A STUDY OF THE CANONICAL FORM FOR A PAIR OF REAL SYMMETRIC MATRICES AND APPLICATIONS TO PENCILS AND TO PAIRS OF QUADRATIC FORMS

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ABSTRACT

A pair of real symmetric matrices S and T is called a nonsingular pair if S is nonsingular. A new treatment for obtaining the classical canonical pair form for a nonsingular pair is obtained by the use of results on commuting matrices and by elementary matrix algebra.

This canonical form is used to obtain formulas for an arbitrary real $n \times n$ matrix A that relate the dimensions of both the space N of real symmetric matrices T such that $AT = TA'$ and the space of products AT such that AT is symmetric to the real Jordan normal form of A. The first formula expresses a previously found result in a simpler way while the second one is new. These formulas are then applied to prove anew the known result that A is nonderogatory iff $\dim N = n$.

Simultaneous diagonalization of two real symmetric matrices has been of interest. For instance it has been shown that if the quadratic forms associated with S and T (of dimensions greater than 2) do not vanish simultaneously, then S and T can be diagonalized simultaneously by a real congruence transformation. This subject is generalized here to the study of the following two problems:

1) The finest simultaneous block diagonal structure for nonsingular pairs,

2) common annihilating vectors of the corresponding quadratic forms.

The proofs are obtained here by algebraic means. <u>Results:</u>

ad 1) A simultaneous block diagonalization $X'TX = \mathrm{diag}(A_1, \ldots, A_k)$ and $X'TX = \mathrm{diag}(B_1, \ldots, B_k)$ with $\dim A_i = \dim B_i$ and X nonsingular

is the finest simultaneous block diagonalization of a nonsingular pair S and T, if k is maximal. In this finest diagonalization the sizes of the blocks A_i are uniquely determined (up to permutations) by any set of generators of the pencil $P(S, T) = \{aS + bT \mid a, b \in \mathbb{R}\}$. The number k and the sizes of the diagonal blocks are also derived from the factorization over $\mathbb{C}$ of $f(\lambda, \mu) = \det(\lambda S + \mu T)$ for $\lambda, \mu \in \mathbb{R}$.

ad 2) Knowing the real Jordan normal form of $S^{-1}T$ for a nonsingular pair S and T we compute the maximal number m of linearly independent vectors that are simultaneously annihilated by the corresponding quadratic forms. Conversely, knowing m for two quadratic forms we deduce the first simultaneous block diagonal structure of S and T, the corresponding pair of real symmetric matrices. This is used to give new sufficient conditions for S and T to be simultaneously diagonalizable.

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INTRODUCTION

The subject studied here concerns pairs of real symmetric matrices one of which is nonsingular and pencils generated by them. Such pairs will be called nonsingular. Canonical forms for such pairs have been derived about 100 years ago mainly by Weierstraß [41] and Kronecker [23]. In Chapter I we give a new derivation of a canonical pair form using results on commuting matrices and elementary matrix algebra.

Nonsingular pairs S_1, S_2 can be studied via the real Jordan normal form of the product $S_1^{-1}S_2$. This Jordan normal form in fact partially defines the canonical pair form.

The aim of this thesis is to study and develop a new presentation for these classical results and to use them to redo and extend rather recent work in the theory of pairs of symmetric matrices. Knowing about pairs of symmetric matrices will also help establish theorems about real matrices in general, since every real matrix is the product of a nonsingular pair of real symmetric matrices and conversely information on a nonsingular pair S_1, S_2 can be obtained by looking at the real Jordan normal form of the real matrix S_1^{-1} S_2. This point was elaborated in Taussky [38].

In Chapter II a formula is given which relates the dimension of the space of matrices

$$\{S_1S \mid S \text{ symmetric}\} \cap \{S_2S \mid S \text{ symmetric}\}$$

for a nonsingular pair of real symmetric matrices S_1, S_2 to the real Jordan normal form of $S_1^{-1} S_2$. This formula is then used to reprove a result of Taussky and Zassenhaus [39] about nonderogatory matrices. Furthermore this same formula simplifies a previous result of Marcus and Khan [25] on the dimension of the space of real symmetric matrices T that transform a given real matrix A into its transpose A': $AT = TA'$.

Chapter III contains a survey of known and new results about pairs and pencils of real symmetric matrices and extends them. Equivalent conditions are given to the conditions that a pencil of real symmetric matrices contains a positive definite matrix, that a real matrix can be factored into a product of a positive definite and a real symmetric matrix and that a pair of real symmetric matrices can be simultaneously diagonalized by a real congruence transformation. The results on the canonical pair form are then used to define the finest simultaneous block diagonal form of a nonsingular pair of real symmetric matrices S and T and to prove that the finest simultaneous block diagonalization can be obtained from the real Jordan normal form of $S^{-1} T$ and that it is an invariant of the pencil generated by S and T rather than the generating pair S and T. Also new equivalences to the above mentioned conditions are derived from the factorization over $\mathbb{C}$ of $f(\lambda, \mu) = \det(\lambda S + \mu T)$ for $\lambda, \mu \in \mathbb{R}$ and a nonsingular pair S and T.

Every real symmetric matrix $S_{n \times n}$ generates a quadratic form $x'Sx$ over $\mathbb{R}^n$. In Chapter IV we compute the maximal number k of linearly independent vectors that are simultaneously annihilated by two quadratic forms which stem from a nonsingular pair of real symmetric matrices S and T. Our results relate k to the Jordan structure of $S^{-1}T$. As a corollary one gets, by algebraic means, a result of Greub and Milnor [16]: if $Q_S \cap Q_T = \{0\}$, then S and T can be simultaneously diagonalized by a real congruence transformation if $n > 2$. Furthermore with k as above let $2 \leq k < n-1$ for a nonsingular pair of real symmetric $n \times n$ matrices S and T, $n > 2$, and assume that $S^{-1}T$ is nonderogatory or that for every eigenvalue λ of $S^{-1}T$ the number of associated linearly independent eigenvectors is no greater than half the algebraic multiplicity of λ, unless both are equal, then S and T can be diagonalized simultaneously by a real congruence transformation.

CHAPTER I

THE CANONICAL FORM OF A NONSINGULAR PAIR OF REAL SYMMETRIC MATRICES

A single real symmetric $n \times n$ matrix is completely classified by Sylvester's law of inertia. A canonical form for a pair of real symmetric matrices was first developed by Weierstraß [41] and Kronecker [23]. Subsequently Muth [26], Trott [40], Ingraham and Wegner [19] (for pairs of hermitian matrices only) and both Dickson [10], Chapter 6, and Gantmacher [15], Vol. 2, Chapter 12 (for complex symmetric matrices, though) have worked on this question. Ostrowski [27], p. 9, gives a historic survey of this area. A summary of these and related results can be found in Pickert [28], §7.

In this chapter we will give a new proof of what has become known as the canonical pair form for a nonsingular pair of real symmetric matrices.

Notation: We will abbreviate "real symmetric" in the following by "r. s.".

Definition 1: Let S, T be two r. s. matrices with S nonsingular, then we call S and T a nonsingular pair of r. s. matrices.

The canonical pair form of a nonsingular pair of r. s. matrices S and T is closely related to the real Jordan normal form of $S^{-1}T$, as we will see in Theorem 1. Thus we need to introduce the machinery of Jordan blocks, Jordan chains and Jordan normal forms.

Definition 2: A square matrix of the form

$$M = \begin{pmatrix} \lambda & e & & 0 \\ & \ddots & \ddots & \\ & & \ddots & e \\ 0 & & & \lambda \end{pmatrix}_{k \times k}$$

is called a Jordan block of type (A), if for $k \geq 2$ we have $\lambda \in \mathbb{R}$ and $e = 1$, while for $k = 1$ we have $M = (\lambda)$ with $\lambda \in \mathbb{R}$. Such a matrix M is called a Jordan block of type (B), if for $k \geq 4$ we have $\lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$, $b \neq 0$ and $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, while for $k = 2$ we have $M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a, b \in \mathbb{R}$, $b \neq 0$.

Notation: Here and in the following we denote a matrix A with n rows and k columns by $A_{n \times k}$. Furthermore a Jordan block of dimension m for an eigenvalue λ is denoted by $J(\lambda, m)$, if $\lambda \in \mathbb{R}$, and by $J(a, b, m)$, if $\lambda = a + bi \notin \mathbb{R}$.

Definition 3: A matrix A is a block matrix and A_{ij} are its blocks, if for some $\ell, k \geq 1$ we write

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{\ell 1} & \cdots & A_{\ell k} \end{pmatrix}$$

where the blocks A_{ij} have the same number of rows for fixed i and $j = 1, \ldots, k$, and the same number of columns for fixed j and $i = 1, \ldots, \ell$. We say that a matrix A is a block diagonal matrix if it is a block matrix and $A_{ij} = 0$ for $i \neq j$. We write a block diagonal matrix A

with k diagonal blocks as $A = \mathrm{diag}(A_1, \ldots, A_k)$.

Now we are ready to quote the real Jordan normal form theorem.

Theorem 0: Every real square matrix A is similar over the reals to a matrix $J = \mathrm{diag}(A_1, \ldots, A_\ell)$, in which each square block A_j corresponds to an eigenvalue λ_j of A. If this eigenvalue λ_j is real, the associated A_j is a Jordan block of type (A); if $\lambda_j = a + bi \notin \mathbb{R}$, then A_j is a Jordan block of type (B). This J is called the real Jordan normal form of A. It is uniquely determined by A, except for the order of its Jordan blocks.

For a proof of this well-known result see e.g., Kowalski [21], p. 248, Theorem 36.2.

Definition 4: Let $J_1, \ldots, J_\ell$ be all the Jordan blocks (of either type) associated with the same eigenvalue λ of a real matrix A. Then

$$C = C(\lambda) = \mathrm{diag}(J_1, \ldots, J_\ell) \text{ with } \dim J_i \geq \dim J_{i+1} \quad \text{for all } i$$

is called the full chain of Jordan blocks or full Jordan chain of length ℓ associated with λ.

If $\lambda_1, \ldots, \lambda_k$ are all the distinct eigenvalues of a real matrix A, then its real Jordan normal form J can be written as $J = \mathrm{diag}(C(\lambda_1), \ldots, C(\lambda_k))$.

Next we define special types of matrices that are essential for the canonical pair form of two r. s. matrices.

Definition 5: A real matrix of the form

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ \vdots & & 0 \\ \vdots & & a_1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & & \vdots \\ a_1 & \cdots & a_r \end{pmatrix}_{k \times r} \quad \text{or} \quad \begin{pmatrix} 0 & \cdots\cdots & 0 & a_1 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & \cdots\ 0 & a_1 \cdots & a_r \end{pmatrix}_{r \times k} \quad \text{for } k \geq r$$

is called lower striped matrix of type (A), if $a_i \in \mathbb{R}$ and lower striped matrix of type (B), if each a_i is a 2×2 matrix of the form $\begin{pmatrix} b & a \\ a & -b \end{pmatrix}$ for $a, b \in \mathbb{R}$ and $a_1 = \begin{pmatrix} y & x \\ x & -y \end{pmatrix}$ with $y \neq 0$.

Analogously one defines upper striped matrices.

Definition 6: A real matrix of the form

$$\begin{pmatrix} t_1 & \cdots & t_r \\ 0 & & \vdots \\ \vdots & \ddots & t_1 \\ \vdots & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{k \times r} \quad \text{or} \quad \begin{pmatrix} 0 & \cdots\ 0 & t_1 \cdots & t_r \\ \vdots & & \ddots & \vdots \\ 0 & \cdots\cdots & 0 & t_1 \end{pmatrix}_{r \times k} \quad \text{for } k \geq r$$

is called (upper) triangularly striped of type (A), if $t_i \in \mathbb{R}$, and (upper) triangularly striped of type (B), if each t_i is a 2×2 matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for $a, b \in \mathbb{R}$ and $t_1 = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ with $y \neq 0$.

Notation: Throughout this thesis the symbols E or E_i will always denote lower striped square matrices of type (A) with $a_1 = 1$, $a_j = 0$ for $j > 1$.

Here are some links between lower striped and triangularly striped matrices:

Proposition 1: Let $A = A_{k \times r}$ be a triangularly striped matrix (of either type) and let E have dimension k. Then the matrix $E \cdot A$ is a lower striped matrix of the proper type.

The proof follows by inspection.

Proposition 2: If the inverse of a lower striped matrix exists, then it is an upper striped matrix of the form

$$\begin{pmatrix} b_n & \cdots & b_1 \\ \vdots & ⋰ & \\ b_1 & \cdots & 0 \end{pmatrix},$$

where either all a_i and b_i are real or all a_i and b_i have the form $\begin{pmatrix} b & a \\ a & -b \end{pmatrix}$.

Proof: One wants to solve

$$I = \begin{pmatrix} 0 & & a_1 \\ & ⋰ & \vdots \\ a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_n & \cdots & b_1 \\ \vdots & ⋰ & \\ b_1 & & 0 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & & 0 \\ \vdots & \ddots & \\ a_1 b_n + \cdots + a_n b_1 & \cdots & a_1 b_1 \end{pmatrix},$$

where the a_i and b_i are all of the same type and a_1 is invertible.

Letting

$$b_1 = a_1^{-1}, \; b_2 = -a_1^{-1}(a_2 b_1) \text{ and } b_k = -a_1^{-1}\left(\sum_{\substack{i+j=k+1 \\ i<k}} a_i b_j\right)$$

for $2 < k \leq n$ proves this proposition.

Proposition 3: Let B be an upper striped matrix and A a lower striped matrix, both of the same type, such that BA is defined.

Then BA is triangularly striped of the same type.

This proposition and the following are proved by inspection.

Proposition 4: If A and B are triangularly striped matrices of the same type and AB is defined, then AB is again triangularly striped of the same type.

Proposition 5: Let $A_{n \times n}$ and $B_{n \times n}$ be block matrices, partitioned conformally, where each block is a triangularly striped matrix of one fixed type.

Then AB is partitioned conformally as A or B and each of its blocks is triangularly striped of the same type.

Proof: Let $A = (A_{ij})$ $i, j = 1, \ldots, k$ for $k \geq 1$ where each A_{ij} is a triangularly striped matrix of the same type. Now $B = (B_{ij})$ has the same block structure as A. And hence $AB = (C_{ij})$ where $C_{ij} = \sum_{\ell} A_{i\ell} B_{\ell j}$ for $i, j = 1, \ldots, k$. By Proposition 4 each of the terms $A_{i\ell} B_{\ell j}$ is triangularly striped of the proper type. And hence AB is partitioned conformally into k^2 triangularly striped matrices. ∎

Now we have developed all the tools needed to state and prove the theorem about the canonical pair form for a nonsingular pair of r. s. matrices.

Theorem 1: Let S and T be a nonsingular pair of r. s. matrices. Let $S^{-1}T$ have real Jordan normal form $\operatorname{diag}(J_1, \ldots, J_r, J_{r+1}, \ldots, J_m)$, where $J_1, \ldots, J_r$ are Jordan blocks of type (A) corresponding to real eigenvalues of $S^{-1}T$ and $J_{r+1}, \ldots, J_m$ are Jordan blocks of type (B) for pairs of complex conjugate roots of $S^{-1}T$.
Then S and T are simultaneously congruent by a real congruence transformation to

$$\operatorname{diag}(\epsilon_1 E_1, \ldots, \epsilon_r E_r, E_{r+1}, \ldots, E_m) \quad \text{and}$$

$$\operatorname{diag}(\epsilon_1 E_1 J_1, \ldots, \epsilon_r E_r J_r, E_{r+1} J_{r+1}, \ldots, E_m J_m), \quad \text{respectively,}$$

where $\epsilon_i = \pm 1$ and E_i denotes the lower striped square matrix $\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ of the same size as J_i for $i = 1, \ldots, m$.

Proof: Let $A = S^{-1}T$. Then by Theorem 0 the matrix A is similar to its real Jordan normal form $J = \operatorname{diag}(J_1, \ldots, J_r, J_{r+1}, \ldots, J_m)$ via a real similarity X.

$$J = X^{-1}AX = X^{-1}S^{-1}TX = X^{-1}S^{-1}(X')^{-1}X'TX = S_1^{-1}S_2 \ .$$

where we set $S_1 = X'SX$, $S_2 = X'TX$ and X' denotes the transpose of the matrix X.
Since S_1 and S_2 are simultaneously congruent to S and T, respectively, it suffices to work on the pair of symmetric matrices S_1 and S_2 such that $S_1^{-1}S_2 = J$. If we know all r. s. nonsingular matrices S_0 such that

S_0J is symmetric, then S_1 will be among them, since $S_1J = S_2$ is symmetric.

Furthermore, given S_1 and S_2 symmetric such that $S_1J = S_2$, then S_1 and S_2 are simultaneously congruent to another pair U_1, U_2 with $U_1J = U_2$ via a matrix Y iff Y commutes with J. The reason for this is: $U_2 = Y'S_2Y = Y'S_1JY = Y'S_1YY^{-1}JY = Y'S_1YJ = U_1J$ holds iff $J = Y^{-1}JY$.

So in order to prove Theorem 1 it suffices to find the general form of r.s. matrices S such that SJ is symmetric and then to show that each of these matrices is congruent to $\text{diag}(\epsilon_1E_1, \ldots, \epsilon_rE_r, E_{r+1}, \ldots, E_m)$ for a specific choice of $\epsilon_1, \ldots, \epsilon_r = \pm 1$ via a matrix which commutes with J, where the dimensions of the E_i are as indicated.

The following two lemmas will complete the proof of Theorem 1.

<u>Lemma 1</u>: Let $J = \text{diag}(C(\lambda_1), \ldots, C(\lambda_k))$ be the real Jordan normal form of a real matrix.

If SJ is symmetric for S symmetric, then S is a block diagonal matrix $S = \text{diag}(A_1, \ldots, A_k)$ with $\dim A_i = \dim C(\lambda_i)$ for $i = 1, \ldots, k$ and $A_i = A_i'$. Here each diagonal block A_i is partitioned in the same way as $C(\lambda_i) = \text{diag}(J_1^i, \ldots, J_\ell^i)$ into ℓ^2 blocks, each of which is a lower striped matrix of type (A), if $\lambda_i \in R$, and of type (B) else. Conversely every such S will make SJ symmetric.

<u>Proof</u>: Let $J = \text{diag}(J_1, \ldots, J_m)$, where each J_i is a Jordan block of either type. Let $H = \text{diag}(E_1, \ldots, E_m)$ with $\dim E_i = \dim J_i$ for all i.

Then $HJ = J'H$ because for each Jordan block J_i we have $E_iJ_i = J_i'E_i$ by inspection. Clearly $H^{-1} = H$ and thus we have for an arbitrary r. s. matrix S that SJ is symmetric iff $SJ = J'S$, and this holds iff $SJ = HJHS$, hence iff $HSJ = JHS$ hence iff HS commutes with J.

The ring of matrices commuting with $J = \mathrm{diag}(C(\lambda_1), \dots, C(\lambda_k))$ is the direct sum of the k rings of matrices commuting with $C(\lambda_i)$ for $i = 1, \dots, k$. (Commuting matrices were first studied by Frobenius [14]. For a modern treatment of this specific result see e.g., Suprunenko and Tyshkevich [32], p. 25, Proposition 6 and Lemma 4.)

So, if HS commutes with J, then HS itself is a block diagonal matrix and hence $H^2S = S$ is also block diagonal: $S = \mathrm{diag}(A_1, \dots, A_k)$ with $\dim A_i = \dim C(\lambda_i)$.

Now for real λ, as can be found in Suprunenko and Tyshkevich [30], p. 28, Theorem 6, all real matrices commuting with a full Jordan chain $C(\lambda) = \mathrm{diag}(J_1, \dots, J_\ell)$ are matrices partitioned conformally into ℓ^2 blocks with each block a triangularly striped matrix of type (A). So from the special nature of H and by Proposition 1 it follows that the diagonal blocks A_i of S which correspond to Jordan chains $C(\lambda)$ for real λ will be block matrices with each block a lower striped matrix of type (A) as claimed in Lemma 1.

It only remains to prove the analogous result for Jordan chains $C(\lambda)$ with $\lambda \notin \mathbb{R}$.

Again we have to find all real matrices A with $A = HS$ such that

$$AC(\lambda) = C(\lambda)A \qquad \text{for } \lambda \notin \mathbb{R} .$$

Since $C(a + bi) = aI + C(bi)$ and I commutes with A we may as well assume that λ is purely imaginary: $\lambda = bi$, $b \neq 0$.

We set $C = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$ and $I_2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

With $C(\lambda) = \text{diag}(J_1, \ldots, J_u)$ we partition $A = (A_{ij})$ conformally into u^2 blocks. It then suffices to find A_{ij} from the equation:

(1) $$A_{ij}J_j = J_iA_{ij}, \qquad i, j = 1, \ldots, u$$

We set

$$A_{ij} = \begin{pmatrix} B_{11} & \cdots & B_{1s} \\ \vdots & & \vdots \\ B_{t1} & \cdots & B_{ts} \end{pmatrix}$$

where each $B_{k\ell}$ is a 2×2 block and $\dim J_j = 2s$, $\dim J_i = 2t$. We thus have to solve the following equivalent matrix equations (1*) for the $(s \cdot t)$ matrices $B_{k\ell}$:

(1*) $$\begin{pmatrix} B_{11}C & B_{11} + B_{12}C & \cdots & B_{1,s-1} + B_{1s}C \\ \vdots & \vdots & & \vdots \\ B_{t-1,1}C & \vdots & & \vdots \\ B_{t1}C & B_{t1} + B_{t2}C & \cdots & B_{t,s-1} + B_{ts}C \end{pmatrix} =$$

$$= \begin{pmatrix} CB_{11} + B_{21} & \cdots\cdots & CB_{1s} + B_{2s} \\ \vdots & & \vdots \\ CB_{t-1,1} + B_{t1} & \cdots & CB_{t-1,s} + B_{t-1,s} \\ CB_{t1} & \cdots & CB_{ts} \end{pmatrix}$$

Comparing entries in the bottom left corner of (1*) one gets $B_{t1}C = CB_{t1}$. Since the minimum polynomial $x^2 + b^2$ of C is irreducible over

$\mathbb{R}$, the matrix B_{t1} must be a polynomial in C. Thus B_{t1} has the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ with $x, y \in \mathbb{R}$.

When further comparing entries in (1*) we will steadily come upon equations in B which have the following form

(2) $$B_1 = CB - BC \quad ,$$

where B_1 is already known to be a 2×2 matrix of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ and C is as above.

We note that if such an equation (2) holds, then B is of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ as well, while B_1 must be zero.

For B_1 commutes with C since it is a polynomial in C. Thus if (2) holds, then

$$B_1C = CBC - BC^2 = C^2B - CBC = CB_1 \quad .$$

Since $C^2 = -b^2I$ we have $-b^2B = CBC$. Now $-b^2BC^{-2} = B$ and thus $B = -b^2BC^{-2} = CBC^{-1}$. Thus C and B commute, so that $B_1 = 0$ and B has the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ with $x, y \in \mathbb{R}$ as claimed above.

Thus when comparing entries in the first column and the last row in (1*), starting from the lower left corner, one gets that $B_{i1} = 0 = B_{tj}$ for $i = 2, \ldots, t$ and $j = 1, \ldots, s-1$. And comparing entries below the diagonal in (1*) one gets $B_{ij} = 0$ for $j - i < 0$, while comparing entries above the diagonal in (1*) yields $B_{ij} = B_{k\ell}$ if $j - i = \ell - k$ and furthermore, as we just remarked, all these B_{ij} are 2×2 matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

So A_{ij} is a triangularly striped matrix of type (B). And thus A itself is composed of blocks that are triangularly striped matrices of type (B).

Now we recall the special nature of H and Proposition 1 again as in the real case and conclude that $HA = H^2S = S$ is a block diagonal matrix whose blocks are lower striped matrices of type (B). This proves Lemma 1. ■

Lemma 2: Let $J = \mathrm{diag}(C(\lambda_1), \ldots, C(\lambda_k)) = \mathrm{diag}(J_1, \ldots, J_m)$ be the real Jordan normal form of a real matrix.

If SJ is symmetric for a nonsingular r. s. matrix S, then there exists a nonsingular real matrix Y with $YJ = JY$ such that

$Y'SY = \mathrm{diag}(\epsilon_1 E_1, \ldots, \epsilon_m E_m)$ with $\dim E_i = \dim J_i$, $\epsilon_i = \pm 1$ and $\epsilon_j = 1$ if J_j is a Jordan block of type (B).

Proof: If SJ is symmetric, then by Lemma 1 the matrix S is a block diagonal matrix: $S = \mathrm{diag}(A_1, \ldots, A_k)$ with $\dim A_i = \dim C(\lambda_i)$ and each block A_i is as described in Lemma 1.

We will use double induction to show that there is a matrix Y commuting with J such that $Y'SY = \mathrm{diag}(\epsilon_i E_i)$. We show that this matrix Y can be written as $Y = \mathrm{diag}(B_1, \ldots, B_k)$ with $\dim B_i = \dim C(\lambda_i)$, where each B_i is a block matrix partitioned conformally with A_i and all of its blocks are triangularly striped matrices of the same type as those of A_i.

First we use induction on the number ℓ of Jordan blocks in one Jordan chain and then we use induction on the number k of Jordan chains in J.

If we have just one $r \times r$ Jordan block J_1 of type (A), then $J = C(\lambda) = J_1$ for $\lambda \in \mathbb{R}$. With S a nonsingular lower striped matrix of type (A) by Lemma 1 we want to find Y as described above that solves

$$Y'SY = \begin{pmatrix} t_1 & & 0 \\ \vdots & \ddots & \\ t_r & \cdots & t_1 \end{pmatrix} \begin{pmatrix} 0 & & a_1 \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ a_1 & \cdots & a_r \end{pmatrix} \begin{pmatrix} t_1 & \cdots & t_r \\ & \ddots & \vdots \\ 0 & & t_1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & & & a_1 t_1^2 \\ & & \cdot^{\cdot^{\cdot}} & a_2 t_1^2 + 2a_1 t_1 t_2 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ a_1 t_1^2 & a_2 t_1^2 + 2a_1 t_1 t_2 & \cdots\cdots & * \end{pmatrix} = \pm E_{r \times r} .$$

If $a_1 > 0$, choose $t_1 = 1/\sqrt{a_1}$, t_2 such that $a_2 t_1^2 + 2a_1 t_1 t_2 = 0$ and t_i similarly such that $Y'SY = E_{r \times r}$.

If $a_1 < 0$, choose $t_1 = 1/\sqrt{-a_1}$, $t_2, \ldots, t_r$ as above such that $Y'SY = -E_{r \times r}$. Note that Y commutes with J.

If we have $J = C(\lambda) = J_1$ for $\lambda \notin \mathbb{R}$, then we start out with $Y_0 = \operatorname{diag}(T, \ldots, T)$ where T is a 2×2 matrix of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, $x, y \in \mathbb{R}$ and

$$S = \begin{pmatrix} 0 & & A_1 \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ A_1 & \cdots & A_r \end{pmatrix}$$

where each A_j has the form

$$\begin{pmatrix} b_j & a_j \\ a_j & -b_j \end{pmatrix}$$

for $a_j, b_j \in \mathbb{R}$ while $A_1 = \begin{pmatrix} b & a \\ a & -b \end{pmatrix}$ with $b \neq 0$ since S is nonsingular.

Then

$$Y_0'SY_0 = \begin{pmatrix} 0 & & T'A_1T \\ & \cdot^{\cdot^{\cdot}} & \\ T'A_1T & & * \end{pmatrix} \qquad \text{and}$$

$$T'A_1T = \begin{pmatrix} bx^2 + 2axy - by^2 & -2bxy + ax^2 - ay^2 \\ -2bxy + ax^2 - ay^2 & by^2 - 2axy - bx^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = F$$

can be solved for real x, y: Let $x = \frac{y}{b}\left(-a - \sqrt{a^2+b^2}\right)$ and let y be a nonzero solution of $2y^2(a^2+b^2)\left(a + \sqrt{a^2+b^2}\right) = 1$. Here $y \in \mathbb{R}$ iff $a + \sqrt{a^2+b^2} > 0$ which is true since $b \neq 0$. So far we have found a matrix Y_0 commuting with J such that

$$S_1 = Y_0'SY_0 = \begin{pmatrix} & 0 & & F \\ & & \cdot^{\cdot^{\cdot}} & F_2 \\ & \cdot^{\cdot^{\cdot}} & & \vdots \\ F & F_2 & \cdots & F_r \end{pmatrix}, \qquad \text{where}$$

$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $F_j = T'A_jT$ are still 2×2 matrices of the form $\begin{pmatrix} b & a \\ a & -b \end{pmatrix}$.

Next let

$$Y_1 = \begin{pmatrix} I_2 & T_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & T_2 \\ 0 & & & I_2 \end{pmatrix},$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and T_2 has the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, $x, y \in \mathbb{R}$.

Then

$$Y_1'S_1Y_1 = \begin{pmatrix} 0 & & & & F \\ & & & \cdot^{\cdot^{\cdot}} & T_2'F+FT_2+F_2 \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ F & T_2'F+FT_2+F_2 & & & * \end{pmatrix}.$$

Now by Proposition 5, both $T_2'F + FT_2$ and F_2 are matrices of the form $\begin{pmatrix} b & a \\ a & -b \end{pmatrix}$. Since $T_2'F = FT_2$, a matrix T_2 of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ can be chosen to solve $T_2'F + FT_2 + F_2 = 0$, namely $T_2 = (-\frac{1}{2}) FF_2$.

The matrix Y_1 commutes with J and so does Y_1Y_0 by Proposition 5.

Continuing this process we finally arrive at a matrix $Y = Y_{r-1} \cdots Y_1Y_0$ which commutes with J such that $Y'SY = E_{2r \times 2r}$.

Assume next that the Jordan chain in question contains ℓ Jordan blocks: $J = C(\lambda) = \mathrm{diag}(J_1, \ldots, J_\ell)$. Then by Lemma 1 every r.s. matrix S such that SJ is symmetric can be written as $S = (S_{ij})$, $i, j = 1, \ldots, \ell$, where each S_{ij} is a lower striped matrix of dimensions $\dim J_i \times \dim J_j$. Note that the argument here and in the following holds for lower striped matrices of either type alike.

If S_{11} is singular and one S_{jj} of the same size as S_{11} is nonsingular, one applies a suitable permutation similarity to S such that the new S_{11} becomes nonsingular. Jordan chains have been defined such that $\dim J_i \geq \dim J_{i+1}$. So if all diagonal blocks of S of the same size as S_{11} are singular, there must be a nonsingular block S_{1i} of the same size as S_{11}, else the first row of S would be zero, contradicting S to be nonsingular.

Then take the following block matrix B composed of triangularly striped blocks

$$B = I + \begin{pmatrix} 0 & \dots & 0 & -F & 0 & \dots \\ \vdots & & & & & \\ 0 & & & & & \\ F & & & & & \\ 0 & & & & 0 & \\ \vdots & & & & & \end{pmatrix},$$

where $F = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ with $\dim F = \dim S_{11}$ and where the block $-F$ appears in position $(1, i)$, while F is in position $(i, 1)$. We have

$$B'SB = \begin{pmatrix} S_{11} + 2S_{1i} + S_{ii} & \\ & \\ & * \end{pmatrix}$$

Now $S_{11} + 2S_{1i} + S_{ii}$ is nonsingular since all terms involved are lower striped matrices and S_{11} and S_{ii} were both assumed to be singular. Thus by replacing S by B^1SB one may WLOG assume that the new S_{11} is nonsingular.

Now we make use of a method employed by Trott [40], p. 362, Lemma 3. For $i = 1, \dots, \ell$ let

$$Y_i = I + \begin{pmatrix} 0 & \dots & 0 & -S_{11}^{-1}S_{1i} & 0 & \dots \\ \vdots & & & & & \\ 0 & & & 0 & & \end{pmatrix},$$

where the block $-S_{11}^{-1}S_{1i}$ appears in position $(1, i)$. Each Y_i is a triangularily striped block matrix by Proposition 3.

Let $W = Y_1Y_2 \dots Y_\ell$. Then W commutes with J by Proposition 5 and we have

$$W'SW = \begin{pmatrix} S_{11} & & 0 & \\ & S^0_{22} & \cdots & S^0_{2\ell} \\ & \vdots & & \\ 0 & & & \\ & S^0_{\ell 2} & \cdots & S^0_{\ell\ell} \end{pmatrix},$$

where all S^0_{ij} are lower striped matrices.
By the earlier part of this proof S_{11} is congruent to $\epsilon_1 E_1$ via a triangularly striped matrix for $\epsilon_1 = \pm 1$. So we have reduced the problem to where we have to deal with Jordan chains of length $\ell - 1$ only. Here we use the induction hypothesis and conclude that the corresponding r.s. matrix S can be brought into the form $\text{diag}(\epsilon_2 E_2, \ldots, \epsilon_\ell E_\ell)$ by a congruence transformation with a real matrix Y commuting with J.

Finally, if J contains k Jordan chains, then every r.s. matrix S such that SJ is symmetric is of the form $S = \text{diag}(A_1, \ldots, A_k)$ with $\dim A_i = \dim C(\lambda_i)$ by Lemma 1. But we just concluded the result for one Jordan chain, and so we use the induction hypothesis on the remaining $(k-1)$ Jordan chains and Lemma 2 is proved. ∎

This concludes the proof of the canonical pair theorem. ∎

The canonical form of a nonsingular pair of r.s. matrices which we have just developed is quite useful, and in the remaining chapters we will often simplify proofs by assuming that the pair of r.s. matrices in question is already in canonical pair form. Then, in Chapter III, Theorem 9, we will find out in which way the canonical pair form of a nonsingular pair of r.s. matrices is "canonical".

CHAPTER II

A FORMULA FOR A NONSINGULAR PAIR OF REAL SYMMETRIC MATRICES AND AN APPLICATION TO NONDEROGATORY REAL MATRICES

In this chapter we will prove a rather technical and complicated looking theorem, Theorem 2, from which we will deduce in Theorem 3 an equivalent condition for a nonsingular real matrix to be nonderogatory, that is, to have only Jordan chains of length one in its Jordan normal form. The formula obtained in Theorem 2 is closely related to results on matrix commutators by Taussky and Zassenhaus [39] and Marcus and Khan [25].

As an introduction we state the following theorem and prove a lemma.

Theorem 1: Every real square matrix is the product of two r. s. matrices, of which either the first or the second factor can be chosen nonsingular.

Proofs are given in [15], [29], [35], [38] and [39].

Lemma 1: Let S be a r. s. $n \times n$ matrix of rank k. Then $N = \{ST \mid T \text{ symmetric}\}$ is a linear space of dimension $\frac{1}{2}(2nk - k^2 + k)$.

Proof: It is obvious that N is a linear space. Let $e_{\ell m}$ be the $n \times n$ matrix which has a 1 in the (ℓ, m) position and zeros everywhere else.

Let $X_{\ell m} = e_{\ell m} + e_{m\ell}$, then $\{X_{\ell m} | \ell \le m\}$ is a basis for the space of $n \times n$ r.s. matrices. Assume that the given matrix S is congruent to $D = \mathrm{diag}(d_1, \ldots, d_k, 0, \ldots, 0)$, with $d_i \in \mathbb{R}$ via a nonsingular congruence transformation $S = BDB'$. If T varies over all symmetric matrices, so does U_T defined by $T = (B^{-1})' U_T B^{-1}$. Then $A = ST = BDU_TB^{-1}$. Putting $A_0 = B^{-1}AB$ we get

$$\dim \{A | A = ST\} = \dim \{A_0 | A_0 = DU_T\} .$$

Thus it suffices to assume that $S = D = \mathrm{diag}(d_1, \ldots, d_k, 0, \ldots, 0)$. Then $DX_{\ell m} = d_\ell e_{\ell m} + d_m e_{m\ell}$ for $\ell \le m$, where we set $d_i = 0$ for $i > k$. Since all $e_{\ell m}$ are lin. indep., the matrices $DX_{\ell m}$ will be lin. indep. as long as one of the coefficients d_ℓ or d_m is not zero. Hence there are as many lin. indep. matrices among the $DX_{\ell m}$ as there are pairs of integers (ℓ, m) with $1 \le \ell \le k$ and $1 \le \ell \le m \le n$. There are $1+2+\ldots$ $\ldots+k+(n-k)k$ such pairs. Hence Lemma 1 follows. ∎

The next question is: given a nonsingular pair S_1, S_2 of r.s. matrices in how many ways can one find r.s. matrices S and T such that $S_1S = S_2T$, or how "big" is the overlap of the two sets $\{S_1S | S \text{ symmetric}\}$ and $\{S_2S | S \text{ symmetric}\}$, if S_1 and S_2 are given.

Theorem 2: Let S_1, S_2 be a nonsingular pair of r.s. matrices. Let $S_1^{-1}S_2$ have real Jordon normal form

$$\begin{aligned} J = \mathrm{diag}\Big(& J(\lambda_1, n_1^1), \ldots, J(\lambda_1, n_{k_1}^1), \ldots, J(\lambda_w, n_{k_w}^w), \\ & J(a_{w+1}, b_{w+1}, n_{k_{w+1}}^{w+1}), \ldots, J(a_{w+1}, b_{w+1}, n_{w+1}^{w+1}), \ldots, \\ & J(a_t, b_t, n_{k_t}^t), J(0, n_1^0), \ldots, J(0, n_{k_0}^0)\Big) \end{aligned}$$

with $\lambda_i, b_j \neq 0$, λ_i distinct, b_j distinct for $i = 1,\ldots,w$; $j = w+1,\ldots,t$. Assume furthermore that $n_1^i \geq n_2^i \geq \ldots \geq n_{k_i}^i$ for $i = 0,1,\ldots,t$ and define $D = k_0(k_0+1)/2$. Let

$$N_i = \sum_{j=1}^{k_i} j n_j^i \qquad \text{for } i = 0,\ldots,t \quad .$$

Then $\dim(\{S_1S \mid \text{symmetric}\} \cap \{S_2S \mid S \text{ symmetric}\}) = \sum_{i=0}^{t} N_i - D.$

In order to prove this theorem we will reduce the problem to the case that S_1 and S_2 are already in canonical pair form. Lemma 3 then further describes the space of matrices in question, so that Theorem 2 can be proved by actually exhibiting a basis of the space $\{S_1S\} \cap \{S_2S\}$. With $A = S_1^{-1}S_2$ the above formula counts the number of lin. ind. matrices AT such that $AT = TA'$ for a given real matrix A. Theorem 2 is thus an extension of the known result (c.f., Taussky and Zassenhaus [39], Theorem 1) that there is always a nonsingular r.s. matrix transforming a given matrix A into its transpose.

Now for the proof of Theorem 2.

Lemma 2: Let T_1, T_2 be a pair of r.s. matrices. For a nonsingular X define $S_i = XT_iX'$ for $i = 1,2$. Then $\dim(\{S_1S \mid S \text{ symmetric}\} \cap \{S_2S \mid S \text{ symmetric}\}) = \dim(\{T_1S \mid S \text{ symmetric}\} \cap \{T_2S \mid S \text{ symmetric}\})$.

Proof: Let $A_1,\ldots,A_k$ be a basis of the first mentioned space. Then $B_1 = X^{-1}A_1X,\ldots,B_k = X^{-1}A_kX$ is a basis of the second space. ∎

Lemma 3: Let S_1, S_2 be a nonsingular pair of r.s. matrices that are in canonical pair form.

Then $N = \{T \text{ symmetric} \mid \text{there exists at least one r. s. matrix } S \text{ with } S_1S = S_2T\}$ is the set of block diagonal matrices

$$\mathrm{diag}(A_1, \ldots, A_r)$$

where each diagonal block A_i is itself a block matrix of the form $A_i = (B^i_{kj})$, $k,j = 1, \ldots, t_i$, and each B^i_{kj} is an upper striped matrix with $B^i_{kj} = (B^i_{jk})'$.

The dimensions of the A_i and B^i_{kj} and the types of B^i_{kj} are determined by the corresponding Jordan chains $C(\lambda_i)$ of length t_i and the Jordan blocks J_k and J_j, respectively in $S_1^{-1}S_2 = J = \mathrm{diag}(C(\lambda_1), \ldots, C(\lambda_r))$ as explained in Lemma 1 of Chapter I.

Proof: (Lemma 3) We have that $S_1S = S_2T$ holds iff $S = S_1^{-1}S_2T$. By assumption $S_1^{-1}S_2 = \mathrm{diag}(J_1, \ldots, J_m)$, where the J_i are various Jordan blocks. Hence N consists of all r. s. matrices T such that $\mathrm{diag}(J_i)T$ is symmetric. Similar arguments as those used for Lemma 1 in Chapter I show that T has the form described in Lemma 3 of this chapter, if "lower striped" is replaced by "upper striped" here. ∎

Before proving Theorem 1 we remark the following: When comparing the formulas for the N_i in Theorem 2 with results about the number of lin. ind. matrices that commute with a full Jordan chain, say with Corollory 9, p. 31 in Suprunenko and Tyshkevich [32], one discovers a great similarity.

In the case that S_1 and S_2 are both nonsingular, the connection is the following:

With $A = S_1S = S_2T$ we have $A' = S_1^{-1}AS_1 = S_2^{-1}AS_2$ and thus

$S_2S_1^{-1}AS_1S_2^{-1} = A$ which says that $S_2S_1^{-1}$ commutes with A. But not all matrices X commuting with A are feasible, namely only those are which make XS_1 symmetric. Looking again at the derivation of the formula in Suprunenko and Tyshkevich [32] on page 31 one sees that in our case we only have (in Suprunenko's notation and ours) $m_1 + 2m_2 + 3m_3 + \ldots = \sum_{j=1}^{k_i} jn_j^i$ lin. ind. matrices at hand for each Jordan chain $C(\lambda_i)$. Hence our formula for the N_i.

We now prove Theorem 2.

Proof: (Theorem 2) In view of Lemma 2 we may WLOG assume that S_1 and S_2 are already in canonical pair form.

Lemma 3 tells us the general form of r. s. matrices T such that there is a r. s. matrix S with $S_1S = S_2T$. These T are block diagonal matrices $T = \mathrm{diag}(T_1, \ldots, T_{t+1})$ and each T_i is associated with a Jordan chain $C(\lambda_i)$ for each of the t + 1 distinct eigenvalues $\lambda_1, \ldots, \lambda_t, 0$ of $S_1^{-1}S_2$. The blocks T_i have the form $T_i = (B_{uv})$, $u, v = 1, \ldots, k_i$, where $B_{kj} = B_{jk}'$ are upper striped matrices of dimensions $n_k^i \times n_j^i$, if we set $C(\lambda_i) = \mathrm{diag}(J(\lambda_i, n_1^i), \ldots, J(\lambda_i, n_{k_i}^i))$. Since by definition $n_k^i \geq n_j^i$ for $k \leq j$, we have n_1^i free parameters in choosing B_{11} and $2n_2^i$ parameters for B_{12} and B_{22}, until finally $k_i n_{k_i}^i$ parameters for the choice of $B_{1k_i}, \ldots, B_{k_ik_i}$. So if the N_i are defined as in Theorem 2, there are $\sum N_i$ lin. ind. r. s. matrices T for which there exists a r. s. matrix S such that $S_1S = S_2T$.

Since S_1 is nonsingular, it suffices to show how many of these $\sum N_i$ matrix products S_2T are lin. ind. in order to complete the proof.

We know that all of the above matrices T form a linear space. From its description in Lemma 3 a basis $T_{\ell rjp}$ for this space is given by the collection of all $n \times n$ matrices of the form

$$T_{\ell rjp} = \mathrm{diag}(T_1^0, \ldots, T_{t+1}^0)$$

for $\ell = 1, \ldots, t+1$; $r, j = 1, \ldots, k_\ell$; $r \le j$ and $p = 1, \ldots, n_j^\ell$, with t, k_ℓ and n_j^ℓ as introduced for J in Theorem 2. Here all diagonal blocks except one are zero and this one exceptional block T_ℓ^0 is a block matrix partitioned conformally into $(k_\ell)^2$ blocks with

$$C(\lambda_\ell) = \mathrm{diag}(J(\lambda_\ell, n_1^\ell), \ldots, J(\lambda_\ell, n_{k_\ell}^\ell))$$

in the following way:

All blocks of T_ℓ^0 are zero, except for the blocks B_{rj}^p and B_{jr}^p in positions (r, j) and (j, r) with $B_{rj}^p = (B_{jr}^p)^1$ for $1 \le r \le j \le k_\ell$.

Here B_{rj}^p denotes the following upper striped matrix:

If $\lambda_\ell \in R$, then for $r \le j$ the matrix B_{rj}^p is upper striped of type (A) of the form

$$\begin{array}{c} \\ \\ \\ p \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots \\ \vdots & & & \cdot^{\cdot^{\cdot}} & & \\ 0 & & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & & \\ 0 & & & & & 0 \\ \vdots & & & & & \end{pmatrix}_{n_r^\ell \times n_j^\ell} \qquad \text{for } 1 \le p \le n_j^\ell .$$

If $\lambda_\ell \notin R$, then for $r \le j$ the block B_{rj}^p is upper striped of type (B) of the form

$$
\begin{array}{r l}
 & \\
 & \\
 & \\
 & \\
 & \\
q \rightarrow & \\
 & \\
 &
\end{array}
\left(\begin{array}{cccccccc}
0 & 0 & \cdots & 0 & 1 & C & 0 & \cdots \\
0 & 0 & \cdots & 0 & C & -1 & 0 & \cdots \\
\vdots & \vdots & & \iddots & & & & \\
0 & 0 & \iddots & & & & & \\
1 & C & & & & & & \\
C & -1 & & & & 0 & & \\
0 & 0 & & & & & & \\
\vdots & \vdots & & & & & &
\end{array}\right)_{n_r^{\ell} \times n_j^{\ell}}
$$

for $1 \le p \le n_j^{\ell}$, where $q = 2[(p+1)/2]$ and $C = 1$ if p is odd, while otherwise $C = -1$. Here $[\cdot\cdot]$ denotes the greatest integer function.

In order to study the dimension of the space S_2T, we want to look at the products

$$S_2 T_{\ell rjp} = \operatorname{diag}(\epsilon_s E_s J_s)\ \operatorname{diag}(T_i^0) = \operatorname{diag}(A_1, \ldots, A_{t+1})$$

where all A_i are zero except A_ℓ and A_ℓ is a block matrix partitioned as T_ℓ^0 with zero blocks except for the block $\epsilon_u E_u J_u B_{rj}^p$ appearing in position (r, j) and $\epsilon_v E_v J_v B_{jr}^p$ appearing in position (j, r), where the index u is such that J_u is the r^{th} Jordan block occurring in the Jordan chain $C(\lambda_\ell)$ and v is such that J_v is the j^{th} Jordan block occurring therein.

It hence suffices to look at products of the form $\epsilon_u E_u J_u B_{rj}^p$ for arbitrary u, r, j, p, where u and r are related as above.

For $\lambda = \lambda_\ell \in \mathbb{R}$ this becomes

$$
\varepsilon_u \begin{pmatrix} & & & & \lambda \\ & 0 & & \cdot^{\cdot^{\cdot}} & 1 \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \vdots \\ \lambda & 1 & 0 & \dots & 0 \end{pmatrix}_{n_r^\ell \times n_r^\ell}
\begin{pmatrix} 0 & \dots & 0 & 1 & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \\ 0 & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & \\ 0 & & & 0 & \\ \vdots & & & & \end{pmatrix}_{n_r^\ell \times n_j^\ell}
\begin{array}{l} \\ \\ \\ \leftarrow p \\ \\ \\ \end{array}
$$

$$
= \varepsilon_u \begin{pmatrix} 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & 0 & & \\ \lambda & & & & & & & \\ 1 & \ddots & & & & & & \\ 0 & \ddots & \ddots & & & & & \\ \vdots & & \ddots & \ddots & & & & \\ 0 & \dots & 0 & 1 & \lambda & 0 & \dots & 0 \end{pmatrix}_{n_r^\ell \times n_j^\ell}
\begin{array}{l} \\ \\ \\ \leftarrow n_r^\ell - p + 1 \\ \\ \\ \\ \\ \end{array}
$$

for $p \leq n_j^\ell \leq n_r^\ell = \dim J_u$.

One sees that all such products will be lin. ind. for different basis elements $T_{\ell r j p}$ as long as the λ_ℓ involved is not zero. The analogous argument holds for Jordan blocks of type (B).

If $\lambda = 0$, then

$$\epsilon_u E_u J_u(0,s) B^p_{rj} =$$

$$\begin{pmatrix} & & & & 0 \\ 0 & & & \cdot & 1 \\ & & \cdot & & 0 \\ & \cdot & & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}_{s \times s} \begin{array}{l} \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots \\ \vdots & & & \cdot & & \\ 0 & \cdot & & & & \\ 1 & & & & & \\ 0 & & & 0 & & \\ \vdots & & & & & \end{pmatrix}_{s \times n_j^{\ell}} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow p \\ \\ \\ \end{array} = 0$$

holds for $n_j^{\ell} \leq S = \dim J_u$ iff $p = 1$. Hence if $C(0)$ contains k_0 Jordan blocks, then $k_0 + (k_0 - 1) + \ldots + 1$ of the above products $S_2 T_{\ell rjp}$ will vanish, since there are that many lin. ind. matrices $T_{t+1,r,j,1}$ for $r, j = 1, \ldots, k_0$ and $r \leq j$. So these $D = k_0(k_0+1)/2$ products $S_2 T_{\ell rjp}$ need not be counted.

Thus the formula in Theorem 2 is proved. ∎

Corollary: Let S and T be a nonsingular pair of r. s matrices. Assume that the nonzero eigenvalues of $S^{-1}T$ have multiplicities $k_1, \ldots, k_{\ell}$. If S and T can be simultaneously diagonalized by a real congruence transformation, then

$$\dim(\{S_1 S \mid S \text{ symmetric}\} \cap \{S_2 S \mid S \text{ symmetric}\}) = \sum_{i=1}^{\ell} k_i(k_i+1)/2$$

Proof: Theorem 1 of Chapter I says that if S and T can be simultaneously diagonalized, then $S^{-1}T$ is similar to a real diagonal matrix. Hence all $n_j^i = 1$ and each

$$N_j = \sum_{i=1}^{k_j} i = k_j(k_j+1)/2 \quad ,$$

while $N_o = D$. ∎

<u>Definition 2</u>: A real square matrix A is <u>nonderogatory</u> if the Jordan normal form of A contains chains of length 1 only.

It can be shown that, for example, A is nonderogatory iff A is similar to a companion matrix or iff the minimal and characteristic polynomial of A coincide.

Now we apply Theorem 2 to another special case.

<u>Theorem 3</u>: Let A be a nonsingular real $n \times n$ matrix. Then A is nonderogatory iff for all symmetric factorizations $A = S_1^{-1} S_2$ we have

$$\dim(\{S_1 S \mid S \text{ symmetric}\} \cap \{S_2 S \mid S \text{ symmetric}\}) = n \ .$$

<u>Proof</u>: Since A is nonsingular, we have $N_0 = D = 0$ in the notation of Theorem 2.

If A is nonderogatory, then all Jordan chains have length 1 and $k_i = 1$ for all i. Hence $N_i = n_1^i$ for all i and $n = \sum_i n_1^i = \sum_i N_i = \dim(\{S_1 S\} \cap \{S_2 S\})$ by Theorem 2.

Conversely if $\sum_i N_i = n$, then we have

$$n = \sum_i N_i = \sum_i \sum_{j=1}^{k_i} j n_j^i \ , \quad \text{while} \ \sum_i \sum_{j=1}^{k_i} n_j^i = n \ , \quad \text{too.}$$

Thus

$$0 = \sum_i \sum_{j=1}^{k_i} (j-1)\, n_j^i \ .$$

Since $n_j^i \geq 1$ for all i, j we must have $k_i = 1$ for all i in order to satisfy the last equation. Hence all Jordan chains of A have length 1 and A is nonderogatory. ∎

Theorem 3 in fact is a slightly weaker result than a theorem of Taussky and Zassenhaus [39] which holds even for nonsingular matrices A:

(*) $\dim\{T \text{ symmetric} \mid AT = TA'\} = n$ iff A is nonderogatory .

For in Theorem 3 we count the number of lin. ind. matrices AS such that AS is symmetric only for nonsingular A and r. s. matrices S. Since A is assumed nonsingular this amounts to counting the r. s. matrices S that make AS symmetric. Hence (*) implies Theorem 3. However Theorem 3 is proved here independently (in the case of non-singular A).

By using Lemma 3 in the proof of Theorem 2 we found a basis $T_{\ell rjp}$ for the space of r. s. matrices T such that AT is symmetric and computed its dimension, so that we can deduce (*) even for nonsingular A in a new way via the canonical pair form.

Now if $A = S_1^{-1}S_2$ is singular and nonderogatory, then the formula in Theorem 2 gives $\dim(\{S_1S\} \cap \{S_2S\}) = n - 1$, while the converse is not true; for example, take n = 4, S_1 and S_2 such that $S_1^{-1}S_2$ has real Jordan normal form diag(J(1, 1), J(1, 1), J(0, 1), J(0, 1)). Then $N_0 = N_1 = 3 = D$ and $\dim(\{S_1S\} \cap \{S_2S\}) = \Sigma N_i - D = 3 = n - 1$, but $S_1^{-1}S_2$ is derogatory.

Finally we are able to express one of Marcus and Khan's [25] results in more simple form.

<u>Theorem 4</u>: Let A be a real matrix with real Jordan normal form J as defined in Theorem 2. Then $\dim\{T \text{ symmetric} \mid AT \text{ symmetric}\} =$

$$\sum_{i=0}^{t} \sum_{j=1}^{k_i} jn_j^i .$$

<u>Proof:</u> In Lemma 3 we described all r. s. T such that AT is symmetric, where $A = S_1^{-1}S_2$. In the proof of Theorem 2 we showed how many such T are lin. indep. Thus Theorem 4 follows. ∎

Marcus and Khan [25], Theorem 1, formula (1. 5), work over fields, though, that contain all the characteristic roots of A.

CHAPTER III

PAIRS AND PENCILS OF REAL SYMMETRIC MATRICES

In this chapter we give a survey of known results on pairs and pencils of r. s. matrices and derive some new ones (as summarized in the introduction). Specifically we treat the cases where a pencil contains a positive definite matrix, when two r. s. matrices can be simultaneously diagonalized or block diagonalized by a real congruence transformation and when a real matrix can be written as a product of two r. s. matrices one of which is positive definite.

Different authors have worked on each of the above questions, some of the theorems are classical, some results first appeared in the thirties, but most of the work quoted here was done in the fifties and sixties.

First we define pencils:

Definition 1: Let S and T be two r. s. $n \times n$ matrices that are lin. indep. Then the pencil generated by S and T is

$$\underline{P(S,T)} = \{aS + bT \mid a, b \in \mathbb{R}\} \quad .$$

Throughout this chapter we will find properties associated with a pencil P that are independent of the generators of P. For example:

Theorem 1: Let S_i, T_i, $i = 1, 2$, be two nonsingular pairs of r. s. matrices. Let J_1 and J_2 be the Jordan normal forms of $S_1^{-1}S_2$ and

$T_1^{-1}T_2$, respectively.

If $P(S_1, S_2) = P(T_1, T_2)$, then each full Jordan chain $C_1(\lambda)$ in J_1 corresponds to a conformally partitioned full Jordan chain $C_2(\mu)$ in J_2 and conversely, such that either λ, μ are both in $\mathbb{R}$ or λ, μ are both in $\mathbb{C} - \mathbb{R}$.

In the following proof we will give the exact formula for this correspondence of the roots.

Proof: Since $T_1, T_2 \in P(S_1, S_2)$, we have $T_1 = aS_1 + bS_2$ and $T_2 = cS_1 + dS_2$ for real constants a, b, c, d. By definition both pairs S_1, S_2 and T_1, T_2 are lin. indep. such that $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Let X transform $S_1^{-1}S_2$ into its Jordan normal form over $\mathbb{C}$:

$$J_1 = X^{-1}S_1^{-1}S_2X$$

Then

$$\begin{aligned} T_1^{-1}T_2 &= (aS_1 + bS_2)^{-1}(cS_1 + dS_2) \\ &= \left(S_1(aI + bS_1^{-1}S_2)\right)^{-1}\left(S_1(cI + dS_1^{-1}S_2)\right) \\ &= (aI + bS_1^{-1}S_2)^{-1}(cI + dS_1^{-1}S_2) \ . \end{aligned}$$

And thus

$$\begin{aligned} X^{-1}T_1^{-1}T_2X &= X^{-1}(aI + bS_1^{-1}S_2)^{-1}XX^{-1}(cI + dS_1^{-1}S_2)X \\ &= \left(X^{-1}(aI + bS_1^{-1}S_2)X\right)^{-1}(cI + dJ_1) \\ &= (aI + bJ_1)^{-1}(cI + dJ_1) \ . \end{aligned}$$

Now $aI + bJ_1$ is a block diagonal matrix. So in order to find its inverse it suffices to look at the inverses of each of its diagonal blocks. These have the form

$$M = \begin{pmatrix} a+b\lambda & b & & 0 \\ & \ddots & \ddots & \\ & & \ddots & b \\ 0 & & & a+b\lambda \end{pmatrix}_{r \times r},$$

if r is the dimension of a Jordan block $J(\lambda, r)$ occurring in J_1.
Note that for all eigenvalues λ of $S_1^{-1}S_2$ we have $a+b\lambda \neq 0$, since $0 \neq \det T_1 = \det(aS_1 + bS_2) = \det S_1 \det(aI + bS_1^{-1}S_2)$. So letting $d_k = (-1)^{k+1} b^{k-1} (a+b\lambda)^{-k}$ for $k = 1, \ldots, r$, we have

$$M \cdot \begin{pmatrix} d_1 & \cdots & d_r \\ & \ddots & \vdots \\ 0 & & d_1 \end{pmatrix} = I_{r \times r}.$$

Thus the matrix $X^{-1}T_1^{-1}T_2X = (aI + bJ_1)^{-1}(cI + dJ_1)$ is a block diagonal matrix with diagonal blocks of the form

$$\begin{pmatrix} d_1 & \cdots & d_r \\ & \ddots & \vdots \\ 0 & & d_1 \end{pmatrix} \begin{pmatrix} c+d\lambda & d & & 0 \\ & \ddots & \ddots & \\ & & \ddots & d \\ 0 & & & c+d\lambda \end{pmatrix} =$$

$$= \begin{pmatrix} d_1(c+d\lambda) & d_1d + d_2(c+d\lambda) & \cdots & d_{r-1}d + d_r(c+d\lambda) \\ & \ddots & \ddots & \vdots \\ & & \ddots & d_1d + d_2(c+d\lambda) \\ 0 & & & d_1(c+d\lambda) \end{pmatrix}_{r \times r}$$

Now $d_1d + d_2(c+d\lambda) = (ad - bc)(a+b\lambda)^{-2} \neq 0$.

Hence a Jordan block $J(\lambda, r)$ in J_1 corresponds to a Jordan block $J(\mu, r)$

in J_2, the Jordan normal form of $T_1^{-1}T_2$, where $\mu = (c+d\lambda)(a+b\lambda)^{-1}$. Moreover full Jordan chains $C_1(\lambda)$ in J_1 correspond to full Jordan chains $C_2(\mu)$ in J_2 with $\mu = (c+d\lambda)(a+b\lambda)^{-1}$ and $\mu \in \mathbb{R}$ iff $\lambda \in \mathbb{R}$. Interchanging the roles of the S's and T's one arrives at the converse correspondence. ∎

The converse to Theorem 1 does not hold. Take $S_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $T_1 = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $T_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, for example.

Definition 2: For a r. s. $n \times n$ matrix S define the quadratic hypersurface as $Q_S = \{x \in \mathbb{R}^n \mid x'Sx = 0\}$.

Lemma 1: Let S, T be r. s. matrices and A be real nonsingular. Then

a) $Q_{A'SA} = A^{-1}Q_S$ and

b) $Q_{A'SA} \cap Q_{A'TA} = A^{-1}(Q_S \cap Q_T)$.

Proof: a) We have $x \in Q_{A'SA}$ iff $x'A'SAx = 0$, hence iff $Ax \in Q_S$, hence iff $x \in A^{-1}Q_S$. For another proof see Kowalski [21], p. 193-194. Statement b) follows from a). ∎

Definition 3: For a r. s. matrix S we define the inertia of S as $\text{in}\,S = (a, b, c)$, if S has a positive, b negative and c zero eigenvalues.

By Sylvester's law of inertia there is a one-to-one correspondence between each class of congruent r. s. $n \times n$ matrices and each of the $(n+1)(n+2)/2$ possible triples (a, b, c), with $a, b, c \in \mathbb{N}$ and $a+b+c = n$, the inertia of the class.

Definition 4: A r. s. $n \times n$ matrix S is called positive definite, if in $S = (n, 0, 0)$; positive semidefinite, if in $S = (k, 0, \ell)$, $k > 0$, $\ell > 0$,; negative definite, if in $S = (0, n, 0)$; negative semidefinite, if in $S = (0, k, \ell)$, $k > 0$, $\ell > 0$; and indefinite, if in $S = (a, b, c)$ with $a, b \neq 0$.

Theorem 2: Let S and T be r. s. $n \times n$ matrices. If $n \geq 3$, then the following are equivalent:

i) There exists a positive definite matrix in $P(S, T)$,

ii) $Q_S \cap Q_T = \{0\}$,

iii) trace YS = trace $YT = 0$ for Y positive semidefinite implies $Y = 0$.

Proof: The equivalence of i) and ii) was proved by Calabi [7], while Berman [5] showed the equivalence of i) and iii). ∎

The question whether a given pencil of r. s. matrices contains a positive definite matrix was treated in chronological order by Finsler [13], Albert [1], Reid [30], Hestenes and McShane [18], Dines [11], Calabi [7], Taussky [36], Hestenes [17], Theorem 3, and Berman [5]. Before Calabi [7] only condition

ii') $\qquad x'Sx = 0 \qquad$ implies $\qquad x'Tx > 0$

was generally used instead of ii). And thus only the fact that ii') implies i) was proved.

Theorem 3: Let $P(S_1, S_2) = P(T_1, T_2)$ for r. s. matrices S_i, T_i, $i = 1, 2$. Then

$$Q_{S_1} \cap Q_{S_2} = Q_{T_1} \cap Q_{T_2} .$$

Proof: We have $T_1 = aS_1 + bS_2$ and $T_2 = cS_1 + dS_2$ for real a, b, c, d. Since the S_i and T_i are lin. indep. by Definition 1 we have $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$. Now $x \in Q_{T_1} \cap Q_{T_2}$ iff

$$x'T_1x = ax'S_1x + bx'S_2x = 0$$

and

$$x'T_2x = cx'S_1x + dx'S_2x = 0 \; .$$

This is a homogeneous linear system in $x'S_1x$ and $x'S_2x$. Since the system's determinant does not vanish it has only the zero solution $x'S_1x = x'S_2x = 0$. And thus $Q_{T_1} \cap Q_{T_2} = Q_{S_1} \cap Q_{S_2}$. ∎

The following is a classical result that goes back to Cauchy, Kronecker, Sylvester and Weierstraß.

Theorem 4: Let S and T be r.s. matrices. If S is positive definite, then S and T can be simultaneously diagonalized by a real congruence transformation.

Corollary 1: Let S and T be r.s. matrices. If there exists a positive definite matrix in P(S, T), then S and T can be simultaneously diagonalized by a real congruence transformation.

Using this and Theorem 2 one gets:

Corollary 2: (Greub and Milnor, [16], p. 256). Let S and T be r.s. $n \times n$ matrices for $n \geq 3$. If $Q_S \cap Q_T = \{0\}$, then S and T can be simultaneously diagonalized by a real congruence transformation.

Greub and Milnor [16] proved Corollary 2 prior and independently of both Calabi's result in Theorem 2 and Theorem 4. They used methods of complex analysis. Later Majindar [24] proved the corresponding result for pairs of hermitian matrices. Kraljevic [22] proved Corollary 2 directly via methods of linear algebra. Wonenburger [42], Theorem 2, extended Corollary 2 to a pair of symmetric matrices with coefficients in a real closed field. Finally Au-Yeung [3] proved Corollary 2 not only for real symmetric matrices but also for pairs of hermitian matrices and "hermitian" matrices with coefficients in the real quaternious. In the hermitian case the dimension of the underlying vector space can even be 2.

We deduced the Greub-Milnor Theorem (Corollary 2) from Theorems 2 and 4. Taussky [36] has shown that the Greub-Milnor Theorem implies Calabi's result in Theorem 2. This was done via Stiemke's Theorem [31].

While $Q_S \cap Q_T = \{0\}$ is sufficient for S and T to be simultaneously diagonalizable, it is not necessary, as shown for example by $S = T = 0$.

The following theorem has been proved by Greub [16], p. 255, Proposition, and Wonenburger [42], Theorem 1.

Theorem 5: Let S and T be a nonsingular pair of r. s. matrices. Then the following are equivalent:

a) S and T can be simultaneously diagonalized by a real congruence transformation, and

b) $S^{-1}T$ is similar to a real diagonal matrix.

We can give a new proof of Theorem 5:

Look at the canonical pair form of S and T as developed in Chapter I. Then the equivalence of a) and b) is obvious. ∎

Gantmacher [15], Vol. 2, p. 43, proves the analogous result for complex symmetric matrices.

Here is a new condition equivalent to both a) and b) in Theorem 5.

Theorem 6: Let S and T be a nonsingular pair of r.s. matrices. Let $f(\lambda, \mu) = \det(\lambda S + \mu T)$ for $\lambda, \mu \in \mathbb{R}$ and write

$$f(\lambda, \mu) = c \prod_{i=1}^{k} (\lambda + \beta_i \mu)^{\gamma_i}$$

for $c \in \mathbb{R}$, $\beta_i \in \mathbb{C}$, $\gamma_i \in \mathbb{Z}^+$, where $\beta_i \neq \beta_j$ for $i \neq j$.

Then the following condition c) is equivalent to a) and b) of Theorem 5:

c) $\beta_i \in \mathbb{R}$ for $i = 1, \ldots, k$ and $\dim \ker(\beta_i S - T) = \gamma_i$ for all i.

We do not give a separate proof for Theorem 6. For this theorem is in fact a special case of Theorem 10 in this chapter, once the finest simultaneous block diagonalization of two r.s. matrices has been defined (Definitions 5 and 6). None of the results up to and including Theorem 10 depends on Theorem 6.

The next theorem sheds some light on the zeros of $f(\lambda, \mu)$ defined above.

Theorem 7: Let S and T be r.s. $n \times n$ matrices. If $f(\lambda, \mu) = \det(\lambda S + \mu T) \neq 0$ for all $(\lambda, \mu) \neq (0,0)$, then $\mathrm{in}(\lambda S + \mu T) = (n/2, n/2, 0)$ with n even for all $(\lambda, \mu) \neq (0,0)$ and $S^{-1}T$ has no real root.

Proof: By assumption $\lambda S + \mu T$ is nonsingular for all choices of $(\lambda, \mu) \neq (0,0)$. So, since $\mathrm{in}(\lambda S + \mu T)$ is a continuous function of (λ, μ), it must be constant for all $(\lambda, \mu) \neq (0,0)$. Let $\operatorname{in}(\lambda S + \mu T) = (a, b, 0)$, then $\operatorname{in}(-\lambda S - \mu T) = (b, a, 0)$. Thus $a = b = n/2$. And n must be even. Now S is nonsingular and thus $\det(\lambda I - S^{-1}T) = \det S^{-1} \det(\lambda S - T) = \det S^{-1} f(\lambda, -1) \neq 0$ for all real λ. So $S^{-1}T$ can have no real eigenvalue. ■

Notice that the converse holds, too; namely, if $S^{-1}T$ has no real root, then $f(\lambda, \mu) \neq 0$ for all $(\lambda, \mu) \neq (0, 0)$.

We will now further investigate symmetric matrix products $S^{-1}T$ such that $S^{-1}T$ is similar to a real diagonal matrix. Taussky [35] has characterized those matrices.

Theorem 8: Let A be a real square matrix. Then the following are equivalent:

r) A is similar to a real diagonal matrix, and

s) A can be factored into a product of two r. s. matrices in which one factor is positive definite.

For further equivalent characterization, and related questions see Taussky [37] and Carlson [8], Theorem 3. Drazin and Haynsworth [12] further extended this problem. They characterize complex matrices that have in their Jordan normal form exactly $m \leq n$ Jordan blocks of type (A).

Next we show how nonsingular pairs of r. s. matrices can be simultaneously block diagonalized by a real congruence transformation and determine what is the finest simultaneous block diagonal form that can be achieved for a given pair. It is here (Theorem 9) that the use of the word "canonical" in the canonical pair form of a nonsingular pair of r. s. matrices is fully understood.

Definition 5: Let S and T be r. s. matrices, X be a nonsingular real matrix and let S_i, T_i be square matrices such that $X'SX = \text{diag}(S_1, \ldots, S_k)$ and $X'TX = \text{diag}(T_1, \ldots, T_k)$ with $\dim T_i = \dim S_i$ for all i. Then we say S and T are simultaneously block diagonalizable into k blocks.

Definition 6: Let S and T be r. s. matrices. Then a simultaneous block diagonalization $X'SX = \text{diag}(S_1, \ldots, S_k)$ and $X'TX = \text{diag}(T_1, \ldots, T_k)$ is called the finest block diagonalization of S and T, if

$$k = \max_{\substack{X \text{ non-} \\ \text{singular}}} \{k(X) \mid X'SX = \text{diag}(S_1^{(X)}, \ldots, S_{k(X)}^{(X)}), \; X'TX = \text{diag}(T_1^{X}, \ldots, T_{k(X)}^{(X)}) \text{ with } \dim S_i^{(X)} = \dim T_i^{(X)} \text{ for all } i\} .$$

For example, if two r. s. $n \times n$ matrices can be simultaneously diagonalized, then their finest simultaneous block diagonalization contains n blocks.

Next we will see that not only the number k of blocks in the finest simultaneous block diagonalization of two r. s. matrices S, T is unique, but that also the k-tuple of block sizes $(\dim S_1, \ldots, \dim S_k)$ is uniquely determined by the pair S and T up to permutations of the integers $\dim S_i$.

Theorem 9: Let S and T be a nonsingular pair of r. s. matrices. Then the finest simultaneous block diagonalization of S and T contains k blocks of dimensions $n_1, \ldots, n_k$ iff the real Jordan normal form of $S^{-1}T$ consists of k Jordan blocks of dimensions $n_1, \ldots, n_k$.

Proof: This is a direct application of Theorem 1 in Chapter I, if one observes the following: The real Jordan normal form represents a linear transformation L relative to a certain basis of $\mathbb{R}^n$. Relative to this basis $\mathbb{R}^n$ is the direct sum of L-invariant subspaces. The number k of summands is maximal and their dimensions $n_1, \ldots, n_k$ are determined by L up to permutations. (See Kowalski [21], p. 242-248 or Jacobsen [20], p. 63-73). Thus we can talk of the "finest" simultaneous block diagonalization in Theorem 9. ∎

A similar theorem about complex symmetric matrices is proved in Gantmacher [15], Vol. II, p. 44.

Theorems 5 and 6 of this chapter characterized pairs S and T which can be simultaneously diagonalized. Next we will deal with the other extreme: no simultaneous reduction at all.

Corollary 3: Let S and T be a nonsingular pair of r. s. matrices. Then the following are equivalent:

p) S and T can not be reduced simultaneously by a real congruence transformation at all, and

q) either $S^{-1}T$ has only one real eigenvalue λ and $\dim \ker (\lambda S - T) = 1$, or $S^{-1}T$ has only a pair of complex conjugate roots λ and $\bar{\lambda}$ with $\dim_{\mathbb{C}} \ker_{\mathbb{C}} (\lambda S - T) = 1$.

Proof: By Theorem 9 condition p) is equivalent to the fact that the real Jordan normal form of $S^{-1}T$ contains exactly one Jordan block, which is stated in condition q). ∎

Now we will relate the number of blocks in the finest simultaneous block diagonalization of a nonsingular pair S and T to the factorization of $f(\lambda, \mu) = \det(\lambda S + \mu T)$ over $\mathbb{C}$.

Theorem 10: Let S and T be a nonsingular pair of r. s. matrices. Let $f(\lambda, \mu) = \det(\lambda S + \mu T)$. If β_i are k distinct complex numbers, c is real and γ_i, η_i, ϵ_i are positive integers, then the following are equivalent:

w) $f(\lambda, \mu) = c \prod_{i=1}^{k} (\lambda + \beta_i \mu)^{\gamma_i}$,

$\eta_i = \dim \ker(\beta_i S - T)$ if $\beta_i \in \mathbb{R}$ and

$\epsilon_i = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\beta_i S - T)$ if $\beta_i \in \mathbb{C} - \mathbb{R}$, and

z) $S^{-1}T$ has eigenvalues β_i of multiplicity γ_i for $i = 1, \ldots, k$, and its real Jordan normal form contains η_i Jordan blocks of type (A) for real eigenvalue β_i and ϵ_i Jordan blocks of type (B) for each pair of complex conjugate eigenvalues β_i, $\overline{\beta}_i$ of $S^{-1}T$.

Moreover w) or z) both imply:

The finest simultaneous block diagonalization of S and T via a non-singular real congruence transformation contains $\sum \eta_i + \sum \epsilon_i$ blocks.

Proof: The last remark follows from Theorem 9.

First we assume that z) holds. Then

$$\det(\lambda I - S^{-1}T) = \prod_{i=1}^{k} (\lambda - \beta_i)^{\gamma_i}$$

and we have

$$f(\lambda, \mu) = \det(\lambda S + \mu T) = \det S \det(\lambda I + \mu S^{-1} T) = \det S \prod_{i=1}^{k} (\lambda + \beta_i \mu)^{\gamma_i} .$$

By assumption there are η_i Jordan blocks of type (A) for a real eigenvalue β_i of $S^{-1}T$, hence there are exactly η_i lin. indep. eigenvectors corresponding to a real eigenvalue β_i and thus $\eta_i = \dim \ker(\beta_i I - S^{-1}T)$ $= \dim \ker(\beta_i S - T)$. The analogous argument holds for complex roots. Thus w) holds.

Now assume w) holds: Then

$$\det(\lambda I - S^{-1}T) = \det S^{-1} \det(\lambda S - T) = \det S^{-1} f(\lambda, -1) =$$

$$= \det S^{-1} \prod_{i=1}^{k} (\lambda - \beta_i)^{\gamma_i} \text{ with } \beta_i \neq \beta_j \text{ for } i \neq j.$$

So the roots β_i of $S^{-1}T$ have the multiplicities claimed in z). Since for real β_i we have $\eta_i = \dim \ker(\beta_i S - T) = \dim \ker(\beta_i I - S^{-1}T)$, there are η_i lin. ind. eigenvectors of $S^{-1}T$ for each real root β_i. The analogous argument holds for complex roots of $S^{-1}T$ and thus $S^{-1}T$ has the real Jordan normal form as stated in z). ■

Finally we prove that the finest simultaneous block diagonal structure of a nonsingular pair of r. s. matrices is a property of their pencil:

Theorem 11: Let S_i and T_i $(i = 1, 2,)$ be two nonsingular pairs of real symmetric matrices.

If $P(S_1, S_2) = P(T_1, T_2)$, then the two pairs S_1, S_2 and T_1, T_2 have the same finest block diagonal structure.

Proof: By Theorem 1 the matrices $S_1^{-1}S_2$ and $T_1^{-1}T_2$ have the same Jordan structure. Then Theorem 11 follows from Theorem 9. ■

CHAPTER IV

ON THE MAXIMAL NUMBER OF LINEARLY INDEPENDENT REAL VECTORS ANNIHILATED SIMULTANEOUSLY BY TWO REAL QUADRATIC FORMS

In this chapter we will compute the maximal number m of lin. ind. vectors simultaneously annihilated by two quadratic forms derived from a nonsingular pair of real symmetric matrices S and T as a function of the real Jordan normal form of $S^{-1}T$. In Theorems 1 and 2 we compute m for all possible real Jordan normal forms of $S^{-1}T$, if S and T have at least dimension 4. Theorem 3 treats the lower dimensional cases.

In Theorem 4 we reverse the arguments and thus can say which real Jordan normal form $S^{-1}T$ must have, if a specific m is the maximal number of lin. indep. vectors simultaneously annihilated by the two quadratic forms $x'Sx$ and $x'Tx$.

From Chapters I and III we know how the real Jordan normal form of $S^{-1}T$ is related to the finest simultaneous block diagonalization of S and T. Using these results Theorem 4 gives a new proof of the Greub-Milnor Theorem, [16], p. 256, as stated in Corollary 2 of Chapter III, namely: if $m = 0$, then S and T can be simultaneously diagonalized provided S and T are more than 2 dimensional matrices. Besides this, Theorem 4 is used to develop a set of new conditions on m and $S^{-1}T$ that assure S and T to be simultaneously diagonalizable by a real congruence transformation.

But before stating and proving these theorems, we will, for completeness, treat just one quadratic form:

Lemma 1: Let S be a real symmetric matrix. Then

a) $Q_S = \ker S$ and $Q_S \neq \{0\}$ iff S is semidefinite, and

b) Let $S \neq 0$. Then S is indefinite iff Q_S contains n linearly independent vectors.

The symbol Q_S was defined in the beginning of Chapter III (Definition 4).

Proof: First we prove in a): if S is semidefinite, then $Q_S = \ker S \neq \{0\}$. Let S be semidefinite. Then there exists a nonsingular real matrix A such that $S = A'DA$. Here $D = \mathrm{diag}(d_1, \ldots, d_k, 0, \ldots, 0)$ is a real diagonal matrix and the d_i are either all positive or all negative.
We know that $Q_S = A^{-1}Q_D$ by Lemma 1 of Chapter III. But

$$Q_D = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{k} d_i x_i^2 = 0 \right\} = \langle e_{k+1}, \ldots, e_n \rangle = \ker D$$

where e_i stands for the i^{th} unit vector and $\langle \cdots \rangle$ denotes the linear span.
Hence $Q_S = A^{-1}\ker D$. And it remains to show that $A^{-1}\ker D = \ker S$.
Let $x \in \ker S$. Then $Sx = A'DAx = 0$, hence $DAx = 0$, since A' is nonsingular. Thus $Ax \in \ker D$ or $x \in A^{-1}\ker D$. Conversely let $x \in A^{-1}\ker D$. Then $x = A^{-1}y$ for some $y \in \ker D$. Since $y = Ax$ we have $Dy = DAx = 0$ and $Sx = A'DAx = 0$, so that $x \in \ker S$. Thus we have $Q_S = \ker S$.
Next we prove in b): if $S \neq 0$ is indefinite, then Q_S contains n lin. ind. vectors.

Let $S \neq 0$ be indefinite. Then there exists a nonsingular A such that $S = A'DA$. Here $D = \mathrm{diag}(a_1, \ldots, a_k, b_1, \ldots, b_j, 0, \ldots, 0)$ with $a_i > 0$, $b_\ell < 0$ and $k, j \neq 0$.

Again we have $Q_S = A^{-1}Q_D$ and hence it suffices to find n lin. ind. vectors in Q_D.

With e_i the i^{th} unit vector we define

$$x_i = \alpha_i e_1 + \beta_i e_{k+i} \qquad \text{for } i = 1, \ldots, j ,$$

where $\alpha_i, \beta_i \neq 0$ are chosen such that $a_1\alpha_i^2 + b_i\beta_i^2 = 0$,

$$x_{j+1} = -\alpha_1 e_1 + \beta_1 e_{k+1} ,$$

$$x_i = \alpha_i e_{i-j} + \beta_i e_{k+1} \qquad \text{for } i = j+2, \ldots, j+k ,$$

where $\alpha_i, \beta_i \neq 0$ are such that $a_{i-j}\,\alpha_i^2 + b_{k+1}\,\beta_i^2 = 0$, and $x_i = e_i$ for $i > j+k$.

Note that these n vectors x_i are lin. ind. and belong to Q_D.

To finish proving a) we note that if S is not semidefinite, then either S is definite or indefinite. If S is definite, then $Q_S = \{0\}$ and a) is proved. If S is indefinite, then by the part of b) that we just proved Q_S contains n lin. ind. vectors and hence $Q_S \neq \ker S$ since $S \neq 0$.

Finally in b) let $S \neq 0$ and assume Q_S contains n lin. ind. vectors. Then S is not definite and clearly $Q_S \neq \ker S$. Thus by a) S is not semidefinite. Hence S must be indefinite.

This finishes the proof. ∎

Now we will state the first two theorems of this chapter in the notations developed in Chapters I and III:

Theorem 1: Let S and T be a nonsingular pair of r. s. $n \times n$ matrices. Let J be the real Jordan normal form of $S^{-1}T$. Let

i) J contains a Jordan block of dimension greater than 3, or

ii) J contains two Jordan blocks of dimension 3 each, or

iii) J contains one Jordan block of dimension 3 and one of dimension 2, or

iv) $n > 3$ and J contains a Jordan block of dimension 3 and 1-dimensional blocks else, but not all eigenvalues of $S^{-1}T$ are the same, or

v) J contains two 2-dimensional Jordan blocks which correspond to different eigenvalues of $S^{-1}T$ if both blocks are of type (A).

Then $Q_S \cap Q_T$ contains n linearly independent vectors.

Theorem 2: Let S and T be a nonsingular pair of r. s. matrices of dimension n. Let J be the real Jordan normal form of $S^{-1}T$. Let

vi) $n > 3$, J contains one 3-dimensional Jordan block, linear blocks else and all eigenvalues of $S^{-1}T$ are the same while inertia $S \neq (n-1, 1, 0), (1, n-1, 0)$; or

vii) $n > 3$ and J contains $k \geq 1$ identical 2-dimensional Jordan blocks $J(\lambda, 2)$ of type (A), linear blocks else for eigenvalues μ_i ($i = 2k+1, \ldots, n$) and the set

$$\{\epsilon_1, \ldots, \epsilon_k, \ \epsilon_i(\mu_i - \lambda) \mid i > 2k\}$$

contains positive as well as negative numbers, where the $\epsilon_j = \pm 1$ are the constants in the canonical pair form of S and T (see Theorem 1, Chapter I), or

viii) $n > 3$, J contains one 2-dimensional block $J(a, b, 2)$ of type (B) and linear blocks else for eigenvalues μ_i, where not all μ_i are the same or* inertia $S \neq (n-1, 1, 0), (1, n-1, 0)$.

Then $Q_S \cap Q_T$ contains n linearly independent vectors.

Let

vi)a) condition vi) holds, except that inertia $S = (n-1, 1, 0)$ or $(1, n-1, 0)$, or

viii)a) condition viii) holds, except that all real eigenvalues μ_i as defined in viii) are the same and inertia $S = (n-1, 1, 0)$ or $(1, n-1, 0)$.

Then $Q_S \cap Q_T$ contains a maximum of $n-1$ lin. indep. vectors only.

If

vii)a) condition vii) holds except that the set $\{\varepsilon_1, \ldots, \varepsilon_k, \varepsilon_i(\mu_i - \lambda) \mid i > 2k\}$ as defined in vii) contains $r \geq 0$ zeros $\mu_{2k+1} - \lambda = \cdots = \mu_{2k+r} - \lambda = 0$ and only positive or only negative numbers else, and

$\varepsilon_{2k+1} = \cdots = \varepsilon_{2k+r}$,

then $Q_S \cap Q_T$ contains a maximum number of k lin. ind. vectors.

If

vii)b) condition vii)a) holds except that not all ε_i are the same for $2k+1 \leq i \leq 2k+r$,

then $Q_S \cap Q_T$ contains a maximum of $k+r$ lin. ind. vectors.

If

ix) $n > 1$ and S and T can be simultaneously diagonalized by a real congruence transformation, then the maximal number k of lin. indep. vectors in $Q_S \cap Q_T$ can be $k=0, 2, \ldots, n$ depending on S and T.

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*This "or" does not mean "either... or".

Theorem 3 will treat n-dimensional r. s. matrix pairs for $n \le 3$:

The following Lemma is useful for the proofs of Theorems 1 and 2.

Lemma 2: Let S and T be real symmetric matrices and A be a real nonsingular matrix.

Then

$$\max\{k \mid \text{there exist } k \text{ lin. indep. vectors in } Q_S \cap Q_T\}$$
$$= \max\{k \mid \text{there exist } k \text{ lin. indep. vectors in } Q_{A'SA} \cap Q_{A'TA}\}.$$

The lemma is a direct application of Lemma 6, Chapter III.

Proof: (Theorem 1) In view of the above lemma we may WLOG assume that S and T are in canonical pair form as developed in Chapter I:

$$S = \mathrm{diag}(\pm E_1, \ldots, \pm E_r, E_{r+1}, \ldots, E_m) ,$$
$$T = \mathrm{diag}(\pm E_1 J_1, \ldots, \pm E_r J_r, E_{r+1} J_{r+1}, \ldots, E_m J_m) ,$$

where $J_1, \ldots, J_r$ are Jordan blocks of type (A) and $J_{r+1}, \ldots, J_m$ are of type (B).

Having S and T in this form is very advantageous. For then we have $(e_i' S e_i)^2 + (e_i' T e_i)^2 \neq 0$ for at most $r + 2(m - r)$ unit vectors e_i. The reason is as follows:

For the Jordan block $J_1 = J(\lambda, k)$ of type (A) we have:

if k is even: $\quad e'_{\frac{k}{2}+1} \, E \, e_{\frac{k}{2}+1} = 0 \quad$ and

$$e'_{\frac{k}{2}+1} \, E J(\lambda, k) \, e_{\frac{k}{2}+1} = 1 ,$$

while for all other $i \le k$: $\quad e_i' E e_i = e_i' E J(\lambda, k) e_i = 0$, if k is odd: $$e'_{\frac{k+1}{2}} E\, e_{\frac{k+1}{2}} = 1 \quad \text{and}$$

$$e'_{\frac{k+1}{2}} EJ(\lambda, k)\, e_{\frac{k+1}{2}} = \lambda ,$$

while for all other $i \le k$: $e_i' E e_i = e_i' EJ(\lambda, k) e_i = 0$.

For the Jordan block $J_1 = J(a,b,k)$ $(b \neq 0)$ of type (B) we have:

if k is divisible by 4: $e_i' E e_i = e_i' EJ(a,b,k) e_i = 0$ for all $i \le k$;

while for a k not divisible by 4 we have

$$e'_{\frac{k}{2}} E\, e_{\frac{k}{2}} = 0,\ e'_{\frac{k}{2}} EJ(a,b,k)\, e_{\frac{k}{2}} = b;\ e'_{\frac{k}{2}+1} E\, e_{\frac{k}{2}+1} = 0 ,$$

$$e'_{\frac{k}{2}+1} EJ(a,b,k)\, e_{\frac{k}{2}+1} = -b \quad \text{and} \quad e_i' E e_i = e_i' EJ(a,b,k)\, e_i = 0$$

for all other $i \le k$.

The same argument holds for each of the Jordan blocks. So there are at most $r + 2(m - r)$ unit vectors not simultaneously annihilated by the two quadratic forms $x'Sx$ and $x'Tx$ if S and T are in canonical pair form. For all i such that $e_i \notin Q_S \cap Q_T$ we will exhibit lin. ind. vectors $Y_i \in Q_S \cap Q_T$ that have a nonzero i^{th} component and hence are also lin. ind. of all e_i with $e_i \in Q_S \cap Q_T$. Then Theorem 1 is proved: There are n lin. ind. vectors in $Q_S \cap Q_T$.

The remainder of this proof will consist of finding these vectors Y_i, one for each Jordan block of type (A), two for each Jordan block of type (B) of dimension not divisible by 4 in each of the cases i),...,v).

From now on we will in general assume that the Jordan blocks of $S^{-1}T$ mentioned in i), ..., v) appear in the first diagonal positions. Before starting on the individual cases we express the quadratic forms corresponding to S and T by only singling out the first block here:

If a Jordan block $J(\lambda, k) = J_1$ of type (A) appears first, let us look at the two quadratic forms $F(x) = x'Sx$ and $G(x) = x'Tx$:

For $\quad S = \mathrm{diag}(\pm E_1, \dots, \pm E_m)$ and $x = (x_1, \dots, x_n)$

we have
$$F(x) = \pm h(x) + f(x) \quad ,$$

where
$$h(x) = x' \,\mathrm{diag}(E_1, 0, \dots, 0)\, x = \sum_{i+j=k+1} x_i x_j$$

and
$$f(x) = x' \,\mathrm{diag}(0, \pm E_2, \dots, \pm E_m)\, x$$

is a quadratic form involving $x_{k+1}, \dots, x_n$ only.

For
$$T = \mathrm{diag}(\pm E_1 J_1, \dots, \pm E_m J_m)$$

we have
$$G(x) = \pm\Big(\lambda h(x) + e(x)\Big) + g(x)$$

where h is as above, $\quad e(x) = \sum_{i+j=k+2} x_i x_j \quad$ for $i, j \le k$

and
$$g(x) = x' \,\mathrm{diag}(0, \pm E_2 J_2, \dots, \pm E_m J_m)\, x$$

involves $x_{k+1}, \dots, x_n$ only.

Now $F(x) = 0$ iff $f(x) = \mp h(x)$. And by definition $x \in Q_S \cap Q_T$ iff $F(x) = G(x) = 0$ hence iff

$$(1) \qquad \pm e(x) + g(x) - \lambda f(x) = 0 \qquad \text{and} \qquad F(x) = 0 \;.$$

If a Jordan block $J(a, b, k) = J_1$ $(b \neq 0)$ of type (B) appears first in $S^{-1}T$, then we define $F(x) = s'Sx = h(x) + f(x)$ with h and f as above and $G(x) = x'Tx = ah(x) + bt(x) + u(x) + g(x)$, where h and g are as above and

$$u(x) = \sum_{\substack{i+j=k+3\\ i,j\le k}} x_i x_j, \quad \text{while} \quad t(x) = \sum_{\substack{i+j=k\\ i,j\ \text{odd}}} x_i x_j - \sum_{\substack{i+j=k+2\\ i,j\ \text{even}\\ i,j\le k}} x_i x_j$$

Thus in this case $x \in Q_S \cap Q_T$ iff $F(x) = G(x) = 0$, hence iff

(2) $\qquad bt(x) + u(x) + g(x) - af(x) = 0 \qquad$ and $\qquad F(x) = 0$.

<u>i)</u>: Assume i) holds with a Jordan block $J(\lambda, k)$ of type (A) for $k \ge 4$. Then from p. 51 there is an i, $2 < i < k$ such that $e_i \notin Q_S \cap Q_T$. For this index i we define $\alpha_i, \beta_i \in \mathbb{R}$ and $y_i = \alpha_i e_1 + \beta_i e_2 + e_i + e_k$ such that (1) holds: $\pm e(y_i) + g(y_i) - \lambda f(y_i) = \pm(2\beta_i + e(e_i)) = 0$ determines β_i and $F(y_i) = 0$ determines α_i.

For $i > k$ such that $e_i \notin Q_S \cap Q_T$ and $g(e_i) - \lambda f(e_i) = 0$, we define the vector $y_i = \alpha_i e_1 + e_k + e_i$, where α_i is such that $F(y_i) = 0$. In the case that $g(e_i) - \lambda f(e_i) \ne 0$ we define $y_i = \alpha_i e_1 + \beta_i e_2 + e_k + e_i$, where $\alpha_i, \beta_i \in \mathbb{R}$ are such that (1) holds: $+2\beta_i + g(e_i) - \lambda f(e_i) = 0$ defined β_i and $F(y_i) = 0$ defines α_i.

Next assume i) holds for a Jordan block $J(a,b,k)$ of type (B) for $k = 2\ell \ge 4$.

First assume $k = 2\ell$ is divisible by 4. Then $e_i \notin Q_S \cap Q_T$ implies $i > k$ as pointed out on pp. 51-52. For such an i define $y_i = \alpha_i e_{\ell-1} + \beta_i e_\ell + e_{\ell+1} + e_i$ where $\alpha_i, \beta_i \in \mathbb{R}$ are such that (2) holds. When checking (2), note that ℓ is even, if k is divisible by 4. $2b\alpha_i + g(e_i) - af(e_i) = 0$ defines α_i and $2\beta_i + h(e_i) = 0$ defines β_i.

Now assume $k = 2\ell$ is not divisible by 4. Then ℓ is odd and we know that both $e_\ell, e_{\ell+1} \notin Q_S \cap Q_T$ from pp. 51-52. If we define

$$y_\ell = e_\ell - \frac{b}{2} e_{\ell+3} \qquad \text{and} \qquad y_{\ell+1} = e_{\ell+1} + \frac{b}{2} e_{\ell+2} \ ,$$

then (2) holds for these two vectors. For $i > k$ such that $e_i \notin Q_S \cap Q_T$ we define as before for the real case $y_i = \alpha_i e_1 + e_k + e_i$ if $g(e_i) - \lambda f(e_i) = 0$ and $y_i = \alpha_i e_1 + \beta_i e_2 + e_k + e_i$ otherwise. This proves i) of Theorem 1.

<u>ii):</u> Assume J contains two Jordan block of dimensions 3 each. Then these must be Jordan blocks of type (A); $J(\lambda, 3)$ and $J(\mu, 3)$ for $\lambda, \mu \in \mathbb{R}$. Define for $x = (x_1, \dots, x_n)$:

$$F(x) = x'Sx = \epsilon(2x_1x_3 + x_2^2) + \delta(2x_4x_6 + x_5^2) + f(x) \qquad \text{and}$$

(3)

$$G(x) = x'Tx = \epsilon(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3) + \delta(\mu(x_4x_6 + x_5^2) + 2x_5x_6) + g(x) \ ,$$

where f and g are quadratic forms not involving $x_1, \dots, x_6$ and $\epsilon, \delta = \pm 1$ independently from the canonical pair form.

Now $e_2, e_5 \notin Q_S \cap Q_T$. And for these indices define the vectors $y_2 = -\frac{1}{2}e_1 - \delta\epsilon e_2 + e_3 - \frac{1}{2}e_4 + e_5 + e_6$ and $y_5 = -\frac{1}{2}e_1 - \delta\epsilon e_2 + e_3 + \frac{1}{2}e_4 - e_5 - e_6$. They are lin. ind. and satisfy $F(y_i) = G(y_i) = 0$ in (3). For $i > 6$ such that $e_i \notin Q_S \cap Q_T$ we define

$$y_i = \alpha_i e_1 + \beta_i e_2 + e_3 + e_i \ ,$$

where β_i is chosen such that

$$2\epsilon\beta_i + g(e_i) - \lambda f(e_i) = 0 \ ,$$

and α_i is such that

$$F(y_i) = \epsilon(2\alpha_i + \beta_i^2) + f(e_i) = 0 \ .$$

Then $G(y_i) = 0$, too.

This completes ii).

iii): Here again the 3-dimensional Jordan block has to be of type (A): $J(\lambda, 3)$, while the 2-dimensional block can be of either type. Let for $x = (x_1, \ldots, x_n)$,

$$F(x) = x'Sx' = \epsilon(2x_1x_3 + x_2^2) + \delta(2x_4x_5) + f(x)$$

and
$$G(x) = x'Tx = \epsilon(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3) +$$

$$+ \delta \begin{Bmatrix} (2\mu x_4x_5 + x_5^2) \\ (2ax_4x_5 + b(x_4^2 - x_5^2)) \end{Bmatrix} + g(x) \qquad \begin{matrix} \text{in case of (A)} \\ \text{in case of (B)} \end{matrix}$$

where $\delta, \epsilon = \pm 1$ from the canonical pair form and f and g do not involve the first five components. If the 2-dimensional Jordan block in question is of type (A), then for $i \leq 5$ we have $e_i \notin Q_S \cap Q_T$ exactly for $i = 2, 5$, while for a Jordan block of type (B) those indices are $i = 2, 4, 5$.

In case of (A) define

$$y_2 = \delta\epsilon e_1 + e_2 - \frac{\delta\epsilon}{2} e_3 + e_5$$

$$y_5 = \delta\epsilon e_1 + e_2 - \frac{\delta\epsilon}{2} e_3 - e_5$$

and one has $y_2, y_5 \in Q_S \cap Q_T$.

In case of a 2-dimensional block $J(a, b, 2)$, $b \neq 0$ of type (B), define

$$y_2 = -\frac{\epsilon}{b} e_1 + e_2 + \frac{b}{2} \epsilon e_3 + e_5$$

$$y_4 = +\frac{\epsilon}{b} e_1 + e_2 - \frac{b}{2} \epsilon e_3 + e_4$$

$$y_5 = -\frac{\epsilon}{b} e_1 + e_2 + \frac{b}{2} \epsilon e_3 - e_5 .$$

Then $y_2, y_4, y_5 \in Q_S \cap Q_T$.

For $i > 5$ such that $e_i \notin Q_S \cap Q_T$, define $y_i = \alpha_i e_1 + \beta_i e_2 + e_3 + e_i$.

where $\alpha_i, \beta_i \in \mathbb{R}$ are such that $2\epsilon\beta_i + g(e_i) - \lambda f(e_i) = 0$ and $F(y_i) = 0$. This concludes part iii).

<u>iv)</u>: Here we have $F(x) = x'Sx = \epsilon(2x_1x_3 + x_2^2) + f(x)$ and $G(x) = x'Tx = \epsilon(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3) + g(x)$ and $F(x) = G(x) = 0$ iff

$$(4) \qquad \epsilon 2x_2x_3 + g(x) - \lambda f(x) = 0 \quad \text{and} \quad F(x) = 0 .$$

By assumption all but the first Jordan block $J(\lambda, 3)$ in $S^{-1}T$ are 1-dimensional blocks $J(\mu_i, 1)$. We assumed $n > 3$, so there exists an $i_0 > 3$ such that $g(e_{i_0}) - \lambda f(e_{i_0}) \neq 0$, for $g(e_i) - \lambda f(e_i) = \pm(\mu_i - \lambda) = 0$ for all $i > 3$ contradicts our assumption.

Now $e_2 \notin Q_S \cap Q_T$ and we define $y_2 = \alpha_2 e_1 + \beta_2 e_2 + e_3 + e_{i_0}$, where $\beta_2 \neq 0$ is such that $2\epsilon\beta_2 + g(e_{i_0}) - \lambda f(e_{i_0}) = 0$ and α_2 is such that $F(y_2) = 0$. For all $i > 3$ we have $e_i \notin Q_S \cap Q_T$ and we define $y_{i_0} = -\alpha_2 e_1 - \beta_2 e_2 - e_3 + e_{i_0}$ and $y_i = \alpha_i e_1 + \beta_i e_2 + e_3 + e_i$ for all other $i > 3$, where the α's and β's are chosen such that (4) holds for all y_i. These n vectors y_i are lin. ind. .

<u>v)</u>: Now only v) remains to be proved. Let us first assume that the two 2-dimensional Jordan blocks in question are both of type (A): $J(\lambda, 2)$, $J(\mu, 2)$, where by assumption $\lambda \neq \mu$. Then $F(x) = x'Sx = \epsilon 2x_1x_2 + \delta 2x_3x_4 + f(x)$ and $G(x) = x'Tx = \epsilon(2\lambda x_1x_2 + x_2^2) + \delta(2\mu x_3x_4 + x_4^2) + g(x)$ where $\epsilon, \delta = \pm 1$ and f and g do not involve the first four components of x. Then $F(x) = G(x) = 0$ is equivalent to

$$(5) \quad F(x) = 0 \quad \text{and} \quad 2\delta(\mu - \lambda)x_3x_4 + \epsilon x_2^2 + \delta x_4^2 + g(x) - \lambda f(x) = 0 .$$

Now if $e_i \notin Q_S \cap Q_T$, then $i = 2$ or $i = 4$, unless $i > 4$. We define

$$y_2 = \alpha e_1 + 2e_2 + \beta e_3 - e_4$$

$$y_4 = \alpha e_1 + 2e_2 + \beta e_3 + e_4$$

and

$$y_i = \alpha_i e_1 + \gamma_i e_2 + \beta_i e_3 + e_4 + e_i$$

for all $i > 4$ with $e_i \notin Q_S \cap Q_T$. Here $\gamma_i \neq 0$ are chosen such that $\epsilon\gamma_i^2 + \delta + g(e_i) - \lambda f(e_i) \neq 0$ while the α's and β's are chosen such that (5) holds.

Next assume, the two 2-dimensional blocks are both of type (B): $J(a,b,2)$, $J(c,d,2)$ where $b, c \neq 0$.

Then $F(x)$ is as above with $\epsilon = \delta = 1$ while

$$G(x) = x'Tx = 2ax_1x_2 + 2cx_3x_4 + b(x_1^2 - x_2^2) + d(x_3^2 - x_4^2) + g(x) \ .$$

and $F(x) = G(x) = 0$ is equivalent to

$$(6) \quad F(x) = 0 \quad \text{and} \quad 2(c-a)x_3x_4 + b(x_1^2 - x_2^2) + d(x_3^2 - x_4^2) + g(x) - af(x) = 0 \ .$$

Here we have $e_i \notin Q_S \cap Q_T$ for all $i \leq 4$.

If $bd > 0$ we define the following four lin. ind. vectors

$$y_1 = \alpha e_1 + \beta e_4 \ , \qquad y_2 = \alpha e_1 - \beta e_4 \ ,$$

$$y_3 = \alpha e_2 + \beta e_3 \ , \qquad y_4 = \alpha e_2 - \beta e_3 \ ,$$

where $\alpha, \beta \neq 0$ are such that $b\alpha^2 - d\beta^2 = 0$ and thus (6) holds for all y_i, $i \leq 4$.

If $bd < 0$, we define y_i as follows:

$$y_1 = \alpha e_1 + \beta e_3 \ , \qquad y_2 = \alpha e_1 - \beta e_3 \ ,$$

$$y_3 = \alpha e_2 + \beta e_4 \ , \qquad y_4 = \alpha e_2 - \beta e_4 \ ,$$

where $\alpha, \beta \neq 0$ satisfy $b\alpha^2 + d\beta^2 = 0$ such that all four y_i satisfy (6) again.

For indices $i > 4$ such that $e_i \notin Q_S \cap Q_T$ we define the corresponding vector y_i as follows:

If $f(e_i) = 0$ and $bd > 0$, let $y_i = \alpha_i e_1 + \beta_i e_4 + e_i$, where α_i, β_i are chosen such that $b\alpha_i^2 - d\beta_i^2 = -g(e_i)$. If $f(e_i) = 0$ and $bd < 0$, let $y_i = \alpha_i e_1 + \beta_i e_3 + e_i$, where $\alpha_i, \beta_i \in \mathbb{R}$ such that $b\alpha_i^2 + d\beta_i^2 = -g(e_i)$. If $f(e_i) \neq 0$ and $g(e_i) - af(e_i) = 0$, then let $y_i = \alpha_i e_1 + \beta_i e_2 + e_i$ where $|\alpha_i| = |\beta_i|$ such that y_i satisfies (6). If $f(e_i) \neq 0$ and $(g(e_i) - af(e_i))d > 0$, let $y_i = \alpha_i e_1 + \beta_i e_2 + \gamma_i e_4 + e_i$, where $|\alpha_i| = |\beta_i|$ and γ_i are chosen such that (6) holds. If $f(e_i) \neq 0$ and $(g(e_i) - af(e_i))d < 0$, let $y_i = \alpha_i e_1 + \beta_i e_2 + \gamma_i e_3 + e_i$, with α_i, β_i, γ_i chosen to satisfy (6).

Finally we prove v) for a Jordan block of type (A) and one of type (B): $J(\lambda, 2)$, $J(a, b, 2)$. Then $F(x)$ is as above with $\epsilon = \pm 1$, $\delta = 1$ while $G(x) = x'Tx = \epsilon(2\lambda x_1 x_2 + x_2^2) + 2ax_3x_4 + b(x_3^2 - x_4^2) + g(x)$ where $g(x)$ does not involve $x_1, \dots, x_4$. And $F(x) = G(x) = 0$ is equivalent to:

$$(7) \quad F(x) = 0 \quad \text{and} \quad 2(a - \epsilon\lambda)x_3x_4 + \epsilon x_2^2 + b(x_3^2 - x_4^2) + g(x) - \lambda f(x) = 0 \ .$$

If $e_i \notin Q_S \cap Q_T$, then $i = 2, 3, 4$ or $i > 4$. We define y_2 and y_3 first:

$$y_2 = e_2 + \beta e_3 + \gamma e_4$$
$$y_3 = e_2 - \beta e_3 - \gamma e_4 \ ,$$

where $\beta = \sqrt{-\epsilon/b}$, $\gamma = 0$, if $\epsilon \cdot b < 0$ and $\beta = 0$, $\gamma = \sqrt{\epsilon/b}$, if $\epsilon b > 0$.

Then e_1, y_2 and y_3 are all lin. ind. and satisfy (7).

If y_4 has all of its first four components nonzeros it will be lin. indep. of e_1, y_2, y_3 and all e_i for $i > 4$. So let $y_4 = \alpha e_1 + \beta e_2 + \gamma e_3 + \eta e_4$ where α, β, γ, η are chosen as follows:

If $a - \epsilon\lambda = 0$, take $\gamma = 1$, $\eta = 2$, $\beta = \sqrt{3b\epsilon}$, if $\epsilon b > 0$ and $\alpha \neq 0$ such that $F(y_4) = 0$; but if $\epsilon b < 0$, choose $\gamma = 2$, $\eta = 1$, $\beta = \sqrt{-3b\epsilon}$ and α as above, and y_4 satisfies (7).

If $a - \epsilon\lambda \neq 0$, choose $\eta \neq 0$, $\gamma = \frac{1}{\eta}$ such that

$$2(a - \epsilon\lambda) + b\left(\frac{1}{\eta^2} - \eta^2\right) < 0 \qquad \text{if } \epsilon = 1 \qquad \text{and}$$

$$2(a - \epsilon\lambda) + b\left(\frac{1}{\eta^2} - \eta^2\right) > 0 \qquad \text{if } \epsilon = -1 .$$

Then choose $\beta \neq 0$ such that the second equation in (7) holds and after letting $\alpha = -\epsilon/\beta$ the vector y_4 again satisfies (7). For $i > 4$ define $y_i = \alpha e_1 + e_2 + \beta e_3 + \gamma e_4 + e_i$ where $\alpha \in \mathbb{R}$ and either $\beta = 0$ or $\gamma = 0$ as before in such a way that (7) holds for each y_i, $i > 4$. This completes the proof of Theorem 1. ■

We now go on to prove Theorem 2:

<u>Proof:</u> (Theorem 2) We use the notation of the previous proof:

<u>vi), vi)a):</u> Let vi) or vi)a) hold. Then the 3 dimensional Jordan block is of type (A): $J(\lambda, 3)$. And we have with $x = (x_1, \ldots, x_n)$

$$F(x) = x'Sx = \epsilon_1(2x_1x_3 + x_2^2) + \sum_{i=4}^{n} \epsilon_i x_i^2$$

and

$$G(x) = x'Tx = \epsilon_1(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3) + \sum_{i=4}^{n} \epsilon_i \lambda x_i^2 \quad ,$$

where $\epsilon_i = \pm 1$. Hence $F(x) = G(x) = 0$ is equivalent to $F(x) = 0$ and $x_2 x_3 = 0$.

If all ϵ_i are the same, then, since inertia $E_{3 \times 3} = (2,1,0)$, we have inertia $S = (n-1,1,0)$ or $(1,n-1,0)$ and vi)a) would hold.

But let us first assume vi) holds. Then for some $\ell \geq 4$ we must have $\epsilon_1 \cdot \epsilon_\ell < 0$. Clearly $e_1, e_3 \in Q_S \cap Q_T$ and for the other indices we define:

$$y_2 = e_2 + e_\ell$$

$$y_i = \frac{\epsilon_1 \epsilon_i}{2} e_1 - e_3 + e_i \qquad \text{for } i \geq 4.$$

Then e_1, e_3, y_2 and y_i $(i \geq 4)$ are in $Q_S \cap Q_T$ and are lin. ind. .

If vi(a) holds, then $F(x) = 0$ and $x_2 x_3 = 0$ implies $x_2 = 0$, such that we cannot find a vector $y_2 \in Q_S \cap Q_T$ with a nonzero second component. Hence $Q_S \cap Q_T$ contains at most $(n-1)$ lin. ind. vectors. But e_1, e_3, $y_i (i \geq 4)$ defined above are linearly independent and belong to $Q_S \cap Q_T$. This proves vi) and vi)a).

<u>vii), vii)a), vii)b)</u>: We define

$$F(x) = x'Sx = 2 \sum_{i=1}^{k} \epsilon_i x_{2i-1} x_{2i} + \sum_{i=2k+1}^{n} \epsilon_i x_i^2 \qquad \text{and}$$

$$G(x) = x'Tx = 2\lambda \sum_{i=1}^{k} \epsilon_i x_{2i-1} x_{2i} + \sum_{i=1}^{k} \epsilon_i x_{2i}^2 + \sum_{i=2k+1}^{n} \epsilon_i \mu_i x_i^2 ,$$

where $\epsilon_i = \pm 1$. Thus $F(x) = G(x) = 0$ is equivalent to $F(x) = 0$ and

$$(8) \qquad \sum_{i=1}^{k} \epsilon_i x_{2i}^2 + \sum_{i=2k+1}^{n} \epsilon_i (\mu_i - \lambda) x_i^2 = 0 .$$

Assuming vii) holds, then the quadratic form in (8) is indefinite, so there must exist an index ℓ such that

$$\epsilon_1 \epsilon_\ell (\mu_\ell - \lambda) < 0 \qquad \text{for some } \ell \geq 2k+1$$

or such that $\quad \epsilon_1 \epsilon_{\ell/2} < 0 \qquad$ for even $\ell \leq 2k$.

We define
$$y_1 = e_1$$
$$y_2 = \alpha_2 e_1 + \beta_2 e_2 + e_\ell \quad \text{for } \beta_2 \neq 0 ,$$
$$y_\ell = \alpha_2 e_1 + \beta_2 e_2 - e_\ell \quad \text{and}$$
$$y_i = \alpha_i e_1 + \beta_i e_2 + \gamma_i e_\ell + e_i \qquad \text{for } i \neq 1, 2, \ell, \beta_i \neq 0 ,$$

where β_i and γ_i are chosen such that y_i satisfies (8), while α_i are chosen such that $F(y_i) = 0$. This proves vii).

To prove vii)a) and vii)b) assume now that the quadratic form in (8) is semidefinite and that the symmetric matrix corresponding to the quadratic form in (8) has rank $n - k - r$, where the r zeros among the $\epsilon_i(\mu_i - \lambda)$ occur for the indices $i = 2k+1, \ldots, 2k+r$. Then by Lemma 1 the only unit vectors satisfying (8) are $e_1, e_3, \ldots, e_{2k-1}, e_{2k+1}, \ldots, e_{2k+r}$. And clearly $e_1, e_3, \ldots, e_{2k-1} \in Q_S \cap Q_T$ in either of the cases vii)a) or vii)b).

In case of vii)a) exactly $e_1, e_3, \ldots, e_{2k-1} \in Q_S \cap Q_T$, because the quadratic form in r variables

$$\sum_{i=2k+1}^{2k+r} \epsilon_i x_i^2$$

appearing in F is definite and $F(x) \neq 0$ for all $x \in \langle e_{2k+1}, \ldots, e_{2k+r} \rangle$. So in this case we conclude that $Q_S \cap Q_T$ contains a maximum of k lin. ind. vectors.

In case of vii)b)

$$\sum_{i=2k+1}^{2k+r} \epsilon_i x_i^2 \tag{9}$$

is an indefinite quadratic form and besides $e_1, e_3, \dots, e_{2k-1}$, r more lin. indep. vectors $y_1, \dots, y_r$ can be found that satisfy $F(x) = 0$ and (8): Choose y_i as follows. Since (9) is indefinite, there are indices $2k < \ell, j \le 2k+r$ with $\epsilon_\ell = 1$, $\epsilon_j = -1$. Then define for $2k < i \le 2k+r$, $i \ne \ell, j$:

$$y_i = e_j + e_i \qquad \text{if } F(e_i) = 1$$

and

$$y_i = e_\ell + e_i \qquad \text{if } F(e_i) = -1$$

while we set

$$y_\ell = e_j + e_\ell$$

and

$$y_j = e_j - e_\ell \;.$$

This proves vii)b).

<u>viii)</u>: Here we define

$$F(x) = x'Sx = 2x_1x_2 + \sum_{i=3}^{n} \epsilon_i x_i^2 \qquad \text{and}$$

$$G(x) = x'Tx = 2ax_1x_2 + b(x_1^2 - x_2^2) + \sum_{i=3}^{n} \epsilon_i \mu_i x_i^2 \;.$$

So $F(x) = G(x) = 0$ is equivalent to

$$F(x) = 0 \qquad \text{and}$$

$$b(x_1^2 - x_2^2) + \sum_{i=3}^{n} \epsilon_i(\mu_i - a)x_i^2 = 0 \;. \tag{10}$$

Now unless viii)a) holds, not all μ_i or* not all ϵ_i are the same for $i \geq 3$. So for some pair of indices $i, j \geq 3$ we must have $\mu_i \neq \mu_j$ or* $\epsilon_i \neq \epsilon_j$. After a suitable index permutation we may start the proof assuming that $\mu_3 \neq \mu_4$ or* $\epsilon_3 \neq \epsilon_4$ already.

We define $y_1 = \alpha_3 e_1 + \beta_3 e_2 - e_3$, $y_2 = \alpha_4 e_1 + \beta_4 e_2 - e_3$ and $y_i = \alpha_i e_1 + \beta_i e_2 + e_i$ for $i \geq 3$, where the α_i, β_i are chosen to satisfy

$$(11) \quad b(\alpha_i^2 - \beta_i^2) + \epsilon_i(\mu_i - a) = 0 \quad \text{and} \quad 2\alpha_i\beta_i + \epsilon_i = 0 \quad \text{for each } i .$$

Then the vectors y_i for $i \leq n$ are lin. ind. iff

$$\det(y_1, \ldots, y_n) = \begin{vmatrix} \alpha_3 & \beta_3 & -1 & 0 \\ \alpha_4 & \beta_4 & 0 & -1 \\ \alpha_3 & \beta_3 & 1 & 0 \\ \alpha_4 & \beta_4 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2\alpha_3 & 2\beta_3 & 0 & 0 \\ 2\alpha_4 & 2\beta_4 & 0 & 0 \\ \alpha_3 & \beta_3 & 1 & 0 \\ \alpha_4 & \beta_4 & 0 & 1 \end{vmatrix} \neq 0 .$$

So the n vectors y_i are lin. dep. iff for the 2-vectors we have

$$(12) \qquad (\alpha_3, \beta_3) = d(\alpha_4, \beta_4)$$

for some real coefficient d. Now (12) holds only if $d = \pm 1$, for (12) implies $\alpha_3\beta_3 = d^2\alpha_4\beta_4$ and we know $\alpha_3\beta_3 = -\frac{\epsilon_3}{2}$, since $F(y_3) = 0$ and $\alpha_4\beta_4 = -\frac{\epsilon_4}{2}$, since $F(y_4) = 0$ and thus $d = \pm 1$. The second equation in (10) gives for y_1, y_2, respectively: $b(\alpha_3^2 - \beta_3^2) + \epsilon_3(\mu_3 - a) = 0 = b(\alpha_4^2 - \beta_4^2) + \epsilon_4(\mu_4 - a)$. Since $d = \pm 1$ in (12) we get $\epsilon_3(\mu_3 - a) = \epsilon_4(\mu_4 - a)$. Now if $\mu_3 = \mu_4$, then by assumption $\epsilon_3 \neq \epsilon_4$. If $\mu_3 \neq \mu_4$, then we again must have $\epsilon_3 \neq \epsilon_4$. In either case the second equation in (11) gives $2\alpha_3\beta_3 = -\epsilon_3 = \epsilon_4 = -2\alpha_4\beta_4$, contradicting $d = \pm 1$ in (12). Thus we found that the n vectors y_i in

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*This "or" does not mean "either... or".

$Q_S \cap Q_T$ are lin. ind. in case of viii).

If viii(a) holds, then $\mu_i = \mu$ and $\epsilon_i = \epsilon$ for all i. We define for $x = (x_1, \ldots, x_n)$

$$F(x) = x'Sx = 2x_1x_2 + \epsilon \sum_{i=3}^{n} x_i^2 \qquad \text{and}$$

(13)

$$G(x) = x'Tx = 2x_1x_2 + b(x_1^2 - x_2^2) + \epsilon\mu \sum_{i=3}^{n} x_i^2 .$$

And $F(x) = G(x) = 0$ is equivalent to

$$F(x) = 0 \qquad \text{and}$$

(14)

$$b(x_1^2 - x_2^2) + \epsilon(\mu - a) \sum_{i=3}^{n} x_i^2 = 0 .$$

We define the following $n - 1$ lin. ind. vectors

$$y_i = \alpha e_1 + \beta e_2 + e_i \qquad \text{for } i \geq 3,$$
$$y_2 = \alpha e_1 + \beta e_2 - e_3 ,$$

where α, β are chosen such that $F(y_i) = G(y_i) = 0$ for all i. Such numbers α, β exist, since they can be chosen as the intersection of the following two hyperbola in α, β:

$$2\alpha\beta + \epsilon = 0 ; \qquad \alpha^2 - \beta^2 = -\frac{\epsilon}{b}(\mu - a) .$$

Now any $w = (\beta_1, \ldots, \beta_n) \in Q_S \cap Q_T$ satisfies (14). We are going to show that if $0 \neq w \in Q_S \cap Q_T$ then the 2-vector (β_1, β_2) can be written as $\pm \|\hat{x}\| (\alpha, \beta)$ with α, β as chosen above and $\hat{x} = (0, 0, \beta_3, \ldots, \beta_n)$.

Now if $\|\hat{x}\| = 0$, i.e., $\beta_i = 0$ for all $i \geq 3$, then by (14) $\beta_1\beta_2 = 0 = \beta_1^2 - \beta_2^2$ so that $w = 0$. If $\beta_1 = 0$, then by (13) we get $w = 0$.

So if $w \neq 0$ belongs to $Q_S \cap Q_T$, then $||\hat{x}|| \neq 0$ and we define d as $d = \beta_1/\alpha$ with α as introduced above. Using the equations $F(w) = F(y_i) = 0$ we get $2\alpha\beta = -\epsilon = 2\beta_1\beta_2/||\hat{x}||^2$ and hence $\beta_2 = ||\hat{x}||^2\beta/d$. The second equation in (14), written out for y_i and w, reads like

$$b(\alpha^2 - \beta^2) + \epsilon(\mu - a) = 0 = b(d^2\alpha^2 - ||\hat{x}||^4\beta^2/d^2) + \epsilon(\mu - a)||\hat{x}||^2 ,$$

and hence

$$\alpha^2 - \beta^2 = d^2\alpha^2/||\hat{x}||^2 - ||\hat{x}||^2\beta^2/d^2 \text{ or } d^4\alpha^2 + d^2||\hat{x}||^2(\beta^2 - \alpha^2) - \beta^2||\hat{x}||^4 = 0 .$$

This last equation in d has only two real roots, namely $d = \pm ||\hat{x}||$.
Hence $\beta_2 = \pm ||\hat{x}|| \beta$, while $\beta_1 = \pm ||\hat{x}|| \alpha$.
So the equation $w = (\beta_1, \ldots, \beta_n) = d(\alpha e_1 + \beta e_2) + (0, 0, \beta_3, \ldots, \beta_n) = \sum_{i=2}^{n} \eta_i y_i$ can be solved for real coefficients η_i, namely by $\eta_i = \beta_i$ for $i > 3$,

$$\eta_2 = \left(d - \sum_{i=3}^{n} \beta_i\right)/2$$

and $\eta_3 = \beta_3 + \eta_2$, where

$$d = \pm \left(\sum_{i=3}^{n} \beta_i^2\right)^{\frac{1}{2}}$$

as we have seen above.
So every $w \in Q_S \cap Q_T$ is lin. dep. of $y_2, \ldots, y_n$ and in this case $n-1$ is the maximal number of lin. ind. vectors in $Q_S \cap Q_T$. This proves viii)a).

ix): It only remains to show ix): Let S and T be simultaneously diagonalizable.
Assume S is positive definite, then $Q_S = \{0\}$ and hence for any symmetric T we have $Q_S \cap Q_T = \{0\}$, hence the case $k = 0$ occurs.

If $S = \mathrm{diag}(1,-1,\ldots,-1,1,\ldots,1)$ and $T = \mathrm{diag}(\lambda,-\lambda,\ldots,-\lambda,0,\ldots,0)$ with $(\ell-1)$ numbers -1 and $-\lambda$ appearing on the diagonals of S and T, then $Q_S \cap Q_T$ contains a maximum of ℓ lin. indep. vectors for $\lambda \neq 0$, $2 \leq \ell \leq n$ as can be seen by inspection. Finally if $x \in Q_S \cap Q_T$, then x can be written as $x = \alpha e_\ell + \beta e_k + y$ for two indices ℓ, k, nonzero constants α, β and y orthogonal to e_ℓ and e_k, because x has to satisfy

$$F(x) = x'Sx = \sum_{i=1}^{n} \epsilon_i x_i^2 = 0 \qquad \text{and}$$

$$G(x) = x'Tx = \sum_{i=1}^{n} \epsilon_i \mu_i x_i^2 = 0, \qquad \text{with } \epsilon_i = \pm 1 .$$

But then $\hat{x} = \alpha e_\ell - \beta e_k + y \in Q_S \cap Q_T$ as well and x and $\hat{x}$ are lin. indep. So in case ix) $Q_S \cap Q_T$ cannot contain just one vector and its multiples.

This proves Theorem 2. ∎

Next we treat nonsingular pairs of real symmetric matrices that have dimensions 2 or 3.

<u>Theorem 3</u>: Let S, T be a nonsingular pair of r. s. matrices of dimension n. Assume that n = 2 or 3. Let the Roman numerals vi),,,. viii) denote the various cases of Theorem 2.

If vii) holds, then $Q_S \cap Q_T$ contains n lin. indep. vectors.

If vi)a) or viii)a) (with n = 3) holds, then $Q_S \cap Q_T$ contains a maximum of n - 1 lin. indep. vectors.

If vii)a) holds, then $Q_S \cap Q_T$ contains a maximum of k lin. indep. vectors, where k is defined as in Theorem 2.

If viii)a) holds with $n = 2$, then $Q_S \cap Q_T = \{0\}$.

Proof: In view of Lemma 2 we can again assume that S and T are already in canonical pair form.

a) Let $n = 3$: If $J = S^{-1}T$ contains just one 3-dimensional block $J(\lambda, 3)$, then inertia $S = (2,1,0)$ or $(1,2,0)$ and we have condition vi)a).

Then $F(x) = x'Sx = \epsilon(2x_1x_3 + x_2^2)$ and $G(x) = x'Tx = \epsilon(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3)$ with $\epsilon = \pm 1$. Hence the only vectors x satisfying $F(x) = G(x) = 0$ are multiples of e_1 and of e_3. Hence there are maximally 2 lin. ind. vectors in $Q_S \cap Q_T$.

If $S^{-1}T$ has a complex root, then we have case viii)a) and the proof of Theorem 2 viii)a) carries over.

If $S^{-1}T$ satisfies condition vii), then we have for $\epsilon_i = \pm 1$, $F(x) = x'Sx = \epsilon_1(2x_1x_2) + \epsilon_3x_3^2$ and $G(X) = x'Tx = \epsilon_1(\lambda 2x_1x_2 + x_2^2) + \epsilon_3\mu x_3^2$ and thus $F(x) = G(x) = 0$ is equivalent to

$$(15) \qquad F(x) = 0 \quad \text{and} \quad \epsilon_1 x_2^2 + \epsilon_3(\mu - \lambda)x_3^2 = 0 \ .$$

If $\lambda = \mu$, then only multiples of e_1 are in $Q_S \cap Q_T$ and if $\lambda \neq \mu$, but $\epsilon_1 \epsilon_3(\mu - \lambda) > 0$, then again only multiples of e_1 are in $Q_S \cap Q_T$. Now Condition vii)a) encompasses exactly these two cases, hence if vii)a) holds, then $Q_S \cap Q_T$ is just a one dimensional space.

If vii) holds, i.e., $\lambda \neq \mu$ and $\epsilon_1 \epsilon_3(\mu - \lambda) < 0$, then we define

$$\begin{aligned} y_1 &= e_1 \\ y_2 &= \alpha e_1 + \beta e_2 - e_3 \qquad \text{and} \\ y_3 &= \alpha e_1 + \beta e_2 + e_3 \qquad \text{where } \alpha, \beta \neq 0 \end{aligned}$$

are such that y_2, y_3 satisfy (15).

b) If $n = 2$, we have in case of just one Jordan block $J(\lambda, 2)$ in $J = S^{-1}T$: $F(x) = x'Sx = \epsilon(2x_1x_2)$ and $G(x) = x'Tx = \epsilon(2\lambda x_1x_2 + x_2^2)$ for $\epsilon = \pm 1$. So $F(x) = G(x) = 0$ holds iff $x = \alpha e_1$. Hence vii)a) is proved.
In case of viii)a) for a Jordan block $J(a, b, 2)$ of type (B), we have $F(x) = 2x_1x_2$ and $G(x) = 2ax_1x_2 + b(x_1^2 - x_2^2)$. And hence $F(x) = G(x) = 0$ holds iff $x = 0$. ■

Let S and T be a nonsingular pair of r. s. matrices of dimension greater than 2. In Theorems 1, 2, and 3 we have seen how the real Jordan normal form of $S^{-1}T$ determines the maximal number of lin. indep. vectors in $Q_S \cap Q_T$. Since we have dealt with all possible real Jordan normal forms, we can reverse the argument and get the following:

Theorem 4: Let S and T be a nonsingular pair of r. s. $n \times n$ matrices where $n > 2$.
Let $m = \max\{\ell \mid \text{there exist } \ell \text{ lin. indep. vectors in } Q_S \cap Q_T\}$.
Let the Roman numerals i), ..., viii) denote the various conditions in Theorem 2.

If $m = 0$, then S and T can be simultaneously diagonalized by a real congruence transformation.

If $m = 1$, then vii)a) holds with $k = 1$.

If $2 \leq m \leq [n/2]$, then vii)a) holds with $k = m$ or vii)b) holds with $r = m - k$ for S and T, or S and T can be diagonalized simultaneously.

If $[n/2] < m < n - 1$, then vii)b) holds with $r = m - k$ where $k \leq [n/2]$ for S and T, or S and T can be diagonalized simultaneously.

If $m = n - 1$, then vi)a) or viii)a) or vii)b) holds with $r = m - k$, where $k \leq [n/2]$ for S and T, or S and T can be diagonalized simultaneously.

If $m = n$, then i), . . . , viii) or vii)b) holds with $r = m - k$ where $k \leq [n/2]$ for S and T, or S and T can be diagonalized simultaneously by a real congruence transformation.

Here [ ] denotes the greatest integer function.

Note that Greub and Milnor's Theorem, [16], p. 256, is a special case of Theorem 4 if $m = 0$.

If m, the maximal number of lin. ind. vectors simultaneously annihilated by two quadratic forms $x'Sx$, $x'Tx$, lies properly between 1 and $n - 1$, and if we can rule out the cases vii)a) or vii)b), then we can conclude that S and T are simultaneously diagonalizable. For example, here are two such conditions that make vii)a) or vii)b) impossible to happen:

Corollary 1: Let S and T be a nonsingular part of r. s. $n \times n$ matrices. Let $m = \max\{\ell \mid \text{there exist } \ell \text{ lin. ind. vectors in } Q_S \cap Q_T\}$.
Assume $1 < m < n - 1$.
If a) $S^{-1}T$ is nonderogatory, or
b) for every eigenvalue λ of $S^{-1}T$ the number of associated lin. ind. eigenvectors is smaller than half the algebraic multiplicity of λ, unless both are the same, then S and T can be diagonalized simultaneously by a real congruence transformation.

Nonderogatory matrices were defined in Definition 2, Chapter II.

NOTATIONS

$A = (a_{ij})_{n \times k}$	matrix A consisting of n rows and k columns
$A^t = A'$	transpose of A
$I_{n \times n}, I_n$	$n \times n$ identity matrix
$E_{n \times n}$	matrix $\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}_{n \times n}$
dim A	size of a square matrix A
$J(\lambda, k)$	Jordan block of type (A) of dimension k for eigenvalue λ
$J(a, b, k)$	Jordan block of type (B) and dimension k for eigenvalue $a + bi$
$A = (A_{ij})$	block matrix A composed of blocks A_{ij}
$A = \mathrm{diag}(A_1, \ldots, A_k)$	block diagonal matrix A with diagonal blocks A_i
$C(\lambda) = \mathrm{diag}(J_1, \ldots, J_k)$	Jordan chain of length k
$e_{\ell, m}$	$n \times n$ matrix with a one in position (ℓ, m) and zeros elsewhere
$D = \mathrm{diag}(d_1, \ldots, d_n)$	$n \times n$ matrix with diagonal elements $d_1, \ldots, d_n$
basis	set of linearly independent generating vectors
$\begin{pmatrix} a & \\ & \end{pmatrix}_{n \times n} \leftarrow p$	a appears in the p^{th} row of an $n \times n$ matrix
$[x]$	greatest integer function
$P(S, T)$	pencil generated by two linearly independent real symmetric matrices S and T
Q_S	quadratic hypersurface generated by a real symmetric matrix S
in S	inertia of a real symmetric matrix S

e_i	i^{th} unit vector
$\langle x, y, \ldots \rangle$	linear span of the vectors $x, y, \ldots$

ABBREVIATIONS

r. s.	real symmetric
lin. ind.	linearly independent
lin. dep.	linearly dependent
WLOG	without loss of generality
iff	if and only if

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