A GENERAL DEPENDENCE RELATION AND ITS APPLICATION TO LATTICE IMBEDDINGS

Thesis by

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SUMMARY

It is well known that a dependence relation defined between the elements and the subsets of an abstract set M can be used to construct a complete lattice L'. The elements of L' are the subsets of M which are closed with respect to the dependence relation. The properties of L' are determined by the set M and the dependence relation. If the set M is taken to be a set of lattice elements, a partial ordering is defined over M by the lattice ordering. In this thesis postulates are given for a generalized dependence relation which takes into account any partial ordering which is defined over M and which reduces to the classical dependence relation if M is not ordered. In particular if M is taken to be the set of join irreducible elements of a lattice L, then the complete lattice L', which is induced by a generalized dependence relation, is such that the set of completely join irreducible elements of L' is isomorphic to M. As the dependence relation is varied, different lattices are obtained, all of which have the same set of join irreducible elements.

Let L be any finite dimensional lattice over which an integral valued semi-modular function σ is defined. In Part II the theory of Part I is applied to imbed L as a sublattice of a semi-modular lattice L' such that if $a \rightarrow a'$, then the ordinary lattice rank of a' equals $\sigma(a)$.

In Part III the following imbedding problem is discussed. If a given lattice L has the property that every quotient lattice u/a for $a \neq z$ in L is distributive (modular, semi-modular), is it always possible to extend L to a distributive (modular, semi-modular) lattice L' by introducing new elements which contain no element of L except z? It is shown that the process is always possible in the finite dimensional distributive case and that the resulting lattice L' is unique

under an additional mild restriction. However, for the modular and semi-modular cases, counter examples are given to prove that in general the imbedding is impossible.

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NOTATION

In this thesis we shall consider lattice elements a,b and sets S,T of lattice elements. Lattice inclusion, union, and intersection will be denoted $a \supseteq b$, $a \cup b$, and $a \cap b$ respectively. Set inclusion, union, and intersection will be denoted $S \supseteq T$, $S \vee T$ and $S \wedge T$. No confusion is likely to arise from the use of the same symbol for lattice and set inclusion. Proper inclusion is denoted $a \supset b$ or $S \supset T$. Set difference will be denoted $S - T$. The lattice union and intersection of all elements of a set S will be denoted $\bigcup S$ and $\bigcap S$. The unit element of a lattice will be denoted by u, the null element by z, and the null set by N.

The symbol Q will denote the set of all join irreducible elements q (see definition 1) of a lattice. The null element z of a lattice is trivially a join irreducible element and is explicitly excluded from Q. If a is a lattice element, S_a denotes the set of all $q \in Q$ such that $q \subseteq a$.

Definitions.

(1) A lattice element q is join irreducible if and only if $q = a \cup b$ implies either $q = a$ or $q = b$.

(2) A lattice element q is completely join irreducible if and only if $q = \bigcup a_\alpha$ implies $q = a_\alpha$ for some α.

(3) A lattice element x is meet irreducible if and only if $x = a \cap b$ implies either $x = a$ or $x = b$.

(4) A lattice element x is completely meet irreducible if and only if $x = \bigcap a_\alpha$ implies $x = a_\alpha$ for some α.

(5) A lattice element a covers the element b, written $a \succ b$, if and only if $a \supset c \supseteq b$ implies $c = b$.

(6) A lattice element p is called a <u>point</u> if and only if $p \succ z$.

(7) A lattice L is upper <u>semi-modular</u> if and only if $a \succ a \cap b$ implies $a \cup b \succ b$ for all $a, b \in L$.

(8) A lattice L is <u>lower semi-modular</u> if and only if $a \cup b \succ b$ implies $a \succ a \cap b$ for all $a, b \in L$.

(9) Any subset L' of the set of elements of a lattice L is said to form a <u>lattice within L</u> if and only if L' forms a lattice with respect to the ordering of L.

(10) Any subset L' of the set of elements of a lattice L is said to be a <u>sublattice of L</u> if and only if L' is a lattice within L, $a \cup_{L'} b = a \cup_{L} b$, and $a \cap_{L'} b = a \cap_{L} b$ for every a, b in L'.

(11) The <u>quotient lattice</u> a/b of L is the sublattice of all elements $c \in L$ such that $a \geq c \geq b$.

(12) L is a <u>point lattice</u> if and only if every element of L can be expressed as the union of points of L.

(13) A lattice L is <u>complete</u> if and only if every set of lattice elements has a least upper bound and a greatest lower bound in L.

(14) A lattice is said to satisfy the <u>descending chain condition</u> if and only if every chain

$$a_1 \supset a_2 \supset \ldots \supset a_n \supset \ldots$$

is finite.

PART I. DEPENDENCE RELATIONS IN A LATTICE

Section 1.1. Introduction.

Let L be a lattice which satisfies the descending chain condition. Every element a of L can be represented as the union of all join irreducible elements contained in a, and therefore the join irreducible elements of L are the building stones of the lattice with respect to the operation of lattice union. Obviously the set of join irreducible elements, which is partially ordered by the lattice inclusion, does not completely determine the lattice structure.

Let M be any set of lattice elements. We define a relation Δ between the elements m and the subsets S of M as follows:

(A) $m \,\Delta\, S$ if and only if $m \leq \bigcup S$,

where $\bigcup S$ denotes the union in L of all elements of S. The notation $m \,\Delta\, S$ is read "m depends on S". This relation has the following properties.

(Δ1) If $m' \leq m$, then $m' \,\Delta\, S \vee m$ for arbitrary $S \leq M$.

(Δ2) If $m \,\Delta\, S$ and $S \,\Delta\, T$(*), then $m \,\Delta\, T$.

(Δ3) If $m' \,\Delta\, m$, then $m' \leq m$.

The relation Δ induces an algebraic closure operation on the subsets of M, where the closure $\overline{S}$ of S is defined to be the set of all $m \in M$ such that $m \,\Delta\, S$. This definition of closure satisfies the following properties.

(*) We define $S \,\Delta\, T$ to mean $m \,\Delta\, T$ for all $m \in S$.

(C1) $\bar{S} \supseteq S$.

(C2) If $S \supseteq T$, then $\bar{S} \supseteq \bar{T}$.

(C3) $\bar{\bar{S}} = \bar{S}$.

A set is called closed if and only if $q \Delta S$ implies $q \in S$. Under any closure operation satisfying C1 - C3, the closed subsets form a complete lattice L' with respect to set inclusion (Birkhoff [1]).

Now consider the set M' of all elements m' of L' such that $m' = \bar{m}$ for $m \in M$, and define $m' \Delta S'$ if and only if $m' \leq \bigcup S$ in L'. Then the closed subsets of M' form a complete lattice L'' which is isomorphic to L'. Hence the structure of L' is determined by the set M and the dependence relation Δ.

Conversely, any relation Δ which is defined over M and satisfies $\Delta 1$ and $\Delta 2$ is called a dependence relation, and the closure operation induced by Δ satisfies C1 - C3. Hence the closed sets under any dependence relation form a complete lattice.

If M is an abstract set, any relation D between the elements and subsets of M is called a dependence relation if D satisfies

(D1) $m \; D \; S \vee m$ for arbitrary $S \subseteq M$,

(D2) If $m \, D \, S$ and $S \, D \, T$, then $m \, D \, T$.

The principal distinction between relations Δ which satisfy $\Delta 1$ - $\Delta 2$ and the relations D which satisfy D1 - D2 is that Δ takes into account the partial ordering of the set M. In the event that M is unordered, $\Delta 1$ and $\Delta 2$ clearly reduce to D1 and D2 respectively.

In this thesis we specialize M to be the set of join irreducible elements of L. The dependence relation (A) then also satisfies

($\Delta 4$) If $m \,\Delta\, S$ and $S \,\Delta\, m$, then $m \in S$.

Conversely, consider any relation Δ which is defined over the set M of join irreducible elements of a lattice and which satisfies $\Delta 1 - \Delta 4$. The closed subsets form a complete lattice L' since $\Delta 1$ and $\Delta 2$ are satisfied. By $\Delta 3$ every irreducible element $m \in M$ induces a closed set $S_m = \{m' \in M \mid m' \leq m\}$. By $\Delta 4$ every S_m is a join irreducible element of L' , and every join irreducible element of L' has the form S_m for some $m \in M$. Hence any such dependence relation can be used to construct a lattice L' whose set of join irreducible elements is isomorphic to the set of join irreducible elements of a given lattice L.

The structure of L' is governed by the particular dependence relation. We shall prove, for example, that L' is upper semi-modular if Δ also satisfies

($\Delta 5$) If $m'' \subset m'$ implies $m'' \,\Delta\, S$, then $m \,\Delta\, S \vee m'$ implies either $m \,\Delta\, S$ or $m' \,\Delta\, S \vee m$.

If the set M is unordered then $\Delta 5$ reduces to

(D3) If $m \; D \; S \vee m'$, then either $m \; D \; S$ or $m' \; D \; S \vee m$. This postulate is analogous to the Steinitz-MacLane exchange axiom which holds in an upper semi-modular point lattice (Birkhoff, [1]). It is well known (MacLane [1]) that if a relation satisfying D1 - D3 is defined over an abstract set M, then the closed subsets form an upper semi-modular point lattice. We shall show that if a relation satisfying $\Delta 1 - \Delta 5$ is defined over the set of join irreducible elements of a lattice L, then the closed subsets form an upper semi-modular lattice L' which has the same number of points as L since the sets of join irreducible elements of L and L' are isomorphic.

Section 1.2. General Properties of Irreducibles.

Before considering the theory of dependence relations we derive some fundamental relationships between the lattice and its irreducibles. Let L be a lattice, and let Q be the set of all join irreducible elements of L. For every $a \in L$ define

$$S_a = \{q \in Q \mid q \leq a\} .$$

Then, if L satisfies the descending chain condition, $a = \bigcup S_a$ (Birkhoff [1]). As a consequence we have $a \leq b$ if and only if $S_a \subseteq S_b$, where if equality holds in either relation, it holds in both.

Lemma 1.1. In any lattice $S_a \wedge S_b = S_{a \cap b}$.

Proof: Let $q \in S_a \wedge S_b$. Then $q \in S_a$ and $q \in S_b$, which imply $q \leq a$ and $q \leq b$. But then $q \leq a \cap b$, and $q \in S_{a \cap b}$. Thus $S_a \wedge S_b \subseteq S_{a \cap b}$. Since each step of the argument reverses, we have $S_a \wedge S_b \supseteq S_{a \cap b}$ and hence $S_a \wedge S_b = S_{a \cap b}$.

Lemma 1.2. In a lattice in which the descending chain condition holds, if $a \succ b$, there exist an element c and a join irreducible element q such that $q \in S_a - S_b$, $q \succ c$, and $c \leq b$.

Proof: If $a \succ b$ then $S_a - S_b$ is non-void. Let q be minimal in $S_a - S_b$. Such a minimal element exists because of the descending chain condition. Let $c = q \cap b$, and let $q \geq d \succ c$. Clearly $c \leq b$, and $S_q \supseteq S_d \supset S_c$. Let $q' \in S_d - S_c$. Then $q' \in S_q$ since $S_q \supseteq S_d$, and hence $q' \leq q$. If $q' \in S_b$ then $q' \in S_b \wedge S_q = S_{b \cap q} = S_c$ which is a contradiction. Hence

$q' \in S_a - S_b$ and by the minimal property of q, $q' \nleq q$. Thus $q' = q \in S_d$, and hence $q = d \succ c$.

Lemma 1.3. Let L be a lattice in which the descending chain condition holds. Let q and q' be join irreducible elements and b an arbitrary element of L. If for all irreducibles $q'' \leq q'$,

$$\text{either } q'' \leq b \quad \text{or} \quad q \leq q'' \cup b,$$

then for every element $a \leq q'$,

$$\text{either } a \leq b \quad \text{or} \quad q \leq a \cup b.$$

Proof: Let $a \leq q'$ and assume $a \nleq b$. Then there exists an irreducible $q'' \in S_a$ such that $q'' \notin S_b$. Hence we have $q'' \leq a \leq q'$ but $q'' \nleq b$. By hypothesis $q \leq q'' \cup b$, and since $q'' \cup b \leq a \cup b$, $q \leq a \cup b$.

The hypothesis of lemma 1.3 is a generalized form of the exchange property, and the lemma illustrates that this exchange property holds in L if and only if the property holds for the join irreducible elements.

Section 1.3. The Generalized Dependence Relation.

Throughout the remainder of Part I we shall assume that the lattice L satisfies the descending chain condition, and that Q, the set of join irreducible elements of L, is partially ordered by the containing relation of the lattice. We exclude from Q the null element of L.

In this thesis we consider dependence relations over Q which take into account the partial ordering of Q and which satisfy the following set of postulates:

(Δ1) If $q'' \leq q'$, then $q'' \,\Delta\, S \vee q'$ for any $S \leq Q$.

(Δ2) If $q \,\Delta\, S$ and $S \,\Delta\, T$ then $q \,\Delta\, T$.

($\Delta 3$) If $q'' \Delta q'$, then $q'' \leq q'$.

($\Delta 4$) If $q \Delta S$ and $S \Delta q$, the $q \in S$.

($\Delta 5$) If $q'' < q'$ implies $q'' \Delta S$, then $q \Delta S \vee q'$ implies either $q \Delta S$ or $q' \Delta S \vee q$.

In the event that Q is unordered these postulates clearly reduce to D1 - D3.

As before we define the closure $\overline{S}$ of a subset S of Q to be the set of all $q \in Q$ such that $q \Delta S$. S will be called closed if and only if $S = \overline{S}$. A series of lemmas will show that $\Delta 1$ and $\Delta 2$ induce a closure relation, $\Delta 3$ insures that S_q is closed for all $q \in Q$, and $\Delta 4$ guarantees that a closed set is join irreducible in the lattice of closed sets if and only if $S = S_q$ for some $q \in Q$. Finally $\Delta 5$ is a form of the exchange property, and it makes the lattice of closed sets upper semi-modular.

<u>Lemma 1.4</u>. If Δ satisfies $\Delta 1$ and $\Delta 2$, then Δ induces a closure operation satisfying C1 - C3.

<u>Proof</u>:

(C1) Let $q \in S$. Then by $\Delta 1$, $q \Delta S \vee q = S$, and $q \in \overline{S}$.

(C2) Let $S \geq T$, and let $q \in \overline{T}$. Then $q \Delta T$, and since $q_T \in S$ for every $q_T \in T$, $T \Delta S$. By $\Delta 2$, $q \Delta S$ and $q \in \overline{S}$. Thus $\overline{S} \geq \overline{T}$.

(C3) Let $q \in \overline{\overline{S}}$. Then $q \Delta \overline{S}$, and by definition of $\overline{S}$, $\overline{S} \Delta S$. Then $q \Delta S$ and $q \in \overline{S}$. Hence $\overline{\overline{S}} \leq \overline{S}$, and using C1 we have $\overline{\overline{S}} = \overline{S}$.

<u>Corollary</u>. The closed subsets induced by any dependence relation form a complete lattice.

Lemma 1.5. If S and T are closed sets, then $S \wedge T$ is closed.

Proof: Let $q \,\Delta\, S \wedge T$. Since $S \wedge T \,\Delta\, S$ and $S \wedge T \,\Delta\, T$, we have by $\Delta 2$, $q \,\Delta\, S$ and $q \,\Delta\, T$. Then $q \in S$ and $q \in T$ since S and T are closed. Hence $q \in S \wedge T$.

Let L' denote the lattice of closed sets. According to lemma 1.5, therefore, lattice intersection in L' coincides with set intersection. However the set union of closed sets is not closed in general, and the lattice union of closed sets is the smallest closed set containing the set union.

Lemma 1.6. If Δ satisfies $\Delta 1$, $\Delta 2$, and $\Delta 3$, S_q is closed for all $q \in Q$.

Proof: Let $q' \,\Delta\, S_q$. By $\Delta 1$, $S_q \,\Delta\, q$, and hence $q' \,\Delta\, q$ by $\Delta 2$. Thus $q' \leq q$ by $\Delta 3$ and $q' \in S_q$.

With each $a \in L$ we associate the set S_a and refer to this correspondence as the "natural" mapping. By lemma 1.6 this mapping takes each irreducible $q \in Q$ into an element S_q of L'. Since the descending chain condition holds in L by hypothesis, $S_q \supset S_{q'}$ if and only if $q > q'$, and $S_q = S_{q'}$ if and only if $q = q'$. Hence L' contains a partially ordered set Q' which is isomorphic to Q.

Lemma 1.7. If Δ satisfies $\Delta 1$, $\Delta 2$, and $\Delta 3$, a closed set S is completely join irreducible in L' only if $S = S_q$ for some $q \in Q$.

Proof: Let $S = \{q_\alpha\}$ be closed. Then $S \supseteq S_{q_\alpha}$ for all α, and so $S \supseteq \bigvee S_{q_\alpha}$. Trivially $S \subseteq \bigvee S_{q_\alpha}$ and we therefore have $S = \bigvee S_{q_\alpha} = \bigcup S_{q_\alpha}$ since S is closed. But if S is completely join irreducible, $S = S_{q_\alpha}$ for some α.

If the descending chain condition holds in a lattice then every element can be expressed as the union of a finite number of join irreducibles, and then complete irreducibility is equivalent to ordinary irreducibility. Consider a lattice L of dimension two, consisting of a denumerable set Q of points, a unit element, and a null element. Define $q \Delta S$ if and only if $q \in S$, making all subsets of Q closed. Then $\Delta 1 - \Delta 5$ are trivially satisfied, but neither chain condition holds in L'. Hence in general we must distinguish between irreducibility and complete irreducibility in L'.

An example considered in Section 1.4 shows that S_q is not necessarily completely join irreducible in L' for every $q \in Q$, even if the dependence relation satisfies $\Delta 5$ as well as $\Delta 1$, $\Delta 2$, and $\Delta 3$. The next lemma gives the precise conditions under which S_q is completely join irreducible in L'. Let $S'_{q'}$ denote the set of all $q \in Q$ which are properly contained in q'. That is, $S'_{q'} = S_{q'} - q'$.

<u>Lemma 1.8</u>. If Δ satisfies $\Delta 1$, $\Delta 2$, and $\Delta 3$, then $S_{q'}$ is completely join irreducible in L' if and only if $S'_{q'}$ is closed.

<u>Necessity</u>: Let $S_{q'}$ be completely join irreducible, and let $q \Delta S'_{q'}$. By $\Delta 1$ $S'_{q'} \Delta S_{q'}$, and $q \Delta S_{q'}$ by $\Delta 2$. Since $S_{q'}$ is closed by lemma 1.6, $q \in S_{q'}$. Hence either $q \in S'_{q'}$ in which case we are through, or $q = q'$. Let $S'_{q'} = \{q_\alpha\}$ where $q_\alpha \subset q'$ for all α. Then

$$S'_{q'} \leq \vee S_{q_\alpha} \leq \cup S_{q_\alpha} \leq S_{q'}$$

since $S_{q'}$ is closed. If $\cup S_{q_\alpha} = S_{q'}$ then $S_{q_\alpha} = S_{q'}$ for some α since $S_{q'}$ is completely irreducible in L'. But then $q_\alpha = q'$ which contradicts $q_\alpha \subset q'$. Hence $\cup S_{q_\alpha} \subset S_{q'}$, which implies $S'_{q'} = \cup S_{q_\alpha}$, and $S'_{q'}$ is closed.

Sufficiency: Let $S'_{q'}$ be closed, and suppose $S_{q'} = \bigcup S_\alpha$ where $S_\alpha \subset S_{q'}$ and S_α is closed for all α. If $q' \in S_\beta$ for some β, then since S_β is closed, $S_\beta = S_{q'}$ which is a contradiction. Hence $S_\alpha \subseteq S'_{q'}$ for all α, and $\bigcup S_\alpha \subseteq S'_{q'} \subset S_{q'}$ which also is a contradiction. Hence $S_\beta = S_{q'}$ for some β, and $S_{q'}$ is completely join irreducible.

In Section 1.4 we give an example of a dependence relation which satisfies $\Delta 1$, $\Delta 2$, $\Delta 3$, and $\Delta 5$ and for which S'_q is not closed for a particular $q \in Q$. As the next lemma proves, the additional restriction required to make all S'_q closed is $\Delta 4$.

Lemma 1.9. If Δ satisfies $\Delta 1 - \Delta 3$, then $S'_{q'}$ is closed for all $q' \in Q$ if and only if $\Delta 4$ also is satisfied.

Necessity: Let $S'_{q'}$ be closed for all $q' \in Q$. Let $q \,\Delta\, S$ and $S \,\Delta\, q$. Now $S \,\Delta\, q$ implies $S \,\Delta\, S_q$, and so by lemma 1.6 $S \subseteq S_q = S'_q \vee q$. If $q \notin S$, then $S \subseteq S'_q$. Then $q \,\Delta\, S$ and $S \,\Delta\, S'_q$ imply $q \,\Delta\, S'_q$, and hence $q \in S'_q$ since S'_q is closed by hypothesis. But this contradicts the definition of S'_q, and hence $q \in S$.

Sufficiency: Let $q \,\Delta\, S'_{q'}$. Then $q \,\Delta\, S_{q'}$ since $S'_{q'} \,\Delta\, S_{q'}$. By lemma 1.6 $q \in S_{q'}$, and hence either $q = q'$ or $q \in S'_{q'}$. Suppose $q = q'$. Then $S'_{q'} \,\Delta\, q$ and $q \,\Delta\, S'_{q'}$. By $\Delta 4$, $q = q' \in S'_{q'}$, contrary to the definition of $S'_{q'}$. Hence $q \in S'_{q'}$ and the proof is complete.

Theorem 1.1. If Δ satisfies $\Delta 1 - \Delta 3$, then the set Q' of completely join irreducible elements of L' is isomorphic to Q under the natural mapping if and only if $\Delta 4$ is also satisfied.

Necessity: If $\Delta 4$ is not satisfied, then S'_q is not closed for some $q \in Q$. Then by lemma 1.8 S_q is not completely join irreducible, and by lemma 1.7, any set not of the form $S_{q'}$ for some $q' \in Q$ must be join reducible. Hence the mapping $q \rightarrow S_q$ takes some irreducible element of L into a reducible element of L', if $\Delta 4$ is not satisfied.

Sufficiency: By lemma 1.9 $S'_{q'}$ is closed for all $q' \in Q$ whenever $\Delta 4$ also holds. Then by lemma 1.8, $S_{q'}$ is completely join irreducible, and any set not of this form is reducible. It was shown previously that the ordering of Q is preserved in Q'.

Lemma 1.10. If Δ satisfies $\Delta 1$, $\Delta 2$, $\Delta 3$, and $\Delta 5$, the lattice L' of closed sets is upper semi-modular.

Proof: Let $S \succ S \cap T$, where S and T are closed sets. Assume for the present that S is completely join irreducible in L'. Then by lemma 1.7 $S = S_{q'}$ for some $q' \in Q$. By lemma 1.8 $S'_{q'}$ is closed; hence $S_{q'} \succ S'_{q'}$ in L', and $S \cap T = S'_{q'}$. Then $q'' \subset q'$ implies $q'' \in S'_{q'}$, which implies $q'' \in T$ and $q'' \Delta T$. Let $\mathbf{R}$ be any closed set such that $S \cup T \supseteq \mathbf{R} \supset T$. For any q such that $q \in \mathbf{R} - T$, $q \in S \cup T$, and hence $q \Delta T \vee q'$ since by definition $S \cup T = \overline{T \vee q'}$. But then by $\Delta 5$ either $q \Delta T$, which implies $q \in T$ and contradicts $q \in \mathbf{R} - T$, or $q' \Delta T \vee q$. Then $q' \in \overline{T \vee q} \subseteq \mathbf{R}$, and hence $S \cup T = \overline{T \vee q'} \subseteq \mathbf{R}$ since $\mathbf{R}$ is closed. Thus $\mathbf{R} = S \cup T$ and $S \cup T \succ T$.

Now consider the general case. Let $S \succ S \cap T$ in L'. Then $S - S \wedge T$ is not void. Let $q' \in S - S \wedge T$ be such that if $q'' \subset q'$ then $q'' \in T$. Such a q' exists since Q, as a subset of L, satisfies the descending chain condition by hypothesis. $S_{q'}$ is closed by lemma 1.6; we also have $S_{q'} \subseteq S$, $S_{q'} \nsubseteq T$, and $S'_{q'} \subseteq T$. Therefore by lemma 1.5 $S'_{q'} = S_{q'} \wedge T$

is closed, and $S_{q'}$ is join irreducible by lemma 1.8. Then $S_{q'} \succ S'_{q'} = S_{q'} \cap T$. Hence by the first part of this proof $\overline{T \vee q'} = T \cup S_{q'} \succ T$. But $q \in S$ implies $q \; \Delta \; [(S \wedge T) \vee q']$ because $q' \in S - S \wedge T$ and $S \succ S \cap T$. Trivially $[(S \wedge T) \vee q'] \Delta (T \vee q')$, and we have $q \Delta T \vee q'$ which implies $q \in \overline{T \vee q'} = T \cup S_{q'}$. Hence $S \subseteq T \cup S_{q'}$, which implies $S \cup T \subseteq T \cup S_{q'}$, and therefore $S \cup T \succ T$, completing the proof of upper semi-modularity.

The results of lemmas 1.5, 1.6, and 1.10 are summarized in the following theorem.

Theorem 1.2. The subsets of Q which are closed with respect to any dependence relation satisfying $\Delta 1$, $\Delta 2$, $\Delta 3$, and $\Delta 5$ form a complete upper semi-modular lattice which contains a partially ordered set isomorphic to Q.

Combining theorems 1.1 and 1.2, we have

Theorem 1.3. Let Q be the partially ordered set of join irreducible elements of a lattice L in which the descending chain condition holds. Let Δ be a dependence relation over Q which satisfies $\Delta 1 - \Delta 5$. Then the closed subsets of Q form a complete upper semi-modular lattice whose completely join irreducible elements form a partially ordered set isomorphic to Q.

Section 1.4. Examples.

We now consider four examples of dependence relations. The first two are of a general nature, and the last two prove the independence of postulates $\Delta 4$ and $\Delta 3$.

Let L be a lattice in which the descending chain condition holds, and let Q be the set of join irreducible elements of L. Define

(A) $\quad q \,\Delta\, S$ if and only if $q \subseteq \bigcup S$,

where $\bigcup S$ denotes the union in L of all elements of S. It is evident that $\Delta 1 - \Delta 3$ are satisfied. Let $q \,\Delta\, S$ and $S \,\Delta\, q$. Then $q \subseteq \bigcup S$ and $q_s \subseteq q$ for all $q_s \in S$. Hence $\bigcup S \subseteq q$, and $q = \bigcup S$. But since q is join irreducible and the descending chain condition holds, q is completely irreducible and $q = q_s$ for some $q_s \in S$. Hence $\Delta 4$ is also satisfied.

Lemma 1.11. Under the dependence relation (A) a set S is closed if and only if $S = S_a$ for some $a \in L$.

Necessity: Let S be closed and let $a = \bigcup S$. If $q \in S_a$, then $q \subseteq \bigcup S$ and $q \,\Delta\, S$. Since S is closed, $q \in S$. Hence $S_a \subseteq S$. But $q \notin S_a$ implies $q \nsubseteq a = \bigcup S$, and therefore $q \not\Delta S$. Hence $S_a = S$.

Sufficiency: Let $q \,\Delta\, S_a$. Then $q \subseteq \bigcup S = a$. Hence $q \in S_a$, and S_a is closed.

Lemma 1.12. The natural mapping $a \longleftrightarrow S_a$ is a lattice isomorphism.

Proof: By lemma 1.11 the correspondence is one to one. Let a and b be elements of L. By lemma 1.5, since S_a and S_b are closed, $S_a \wedge S_b$ is closed. Clearly $q \in S_a \wedge S_b$ implies $q \subseteq a \cap b$, and hence $q \in S_{a \cap b}$. The argument reverses, so $S_{a \cap b} = S_a \wedge S_b = S_a \cap S_b$.

Let $q \in \overline{S_a \vee S_b} = S_a \cup S_b$. Then $q \,\Delta\, S_a \vee S_b$, which implies $q \subseteq \bigcup(S_a \vee S_b) = \bigcup S_a \cup \bigcup S_b = a \cup b$. Thus $q \in S_{a \cup b}$. Again the argument reverses, so $S_a \cup S_b = S_{a \cup b}$.

Thus we have proved

Theorem 1.4. The dependence relation (A) induces an isomorphic mapping of any lattice L onto itself, provided L satisfies the descending chain condition.

From theorems 1.3 and 1.4 it is clear that if the dependence relation (A) satisfies $\Delta 5$, L must be upper semi-modular. We shall show that the converse is true, after first proving a characterization of upper semi-modularity which is stated in terms of irreducibles and hence is more convenient to apply here than is definition 7.

Theorem 1.5. Let L be a lattice in which the descending chain condition holds, and let Q be the set of join irreducible elements of L. Then L is upper semi-modular if and only if $q \succ q \cap b$ implies $q \cup b \succ b$ for every $q \in Q$ and $b \in L$.

Proof: The necessity is obvious from definition 7. Hence let $a \succ a \cap b$. If a is join irreducible, then $a \cup b \succ b$ by hypothesis, and we are through. If a is join reducible, then by lemma 1.2 there exists an irreducible $q \in S_a - S_b$ such that $q \succ c$ where $c \subset a \cap b$. Then $c = q \cap b$, and by hypothesis $q \cup b \succ b$. Also $a = q \cup (a \cap b)$, and hence we have

$$a \cup b = q \cup (a \cap b) \cup b = q \cup b \succ b.$$

Thus L is upper semi-modular.

The dual statement is an immediate

Corollary. Let L be a lattice in which the ascending chain condition holds, and let X be the set of meet irreducible elements of L. Then L is lower semi-modular if and only if $a \cup x \succ x$ implies $a \succ a \cap x$ for every $x \in X$ and $a \in L$.

Now let L be upper semi-modular; we prove that (A) satisfies $\Delta 5$. Let $q'' \,\Delta\, S$ whenever $q'' \subset q'$, and let $q \,\Delta\, S \vee q'$. Since by lemma 1.9 $S'_{q'}$ is closed, we have $S_{q'} \succ S'_{q'}$ in L'. Then by the isomorphism a exists in L such that $q' \succ a$ and $a = \bigcup S'_{q'}$. Hence by hypothesis, $q'' \Delta S$ for all $q'' \in S'_{q'}$. Then $q'' \subseteq \bigcup S$ for all $q'' \in S'_{q'}$, and $a = \bigcup S'_{q'} \subseteq \bigcup S$. If $a = \bigcup S$, then $\bigcup (S \vee q') = a \cup q' = q'$. Then $q \Delta (S \vee q')$ implies $q \subseteq q'$, which implies either $q = q'$ or $q \subseteq a$. In the former case $q' \Delta (S \vee q)$ trivially, and in the latter case $q \Delta S$ since $q \subseteq \bigcup S = a$. Therefore we need only consider $a \subset \bigcup S$. If $q' \subseteq \bigcup S$, then $(S \vee q') \Delta S$, and hence $q \Delta S \vee q'$ implies $q \Delta S$. Hence we have $q' \succ a$, $a \subset \bigcup S$, and $q' \nsubseteq \bigcup S$, and therefore $q' \cup \bigcup S = \bigcup (S \vee q') \succ \bigcup S$ since L is upper semi-modular. Then $q \Delta (S \vee q')$ implies $q \subseteq \bigcup (S \vee q')$, which implies $\bigcup S \subseteq \bigcup (S \vee q) \subseteq \bigcup (S \vee q')$. But since $\bigcup (S \vee q') \succ \bigcup S$, either $\bigcup S = \bigcup (S \vee q)$ or $\bigcup (S \vee q) = \bigcup (S \vee q')$. In the first case $q \subseteq \bigcup S$, and so $q \Delta S$, while in the second case $q' \subseteq \bigcup (S \vee q)$, and hence $q' \Delta S \vee q$ which concludes the proof.

Our second example of a dependence relation satisfies $\Delta 1 - \Delta 5$ when defined over the set Q of join irreducible elements of any lattice L in which the descending chain condition holds. We define

(B) $q \Delta S$ if and only if $q \subseteq q_s$ for some $q_s \in S$.

From this definition, $\Delta 1 - \Delta 4$ follow trivially. To verify 5 we assume that $q \Delta S \vee q'$ and that $q'' \Delta S$ whenever $q'' \subset q'$. Then either $q \subseteq q_s$ for some $q_s \in S$, or $q \subset q'$, or $q = q'$. In the first two cases we have $q \Delta S$, and if $q = q'$, the exchange property holds trivially. Hence the dependence relation (B) satisfies $\Delta 1 - \Delta 5$.

Lemma 1.13. For every $a \in L$, S_a is closed under the dependence relation (B).

Proof: Let $q \, \Delta \, S_a$. Then $q \leq q_a$ for some $q_a \in S_a$. But $q_a \leq a$, and hence $q \leq q_a \leq a$. Thus $q \in S_a$.

Hence the natural mapping imbeds L as a lattice within the lattice L' of sets which are closed under (B).

Lemma 1.14. L' is completely distributive.

Proof: Let S and T be closed sets. By lemma 1.5 $S \cap T = S \wedge T$. Let $q \, \Delta \, S \vee T$. Then either $q \leq q_S$ for some $q_S \in S$, or $q \leq q_T$ for some $q_T \in T$. Since S and T are closed, either $q \in S$ or $q \in T$. Hence $q \in S \vee T$, and $S \vee T$ is closed. Therefore, $S \cup T = S \vee T$. Since the lattice operations coincide with set operations, L' is completely distributive.

Consider the natural mapping $a \rightarrow S_a$ of L into L'. An immediate consequence of lemma 1.13, lemma 1.14 and theorem 1.3 is

Theorem 1.6. Under the dependence relation (B) the natural mapping imbeds any lattice L, in which the descending chain condition holds, as a lattice within a completely distributive lattice, L'. Furthermore an element of L' is completely join irreducible if and only if it is the image of a join irreducible element of L.

Theorem 1.7. Let Δ_B be the dependence relation defined by (B), and let Δ be any other dependence relation satisfying $\Delta 1$ and $\Delta 2$. Then $q \, \Delta_B \, S$ implies $q \, \Delta \, S$.

Proof: Let $q \, \Delta_B S$. Then $q \leq q_S$ for some $q_S \in S$, and hence by $\Delta 1$, $q \, \Delta \, S \vee q_S = S$.

Therefore (B) is the strongest dependence relation which satisfies $\Delta 1 - \Delta 5$, in the sense that it induces the greatest number of closed sets and hence the largest upper semi-modular lattice whose completely join irreducible elements form a partially ordered set isomorphic to Q.

The following example shows that S_q is not necessarily join irreducible in L' even though the dependence relation satisfies $\Delta 1$, $\Delta 2$, $\Delta 3$, and $\Delta 5$. Consider the three element set Q, ordered as indicated in figure 1.

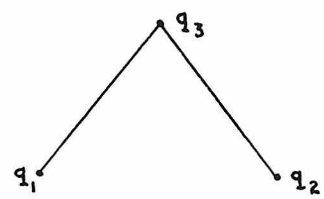

Figure 1.

To every subset $S \subseteq Q$ there corresponds a unique minimal element $q_S \in Q$ such that $q_S \geq q$ for all $q \in S$, and such that if $q'_S \geq q$ for all $q \in S$ then $q'_S \geq q_S$. Define $q \,\Delta\, S$ if and only if $q \leq q_S$. This dependence relation satisfies $\Delta 1$, $\Delta 2$, $\Delta 3$, and $\Delta 5$ as we now verify.

($\Delta 1$). Let $q'' \leq q'$. Then if q'_S is the minimal element corresponding to $S \vee q'$, $q'_S \geq q'$. Hence $q'' \leq q'_S$, and $q'' \,\Delta\, S \vee q'$.

($\Delta 2$). Let $q'' \,\Delta\, S$. Then $q'' \leq q_S$ where q_S is minimal such that $q \leq q_S$ for all $q \in S$. $S \,\Delta\, T$ implies that for all $q \in S$, $q \leq q_T$, where $q_T \geq q'$ for all $q' \in T$. By the minimal property $q_S \leq q_T$. Hence $q'' \leq q_S \leq q_T$ and $q'' \,\Delta\, T$.

($\Delta 3$). Let $q'' \,\Delta\, q'$. Then q' itself is the distinguished element and $q'' \leq q'$.

($\Delta 5$). In the statement of $\Delta 5$ (page 8), if $q = q_3$, then $q' \Delta S \vee q$ for any choice of q' and S. The cases of $q = q_1$ and $q = q_2$ are clearly symmetric, so let $q = q_1$. Then if $q_1 \in S$, $q \Delta S$. Also if $q_3 \in S$, $q \Delta S$, and so we need consider only $S = q_2$.

(a.) Let $q' = q_1$. Then $q' = q$, and the exchange property holds trivially.

(b.) Let $q' = q_2$. Then $q = q_1 \not\Delta S \vee q_2 = S$, and the hypothesis of $\Delta 5$ is not satisfied.

(c.) Let $q' = q_3$, and consider q_1. We have $q_1 \subset q_3$ but $q_1 \not\Delta q_2 = S$. Hence the hypothesis of $\Delta 5$ is not satisfied.

Therefore, Δ satisfies $\Delta 1$, $\Delta 2$, $\Delta 3$, and $\Delta 5$, but $q_3 \Delta \{q_1, q_2\} = S'_{q_3}$ so that S'_{q_3} is not closed. Then by lemma 1.8 S_{q_3} is not irreducible in L'. Hence $\Delta 4$ is independent of the other four postulates.

We next consider the independence of $\Delta 3$. From $\Delta 1$, $\Delta 2$, $\Delta 4$, and $\Delta 5$ we can derive the following weak form of $\Delta 3$ which becomes equivalent to $\Delta 3$ if $S'_{q'}$ is closed for all $q' \in Q$.

($\Delta 3'$.) If $q \Delta q'$, then either $q = q'$ or $q \Delta S'_{q'}$. By $\Delta 2$, $q \Delta q'$ implies $q \Delta S_{q'} = S'_{q'} \vee q'$. Also $q'' \subset q'$ implies $q'' \Delta S'_{q'}$, and hence by $\Delta 5$ either $q \Delta S'_{q'}$ or $q' \Delta S'_{q'} \vee q$. Since $(S'_{q'} \vee q) \Delta q'$, the latter alternative implies $q' \in S'_{q'} \vee q$ by $\Delta 4$, from which it follows that $q' = q$ since $q' \notin S'_{q'}$. Hence either $q = q'$ or $q \Delta S'_{q'}$ which is the conclusion of $\Delta 3$.

The following example shows that this result cannot be sharpened, because $\Delta 1$, $\Delta 2$, $\Delta 3'$, $\Delta 4$, and $\Delta 5$ are satisfied but $\Delta 3$ is not. Consider the lattice shown in figure 2 and the set Q of join irreducibles as indicated.

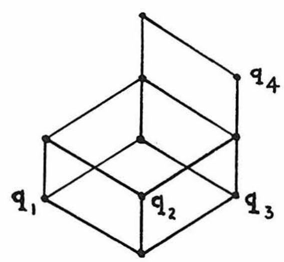

Figure 2.

Let ρ be the usual rank function, defined as follows:

(1). $\rho(z) = 0$.

(2). $a \succ b$ implies $\rho(a) = \rho(b) + 1$.

For any subset $S \subseteq Q$, let $q \,\Delta\, S$ if and only if either $q \in S$ or $\rho(q) < \rho(\cup S)$. We show that $\Delta 1$, $\Delta 2$, $\Delta 4$, and $\Delta 5$ are satisfied; consequently $\Delta 3'$ also holds.

(Δ 1). Let $q'' \leq q'$. Then either $q'' = q'$ in which case $q'' \in S \vee q'$, or $q'' < q'$ in which case $\rho(q'') < \rho(q') \leq \rho(\cup(S \vee q'))$. In either case, $q'' \,\Delta\, S \vee q'$.

(Δ 2). Let $q \,\Delta\, S$ and $S \,\Delta\, T$. If $q \in S$ or $q \in T$, then $q \,\Delta\, T$. Likewise if $q_4 \in S$, then $q_4 \in T$ because $\rho(q_4) \geq \rho(\cup(Q - q_4))$. But $q_4 \in T$ implies $q \,\Delta\, T$. Hence we may assume $q \notin S \vee T$ and $q_4 \notin S \vee T$.

If $q = q_1$, then either $q_1 \in S$ which implies $q \,\Delta\, T$, or $\rho(\cup S) > 1$ which implies $\{q_2, q_3\} \subseteq S$, which implies $q_2 \,\Delta\, T$ and $q_3 \,\Delta\, T$, which implies either $\{q_2, q_3\} \subseteq T$ or $\rho(\cup T) > 1$. In either case, then, $\rho(\cup T) > 1 = \rho(q_1)$ and $q_1 \,\Delta\, T$.

If $q = q_2$ or $q = q_3$, the argument is parallel to $q = q_1$.

If $q = q_4$, then $q_4 \Delta S$ implies $q_4 \in S$. Then $S \Delta T$ implies $q_4 \Delta T$. Hence $\Delta 2$ holds for the lattice considered.

($\Delta 4$). In the statement of $\Delta 4$ if $q = q_1$ then $S \Delta q$ implies $S = q_1$. Hence $q \in S$. The cases for $q = q_2$ and $q = q_3$ are similar. If $q = q_4$, $q \Delta S$ implies $q \in S$ since $q_4 \not\Delta Q - q_4$. Hence $\Delta 4$ holds.

($\Delta 5$). In the statement of $\Delta 5$ if $q = q_4$, then $q' \Delta S \vee q$ for any q' and any S. If $q_4 \in S$, $q \Delta S$ for any $q \in Q$. If $q' = q_4$, the hypothesis of $\Delta 5$ requires that $q_i \Delta S$ for $i = 1, 2, 3$. Hence $q \Delta S$ if $q \neq q_4$, and if $q = q_4 = q'$ the exchange property holds trivially. Hence we may exclude all these possibilities in the remaining cases.

Let $q = q_1$. If $q_1 \in S$, then $q \Delta S$. If $q' = q_1$, $q' \Delta S \vee q$. If $q' = q_2$, then $q \Delta S \vee q'$ implies $q_3 \in S$, which implies $q' \Delta S \vee q$. If $q' = q_3$, then $q \Delta S \vee q'$ implies $q_2 \in S$, which implies $q' \Delta S \vee q$.

The cases for $q = q_2$ and $q = q_3$ are similar to $q = q_1$, and hence $\Delta 5$ holds.

However, $q_1 \Delta q_4$ but $q_1 \not\leq q_4$, and hence $\Delta 3$ does not hold. The lattice of closed subsets is illustrated in figure 3.

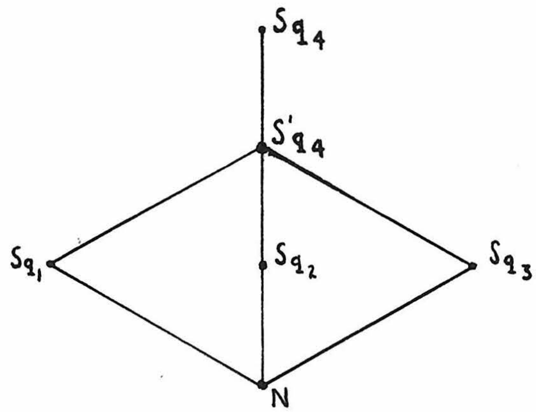

Figure 3.

PART II. AN IMBEDDING THEOREM

Section 2.1. Introduction.

Let L be a finite dimensional lattice, and let Q be the set of join irreducible elements of L. In part I we showed that any dependence relation Δ, defined over Q and satisfying $\Delta 1 - \Delta 5$, can be used to construct an upper semi-modular lattice L' whose set Q' of completely join irreducible elements is isomorphic to Q. In general S_a is not a closed set under the dependence relation for every $a \in L$, and hence L is not imbedded in L'. However dependence relation (A) does imbed L as a lattice within L'.

We now determine a necessary and sufficient condition that a finite dimensional lattice L can be imbedded as a sublattice of an upper semi-modular lattice L', where Q' is isomorphic to Q. Dilworth (Dilworth [1]) has shown that L can be imbedded as a sublattice of an upper semi-modular point lattice M, where the usual lattice rank in M of the image of an element of L is predetermined by an integral valued semi-modular function defined over L. The imbedding lattice L' which we shall define is a sublattice of M, and therefore is a refinement of Dilworth's imbedding process. As we shall see, L' contains the same number of points as L.

Section 2.2. The Dependence Relation.

Let L be any finite dimensional lattice over which an integral valued function σ is defined with the properties:

(σ 1) $\sigma(z) = 0$.

(σ 2) $a \supset b$ implies $\sigma(a) > \sigma(b)$.

(σ 3) $\sigma(a) + \sigma(b) \geq \sigma(a \cup b) + \sigma(a \cap b)$.

In the paper mentioned in the last section, Dilworth associated with each join irreducible element $q \in L$ an abstract unordered set S_q of $\sigma(q) - \sigma(a)$ elements p, where $q \succ a$ in L. Let P be the collection of all such sets, it being understood that the sets associated with distinct irreducibles are disjoint. With each element $a \in L$ associate the set $S_a = \bigvee_{q \leq a} S_q$. For any subset T of P let n(T) denote the number of elements in T.

Definition 2.1. A subset $S \subseteq P$ is independent if and only if, for every subset $T \subseteq S$ and every element $a \in L$ such that $T \subseteq S$, $n(T) \leq \sigma(a)$.

Corollary. Any subset of an independent set is independent.

Definition 2.2. A subset $S \subseteq P$ which is not independent is said to be dependent.

Hence S is dependent if and only if there exist a subset $T \subseteq S$ and an element $a \in L$ such that $T \subseteq S_a$ and $n(T) > \sigma(a)$.

The dependence relation which Dilworth used is defined between the elements p and the subsets S of P as follows:

Definition 2.3. p D S if and only if

either (1) $p \in S$,

or (2) there exists an independent subset $T \subseteq S$ such that $T \vee p$ is dependent.

In the paper mentioned it is proved that D satisfies D1 - D3 of section 1.1, and hence the closed subsets of P form an upper semi-modular point lattice L'. Furthermore Dilworth proved that the natural mapping $a \rightarrow S_a$ imbeds L as a sublattice within L', and that if r is the usual rank function in L', then $\sigma(a) = r(S_a)$.

The definitions of independent set and the dependence relation have the desired property that if $p \in S$ and S is independent, p does not depend on $S - p$. That is, no member of an independent set depends on the remainder of the set. We shall now modify this dependence relation to acquire a smaller imbedding lattice.

First enlarge L to a lattice $\bar{L}$ as follows: between every pair of elements $q, b \in L$, such that q is join irreducible and $q \succ b$, introduce a complete chain of $\sigma(q) - \sigma(b) - 1$ new elements q_i which are to be join and meet irreducible in $\bar{L}$. Then we have

$$q \succ q_{k-1} \succ \dots \succ q_1 \succ b$$

where $k = \sigma(q) - \sigma(b)$. These chains we shall call construction chains. Only the maximal and minimal elements of each chain are elements of L, and if two chains are distinct they have distinct maximal elements but possibly the same minimal element. The set $\bar{L}$ is the set sum of L and the elements of the construction chains. With each element $a \in \bar{L}$ we can associate uniquely two elements a_1 and a_2 of L as follows. If $a \notin L$, let a_1 be the minimal element of the construction chain in which a appears, and let a_2 be the maximal element of the same chain. If $a \in L$, let $a_2 = a = a_1$.

Define over $\bar{L}$ a partial ordering in the following manner. If a and b are elements of L which are not in the same construction chain, $a \geq b$ in $\bar{L}$ if and only if $a_1 \geq b_2$ in L. If a and b are in the same construction chain, $a \geq b$ in $\bar{L}$ if and only if $a \succ \dots \succ b$ in that chain. It is trivially true that $a \geq a$ in $\bar{L}$ and that $a \geq b$ and $b \geq a$ implies $a = b$.

Let $a \geq c \geq b$. If a, c, and b are not all in the same construction chain, we have $a_1 \geq c_2 \geq c_1 \geq b_2$ in L, and therefore $a \geq b$ in $\bar{L}$. If a and c are not in the same chain but b and c are, then we have $a_1 \geq c_2 = b_2$, and $a \geq b$ in $\bar{L}$. If a and c are in the same chain but b and c are not, we have $a \succ \ldots \succ c \succ \ldots \succ a_1 = c_1 \geq b_2$, and $a \geq b$ in $\bar{L}$. Finally if a, b, and c are in the same chain $a \succ \ldots \succ c \succ \ldots \succ b$ and $a \geq b$ in $\bar{L}$. Hence transitivity holds and a partial ordering of $\bar{L}$ is defined.

We now show that for $a, b \in L$ there exist a unique minimal element c and a unique maximal element d such that $c \geq a$, $c \geq b$, $a \geq d$, and $b \geq d$. If $a \geq b$ take $c = a$ and $d = b$, which are clearly the unique elements desired. If a and b are unrelated let $c = a_2 \underset{L}{\cup} b_2$. Suppose $x \geq a$ and $x \geq b$ in $\bar{L}$. Since a and b are unrelated, they cannot appear in the same construction chain. Hence $a_2 \neq b_2$ since otherwise a_2 would be the maximal element of two distinct construction chains. But by construction the maximal element of each chain is join irreducible in L and hence is maximal in only one chain. Then $x \geq a_2$ and $x \geq b_2$, which implies $x_1 \geq a_2$ and $x_1 \geq b_2$ in L. Then $x \geq x_1 \geq a_2 \underset{L}{\cup} b_2 = c$, and c is minimal in $\bar{L}$ containing a and b. Let $d = a_1 \underset{L}{\cap} b_1$. Suppose $a \geq x$ and $b \geq x$ in $\bar{L}$. Since a and b are unrelated, they are not in the same construction chain, and hence $a_1 \geq x_2$ and $b_1 \geq x_2$ in L. Then $d = a_1 \underset{L}{\cap} b_1 \geq x_2 \geq x$, and hence d is maximal. Hence $\bar{L}$ is a lattice. Furthermore it is clear that the elements introduced by the construction are both join and meet irreducible in $\bar{L}$, and that if q is join irreducible in L, q is join irreducible in $\bar{L}$.

From the definitions it is clear that the set of all elements of $\bar{L}$ except those introduced by the construction forms a sublattice of $\bar{L}$ which is isomorphic to L. For simplicity we call this sublattice L and observe that L and $\bar{L}$ have the same number of points.

Over $\bar{L}$ define the functional ρ as follows:

(1). $\rho(a) = \sigma(a)$ if $a \in L$

(2). $\rho(a) = \sigma(a_1) + k$ if $a \notin L$, where a_1 is the minimal element of the construction chain and k is the length of the complete chain from a to a_1. Clearly ρ has the property (σ3) when a and b are elements of L.

Let $\bar{Q}$ be the set of join irreducible elements of $\bar{L}$, excluding the null element z of $\bar{L}$. With every element $a \in \bar{L}$ associate the set S_a of all $q \in \bar{Q}$ such that $q \leq a$ in $\bar{L}$. The null set is associated with z. For any subset $S \subseteq \bar{Q}$ let $n(S)$ denote the number of elements in S.

Definition 2.4. A set $S \subseteq \bar{Q}$ will be called quasi-independent if and only if for every subset $T \subseteq S$ and every element $a \in L$ such that $T \subseteq S_a$, $n(T) \leq \rho(a)$.

Definition 2.5. A set which is not quasi-independent will be called quasi-dependent.

Let $\{S\}$ denote the set of all $q \in \bar{Q}$ such that $q \leq q_s$ in $\bar{L}$ for some $q_s \in S$.

Definition 2.6. $q \,\Delta\, S$ if and only if

either (1) $q \in \{S\}$,

or (2) there exists a quasi-independent subset $T \subseteq \{S\}$ such that $T \vee q$ is quasi-dependent.

From the definition it is clear that an element q of a quasi-independent set S will depend upon $S - q$ if $q \subset q_s$ for some $q_s \in S$. Therefore we adopt the term quasi-independent to emphasize that "independence" has a slightly different meaning here than in the previous definition. Obviously D is a stronger definition of dependence than Δ in that there will be fewer closed sets under the latter relation. This is a consequence of the partial ordering imposed on $\bar{Q}$ by the construction of $\bar{L}$ from L.

By referring to definition 2.3 we see that definition 2.6 could be stated, equivalently, $q \, \Delta \, S$ if and only if $q \, D \, \{S\}$, since only the terminology of the definitions differs, and not the actual conditions. We now verify that Δ satisfies $\Delta 1 - \Delta 5$.

($\Delta 1$). If $q \subseteq q'$, then $q \in \{S \vee q'\}$, and hence $q \, \Delta \, S \vee q'$.

($\Delta 2$). Let $q \, \Delta \, S$ and $S \, \Delta \, T$. Let $q' \in \{S\}$. Then by definition $q' \subseteq q_s$ for some $q_s \in S$, and by hypothesis $q_s \, \Delta \, T$. If $q_s \in \{T\}$, then $q' \in \{T\}$, and $q' \, \Delta \, T$. Otherwise there exists a quasi-independent subset $T' \subseteq \{T\}$, such that $T' \vee q_s$ is quasi-dependent, and hence there exist a subset $T'' \subseteq T' \vee q_s$ and an element $a \in L$ such that $T'' \subseteq S_a$ and $n(T'') > \rho(a)$. Then T'' is quasi-dependent, and hence $q_s \in T''$, since otherwise T'' is a subset of T' and hence would be quasi-independent. Write $T'' = R \vee q_s$, where R as a subset of T' is quasi-independent. Then we have

$$n(T'') = n(R \vee q_s) = n(R) + 1 > \rho(a).$$

Then $n(R \vee q') = n(R) + 1 > \rho(a)$, and since $q' \subseteq q_s$ we have $R \vee q' \subseteq S_a$.

Hence $R \vee q'$ is quasi-dependent, and so $q' \Delta T$ for all $q' \in \{S\}$. Thus $S \Delta T$ implies $\{S\} \Delta T$, which implies $\{S\} D \{T\}$.

But $q \Delta S$ implies $q D \{S\}$, and since D satisfies D2 we have $q D \{T\}$, and hence $q \Delta T$. Hence the relation Δ is transitive.

(Δ3). Let $q'' \Delta q'$. Then $q'' D \{q'\} = S_{q'}$. If $q'' \in S_{q'}$, we have $q'' \leq q'$ and are through. Otherwise there exists a quasi-independent subset $T \subseteq S_{q'}$ such that $T \vee q''$ is quasi-dependent. Then there exist a subset $T' \subseteq T \vee q''$ and an element $a \in L$ such that $T' \subseteq S_a$ but $n(T') > \rho(a)$. As before $q'' \in T'$, since otherwise T' is quasi-independent which contradicts $n(T') > \rho(a)$. Write $T' = R \vee q''$, where R is quasi-independent. Then since $R \subseteq T \subseteq S_{q'}$, we have

$$n(R) \leq \rho(q') \text{ and } n(R) \leq \rho(a).$$

Also $n(R \vee q'') = n(R) + 1 > \rho(a)$, and hence

$$\rho(a) = n(R) \leq \rho(q').$$

Consider $\bigcup R$ in $\bar{L}$. Clearly $a \supseteq \bigcup R$. Let c be minimal in L such that $\bigcup R \subseteq c$. Then $R \subseteq S_c$, and hence $\rho(c) \geq n(R) = \rho(a)$. But since c is minimal containing $\bigcup R$, $c \subseteq a$, and we have $\rho(c) \leq \rho(a)$ which implies $c = a$.

If $c = \bigcup R$, then $q' \supseteq c = a \supseteq q''$, and we are through. Otherwise $\bigcup R \notin L$, and hence $\bigcup R$ is both join and meet irreducible by construction. Let $q = \bigcup R$. Then every element of $\bar{L}$ in the chain from q to a must also be join and meet irreducible. But $q \subseteq q \cup q'' \subseteq a$, and hence either $q \cup q'' = q$, in which case $q' \supseteq q \supseteq q''$ and we are through, or $q \cup q'' = q''$, in which case we have $q \subseteq q'' \subseteq a$.

But also $q \subseteq a \cap q' \subseteq a$, and hence either $a \cap q' = a$, in which case $q'' \subseteq a \subseteq q'$ and we are through, or $a \cap q' = q'$ in which case $q \subseteq q' \subseteq a$. Hence either $q'' \subseteq q'$ or $q'' \supset q'$. The first statement is the conclusion of $\Delta 3$, and we show next that the second statement leads to a contradiction.

Let $b \in L$ be maximal such that $b \subseteq q'$, and let $\rho(a) = \rho(b) + m$. Let $\rho(q') = \rho(b) + k$, where $m \geq k \geq 0$. Then $\rho(b) = \rho(q')-k < \rho(q'')-k \leq \rho(a)-k$. Since $R \subseteq S_{q'}$, there are at most k elements in R which are not in S_b. Hence $n(R) - k \leq n(R \wedge S_b)$. Thus we have

$$n(R \wedge S_b) \geq n(R) - k = \rho(a) - k > \rho(b).$$

But $R \wedge S_b \subseteq S_b$ and $R \wedge S_b$ is quasi-independent as a subset of R. Hence $n(R \wedge S_b) \leq \rho(b)$, which is a contradiction. Thus $\Delta 3$ is satisfied.

($\Delta 4$). Let $q \Delta S$ and $S \Delta q$. By $\Delta 3$ $S \Delta q$ implies $S \subseteq S_q$, and hence $q \supseteq q_s$ for all $q_s \in S$. Suppose $q \Delta S$ implies $q \in \{S\}$. Then $q \subseteq q_s$ for some $q_s \in S$ and hence $q_s = q$, and we are through. We may assume therefore that there exists a quasi-independent subset $T \subseteq \{S\}$ such that $T \vee q$ is quasi-dependent. Then, as before, there exists a quasi-independent set $R \subseteq T$ such that $n(R \vee q) > \rho(a)$ for some $a \in L$ such that $R \vee q \subseteq S_a$.

Let $c \in L$ be minimal such that $q \subseteq c$, and let $b \in L$ be maximal such that $q \supset b$. Then $S_c \subseteq S_a$ since $q \subseteq c \subseteq a$, and we have $n(R \vee q) > \rho(a) \geq \rho(c)$. Also $T \subseteq \{S\} \subseteq S_q \subseteq S_c$, and hence $n(T) \leq \rho(c)$ since T is quasi-indedependent. Therefore

$$\rho(c) \leq \rho(a) < n(R \vee q) \leq n(T) + 1 \leq \rho(c) + 1,$$

and hence $$n(T) = \rho(c).$$

Let $\rho(q) = \rho(b) + k$ and $\rho(c) = \rho(b) + m$ where $m \geq k \geq 0$. Then $\rho(b) = \rho(c) - m \leq \rho(c) - k$. Since $T \subseteq S_q$, T contains at most k elements which are not in S_b. Hence $n(T \wedge S_b) \geq n(T) - k$ where the equality holds only if $q \in T$. But $T \wedge S_b$ is quasi-independent as a subset of T. Hence we have

$$n(T \wedge S_b) \leq \rho(b) \leq \rho(c) - k = n(T) - k \leq n(T \wedge S_b),$$

which implies $n(T \wedge S_b) = n(T) - k$, and $q \in T \subseteq \{S\}$. Then $q \leq q_s$ for some $q_s \in S$. Thus $q = q_s$, and $q \in S$.

($\Delta 5$). Let $q \,\Delta\, S \vee q'$ where q' is such that $q'' \subset q'$ implies $q'' \,\Delta\, S$. If $q \,\Delta\, S$ we are through, so we may assume $q \not\Delta S$. Then $q \not\leq q'$, and $q \not\leq q_s$ for all $q_s \in S$. If $q' \leq q$, then $q' \,\Delta\, S \vee q$ and we are through.

Under the assumption that none of these cases occur, there exists a quasi-independent subset $T \subseteq \{S \vee q'\}$ such that $T \vee q$ is quasi-dependent. If $T \subseteq \{S\}$ then $q \,\Delta\, S$, contrary to assumption. Hence there is an element $q'' \in T$ such that $q'' \subset q', q'' \notin \{S\}$. Write $T = S^* \vee S^*_{q'}$ where $S^* \subseteq \{S\}$ and $S^*_{q'} = T - S^* \subseteq S_{q'}$. If $q' \notin S^*_{q'}$ then $S^*_{q'} \,\Delta\, S$ by hypothesis, and hence $T \,\Delta\, S$. But since T is quasi-independent and $T \vee q$ is quasi-dependent, $q \,\Delta\, T$. Then by $\Delta 2$ $q \,\Delta\, S$, again contrary to assumption. Hence we may assume $q' \in T$. Now write $T = R \vee q'$ where $R \,\Delta\, S$ and R is quasi-independent. If $R \vee q$ is quasi-dependent, $q \,\Delta\, R$ and hence $q \,\Delta\, S$, a contradiction. Then since $T \vee q = R \vee q \vee q'$ is quasi-dependent while $R \vee q$ is quasi-independent, $q' \,\Delta\, R \vee q$. But $R \vee q \,\Delta\, S \vee q$, and hence $q' \,\Delta\, S \vee q$.

Therefore Δ satisfies $\Delta 1 - \Delta 5$, and by theorem 1.3 the closed subsets of $\bar{Q}$ form a complete upper semi-modular lattice, L', whose completely join irreducible elements form a partially ordered set isomorphic to $\bar{Q}$.

Lemma 2.1. For every $b \in \bar{L}$, S_b is closed.

Proof: If b is join irreducible then S_b is closed by lemma 1.6. Hence we consider $b \in L$ and let $q \Delta S_b$. If $q \subseteq \{S_b\} = S_b$, we are through. Otherwise there exists a quasi-independent subset $T \subseteq S_b$ such that $T \vee q$ is quasi-dependent. Then $T' \subseteq T \vee q$ exists and $a \in L$ exists such that $T' \subseteq S_a$ but $n(T') > \rho(a)$. We may assume $q \in T'$, since otherwise T' is quasi-independent and $n(T') \leq \rho(a)$, which contradicts $n(T') > \rho(a)$. Write $T' = R \vee q$, where R is quasi-independent. Then we have $n(R \vee q) = n(R) + 1 > \rho(a)$ or $n(R) \geq \rho(a)$. But since $R \subseteq S_a$ and $R \subseteq S_b$, $R \subseteq S_a \wedge S_b = S_{a \cap b}$, and hence $n(R) \leq \rho(a \cap b)$. Thus we have $\rho(a) \leq n(R) \leq \rho(a \cap b)$, and therefore $a = a \cap b$. But this implies $a \subseteq b$, and since $q \subseteq a$ we have $q \subseteq b$ and $q \in S_b$.

Lemma 2.2. If $a \supseteq b$ in $\bar{L}$, then $n(S_a) - \rho(a) \geq n(S_b) - \rho(b)$.

Proof: Let a_1 and b_1 be maximal elements of L such that $a \supseteq a_1$ and $b \supseteq b_1$. By the construction of $\bar{L}$ from L, $S_a = S_{a_1} \vee S^*$ where S^* is the set of all elements in the chain from a to a_1, excluding a_1. Also by definition $\rho(a) = \rho(a_1) + n(S^*)$. Hence we have

$$n(S_a) - \rho(a) = n(S_{a_1} \vee S^*) - \rho(a) = n(S_{a_1}) + n(S^*) - \rho(a)$$
$$= n(S_{a_1}) + n(S^*) - \rho(a_1) - n(S^*) = n(S_{a_1}) - \rho(a_1).$$

Thus the lemma holds in $\bar{L}$ if it holds in L, and we may assume $a, b \in L$.

The conclusion is trivial for a = z, so assume it holds for all a such that $\rho(a) < k$. Let $\rho(a) = k$. If a = b the lemma follows trivially. Hence we may assume that in L $a \succ a_1 \supseteq b$. If a is join irreducible then $S_a = S_{a_1} \vee S^* \vee a$, where S^* is the set of $\rho(a) - \rho(a_1) - 1$ irreducibles introduced by construction between a and a_1. Then we have $n(S_a) = n(S_{a_1}) + \rho(a) - \rho(a_1)$. By the induction hypothesis $n(S_{a_1}) - \rho(a_1) \geq n(S_b) - \rho(b)$ and hence $n(S_a) - \rho(a) \geq n(S_b) - \rho(b)$.

If a is join reducible, let $a \succ a_2 \neq a_1$. Then we have
$n(S_{a_2}) - n(S_{a_2 \cap a_1}) \geq \rho(a_2) - \rho(a_2 \cap a_1) \geq \rho(a_2 \cup a_1) - \rho(a_1) = \rho(a) - \rho(a_1)$.
If $q \subseteq a_2$ but $q \nsubseteq a_2 \cap a_1$, then $q \subseteq a$ but $q \nsubseteq a_1$. Hence

$$n(S_a) - n(S_{a_1}) \geq n(S_{a_2}) - n(S_{a_2 \cap a_1}) \geq \rho(a) - \rho(a_1).$$

Then $n(S_a) - \rho(a) \geq n(S_{a_1}) - \rho(a_1) \geq n(S_b) - \rho(b)$, where the last inequality holds by the induction hypothesis. Since L is finite dimensional, the lemma holds by induction.

Corollary. For any $q \in \bar{L}$, $n(S_a) - \rho(a) \geq 0$.

Proof: Choose b = z and apply lemma 2.2.

Lemma 2.3. For every $a \in \bar{L}$ there exists a quasi-independent set $T \subseteq S_a$ such that $n(T) = \rho(a)$.

Proof: First assume $a \in L$ and let R be a maximal quasi-independent subset of S_a. Then $n(R) \leq \rho(a)$, and we assume $n(R) < \rho(a)$. Let B be the set of all $b \in L$ such that $n(R_b) = \rho(b)$ for some $R_b \subseteq R \wedge S_b$, and let $d = \cup B$. Then since R_b is quasi-independent and since $R_b \subseteq S_a \wedge S_b = S_{a \cap b}$, we have $\rho(b) = n(R_b) \leq \rho(a \cap b)$. Hence $b = a \cap b$ and $b \subseteq a$ for all $b \in B$. Thus $d = \cup B \subseteq a$.

Let b_1 and b_2 be elements of B, and let R_{b_1} and R_{b_2} be corresponding quasi-independent subsets of R such that $\rho(b_1) = n(R_{b_1})$ and $\rho(b_2) = n(R_{b_2})$. Then $R_{b_1} \vee R_{b_2} \subseteq S_{b_1 \cup b_2}$.
But $R_{b_1} \vee R_{b_2}$ is quasi-independent as a subset of R, so

$$n(R_{b_1} \vee R_{b_2}) \leq \rho(b_1 \cup b_2).$$

However we have

$$\begin{aligned} n(R_{b_1} \vee R_{b_2}) &= n(R_{b_1}) + n(R_{b_2}) - n(R_{b_1} \wedge R_{b_2}) \\ &\geq \rho(b_1) + \rho(b_2) - \rho(b_1 \cap b_2) \\ &\geq \rho(b_1 \cup b_2). \end{aligned}$$

Hence if b_1 and b_2 are in B so is $b_1 \cup b_2$, and therefore $d \in B$. Let R_d be a corresponding subset of R such that $\rho(d) = n(R_d)$.

Suppose $d = a$. Then a is the union of a finite number of the b's since $\bar{L}$ is finite dimensional, and hence $a \in B$ which contradicts $n(R) < \rho(a)$.

Hence we may assume $d \subset a$. Then there exists a $q_a \leq a$ such that $q_a \nleq d$, and hence $q_a \notin R_b$ for every $b \in B$. Then we have $n(R \vee q_a) \leq \rho(a)$. Let c be any element of L such that $q_a \leq c$. Then $c \notin B$, since $q_a \nleq d = \bigcup B$. Hence for any $R_c \subseteq R \wedge S_c$ we have $n(R_c) < \rho(c)$, and therefore $n(R_c \vee q_a) \leq \rho(c)$. Hence $R \vee q_a$ is quasi-independent, and the maximal property of R implies that $q_a \in R$ for every $q_a \leq a$ such that $q_a \nleq d$. Hence we have $S_a = R \vee S_d$. By lemma 2.2 we have

$$n(S_d) - \rho(d) \leq n(S_a) - \rho(a),$$

and by assumption $\quad n(R) < \rho(a).$

Then

$$\begin{aligned} n(S_d) + n(R) - \rho(d) < n(S_a) &= n(R \vee S_d) \\ &= n(R) + n(S_d) - n(R \wedge S_d), \end{aligned}$$

which implies $\quad \rho(d) > n(R \wedge S_d).$

But we have $R_d \leq R \wedge S_d$, which implies

$$\rho(d) = n(R_d) \leq n(R \wedge S_d),$$

which is a contradiction. Hence the assumption $n(R) < \rho(a)$ must be false and R itself had the desired property $n(R) = \rho(a)$. Thus the lemma holds in L.

If $a \notin L$, let a_1 be the minimal element of the construction chain in which a appears. Then there exists a quasi-independent set T_{a_1} such that $n(T_{a_1}) = \rho(a_1)$ and $T_{a_1} \leq S_{a_1}$. Let $T = T_a \vee T^*$ where T^* is the set of all elements in the chain from a to a_1, excluding a_1. Then $n(T^*) = \rho(a) - \rho(a_1) \leq \rho(c) - \rho(a_1)$ for any $c \in L$ such that S contains elements of T^*. Hence T is quasi-independent, and

$$n(T) = n(T_{a_1}) + n(T^*) = \rho(a_1) + \rho(a) - \rho(a_1) = \rho(a).$$

Therefore the lemma holds in $\bar{L}$.

<u>Lemma 2.4.</u> To each pair of elements $a,b \in L$ there corresponds a quasi-independent set M such that $M \leq S_a \vee S_b$, $n(M) = \rho(a \cup b)$, and $n(M \wedge S_a \wedge S_b) = \rho(a \cap b)$.

<u>Proof</u>: We have from lemma 2.2

$$\begin{aligned} n(S_a \vee S_b) - \rho(a \cup b) &= n(S_a) + n(S_b) - n(S_a \wedge S_b) - \rho(a \cup b) \\ &\geq n(S_a) + n(S_b) - n(S_{a \cap b}) - \rho(a) - \rho(b) + \rho(a \cap b) \\ &\geq n(S_a) - \rho(a) + \{[n(S_b) - \rho(b)] - [n(S_{a \cap b}) - \rho(a \cap b)]\} \\ &\geq n(S_a) - \rho(a) \geq 0. \end{aligned}$$

Let M_1 be a maximal quasi-independent set of $S_{a \cap b}$. By lemma 2.3 $n(M_1) = \rho(a \cap b)$. Let C_1 be the set of all $c \in L$ such that, for some $M'_1 \leq S_c \wedge M_1$, $n(M'_1) = \rho(c)$. Then as in lemma 2.3, $c \leq a \cap b$ for all $c \in C_1$, and the set C_1 is algebraically closed under lattice union.

Now adjoin to M_1 any element $q_1 \in S_a - S_{a \cap b}$. Let $T_2 = M_1 \vee q_1$, and consider $R \subseteq S_c \wedge T_2$ for any $c \in L$. If $q_1 \notin R$, then $n(R) \leq \rho(c)$ since R is then a subset of the quasi-independent set M_1. Write $R = M'_1 \vee q_1$. Then $c \nsubseteq a \cap b$, since $q_1 \nsubseteq a \cap b$, and hence $c \notin C$. But then $n(M'_1) < \rho(c)$ and $n(R) = n(M'_1) + 1 \leq \rho(c)$. Hence T_2 is quasi-independent. Let $a_2 = q_1 \cup (a \cap b)$. Then $a_2 \subseteq a$, and $T_2 \subseteq S_{a_2}$. Let C_2 be the set of all $c \in L$ such that $\rho(c) = n(R_2)$ for some corresponding subset $R_2 \subseteq T_2 \wedge S_c$. Since R_2 is quasi-independent and $R_2 \subseteq S_{a_2} \wedge S_c$, we have $\rho(c) = n(R_2) \leq \rho(a_2 \cap c)$ and hence $c \subseteq a_2$ for all $c \in C_2$. Extend T_2 to a maximal quasi-independent set $M_2 \subseteq S_{a_2}$. Then $n(M_2) = \rho(a_2) \leq \rho(a)$.

If $\rho(a_2) < \rho(a)$, we adjoin to M_2 an element $q_2 \in S_a - S_{a_2}$, getting a quasi-independent set $T_3 = M_2 \vee q_2$. Let $a_3 = a_2 \cup q_2$. Then $a_3 \subseteq a$, and we can extend T_3 to M_3 which is maximal quasi-independent in S_{a_3}. Then $n(M_3) = \rho(a_3) \leq \rho(a)$. In a finite number of extensions we thus construct a maximal quasi-independent set $M_k \subseteq S_{a_k} = S_a$. Then $n(M_k) = \rho(a) \leq \rho(a \cup b)$. Let C_k be the set of all $c \in L$ such that $\rho(c) = n(R_k)$ for some corresponding subset $R_k \subseteq M_k \wedge S_c$. Then as before $c \subseteq a_k = a$ for all $c \in C_k$.

Let q_k be any element in $S_b - S_{a \cap b}$, and consider $T_{k+1} = M_k \vee q_k$. By the same argument as before, T_{k+1} is quasi-independent, since $q_k \subseteq c$ implies $c \notin C_k$. Let $a_{k+1} = a \cup q_k \subseteq a \cup b$, and if T_{k+1} is not maximal in $S_{a_{k+1}}$, enlarge T_{k+1} to a maximal quasi-independent subset $M_{k+1} \subseteq S_{a_{k+1}}$. Then $n(M_{k+1}) = \rho(a_{k+1}) \leq \rho(a \cup b)$.

We thus continue, always adjoining to M_j an element of $S_b - S_{a_j}$, until for some quasi-independent set M either $n(M) = \rho(a \cup b)$ or $n(M \wedge S_b) = \rho(b)$. In the first case we are through since $M \subseteq S_a \vee S_b$ by construction. Otherwise we have

$$n(M) = n(M \wedge S_a) + n(M \wedge S_b) - n(M \wedge S_a \wedge S_b)$$
$$= \rho(a) + \rho(b) - \rho(a \cap b) \geq \rho(a \cup b).$$

Since M is quasi-independent and $M \subseteq S_{a \cup b}$, we have $n(M) \leq \rho(a \cup b)$. Hence $n(M) = \rho(a \cup b)$, and the lemma holds.

Lemma 2.5. In the lattice L' of closed subsets of $\bar{Q}$ $S_a \cup S_b = S_{a \cup b}$ for $a, b \in L$.

Proof: $S_a \cup S_b$ is defined to be $\overline{S_a \vee S_b}$. Clearly $S_a \cup S_b \subseteq S_{a \cup b}$. Let $q \in S_{a \cup b}$, and let $M \subseteq S_a \vee S_b$ be such that $n(M) = \rho(a \cup b)$. M exists by lemma 2.4. Then either $n(M \vee q) = n(M)$ in which case $q \in M \subseteq S_a \vee S_b$, or $n(M \vee q) = n(M) + 1 > \rho(a \cup b)$, and hence $q \,\Delta\, M$. But then $q \,\Delta\, S_a \vee S_b$. Hence $q \in S_{a \cup b}$ implies $q \,\Delta\, S_a \vee S_b$, which implies $q \in \overline{S_a \vee S_b} = S_a \cup S_b$. Therefore $S_a \cup S_b \supseteq S_{a \cup b}$ and $S_a \cup S_b = S_{a \cup b}$.

By lemma 2.1 the natural mapping $a \rightarrow S_a$ imbeds $\bar{L}$ as a lattice within L'. Lemmas 1.5 and 2.5 then imply that the sublattice L of $\bar{L}$ is imbedded as a sublattice of L'. Since L' is upper semi-modular, the chain law holds in L', and the usual rank function r can be defined in L'.

We now show that L' is an isometric sublattice of D', the lattice of closed sets in Dilworth's imbedding. The abstract set over which the relation D (definition 2.3) is defined may be taken to be $\bar{Q}$. We recall from the definitions that $q \,\Delta\, S$ if and only if $q \, D \, \{S\}$.

Let $q \, D \, S$. Then $q \, D \, \{S\}$, since $S \subseteq \{S\}$, and $q \,\Delta\, S$. Hence $q \, D \, S$

implies $q \,\Delta\, S$. It follows that if S is Δ-closed then S is D-closed. Hence L' is a lattice within D'. Let S and T be elements of L'. Since lattice intersection in each case is set intersection, $S \cap_{L'} T = S \cap_{D'} T$. Let $q \,\Delta\, S \vee T$. Then $q \; D \; \{S \vee T\} = \{S\} \vee \{T\} = S \vee T$. Hence $q \,\Delta\, S \vee T$ if and only if $q \; D \; S \vee T$, and therefore $S \cup_{L'} T = S \cup_{D'} T$, which proves that L' is a sublattice of D'.

Suppose S covers T in L', and let R be an element of D' such that $S \supseteq R \supset T$. Let $q' \in R$ be minimal in $\bar{Q}$ such that $q' \notin T$. Then for any $q \in S$, $q \,\Delta\, T \vee q'$ since $S \succ T$ in L'. Hence $q \; D \; \{T \vee q'\} = \{T\} \vee \{q'\} = T \vee q'$, since q' is minimal not in T and T is closed. But $q \; D \; T \vee q'$ for all $q \in S$ implies that the D-closure of $T \vee q'$ contains S. Hence $R \supseteq S$ since R is closed and $R \supseteq T \vee q'$. Then $R = S$, and whenever $S \succ T$ in L', $S \succ T$ in D'.

Now for any $a \in L$, S_a is D-closed. Let T be a maximal independent subset of S_a. Then $n(T) = \sigma(a)$, and we write $T = q_1 \vee q_2 \vee \ldots \vee q_{\sigma(a)}$. Let $T_i = q_1 \vee \ldots \vee q_i$, and let C_i be the D-closure of T_i. Then for $j > i$, $q_j \in C_i$ implies there exists an independent subset $R_i \subseteq T_i$ such that $R_i \vee q_j$ is dependent, contradicting the independence of T_i. Hence if N is the null set, we have the complete chain

$$N \prec C_1 \prec C_2 \prec \ldots \prec C_{\sigma(a)} = S_a.$$

If r is the ordinary rank function in D', then $r(S_a) = \sigma(a)$. Since L' is imbedded isometrically in D', $r(S_a) = \sigma(a)$ in L', for every $a \in L$. Now let $q \in \bar{L}$ but $q \notin L$, and let a be the maximal element of L such that

$q \supset a$. Then corresponding to the construction chain

$$q = q_k \succ q_{k-1} \succ \dots \succ q_1 \succ a,$$

we have in L' the complete chain

$$S_q = S_{q_k} \succ S_{q_{k-1}} \succ \dots \succ S_{q_1} \succ S_a,$$

where $S_{q_i} = S_{q_{i-1}} \vee q_i$. Hence $r(S_q) = \sigma(a) + k = \rho(q)$.

Hence for all $a \in \bar{L}$, $r(S_a) = \rho(a)$.

Combining the results of this section we have

Theorem 2.1. A finite dimensional lattice L can be imbedded as a sub-lattice of an upper semi-modular lattice L', such that the set of completely join irreducible elements of L' is isomorphic to the set of join irreducible elements of L, if and only if it is possible to define over L an integral-valued function σ which satisfies

(σ1) $\sigma(z) = 0$,

(σ2) $a \supset b$ implies $\sigma(a) > \sigma(b)$,

(σ3) $\sigma(a) + \sigma(b) \geq \sigma(a \cup b) + \sigma(a \cap b)$.

If such a function can be defined over L, then an imbedding into L' exists such that if $a \longrightarrow a'$ then $\sigma(a) = r(a')$ where r is the usual rank function of L'.

PART III. A GENERAL IMBEDDING PROBLEM

Section 3.1. Introduction.

Let π be any lattice property (such as distributivity, modularity, upper or lower semi-modularity, etc.) which holds in L if and only if π holds in every quotient lattice a/b of L.

Definition 3.1. π is said to hold weakly in L if and only if π holds in every quotient lattice a/b where $b \neq z$ in L.

Starting with a lattice L' in which π holds, we can easily construct a lattice L in which π holds weakly such that a/b in L is isomorphic to a'/b' in L' where $b \neq z$. To do so, select any set S of elements c' of L', and remove from L' all elements d' such that $d' \leq c'$. Then adjoin a null element z, and in the resulting lattice π holds weakly.

We consider the converse problem of whether any lattice in which π holds weakly can be constructed in this way from a lattice in which π holds. Stated explicitly, if π holds weakly in L, can L be imbedded in a lattice L' such that π holds in L' and such that every quotient lattice a/b for $b \neq z$ in L is isomorphic to a'/b' in L'?

Section 3.2. Distributivity.

We first consider a finite dimensional lattice L which is weakly distributive. We shall prove that there exists a distributive lattice L' such that for $a \neq z$ in L every quotient lattice u/a in L is isomorphic to a corresponding quotient lattice u'/a' in L'. Furthermore L' is unique if every completely meet irreducible element of L' is the image of a meet irreducible element of L, and if every element of L' has a reduced representation as an intersection of completely meet irreducible elements.

Let D be the dual lattice of L. Then the descending chain condition holds in D, and every element of D can be written as the union of a finite number of join irreducible elements. The join irreducible elements of D are simply the meet irreducible elements of L. Every quotient lattice a/z for $a \neq u$ in D is distributive. Let Q be the set of join irreducible elements of D, and define over Q the dependence relation (B) of page 16,

(B) $q \Delta S$ if and only if $q \leq q_s$ for some $q_s \in S$.

By theorem 1.6 the natural mapping induced by this dependence relation imbeds D as a lattice within a completely distributive lattice D', and an element of D' is completely join irreducible if and only if it is the image of a join irreducible element of D.

Lemma 3.1. Let S_a denote the set of all join irreducible elements contained in a. In any distributive lattice $S_{a \cup b} = S_a \vee S_b$.

Proof: Let $q \in S_{a \cup b}$. Then $q = q \cap (a \cup b) = (q \cap a) \cup (q \cap b)$, which implies either $q = q \cap a$ or $q = q \cap b$, since q is join irreducible. Hence either $q \leq a$, or $q \leq b$, and therefore $S_{a \cup b} \leq S_a \vee S_b$. The opposite containing relation is obvious, and the lemma is proved.

We now prove that every quotient lattice a/z, $a \neq u$ in D, is isomorphic to the quotient lattice S_a/N in D'. Let $a \neq u$ be in D, let $S_a \supset S$ in D', and let $b = \bigcup S$ in D. Since the ascending chain condition holds in D, we have $b = q_1 \cup \ldots \cup q_n$, where all $q_i \in S$. Assume that there exists a join irreducible element q such that $q \in S_b - S$. Then we have

$$q = q \cap b = q \cap \bigcup q_i = \bigcup (q \cap q_i),$$

since a/z is distributive. But since q is join irreducible, $q = q \cap q_k$ for some $q_k \in S$, $k \leq n$. Hence $q \leq q_k$ and $q \Delta S$, which implies $q \in S$. This contradicts $q \in S_b - S$, and hence $S = S_b$. Thus every element of D' which is contained in S_a is the image of an element b which is contained in a. Within the quotient lattice S_a/N we have $S_{b \cup c} = S_b \vee S_c$ by lemma 3.1, and $S_{b \cap c} = S_b \wedge S_c$ by the properties of the dependence relation. Hence a/z is isomorphic to S_a/N in D'.

Clearly the dual lattice L' of D' is distributive, an element of L' is completely meet irreducible if and only if it is the image of a meet irreducible element of L, and every quotient lattice u/a, $a \neq z$ in L, is isomorphic to the quotient lattice u/a' in L'.

We next prove the uniqueness of L' which follows from the fact that the reduced representation of an element of a completely distributive lattice as an intersection of completely meet irreducible elements is unique. Let $a = \bigcap x_\alpha$ be any representation of a as an intersection of completely meet irreducible elements. The representation is said to be reduced if no x_α is superfluous.

Lemma 3.2. Any reduced representation of an element of a completely distributive lattice as an intersection of completely meet irreducible elements is unique.

Proof: Let $a = \bigcap x_\alpha = \bigcap x'_\beta$ be two reduced representations of a. Then we have for any x_α,

$$x_\alpha = x_\alpha \cup a = x_\alpha \cup \bigcap x'_\beta = \bigcap (x_\alpha \cup x'_\beta) = x_\alpha \cup x'_\beta \text{ for some } x'_\beta$$

since x_α is meet irreducible. Hence $x_\alpha \geq x'_\beta$ for some x'_β. Likewise for that x'_β we have $x'_\beta \geq x_\gamma$ for some x_γ. But since the representations are reduced, $x_\alpha = x_\gamma = x'_\beta$. Thus each element of either representation is in the other, and the two representations are identical.

Now suppose there are two distributive imbedding lattices L' and L" of L which preserve quotient lattices of L, are such that the completely meet irreducible elements of L' and L" are precisely the images of meet irreducible elements of L, and are such that every element of L' and L" has a reduced representation as an intersection of completely meet irreducible elements. Let S' be the set of unique reduced representations of elements of L' as intersections of completely meet irreducible elements x'_α, and let S" be the corresponding set of representations of elements of L". The completely meet irreducible elements of L' and L" are in a natural one-to-one correspondence since both are in one-to-one correspondence with the meet irreducibles of L. Suppose that $s'' = \bigcap x''_\gamma$ is a member of S" such that the corresponding representation $s' = \bigcap x'_\gamma$ is not in S'. Then $a' = \bigcap x'_\gamma$ is an element of L', and thus has a reduced representation $a' = \bigcap x'_\beta$ in S'. Then $a' = \bigcap x'_\beta = \bigcap x'_\gamma$, which is a contradiction since the representation is unique. Hence the set of meet irreducible elements of L completely determine the imbedding lattice, and any two imbedding lattices of the stated form are isomorphic. We have therefore proved

<u>Theorem 3.1</u>. Any finite dimensional weakly distributive lattice L can be imbedded in a distributive lattice L' such that the following conditions hold.

(1) An element of L' is completely meet irreducible if and only if it is the image of a meet irreducible element of L.

(2) For any elements a and $b \neq z$ in L, the quotient lattice a/b is isomorphic to a'/b' in L'.

(3) If every element of L' has a reduced representation as an intersection of completely meet irreducible elements, the imbedding is unique.

Section 3.3. Modularity.

In the construction of a counter example to the desired imbedding when the property π is modularity we use the following theorem (Birkhoff [1]).

Theorem. Any finite dimensional modular lattice is the direct product of a finite number of projective geometries, and any projective geometry is a complemented modular lattice.

Let P and Q be two finite projective planes with coordinatizing fields of characteristic p and q respectively, where p and q are distinct primes. Then consider the corresponding lattices L_P and L_Q joined as in figure 4 with one maximal element m in common and with a common null element added.

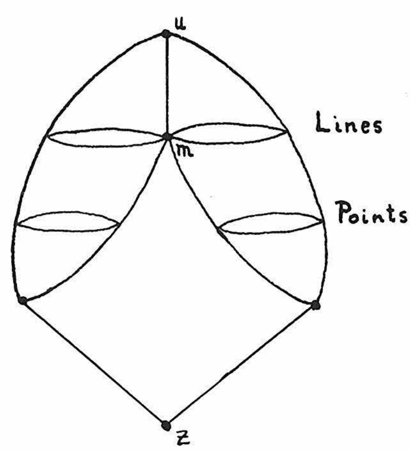

Figure 4.

This configuration forms a weakly modular lattice L if we define

(1) If $a,b \in L_p$, $\quad a \cup b = a \underset{L_p}{\cup} b$

$a \cap b = a \underset{L_p}{\cap} b.$

(2) If $a,b \in L_Q$, $\quad a \cup b = a \underset{L_Q}{\cup} b$

$a \cap b = a \underset{L_Q}{\cap} b.$

(3) If $a \in L_p$, $b \in L_Q$, $\quad a \cup b = (a \underset{L_p}{\cup} m) \underset{L_p}{\cup} (a \underset{L_Q}{\cup} m)$

$a \cap b = z.$

(4) If $a = z$, $\quad a \cup b = b$

$a \cap b = z.$

Assume that L can be imbedded in a modular lattice L' which preserves the quotient lattices of L and is finite dimensional. Then every quotient a'/b' where $a' \succ b'$ in L' is projective to u'/m', and therefore L' is simple. Thus L' is a projective geometry G' of dimension at least three, and L' has a coordinatizing field of some characteristic k. Since the original projective planes are subspaces of G', k must equal p; likewise, k must equal q, which contradicts the distinctness of p and q. Hence the imbedding problem is not possible for weakly modular lattices.

This example suggests that the imbedding problem for a weakly modular lattice is closely connected with projective geometry and that precise conditions under which the imbedding is possible are quite complex. Problems of this nature have been discussed by Hall and Dilworth (Hall and Dilworth [1]), and in this paper we do not pursue the question further.

Section 3.4. Semi-modularity.

As in the case of modularity, the desired imbedding is in general impossible for weakly upper semi-modular and weakly lower semi-modular lattices. We do not obtain precise criteria for determining when a given weakly semi-modular lattice L can be imbedded in a semi-modular lattice so as to preserve quotient lattices. The counter examples we shall consider show that such conditions are intricately related to the entire structure of L.

We first consider a weakly upper semi-modular lattice L, every element of which contains at least one point. Let P be the union of all points in L.

Theorem 3.2. L can be imbedded in an upper semi-modular lattice L' such that, for $a \neq z$ in L, u/a is isomorphic to u'/a', if and only if the quotient lattice P/z of L can be imbedded in the same way. Furthermore if L_p denotes an imbedding lattice for P/z, then the set union of L and L_p is an imbedding lattice L' of L.

Proof: The necessity is obvious. Hence we assume that P/z is imbedded in L_p in such a way that all quotient lattices P/b, $b \neq z$, are preserved. Suppose this imbedding takes $a \in P/z$ into $a' \in L_p$. Consider the set of elements $L' = L \vee L_p$, where a and a' are identified for every $a \in P/z$. Define a partial ordering $\supseteq$ over L' as follows:

(1) If $a \notin L$ and $b \notin L$, then $a \supseteq b$ if and only if $a \underset{L_p}{\geq} b$.

(2) If $a \in L$ and $b \in L$, then $a \supseteq b$ if and only if $a \underset{L}{\geq} b$.

(3) If $a \in L$ and $b \notin L$, then $a \supseteq b$ if and only if there exists an element $c \in L \wedge L_p$ such that $a \underset{L}{\geq} c \underset{L_p}{\geq} b$.

We verify that this definition produces a partial ordering. Trivially $a \supseteq a$. Assume $a \supseteq b$ and $b \supseteq a$. If $a \in L$, then $b \in L$ since otherwise $b \supseteq a$ is not defined. If $a \notin L$, then $b \notin L$ since otherwise $a \supseteq b$ is not defined. Hence $a = b$ since either $a \underset{L}{=} b$ or $a \underset{L_p}{=} b$. To prove the transitivity we assume $a \supseteq b$ and $b \supseteq c$ and consider three cases.

(1) If $a \notin L$, then $b \notin L$ since $a \supseteq b$, and similarly $c \notin L$. Hence $a \supseteq c$ by the transitivity of the partial ordering of L_p.

(2) If $a \in L$ and $b \in L$ then either $c \in L$ or $c \notin L$. But $c \in L$ implies $a \supseteq c$ by the transitivity of the ordering in L. Also $c \notin L$ implies that $d \in L_p \wedge L$ exists such that $b \underset{L}{\geq} d \underset{L_p}{\geq} c$, which implies $a \underset{L}{\geq} d \underset{L_p}{\geq} c$. Hence $a \supseteq c$.

(3) If $a \in L$ and $b \notin L$, then $c \notin L$ since $b \supseteq c$. Also d exists such that $a \underset{L}{\geq} d \underset{L_p}{\geq} b \underset{L_p}{\geq} c$, and hence $a \supseteq c$. Therefore transitivity is established, and L' is partially ordered.

Given any two elements a and b of L' we now exhibit a unique minimal element c containing a and b and a unique maximal element d contained in a and b. Again there are three main cases.

(1) If $a \notin L$ and $b \notin L$, clearly $c = a \underset{L_p}{\cup} b$ and $d = a \underset{L_p}{\cap} b$.

(2) If $a \in L$ and $b \in L$, then $c = a \underset{L}{\cup} b$. If $a \underset{L}{\cap} b \neq z$ then $d = a \underset{L}{\cap} b$. If $a \underset{L}{\cap} b = z$, consider $d = (a \underset{L}{\cap} P) \underset{L_p}{\cap} (b \underset{L}{\cap} P)$. Then we have

$$a \underset{L}{\geq} (a \underset{L}{\cap} P) \underset{L_p}{\geq} (a \underset{L}{\cap} P) \underset{L_p}{\cap} (b \underset{L}{\cap} P) = d,$$

and

$$b \underset{L}{\geq} (b \underset{L}{\cap} P) \underset{L_p}{\geq} (a \underset{L}{\cap} P) \underset{L_p}{\cap} (b \underset{L}{\cap} P) = d.$$

Hence

$$a \supseteq d \quad \text{and} \quad b \supseteq d.$$

Let x be any element such that $a \supseteq x$ and $b \supseteq x$. If $x \in L$, then $z = a \underset{L}{\cap} b \underset{L}{\geq} x$, and hence $d \supseteq x = z$. If $x \notin L$, we have a' and b' in $L \wedge L_p$

We verify that this definition produces a partial ordering. Trivially $a \supseteq a$. Assume $a \supseteq b$ and $b \supseteq a$. If $a \in L$, then $b \in L$ since otherwise $b \supseteq a$ is not defined. If $a \notin L$, then $b \notin L$ since otherwise $a \supseteq b$ is not defined. Hence $a = b$ since either $a \underset{L}{=} b$ or $a \underset{L_p}{=} b$. To prove the transitivity we assume $a \supseteq b$ and $b \supseteq c$ and consider three cases.

(1) If $a \notin L$, then $b \notin L$ since $a \supseteq b$, and similarly $c \notin L$. Hence $a \supseteq c$ by the transitivity of the partial ordering of L_p.

(2) If $a \in L$ and $b \in L$ then either $c \in L$ or $c \notin L$. But $c \in L$ implies $a \supseteq c$ by the transitivity of the ordering in L. Also $c \notin L$ implies that $d \in L_p \wedge L$ exists such that $b \underset{L}{\geq} d \underset{L_p}{\geq} c$, which implies $a \underset{L}{\geq} d \underset{L_p}{\geq} c$. Hence $a \supseteq c$.

(3) If $a \in L$ and $b \notin L$, then $c \notin L$ since $b \supseteq c$. Also d exists such that $a \underset{L}{\geq} d \underset{L_p}{\geq} b \underset{L_p}{\geq} c$, and hence $a \supseteq c$. Therefore transitivity is established, and L' is partially ordered.

Given any two elements a and b of L' we now exhibit a unique minimal element c containing a and b and a unique maximal element d contained in a and b. Again there are three main cases.

(1) If $a \notin L$ and $b \notin L$, clearly $c = a \underset{L_p}{\cup} b$ and $d = a \underset{L_p}{\cap} b$.

(2) If $a \in L$ and $b \in L$, then $c = a \underset{L}{\cup} b$. If $a \underset{L}{\cap} b \neq z$ then $d = a \underset{L}{\cap} b$. If $a \underset{L}{\cap} b = z$, consider $d = (a \underset{L}{\cap} P) \underset{L_p}{\cap} (b \underset{L}{\cap} P)$. Then we have

$$a \underset{L}{\geq} (a \underset{L}{\cap} P) \underset{L_p}{\geq} (a \underset{L}{\cap} P) \underset{L_p}{\cap} (b \underset{L}{\cap} P) = d,$$

and

$$b \underset{L}{\geq} (b \underset{L}{\cap} P) \underset{L_p}{\geq} (a \underset{L}{\cap} P) \underset{L_p}{\cap} (b \underset{L}{\cap} P) = d.$$

Hence $\qquad a \supseteq d \quad$ and $\quad b \supseteq d$.

Let x be any element such that $a \supseteq x$ and $b \supseteq x$. If $x \in L$, then $z = a \underset{L}{\cap} b \underset{L}{\geq} x$, and hence $d \supseteq x = z$. If $x \notin L$, we have a' and b' in $L \wedge L_p$

such that $a \underset{L}{\geqq} a' \underset{L_p}{\geqq} x$ and $b \underset{L}{\geqq} b' \underset{L_p}{\geqq} x$. Also $P \underset{L}{\geqq} a'$ and $P \underset{L}{\geqq} b'$. Hence $(a \underset{L}{\cap} P) \underset{L}{\geqq} a'$, which implies $(a \underset{L}{\cap} P) \underset{L_p}{\geqq} a'$ since both elements are common to L and L_p. Similarly $(b \underset{L}{\cap} P) \underset{L_p}{\geqq} b'$. Then we have $(a \underset{L}{\cap} P) \underset{L_p}{\cap} (b \underset{L}{\cap} P) = d \underset{L_p}{\geqq} a' \underset{L_p}{\cap} b' \underset{L_p}{\geqq} x$, and thus $d \supseteq x$, which establishes d as the maximal element contained in a and b.

(3) If $a \in L$ and $b \notin L$, let $c = [(a \underset{L}{\cap} P) \underset{L_p}{\cup} b] \underset{L}{\cup} a$.
Now $(a \underset{L}{\cap} P) \in L \wedge L_p$, and hence $(a \underset{L}{\cap} P) \underset{L_p}{\cup} b \in L \wedge L_p$, since $P/(a \cap P)$ in L_p is precisely $P/(a \cap P)$ in L by hypothesis. Then since $c \underset{L}{\geqq} a$, we have $c \supseteq a$. Also $c \underset{L}{\geqq} [(a \underset{L}{\cap} P) \underset{L_p}{\cup} b] \underset{L_p}{\geqq} b$. Hence $c \supseteq b$. Let x be any element such that $x \supseteq a$ and $x \supseteq b$. Then $x \underset{L}{\geqq} a$ and b' exists such that $x \underset{L}{\geqq} b' \underset{L_p}{\geqq} b$. Since $a \underset{L}{\geqq} (a \underset{L}{\cap} P)$, we have

$$x \underset{L}{\geqq} [(a \underset{L}{\cap} P) \underset{L}{\cup} b'] ,$$

$$x \supseteq [(a \underset{L}{\cap} P) \underset{L}{\cup} b'] ,$$

and $$x \underset{L}{\geqq} x' \underset{L}{\geqq} [(a \underset{L}{\cap} P) \underset{L}{\cup} b'] \underset{L_p}{\geqq} [(a \underset{L}{\cap} P) \underset{L_p}{\cup} b] ,$$

for some $x' \in L \wedge L_p$.

Hence $$x \supseteq [(a \underset{L}{\cap} P) \underset{L_p}{\cup} b] ,$$

$$x \underset{L}{\geqq} [(a \underset{L}{\cap} P) \underset{L_p}{\cup} b] \underset{L}{\cup} a = c,$$

and $$x \supseteq c,$$

which establishes c as the minimal element.

Now let $d = (a \underset{L}{\cap} P) \underset{L_p}{\cap} b$. Then we have

$$a \underset{L}{\geqq} (a \underset{L}{\cap} P) \underset{L_p}{\geqq} (a \underset{L}{\cap} P) \underset{L_p}{\cap} b,$$

and hence $a \supseteq d$. Likewise $b \supseteq d$. Let x be any element such that $a \supseteq x$ and $b \supseteq x$. Then $x \in L_p$, since $b \notin L$. Also a' exists in $L \wedge L_p$

such that $a \underset{L}{\geq} a' \underset{L_p}{\geq} x$. Since $P \underset{L}{\geq} a'$, $(a \underset{L}{\cap} P) \underset{L}{\geq} a'$, which implies $(a \underset{L}{\cap} P) \supseteq a'$, and hence $(a \underset{L}{\cap} P) \underset{L_p}{\geq} a' \underset{L_p}{\geq} x$. Since $b \underset{L_p}{\geq} x$, we have $d = (a \underset{L}{\cap} P) \underset{L_p}{\cap} b \underset{L_p}{\geq} x$, and $d \supseteq x$. Hence d is maximal.

Therefore $L' = L \vee L_p$ is a lattice, and we have the following definitions for union $\cup$ and intersection $\cap$ in L'.

(1) If $a,b \notin L$, $\quad a \cup b = a \underset{L_p}{\cup} b$

$a \cap b = a \underset{L_p}{\cap} b.$

(2) If $a,b \in L$ and $a \cap b \neq z$, $\quad a \cup b = a \underset{L}{\cup} b$

$a \cap b = a \underset{L}{\cap} b.$

(3) If $a,b \in L$ and $a \cap b = z$, $\quad a \cup b = a \underset{L}{\cup} b$

$a \cap b = [(a \underset{L}{\cap} P) \underset{L_p}{\cap} (b \underset{L}{\cap} P)].$

(4) If $a \in L$ and $b \notin L$, $\quad a \cup b = [(a \underset{L}{\cap} P) \underset{L_p}{\cup} b] \underset{L}{\cup} a$

$a \cap b = (a \underset{L}{\cap} P) \underset{L_p}{\cap} b.$

Given any element $a \neq z$ in L, consider the corresponding element a' in L'. If b' and c' are such that $a' \subseteq b'$ and $a' \subseteq c'$, then b' and c' are images of elements b and c in L, since neither the imbedding of P/z nor the construction of L' introduces elements containing a' which are not of this form. But then since $b \underset{L}{\cap} c \underset{L}{\geq} a \neq z$ we have $b \cup c = b \underset{L}{\cup} c$ and $b \cap c = b \underset{L}{\cap} c$, and hence the quotient lattice u'/a' is isomorphic to u/a for all $a \neq z$ in L.

We now prove that L' is upper semi-modular. In L' let $b \succ a$, $c \supset a$ and $c \not\supset d$. There are four possibilities to consider.

(1) If $a \in L$ and $a \neq z$, then u/a in L' is upper semi-modular because of the weak upper semi-modularity of L and the preservation of quotient lattices in L'. Hence $b \cup c \succ c$.

(2) Let $a \notin L$, $b \in L$, and $c \in L$. Then $b \succ a$ implies $b \in L_p$, because otherwise a is common to L_p and L. If $c \notin L_p$ this case reduces to case 3, and if $c \in L_p$ it reduces to case 4.

(3) Let $a \notin L$, $b \notin L$, and $c \in L$. If $c \in L_p$ we have case 4, so consider $c \notin L_p$. We have $c \supset a$ and $P \supset a$ which imply $c \cap P \supseteq a$, and hence $c \underset{L}{\cap} P \underset{L_p}{\supseteq} a$. If $c \cap P \supseteq b$ then $c \supseteq b$ which is a contradiction. Hence $(c \underset{L}{\cap} P) \underset{L_p}{\not\supseteq} b$. Then $(c \underset{L}{\cap} P) \underset{L_p}{\cup} b \underset{L_p}{\succ} c \underset{L}{\cap} P$ by the upper semi-modularity of L_p, and therefore $(c \underset{L}{\cap} P) \underset{L_p}{\cup} b \underset{L}{\succ} c \cap P$ since the element on each side of the relation is common to L and L_p. Now $c \underset{L}{\supseteq} c \underset{L}{\cap} P$. But if $c \underset{L}{\supset} (c \underset{L}{\cap} P) \underset{L_p}{\cup} b$, then $c \supset (c \cap P) \cup b$, which implies $c \supset b$ contradicting $c \not\supset b$. Hence by the weak upper semi-modularity of L we have

$$\left\{\left[(c \underset{L}{\cap} P) \underset{L_p}{\cup} b\right] \underset{L}{\cup} c\right\} \underset{L}{\succ} c.$$

But the left side is precisely $c \cup b$, and hence $c \cup b \succ c$.

(4) If a, b, and c are all elements of L_p, then $c \underset{L_p}{\cup} b \underset{L_p}{\succ} c$ by the upper semi-modularity of L_p, and therefore $c \cup b \succ c$.

Hence L' is upper semi-modular, and the proof of theorem 3.2 is complete.

We now consider an example which shows that the corresponding theorems for lower semi-modularity, modularity, and distributivity are not valid. The lattice of figure 5a is weakly lower semi-modular, weakly modular, and weakly distributive. Figure 5b shows a possible imbedding of P/z which is distributive and hence modular and semi-modular. However, L' which is shown in figure 5c is not lower semi-modular, and hence neither modular nor distributive.

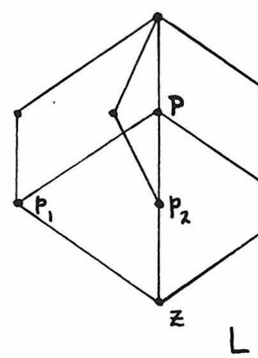

Figure 5a.

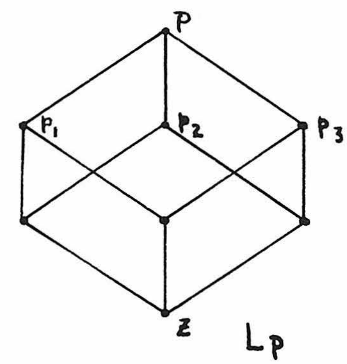

Figure 5b.

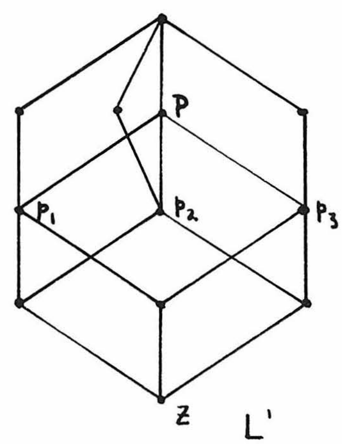

Figure 5c.

Next we prove that if the upper semi-modular imbedding can be made such that the descending chain condition holds in the imbedding lattice, then it is not necessary to introduce new meet irreducible elements in order to perform the imbedding.

Theorem 3.3. If a weakly upper semi-modular lattice L can be imbedded in an upper semi-modular lattice L' such that L' satisfies the ascending chain condition and for $a \neq z$ in L u/a is isomorphic to u'/a' in L', then there exists an imbedding of L into an upper semi-modular lattice L'' such that the quotient lattices are preserved and every meet irreducible element of L'' is the image of a meet irreducible element of L.

Proof: We assume that L is imbedded in L' with the properties stated in the theorem. Then the image x' of a meet irreducible element $x \in L$ is also meet irreducible since the quotient lattices of L and L' are isomorphic. Let Q' be the set of all such irreducible images in L', and let $Q' \vee P'$ be the set of all meet irreducible elements of L', where P' contains every meet irreducible element of L' which is not the image of an irreducible of L.

For any $a' \in L'$ let $S_{a'}$ be the set of all $q' \in Q'$ such that $q' \geq a$. Let L'' be the lattice of all such sets, ordered by set inclusion. Since L' is assumed to satisfy the ascending chain condition, if $a' \supset b'$ then $S_{a'} \subseteq S_{b'}$. Clearly $S_{a'} \subset S_{b'}$ implies $a' \supset b'$.

Lemma 3.3. In L'' $S_{a' \cap b'} = S_{a'} \cup S_{b'}$ and $S_{a' \cup b'} = S_{a'} \cap S_{b'}$.

Proof: By definition $S_{a'} \cup S_{b'}$ is the smallest set in L'' which contains $S_{a'} \vee S_{b'}$. Clearly $S_{a' \cap b'} \geq S_{a'} \vee S_{b'}$. Let $S_c \in$ L'' be such that $S_{c'} \geq S_{a'} \vee S_{b'}$. Then $S_{c'} \geq S_{a'}$ and $S_{c'} \geq S_{b'}$, which imply $c' \leq a'$ and

$c' \leq b'$. Hence $c' \leq a' \cap b'$ and $S_{c'} \geq S_{a' \cap b'}$. Hence $S_{a' \cap b'}$ is the smallest set containing $S_{a'} \vee S_{b'}$, which concludes the proof of the first statement.

By definition $S_{a'} \cap S_{b'}$ is the largest set in L" which is contained in $S_{a'} \wedge S_{b'}$. But for $q' \in S_{a'} \wedge S_{b'}$ we have $q' \geq a'$ and $q' \geq b'$. Hence $q' \geq a' \cup b'$, and $q' \in S_{a' \cup b'}$. Thus we have $S_{a'} \wedge S_{b'} \leq S_{a' \cup b'}$, and since the opposite inclusion is trivial, we have $S_{a'} \wedge S_{b'} = S_{a' \cup b'} = S_{a'} \cap S_{b'}$. Hence the lemma holds, and in L" lattice intersection coincides with set intersection.

By lemma 3.3 the mapping $L' \longrightarrow L''$ is a dual lattice homomorphism. A set $S \in L''$ may be the image of more than one element of L', because any element a' of L' can be expressed as the intersection of all meet irreducibles containing a'. Hence if $a'_1 = \bigcap (T_1 \vee S_1)$ and $a'_2 = \bigcap (T_1 \vee S_2)$ where $T_1 \leq Q'$, $S_1 \leq P'$, and $S_2 \leq P'$, then for $S_1 \neq S_2$ we have $a'_1 \neq a'_2$. Underthe mapping, T_1 is the image of both a'_1 and a'_2. However to each $T \in L''$ we can associate the element $t' = \bigcap T$ in L'. Let $T \vee P_{t'}$ be the set of all meet irreducibles containing t', where $P_{t'} \leq P'$. Then every element t'_1 whose image is T has the representation $t'_1 = \bigcap (T \vee P_{t'_1})$ where $P_{t'} \leq P_{t'_1}$, and hence t' is the maximal element in L' whose image is T. Clearly distinct elements in L" are associated with distinct elements of L'.

We now show that L" is lower semi-modular. Let R, S, and T be elements of L" and let r', s', and t' be the associated elements of L'. Let $S \succ T$, $R \subset S$, and $R \not\subset T$. Then we have $s' \subset t'$, $r' \supset s'$, and $r' \not\supset t'$. Let $t' \geq w' \succ s'$ in L'. Then the image of w' in L" is T since s' is the maximal element in L' whose image is S. If $r' \supset w'$, then by the homomorphism $R \subseteq T$ which contradicts $R \nsubseteq T$. Hence by the upper semi-modularity of L', $r' \cup w' \succ r'$. By lemma 3.3 the image of $r' \cup w'$ is $R \wedge T = R \cap T$, and since coverings in L' are either collapsed or inverted by the mapping to L'', we have either $R = R \cap T$ or $R \succ R \cap T$. But $R = R \cap T$ implies $R \subseteq T$ which is a contradiction. Hence $R \succ R \cap T$, and L'' is lower semi-modular.

Next we prove that every join irreducible element S of L'' is the image of a meet irreducible element of L. Let $S = T \cup R$. Then in L', $s' = t' \cap r'$. If S is join irreducible either $S = T$, which implies $s' = t'$, or $S = R$, which implies $s' = r'$. Hence s' is meet irreducible in L'. But $S \subseteq Q'$ and $s' = \bigcap S$ imply $s' = q'$ for some $q' \in Q'$. Hence by the definition of Q', s' is the image of a meet irreducible element of L. But then under the repeated mapping $L \longrightarrow L' \longrightarrow L''$, S is the image of a meet irreducible element of L.

Conversely, the image in L'' of every meet irreducible element of L is join irreducible, for if $x \in L$ is meet irreducible, then its image x' in L' is meet irreducible by hypothesis. Then $S_{x'}$ is the image in L'' of x', and x' is the unique element associated with $S_{x'}$. Suppose $S_{x'} = S_{a'} \cup S_{b'}$. Then by lemma 3.3 $S_{x'} = S_{a' \cup b'} \supseteq S_{a'} \vee S_{b'}$, and we have $S_{x'} \supseteq S_{a'}$ and $S_{x'} \supseteq S_{b'}$. This implies $x' \subseteq a'$ and $x' \subseteq b'$, and hence $x' \subseteq a' \cap b'$. But since x' is the unique element associated with $S_{x'}$, $x' \supseteq a' \cap b'$. Hence $x' = a' \cap b'$ which implies either $x' = a'$ or $x' = b'$. Then either $S_{x'} = S_{a'}$ or $S_{x'} = S_{b'}$, and hence $S_{x'}$ is join irreducible.

Consider any quotient lattice u/a for $a \neq z$ in L. By hypothesis u/a is isomorphic to u'/a' in L', and for any b' in u'/a' the representation of b' as the intersection of all meet irreducibles in L' containing b' is in terms of elements of Q' and completely free from any element of P'. Hence u'/a' is dually isomorphic to $S_{a'}/z''$. But the dual lattice of L" is upper semi-modular, its meet irreducible elements are precisely the images of meet irreducible elements of L, and the quotient lattices of L are preserved. Hence the proof of theorem 3.3 is complete.

The example shown in figure 6 proves that the imbedding problem stated in section 3.1 is in general impossible for weakly upper semi-modular lattices. By theorem 3.3 the imbedding can be made, if at all, in terms of four meet irreducible elements of L. If the imbedding lattice L' is upper semi-modular, the chain law must hold between z' and x'_1. Since the quotient lattice $u/(x_1 \cap x_4)$ is to be preserved, we must introduce an element between z and $x_1 \cap x_4$. But $x_2 \geq x_1 \cap x_3 \geq x_1 \cap x_3 \cap x_4$ and $x_3 \geq x_2 \cap x_4 \geq x_1 \cap x_2 \cap x_4$. Therefore any element contained by $x_1 \cap x_4$ must be $x_1 \cap x_2 \cap x_3 \cap x_4 = z$, and the imbedding preserving quotient lattices is impossible.

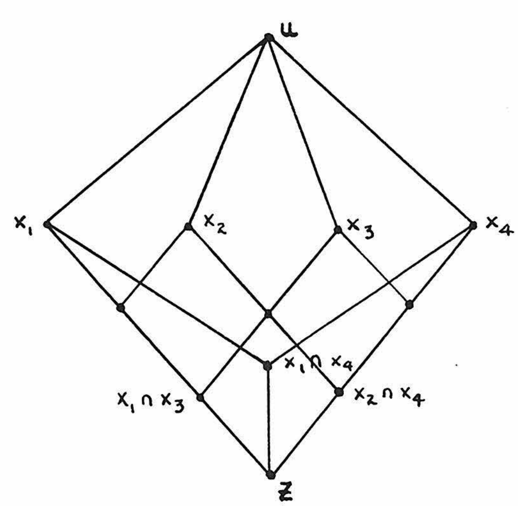

Figure 6.

The final example demonstrates a weakly lower semi-modular lattice which cannot be imbedded to preserve quotient lattices without introducing new meet irreducible elements, and is shown in figure 7. Notice that $x_2 = x_1 \cap x_2 \succ x_1 \cap x_2 \cap x_5$ and $x_1 \cap x_2 \succ x_1 \cap x_2 \cap x_3 \cap x_4$. Thus in a lower semi-modular imbedding lattice which preserves quotients we must have

$x_1 \cap x_2 \cap x_5 \succ (x_1 \cap x_2 \cap x_3 \cap x_4) \cap (x_1 \cap x_2 \cap x_5) = a.$

But $x_6 \geq x_4 \cap x_5 \geq a$, and hence $a = z$. However

$x_1 \cap x_2 \cap x_5 \succ a = z$ is impossible since

$x_1 \cap x_2 \cap x_5 \succ x_1 \cap x_2 \cap x_5 \cap x_6 \supset z.$

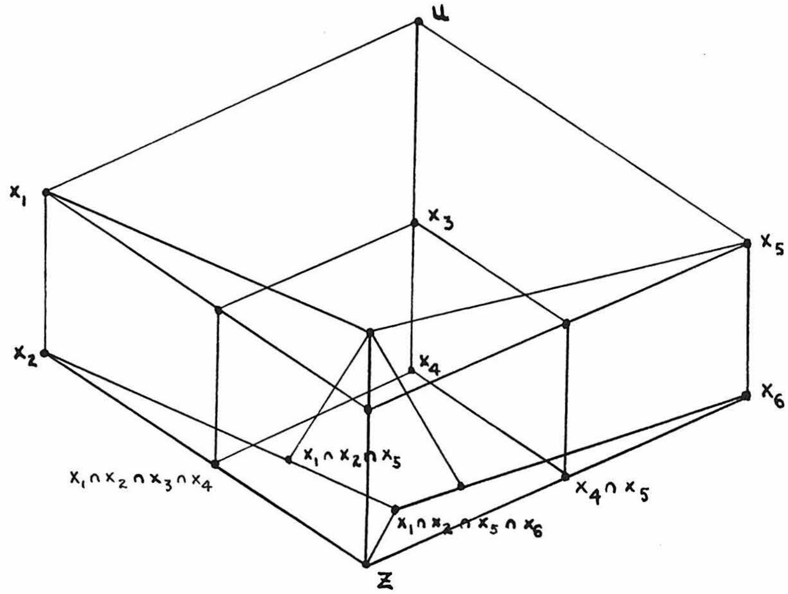

Figure 7.

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