COMBINATORIAL INEQUALITIES FOR GEOMETRIC LATTICES

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ABSTRACT

A geometric lattice is a semimodular point lattice L. The i^{th} Whitney number of L is the number of elements of rank i in L. The logarithmic concavity conjecture states that

$$\frac{W_i(L)^2}{W_{i-1}(L)W_{i+1}(L)} \geqslant 1$$

for any finite geometric lattice L.

In a finite nondirected graph without loops or double edges, a set of edges is closed if whenever it contains all but one edge of a cycle, it contains the whole cycle. With set containment as the order relation, the closed sets of such a graph form a geometric lattice. It is shown that any such lattice satisfies the first nontrivial case of the logarithmic concavity conjecture. In fact,

$$\frac{W_2(L)^2}{W_1(L)W_3(L)} \geqslant \frac{3}{2} \cdot \frac{(W_1(L)-1)}{(W_1(L)-2)} \quad .$$

This is a best possible result since equality holds for graphs without cycles.

The cut-contraction of a geometric lattice L with respect to a modular cut Q of L is the geometric lattice L - T where $T = \{x \in L : x \notin Q, \ \exists q \in Q \ni x \prec q\}$. It is shown that any geometric lattice L can be obtained from the Boolean algebra with $W_1(L)$ points by means of a sequence of $k = W_1(L) - \dim(L)$ cut-contractions.

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INTRODUCTION

Many combinatorial problems can be formulated in the framework of a collection $\mathcal{C}$ of subsets of a finite set S, where this collection contains the whole set and is closed under intersections. With set containment as the partial ordering, $\mathcal{C}$ is a lattice. For two subsets $A, B \in \mathcal{C}$, the meet $A \wedge B$ is set intersection, and the join $A \vee B$ is the intersection of all subsets in $\mathcal{C}$ which contain both A and B. The systems considered in this thesis are of this type with two restrictions: (i) each subset in $\mathcal{C}$ can be expressed as the set union of minimal subsets of $\mathcal{C}$, and (ii) for $A, P \in \mathcal{C}$, if $P \nleq A$ and P is minimal in $\mathcal{C}$, then there is no subset $B \in \mathcal{C}$ properly between A and $A \vee P$. The lattice theoretic analog of such a system is called a <u>geometric lattice</u>. It is from the lattice perspective that we shall study some combinatorial problems.

Projective planes have been the source of many combinatorial problems. They also provide examples of geometric lattices. The elements of the geometric lattice associated with a projective plane are the null set, the points and lines, and the plane itself. The order relation is set containment. Similarly, the subspaces of a projective geometry form a geometric lattice. In fact, any abstract finite geometry can be represented as a geometric lattice, and vice-versa.

Geometric lattices arise naturally in other areas of mathematics. For example, the collection of partitions of a finite set S form a geometric lattice under the partial ordering $\pi_1 \leq \pi_2$ if π_1 is a refinement of π_2.

This thesis will be concerned primarily with geometric lattices which arise from finite, nondirected graphs. A set S of edges is said to be closed if whenever S contains all but one edge of a cycle, it contains the whole cycle. The collection of closed sets of edges of a graph G forms a geometric lattice which will be denoted by L(G) and will be called the edge lattice of the graph G.

A set of numerical invariants of particular combinatorial interest are the Whitney numbers W_i of L. W_i is defined as the number of elements of rank i in L. Rota has proposed two important conjectures regarding these Whitney numbers. The first is that they are unimodal, i.e., for $i \leqslant j \leqslant k$, $W_j \geqslant \min(W_i, W_k)$ [7]. The second is that they are logarithmically concave, i.e.,

$$\frac{W_i}{W_{i-1}} \geqslant \frac{W_{i+1}}{W_i} \quad \text{or} \quad \frac{W_i^2}{W_{i-1}W_{i+1}} \geqslant 1 \ [5] \ .$$

These two conjectures are related. In fact, lattices which are logarithmically concave are unimodal. For suppose L is not unimodal; then there exist $i < j < k$ such that $W_j < \min(W_i, W_k)$. Let

$$W = \min_{i<m<k} (W_m) , \quad \text{and let} \quad n = \max_{\substack{W_m = W \\ i<m<k}} (m) \ .$$

Then $W_{n+1} > W_n \leqslant W_{n-1}$, and $\dfrac{W_n^2}{W_{n-1}W_{n+1}} < 1$. Hence L is not logarithmically concave.

If the geometric lattice is a Boolean algebra, then W_i is the binomial coefficient $\binom{W_1}{i}$. The logarithmic concavity conjecture holds in this case since

$$\frac{W_i^{\,2}}{W_{i-1}W_{i+1}} = \frac{\binom{W_1}{i}^2}{\binom{W_1}{i-1}\binom{W_1}{i+1}} = \frac{i+1}{i} \cdot \frac{W_1 - i+1}{W_1-i} > 1 \quad .$$

If the geometric lattice is derived from a finite projective plane, then $W_0 = 1$, $W_1 = W_2$ and $W_3 = 1$. These numbers are clearly logarithmically concave.

The Whitney numbers of the lattice of subspaces of a projective geometry of dimension three or greater are the Gaussian coefficients:

$$W_i = \frac{(q^n-1)(q^n-q)\ldots(q^n-q^{i-1})}{(q^i-1)(q^i-q)\ldots(q^i-q^{i-1})}$$

where n is the dimension of the lattice, and q is the order of the field used to coordinatize the geometry [7]. Hence

$$\frac{W_i^{\,2}}{W_{i-1}W_{i+1}} = \frac{\dfrac{(q^n-1)^2\ldots(q^n-q^{i-1})^2}{(q^i-1)^2\ldots(q^i-q^{i-1})^2}}{\dfrac{(q^n-1)\ldots(q^n-q^{i-2})}{(q^{i-1}-1)\ldots(q^{i-1}-q^{i-2})} \cdot \dfrac{(q^n-1)\ldots(q^n-q^i)}{(q^{i+1}-1)\ldots(q^{i+1}-q^i)}}$$

$$= \frac{(q^n-q^{i-1})}{(q^n-q^i)} \cdot \frac{(q^{i+1}-1)}{(q^i-1)} \cdot q$$

$$= \frac{(q^{n-i+1}-1)}{(q^{n-i}-1)} \cdot \frac{(q^{i+1}-1)}{(q^i-1)}$$

$$> 1 \quad .$$

Finally, the Whitney numbers of a partition lattice are the Stirling numbers of the second kind. This sequence of numbers has also been shown to be logarithmically concave [6].

The following theorem, whose proof is given in the appendix, shows that logarithmic concavity is preserved under direct products.

Theorem A.1: Let L_1 and L_2 be logarithmically concave geometric lattices. Then $L = L_1 \times L_2$ is logarithmically concave.

Since Birkhoff [1] has proved that any modular geometric lattice is the direct product of projective geometries and a Boolean algebra, the results above show that any modular geometric lattice is logarithmically concave.

There are two further results which support these conjectures. A theorem of Greene [4] states that for a geometric lattice L, $W_1 \leq W_m$ for $m \neq 0, n$ where $n = \dim(L)$. A generalization due to Dowling and Wilson [3] asserts that

$$\sum_{i=1}^{k} W_i \leq \sum_{i=1}^{k} W_{m-i} \quad \text{for} \quad n \geq m > k \ .$$

For an arbitrary geometric lattice, only two cases of the logarithmic concavity conjecture have been proved. These cases are $i=1$ and $i=n-1$ where n is the dimension of the lattice. To prove that this is true for $i=1$, we observe that each element of rank 2 can be represented as the join of two points. Since the join operation is unique, it follows that $W_2 \leq \binom{W_1}{2}$. Hence

$$\frac{W_1^{\,2}}{W_0W_2} \geqslant \frac{W_1^{\,2}}{\binom{W_1}{2}} = 2 \cdot \frac{W_1}{W_1-1} > 1 \quad .$$

The proof for i=n-1 is the dual of this proof.

The main theorem of Chapter I treats the next case of logarithmic concavity, namely $i = 2$.

Theorem 1.7: For any finite graph G,

$$\frac{W_2(G)^2}{W_1(G)W_3(G)} \geqslant \frac{3}{2} \cdot \frac{(W_1(G)-1)}{(W_1(G)-2)} \quad ,$$

where $W_i(G)$ is the i^{th} Whitney number of L(G).

This is a best possible result since equality holds for all graphs without cycles.

CHAPTER 1

EDGE LATTICES OF GRAPHS

We begin with some definitions. A <u>graph</u> G consists of a set V of <u>vertices</u> and a set E of unordered pairs of vertices called <u>edges</u>. The vertices determining an edge are its <u>endpoints</u>. An edge will be denoted by e or by (v_1, v_2) where v_1 and v_2 are the endpoints of the edge. All graphs will be finite and without loops or double edges, i.e., there is no edge (v, v) for any $v \in V$, and there is at most one edge of the form (v_1, v_2) for $v_1, v_2 \in V$.

A <u>subgraph</u> G' of a graph G is a graph whose vertices are vertices of G and whose edges are edges of G such that if (v_1, v_2) is an edge of G', then v_1 and v_2 are vertices of G'. There is a natural correspondence between sets of edges and subgraphs of a graph G. With a set of edges, associate the subgraph whose edges are the edges of the set and whose vertices are the endpoints of the edges in the set. When no confusion exists, a subgraph will be designated by the corresponding set of edges.

A <u>path</u> is an ordered set of edges such that any edge of this set has one endpoint in common with the preceding edge and the other endpoint in common with the following edge. Paths will be denoted by either the edges or the vertices. Note that a path P from v_1 to v_2 may be shortened to a path P' from v_1 to v_2 such that $P' \subseteq P$ and P' has no repeated edges. A <u>cycle</u> is a closed path with no repeated edges. Finally, a set S of edges is said to be <u>closed</u> if whenever S contains all but one edge of a cycle, S contains the whole cycle.

Let S_1 and S_2 be closed sets. If $S_1 \cap S_2$ contains all but one edge of a cycle, then each of S_1 and S_2 contains the whole cycle, and so $S_1 \cap S_2$ contains the whole cycle. Thus the intersection of two closed sets is closed. Now, for an arbitrary set R of edges define the <u>closure</u> of R as the intersection of all closed sets containing R. This is the smallest closed set containing R, and it will be denoted by $c\ell(R)$. The following lemma gives an alternate characterization of the closure of a set.

<u>Lemma 1.1</u>: Let G be a graph, and let R be a set of edges in G. Let $\overline{R}$ be the set of edges e such that there is a path $P_e \subseteq R$ from one endpoint of e to the other. Then $\overline{R} = c\ell(R)$.

<u>Proof</u>: If $e \in R$, then $\{e\}$ is a suitable choice for P_e, and hence $R \subseteq \overline{R}$. By the definition of closure, it is clear that $\overline{R} \subseteq c\ell(R)$. Therefore it is sufficient to prove that $\overline{R}$ is closed. Let $\{(v_0, v_1), (v_1, v_2), \ldots, (v_n, v_0)\}$ be a cycle such that $\{(v_0, v_1), \ldots, (v_{n-1}, v_n)\} \subseteq \overline{R}$. Then there exist paths $P_0, P_1, \ldots, P_{n-1}$ such that P_i is a path from v_i to v_{i+1} and $P_i \subseteq R$. Now the ordered sequence $P_0, P_1, \ldots, P_{n-1}$ is a path from v_0 to v_n, and it is a path in R. Hence $(v_n, v_0) \in \overline{R}$, and $\overline{R}$ is closed. Thus $\overline{R} = c\ell(R)$.

<u>Lemma 1.2</u>: Let G be a graph, and let S be a closed set of edges in G. Given an edge $e_0 \notin S$, let $e \in c\ell(S \cup \{e_0\}) - S$. Then there exist paths $P_1, P_2 \subseteq S$ such that the ordered sequence e, P_1, e_0, P_2 is a cycle in G.

<u>Proof</u>: Since $e \in c\ell(S \cup \{e_0\})$, by Lemma 1.1 there exists a path $P \subseteq S \cup \{e_0\}$ from one endpoint of e to the other with no repeated edges. Since $e \notin S$, $P \not\subseteq S$; this implies $e_0 \in P$. Denote by P_1 that part of P preceding e_0, and by P_2, that part of P following e_0. This choice for P_1 and P_2 satisfies the lemma.

<u>Theorem 1.3:</u> Let G be a graph. Then $L(G)$, the collection of closed sets of edges of G, is a semimodular point lattice.

<u>Proof</u>: With the order relation defined by $S \leq T$ if and only if $S \subseteq T$, it is clear that $L(G)$ is a partially ordered set. Also ϕ, the empty set, and E, the set of all edges, are both closed. It has been shown that if S and T are closed, then so is $S \cap T$. Thus $S \cap T$ is the greatest lower bound of S and T in $L(G)$, i.e., $S \cap T = S \wedge T$. This is sufficient to show that $L(G)$ is a lattice and hence has a join operation. The smallest closed set containing S and T has already been denoted by $c\ell(S \cup T)$. Hence $S \vee T = c\ell(S \cup T)$.

Sets which consist of a single edge are closed; these are exactly the points of $L(G)$. Since each closed set S is a set of edges, S is the join of the set of points contained in S. Hence $L(G)$ is a point lattice.

To prove semimodularity, first consider the case where S is a closed set and $P = \{e_0\}$ where $e_0 \notin S$. If there exists a closed set S' such that $S < S' \leq S \vee P$, let $e \in S' - S$. Then $e \in S \vee P$. By Lemma 1.2, there is a cycle through e and e_0 with all other edges in S. Using the same cycle, we see that $e_0 \in S \vee \{e\} \leq S'$. Hence $S \vee P \leq S'$, and $S \prec S \vee P$.

Now let S and T be any two closed sets such that $S \wedge T \prec T$. Let P be a point in L(G) such that $P \leq T$ but $P \nleq S$. Then $P \nleq S \wedge T$, and hence $(S \wedge T) \vee P = T$. We thus have

$$S \vee T = S \vee ((S \wedge T) \vee P) = (S \vee (S \wedge T)) \vee P = S \vee P \succ S \quad .$$

Hence L(G) is semimodular.

Since L(G) is a finite semimodular point lattice, it has a well defined rank function which can be characterized in terms of independent sets of edges.

<u>Definition:</u> A set A of edges is <u>independent</u> if $e \notin c\ell(A-\{e\})$ for all $e \in A$.

<u>Lemma 1.4</u>: A is an independent set of edges if and only if A contains no cycles.

<u>Proof</u>: If A contains a cycle, let e be an edge of that cycle. Then $e \in c\ell(A-\{e\})$, and A is not independent.

If A is not independent, let $e \in A$ be such that $e \in c\ell(A-\{e\})$. By Lemma 1.1, there is a path P from one endpoint of e to the other such that $P \subseteq A - \{e\}$ and P has no repeated edges. Then $P \cup \{e\}$ is a cycle in A.

<u>Lemma 1.5</u>: If A is an independent set of edges, then $r(c\ell(A)) = |A|$.

<u>Proof</u>: Let $A = \{e_1, e_2, \ldots, e_n\}$. By semimodularity

$$\phi \prec \{e_1\} \prec \{e_1\} \vee \{e_2\} \prec \ldots \prec \{e_1\} \vee \{e_2\} \vee \ldots \vee \{e_n\}$$

$$= c\ell(\{e_1, e_2, \ldots, e_n\}) = c\ell(A) \quad .$$

This chain has length n, so $r(c\ell(A)) = n = |A|$.

Corollary 1.6: If A is a maximal independent subset of S, a closed set, then $S = c\ell(A)$ and $r(S) = |A|$.

Now let $W_i(G)$ represent the i^{th} Whitney number of $L(G)$; that is, $W_i(G) = |\{x \in L(G) : r(x) = i\}|$. The main result can be stated as follows.

Theorem 1.7: For any graph G,

$$\frac{W_2(G)^2}{W_1(G)W_3(G)} \geqslant \frac{3}{2} \cdot \frac{(W_1(G)-1)}{(W_1(G)-2)} \quad .$$

It will be shown in the proof that equality holds if G has disconnected edges, i.e., if no two edges of G have a common endpoint. Hence this theorem is a best possible result.

Proof: Throughout the proof, G will be some fixed graph. The proof is inductive and is based upon a method for constructing G from a graph G_0 with the same number of edges as G, but with disconnected edges.

Pick any vertex of G of degree greater than 1. One by one, separate the edges at this vertex. Continue this process until all vertices are of degree 1. This is G_0. Reversing the procedure provides a method for constructing G from G_0. In each step of the reverse process, two vertices of G_0 are identified. Any order of the identifications will produce G, but the proof will require a special order. Let v be a vertex in G of maximal degree. From among all vertices in G_0 which are eventually to be identified to give v, select one and label it v. One by one, identify the other appropriate vertices with v. When the vertex v is completed, continue this procedure at the vertex in G of next highest degree. Continue in this manner until the graph G is obtained.

The following lemma is obvious from this construction, but it is of special importance.

Lemma 1.8: Let H and H' be consecutive graphs in this construction of G. If H' is obtained from H by identifying a vertex v^* of degree 1 with the vertex v of degree n, then all other vertices in H are of degree 1 or of degree larger than n.

Now let H and H' be any two consecutive graphs in a construction of G from G_0. If we can show

$$\frac{W_2(G_0)^2}{W_1(G_0)W_3(G_0)} = \frac{3}{2} \cdot \frac{(W_1(G_0)-1)}{(W_1(G_0)-2)}$$

and

$$\frac{W_2(H')^2}{W_1(H')W_3(H')} \geq \frac{W_2(H)^2}{W_1(H)W_3(H)} \quad ,$$

then by repeated application of the second formula, we have

$$\frac{W_2(G)^2}{W_1(G)W_3(G)} \geq \frac{W_2(G_0)^2}{W_1(G_0)W_3(G_0)} = \frac{3}{2}\,\frac{(W_1(G_0)-1)}{(W_1(G_0)-2)} = \frac{3}{2}\,\frac{(W_1(G)-1)}{(W_1(G)-2)} \quad ,$$

and the theorem follows.

In order to establish the first formula, consider any set S of edges in G_0. S is trivially closed in G_0 since there are no cycles in G_0. Therefore, all sets of edges are closed in G_0, and $L(G_0)$ is a Boolean algebra. Hence

$$W_2(G_0) = \binom{W_1(G_0)}{2} = \frac{W_1(G_0)(W_1(G_0)-1)}{2}$$

and

$$W_3(G_0) = \binom{W_1(G_0)}{3} = \frac{W_1(G_0)(W_1(G_0)-1)(W_1(G_0)-2)}{6} \quad .$$

But then

$$\frac{W_2(G_0)^2}{W_1(G_0)W_3(G_0)} = \frac{3}{2}\,\frac{(W_1(G_0)-1)}{(W_1(G_0)-2)} \quad .$$

In establishing the second formula for the graphs H and H′, it will be convenient to set $W_i = W_i(H)$ and $W_i' = W_i(H')$. Since $W_1 = W_1'$, it will suffice to show that

$$W_2'^2 W_3 - W_2^2 W_3' \geqslant 0 \quad .$$

In order to relate W_2' and W_3' to W_2 and W_3, it is necessary to examine closely the relationship of the closed sets in H to the closed sets in H′. Note first that any set S which is closed in H′ is also closed in H. For any two edges with a common vertex in H also have this vertex in common in H′, and hence any cycle in H is also a cycle in H′.

On the other hand, there may be sets which are closed in H but not closed in H′. These sets will be characterized in the next lemma. Let v* denote the vertex of degree 1 in H which is identified with the vertex v in H to give the graph H′. Also let v′ be the vertex which, in H, has an edge in common with v*, and let this edge be denoted by e* in both H and H′.

Lemma 1.9: Let S be closed in H. Then S is not closed in H′ if and only if (i) S is covered in L(H) by a set S′ which contains a path P in H from v^* to v such that P has no repeated edges and (ii) S does not contain a path in H from v^* to v.

Proof: Let S be closed in H but not closed in H′. Then there exist an edge e and a cycle C in H′ containing e such that $e \notin S$ and $C - \{e\} \subseteq S$. C is not a cycle in H since S is closed in H and $e \notin S$. It follows that $e^* \in C$. Hence, in H, C forms a path P from v^* to v with no repeated edges. Condition (i) follows by semimodularity since $S \prec c\ell_H(S \cup \{e\})$ in L(H) and $P = C \subseteq c\ell_H(S \cup \{e\})$. Assume that S contains a path P′ in H from v^* to v. Then $e \neq e^*$ and $(P-\{e\}) \cup P'$ is a path in H, contained in S, from one endpoint of e to the other. Hence $e \in c\ell_H(S) = S$, a contradiction. Thus S does not contain a path in H from v^* to v; this is condition (ii).

Let S be a closed set in H which satisfies (i) and (ii). Let S′ and P be the sets guaranteed by (i). P is a path in H from v^* to v with no repeated edges, and S′ is a closed set in H such that $P \subseteq S'$ and $S \prec S'$. Assume S is also closed in H′. Since by (ii) $P \not\subseteq S$, and P is a cycle in H′, there must be at least two edges of P which are not in S. Let the path P be described by its vertices

$$x_0 = v^*, x_1 = v', x_2, \ldots, x_{n-1}, x_n = v \quad ,$$

let (x_i, x_{i+1}) be the first edge which is not in S, and let (x_j, x_{j+1}) be the last edge which is not in S. Since there are at least two edges of P not in S, $(x_i, x_{i+1}) \neq (x_j, x_{j+1})$. Then $S' = c\ell_H(S \cup \{(x_j, x_{j+1})\})$ because $S \prec S'$

and $(x_j, x_{j+1}) \in S' - S$. If $(x_i, x_{i+1}) = (v^*, v')$, then $(v^*, v') \notin S$, and $(x_i, x_{i+1}) = (v^*, v') \notin c\ell_H(S \cup \{(x_j, x_{j+1})\}) = S'$ because (v^*, v') is not on any cycle in H. This contradicts $(x_i, x_{i+1}) \in S'$, so $(x_i, x_{i+1}) \neq (v^*, v')$. Now since $(x_i, x_{i+1}) \in c\ell_H(S \cup \{(x_j, x_{j+1})\})$, Lemma 1.2 guarantees that there exist paths $P_1, P_2 \subseteq S$ such that the ordered sequence $(x_i, x_{i+1}), P_1, (x_j, x_{j+1}), P_2$ is a cycle in H. If x_i is connected to x_j by P_1, then the ordered sequence $(x_0, x_1), \ldots, (x_{i-1}, x_i), P_1, (x_j, x_{j+1}), \ldots, (x_{n-1}, x_n)$, when appropriately shortened, is a cycle in H′ with all but one edge, namely (x_j, x_{j+1}), in S. This contradicts the assumption that S is closed in H′. If, on the other hand, x_i is connected to x_{j+1}, by P_1, then the ordered sequence $(x_0, x_1), \ldots, (x_{i-1}, x_i), P_1, (x_{j+1}, x_{j+2}), \ldots, (x_{n-1}, x_n)$ is a path in S from v^* to v. This contradicts (ii). Likewise the assumption that P_1 is a path from x_{i+1} to x_j or x_{j+1} leads to a similar contradiction. Hence S is not closed in H′.

If S″ is another set which satisfies (i), then by this lemma, both S′ and S″ are closed in H′. Therefore S, which is the meet of S′ and S″ in L(H) and also the set intersection of S′ and S″, is closed in H′. Since this is not true, S′ must be unique. It also follows that $c\ell_{H'}(S) = S'$.

<u>Lemma 1.10</u>: Let S be a set of edges which is closed in both H and H′. Let r(S) and r′(S) denote the ranks of S in L(H) and L(H′), respectively. Then $r'(S) = r(S)$ or $r'(S) = r(S) - 1$. Furthermore $r(S) = r'(S)$ if and only if S does not contain a path in H from v^* to v.

<u>Proof</u>: Let A be a maximal independent subset of S in H. Then $|A| = r(S)$. If S does not contain a path in H from v^* to v, then A does

not contain a cycle in H' through e^*. Since these are the only cycles in H' which are not cycles in H, A does not contain any cycles in H'. Hence A is independent in H'. Since $c\ell_{H'}(A)$ is clearly S, $r'(S) = |A| = r(S)$.

If S does contain a path in H from v^* to v, we may choose A such that $e^* \in A$. Again, since all cycles in H' either are cycles in H or contain e^*, the set $A - \{e^*\}$ does not contain any cycles in H'. Hence $A - \{e^*\}$ is independent in H'. To prove that $r'(S) = |A - \{e^*\}| = |A| - 1 = r(S) - 1$, it is sufficient to show that $c\ell_{H'}(A - \{e^*\}) = S$.

Since $A - \{e^*\}$ is independent in H, $r(c\ell_H(A - \{e^*\})) = |A - \{e^*\}| = r(S) - 1$. $S - \{e^*\}$ is closed in H since no cycle in H contains e^*. But $A - \{e^*\} \subseteq S - \{e^*\}$ and $r(S - \{e^*\}) = r(S) - 1 = r(c\ell_H(A - \{e^*\}))$, so that $c\ell_H(A - \{e^*\}) = S - \{e^*\}$. Hence

$$c\ell_{H'}(A - \{e^*\}) = c\ell_{H'}(c\ell_H(A - \{e^*\})) = c\ell_{H'}(S - \{e^*\}) = S$$

since S contains a path in H' from one endpoint of e^* to the other, i.e., from v' to v.

The following factors contribute to the relationship of the W_i to the W_i'. Some closed sets in H of rank i may remain closed and of rank i. Some of these sets may drop to rank $i-1$ in $L(H')$, or they may not be closed in H'. Finally, some closed sets of rank $i+1$ in $L(H)$ may drop to rank i in $L(H')$. In view of the foregoing discussion, each set of rank i in $L(H)$ which is not closed in H' must be covered, in $L(H)$, by a unique set of rank $i+1$ which drops to rank i in $L(H')$. Conversely, if a set of rank $i+1$ in $L(H)$ drops to rank i in $L(H')$, then any closed set covered by this set in L(H) either is not closed in H′ or drops to rank i - 1 in L(H′).

Since W_1 and W_1' each represent the number of edges in G, $W_1 = W_1'$ and no closed set of rank 2 in L(H) drops to rank 1 in L(H′). Thus there are only two ways to change the number of sets of rank 2: closed sets of rank 3 in L(H) may drop to rank 2 in L(H′), and closed sets of rank 2 in L(H) may be deleted because they are not closed in H′. Let k denote the number of sets of rank 3 in L(H) which drop to rank 2 in L(H′). Now, since subgraphs corresponding to closed sets of rank 2 can only take the forms

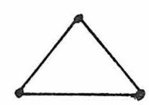
 ,  and  ,

Figure 1

each of the k subgraphs corresponding to these k sets of rank 3 in L(H) must have been isomorphic to

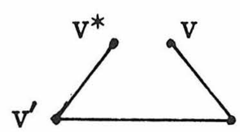


Figure 2

in H. Note that each of these k subgraphs contains three subgraphs which correspond to sets of rank 2 in L(H), and none of these sets is closed in H′. Thus

$$W_2' = W_2 + k - 3k = W_2 - 2k \quad .$$

Since for each new triangle formed, the number of closed sets of rank 2 decreases by two, $W_2 = \binom{W_1}{2} - 2t$ where t is the number of triangles in H.

We also need to know how the number of sets of rank 3 is changed. The number of sets of rank 3 in L(H) which drop to rank 2 in L(H′) has already been denoted by k. Closed sets of rank 3 in L(H′) must correspond to subgraphs which are isomorphic to one of the following graphs:

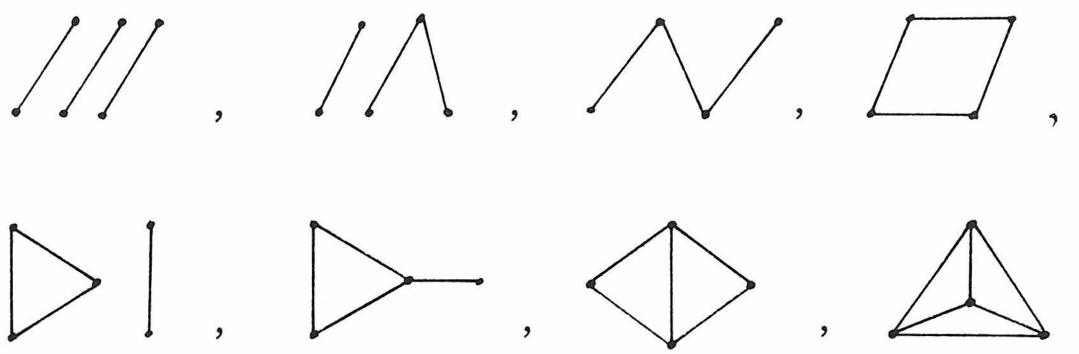


Figure 3

The first three graphs can not be derived from closed sets of rank 4 in L(H) since they have only three edges. However, the five remaining graphs can be derived from closed sets of rank 4 in L(H). There the corresponding subgraphs would have been isomorphic to one of the following:

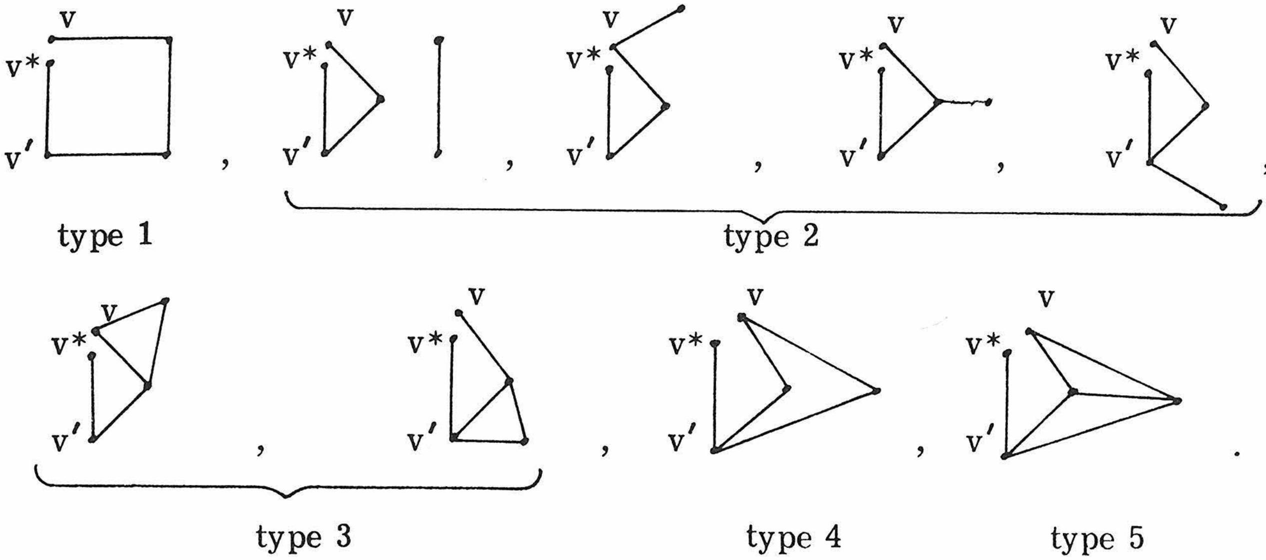


Figure 4

Any set which is deleted must be covered by exactly one set corresponding to one of these subgraphs, and the corresponding subgraph must not contain a path from v* to v. The numbers of such sets are 4; 3, 3, 3, 3; 4, 4; 5; 5, respectively.

As an example, consider the graph of type 3

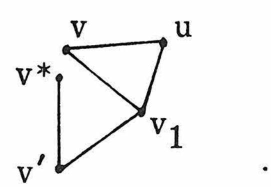


Figure 5

The four subgraphs of this graph which do not contain a path from v* to v and also correspond to closed sets of rank 3 are

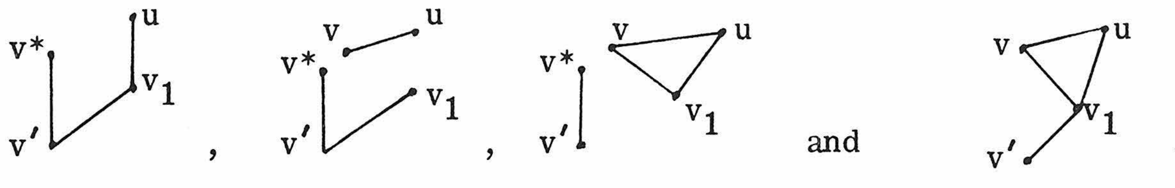


Figure 6

Since each of the k subsets which drop from rank 3 in $L(H)$ to rank 2 in $L(H')$ must correspond to a graph which contains the edge (v^*, v') and some edge with v as an endpoint, the k subgraphs form the following configuration:

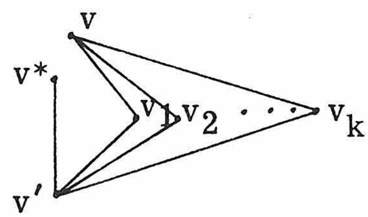


Figure 7

Let a_i be the number of edges connecting v_i to the other v_j's. Let

$$a = \frac{1}{2} \sum_{i=1}^{k} a_i \quad ;$$

this is the number of edges of the form (v_i, v_j). Let $u_1, u_2, \ldots, u_m$ be vertices which have edges in common with some v_i and also either v or v' but not both. (If a vertex has edges in common with both v and v', then that vertex is some v_i.) Let b_i be the number of edges from v_i to the u's, and let

$$b = \sum_{i=1}^{k} b_i \quad .$$

Let c be the number of edges which do not occur in Figure 7 and are not counted in a or b. Since c does include the edges (v, u_i) and (v', u_i), it follows that $kc \geq b$ since u_i can have at most k edges which contribute to b. Applying Lemma 1.8 to each v_i and summing over i gives $2a + b + c \geq k^2 - k$.

Now let d be the number of sets of type 1. There are b of type 3, a of type 5 and $\binom{k}{2}$ - a of type 4. There must be

$$\sum_{i=1}^{k} (W_1-2k-1-a_i-2b_i) = kW_1 - 2k^2 - k - 2a - 2b$$

of type 2. Thus the total number of sets of rank 4 in L(H) which drop to rank 3 in L(H′) is

$$(d) + (kW_1-2k^2-k-2a-2b) + (b) + \left(\binom{k}{2} - a\right) + (a) \quad ,$$

and the total number of sets of rank 3 in L(H) which are not closed in H′ is

$$4(d) + 3(kW_1-2k^2-k-2a-2b) + 4(b) + 5\left(\binom{k}{2} - a\right) + 5(a) \quad .$$

Finally,

$$\begin{aligned} W_3' &= W_3 - k + \left((d) + (kW_1-2k^2-k-2a-2b) + (b) + \left(\binom{k}{2} - a\right) + (a)\right) \\ &\quad - \left(4(d) + 3(kW_1-2k^2-k-2a-2b) + 4(b) + 5\left(\binom{k}{2} - a\right) + 5(a)\right) \\ &= W_3 - 2kW_1 + 2k^2 + 3k + 4a + b - 3d \quad . \end{aligned}$$

Some upper bounds for W_3 will be needed. The induction hypothesis

$$\frac{{W_2}^2}{W_1W_3} \geqslant \frac{3}{2}\,\frac{(W_1-1)}{(W_1-2)}$$

implies

$$W_3 \leqslant \frac{2}{3}\frac{W_2^{\,2}}{W_1}\;\frac{(W_1-2)}{(W_1-1)}$$

$$= \frac{2}{3}\frac{W_2}{W_1}\;\frac{(W_1-2)}{(W_1-1)}\left(\frac{W_1(W_1-1)}{2} - 2t\right)$$

$$= \frac{W_1W_2}{3} - \frac{2W_2}{3} - \frac{4}{3}\frac{W_2}{W_1}\;\frac{(W_1-2)}{(W_1-1)}\;t \quad .$$

With these estimates,

$$W_2'^{\,2}W_3 - W_2^{\,2}W_3' = (W_2-2k)^2W_3 - W_2^{\,2}(W_3-2kW_1+2k^2+3k+4a+b-3d)$$

$$\geqslant (-4kW_2+4k^2)W_3 - W_2^{\,2}(-2kW_1+2k^2+3k+4a+b)$$

$$\geqslant (-4kW_2)\left(\frac{W_1W_2}{3} - \frac{2W_2}{3} - \frac{4}{3}\frac{W_2}{W_1}\;\frac{(W_1-2)}{(W_1-1)}\;t\right) + (4k^2)\left(\frac{2}{3}\frac{W_2^{\,2}}{W_1}\;\frac{(W_1-2)}{(W_1-1)}\right)$$
$$- W_2^{\,2}(-2kW_1+2k^2+3k+4a+b)$$

$$= W_2^{\,2}\left(\frac{2}{3}kW_1 - 2k^2 - \frac{1}{3}k - 4a - b + \frac{8}{3}\frac{k(W_1-2)}{W_1(W_1-1)}\;(2t+k)\right)$$

$$\geqslant W_2^{\,2}\left(\frac{2}{3}kW_1 - 2k^2 - \frac{1}{3}k - 4a - b + \frac{8}{3}\frac{k(2k-1)}{W_1(2k)}\;(2t+k)\right)$$

(1) $$= W_2^{\,2}\left(\frac{2}{3}kW_1 - 2k^2 - \frac{1}{3}k - 4a - b + \frac{4}{3}\frac{(2k-1)}{W_1}\;(2t+k)\right)$$

$$= W_2^{\,2}\left(\frac{2}{3}k(2k+1+a+b+c) - 2k^2 - \frac{1}{3}k - 4a - b + \frac{4(2k-1)(2t+k)}{3(2k+1+a+b+c)}\right)$$

(2) $$= W_2^{\,2}\left(-\frac{2}{3}k^2 + \frac{1}{3}k + (\frac{2}{3}k-4)a + (\frac{2}{3}k-1)b + \frac{2}{3}kc + \frac{(8k-4)(2t+k)}{3(2k+1+a+b+c)}\right)$$

(3) $$\geqslant W_2^{\,2}\left(-\frac{2}{3}k^2 + \frac{1}{3}k + (\frac{2}{3}k-4)a + (\frac{2}{3}k-1)b + \frac{2}{3}kc + \frac{(8k-4)(4a+2b+k)}{3(2k+1+a+b+c)}\right)$$

$$(4) \qquad = \frac{W_2^{\,2}}{3W_1}\left((-2k^2+k+(2k-12)a+(2k-3)b+2kc)(2k+1+a+b+c)+(8k-4)(4a+2b+k)\right)$$

From here, the proof must be broken into cases: $k \geqslant 6$, $k = 0$, $k = 1$ and $2 \leqslant k \leqslant 5$.

Let $k \geqq 6$: If also $b+c \geqslant k+1$, then $c \geqslant 1$ and by (3),

$$\begin{aligned}
W_2'^{\,2}W_3 - W_2^{\,2}W_3' &\geqslant W_2^{\,2}\Big(-\tfrac{2}{3}k^2+\tfrac{1}{3}k+(\tfrac{2}{3}k-4)a+(\tfrac{2}{3}k-1)b+\tfrac{2}{3}kc \\
&\qquad\qquad + \frac{(8k-4)(4a+2b+k)}{3(2k+1+a+b+c)}\Big) \\
&> W_2^{\,2}(-\tfrac{2}{3}k^2+\tfrac{1}{3}k+(\tfrac{2}{3}k-1)(b+c)+c) \\
&\geqslant W_2^{\,2}(-\tfrac{2}{3}k^2+\tfrac{1}{3}k+(\tfrac{2}{3}k-1)(k+1)+1) \\
&= 0 \quad .
\end{aligned}$$

If $b+c \leqslant k$, then by Lemma 1.8, the only vertices of degree greater than k are $v, v', v_1, v_2, \ldots, v_k$, and all others are of degree 1. Since any u_i must have degree at least 2, there are no u_i's. Hence $b = 0$. Therefore, $b+c \leqslant k$ and $2a+b+c \geqslant k^2-k$ reduce to $c \leqslant k$ and $2a+c \geqslant k^2-k$. These imply $2a \geqslant k^2-k-c \geqslant k^2-2k$. Using these inequalities, $t \geqslant 2a$ and $a \leqslant \frac{k^2-k}{2}$, (2) becomes

$$\begin{aligned}
W_2'^{\,2}W_3 - W_2^{\,2}W_3' &\geqslant W_2^{\,2}\Big(-\tfrac{2}{3}k^2+\tfrac{1}{3}k+(\tfrac{2}{3}k-4)a+(\tfrac{2}{3}k-1)b+\tfrac{2}{3}kc \\
&\qquad\qquad + \frac{(8k-4)(2t+k)}{3(2k+1+a+b+c)}\Big) \\
&\geqslant W_2^{\,2}\left(-\tfrac{2}{3}k^2+\tfrac{1}{3}k+(\tfrac{2}{3}k-4)(\frac{k^2-2k}{2})+\frac{(8k-4)(2(k^2-2k)+k)}{3(2k+1+\frac{k^2-k}{2}+k)}\right) \\
&= \frac{W_2^{\,2}}{3(k^2+5k+2)}\,(k^5-5k^4-3k^3-19k^2+50k) \\
&> 0 \qquad \text{for } k \geqslant 6.
\end{aligned}$$

Let $k = 0$: Here

$$W_2' = W_2 - 2k = W_2$$

and

$$W_3' = W_3 - 2kW_1 + 2k^2 + 3k + 4a + b - 3d = W_3 - 3d \quad .$$

So

$$W_2'^2 W_3 - W_2^2 W_3' = W_2^2 W_3 - W_2^2(W_3 - 3d) = 3dW_2^2 \geqslant 0 \quad .$$

Let $k = 1$: Here $a = 0$, and by (4)

$$\begin{aligned} W_2'^2 W_3 - W_2^2 W_3' &\geqslant \frac{W_2^2}{3W_1}\Big((-2k^2+k+(2k-12)a+(2k-3)b+2kc)(2k+1+a+b+c) \\ &\qquad\qquad +(8k-4)(4a+2b+k)\Big) \\ &= \frac{W_2^2}{3W_1}\Big((-2+1-b+2c)(3+b+c)+4(2b+1)\Big) \\ &= \frac{W_2^2}{3W_1}\,(1+4b+5c-b^2+bc+2c^2) \\ &> 0 \end{aligned}$$

since $c = kc \geqslant b$ as showed earlier.

Let $2 \leqslant k \leqslant 5$: For these k, use (4) and minimize the factor

$$(-2k^2+k+(2k-12)a+(2k-3)b+2kc)(2k+1+a+b+c) + (8k-4)(4a+2b+k)$$

with respect to the variable b.

$$\frac{\partial}{\partial b}\left((-2k^2+k+(2k-12)a+(2k-3)b+2kc)(2k+1+a+b+c)+(8k-4)(4a+2b+k)\right)$$

$$= (-2k^2+k+(2k-12)a+(2k-3)b+2kc)+(2k-3)(2k+1+a+b+c)+(16k-8)$$

$$= 2k^2+13k-11+(4k-15)a+(4k-6)b+(4k-3)c$$

$$\geqslant 2k^2+13k-11+(4k-15)a$$

$$> 0 \quad \text{for} \quad k = 4, 5.$$

For $k = 2, 3$,

$$2k^2+13k-11+(4k-15)a$$

$$\geqslant 2k^2+13k-11+(4k-15)\frac{(k^2-k)}{2}$$

$$= \frac{1}{2}(4k^3-15k^2+41k-22)$$

$$= \frac{1}{2}(4k(k-2)^2+k^2+25k-22)$$

$$> 0 \quad .$$

Thus this factor is minimized at $b = 0$. Now minimize

$$(-2k^2+k+(2k-12)a+2kc)(2k+1+a+c) + (8k-4)(4a+k)$$

with respect to the variable c.

$$\frac{\partial}{\partial c}\left((-2k^2+k+(2k-12)a+2kc)(2k+1+a+c) + (8k-4)(4a+k)\right)$$

$$= (-2k^2+k+(2k-12)a+2kc) + 2k(2k+1+a+c)$$

$$= 2k^2+3k+(4k-12)a+4kc$$

$$\geqslant 0 \quad \text{for} \quad k \geqslant 3.$$

For $k = 2$, $a = 0$ or 1, and

$$2k^2 + 3k + (4k-12)a + 4kc$$

$$\geqslant 10 + 8c$$

$$> 0 \quad .$$

Since $2a+b+c \geqslant k^2-k$ and $kc \geqslant b$, it follows that $c \geqslant \frac{k^2-k-2a}{k+1}$. Substituting $b = 0$ and $c = \frac{k^2-k-2a}{k+1}$ into (4) gives

$$
\begin{aligned}
W_2'^2 W_3 - W_2^2 W_3' &\geqslant \frac{W_2^2}{3W_1}\Big((-2k^2+k+(2k-12)a+(2k-3)b+2kc)(2k+1+a+b+c) \\
&\qquad + (8k-4)(4a+2b+k)\Big) \\
&\geqslant \frac{W_2^2}{3W_1}\Big((-2k^2+k+(2k-12)a+2k\frac{k^2-k-2a}{k+1})(2k+1+a+\frac{k^2-k-2a}{k+1}) \\
&\qquad + (8k-4)(4a+k)\Big) \\
&= \frac{W_2^2}{3(k+1)^2W_1}\Big[\Big((-2k^2+k+(2k-12)a)(k+1) \\
&\qquad + (2k^3-2k^2-4ka)\Big)\Big((2k+1+a)(k+1) \\
&\qquad + (k^2-k-2a)\Big) + (8k-4)(4a+k)(k+1)^2\Big] \\
&= \frac{W_2^2}{3(k+1)^2W_1}\Big(((2k^2-14k-12)a-3k^2+k)((k-1)a+3k^2+2k+1) \\
&\qquad + ((32k-16)a+8k^2-4k)(k^2+2k+1)\Big) \\
&= \frac{W_2^2}{3(k+1)^2W_1}\Big((2k^3-16k^2+2k+12)a^2+(6k^4-9k^3-10k^2-39k-28)a \\
&\qquad + (-k^4+9k^3-k^2-3k)\Big)
\end{aligned}
$$

Since $2k^3 - 16k^2 + 2k + 12 < 0$ and $-k^4 + 9k^3 - 9k^2 - 3k > 0$ for $2 \leqslant k \leqslant 5$, this function of a is a parabola opening downward which is positive at $a = 0$. Now we choose maximal values of a such that this quadratic is positive. Then it will be positive for all smaller positive values of a.

For k = 5, let a = 10. Then

$$W_2'^2 W_3 - W_2^2 W_3' \geq \frac{W_2^2}{3(5+1)^2(W_1)}\Big((2\cdot 125 - 16\cdot 25 + 2\cdot 5 + 12)\cdot 100$$
$$+ (6\cdot 625 - 9\cdot 125 - 10\cdot 25 - 39\cdot 5 - 28)\cdot 10$$
$$+ (-625 + 9\cdot 125 - 25 - 3\cdot 5)\Big)$$
$$= \frac{W_2^2}{108(W_1)}\ (7180) > 0\ .$$

For k = 4, let a = 6. Then

$$W_2'^2 W_3 - W_2^2 W_3' \geq \frac{W_2^2}{3(4+1)^2(W_1)}\Big((2\cdot 54 - 16\cdot 16 + 2\cdot 4 + 12)\cdot 36$$
$$+ (6\cdot 256 - 9\cdot 64 - 10\cdot 16 - 39\cdot 4 - 28)\cdot 6$$
$$+ (-256 + 9\cdot 64 - 16 - 3\cdot 4)\Big)$$
$$= \frac{W_2^2}{75(W_1)}\ (100) > 0\ .$$

For k = 3, let a = 1. Then

$$W_2'^2 W_3 - W_2^2 W_3' \geq \frac{W_2^2}{3(3+1)^2(W_1)}\Big((2\cdot 27 - 16\cdot 9 + 2\cdot 3 + 12)$$
$$+ (6\cdot 81 - 9\cdot 27 - 10\cdot 9 - 39\cdot 3 - 28) + (-81 + 9\cdot 27 - 9 - 3\cdot 3)\Big)$$
$$= \frac{W_2^2}{48(W_1)}\ (80) > 0.$$

There are still three cases remaining: k = 3, a = 3; k = 3, a = 2; and k = 2, a = 1. For k = 3 and a = 3, t ⩾ 7. By (2)

$$W_2'^2 W_3 - W_2^2 W_3' \geqslant W_2^2 \Big(-\tfrac{2}{3} k^2 + \tfrac{1}{3} k + (\tfrac{2}{3} k - 4)a + (\tfrac{2}{3} k - 1)b + \tfrac{2}{3} kc + \frac{(8k-4)(2t+k)}{3(2k+1+a+b+c)} \Big)$$

$$\geqslant W_2^2 \Big(-6+1-6+b+2c + \frac{(20)(14+3)}{3(10+b+c)} \Big)$$

$$= \frac{W_2^2}{3(10+b+c)} (10+37b+27c+3b^2+9bc+6c^2)$$

$$> 0 \quad .$$

For $k = 3$, $a = 2$, by (4)

$$W_2'^2 W_3 - W_2^2 W_3' \geqslant \frac{W_2^2}{3W_1} \Big((-2k^2+k+(2k-12)a + (2k-3)b + 2kc)\,(2k+1+a+b+c) + (8k-4)(4a+2b+k) \Big)$$

$$= \frac{W_2^2}{3W_1} \Big((-18+3-12+3b+6c)(9+b+c) + (20)(11+2b) \Big)$$

$$= \frac{W_2^2}{3W_1} (-23+40b+27c+3b^2+9bc+6c^2)$$

$$> 0 \quad \text{since} \quad c \geqslant 1 \ .$$

For $k = 2$, $a = 1$, by (4)

$$W_2'^2 W_3 - W_2^2 W_3' \geqslant \frac{W_2^2}{3W_1} \Big((-2k^2+k+(2k-12)a + (2k-3)b + 2kc)\,(2k+1+a+b+c) + (8k-4)(4a+2b+k) \Big)$$

$$= \frac{W_2^2}{3W_1} \Big((-8+2-8+b+4c)(6+b+c) + (12)(6+2b) \Big)$$

$$= \frac{W_2^2}{3W_1} (-12+16b+10c+b^2+5bc+4c^2)$$

$$> 0 \quad \text{if} \ b \geqslant 1 \ \text{or} \ c \geqslant 1 \ .$$

The case $k = 2$, $a = 1$, $b = 0$, $c = 0$ is

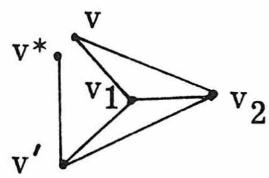


Figure 8

Here $W_2 = 11$, $W_3 = 7$, $W_2' = 7$, $W_3' = 1$, so

$$W_2'^2 W_3 - W_2^2 W_3' = 49 \cdot 7 - 121 \cdot 1 = 222 > 0 \quad .$$

This completes the proof of Theorem 1.7.

CHAPTER II

CUT-CONTRACTIONS

The previous chapter gave a characterization of the closed sets of H′ in terms of closed sets in H. The effect was that of deleting certain elements from L(H) to produce L(H′). In this chapter it will be shown that a generalization of this procedure can be carried out for arbitrary geometric lattices, and that any geometric lattice can be constructed from a Boolean algebra in this way.

Some preliminary definitions and a lemma will be needed.

Definition: Let x, y be elements of a geometric lattice. Then (x, y) is a modular pair if $r(x) + r(y) = r(x \vee y) + r(x \wedge y)$ [8].

Definition: A modular cut Q of a geometric lattice L is a subset of L such that (i) if $x \in Q$ and $y \geqslant x$, then $y \in Q$ and (ii) if $x, y \in Q$ and $x \succ x \wedge y$, then $x \wedge y \in Q$ [2].

Lemma 2.1: If (x, y) is a modular pair in a modular cut Q of a geometric lattice L, then $x \wedge y \in Q$.

Proof: Let $x \wedge y = x_0 < x_1 < \dots < x_n = x$, and let $y_i = x_i \vee y$. By semimodularity, $y = y_0 \leq y_1 \leq \dots \leq y_n = x \vee y$, but since $n = r(x) - r(x \wedge y) = r(x \vee y) - r(y)$, we must have $y = y_0 < y_1 < \dots < y_n = x \vee y$. Hence $r(y_j) - r(y_i) = j - i = r(x_j) - r(x_i)$. From $x_i \leqslant x_{i+1} \wedge y_i$ and

$$\begin{aligned} r(x_{i+1} \wedge y_i) &\leq r(x_{i+1}) + r(y_i) - r(x_{i+1} \vee y_i) \\ &= r(x_{i+1}) + r(y_i) - r(y_{i+1}) + (r(y_{i+1}) - r(y_n)) - (r(x_{i+1}) - r(x_n)) \\ &= r(y_i) - r(y_n) + r(x_n) \\ &= r(x_i) \quad , \end{aligned}$$

it follows that $x_i = x_{i+1} \wedge y_i$ for all i. Let x_j be maximal such that $x_j \notin Q$. Then $j \neq n$, and $x_{j+1} \in Q$. Since Q is a modular cut, $y_j \in Q$ and $x_j = x_{j+1} \wedge y_j \in Q$ which is a contradiction. Thus $x_i \in Q$ for all i, and in particular, $x \wedge y = x_0 \in Q$.

Theorem 2.2: Let L be a geometric lattice, let Q be a modular cut of L and let $T = \{x \in L : x \notin Q, \exists\, q \in Q \ni x \lessdot q\}$. Then $L - T$ is a geometric lattice.

Before proving the theorem, we shall show that this construction is indeed a generalization of the basic reduction process used in Chapter I for geometric lattices associated with a graph. Let the graph H' be derived from the graph H by identifying two vertices v^* and v in H. Let $Q = \{S \in L(H) : \exists$ a path P in H from v^* to v, $P \subseteq S\}$. Clearly if $S_1, S_2 \in L(H)$ are such that $S_1 \in Q$ and $S_1 \leq S_2$, then $S_2 \in Q$. Now let $S_1, S_2 \in Q$ be such that $S_1 \gtrdot S_1 \wedge S_2$ in $L(H)$. Then $S_1 \wedge S_2 \in L(H')$ since $S_1, S_2 \in Q \subseteq L(H')$. But then by Lemma 1.9, $S_1 \wedge S_2$ must contain a path in H from v^* to v, i.e., $S_1 \wedge S_2 \in Q$. Thus Q is a modular cut. By Lemma 1.9 again, the set T corresponding to Q is exactly the collection of closed sets of H which are not closed in H'. Hence $L(H) - T$ is isomorphic to $L(H')$.

The following three lemmas will be useful in proving the theorem.

Lemma 2.3: If $x \in T$ and $y \geq x$, then $y \in T \cup Q$.

Let $q \in Q$ be such that $q \succ x$. If $y \geq q$, then $y \in Q$. If $y \not\geq q$, then $y \wedge q = x \prec q$, and $y \vee q \succ y$. But $y \vee q \in Q$ since $y \vee q \geq q$. Hence if $y \notin Q$, then $y \in T$.

Lemma 2.4: Let $x \succ y \succ z$ in L with $x \in Q$, $z \in L - Q - T$. Then $y \in T$, and hence $x \succ z$ in $L - T$.

Since $x \succ y$ in L and $x \in Q$, either $y \in Q$ or $y \in T$. If $y \in Q$, then $z \in Q$ or $z \in T$. That contradicts $z \in L - Q - T$, so $y \in T$. Since $y \notin L - T$ for all y such that $x \succ y \succ z$ in L, it follows that $x \succ z$ in $L - T$.

Lemma 2.5: If $x \succ z$ in $L - T$, then either $x \succ z$ in L, or there exists $y \in T$ such that $x \succ y \succ z$ in L.

Let $x \succ z$ in $L - T$. If $x \not\succ z$ in L, then there exists $y \in T$ such that $x \succ y > z$ in L. Since $y \in T$, $x \in (L - T) \cap (Q \cup T) = Q$. If $y \not\succ z$ in L, then there exists $t \in T$ such that $x \succ y \succ t > z$ in L. Let $q \in Q$ be such such that $q \succ t$. Now $q \not\leq x$ since otherwise $x > q > z$ in $L - T$. Hence $t = x \wedge q \prec q$, and $t \in Q$ which is impossible. Thus $y \succ z$ in L.

The proof of Theorem 2.2 has three steps. It must be shown that $L - T$ is a lattice, that it is a point lattice, and that it is semi-modular.

Proof: In order to show that $L - T$ is a lattice, we show that $L - T$ is closed with respect to meet. Let $x, y \in L - T$, and assume that

$x \wedge y \in T$. By Lemma 2.3, $x, y \in T \cup Q$. But $x, y \in L - T$, so $x, y \in Q$. Let $q \in Q$ be such that $x \wedge y \prec q$; such a q exists since $x \wedge y \in T$. Either $x \not\geq q$ or $y \not\geq q$, so without loss of generality, it may be assumed that $x \not\geq q$. Then $x \wedge q = x \wedge y \prec q$. Since Q is a modular cut, $x \wedge q \in Q$, and so $x \wedge y \in Q$. This contradicts $x \wedge y \in T$.

An immediate consequence of $L - T$ being closed with respect to meet is that for $t \in T$, there is a unique $q \in Q$ such that $q \succ t$. For assume this is not so, and let $t \in T$ and $q_1, q_2 \in Q$ be such that $q_1 \neq q_2$, $q_1 \succ t$ and $q_2 \succ t$ in L. Then $t = q_1 \wedge q_2 \prec q_1$; so $t \in Q$, a contradiction.

Since $L - T$ contains the unit element of L and is closed with respect to meet, it follows that $L - T$ is a lattice. Let $\vee'$ denote the join operation in $L - T$. Let $x, y \in L - T$. If $x \vee y \in L - T$, then it is clear that $x \vee' y = x \vee y$. If $x \vee y \notin L - T$, i.e., $x \vee y \in T$, then there exists a unique $q \in Q$ such that $x \vee y \prec q$ in L. Let $z \in L - T$ be such that $z \geq x, y$. Then $q \geq q \wedge z \geq x \vee y$. Since $q \wedge z \in L - T$ and $x \vee y \in T$, it follows that $q = q \vee z \leq z$. Thus $q = x \vee' y$.

In order to show that $L - T$ is a point lattice, let $x \in L - T$, and let $P = \{p \in L : x \geq p \succ 0\}$. Then $\bigvee P = x$ since L is a point lattice. There are two cases to consider. First, if $0 \in T$, let $q \in Q$ be such that $q \succ 0$. Then q is the 0 element of $L - T$. Since $q \vee x \in Q$, $q \vee x \geq x$ and $x \notin T$, it follows that $x \in Q$. But now $x \geq q$ or else $0 = x \wedge q \prec q \in Q$ and $0 \in Q$, a contradiction. Thus $x = q \vee \bigvee P$ $= \bigvee\{p \vee q : p \in P\} = \bigvee'\{p \vee q : p \in P\}$ which is a representation of x as a join of points in $L - T$.

If $0 \in L - T$, let $P' = \{p' \in L - T : x \geq p' \succ 0 \text{ in } L - T\}$. For $p \in P$, if $p \in L - T$, then $p \in P'$ and $\bigvee P' \geq p$. If $p \in T$, then there exists

$q \in Q$ such that $q \succ p$ in L. By Lemma 2.4, $q \succ 0$ in L - T. By the argument of the preceding paragraph, $x \geqslant q$. Hence $q \in P'$ and $\bigvee P' \geqslant q > p$. Since $\bigvee P' \geqslant p$ for all $p \in P$, it follows that $\bigvee P' \geqslant \bigvee P$. Thus

$$\bigvee' P' \geqslant \bigvee P' \geqslant \bigvee P = x \geqslant \bigvee' P' \quad ,$$

and hence $x = \bigvee' P'$.

Finally it must be shown that L - T is semimodular. Let $x \succ x \wedge y$ in L - T. Then $x \succ x \wedge y$ in L or there exists $t \in T$ such that $x \succ t \succ x \wedge y$ in L by Lemma 2.5. If $x \succ x \wedge y$ in L, then $x \vee y \succ y$ in L. If in addition $x \vee y \in L - T$, then $x \vee' y = x \vee y \succ y$. Thus we may assume that $x \vee y \in T$. Then $x \vee' y \in Q$, and $x \vee' y \succ x \vee y$. Since y must be in L - Q - T, Lemma 2.4 implies that $x \vee' y \succ y$ in L - T.

On the other hand, if there exists $t \in T$ such that $x \succ t \succ x \vee y$ in L, then $x \in Q$, $x \wedge y \notin Q$, $x \vee y \in Q$ and $x \vee' y = x \vee y$. Clearly if $x \vee y \succ y$ in L, then $x \vee' y = x \vee y \succ y$ in L - T. Otherwise, if $x \vee y \not\succ y$ in L, then there exists z such that $x \vee y \succ z \succ y$ in L, and (x, y) is a modular pair. Since $x \in Q$ and $x \wedge y \notin Q$, it follows that $y \notin Q$, i.e., $y \in L - Q - T$. Then by Lemma 2.4, $x \vee' y \succ y$ in L - T. This completes the proof of Theorem 2.2.

Since L - T is a geometric lattice, it has a well defined rank function r'. The following lemma will be useful in characterizing r' in terms of the rank function r on L.

<u>Lemma 2.6</u>: Let $x, y \in L - T$ be such that $x \geqslant y$. If $x, y \in Q$ or $x, y \in L - Q - T$, then $x \geqslant z \geqslant y$ implies $z \notin T$.

Proof: Assume that there exists $z \in T$ such that $x \geqslant z \geqslant y$. Then $x > z > y$ since $x, y \in L - T$. By Lemma 2.3, $x \in Q$ and $y \in L - Q - T$. Thus the contrapositive of Lemma 2.6 is established.

Theorem 2.7: For $x \in L - T$,

$$r'(x) = \begin{cases} r(x) & \text{if } x \notin Q \\ r(x) - 1 & \text{if } x \in Q \end{cases} .$$

Proof: If $x \notin Q$, i.e., $x \in L - Q - T$, then $0 \in L - Q - T$ and $z \in L - T$ for all $z \in L$ such that $z \leqslant x$ by Lemma 2.6. Hence any maximal chain in $L - T$ from 0 to x is a maximal chain in L from 0 to x. Thus $r'(x) = r(x)$.

If $x \in Q$, then let $0 = x_0, x_1, \ldots, x_k = x$ be a maximal chain in $L - T$ from 0 to x such that $x_0, \ldots, x_{i-1} \in L - Q - T$, and $x_i, \ldots, x_k \in Q$. By Lemma 2.6, $x_0, \ldots, x_{i-1}$ and $x_i, \ldots, x_k$ are maximal chains in L. Since $x_i \in Q$ and $x_{i-1} \notin T$, there must exist $t \in T$ such that $x_i \succ t > x_{i-1}$ in L. Lemma 2.5 then implies that $x_i \succ t \succ x_{i-1}$ in L. Hence $0 = x_0, \ldots, x_{i-1}, t, x_i, \ldots, x_k = x$ is a maximal chain in L from 0 to x, and $r'(x) = r(x) - 1$.

The geometric lattice $L - T$ will be called the cut-contraction of L with respect to Q. A cut-contraction $L - T$ of L will be called a trivial cut-contraction of L if T is empty; this occurs if and only if Q is empty or $Q = L$. The next corollary is an immediate consequence of Theorem 2.7.

Corollary 2.8: If L' is a nontrivial cut-contraction of L, then $\dim(L') = \dim(L) - 1$.

Thus by forming cut-contractions, we can produce new, smaller geometric lattices from other geometric lattices. In fact, it will be shown that any geometric lattice can be obtained by means of a sequence of cut-contractions, starting from a suitable Boolean algebra.

Every geometric lattice L has a canonical representation in terms of closed subsets of the set of points of L. In this representation, each element $x \in L$ is associated with the closed set $S_x = \{p \in L : x \geqslant p > 0\}$. The element $x \wedge y$ is associated with $S_{x \wedge y} = S_x \cap S_y$, and the element $x \vee y$ is associated with $S_{x \vee y} = \cap \{S_z : S_z \supseteq S_x \text{ and } S_z \supseteq S_y\}$. The next theorem will be formulated in terms of this canonical representation.

Theorem 2.9: Let L and L' be geometric lattices. If there exists a one-to-one map from the points of L' onto the points of L such that closed sets in L' correspond to closed sets in L, then L' can be obtained from L by a sequence of $k = \dim(L) - \dim(L')$ cut-contractions.

The hypothesis of Theorem 2.9 is equivalent to the assumption that there exists a meet preserving embedding of L' into L which maps the points of L' onto the points of L.

Proof: First note that for any pair $x, y \in L'$ with $y \geqslant x$ that $r(y) - r(x) \geqslant r'(y) - r'(x)$. For let $x = x_0 \prec x_1 \prec \ldots \prec x_m = y$ be a maximal chain in L'. Then each x_i is closed in L, and they are distinct. Hence $x = x_0 < x_1 < \ldots < x_m = y$ in L, and $r(y) - r(x) \geqslant m = r'(y) - r'(x)$.

Let $Q = \{x \in L' : \dim(L) - r(x) = \dim(L') - r'(x)\}$. It will be shown that Q is a modular cut of L and that L' is contained in the cut-contraction of L with respect to Q.

In order to show that Q is a modular cut of L, let $x \in Q$, and let $y \in L$ be minimal such that $y > x$ and $y \notin Q$. If $y \in L'$, then

$$\begin{aligned}\dim(L) - r(y) &\geqslant \dim(L') - r'(y)\\ &= (\dim(L') - r'(x)) - (r'(y) - r'(x))\\ &\geqslant (\dim(L) - r(x)) - (r(y) - r(x))\\ &= \dim(L) - r(y).\end{aligned}$$

Hence $\dim(L) - r(y) = \dim(L') - r'(y)$, and $y \in Q$. If $y \notin L'$, then let $y \succ z \geqslant x$ in L. By the minimality of y, $z \in Q$. Let p be a point in L (and in L') such that $y = z \vee p$. But $z \vee' p$ is a closed set in L', and hence $z \vee' p$ is closed in L such that $z \vee' p \geqslant z \vee p = y$. By semi-modularity in L', $r'(z \vee' p) = r'(z) + 1$. Then

$$\begin{aligned}r(z \vee' p) &= \dim(L) - \dim(L') + r'(z \vee' p)\\ &= \dim(L) - (\dim(L') - r'(z)) + 1\\ &= \dim(L) - (\dim(L) - r(z)) + 1\\ &= r(z) + 1\\ &= r(y) \quad .\end{aligned}$$

Hence $y = z \vee' p$, and $y \in L'$, a contradiction. Thus $x \in Q$, $y \in L$ and $y \geqslant x$ imply $y \in Q$.

Since $x \wedge y$ corresponds to the intersection of the sets of points contained in x and y, $x \wedge y = x \wedge' y$. Let $x, y \in Q$ be such that

$x \succ x \wedge y$ in L. Then since $1 = r(x) - r(x \wedge y) \geq r'(x) - r'(x \wedge y) \geq 1$, it follows that $r(x) - r(x \wedge y) = r'(x) - r'(x \wedge y)$ and

$$\begin{aligned} \dim(L) - r(x \wedge y) &= (\dim(L) - r(x)) + (r(x) - r(x \wedge y)) \\ &= (\dim(L') - r'(x)) + (r'(x) - r'(x \wedge y)) \\ &= \dim(L') - r'(x \wedge y) \quad . \end{aligned}$$

Hence $x \wedge y \in Q$, and Q is a modular cut.

If $\dim(L) = \dim(L')$, then $0 \in Q$, and $L - T$ is a trivial cut-contraction of L. All closed sets of L are in Q, and hence they are closed in L′. Thus L and L′ have the same closed sets. Since L and L′ have the same order relation, they are isomorphic. If $\dim(L) > \dim(L')$, then $0 \notin Q$. However, the unit element of L is in Q, so $L - T$ is a non-trivial cut-contraction of L. Then by Corollary 2.8, $\dim(L) = \dim(L - T) + 1$.

Let $L_1 = L - T$ denote the cut-contraction of L with respect to Q. By Theorem 2.2, L_1 is a geometric lattice. There is an induced map of points of L to the points of L_1. In order to see that all closed sets in L′ are closed in L_1, let $t \in L$, $q \in Q$ be such that $t \notin Q$ and $t \prec q$ in L. If t were closed in L′, then $t \prec q$ in L′ and $\dim(L) - r(t) = \dim(L) - r(q) - 1 = \dim(L') - r'(q) - 1 = \dim(L') - r'(t)$. Hence $t \in Q$, a contradiction. Thus closed sets in L′ are closed in L_1; in particular, each point in L′ is a point in L_1.

To see that the induced map from the points of L′ to the points of L_1 is onto, let q be a point of L which does not correspond to a point in L'. Then q does not correspond to a point in L. By Lemma 2.5, with $x = q$ and $z = 0$, there exists an element $y \in T$ such that $q \succ y \succ 0$ in L. But y is a point in L, and hence is a point in L_1. This contradicts $y \in T$. Thus all points in L_1 correspond to points in L', i.e., the induced map from points of L' to points of L_1 is onto. The previous paragraph showed that it is one-to-one. Hence L_1 and L' satisfy the conditions of the theorem.

Repeating this construction k times, where $k = \dim(L) - \dim(L')$, gives the sequence of cut-contractions

$$L, L_1, L_2, \ldots, L_k \quad .$$

As shown above, the lattices L_k and L' satisfy the conditions of the theorem, and $\dim(L_k) = \dim(L) - k = \dim(L')$. It follows that L_k and L' are isomorphic. This completes the proof of Theorem 2.9.

If L' is an arbitrary geometric lattice, let L be the Boolean algebra of all subsets of the set of points of L'. Then L and L' satisfy the conditions of Theorem 2.9. Thus we get the following corollary.

Corollary 2.10: Let L be a geometric lattice with W_1 points, and let B be the Boolean algebra having W_1 points. Then L can be obtained from B by a sequence of $k = W_1 - \dim(L)$ cut-contractions.

APPENDIX

Theorem A.1: Let L_1 and L_2 be logarithmically concave geometric lattices. Then $L = L_1 \times L_2$ is logarithmically concave also.

Proof: Let U_i, V_i and W_i be the Whitney numbers of L_1, L_2 and L, respectively. Then

$$W_i = \sum_{j=0}^{i} U_j V_{i-j} \quad ,$$

and

$$W_i^2 - W_{i-1}W_{i+1} = \left(\sum_{j=0}^{i} U_j V_{i-j}\right)\left(\sum_{k=0}^{i} U_k V_{i-k}\right) - \left(\sum_{j=0}^{i-1} U_j V_{i-1-j}\right)\left(\sum_{k=0}^{i+1} U_k V_{i+1-k}\right)$$

$$= \sum_{j=0}^{i}\sum_{k=0}^{i} U_j U_k V_{i-j} V_{i-k} - \sum_{j=0}^{i-1}\sum_{k=0}^{i+1} U_j U_k V_{i-1-j} V_{i+1-k}$$

$$= \sum_{k=0}^{i} U_i U_k V_0 V_{i-k} + \sum_{j=0}^{i-1}\sum_{k=0}^{i} U_j U_k V_{i-j} V_{i-k}$$

$$- \sum_{j=0}^{i-1} U_j U_{i+1} V_{i-1-j} V_0 - \sum_{j=0}^{i-1}\sum_{k=0}^{i} U_j U_k V_{i-1-j} V_{i+1-k}$$

$$= U_i U_0 V_0 V_i + \sum_{k=1}^{i} U_i U_k V_0 V_{i-k} + \sum_{j=0}^{i-1} U_j U_0 V_{i-j} V_i$$

$$+ \sum_{j=0}^{i-1} U_j U_{j+1} V_{i-j} V_{i-j-1} + \sum_{j=1}^{i-1} \sum_{k=1}^{j} U_j U_k V_{i-j} V_{i-k}$$

$$+ \sum_{j=0}^{i-2} \sum_{k=j+2}^{i} U_j U_k V_{i-j} V_{i-k}$$

$$- \sum_{j=0}^{i-1} U_j U_{i+1} V_{i-1-j} V_0 - \sum_{j=0}^{i-1} U_j U_0 V_{i-1-j} V_{i+1}$$

$$- \sum_{j=0}^{i-1} U_j U_{j+1} V_{i-1-j} V_{i-j}$$

$$- \sum_{j=1}^{i-1} \sum_{k=1}^{j} U_j U_k V_{i-1-j} V_{i+1-k} - \sum_{j=0}^{i-2} \sum_{k=j+2}^{i} U_j U_k V_{i-1-j} V_{i+1-k}$$

$$= \left(\sum_{k=1}^{i} U_i U_k V_0 V_{i-k} - \sum_{j=0}^{i-1} U_j U_{i+1} V_{i-1-j} V_0 \right)$$

$$+ \left(\sum_{j=0}^{i-1} U_j U_0 V_{i-j} V_i - \sum_{j=0}^{i-1} U_j U_0 V_{i-1-j} V_{i+1} \right)$$

$$+ \left(\sum_{j=0}^{i-1} U_j U_{j+1} V_{i-j} V_{i-j-1} - \sum_{j=0}^{i-1} U_j U_{j+1} V_{i-1-j} V_{i-j} \right)$$

$$+ \left(\sum_{j=1}^{i-1} \sum_{k=1}^{j} U_j U_k V_{i-j} V_{i-k} - \sum_{j=1}^{i-1} \sum_{k=1}^{j} U_j U_k V_{i-1-j} V_{i+1-k} \right)$$

$$+ \left(\sum_{j=0}^{i-2} \sum_{k=j+2}^{i} U_j U_k V_{i-j} V_{i-k} - \sum_{j=0}^{i-2} \sum_{k=j+2}^{i} U_j U_k V_{i-1-j} V_{i+1-k} \right)$$

$$+ U_i U_0 V_0 V_i$$

$$= \sum_{k=1}^{i} (U_k U_i - U_{k-1} U_{i+1}) V_0 V_{i-k}$$

$$+ \sum_{j=0}^{i-1} U_0 U_j (V_{i-j} V_i - V_{i-j-1} V_{i+1}) + 0$$

$$+ \sum_{j=1}^{i-1} \sum_{k=1}^{j} U_j U_k (V_{i-j} V_{i-k} - V_{i-j-1} V_{i-k+1})$$

$$+ \sum_{j=0}^{i-2} \sum_{k=j+2}^{i} U_j U_k (V_{i-j} V_{i-k} - V_{i-j-1} V_{i-k+1})$$

$$+ U_0 U_i V_0 V_i$$

$$\geqslant \sum_{m=1}^{i-1} \sum_{n=m}^{i-1} U_n U_m (V_{i-n} V_{i-m} - V_{i-n-1} V_{i-m+1})$$

$$+ \sum_{m=1}^{i-1} \sum_{n=m}^{i-1} U_{m-1} U_{n+1} (V_{i-m+1} V_{i-n-1} - V_{i-m} V_{i-n})$$

$$= \sum_{m=1}^{i-1} \sum_{n=m}^{i-1} (U_m U_n - U_{m-1} U_{n+1})(V_{i-n} V_{i-m} - V_{i-n-1} V_{i-m+1})$$

$$\geqslant 0$$

since $U_k U_j - U_{k-1} U_{j+1} \geqslant 0$ for $k \leqslant j$, and $V_k V_j - V_{k-1} V_{j+1} \geqslant 0$ for $k \leqslant j$.

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