To Professor McEliece
Andre Tkacenko
Thanks for all of your support!

The Fractional Discrete Fourier Transform

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Abstract

A fractional version of the Discrete Fourier Transform or DFT, denoted by the Fractional Discrete Fourier Transform or FDFT for short, is discussed here. First, results of a fractional version of the continuous-time Fourier Transform or CTFT are explored and then parallels are made between the DFT and the CTFT. Using the method of spectral decomposition [1], an expression for the FDFT is then derived which satisfies properties analogous to the fractional CTFT. Afterwards, properties of the FDFT are discovered and proven, and an example of an FDFT pair is given. Finally, various applications of the FDFT in signal processing in areas such as allpass filter networks and the M-channel maximally decimated filter bank are discussed.

Contents

Introduction

This thesis is organized into three chapters. In the first chapter, an expression for the FDFT is derived. In order to properly introduce the notion of a fractional DFT, the CTFT and its duality properties are discussed. At this point, a fractional version of the CTFT is mentioned. Then, analogies are made between the CTFT and the DFT, such as duality properties. This then serves as the basis upon which we derive an expression for the FDFT. Using the method of spectral decomposition, which itself is proven here, a closed form expression for the FDFT is derived.

In the second chapter, properties of the newly derived FDFT are stated and proven. Tables of properties are included at the end of this section, as well as a very simple example of an FDFT pair.

Applications of the FDFT in signal processing are discussed in the third chapter. First, we explore the computational complexity of the FDFT and then consider implementing it in a simple digital allpass filter network. Finally, the implementation of the FDFT in the M-channel maximally decimated filter bank is analyzed and the example of the classical 2-channel QMF bank is discussed.

Chapter 1

Derivation of the Fractional Discrete Fourier Transform

1.1 The Continuous-Time Fourier Transform as a Rotational Operator

In many classical engineering publications, the Fourier Transform of a continuous-time signal $x(t)$, called the CTFT for short, is defined as follows.

$$X(\omega) = \mathcal{F}\{x(t)\} \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

One particular alternate definition of the integral transform from above is given below.

$$X(\omega) = \mathcal{F}\{x(t)\} \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{1.1}$$

In addition to retaining all of the important properties from the transform introduced originally, this slightly modified transform has the following duality properties.

$$\mathcal{F}\{\mathcal{F}\{x(t)\}\} = x(-t) \tag{1.2}$$

$$\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{x(t)\}\}\} = X(-\omega) \tag{1.3}$$

$$\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{x(t)\}\}\}\} = x(t) \tag{1.4}$$

As we can see from above, by applying the linear Fourier Transform operator $\mathcal{F}$ twice to the signal $x(t)$, we get a reversed version of $x(t)$, namely $x(-t)$. Applying $\mathcal{F}$ three times, we get the reversed version of $X(\omega)$, namely $X(-\omega)$, whereas by applying $\mathcal{F}$ four times, we get back our original signal $x(t)$. With these particular properties satisfied, the version of the Fourier Transform in (1.1) came to be viewed as a rotational operator [2] in a ficticious time-frequency plane as is shown in Figure 1.1.

Each application of the Fourier Transform would rotate the signal $x(t)$ by 90° in this plane. Further applications of the Fourier Transform would result in a net angular measure equal to the sum of the number of rotations times the angular measure of each rotation. The question then arose as to how one could rotate by an arbitrary angle, say α, in this time-frequency plane. This question was resolved by first considering how to represent any point in the plane with coordinates (t, ω) in terms of a new set of variables (u, v), (see Figure 1.1). The new coordinates, (u, v), are related to the old ones, (t, ω), through the following unitary transformation [3].

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} t \\ \omega \end{bmatrix}$$

With this, the question then arose as to how to transform a time-signal, $x(t)$, into a signal as a function of u. This transformation would have the properties that when $\alpha = 0$, we get $x(u)$, whereas when $\alpha = \frac{\pi}{2}$, we

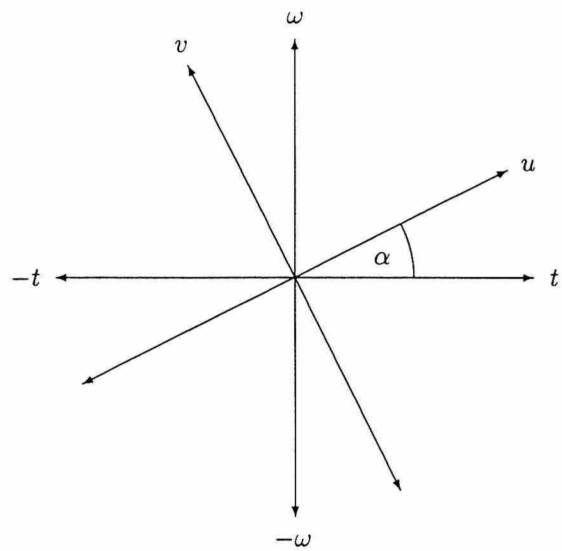

Figure 1.1: The Time-Frequency Plane

get $X(u)$. In addition, the transformation operator would act as a rotational operator in the sense that a rotation by an angle of α followed by another by an angle of β would result in the original signal rotated by a net angle of $(\alpha + \beta)$. One such operator, denoted by $\mathcal{F}_\alpha$, returns a signal $X_\alpha(u)$ given the signal $x(t)$ and is given by [2] as follows.

$$X_\alpha(u) = \mathcal{F}_\alpha\{x(t)\} = \int_{-\infty}^{\infty} K_\alpha(t,u)x(t)dt \tag{1.5}$$

$$\text{where } K_\alpha(t,u) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}}e^{j\left(\frac{t^2+u^2}{2}\cot\alpha - ut\csc\alpha\right)}$$

Here, $K_\alpha(t,u)$ is called the kernel of the integral operator $\mathcal{F}_\alpha$. Indeed as desired, the following properties hold.

$$X_0(u) = \int_{-\infty}^{\infty} x(t)\delta(t-u)dt = x(u) \tag{1.6}$$

$$X_{\frac{\pi}{2}}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)e^{-jut}dt = X(u) \tag{1.7}$$

$$\mathcal{F}_\beta\{\mathcal{F}_\alpha\{x(t)\}\} = \mathcal{F}_{\alpha+\beta}\{x(t)\} \tag{1.8}$$

Because of these properties, the transform operator $\mathcal{F}_\alpha$ was christened the continuous-time fractional Fourier Transform, or FRFT for short. As we will soon see, this concept of viewing the CTFT as a rotational operator can be extended to the Discrete Fourier Transform, or DFT, as well.

1.2 The Discrete Fourier Transform

Classically, the DFT of a discrete-time sequence of length N, say $x[n]$, defined over the interval $0 \leq n \leq N-1$ is given below by the following formula.

$$X[k] = \mathcal{DFT}\{x[n]\} \triangleq \sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{2\pi nk}{N}\right)},\ 0 \leq k \leq N-1$$

As with the CTFT, an alternate definition of the DFT which will be important for reasons which will soon be evident, is given below.

$$X[k] = \mathcal{DFT}\{x[n]\} \triangleq \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{2\pi nk}{N}\right)},\ 0 \leq k \leq N-1 \tag{1.9}$$

1.2.1 Duality

Analogous to the duality properties of the CTFT given in (1.2),(1.3), and (1.4), we have the following.

$$\mathcal{DFT}\{\mathcal{DFT}\{x[n]\}\} = x[-n] \tag{1.10}$$
$$\mathcal{DFT}\{\mathcal{DFT}\{\mathcal{DFT}\{x[n]\}\}\} = X[-k] \tag{1.11}$$
$$\mathcal{DFT}\{\mathcal{DFT}\{\mathcal{DFT}\{\mathcal{DFT}\{x[n]\}\}\}\} = x[n] \tag{1.12}$$

(It should be noted here that by $x[-n]$, we really mean $x[-n \pmod{N}]$. For the remainder of this text, whenever we write, say $y[m]$ as a discrete-time signal of length N, we will really mean $y[m \pmod{N}]$.)

So, the DFT can be regarded as a 90° rotational operator in a discrete $[n, k]$ time-frequency plane.

1.2.2 Matrix Form of the DFT

One advantage of the DFT over the CTFT is that the input and output sequences of the DFT are of finite length (length N in fact), while the input and output functions of the CTFT have infinite length (length $2^{\aleph_0}$ in fact). As a result, we can use matrices to simplify our notation as well as analysis. If we define,

$$\mathbf{x} \triangleq \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}, \ \mathbf{X} \triangleq \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}, \ \mathbf{W} \triangleq \left[\frac{1}{\sqrt{N}} e^{-j\left(\frac{2\pi nk}{N}\right)}\right]_{N\times N},$$

then we have from (1.9),

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

So, the vector $\mathbf{X}$ is viewed as a linear transformation of the original vector $\mathbf{x}$ by the matrix transform $\mathbf{W}$, where the linear transformation here is the DFT. For this reason, $\mathbf{W}$ is called the DFT matrix. Using this matrix notation, we can see that applying the DFT operator p times to the signal $x[n]$ is equivalent to raising $\mathbf{W}$ to the p-th power. So from (1.12), we get,

$$\mathbf{W}^4\mathbf{x} = \mathbf{x}$$

So, we conclude,

$$\mathbf{W}^4 = \mathbf{I}, \tag{1.13}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix.

1.3 Desired Properties of a Fractional DFT

In an analogous fashion to the continuous-time case, let $\mathcal{DFT}_\alpha$ denote our fractional DFT operator, which we will call the FDFT operator for short. When this operator acts on a finite length-N sequence $x[n]$, it returns a finite length-N sequence $X_\alpha[l]$ as follows.

$$X_\alpha[l] = \mathcal{DFT}_\alpha\{x[n]\}$$

We wish to express this transformation in a form similar to that of the FRFT given in (1.5). This is done as follows.

$$X_\alpha[l] = \sum_{n=0}^{N-1} K_\alpha[l,n]x[n], \ 0 \le l \le N-1 \tag{1.14}$$

Here, $K_\alpha[l, n]$ is a two-dimensional, finite-length, discrete-time scalar field called the kernel of the FDFT. If we define the following vectors, then we can simplify notation here by writing (1.14) as a matrix product.

$$\mathbf{x} \triangleq \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}, \ \mathbf{X}_\alpha \triangleq \begin{bmatrix} X_\alpha[0] \\ \vdots \\ X_\alpha[N-1] \end{bmatrix}, \ \mathbf{A}(\alpha) \triangleq [K_\alpha[l,n]]_{N\times N}$$

With these definitions, (1.14) becomes,

$$\mathbf{X}_\alpha = \mathbf{A}(\alpha)\mathbf{x} \tag{1.15}$$

Because of its definition, the matrix $\mathbf{A}(\alpha)$ will be called the FDFT matrix. To satisfy the properties of a rotational operator in the $[n, k]$ plane, we want the following properties satisfied.

$$\mathcal{DFT}_0\{x[n]\} = x[l]$$

$$\mathcal{DFT}_{\frac{\pi}{2}}\{x[n]\} = X[l]$$

$$\mathcal{DFT}_\beta\{\mathcal{DFT}_\alpha\{x[n]\}\} = \mathcal{DFT}_\alpha\{\mathcal{DFT}_\beta\{x[n]\}\} = \mathcal{DFT}_{\alpha+\beta}\{x[n]\}$$

Thus, we want the FDFT matrix $\mathbf{A}(\alpha)$ to satisfy the following properties.

$$\mathbf{A}(0) = \mathbf{I} \tag{1.16}$$

$$\mathbf{A}\left(\frac{\pi}{2}\right) = \mathbf{W} \tag{1.17}$$

$$\mathbf{A}(\beta)\mathbf{A}(\alpha) = \mathbf{A}(\alpha)\mathbf{A}(\beta) = \mathbf{A}(\alpha+\beta) \tag{1.18}$$

1.4 The FDFT Matrix as a Fractional Power of the DFT Matrix

One way of determining $\mathbf{A}(\alpha)$ is to express $\mathbf{A}(\alpha)$ as a fractional power of the DFT matrix $\mathbf{W}$. This method was first considered by Santhanam and McClellan [3]. If we set,

$$\mathbf{A}(\alpha) = \mathbf{W}^{\frac{2\alpha}{\pi}},$$

then indeed the properties desired for our transform, notably (1.16), (1.17), and (1.18), are satisfied, provided that exponents satisfy the same relations for matrices as they do for complex numbers. That is, we will assume for now that we have,

$$\mathbf{W}^\mu \mathbf{W}^\nu = \mathbf{W}^{\mu+\nu} \text{ for some } \mu, \nu \in \mathbb{C} \tag{1.19}$$

(We will later show that for a special class of matrices, this property does indeed hold to be true. In fact, $\mathbf{W}$ belongs to this class.)

To express a fractional power of the DFT matrix, Santhanam and McClellan considered diagonalizing the DFT matrix to obtain the fractional power they desired as follows.

$$\mathbf{A}(\alpha) = \mathbf{T}\mathbf{\Lambda}^{\frac{2\alpha}{\pi}}\mathbf{T}^\dagger$$

Here, $\mathbf{T}$ is a matrix of the eigenvectors of the DFT matrix, while $\mathbf{\Lambda}$ is a diagonal matrix of its eigenvalues. In this case, we can define $\mathbf{\Lambda}^{\frac{2\alpha}{\pi}}$ to be the matrix formed by taking each diagonal element to the $\frac{2\alpha}{\pi}$-th power. Then indeed we have,

$$\mathbf{\Lambda}^\mu \mathbf{\Lambda}^\nu = \mathbf{\Lambda}^{\mu+\nu} \text{ for some } \mu, \nu \in \mathbb{C}$$

as desired. While this method works, I have opted to use a different method which appears to be less entropic than that which Santhanam and McClellan used. In order, though, to properly introduce my method, let us digress for a while before returning to the task at hand.

1.5 The Method of Spectral Decomposition

From Hsu [1], we have the following remarkable theorem. For this, recall that the minimal polynomial, denoted here as $m(\lambda)$, of an $N \times N$ matrix $\mathbf{C}$ is the smallest degree polynomial such that $m(\mathbf{C}) = \mathbf{0}$. By the Cayley-Hamilton Theorem, we know that such a polynomial always exists for every $N \times N$ matrix $\mathbf{C}$.

Theorem 1 (Spectral Decomposition) *If* $\mathbf{C}$ *is any* $N \times N$ *matrix with a minimal polynomial of the form,*

$$m(\lambda) = \prod_{i=1}^{p} (\lambda - \lambda_i) \text{ where } \lambda_k \neq \lambda_l \ \forall\ k \neq l\,,$$

then, we can express $\mathbf{C}$ *in the following way, called the spectral decomposition of* $\mathbf{C}$*.*

$$\mathbf{C} = \sum_{i=1}^{p} \lambda_i \mathbf{E}_i\,,$$

where the $\mathbf{E}_i$*'s are called constituent matrices and are obtained by the formula,*

$$\mathbf{E}_i = \frac{\displaystyle\prod_{\substack{m=1\\ m\neq i}}^{p} (\mathbf{C} - \lambda_m \mathbf{I})}{\displaystyle\prod_{\substack{m=1\\ m\neq i}}^{p} (\lambda_i - \lambda_m)} \tag{1.20}$$

Furthermore, the constituent matrices satisfy the following properties.

1. $\sum_{i=1}^{p} \mathbf{E}_i = \mathbf{I}$
2. $\mathbf{E}_m \mathbf{E}_k = \mathbf{0} \ \forall\ m \neq k$ *(Orthogonality)*
3. $\mathbf{E}_k^2 = \mathbf{E}_k$ *(Idempotency)*
4. $\mathbf{C}\mathbf{E}_k = \mathbf{E}_k \mathbf{C} = \lambda_k \mathbf{E}_k$

Oddly enough, this incredible theorem is not in some of the more classical books on matrix theory, such as Horn and Johnson [4]. In addition to this, Hsu offers no proof of this theorem. So, we will take the time here to prove this theorem. First we will show that by defining $\mathbf{E}_i$ as stated in the theorem, then indeed properties 1 and 2 hold. With this, we will show that the spectral decomposition of $\mathbf{C}$ is valid. Finally, we will prove that properties 3 and 4 are true.

Proof of Theorem 1: Suppose $\mathbf{C}$ has a minimal polynomial, say $m(\lambda)$, of the form stated in the theorem. (Recall that the minimal polynomial of $\mathbf{C}$ is the smallest degree polynomial such that $m(\mathbf{C}) = \mathbf{0}$.) Then we have,

$$\frac{1}{m(\lambda)} = \frac{1}{\displaystyle\prod_{i=1}^{p} (\lambda - \lambda_i)}$$

Using a partial-fraction expansion, we get the following, since the λ_i's are distinct.

$$\frac{1}{m(\lambda)} = \frac{k_1}{\lambda - \lambda_1} + \frac{k_2}{\lambda - \lambda_2} + \cdots + \frac{k_p}{\lambda - \lambda_p} = \sum_{i=1}^{p} \frac{k_i}{\lambda - \lambda_i}$$

Here, we have,

$$k_i = \frac{1}{\displaystyle\prod_{\substack{m=1\\ m\neq i}}^{p} (\lambda_i - \lambda_m)}$$

Adding up each term in the summation above yields,

$$\frac{1}{m(\lambda)} = \frac{k_1 g_1(\lambda) + k_2 g_2(\lambda) + \cdots + k_p g_p(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_p)},$$

$$\text{where } g_i(\lambda) = \prod_{\substack{m=1 \\ m \neq i}}^{p} (\lambda - \lambda_m)$$

We have now,

$$\frac{1}{m(\lambda)} = \frac{k_1 g_1(\lambda) + k_2 g_2(\lambda) + \cdots + k_p g_p(\lambda)}{m(\lambda)}$$

$$\therefore \sum_{i=1}^{p} k_i g_i(\lambda) = 1\,.$$

Here, we have,

$$k_i g_i(\lambda) = \frac{\displaystyle\prod_{\substack{m=1 \\ m \neq i}}^{p} (\lambda - \lambda_m)}{\displaystyle\prod_{\substack{m=1 \\ m \neq i}}^{p} (\lambda_i - \lambda_m)},$$

and so $k_i g_i(\lambda)$ is a polynomial in λ of finite degree $(p-1)$. Let us define $f(\lambda)$ as follows.

$$f(\lambda) \triangleq \sum_{i=1}^{p} k_i g_i(\lambda) = 1$$

Since $k_i g_i(\lambda)$ is a finite degree polynomial for all i, it then follows that $f(\lambda)$ is also a finite degree polynomial. Since polynomials are well-defined functions for square matrices, we have,

$$f(\mathbf{C}) = \sum_{i=1}^{p} k_i g_i(\mathbf{C}) = \mathbf{I} \tag{1.21}$$

So, let us now define $\mathbf{E}_i$ as follows as suggested by the theorem.

$$\mathbf{E}_i \triangleq k_i g_i(\mathbf{C}) = \frac{\displaystyle\prod_{\substack{m=1 \\ m \neq i}}^{p} (\mathbf{C} - \lambda_m \mathbf{I})}{\displaystyle\prod_{\substack{m=1 \\ m \neq i}}^{p} (\lambda_i - \lambda_m)} \tag{1.22}$$

Note that by (1.21), we have,

$$\sum_{i=1}^{p} \mathbf{E}_i = \mathbf{I},$$

and so by construction, property 1 is valid. From (1.22), we have, for $i \neq j$,

$$\mathbf{E}_i \mathbf{E}_j = \frac{\overbrace{\prod_{m=1}^{p} (\mathbf{C} - \lambda_m \mathbf{I})}^{m(\mathbf{C})=\mathbf{0}} \displaystyle\prod_{\substack{m=1 \\ m \neq i,j}}^{p} (\mathbf{C} - \lambda_m \mathbf{I})}{\displaystyle\prod_{\substack{m=1 \\ m \neq i}}^{p} (\lambda_i - \lambda_m) \prod_{\substack{m=1 \\ m \neq j}}^{p} (\lambda_j - \lambda_m)} = \mathbf{0}$$

This result follows because the following factors necessarily commute for all k, l.

$$(\mathbf{C} - \lambda_k\mathbf{I})(\mathbf{C} - \lambda_l\mathbf{I}) = (\mathbf{C} - \lambda_l\mathbf{I})(\mathbf{C} - \lambda_k\mathbf{I}) = \mathbf{C}^2 - (\lambda_k + \lambda_l)\mathbf{C} + \lambda_k\lambda_l\mathbf{I}$$

This proves that property 2 is valid. With these two properties proven, we can show that we indeed have,

$$\mathbf{C} = \sum_{i=1}^{p} \lambda_i \mathbf{E}_i$$

From (1.22), we have,

$$(\mathbf{C} - \lambda_i\mathbf{I})\mathbf{E}_i = \frac{m(\mathbf{C})}{\prod_{\substack{m=1 \\ m \neq i}}^{p} (\lambda_i - \lambda_m)} = \mathbf{0}$$

$$\mathbf{C}\mathbf{E}_i - \lambda_i\mathbf{E}_i = \mathbf{0}$$

$$\therefore\ \mathbf{C}\mathbf{E}_i = \lambda_i\mathbf{E}_i\,.$$

Summing both sides over all i yields the following.

$$\sum_{i=1}^{p} \mathbf{C}\mathbf{E}_i = \sum_{i=1}^{p} \lambda_i\mathbf{E}_i$$

$$\mathbf{C}\left(\sum_{i=1}^{p} \mathbf{E}_i\right) = \sum_{i=1}^{p} \lambda_i\mathbf{E}_i$$

Thus, using the result of property 1, we have,

$$\mathbf{C}\left(\sum_{i=1}^{p} \mathbf{E}_i\right) = \mathbf{C}(\mathbf{I}) = \mathbf{C} = \sum_{i=1}^{p} \lambda_i\mathbf{E}_i\,,$$

as desired. We must finally prove that properties 3 and 4 are valid. From property 1, we have again,

$$\sum_{i=1}^{p} \mathbf{E}_i = \mathbf{I}$$

Premultiplying both sides by $\mathbf{E}_k$, we get,

$$\sum_{i=1}^{p} \mathbf{E}_k\mathbf{E}_i = \mathbf{E}_k$$

Applying the result from property 2, we have,

$$\sum_{i=1}^{p} \mathbf{E}_k\mathbf{E}_i = \mathbf{E}_k^2 = \mathbf{E}_k\,,$$

which proves property 3. In a similar way, we can prove property 4. We showed that,

$$\mathbf{C} = \sum_{i=1}^{p} \lambda_i\mathbf{E}_i$$

So, by postmultiplying both sides by $\mathbf{E}_k$, we get,

$$\mathbf{C}\mathbf{E}_k = \sum_{i=1}^{p} \lambda_i\mathbf{E}_i\mathbf{E}_k = \lambda_k\mathbf{E}_k^2 = \lambda_k\mathbf{E}_k\,,$$

where the last equality is a result of property 3. Similarly, by premultiplying, we get,

$$\mathbf{E}_k\mathbf{C} = \sum_{i=1}^{p} \lambda_i \mathbf{E}_k \mathbf{E}_i = \lambda_k \mathbf{E}_k^2 = \lambda_k \mathbf{E}_k$$

$$\therefore \ \mathbf{C}\mathbf{E}_k = \mathbf{E}_k\mathbf{C} = \lambda_k \mathbf{E}_k \, .$$

Thus, property 4 is valid. This completes the proof.

As a result of this amazing theorem, we have the following important corollary.

Corollary 1 *Let* $\mathbf{C}$ *have a spectral decomposition as described in Theorem 1. Then we have,*

$$\mathbf{C}^n = \sum_{i=1}^{p} \lambda_i^n \mathbf{E}_i \quad \forall \ n \in \mathbb{Z}_+$$

Proof: This result follows by induction. We already know that the corollary holds for $n = 0$ and $n = 1$. Assume now that it holds for $n = k$ say. Then, we have,

$$\mathbf{C}^k = \sum_{i=1}^{p} \lambda_i^k \mathbf{E}_i$$

So,

$$\mathbf{C}^k\mathbf{C} = \mathbf{C}^{k+1} = \left(\sum_{i=1}^{p} \lambda_i^k \mathbf{E}_i\right) \mathbf{C} = \sum_{i=1}^{p} \lambda_i^k (\mathbf{E}_i\mathbf{C})$$

But, by property 4, we have, $\mathbf{E}_i\mathbf{C} = \lambda_i \mathbf{E}_i$. So we get,

$$\mathbf{C}^{k+1} = \sum_{i=1}^{p} \lambda_i^k (\lambda_i \mathbf{E}_i) = \sum_{i=1}^{p} \lambda_i^{k+1} \mathbf{E}_i$$

So, if the proof is valid for $n = k$, then it is valid for $n = k + 1$. Since it is valid for $n = 0$, we conclude that it is valid $\forall \ n \in \mathbb{Z}_+$.

This is where the extension into fractional powers comes into being. We can define the fractional power, say μ, of a matrix $\mathbf{C}$ with a spectral decomposition as follows.

Definition 1 (Fractional Power of a Matrix) *Let* $\mathbf{C}$ *be any* $N \times N$ *matrix with a spectral decomposition as above. Then, the* μ*-th power of* $\mathbf{C}$ *is defined as follows.*

$$\mathbf{C}^\mu \triangleq \sum_{i=1}^{p} \lambda_i^\mu \mathbf{E}_i \ \textit{for some} \ \mu \in \mathbb{C}$$

It should be noted that the fractional power of a matrix as defined above is not unique in general. We will say more about this later when we apply this definition to the DFT matrix. From this definition and from Theorem 1, we have the following important property.

Property 1 (Additivity of Exponents of Matrices)

$$\mathbf{C}^\mu \mathbf{C}^\nu = \mathbf{C}^{\mu+\nu} \ \textit{for some} \ \mu, \nu \in \mathbb{C}$$

Proof: We have the following.

$$\mathbf{C}^\mu \mathbf{C}^\nu = \left(\sum_{i=1}^{p} \lambda_i^\mu \mathbf{E}_i\right) \left(\sum_{k=1}^{p} \lambda_k^\nu \mathbf{E}_k\right) = \sum_{i=1}^{p} \lambda_i^\mu \left(\sum_{k=1}^{p} \lambda_k^\nu \mathbf{E}_i \mathbf{E}_k\right)$$

Applying properties 2 and 3, we get,

$$\mathbf{C}^{\mu}\mathbf{C}^{\nu} = \sum_{i=1}^{p} \lambda_i^{\mu}(\lambda_i^{\nu}\mathbf{E}_i) = \sum_{i=1}^{p} \lambda_i^{(\mu+\nu)}\mathbf{E}_i$$

$$\therefore\ \mathbf{C}^{\mu}\mathbf{C}^{\nu} = \mathbf{C}^{\mu+\nu}\,.$$

Thus, the extension of integral powers of matrices such as $\mathbf{C}$ to include fractional powers is well justified. We can now apply the mathematical tools derived here to address the problem at hand, namely obtaining a fractional power of the DFT matrix.

1.6 Spectral Decomposition of the DFT Matrix

From McClellan and Parks [5], we know that the DFT matrix $\mathbf{W}$ has four distinct eigenvalues, namely $1, j, -1$, and $-j$. Hence, the characteristic polynomial of $\mathbf{W}$, denoted by $c(\lambda)$, has the following form.

$$c(\lambda) = (\lambda - 1)^{\mu_1}(\lambda - j)^{\mu_2}(\lambda + 1)^{\mu_3}(\lambda + j)^{\mu_4} \tag{1.23}$$

Here, the μ_i's represent the multiplicities of the eigenvalues $1, j, -1$, and $-j$ respectively. Also from [5], we know that the multiplicities are distributed as in Table 1.1 when $\mathbf{W}$ is $N \times N$.

N	μ_1	μ_2	μ_3	μ_4
$4m$	$m+1$	$m-1$	m	m
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	m	$m+1$	m
$4m+3$	$m+1$	m	$m+1$	$m+1$

Table 1.1: Distribution of the Multiplicities of the Eigenvalues of the DFT Matrix

Consider the polynomial defined as follows.

$$m(\lambda) \triangleq (\lambda - 1)(\lambda - j)(\lambda + 1)(\lambda + j) = \lambda^4 - 1 \tag{1.24}$$

We can now show that $m(\lambda)$ is indeed the minimal polynomial of $\mathbf{W}$. To prove this, we need to use two properties of minimal polynomials in general which come from Cullen [6].

Property 2 *If $p(\lambda)$ is the minimal polynomial of a matrix $\mathbf{A}$, then every eigenvalue of $\mathbf{A}$ is a zero of $p(\lambda)$.*

Property 3 *If $p(\lambda)$ is the minimal polynomial of a matrix $\mathbf{A}$ with characteristic polynomial $f(\lambda)$, then $p(\lambda)|f(\lambda)$. That is, $p(\lambda)$ divides $f(\lambda)$.*

Given the form of (1.23), the polynomial with the smallest possible order which satisfies these necessary conditions for a minimal polynomial is that given in (1.24). Hence, it follows that if $m(\mathbf{W}) = \mathbf{0}$, then $m(\lambda)$ is indeed the minimal polynomial by virtue of its order. We have,

$$m(\mathbf{W}) = \mathbf{W}^4 - \mathbf{I}$$

But recall from (1.13) that we have,

$$\mathbf{W}^4 = \mathbf{I}$$

Therefore, we conclude,

$$m(\mathbf{W}) = \mathbf{I} - \mathbf{I} = \mathbf{0}\,,$$

and so $m(\lambda)$ is indeed the minimal polynomial of $\mathbf{W}$.

Now, $m(\lambda)$ as defined here has the form required for spectral decomposition, since all of its factors are distinct. Without loss of generality, we can set $\lambda_1 = 1$, $\lambda_2 = j$, $\lambda_3 = -1$, and $\lambda_4 = -j$. Applying the method of spectral decomposition, we have,

$$\mathbf{W} = \mathbf{E}_1 + j\mathbf{E}_2 - \mathbf{E}_3 - j\mathbf{E}_4 \tag{1.25}$$

The constituent matrices $\mathbf{E}_i$ for $i = 1, 2, 3$, and 4 are given below as follows from (1.20).

$$\begin{aligned}
\mathbf{E}_1 &= \tfrac{1}{4}(\mathbf{I} + \mathbf{W} + \mathbf{W}^2 + \mathbf{W}^3) \\
\mathbf{E}_2 &= \tfrac{1}{4}(\mathbf{I} - j\mathbf{W} - \mathbf{W}^2 + j\mathbf{W}^3) \\
\mathbf{E}_3 &= \tfrac{1}{4}(\mathbf{I} - \mathbf{W} + \mathbf{W}^2 - \mathbf{W}^3) \\
\mathbf{E}_4 &= \tfrac{1}{4}(\mathbf{I} + j\mathbf{W} - \mathbf{W}^2 - j\mathbf{W}^3)
\end{aligned}$$

We can express this more compactly in matrix form as follows.

$$\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{j}{2} & -\frac{1}{2} & \frac{j}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{j}{2} & -\frac{1}{2} & -\frac{j}{2} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{W} \\ \mathbf{W}^2 \\ \mathbf{W}^3 \end{bmatrix}$$

$$\text{or } \vec{\mathbf{E}} = \frac{1}{2}\mathbf{W}_4\vec{\mathbf{W}} \text{ , where we have,} \tag{1.26}$$

$$\vec{\mathbf{E}} \triangleq \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix}, \quad \vec{\mathbf{W}} \triangleq \begin{bmatrix} \mathbf{I} \\ \mathbf{W} \\ \mathbf{W}^2 \\ \mathbf{W}^3 \end{bmatrix}$$

Here, $\mathbf{W}_4$ is the $N = 4$ point DFT matrix. We will now proceed to derive the matrix form of the FDFT by taking a fractional power of $\mathbf{W}$.

1.7 Matrix Form of the FDFT

From (1.25), we have,

$$\mathbf{W} = \mathbf{E}_1 + e^{j\frac{\pi}{2}}\mathbf{E}_2 + e^{j\pi}\mathbf{E}_3 + e^{j\frac{3\pi}{2}}\mathbf{E}_4 \tag{1.27}$$

So, we now have the following for the FDFT matrix, $\mathbf{A}(\alpha)$, by using the definition of a fractional power of a matrix considered in Section 1.5.

$$\mathbf{A}(\alpha) = \mathbf{W}^{\frac{2\alpha}{\pi}} = \mathbf{E}_1 + e^{j\alpha}\mathbf{E}_2 + e^{j2\alpha}\mathbf{E}_3 + e^{j3\alpha}\mathbf{E}_4 \tag{1.28}$$

Since the $\mathbf{E}_i$'s are simply linear combinations of the first four powers of the DFT matrix starting from the zeroth power of $\mathbf{W}$, we can express $\mathbf{A}(\alpha)$ in terms of these matrices. This will be more convenient here, since we have more knowledge about the DFT matrix than we do of the constituent matrices. We have,

$$\mathbf{A}(\alpha) = \sum_{i=0}^{3} a_i(\alpha)\mathbf{W}^i \text{ , where we have,}$$

$$a_0(\alpha) = \frac{1}{4}\left(1 + e^{j\alpha} + e^{j2\alpha} + e^{j3\alpha}\right) \tag{1.29}$$

$$a_1(\alpha) = \frac{1}{4}\left(1 - je^{j\alpha} - e^{j2\alpha} + je^{j3\alpha}\right) \tag{1.30}$$

$$a_2(\alpha) = \frac{1}{4}\left(1 - e^{j\alpha} + e^{j2\alpha} - e^{j3\alpha}\right) \tag{1.31}$$

$$a_3(\alpha) = \frac{1}{4}\left(1 + je^{j\alpha} - e^{j2\alpha} - je^{j3\alpha}\right) \tag{1.32}$$

We can also express this in terms of matrices in a form identical to the one we had previously for the $\mathbf{E}_i$'s. This is shown below.

$$\begin{bmatrix} a_0(\alpha) \\ a_1(\alpha) \\ a_2(\alpha) \\ a_3(\alpha) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{j}{2} & -\frac{1}{2} & \frac{j}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{j}{2} & -\frac{1}{2} & -\frac{j}{2} \end{bmatrix} \begin{bmatrix} 1 \\ e^{j\alpha} \\ e^{j2\alpha} \\ e^{j3\alpha} \end{bmatrix}$$

$$\text{or } \vec{a}(\alpha) = \frac{1}{2}\mathbf{W}_4\vec{e}(\alpha) \text{ , where we have,}$$

$$\vec{a}(\alpha) \triangleq \begin{bmatrix} a_0(\alpha) \\ a_1(\alpha) \\ a_2(\alpha) \\ a_3(\alpha) \end{bmatrix}, \quad \vec{e}(\alpha) \triangleq \begin{bmatrix} 1 \\ e^{j\alpha} \\ e^{j2\alpha} \\ e^{j3\alpha} \end{bmatrix}$$

It should be noted that the FDFT matrix obtained here is not unique. In fact, this version of the FDFT is slightly different from that obtained by Santhanam and McClellan [3] in that the coefficients $a_i(\alpha)$ are different. They obtained the following results for these coefficients.

$$\begin{aligned} a_0(\alpha) &= \frac{1}{4}\left(1 + e^{j\alpha} + e^{j2\alpha} + e^{-j\alpha}\right) \\ a_1(\alpha) &= \frac{1}{4}\left(1 - je^{j\alpha} - e^{j2\alpha} + je^{-j\alpha}\right) \\ a_2(\alpha) &= \frac{1}{4}\left(1 - e^{j\alpha} + e^{j2\alpha} - e^{-j\alpha}\right) \\ a_3(\alpha) &= \frac{1}{4}\left(1 + je^{j\alpha} - e^{j2\alpha} - je^{-j\alpha}\right) \end{aligned}$$

The reason for this discrepancy is because the fractional power of a matrix defined in Definition 1 does not yield a unique matrix. To see this, let $\mathbf{C}$ be a matrix with a spectral decomposition as follows.

$$\mathbf{C} = \sum_{i=1}^{p} \lambda_i \mathbf{E}_i$$

In general, we can write λ_i in polar form as $\lambda_i = \rho_i e^{j\theta_i}$. So certainly, we have $\lambda_i = \rho_i e^{j(\theta_i + 2\pi n)}$ for any $n \in \mathbb{Z}$. Now, by definition, the μ-th power of $\mathbf{C}$ is given as,

$$\mathbf{C}^\mu = \sum_{i=1}^{p} \lambda_i^\mu \mathbf{E}_i = \sum_{i=1}^{p} \left(\rho_i e^{j(\theta_i + 2\pi n)}\right)^\mu \mathbf{E}_i$$

$$\mathbf{C}^\mu = \sum_{i=1}^{p} \rho_i^\mu e^{j(\mu\theta_i + 2\pi\mu n)} \mathbf{E}_i$$

Now suppose that μ is real. We can see now that for different choices of n, we will have different possible values for $\mathbf{C}^\mu$ depending on the value of μ. If μ is rational, we will have finitely many different possibilities for $\mathbf{C}^\mu$, whereas if μ is irrational, then there will be infinitely many ones.

As we can see from (1.27), in my analysis of the DFT matrix, I chose arbitrarily all of the arguments of the eigenvalues of the DFT to be in the interval $[0, 2\pi)$. In particular, since $-j = e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}}$, each of the factors of $e^{j3\alpha}$ in (1.30)–(1.32) could have been replaced by $e^{-j\alpha}$, which seems to be what Santhanam and McClellan did. This is the reason for the discrepancy between the version of the FDFT matrix obtained by Santhanam and McClellan and the one which was derived here.

1.8 Kernel Form of the FDFT

Recall that the kernel of the FDFT summation operator $K_\alpha[l,n]$ is simply the (l,n)-th element of $\mathbf{A}(\alpha)$. That is,

$$\mathbf{A}(\alpha) = [K_\alpha[l,n]]_{N\times N}$$

But, we now have the following for the (l,n)-th element of the matrices $\mathbf{I}$, $\mathbf{W}$, $\mathbf{W}^2$, and $\mathbf{W}^3$.

$$\begin{aligned}
\mathbf{I} &= [\delta[l-n]]_{N\times N} \\
\mathbf{W} &= \left[\frac{1}{\sqrt{N}}e^{\left(-j\frac{2\pi l n}{N}\right)}\right]_{N\times N} \\
\mathbf{W}^2 &= [\delta[l+n]]^{1}_{N\times N} \\
\mathbf{W}^3 &= \left[\frac{1}{\sqrt{N}}e^{\left(j\frac{2\pi l n}{N}\right)}\right]_{N\times N}
\end{aligned}$$

Therefore, we have the following expression for the kernel of the FDFT, where the $a_i(\alpha)$'s are given in (1.30)–(1.32).

$$K_\alpha[l,n] = a_0(\alpha)\delta[l-n] + \frac{a_1(\alpha)}{\sqrt{N}}e^{\left(-j\frac{2\pi l n}{N}\right)} + a_2(\alpha)\delta[l+n] + \frac{a_3(\alpha)}{\sqrt{N}}e^{\left(j\frac{2\pi l n}{N}\right)}$$

1.9 Summary of Results

Through the method of spectral decomposition, we were able to derive a matrix form for the FDFT, which we denoted here by $\mathbf{A}(\alpha)$. From there, we were able to determine an expression for the kernel of the FDFT summation operator, which we denoted here by $K_\alpha[l,n]$. To recap, we present these results below.

$$\boxed{\mathbf{A}(\alpha) = \sum_{i=0}^{3} a_i(\alpha)\mathbf{W}^i}$$

$$\boxed{K_\alpha[l,n] = a_0(\alpha)\delta[l-n] + \frac{a_1(\alpha)}{\sqrt{N}}e^{\left(-j\frac{2\pi l n}{N}\right)} + a_2(\alpha)\delta[l+n] + \frac{a_3(\alpha)}{\sqrt{N}}e^{\left(j\frac{2\pi l n}{N}\right)}}$$

The coefficients $a_i(\alpha)$ are given below as follows.

$$\boxed{\begin{aligned}
a_0(\alpha) &= \tfrac{1}{4}\left(1 + e^{j\alpha} + e^{j2\alpha} + e^{j3\alpha}\right) \\
a_1(\alpha) &= \tfrac{1}{4}\left(1 - je^{j\alpha} - e^{j2\alpha} + je^{j3\alpha}\right) \\
a_2(\alpha) &= \tfrac{1}{4}\left(1 - e^{j\alpha} + e^{j2\alpha} - e^{j3\alpha}\right) \\
a_3(\alpha) &= \tfrac{1}{4}\left(1 + je^{j\alpha} - e^{j2\alpha} - je^{j3\alpha}\right)
\end{aligned}}$$

[1] We actually have $\mathbf{W}^2 = [\delta[l+n \pmod N)]]_{N\times N}$.

Here, $\delta[l+n \pmod N)] = \begin{cases} \delta[n], & l=0 \\ \delta[l-(N-n)], & l=1,2,\ldots,N-1 \end{cases}$.

Classically, $\mathbf{W}^2$ is referred to as the circular flip matrix.

Chapter 2

Properties of the FDFT

In this chapter, we will explore the various properties of the FDFT. We will start by discussing the matrix form of the FDFT and then we will consider the kernel form. Finally, we will summarize the results and conclude with an example of an FDFT pair.

2.1 Properties of the FDFT Matrix

2.1.1 Angle Additivity

By construction, $\mathbf{A}(\alpha)$ satisfies the following properties.

$$\mathbf{A}(0) = \mathbf{I}$$

$$\mathbf{A}\left(\frac{\pi}{2}\right) = \mathbf{W}$$

$$\mathbf{A}(\alpha)\mathbf{A}(\beta) = \mathbf{A}(\alpha + \beta)$$

This last property is called the angle additivity property, since applying a rotation by an angle of α to a signal already rotated by an angle of β results in a signal rotated by a net angle of $(\alpha + \beta)$. This property is the foundation upon which we regard the FDFT operator as a rotation operator.

2.1.2 Periodicity and Multiple Rotations

We have,

$$\mathbf{A}(\alpha + 2\pi n) = \mathbf{A}(\alpha) \quad \forall\, n \in \mathbb{Z}$$

Thus, $\mathbf{A}(\alpha)$ is periodic with respect to the angular parameter α with period 2π.

Proof: From (1.28), we have,

$$\mathbf{A}(\alpha) = \mathbf{E}_1 + e^{j\alpha}\mathbf{E}_2 + e^{j2\alpha}\mathbf{E}_3 + e^{j3\alpha}\mathbf{E}_4$$

So, we get,

$$\mathbf{A}(\alpha + 2\pi n) = \mathbf{E}_1 + e^{j(\alpha+2\pi n)}\mathbf{E}_2 + e^{j2(\alpha+2\pi n)}\mathbf{E}_3 + e^{j3(\alpha+2\pi n)}\mathbf{E}_4$$

$$\mathbf{A}(\alpha + 2\pi n) = \mathbf{E}_1 + e^{j\alpha}\mathbf{E}_2 + e^{j2\alpha}\mathbf{E}_3 + e^{j3\alpha}\mathbf{E}_4 = \mathbf{A}(\alpha)$$

This completes the proof.

We also have,

$$\mathbf{A}^m(\alpha) = \mathbf{A}(m\alpha) \quad \forall\, m \in \mathbb{N}$$

This is called the property of multiple rotations. From this property, we see that multiple applications of the FDFT operator result in the angular measure of the signal operated on to be multiplied. This property can be easily proven by induction using the angle additivity property.

2.1.3 Unitarity

The FDFT matrix is unitary. That is,

$$\mathbf{A}^{\dagger}(\alpha)\mathbf{A}(\alpha) = \mathbf{I}$$

Proof: Recall from (1.26) that the constituent matrices, $\mathbf{E}_i$, for the FDFT matrix are related to the first four powers of the DFT matrix (starting from the zeroeth power) as follows.

$$\vec{\mathbf{E}} = \frac{1}{2}\mathbf{W}_4\vec{\mathbf{W}}$$

Here, $\vec{\mathbf{E}}$ is a vector of the constituent matrices while $\vec{\mathbf{W}}$ is a vector of the first four powers of the DFT matrix. So, we have,

$$\vec{\mathbf{E}}^{\dagger} = \frac{1}{2}\vec{\mathbf{W}}^{\dagger}\mathbf{W}_4^{\dagger}$$

Thus, we get,

$$\vec{\mathbf{E}}^{\dagger}\vec{\mathbf{E}} = \frac{1}{4}\vec{\mathbf{W}}^{\dagger}\mathbf{W}_4^{\dagger}\mathbf{W}_4\vec{\mathbf{W}}$$

Now, we know that the $N \times N$ DFT matrix $\mathbf{W}$ is unitary for any N. So, we have,

$$\mathbf{W}^{\dagger}\mathbf{W} = \mathbf{I}$$

Substituting this into the equation above yields,

$$\vec{\mathbf{E}}^{\dagger}\vec{\mathbf{E}} = \frac{1}{4}\vec{\mathbf{W}}^{\dagger}\mathbf{I}\vec{\mathbf{W}} = \frac{1}{4}\vec{\mathbf{W}}^{\dagger}\vec{\mathbf{W}}$$

But, we have,

$$\vec{\mathbf{E}}^{\dagger}\vec{\mathbf{E}} = \sum_{i=1}^{4}\mathbf{E}_i^{\dagger}\mathbf{E}_i \text{ and } \vec{\mathbf{W}}^{\dagger}\vec{\mathbf{W}} = \sum_{i=0}^{3}\left(\mathbf{W}^i\right)^{\dagger}\mathbf{W}^i$$

Now, we have,

$$\vec{\mathbf{W}}^{\dagger}\vec{\mathbf{W}} = \mathbf{I}^{\dagger}\mathbf{I} + \mathbf{W}^{\dagger}\mathbf{W} + \left(\mathbf{W}^2\right)^{\dagger}\mathbf{W}^2 + \left(\mathbf{W}^3\right)^{\dagger}\mathbf{W}^3$$

$$\vec{\mathbf{W}}^{\dagger}\vec{\mathbf{W}} = \mathbf{I} + \mathbf{I} + \mathbf{W}^{\dagger}\mathbf{W}^{\dagger}\mathbf{W}\mathbf{W} + \mathbf{W}^{\dagger}\mathbf{W}^{\dagger}\mathbf{W}^{\dagger}\mathbf{W}\mathbf{W}\mathbf{W}$$

$$\vec{\mathbf{W}}^{\dagger}\vec{\mathbf{W}} = \mathbf{I} + \mathbf{I} + \mathbf{I} + \mathbf{I} = 4\mathbf{I}$$

So, we conclude,

$$\vec{\mathbf{E}}^{\dagger}\vec{\mathbf{E}} = \sum_{i=1}^{4}\mathbf{E}_i^{\dagger}\mathbf{E}_i = \frac{1}{4}\vec{\mathbf{W}}^{\dagger}\vec{\mathbf{W}} = \frac{1}{4}(4\mathbf{I}) = \mathbf{I}$$

$$\therefore \sum_{i=1}^{4}\mathbf{E}_i^{\dagger}\mathbf{E}_i = \mathbf{I}.$$

Postmultiplying both sides by $\mathbf{E}_j$ for some $j = 1, 2, 3$, or 4, we get the following, by applying properties 2 and 3 of Theorem 1 for the constituent matrices.

$$\sum_{i=1}^{4}\mathbf{E}_i^{\dagger}\mathbf{E}_i\mathbf{E}_j = \mathbf{E}_j$$

$$\sum_{i=1}^{4}\mathbf{E}_i^{\dagger}(\mathbf{E}_i\mathbf{E}_j) = \mathbf{E}_j^{\dagger}\mathbf{E}_j^2 = \mathbf{E}_j^{\dagger}\mathbf{E}_j = \mathbf{E}_j$$

Taking the transpose conjugate of both sides yields,

$$\mathbf{E}_j^{\dagger} = \left(\mathbf{E}_j^{\dagger}\mathbf{E}_j\right)^{\dagger} = \mathbf{E}_j^{\dagger}\mathbf{E}_j = \mathbf{E}_j$$

$$\therefore\ \mathbf{E}_j^\dagger = \mathbf{E}_j\,.$$

Thus, the constituent matrices for the FDFT matrix are Hermitian. With this, we can now easily prove that $\mathbf{A}(\alpha)$ is unitary. We have, from (1.28),

$$\mathbf{A}(\alpha) = \mathbf{E}_1 + e^{j\alpha}\mathbf{E}_2 + e^{j2\alpha}\mathbf{E}_3 + e^{j3\alpha}\mathbf{E}_4$$

By taking the transpose conjugate of both sides yields,

$$\mathbf{A}^\dagger(\alpha) = \mathbf{E}_1^\dagger + e^{-j\alpha}\mathbf{E}_2^\dagger + e^{-j2\alpha}\mathbf{E}_3^\dagger + e^{-j3\alpha}\mathbf{E}_4^\dagger$$

Since the constituent matrices are Hermitian, we get,

$$\mathbf{A}^\dagger(\alpha) = \mathbf{E}_1 + e^{-j\alpha}\mathbf{E}_2 + e^{-j2\alpha}\mathbf{E}_3 + e^{-j3\alpha}\mathbf{E}_4$$

Postmultiplying by $\mathbf{A}(\alpha)$, we get, by properties 2 and 3,

$$\mathbf{A}^\dagger(\alpha)\mathbf{A}(\alpha) = \mathbf{E}_1^2 + \mathbf{E}_2^2 + \mathbf{E}_3^2 + \mathbf{E}_4^2 = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4 = \sum_{i=1}^{4}\mathbf{E}_i$$

But, by property 1, we have,

$$\mathbf{A}^\dagger(\alpha)\mathbf{A}(\alpha) = \sum_{i=1}^{4}\mathbf{E}_i = \mathbf{I}$$

$$\therefore\ \mathbf{A}^\dagger(\alpha)\mathbf{A}(\alpha) = \mathbf{I}\,.$$

This completes the proof. So, much like the DFT matrix and the identity matrix, the FDFT matrix is unitary for any angle α.

2.1.4 Symmetry

The FDFT matrix is symmetric, that is,

$$\mathbf{A}^T(\alpha) = \mathbf{A}(\alpha)$$

Proof: In Chapter 1, we found,

$$\mathbf{A}(\alpha) = \sum_{i=0}^{3} a_i(\alpha)\mathbf{W}^i$$

Thus, we get,

$$\mathbf{A}^T(\alpha) = \left(\sum_{i=0}^{3} a_i(\alpha)\mathbf{W}^i\right)^T = \sum_{i=0}^{3}\left(a_i(\alpha)\mathbf{W}^i\right)^T = \sum_{i=0}^{3} a_i(\alpha)\left(\mathbf{W}^i\right)^T = \sum_{i=0}^{3} a_i(\alpha)\left(\mathbf{W}^T\right)^i$$

But, we know that the DFT matrix is symmetric, so that $\mathbf{W}^T = \mathbf{W}$. Applying this, we get,

$$\mathbf{A}^T(\alpha) = \sum_{i=0}^{3} a_i(\alpha)\mathbf{W}^i = \mathbf{A}(\alpha)\,,$$

as desired. Thus, much like the DFT matrix, the FDFT matrix is symmetric for any angle α.

2.1.5 Inversion and Hermitian Symmetry

The inverse of the FDFT matrix is given as follows.

$$\mathbf{A}^{-1}(\alpha) = \mathbf{A}(-\alpha) = \mathbf{A}^\dagger(\alpha)$$

Proof: From angle additivity, we have, by setting $\beta = -\alpha$,

$$\mathbf{A}(\alpha)\mathbf{A}(-\alpha) = \mathbf{A}(\alpha - \alpha) = \mathbf{A}(0) = \mathbf{I}$$

Since $\mathbf{A}(\alpha)$ is unitary, we know that $|\det(\mathbf{A}(\alpha))| = 1$. Because we have, $\det(\mathbf{A}(\alpha)) \neq 0$, we know that $\mathbf{A}^{-1}(\alpha)$ exists. So, by premultiplying by $\mathbf{A}^{-1}(\alpha)$, we get,

$$\mathbf{A}^{-1}(\alpha) = \mathbf{A}(-\alpha)$$

Now, from the fact that $\mathbf{A}(\alpha)$ is unitary, we have,

$$\mathbf{A}^{\dagger}(\alpha)\mathbf{A}(\alpha) = \mathbf{I}$$

Postmultiplying by $\mathbf{A}^{-1}(\alpha)$, we get,

$$\mathbf{A}^{-1}(\alpha) = \mathbf{A}^{\dagger}(\alpha)$$

So, we conclude,

$$\mathbf{A}^{-1}(\alpha) = \mathbf{A}(-\alpha) = \mathbf{A}^{\dagger}(\alpha)\,,$$

as desired. Thus, the calculation of the inverse of the FDFT matrix is a trivial task. From above, we have,

$$\mathbf{A}(\alpha) = \mathbf{A}^{\dagger}(-\alpha)$$

Hence, the FDFT is Hermitian symmetric with respect to the angular parameter α.

2.1.6 Parseval's Relation

Recall from (1.15) that we have,

$$\mathbf{X}_\alpha = \mathbf{A}(\alpha)\mathbf{x}$$

Here, $\mathbf{X}_\alpha$ is the vector form of the FDFT of the finite length sequence $x[n]$ and $\mathbf{x}$ is the vector form of $x[n]$. We then have a Parseval's relation as follows.

$$||\mathbf{X}_\alpha||^2 = ||\mathbf{x}||^2$$

Proof: Taking the transpose conjugate of (1.15), we get,

$$\mathbf{X}_\alpha^{\dagger} = (\mathbf{A}(\alpha)\mathbf{x})^{\dagger} = \mathbf{x}^{\dagger}\mathbf{A}^{\dagger}(\alpha)$$

So, since $\mathbf{A}(\alpha)$ is unitary, we get,

$$\mathbf{X}_\alpha^{\dagger}\mathbf{X}_\alpha = \left(\mathbf{x}^{\dagger}\mathbf{A}^{\dagger}(\alpha)\right)\left(\mathbf{A}(\alpha)\mathbf{x}\right) = \mathbf{x}^{\dagger}\left(\mathbf{A}^{\dagger}(\alpha)\mathbf{A}(\alpha)\right)\mathbf{x} = \mathbf{x}^{\dagger}\mathbf{I}\mathbf{x} = \mathbf{x}^{\dagger}\mathbf{x}$$

So, we conclude,

$$||\mathbf{X}_\alpha||^2 = ||\mathbf{x}||^2\,,$$

as desired. Because of this property, we can view the FDFT as an isometric operator which preserves the energy of a signal.

2.2 Properties of the FDFT Kernel

2.2.1 Angle Additivity

We have,

$$\sum_{m=0}^{N-1} K_\alpha[l,m]K_\beta[m,n] = K_{\alpha+\beta}[l,n]$$

Proof: In Section 2.1.1, we showed that,

$$\mathbf{A}(\alpha)\mathbf{A}(\beta) = \mathbf{A}(\alpha+\beta)$$

But since we have, $\mathbf{A}(\alpha) = [K_\alpha[l,n]]_{N\times N}$, we get, by equating corresponding elements of the matrices above,

$$\sum_{m=0}^{N-1} K_\alpha[l,m]K_\beta[m,n] = K_{\alpha+\beta}[l,n]\,,$$

as desired.

2.2.2 Periodicity

The kernel satisfies,

$$K_{\alpha+2\pi m}[l,n] = K_\alpha[l,n] \quad \forall\ m \in \mathbb{Z}$$

This follows directly from the periodicity of $\mathbf{A}(\alpha)$. Thus, the kernel of the FDFT is periodic with respect to its angular argument α with period 2π.

2.2.3 Symmetry of Discrete Arguments

We have,

$$K_\alpha[l,n] = K_\alpha[n,l]$$

This follows directly from the fact that $\mathbf{A}(\alpha)$ is symmetric. It can also be easily verified by considering the explicit form of $K_\alpha[l,n]$ derived in Chapter 1.

2.2.4 Hermitian Symmetry of Angular Argument

From the Hermitian symmetry of $\mathbf{A}(\alpha)$, we immediately conclude,

$$K_\alpha[l,n] = K^*_{-\alpha}[n,l]$$

By using the symmetry property discussed in Section 2.2.3, we have,

$$K^*_{-\alpha}[n,l] = K^*_{-\alpha}[l,n]$$

So, we conclude,

$$K_\alpha[l,n] = K^*_{-\alpha}[l,n]$$

Thus, the kernel of the FDFT is Hermitian symmetric with respect to the angular parameter α.

2.2.5 Orthonormality

The FDFT kernel satisfies,

$$\sum_{m=0}^{N-1} K_\alpha[m,n]K^*_\alpha[m,p] = \delta[n-p]$$

Proof: By angle additivity, we get, by setting $\beta = -\alpha$,

$$\sum_{m=0}^{N-1} K_\alpha[n,m]K_{-\alpha}[m,p] = K_0[n,p]$$

But since $\mathbf{A}(0) = \mathbf{I}$, we have $K_0[n,p] = \delta[n-p]$. Also, by Hermitian symmetry, $K_{-\alpha}[m,p] = K^*_\alpha[m,p]$. So, we get,

$$\sum_{m=0}^{N-1} K_\alpha[n,m]K^*_\alpha[m,p] = \delta[n-p]$$

Now, by symmetry, we have, $K_\alpha[n,m] = K_\alpha[m,n]$. So, we conclude,

$$\sum_{m=0}^{N-1} K_\alpha[m,n]K^*_\alpha[m,p] = \delta[n-p]\,,$$

as desired. Thus, for a given angle α, the kernel is orthonormal with respect to its discrete arguments.

2.2.6 Decomposition

For a given angle α, the FDFT kernel is a weighted average of the following four kernels as is shown below.

$$K_\alpha[l,n] = a_0(\alpha)K_0[l,n] + a_1(\alpha)K_{\frac{\pi}{2}}[l,n] + a_2(\alpha)K_\pi[l,n] + a_3(\alpha)K_{\frac{3\pi}{2}}[l,n]$$

This follows directly from the form of $\mathbf{A}(\alpha)$ derived in Chapter 1.

2.3 Summary of the Properties of the FDFT

A list of the properties of the FDFT matrix and kernel derived in this chapter is given in the tables below.

Properties of $\mathbf{A}(\alpha)$	
Name	Expression
Angle Additivity	$\mathbf{A}(\alpha)\mathbf{A}(\beta) = \mathbf{A}(\alpha+\beta)$
Periodicity	$\mathbf{A}(\alpha + 2\pi n) = \mathbf{A}(\alpha)$
Unitary	$\mathbf{A}^{\dagger}(\alpha)\mathbf{A}(\alpha) = \mathbf{I}$
Symmetry	$\mathbf{A}^{T}(\alpha) = \mathbf{A}(\alpha)$
Inverse	$\mathbf{A}^{-1}(\alpha) = \mathbf{A}(-\alpha) = \mathbf{A}^{\dagger}(\alpha)$
Parseval's Relation	$\lvert\lvert\mathbf{X}_{\alpha}\rvert\rvert^2 = \lvert\lvert\mathbf{x}\rvert\rvert^2$

Table 2.1: Properties of the Matrix Form of the FDFT

Properties of $K_{\alpha}[l,n]$	
Name	Expression
Angle Additivity	$\sum_{m=0}^{N-1} K_{\alpha}[l,m]K_{\beta}[m,n] = K_{\alpha+\beta}[l,n]$
Periodicity	$K_{\alpha+2\pi m}[l,n] = K_{\alpha}[l,n]$
Symmetry	$K_{\alpha}[l,n] = K_{\alpha}[n,l]$
Hermitian Symmetry	$K_{\alpha}[l,n] = K^{*}_{-\alpha}[l,n]$
Orthonormality	$\sum_{m=0}^{N-1} K_{\alpha}[m,n]K^{*}_{\alpha}[m,p] = \delta[n-p]$
Decomposition	$K_{\alpha}[l,n] = a_0(\alpha)K_0[l,n] + a_1(\alpha)K_{\frac{\pi}{2}}[l,n]$ $+a_2(\alpha)K_{\pi}[l,n] + a_3(\alpha)K_{\frac{3\pi}{2}}[l,n]$

Table 2.2: Properties of the Kernel Form of the FDFT

2.4 Example of an FDFT Pair

Now, let $x[n]$ denote a finite length sequence of length N. Also, let $X[k]$ denote the DFT of $x[n]$ and $X_{\alpha}[l]$ the FDFT of $x[n]$ of angle α. By the decomposition property, we have,

$$X_{\alpha}[l] = a_0(\alpha)x[l] + a_1(\alpha)X[l] + a_2(\alpha)x[-l] + a_3(\alpha)X[-l]$$

Note that here, we have, $x[-l] = x[-l \pmod N]$ and $X[-l] = X[-l \pmod N]$.

As a rudimentary example of the FDFT, consider the following finite length sequence $x[n]$.

$$x[n] = u[n] - u[n-N]$$

From our knowledge of the DFT, we know that,

$$X[k] = \sqrt{N}\,\delta[k]$$

Indeed $x[-n] = x[n]$ and $X[-k] = X[k]$ here, so we get,

$$X_\alpha[l] = (a_0(\alpha) + a_2(\alpha))\,(u[l] - u[l-N]) + (a_1(\alpha) + a_3(\alpha))\left(\sqrt{N}\,\delta[l]\right)$$

On the following page, we have plotted the magnitude and phase of $X_\alpha[l]$ for $N = 16$ for various angles α as can be seen in Figure 2.1. From these plots, we can see that as $\alpha \to \frac{\pi}{2}$, we have $X_\alpha[l] \to X[l]$.

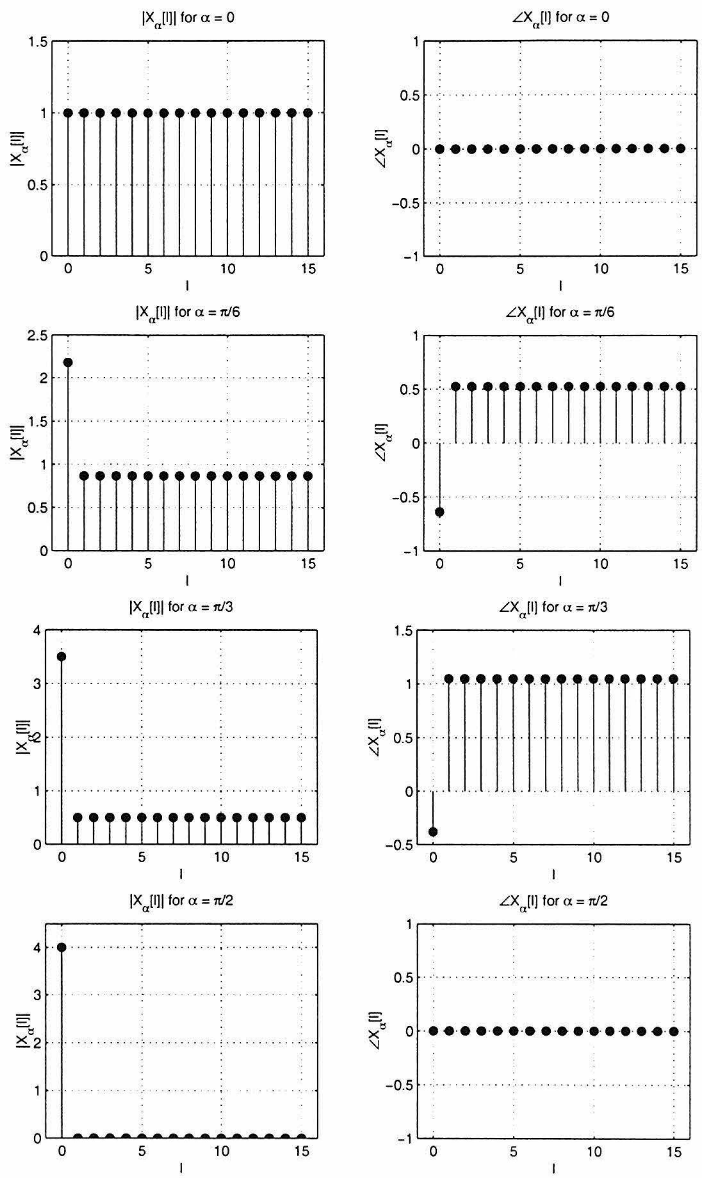

Figure 2.1: Magnitude and phase plots of $X_\alpha[l]$ for various angles α.

Chapter 3

Applications of the FDFT in Signal Processing

In this chapter, we will explore some of the possible uses of the FDFT in digital signal processing. As we will soon see, the FDFT is not computationally efficient. However, it does offer an extra degree of freedom which we will try to exploit.

3.1 The Not So Fast Fourier Transform

The original goal after deriving an expression for the FDFT was to offer an improvement to the Fast Fourier Transform (FFT) algorithm brought about by Cooley and Tukey [7]. They derived an algorithm such that if the dimension of the DFT matrix, N, was of the form $N = 2^m$, then the DFT of a given signal could be performed with a number of operations on the order of $N \log_2 N$, instead of N^2, as would be the case for a general $N \times N$ linear transformation. This has been found to be useful in many applications in signal processing for the following reasons. In many cases, we wish to calculate the output $y[n]$ of an LTI system with impulse response $h[n]$ and input $x[n]$. We have the following linear convolution formula.

$$y[n] = h[n] * x[n] = \sum_k x[k]h[n-k]$$

If the impulse response $h[n]$ is of finite length N_1, i.e, it represents an FIR filter, and $x[n]$ is a finite length input signal of length N_2, then the output $y[n]$ has length N_1+N_2-1. So in this case, $y[n]$ can be equivalently obtained by circular convolution,

$$y[n] = h[n] \otimes x[n] = \sum_{i=0}^{N-1} h[i]x[n-i \pmod{N}], \tag{3.1}$$

provided that $N \geq N_1 + N_2 - 1$. From the circular convolution theorem for the DFT, we have, by taking the N-point DFT of the equation above,

$$Y[k] = \sqrt{N} H[k]X[k]$$

Here is where the advantage of the FFT can be seen. Since the DFT can be implemented with reduced complexity using the FFT, we can compute $y[n]$ by first finding the DFT of $h[n]$ and $x[n]$, perform a point to point multiplication of $H[k]$ and $X[k]$ (with a scale factor of $\sqrt{N}$ included of course), and then take the inverse DFT of the result, which can also be done with the FFT algorithm. This abstract method is actually less complex to implement than direct circular convolution. From Cochran et al. [8], the approximate number of multiplications required to implement $y[n]$ directly by circular convolution is N^2, whereas by using the FFT algorithm, only $3N \log_2 N$ such multiplications are required. So, for N large, we will obtain greater computational efficiency by using the FFT algorithm. As was mentioned earlier, the FDFT unfortunately does not share the same computational efficiency as the DFT. If $Y_\alpha[l]$ denotes the FDFT of $y[n]$ with angle α, then we have by the decomposition property,

$$Y_\alpha[l] = a_0(\alpha)y[l] + a_1(\alpha)Y[l] + a_2(\alpha)y[-l] + a_3(\alpha)Y[-l]$$

Applying (3.1), we get,

$$\begin{aligned} Y_\alpha[l] &= a_0(\alpha)\left(h[l] \otimes x[l]\right) + a_1(\alpha)\left(\sqrt{N}H[l]X[l]\right) \\ &\quad + a_2(\alpha)\left(h[-l] \otimes x[-l]\right) + a_3(\alpha)\left(\sqrt{N}H[-l]X[-l]\right) \end{aligned}$$

As we can see from above, in order to calculate $Y_\alpha[l]$ for some α in general, we must calculate the circular convolution $h[l] \otimes x[l]$, the DFTs $H[l]$ and $X[l]$ as well as their product. Then, we need to reverse these sequences modulo N, multiply each sequence by a *complex* number, and then add everything up. Whereas the DFT offers an alternative to circular convolution, the FDFT does not in general. It is precisely for this reason that the FDFT should not be used for the processing of finite length signals through FIR LTI systems. However, the FDFT does have other uses as we shall soon see.

3.2 Sum and Difference of Allpass Filters

Whereas the FDFT is not in general computationally efficient, the presence of the parameter α does offer an extra degree of freedom in the design of digital filters.

For example, in Vaidyanathan's work on allpass filters [9], the sum and difference of two allpass filters was considered. Using the matrix notation from this correspondence, we have the following structure shown below in Figure 3.1.

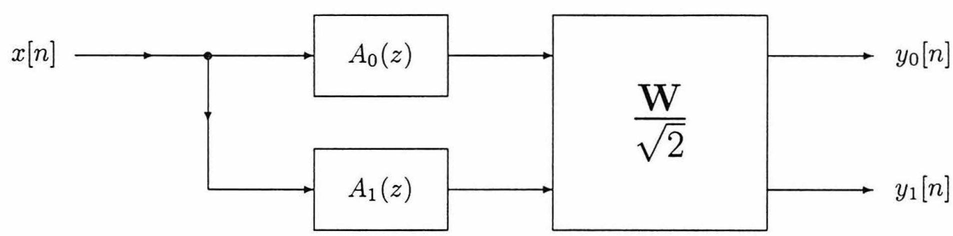

Figure 3.1: Implementation of two transfer functions using the sum and difference of two allpass filters (DFT)

Here, $A_0(z)$ and $A_1(z)$ denote allpass filters. Indeed, if we denote $H_0(z)$ and $H_1(z)$ as the transfer function between $y_0[n]$ and $x[n]$ and between $y_1[n]$ and $x[n]$, respectively, then we have the following.

$$H_0(z) = \frac{Y_0(z)}{X(z)} \quad \text{and} \quad H_1(z) = \frac{Y_1(z)}{X(z)}$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} A_0(z) \\ A_1(z) \end{bmatrix}$$

$$\therefore\ H_0(z) = \frac{A_0(z) + A_1(z)}{2} \quad \text{and} \quad H_1(z) = \frac{A_0(z) - A_1(z)}{2}.$$

So, $H_0(z)$ and $H_1(z)$ are respectively the sum and difference of two allpass filters (scaled by $\frac{1}{2}$ of course). What is interesting about this structure is that the coefficients of the allpass filters can be chosen such that the overall transfer functions $H_0(z)$ and $H_1(z)$ are respectively N-th order lowpass and highpass elliptic filters. If N is chosen to be odd so that $A_0(z)$ and $A_1(z)$ have real coefficients, then we have a computationally efficient structure. For example, a seventh order elliptic lowpass filter normally requires 11 multipliers if the direct form structure is implemented, as opposed to the structure from Figure 3.1, which requires only 7 multipliers. In addition, since the allpass filters can be implemented by lattice structures, the filters $H_0(z)$ and $H_1(z)$ are free from zero input limit cycles. Furthermore, the filter $H_1(z)$ is the power complement of $H_0(z)$, i.e., $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1 \; \forall \; \omega$, and comes as a bonus from this structure. So, we can get, for example, a seventh order elliptic lowpass and highpass filter with a total of only 7 multipliers. With all of these advantages, the question then arises as to what will happen if we replace the 2×2 DFT matrix in the structure above with the 2×2 FDFT matrix, which is a matter that we will now address.

3.3 The FDFT as Applied to the Allpass Filter System

By replacing $\mathbf{W}$ with $\mathbf{A}(\alpha)$ in Figure 3.1, we get the following structure shown in Figure 3.2. Note that to avoid confusion between the allpass filters $A_0(z)$ and $A_1(z)$ and the FDFT matrix $\mathbf{A}(\alpha)$, we have renamed $A_0(z)$ and $A_1(z)$ by $G_0(z)$ and $G_1(z)$, respectively.

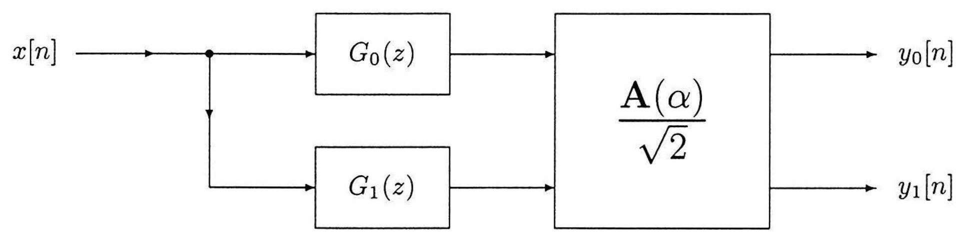

Figure 3.2: Implementation of two transfer functions by applying the FDFT to two allpass filters

Define the following quantities.

$$\mathbf{G}(z) \triangleq \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} \quad \text{and} \quad \mathbf{H}(z) \triangleq \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}$$

As before, we have,

$$H_0(z) = \frac{Y_0(z)}{X(z)} \quad \text{and} \quad H_1(z) = \frac{Y_1(z)}{X(z)}$$

We have then,

$$\mathbf{H}(z) = \frac{1}{\sqrt{2}}\mathbf{A}(\alpha)\mathbf{G}(z) \quad \text{or also} \quad \mathbf{H}(e^{j\omega}) = \frac{1}{\sqrt{2}}\mathbf{A}(\alpha)\mathbf{G}(e^{j\omega}) \tag{3.2}$$

Here, $\mathbf{A}(\alpha)$ is the 2×2 FDFT matrix. We have,

$$\mathbf{A}(\alpha) = \begin{bmatrix} \left(\frac{2+\sqrt{2}}{4}\right) + \left(\frac{2-\sqrt{2}}{4}\right)e^{j2\alpha} & \left(\frac{\sqrt{2}}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)e^{j2\alpha} \\ \left(\frac{\sqrt{2}}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)e^{j2\alpha} & \left(\frac{2-\sqrt{2}}{4}\right) + \left(\frac{2+\sqrt{2}}{4}\right)e^{j2\alpha} \end{bmatrix}$$

So, we get,

$$\frac{1}{\sqrt{2}}\mathbf{A}(\alpha) = \begin{bmatrix} \left(\frac{\sqrt{2}+1}{4}\right) + \left(\frac{\sqrt{2}-1}{4}\right)e^{j2\alpha} & \left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)e^{j2\alpha} \\ \left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)e^{j2\alpha} & \left(\frac{\sqrt{2}-1}{4}\right) + \left(\frac{\sqrt{2}+1}{4}\right)e^{j2\alpha} \end{bmatrix}$$

So, from (3.2), we get,

$$H_0(z) = \left[\left(\frac{\sqrt{2}+1}{4}\right) + \left(\frac{\sqrt{2}-1}{4}\right)e^{j2\alpha}\right]G_0(z) + \left[\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)e^{j2\alpha}\right]G_1(z) \tag{3.3}$$

$$H_1(z) = \left[\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)e^{j2\alpha}\right]G_0(z) + \left[\left(\frac{\sqrt{2}-1}{4}\right) + \left(\frac{\sqrt{2}+1}{4}\right)e^{j2\alpha}\right]G_1(z) \tag{3.4}$$

Indeed, when $\alpha = \frac{\pi}{2}$, we obtain the same structure considered by Vaidyanathan. From (3.2), we can show that $H_0(z)$ and $H_1(z)$ form a power complementary pair. Taking the transpose conjugate of (3.2) yields,

$$\mathbf{H}^{\dagger}(e^{j\omega}) = \frac{1}{\sqrt{2}}\mathbf{G}^{\dagger}(e^{j\omega})\mathbf{A}^{\dagger}(\alpha)$$

Hence, because $\mathbf{A}(\alpha)$ is unitary, we have,

$$\mathbf{H}^{\dagger}(e^{j\omega})\mathbf{H}(e^{j\omega}) = \frac{1}{2}\mathbf{G}^{\dagger}(e^{j\omega})\mathbf{A}^{\dagger}(\alpha)\mathbf{A}(\alpha)\mathbf{G}(e^{j\omega}) = \frac{1}{2}\mathbf{G}^{\dagger}(e^{j\omega})\mathbf{I}\mathbf{G}(e^{j\omega}) =$$

$$\therefore\ \mathbf{H}^{\dagger}(e^{j\omega})\mathbf{H}(e^{j\omega}) = \frac{1}{2}\mathbf{G}^{\dagger}(e^{j\omega})\mathbf{G}(e^{j\omega})\,.$$

But, $\mathbf{G}^{\dagger}(e^{j\omega})\mathbf{G}(e^{j\omega}) = \left|G_0(e^{j\omega})\right|^2 + \left|G_1(e^{j\omega})\right|^2 = 1+1 = 2$, since $G_0(z)$ and $G_1(z)$ are allpass filters. Thus,

$$\mathbf{H}^{\dagger}(e^{j\omega})\mathbf{H}(e^{j\omega}) = \left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = \frac{1}{2}(2) = 1\,,$$

and so $H_0(z)$ and $H_1(z)$ form a power complementary pair.

Suppose that the coefficients of $G_0(z)$ and $G_1(z)$ are chosen such that $H_0(z)$ and $H_1(z)$ are respectively elliptic lowpass and highpass filters when $\alpha = \frac{\pi}{2}$. From a heuristic viewpoint, it is interesting to see what will happen to $H_0(z)$ and $H_1(z)$ for different angles. In particular, we wish to consider what happens for α near $\frac{\pi}{2}$.

One possible reason for this is to make the phase of the filters $H_0(z)$ and $H_1(z)$ more linear in the passband region because in many applications such as image processing, it is desirable to have linear phase in the passband region of a frequency selective filter. The problem is that while the elliptic filter offers the best desired magnitude performance for a given order, its phase response is highly nonlinear in the passband. From the McClellan-Parks algorithm [10], we know that we can design an equiripple FIR approximation to an elliptic filter that has the same magnitude respose performance as the elliptic filter, but with exactly linear phase. The problem here, however, is that the order of the resulting filter is typically much larger than the corresponding elliptic filter's order (sometimes as much as 10 to 15 times as large as the order of the elliptic filter). So, heuristically, it is worthwhile to see if we can possibly "linearize" the phase response of $H_0(z)$ say, by varying the angle α near $\frac{\pi}{2}$. If this can be done without significantly altering the magnitude response of the original elliptic filter, then we will have a better filter overall. To compensate for any digressions from the desired magnitude response, we can then increase slightly the order of the original elliptic filter.

If indeed varying α near $\frac{\pi}{2}$ results in a linearization of the phase in the passband without significantly degrading the magnitude response of the new filter, we will have effectively found a tradeoff between the elliptic filter with a small order but highly nonlinear phase and the equiripple FIR filter with exactly linear phase but with a much higher order. Continuing with our analysis, we have, from (3.3) and (3.4),

$$H_0(z) = \frac{1}{2}e^{j\alpha}\left[\left(\sqrt{2}\cos\alpha - j\sin\alpha\right)G_0(z) - j\sin\alpha\ G_1(z)\right] \tag{3.5}$$

$$H_1(z) = \frac{1}{2}e^{j\alpha}\left[\left(\sqrt{2}\cos\alpha + j\sin\alpha\right)G_1(z) - j\sin\alpha\ G_0(z)\right]$$

For brevity, we will only consider $H_0(z)$ from now on. On the unit circle, i.e., when $z = e^{j\omega}$, we have,

$$G_0(e^{j\omega}) = e^{j\phi_0(\omega)} \quad \text{and} \quad G_1(e^{j\omega}) = e^{j\phi_1}(\omega)\,,$$

since $G_0(z)$ and $G_1(z)$ are allpass filters. So, from (3.5), we get,

$$H_0(e^{j\omega}) = \frac{1}{2}e^{j\alpha}\left[\left(\sqrt{2}\cos\alpha - j\sin\alpha\right)e^{j\phi_0(\omega)} - j\sin\alpha\; e^{j\phi_1(\omega)}\right]$$

After much laborious algebra, we conclude the following.

$$\left|H_0(e^{j\omega})\right| = \sqrt{\frac{1}{2}\left[1 + \sin\alpha\left(\sin\alpha\cos\left(\phi_0(\omega) - \phi_1(\omega)\right) - \sqrt{2}\cos\alpha\sin\left(\phi_0(\omega) - \phi_1(\omega)\right)\right)\right]}$$

$$\measuredangle H_0(e^{j\omega}) = \arctan\left[\frac{\sqrt{2}\cos\alpha\sin\phi_0(\omega) - \sin\alpha\left(\cos\phi_0(\omega) + \cos\phi_1(\omega)\right)}{\sqrt{2}\cos\alpha\cos\phi_0(\omega) + \sin\alpha\left(\sin\phi_0(\omega) + \sin\phi_1(\omega)\right)}\right] + \alpha$$

The following Matlab plots from Figure 3.3 show the effects of variations of α near $\frac{\pi}{2}$ on the magnitude and phase of $H_0(e^{j\omega})$. To illustrate the phase nonlinearity, we have opted to plot the group delay of $H_0(e^{j\omega})$, which is simply $\tau(\omega) \triangleq -\frac{d}{d\omega}\left(\measuredangle H_0(e^{j\omega})\right)$.

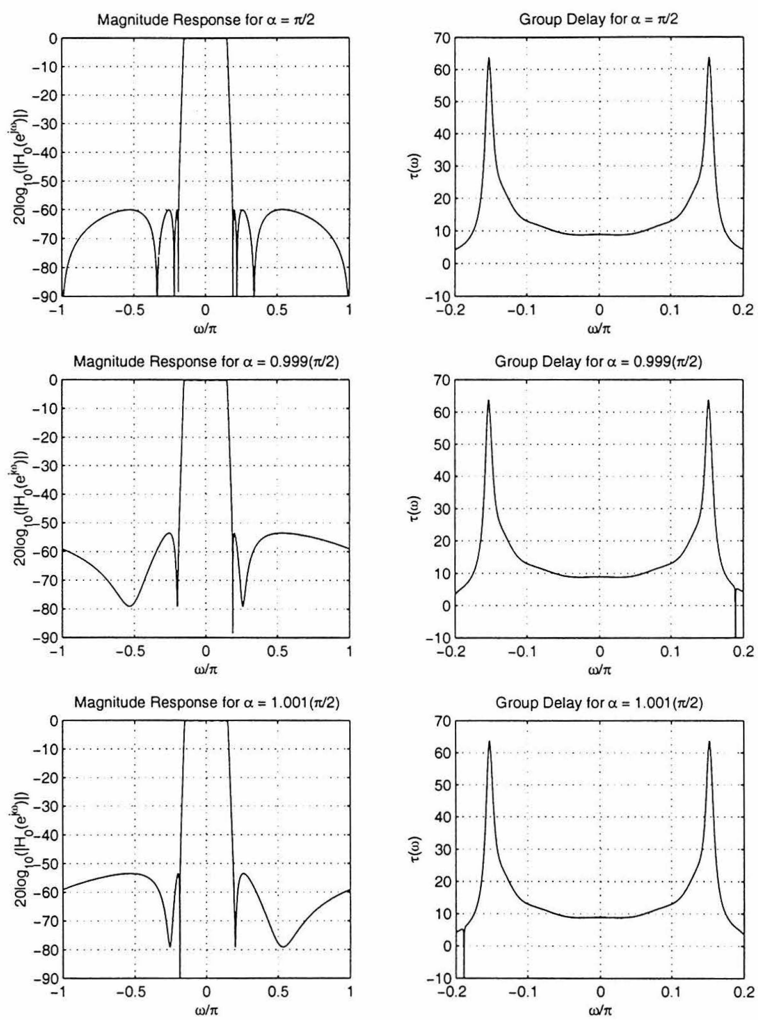

Figure 3.3: Magnitude and group delay plots of $H_0(z)$ as α varies near $\frac{\pi}{2}$.

Unfortunately, as we can see, this heuristic argument doesn't hold much water, since as we can see, by varying α close to $\frac{\pi}{2}$, we get more degradation in our desired magnitude response and virtually no change in the nonlinearity of the phase in the passband. Furthermore, the filters $H_0(z)$ and $H_1(z)$ are now complex and hence more difficult to implement than real filters. With these drawbacks, it is no wonder as to why an equiripple FIR filter designed with the McClellan-Parks algorithm is worth the cost of the increased order, since the phase of such filters is *exactly* linear. Thus, unfortunately, varying the parameter α does not appear to improve the phase response of the overall filters $H_0(z)$ and $H_1(z)$.

3.4 The FDFT as Applied to Filter Banks

In the spirit of exploiting the degree of freedom offered by the FDFT, we can consider applying it to filter bank structures. Here, we will apply the FDFT matrix to the M-channel maximally decimated filter bank, also referred to as the M-channel QMF bank. This particular kind of filter bank is covered at length by Vaidyanathan [11]. In Figure 3.4, we have illustrated the M-channel QMF bank implemented with the FDFT matrix.

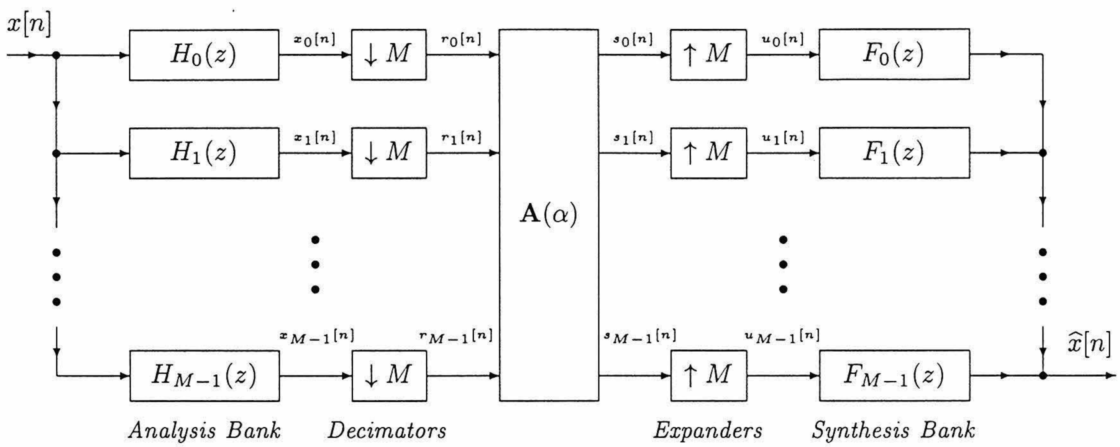

Figure 3.4: M-channel maximally decimated filter bank implemented with the FDFT

In his book on multirate systems, Vaidyanathan considers the special case where $\alpha = 0$, i.e. $\mathbf{A}(\alpha) = \mathbf{I}$. We wish to express $\widehat{X}(z)$ in terms of $X(z)$. The Z-Transform of the subband signal $x_k[n]$ is given as,

$$X_k(z) = H_k(z)X(z)$$

Now, the outputs of the decimators, namely the $r_k[n]$'s, have the following Z-Transform.

$$R_k(z) = \frac{1}{M}\sum_{l=0}^{M-1} X_k\left(z^{\frac{1}{M}}W^l\right)$$

$$R_k(z) = \frac{1}{M}\sum_{l=0}^{M-1} H_k\left(z^{\frac{1}{M}}W^l\right) X\left(z^{\frac{1}{M}}W^l\right)$$

Note that here, W denotes the M-th root of unity, that is $W = W_M \triangleq e^{-j\frac{2\pi}{M}}$. The finite length sequence of Z-Transforms, $S_k(z)$, is the FDFT of the sequence $R_k(z)$. So, we have,

$$S_k(z) = \sum_{m=0}^{M-1} K_\alpha[k,m]R_m(z)$$

$$S_k(z) = \sum_{m=0}^{M-1} K_\alpha[k,m] \left(\frac{1}{M} \sum_{l=0}^{M-1} H_m\left(z^{\frac{1}{M}} W^l\right) X\left(z^{\frac{1}{M}} W^l\right) \right)$$

The outputs of the expanders, namely the $u_k[n]$'s, have the following Z-Transform.

$$U_k(z) = S_k\left(z^M\right)$$

$$U_k(z) = \sum_{m=0}^{M-1} K_\alpha[k,m] \left(\frac{1}{M} \sum_{l=0}^{M-1} H_m\left(zW^l\right) X\left(zW^l\right) \right)$$

Finally, the Z-Transform of the reconstructed signal $\widehat{x}[n]$ is given as,

$$\widehat{X}(z) = \sum_{k=0}^{M-1} F_k(z) U_k(z)$$

$$\widehat{X}(z) = \sum_{k=0}^{M-1} F_k(z) \left(\sum_{m=0}^{M-1} K_\alpha[k,m] \left(\frac{1}{M} \sum_{l=0}^{M-1} H_m\left(zW^l\right) X\left(zW^l\right) \right) \right)$$

$$\widehat{X}(z) = \sum_{l=0}^{M-1} \left[\frac{1}{M} \sum_{k=0}^{M-1} F_k(z) \sum_{m=0}^{M-1} K_\alpha[k,m] H_m\left(zW^l\right) \right] X\left(zW^l\right)$$

We can express this more compactly as follows.

$$\widehat{X}(z) = \sum_{l=0}^{M-1} A_l(z) X\left(zW^l\right) ,$$

$$\text{where } A_l(z) \triangleq \frac{1}{M} \sum_{k=0}^{M-1} F_k(z) \sum_{m=0}^{M-1} K_\alpha[k,m] H_m\left(zW^l\right)$$

Notice that because of the presence of the decimators and expanders, the M-channel QMF is a linear system, but not time-invariant. This is evidenced by the fact that the expression for $\widehat{X}(z)$ above consists of linear combinations of alias terms $X\left(zW^l\right)$ for $l \neq 0$. However, under certain conditions which will soon be discussed, these aliasing terms can be cancelled so that we have effectively an LTI system. In this case, we have,

$$\widehat{X}(z) = T(z) X(z) ,$$

where $T(z)$ is the effective overall transfer function given by $T(z) = A_0(z)$. To properly address the issue of alias cancellation, we must first simplify the structure of our M-channel QMF bank by introducing the polyphase representation of our anaysis bank filters $H_k(z)$ and our synthesis bank filters $F_k(z)$.

3.4.1 Polyphase Representation of the M-Channel QMF Bank

The anaysis bank filters $H_k(z)$ can be expressed in the Type 1 polyphase form as follows.

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{k,l}\left(z^M\right)$$

Defining the following vectors and matrices,

$$\mathbf{h}(z) \triangleq \begin{bmatrix} H_0(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} , \quad \mathbf{e}(z) \triangleq \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix} ,$$

$$\text{and} \quad \mathbf{E}(z) \triangleq \begin{bmatrix} E_{0,0}(z) & E_{0,1}(z) & \cdots & E_{0,M-1}(z) \\ E_{1,0}(z) & E_{1,1}(z) & \cdots & E_{1,M-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ E_{M-1,0}(z) & E_{M-1,1}(z) & \cdots & E_{M-1,M-1}(z) \end{bmatrix},$$

then we have in a more compact form,

$$\mathbf{h}(z) = \mathbf{E}(z^M)\,\mathbf{e}(z)$$

Similarly, the synthesis bank filters $F_k(z)$ can be expressed in the Type 2 polyphase form as follows.

$$F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_{l,k}(z^M)$$

Defining the following vectors and matrices,

$$\mathbf{f}(z) \triangleq \begin{bmatrix} F_0(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix} \quad \text{and} \quad \mathbf{R}(z) \triangleq \begin{bmatrix} R_{0,0}(z) & R_{0,1}(z) & \cdots & R_{0,M-1}(z) \\ R_{1,0}(z) & R_{1,1}(z) & \cdots & R_{1,M-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ R_{M-1,0}(z) & R_{M-1,1}(z) & \cdots & R_{M-1,M-1}(z) \end{bmatrix},$$

we then have in a more compact form,

$$\mathbf{f}(z) = z^{-(M-1)}\widetilde{\mathbf{e}}(z)\mathbf{R}(z^M)$$

Note that here, we have, $\widetilde{\mathbf{e}}(z) \triangleq \mathbf{e}^{\dagger}\left(\frac{1}{z^*}\right)$. With these definitions, we can now implement the M-channel QMF bank as follows below from Figure 3.5.

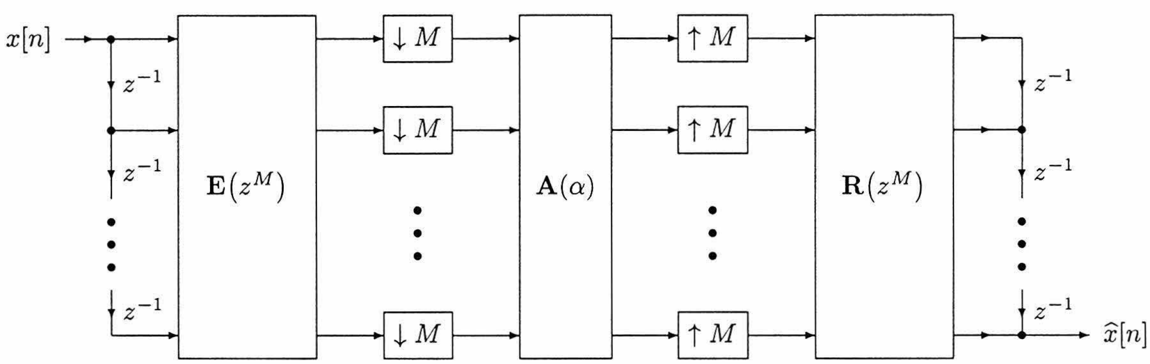

Figure 3.5: Polyphase form of the M-channel maximally decimated filter bank implemented with the FDFT

We can now apply the noble identities [11] to the matrices $\mathbf{E}(z^M)$ and $\mathbf{R}(z^M)$. This simplifies our system. Now, the matrix $\mathbf{A}(\alpha)$ can be lumped together with $\mathbf{E}(z)$ to create a new effective polyphase matrix $\mathcal{E}(z)$, where we have,

$$\mathcal{E}(z) = \mathbf{A}(\alpha)\mathbf{E}(z)$$

With this new matrix, we have the following structure as shown in Figure 3.6, after applying the noble identities.

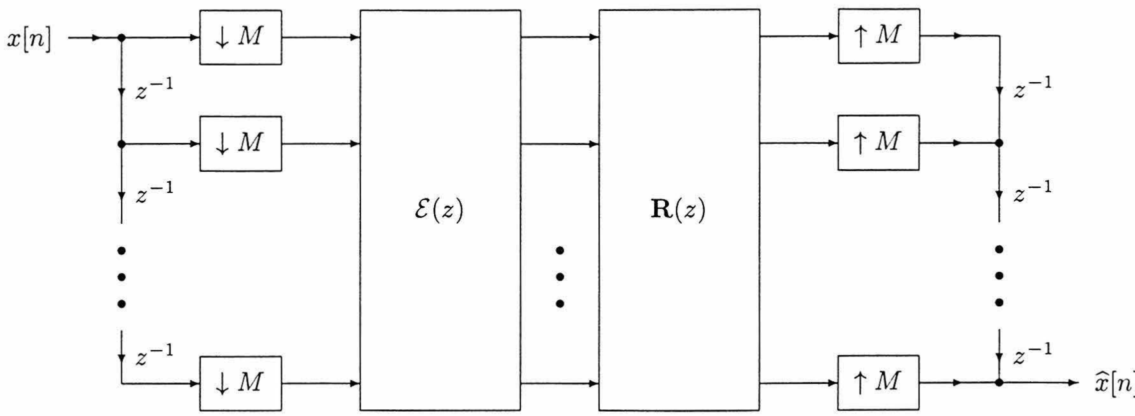

Figure 3.6: Simplification of the polyphase form of the M-channel maximally decimated filter bank using the noble identities

Finally, we can lump $\mathcal{E}(z)$ and $\mathbf{R}(z)$ together to create the matrix $\mathbf{P}(z)$, where we have,

$$\mathbf{P}(z) = \mathbf{R}(z)\mathcal{E}(z) = \mathbf{R}(z)\mathbf{A}(\alpha)\mathbf{E}(z)$$

With this final simplification, the M-channel QMF bank has the following canonical form as can be seen in Figure 3.7.

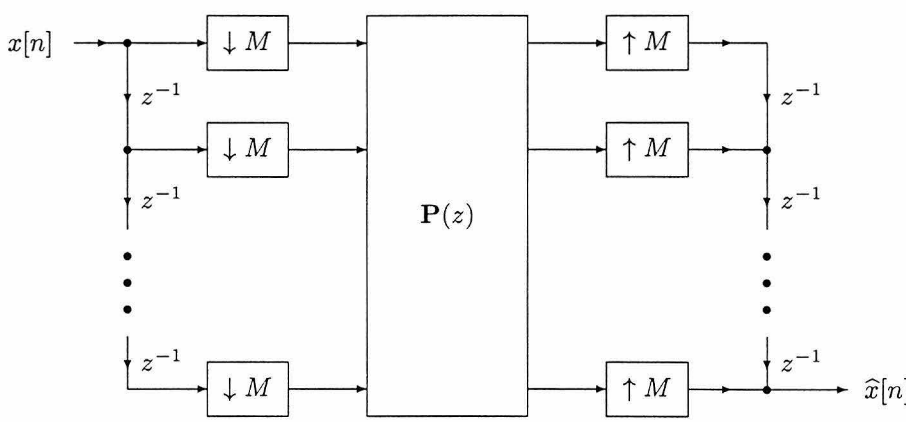

Figure 3.7: Canonical structure of the polyphase form of the M-channel maximally decimated filter bank

From Vaidyanathan [11], the M-channel maximally decimated filter bank is free from aliasing iff $\mathbf{P}(z)$ is pseudocirculant. Recall that a circulant matrix is a matrix of the form,

$$\begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ P_2(z) & P_0(z) & P_1(z) \\ P_1(z) & P_2(z) & P_0(z) \end{bmatrix},$$

where each row of the matrix is a circular shifted version of the previous row. The shift here is by one element and to the right. A psuedocirculant matrix is a matrix of the following form.

$$\begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ z^{-1}P_2(z) & P_0(z) & P_1(z) \\ z^{-1}P_1(z) & z^{-1}P_2(z) & P_0(z) \end{bmatrix}$$

In other words, a pseudocirculant matrix is a circulant matrix with all elements below the main diagonal multiplied by z^{-1}. With the FDFT implemented here, we have an extra degree of freedom in our choices of the filters $H_k(z)$ and $F_k(z)$.

3.4.2 An Insightful Example: The Classical QMF Bank

When $M = 2$, we have the classical QMF bank. In this case, the most general criteria for an alias-free system are as follows.

$$F_0(z) = \left[\left(\frac{\sqrt{2}}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)e^{j2\alpha}\right]H_0(-z)g(z) + \left[\left(\frac{2-\sqrt{2}}{4}\right) + \left(\frac{2+\sqrt{2}}{4}\right)e^{j2\alpha}\right]H_1(-z)g(z)$$

$$F_1(z) = -\left[\left(\frac{2+\sqrt{2}}{4}\right) + \left(\frac{2-\sqrt{2}}{4}\right)e^{j2\alpha}\right]H_0(-z)g(z) - \left[\left(\frac{\sqrt{2}}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)e^{j2\alpha}\right]H_1(-z)g(z)$$

Here, $g(z)$ is an arbitrary function of z. With this, we now see that we have more freedom in the choice of the implementation of $F_0(z)$ and $F_1(z)$ given $H_0(z)$ and $H_1(z)$. Note that when $\alpha = 0$ and $g(z) = 1$, we get,

$$F_0(z) = H_1(-z)\,,\ F_1(z) = -H_0(-z)\,,$$

which is the same result derived by Vaidyanathan [11]. The corresponding overall transfer function, $T(z)$, is given below as follows.

$$T(z) = \frac{1}{2}e^{j2\alpha}[H_0(z)H_1(-z) - H_0(-z)H_1(z)]g(z)$$

So, we can see that by varying α, the magnitude of $T(z)$ remains unchanged, while the phase of $T(z)$ has a constant offset of 2α for all ω. Thus, varying α doesn't essentially change $T(z)$, but it does change the synthesis filters $F_0(z)$ and $F_1(z)$ needed for alias cancellation. This could be useful in many applications, since it may be easier to implement these filters for some angle, say $\alpha = \alpha_0 \neq 0$, than it may be to implement them when $\alpha = 0$, which corresponds to the case analyzed by Vaidyanathan.

Conclusions

Though the FDFT is an elegant generalization of the DFT from a mathematical point of view, it does not seem to be as useful for signal processing as the DFT. In general, it is much more complex to implement than the DFT, and, unfortunately, doesn't appear to be very useful for the allpass filter network considered here. There is, however, what appears to be a glimpse of hope despite all of this.

The extra degree of freedom offered to us by the FDFT could to be useful in its implementation in the M-channel maximally decimated filter bank. As we saw with the example of the classical QMF bank, the choice of α gives us more freedom to design synthesis filters to cancel aliasing given the analysis filters, without significantly altering the overall transfer function. Because of this, the FDFT may yet be useful in its implementation in filter banks.

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