0 \leq G_R(P,Q)$

b) $R_1 \subseteq R_2 \Rightarrow G_{R_1} \leq G_{R_2}$ 7

c) $P \in R \Rightarrow \frac{\partial}{\partial n} G_R(P,Q)$ exists and is continuous on S, where $\frac{\partial}{\partial n} \equiv$ the inner normal derivative. [6]

d) The Green's function for the upper half plane is $\frac{1}{2\pi\sqrt{\epsilon}} \left\{ K_0\left(\frac{ar}{\sqrt{\epsilon}}\right) - K_0\left(\frac{a\bar{r}}{\sqrt{\epsilon}}\right) \right\}$ where $K_0(z)$ is Bessel's function of the second kind with imaginary argument, and

$$r^2 = (x - \xi)^2 + \frac{1}{\epsilon}(y - \eta)^2$$

$$\bar{r}^2 = (x - \xi)^2 + \frac{1}{\epsilon}(y + \eta)^2$$

Lemma 3.3

Let $\Delta u_i - f_i u_i = 0$ $(i = 1, 2)$ in $\mathcal{D}$, a bounded open domain in the x - y plane, where $0 < \delta_1 \leq f_1 < f_2 \leq \delta_2 < \infty$, in $\mathcal{D}$; $u_i = \phi_0$ on S, the boundary of $\mathcal{D}$, and $\phi_0 > 0$; then $u_i \geq 0$ in $\mathcal{D}$ and $u_1 \geq u_2$ in $\mathcal{D}$.

$$\left(\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Proof:

$u_i \geq 0$ from lemma 3.1).

$\Delta(u_1 - u_2) - f_1(u_1 - u_2) = (f_1 - f_2)\, u_2$ in $\mathcal{D}$

and $u_1 - u_2 = 0$ on S.

Hence

$u_1 - u_2 = \iint_{\mathcal{D}} G_{\mathcal{D}}(P,Q) \;\; (f_2 - f_1) \cdot u_2(Q)dQ \geq 0$ in $\mathcal{D} + S$ since $f_2 > f_1$.

$G_{\mathcal{D}}(P,Q)$ is the Green's function for $\Delta u - f_1 u = 0$ in $\mathcal{D}$.

Lemma 3.4:

Let K_1, K_2 be the Green's functions for $\Delta u - f_1 u = 0$, $\Delta u - f_2 u = 0$ respectively, in $\mathcal{D}$, using the notation of lemma 3.3). Then

$K_i = \log \frac{1}{r} + \gamma_i$, and

$$\Delta \gamma_i - f_i \gamma_i = f_i \log \frac{1}{r}$$

$$\Delta(\gamma_1 - \gamma_2) - f_1(\gamma_1 - \gamma_2) = (f_1 - f_2)\left(\gamma_2 + \log \frac{1}{r}\right)$$

in $\mathcal{D}$, $\gamma_1 - \gamma_2 = 0$ on S

and hence, since the same integral equation used in lemma 3.3 is valid [6] we have

$$\gamma_1 - \gamma_2 \geq 0 \text{ in } \mathcal{D}$$

or

$$0 \leq K_2 \leq K_1 \text{ in } \mathcal{D}.$$

We obtain here a bound on the Green's function for equation 3.1) for points in R near S.

First we transform 3.1) into $\Delta u - \lambda^2 u = 0$ in R_1, by means of the classical change of independent and dependend variables, where R_1 is the new region corresponding to R. Next, select any point P_1 on S_1, and construct a circle, exterior to R_1, but tangent to S_1 at P_1. Now we map the interior of this circle into the lower half plane in such a way that P_1 goes into the origin, and the centre of the circle goes into the point at ∞. Let J be the Jacobian of this transformation, which maps R_1 into R_2, a bounded domain in the upper half plane, whose boundary S_2 is tangent to the real axisat the origin. Then $\Delta u - \lambda^2 Ju = 0$ in R_2. By construction, there exist constants δ_1, δ_2, such that

$$0 < \delta_1 \leq J \leq \delta_2 < \infty \quad \text{in } R_2 + S_2.$$

Hence by lemma 3.4), there exists a constant μ such that $0 < \mu \leq \min_{R_2+S_2} J$, so that the Green's functions

$$\begin{aligned} &\bar{K}_1 \text{ for } \Delta u - \lambda^2 Ju = 0 \\ &\bar{K}_2 \text{ for } \Delta u - \mu^2 u = 0 \qquad \text{in } R_2 \end{aligned}$$

have the property

$$0 \leq K_1 \leq K_2.$$

Then from lemma 3.2), we have

$$\bar{K}_2 \leq \frac{1}{2\pi} \Big\{ K_0(\mu|z-\zeta|) \quad - \qquad) - K_0(\mu|z-\bar{\zeta}|) \quad ,$$

where the right hand side is the Green's function for the upper half plane for the equation $\Delta u - \mu^2 u = 0$. Finally, transforming back to the original variables, we obtain the desired bound, which tends to a limit as $\epsilon \to 0$.

We now proceed in a purely formal manner to examine equation 3.8) . Let us put

3.9) $\phi(P,\epsilon) = u(P) + e^{-\frac{g(P)}{\epsilon}} h(P) + \epsilon v(P,\epsilon)$

where ϵ appears only where explicitly exhibited, and substitute into 3.1) . This yields

3.10) $$\epsilon^{-2} e^{-\frac{g(P)}{\epsilon}} h(P) g_x^2(P) + e^{-\frac{g}{\epsilon}} \epsilon^{-1}\{hg_{xx} - 2h_x g_x + hg_y^2 + 2a\, hg_y\}$$
$$+ e^{-\frac{g}{\epsilon}} (h_{xx} + hg_{yy} - 2h_y g_y - 2ah_y) + u_{xx} - 2au_y$$
$$+ \epsilon(v_{xx} + \epsilon v_{yy} - 2av_y + u_{yy} + e^{-\frac{g}{\epsilon}} h_{yy}) = 0.$$

If this equation is to remain valid for all ϵ, sufficiently small, then since we are requiring h, g, u, to be independent of ϵ, we must have

a) $hg_x^2 = 0$

b) $hg_{xx} - 2h_x g_x + hg_y^2 + 2a\, hg_y = 0$

c) $h_{xx} + hg_{yy} - 2h_y g_y - 2a\, h_y = 0$

d) $u_{xx} - 2au_y = 0$

e) $L v = -u_{yy} - e^{-\frac{g}{\epsilon}} h_{yy}$.

If we can pose proper boundary value problems for this system a)...e) in such a way that $|v| \le m_1(\epsilon)$ in R + S, then we have the desired result.

Equations a), b) give

$$g_y(g_y + 2a) = 0; \quad g_x = 0.$$

The condition $g_y = 0 \Rightarrow g = 0$ essentially, which leads to the reduced equation, so we take

3.11) $$g = 2a(y_1 - y)$$

which is precisely the form suggested by 3.6). The remaining equations become

c) $h_{xx} + 2a\, h_y = 0$

d) $u_{xx} - 2a\, u_y = 0$

e) $Lv = -u_{yy} - e^{\frac{-g(P)}{\epsilon}} h_{yy}$.

In order to investigate the boundary value problems associated with c), d) and e), let us first construct one boundary layer term, and the corresponding solution to 3.1) associated with it. Thus we consider

$$3.12) \qquad \phi_{xx} + \epsilon \phi_{yy} - 2a\, \phi_y = 0 \text{ in } R$$

$$\phi(P) = \begin{cases} \phi_0 \text{ on } \bar{S}_k \\ 0 \text{ on } S - \bar{S}_k \end{cases}$$

where $\bar{S}_k$ is one of the arcs of S^*. Setting $\phi(P, \epsilon) = e^{\frac{-g(P)}{\epsilon}} h(P) + \epsilon \sigma(P, \epsilon)$, we get from 3.10)

$$3.13) \qquad \begin{aligned} h_{xx} + 2a\, h_y &= 0 \\ L\sigma &= -e^{\frac{-g(P)}{\epsilon}} h_{yy} \end{aligned}$$

In order that

$$e^{\frac{-g(P)}{\epsilon}} h(P) = \phi_0(P) \text{ on } \bar{S}_k,$$

independent of ϵ, we must take

$$3.14) \qquad \left.\begin{aligned} g(P) &= 0 \\ h(P) &= \phi_0(P) \end{aligned}\right\} \text{ on } \bar{S}_k$$

Hence 3.11) gives $y_i = y_k$, where $y = y_k$ is the equation of $\bar{S}_k$. Then taking $h(P) = 0$ on the remainder of $S - (\bar{S} - S^*)$, we find $h(P) \equiv 0$ for $y > y_k$, and $h(P)$ is uniquely determined in R, and is of class $\mathcal{C}^2(P)$ in R + S except at a finite number of points of S [7] . The boundary conditions of 3.12) being satisfied for $y \geq y_k$, we take $\sigma(P, \epsilon) = 0$ for

$y \geq y_k$, and $v(P, \epsilon) = \frac{-1}{\epsilon} e^{\frac{-g(P)}{\epsilon}} h(P)$ on S for $y < y_k$.

Now $g(P) = 2a(y_k - y)$, and so on $\overline{S}$ for $y < y_k$, we have $g(P) \geq \delta > 0$. Hence on S, $|v| = 0(1)$ for ϵ small. Writing $v = v_1 + v_2$ where

$$L v_1 = 0 \text{ in } R$$

$$v_1 = \frac{-1}{\epsilon} e^{\frac{-g}{\epsilon}} h \text{ on } S$$

$$L v_2 = -e^{\frac{-g(P)}{\epsilon}} h_{yy} \text{ in } R$$

$$v_2 = 0 \text{ on } S$$

we have $|v_1| \leq m_1$ by lemma 3.1).

In order to establish the boundedness of v_2, we proceed as follows.

<u>Lemma 3.5</u>

$$|v_2(P, \epsilon)| \leq m_1(\epsilon) \text{ in } R + S.$$

Proof: From 3.12) and lemma 3.1)

$$|\phi(P, \epsilon)| \leq m_1 \text{ in } R + S.$$

$$|h(P)| \leq m_2 \text{ in } R + S \qquad [7]$$

Hence

$$|\epsilon v| \leq |e^{\frac{-g}{\epsilon}} h(P)| + |\phi(P, \epsilon)| \leq m_1 + m_2 = m_3$$

Take P in R, Q in R close to S, so that the minimum distance from Q to the boundary, $\rho(Q,S) \leq \delta$. Then $G_R(P,Q)$ being the Green's function for 3.1) in R, we have $\frac{G_R(P,Q)}{\rho(Q,S)} \leq m_1$ for δ sufficiently small. This follows from lemma 3.2). Now the only possible discontinuities of h(P) in R + S occur on S, and these at a junction of one of the arcs S_i with one of the arcs of $\overline{S_i}$ [7]. Since there are but a finite number of such points, we may as well assume only one, since we can write $h(P)$ as a sum of such functions. In order to examine this discontinuity, we assume the point in question to be at the origin, and let N_0 be a neighborhood of the origin contained in R. Then we can construct a local representation of $h(P)$ in N_0 exhibiting the discontinuity explicitly. Thus let $h(P) = h_1 + h_2$ where

$h_2(P)$ is of class $C^2(P)$ in $R + S$

$h_1(P)$ is of class $C^2(P)$ in $R + S - N_0$

$h_{1_{xx}} + 2a\, h_{1_y} = 0$ in N_0

$h_1 = 0$ on $y = 0$, $-\delta \leq x < 0 = \psi(x)$ on $y = 0$, $0 < x \leq \delta$

where $\psi(0) \neq 0$, and $\psi(x)$ is of class $C^4(x)$, $0 \leq x \leq \delta$.

Then

$$h_1(P) = \lambda_1 \int_0^\delta \psi(\xi) e^{\frac{a(x-\xi)^2}{2y}} \frac{d\xi}{(-y)^{1/2}} \quad \text{in } N_0$$

where λ_1, λ_2, ... are constants. By explicit construction, we see that $\frac{\partial h_1}{\partial y}$ is Lebesque integrable in N_0, $y\frac{\partial h_1}{\partial y}$ is continuous in the closure of N_0, and $\frac{\partial}{\partial y}(y\frac{\partial h_1}{\partial y})$ is Lebesque integrable in N_0. Hence $y\frac{\partial^2 h_1}{\partial y^2}$ is Lebesque integrable in N_0, and thus we have $\frac{G_R(P,Q)}{\rho(Q,S)} \cdot \rho(Q,S)\, h_{yy}$ is Lebesque integrable in R.

We now show that

$$v_2(P,\epsilon) = \int_R\int G_R(P,Q)\, e^{\frac{-g(Q)}{\epsilon}} h_{yy}(Q)dQ$$

Consider any subregion R', completely contained in R, with $G_{R'}(P,Q)$ as the corresponding Green's function.

Then the solution to $Lv = -e^{\frac{-g(P)}{\epsilon}} h_{yy}$ in R',

$v = 0$ on S'

is given by

$$\sigma' = \int_{R'}\int G_{R'}(P,Q)\, e^{-\frac{g(Q)}{\epsilon}}\, h_{yy}(Q)dQ \qquad [6]$$

Thus for the sequence $R' \subset R'' \subset \ldots \subset R$, we have

$$0 \leq G_{R'} \leq G_{R''} \leq \ldots \leq G_R$$

Hence each $G_{R^{(n)}}(P,Q)\, e^{-\frac{g(Q)}{\epsilon}}\, h_{yy}(Q)$ is bounded by the integrable function $G_R(P,Q)\, e^{-\frac{g(Q)}{\epsilon}} h_{yy}(Q)$ and hence

$$G_{R^{(n)}}(P,Q)\, e^{-\frac{g(Q)}{\epsilon}}\, h_{yy}(Q) \rightarrow G_R(P,Q)\, e^{-\frac{g(Q)}{\epsilon}}\, h_{yy}(Q)$$

a.e. as $R^{(n)} \rightarrow R$, and more over

$$\lim \int G_{R^{(n)}} e^{-\frac{g}{\epsilon}}\, h_{yy} = \int G_R\, e^{-\frac{g}{\epsilon}}\, h_{yy}.$$

To show $\epsilon^{\delta}\sigma_2$ bounded independent of ϵ, we set

$$\sigma_2 = \sigma_3 + \sigma_4 \text{ where } \sigma_3 = \int_{R-N_o}\int, \quad \sigma_4 = \int_{N_o}\int$$

Then $|\sigma_3| \leq m_1$ by lemma 3.1).

Finally

$$\left|\epsilon^{\delta}\sigma_4(P,\epsilon)\right| = \left|\epsilon^{\delta}\int_{N_o}\int e^{-\frac{g(Q)}{\epsilon}} h_{yy}(Q)\, G_R(P,Q)dQ\right| \leq m_1$$

by the discussion following lemma 3.4).

Thus $|\epsilon^{\delta}\sigma(P,\epsilon)|$ is bounded independent of ϵ in $R + S$, and we have constructed one boundary layer term. In an obvious way, the solution to

$$L\phi = 0 \text{ in } R$$

$$\phi = \phi_o \text{ on } S^*$$

$$0 \text{ on } S - S^*$$

is a finite sum of such boundary layer terms plus a similar correction term, namely

$$\phi(P,\epsilon) = \sum_i e^{-\frac{g_i(P)}{\epsilon}}\, h_i(P) + \epsilon\sigma(P,\epsilon).$$

where

$$|\epsilon^{\delta}\sigma(P,\epsilon)| \leq m_1$$

The proof of the main theorem now follows immediately; we have

3.15) $$L\phi = 0 \text{ in } R$$
$$\phi = \phi_o \text{ on } S.$$

Then $\phi(P,\epsilon) = u(P) + \sum_i e^{\frac{-g_i(P)}{\epsilon}} h_i(P) + \epsilon\sigma(P,\epsilon)$

where

3.16) $$u_{xx} - 2au_y = 0 \text{ in } R$$
$$u = \phi_o \text{ on } S - S^*$$

3.17) $$h_{i_{xx}} + 2ah_{i_y} = 0 \text{ in } R$$
$$h_i = \begin{cases} \phi_o - u(P) \text{ on } \overline{S}_i \subset S^* \\ 0 \text{ on } S - (\overline{S} - S^*) \text{ for } y \neq y_k \end{cases}$$

3.18) $$L\sigma = -u_{yy} - \sum_i e^{-\frac{g_i}{\epsilon}} h_{i_{yy}} \text{ in } R$$
$$\sigma = \frac{\phi - u - \sum e^{-\frac{g_i}{\epsilon}} h}{\epsilon} \quad \text{on } S$$

and by construction, $|\sigma| \leq m_1$ on S and by an analogous procedure, we find $|\epsilon^\delta \sigma| \leq m_1$ in R + S.

To obtain more terms of the asymptotic expansion, we take

$$\phi(P,\epsilon) = u_o(P) + \epsilon u_1(P) + \ldots + \epsilon^r u_r(P)$$
$$+ \sum_i e^{\frac{-g_i(P)}{\epsilon}} \left\{ h_{i_o} + \epsilon h_{i_1} + \ldots + \epsilon^r h_{i_r}(P) \right\}$$
$$+ \epsilon^{r+1} \sigma(P,\epsilon)$$

and we find, as above, that $|\epsilon^\delta \sigma(P,\epsilon)| \to 0$ as $\epsilon \to 0$ for every $\delta > 0$ and every $P \varepsilon R$.

It is probably true that we may take $\delta = 0$ in this example, although computational difficulties have led us to the weaker result. It is, however, a fairly simple matter to prove $\lim_{\epsilon\to 0} \epsilon^\delta v(P,\epsilon) = 0$ for all P in R, and every $\delta > 0$.

A typical calculation is the following:

$$3.19)\quad |\sigma(P,\epsilon)| \leq \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\zeta}{\eta^{5/2}}\, e^{-\frac{\zeta^2}{4\eta}}\; \frac{e^{\frac{g}{\epsilon}(y-\eta)}}{2\pi\sqrt{\epsilon}} \left\{ K_0\left(\frac{a r}{\sqrt{\epsilon}}\right) - K_0\left(\frac{a \bar{r}}{\sqrt{\epsilon}}\right) \right\} d\eta\, d\zeta$$

This can be transformed into

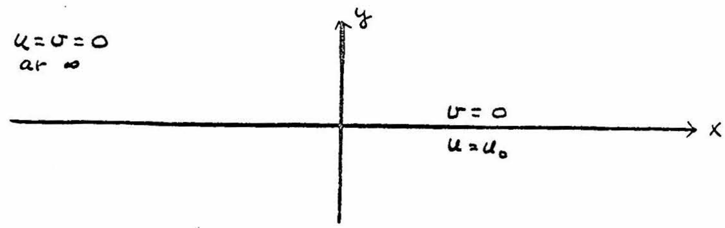
$$3.20)\quad |\sigma(P,\epsilon)| \leq 4\sqrt{\epsilon}\,\frac{x}{y}\, e^{\frac{y}{2\epsilon}} \int_{0}^{\infty} \frac{\sinh\frac{y\zeta}{2\epsilon}}{\zeta}\; \frac{K_1\left(\frac{1}{2\epsilon}\sqrt{\epsilon x^2(1+\zeta) + y^2(1+\zeta)^2}\right)}{\left(\epsilon x^2 + y^2\,\overline{1+\zeta}\right)^{1/2}}\, d\zeta$$

and using the asymptotic expansion of the Bessel function in the integrand, the right hand side is $O(\epsilon^{\delta})$ for every $\delta > 0$. We have so far been unable to prove the boundedness of the right hand side of 3.20) as $\epsilon \to 0$, although it certainly appears to be so.

Sec. 4 Oseen Flow Past a Semi-infinite Flat Plate at Zero Angle of Attack, for a Viscous Incompressible Fluid.

In this section, we study the boundary layer terms associated with a system of equations, and obtain an asymptotic expansion of the solution to a fluid flow problem; it turns out that the expansion terminates, so we actually obtain the exact solution to the equations.

It has been shown [8] that there is a decomposition of certain flow fields into longitudinal and transverse waves. These concepts are assumed known here, and we merely give the equations for, and study, the mathematical problem at hand.



Let the plate coincide with the x-axis, $0 \le x < \infty$ and let u, v be the perturbation velocities of the flow field, where we have $u = v = 0$ at upstream infinity. Then we have $\vec{q} = (u,v)$, $\vec{q} = \vec{q}_L + \vec{q}_T$, a longitudinal plus a transverse wave where

4.1) $$U u_x = \nu \Delta u$$
$$U v_x = \nu \Delta v \qquad \text{for the transversal waves}$$
$$u_x + v_y = 0$$

4.2) $$\vec{q}_L = \operatorname{grad} \phi$$
$$\Delta \phi = 0 \qquad \text{for the longitudinal waves}$$

ν = coefficient of viscosity, assumed small;

$\mathcal{U}$ = free stream velocity = constant.

We set $\frac{\nu}{\mathcal{U}} = \epsilon$. The boundary conditions are the following:

4.3) $$u = u_0 \quad \text{on the plate}$$
$$v = 0$$
$$u = v = 0 \text{ at upstream infinity.}$$

It turns out to be simpler to make a transformation of coordinates before proceeding with the investigation of the boundary layer.(1) The plate is actually a limiting case of a body of finite thickness, and the boundary condition $u = u_0$ on the plate actually means $u = u_0$ on $y = 0^+$, $y = 0^-$, or both sides of the plate.

Let us map the x-y plane, cut along the positive x-axis, into the upper half of the w-plane, by the transformation $z = x + iy = w^2 = (\xi + i\eta)^2$. This "unfolds" the plate into a single boundary, namely, the real axis of the w-plane.

Then equations 4.1), 4.2) become

4.4) $$\epsilon \Delta u - 2\xi u_\xi + 2\eta u_\eta = 0$$
$$\epsilon \Delta v - 2\xi v_\xi + 2\eta v_\eta = 0 \text{ for the transverse waves}$$
$$\xi u_\xi - \eta u_\eta + \xi v_\eta + \eta v_\xi = 0$$

4.5) $$\vec{\delta}_L = \text{grad}\,\phi$$
$$\Delta\phi = 0 \quad \text{for the longitudinal waves}$$

with the boundary conditions

(1) Proceeding in rectangular co-ordinates, we are led to parabolic co-ordinates in a natural way.

$$4.6) \qquad u = u_0 \quad \text{on} \quad \eta = 0$$
$$v = 0 \quad \text{on} \quad \eta = 0$$
$$u = v = 0 \quad \text{when} \quad \eta = \infty .$$

Note: we take $\xi \geq 0$.

(As previously $\Delta \equiv \dfrac{\partial^2}{\partial \xi^2} + \dfrac{\partial^2}{\partial \eta^2}$).

An examination of 4.4) shows that in letting $\epsilon \to 0$ we lose one derivative with respect to η, and one in ξ. The boundary conditions in ξ are the same for the reduced equation as for the system 4.4), 4.5) and hence there is no boundary layer term at $\xi = -\infty$ as $\epsilon \to 0$. On the other hand, if $\epsilon = 0$ we have potential flow, and the solution is merely $u = v = 0$. Hence we expect to find the boundary layer terms for small values of η. Setting $\eta = \sqrt{\epsilon}\,\sigma$, substituting in 4.4), 4.5) and expanding about $\epsilon = 0$, we obtain

$$4.7) \qquad u_{\sigma\sigma} - 2\xi u_\xi + 2\sigma u_\sigma = 0$$
$$v_{\sigma\sigma} - 2\xi v_\xi + 2\sigma v_\sigma = 0 \quad \text{for the transverse wave}$$
$$v_\sigma = 0$$

$$4.8) \qquad \phi_{\sigma\sigma} = 0 \quad \text{for the longitudinal wave.}$$

Denoting $\lim\limits_{\epsilon \to 0} \vec{q}\,(\xi, \sqrt{\epsilon}\,\sigma, \epsilon) \equiv \vec{q}^{\,*}(\xi, \sigma)$ we have, then

$$v_T^* = 0, \Rightarrow v_L^* = 0 \text{ and thus } v^* = 0 .$$

u^* is clearly independent of ξ, and hence

$u^*_{\sigma\sigma} + 2\sigma u^*_\sigma = 0 \qquad u^* = u_0$ when $\sigma = 0$ or

$$4.9) \qquad u^* = u_0 \operatorname{erfc} \sigma .$$

This is the "classical boundary layer" solution.

(note: we use $\sqrt{\epsilon}$ instead of ϵ since the coefficients are variable, and vanish on the boundary of the region).

Proceeding as in Sec. 3, we try, instead of 4.9), the function $u_T^{(1)}(\xi,\eta)\,\mathrm{erfc}\left\{\frac{g(P)}{\epsilon}\right\}^{\frac{1}{2}}$ for the boundary layer term, where

$$\lim_{\epsilon\to 0}\frac{g(\xi,\sqrt{\epsilon}\,\sigma)}{\epsilon}=\sigma^2 .$$

The solution of the reduced equation here is identically zero. Hence we try

$$4.10)\quad \begin{aligned} u(P,\epsilon) &= u^{(0)}(P) + u_T^{(1)}\mathrm{erfc}\sqrt{\tfrac{g}{\epsilon}} + u_T^{(2)} e^{-\frac{g}{\epsilon}}\sqrt{\epsilon} \\ &\quad + \sqrt{\epsilon}\; u_T^{(3)} + \sqrt{\epsilon}\; u_L^{(4)} + \epsilon\,(U_T + U_L) \\ v(P,\epsilon) &= v^{(0)}(P) + \left\{ v_T^{(2)} e^{-\frac{g}{\epsilon}} + v_T^{(3)} \right. \\ &\quad \left. + v_L^{(4)} \right\}\sqrt{\epsilon} + \epsilon\,(V_T + V_L) \end{aligned}$$

where $u^{(0)}(P)$, $v^{(0)}(P)$ are included merely for completeness of argument, since they are both identically zero (the solution to the reduced equation). We include the exponential terms since we intend to equate terms of like order of magnitude, for ϵ small, and look for an expansion in terms of $\sqrt{\epsilon}$ rather than ϵ.

Substituting 4.10) into 4.4), 4.5) and equating to zero, terms of the same magnitude, we arrive at the following system.

a) $\quad g_\xi^{\,2} + g_\eta^{\,2} + 2\xi g_\xi - 2\eta g_\eta = 0$

b) $\quad \frac{1}{\sqrt{\pi}}\, u_T^{(1)} + \eta\, u_T^{(2)} - \xi\, v_T^{(2)} = 0$

c) $\xi\, u^{(1)}_{T_\xi} \quad - \eta\, u^{(1)}_{T_\eta} \quad = \quad 0$

d) $\frac{1}{\sqrt{\pi}}\, u^{(1)}_{T_\eta} \; + \eta\, u^{(2)}_{T_\eta} \quad + \xi\, u^{(2)}_{T_\xi} \quad + u^{(2)}_{T} \quad = \quad 0$

e) $\eta\, v^{(2)}_{T_\eta} \quad + \xi\, v^{(2)}_{T_\xi} \quad + v^{(2)}_{T} \quad = \quad 0$

f) $\xi\, u^{(2)}_{T_\xi} \quad - \eta\, u^{(2)}_{T_\eta} \quad + \xi\, v^{(2)}_{T_\eta} \quad + \eta\, v^{(2)}_{T_\xi} \quad = \quad 0$

g) $- \xi\, u^{(3)}_{T_\xi} \quad + \eta\, u^{(3)}_{T_\eta} \quad = \quad 0$

h) $- \xi\, v^{(3)}_{T_\xi} \quad + \eta\, v^{(3)}_{T_\eta} \quad = \quad 0$

i) $\xi\, u^{(3)}_{T_\xi} \quad - \eta\, u^{(3)}_{T_\eta} \quad + \xi\, v^{(3)}_{T_\eta} \quad + \eta\, v^{(3)}_{T_\xi} \quad = \quad 0$

j) $\Delta u^{(4)}_{L} \quad = \Delta v^{(4)}_{L} \quad = \quad 0$

k) $\epsilon \Delta U_T \; - 2\xi\, U_{T_\xi} \; + 2\eta\, U_{T_\eta} \quad = \quad - \Delta u^{(1)}_{T} \operatorname{erfc} \frac{\eta}{\sqrt{\epsilon}} \quad - \Delta u^{(2)}_{T} \sqrt{\epsilon}\;{}^{-\frac{\eta^2}{\epsilon}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \Delta u^{(3)}_{T}$

l) $\epsilon \Delta V \quad - 2\xi\, V_{T_\xi} \quad + 2\eta\, V_{T_\xi} \quad = \quad -\sqrt{\epsilon}\, e^{-\frac{\eta^2}{\epsilon}} \Delta v^{(2)}_{T} \quad - \Delta v^{(3)}_{T}$

m) $\xi\, U_{T_\xi} \; - \eta\, U_{T_\eta} \; + \xi\, V_{T_\eta} \; + \eta\, V_{T_\xi} \quad = \quad 0$

n) $\Delta U_L \quad = \quad \Delta V_L \quad = \quad 0 .$

where we have already used the solution to a) with the condition $\quad g = 0 \quad \text{on} \quad \eta = 0$

$$g \geq 0 \quad \text{in}\ \eta \geq 0 , \ g \not\equiv 0 .$$

This turns out to be $\quad g(P) = \eta^2 .$

Next, we can solve for $v^{(3)}_{T}$ and obtain $\quad \alpha)\ v^{(3)}_{T} = \text{constant}$

Then, solving for $v^{(2)}_{T}$, $u^{(2)}_{T}$, we obtain

$\beta) \quad v^{(2)}_{T} \quad = \frac{u_0}{\sqrt{\pi}} \; \phi\left(\frac{\eta}{\xi}\right)$ where ϕ is an arbitrary function of class C^2 in $(-\infty, \infty)$. Then

$\gamma) \quad u^{(2)}_{T} \quad = \frac{u_0}{\sqrt{\pi}\ \eta}\left\{\phi\left(\frac{\eta}{\xi}\right) - 1\right\}$

The solution to c) is merely $u_T^{(1)} = \phi_2(\xi\eta)$ subject to the condition that $\phi_2(0) = u_0 =$ constant. It appears that there is no unicity of solution to this problem, but we shall show later that if we use symmetry arguments, we still obtain the same asymptotic expansion what ever function of class $\mathcal{C}^2$ we take. For the moment, let us be guided by simplicity, and choose $\phi_2(\xi\eta) \equiv u_0$, as this simplifies equation k).[1]

Now the boundary conditions on the plate, from 4.6) are

$$u_0 = u_0 + \sqrt{\epsilon}\,(u_T^{(2)} + u_T^{(3)} + u_L^{(4)}) + \epsilon\,(U_T + U_L)$$

4.11) $$0 = \sqrt{\epsilon}\,(v_T^{(2)} + v_T^{(3)} + v_L^{(4)}) + \epsilon\,(V_T + V_L)$$

and at $\eta = \infty$ we have $u = v = 0$, or

4.12) $$0 = (u_T^{(3)} + u_L^{(4)}) + \sqrt{\epsilon}\,(U_T + U_L)$$

$$0 = (v_T^{(3)} + v_L^{(4)}) + \sqrt{\epsilon}\,(V_L + V_T)$$

Taking $v_T^{(3)} \equiv K$, a constant, we have then

$$v_L^{(4)} = -K \quad \text{at} \quad \eta = \infty$$

$$\Delta v_L^4 = 0$$

$$v_L^{(4)} = -K - v_T^{(2)} \quad \text{on} \quad \eta = 0$$

$$= -K - \frac{u_0}{\sqrt{\pi}\,\xi}\,\phi(0)$$

Hence $$v_L^{(4)} = -K - \frac{u_0\,\xi\,\phi(0)}{\sqrt{\pi}\,(\xi^2+\eta^2)} + \frac{\lambda\,\eta}{\xi^2+\eta^2}$$ where λ is an arbitrary constant.

(1) If we use dimensional arguments, we find that u^1 must be homogeneous of degree zero, and hence a constant. Accepting the validity of this argument, many of the subsequent computations are simplified.

If we require v to be finite in a neighborhood of the origin, both as ξ, η tend to zero, and for ϵ small, we must take $\lambda = 0$, $K = 0$ and hence

$$\delta) \quad v_T^{(2)} = \frac{u_0 \; \xi \phi(0)}{\sqrt{\pi} \quad \xi^2 + \eta^2}$$

From $\gamma)$ we get $u_T^{(2)} = \dfrac{u_0}{\sqrt{\pi}\,\eta} \left\{ \phi(0) \dfrac{\xi^2}{\xi^2 + \eta^2} - 1 \right\}$

and again boundedness at the origin gives $\phi(0) = 1$. Thus we have so far

$$v_T^{(2)} = - v_L^4 = \frac{u_0}{\sqrt{\pi}(\xi^2 + \eta^2)}$$

$$u_T^{(2)} = - \frac{u_0 \eta}{\sqrt{\pi}(\xi^2 + \eta^2)}$$

Now $\quad u_T^{(3)} = \phi_4(\xi \eta)$

$$\Delta u_L^4 = 0$$

Thus using the boundary conditions on u, we have

$$u_T^{(3)} + u_L^{(4)} = 0 \quad \text{at} \quad \eta = \infty$$

$$u_T^{(2)} + u_T^{(3)} + u_L^{(4)} = 0 \quad \text{on} \quad \eta = 0$$

and as above we obtain

$$u_L^{(4)} = \frac{u_0 \eta}{\sqrt{\pi}(\xi^2 + \eta^2)} + \lambda \quad \text{where } \lambda \text{ is a constant}$$

$$u_T^{(3)} = - \lambda$$

Since we are only interested in the sum of these terms, the ambiguity is only superficial, and we may as well take $\lambda = 0$. Hence

$$u_L^{(4)} = -u_T^{(2)} = \frac{u_0 \eta}{\sqrt{\pi}(\xi^2 + \eta^2)} \quad ; \quad u_T^{(3)} = 0 .$$

We now have the equations k), ... n) with zero right hand sides, and homogeneous boundary conditions.

Hence $\quad U_T = U_L = V_T = V_L = 0$

The appearance of non unique solutions to these equations is coupled with the vanishing of the coefficients of the reduced equation on the boundary of the region. But for this complication, we should have a particular case of Levinson's [3] equation. However, this shows that the boundary plays a major role in the determination of the form of the boundary layer terms, and instead of continuous dependence on the boundary curves, we get a discontinuous jump comparable to the transition from a non-characteristic to a characteristic boundary for first order linear partial differential equations. In our case, the boundary coincides with one of the characteristics of the reduced equation, and this complicates the problem considerably. It also necessitates the introduction of $\sqrt{\epsilon}$ instead of powers of ϵ alone, which suffices in Levinson's case.

Collecting our results, we have

4.13) $$u = u_0 \operatorname{erfc} \frac{\eta}{\sqrt{\epsilon}} + \left(\frac{\epsilon}{\pi}\right)^{\frac{1}{2}} \left\{ 1 - e^{\frac{-\eta^2}{\epsilon}} \right\} \frac{u_0 \eta}{\xi^2 + \eta^2}$$

$$v = \frac{\sqrt{\epsilon}}{\sqrt{\pi}} \frac{\xi}{\xi^2 + \eta^2} \left\{ e^{\frac{-\eta^2}{\epsilon}} - 1 \right\}$$

Let us consider now the alternate choices for $u_T^{(1)}$ instead of u_0. We have $2\xi\eta = y$, so that $u_T^{(1)} = \phi_2(y)$, and from symmetry, this must be an even function of y. We have also required the solutions of a), ... n) to be of class C^2, so that using $u_T^{(1)}(0) = u_0$, we have

$$\begin{aligned} u_T^{(1)} &= u_0 + y^2\phi_3(y) \\ &= u_0 + \xi^2\eta^2\,\phi_4(\xi\eta) \end{aligned}$$

Hence the difference between any other admissible function $u_T^{(1)}$ and u_0 is of the form $\xi^2\eta^2\phi_4(\xi\eta)$. This leads to $\xi^2\eta^2 \operatorname{erfc}\dfrac{\eta}{\sqrt{\epsilon}}\,\phi_4(\xi\eta) = O(\epsilon)$, and hence can be put into the correction terms $\epsilon\,(U_T + U_L)$. Thus we are led to a unique asymptotic expansion.

One further remark should be made. It is mere coincidence that the first term in the expansion for u is precisely the "classical boundary layer solution". Thus for the finite flat plate, we can no longer conclude that the boundary layer terms are independent of ξ (4.7). On the strength of the above, we offer the following conjecture: Prandtl's boundary layer solution is not necessarily the first term of any asymptotic expansion, but in general is only an approximation to the first term of such an expansion.

This is borne out in investigations now under way on the Oseen flow past a finite flat plate.

Appendix I

If $\alpha_1 \neq 0$, the matrix $\begin{pmatrix} 0 & \epsilon & 0 & \cdots & - \\ & & \ddots & & \\ & & & & \epsilon \\ -\alpha_n & & \cdots & & -\alpha_1 \end{pmatrix} = A + B\epsilon$

has no elementary divisors for $0 \leq \epsilon \leq \epsilon_0$, for ϵ_0 sufficiently small ; moreover, for $0 < \epsilon < \epsilon_0$, the characteristic roots are all distinct.

Proof:- The first statement follows by direct computation.

Suppose next that there is a double root of

$\left| \frac{A}{\epsilon} + B - \lambda I \right| = 0$, say $\lambda = \lambda_1$, for some particular value of ϵ in $(0, \epsilon_0)$.

1.1) Then since $\epsilon \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \dots + \alpha_n = 0$,

the resultant of 1.1) and its derivative must vanish.

1.2)
$$R = \begin{vmatrix} n\epsilon & (n-1)\alpha_1 & \cdots & \alpha_{n-1} & 0 & \cdots & 0 \\ & \ddots & & & & & \\ & & n\epsilon & (n-1)\alpha_1 & \cdots & & \alpha_{n-1} \\ \epsilon\alpha_1 & \cdots & & \alpha_n & 0 & \cdots & 0 \\ & & & & \ddots & & \\ & & \epsilon\alpha_1 & & \cdots & & \alpha_n \end{vmatrix} = 0$$

Assume moreover, that $\alpha_n \neq 0$.

Subtracting n times row $n + i$ from i

$i = 1, 2, \dots n-1$, we obtain

$$R = \left|\begin{array}{cccccccc} n\epsilon & (n-1)\alpha_1 & \cdots & \alpha_{n-1} & 0 & \cdots & & 0 \\ & \ddots & & & & & & \\ & & & n\epsilon & (n-1)\alpha_1 & \cdots & & \alpha_n \\ 0 & -\alpha_1 & \cdots & -n\alpha_n & & & & \\ & & & & \ddots & & & \\ & & & & 0 & -\alpha_1 & \cdots & -n\alpha_n \end{array}\right|$$

The determinant of order n in the lower right hand corner is non vanishing, and equals $(-1)^n \; n^n \alpha_n^{\,n}$ for $\alpha_n \neq 0$, and hence the leading coefficient of ϵ given by $R = 0$ is $(-n\alpha_n)^n \; n^{2n-1}$.

If $R = 0$ for more than a finite number of values of ϵ as $\epsilon \to 0$, it must vanish identically, since R is a polynomial of degree $n-1$ in ϵ. Hence if $\lambda = \lambda_1$ is a double root for arbitrarily small ϵ, it must be so for all $0 < \epsilon < \epsilon_0$.

On the other hand, λ_1 satisfies

1.3) $\epsilon\lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_n = 0$

1.4) $n\epsilon\lambda^{n-1} + \ldots + \alpha_1 = 0$

Multiply 1.3) by n, 1.4) by λ and subtract.

This gives

1.5) $Q(\lambda) = \sum \beta_i \lambda^i$ $\qquad$ β_i independent of ϵ

so that every double root is fixed independent of ϵ.

Thus 1.1) can be written

1.6) $P_k(\lambda)\, Q_{n-k}(\lambda) = 0$

where $P_k(\lambda)$ contains all the multiple roots, and is thus independent of ϵ. This implies $\frac{\partial P_k(\lambda)}{\partial \epsilon} = 0$ or $\frac{\partial \lambda}{\partial \epsilon} = 0$, whence from 1.1), $\lambda^n = 0$ and $\lambda = 0$. But $\alpha_n \neq 0$ so that there are no roots independent of ϵ, and hence no multiple roots for $0 < \epsilon < \epsilon_0$.

For the case $\alpha_n = 0$, $\alpha_{n-k} \neq 0$, similar results hold, namely, there are k roots $\lambda = 0$, and no others are independent of ϵ or multiple.

Appendix 2

$A + B\epsilon$ has no elementary divisors, and neither has A. Thus we can find non singular matrices $C(\epsilon)$, $C^{-1}(\epsilon)$ such that

$$A + B\epsilon = C^{-1}(\epsilon) \begin{pmatrix} \epsilon\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \epsilon\lambda_n \end{pmatrix} C(\epsilon) \quad \text{where}$$

the λ_i are the roots of 1.1). Moreover, $C(\epsilon)$ can be chosen so that $\lim_{\epsilon \to 0} C(\epsilon)$ exists, is finite, and non zero.

<u>Proof</u>:-

consider

$$e^{A + B\epsilon} = C^{-1}(\epsilon) \begin{pmatrix} e^{\epsilon\lambda_1} & & \\ & \ddots & \\ & & e^{\epsilon\lambda_n} \end{pmatrix} C(\epsilon)$$

For $\epsilon = 0$, $e^{A} = D^{-1} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & e^{-\alpha_1} \end{pmatrix} D$

so that $|D|$ is finite and non zero.

Hence we may choose $C(\epsilon)$ so that $C(\epsilon) \rightarrow D$ as $\epsilon \rightarrow 0$, and thus for $0 \leq \epsilon < \epsilon_0$, $0 < |C(\epsilon)| < \infty$.

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