STRUCTURE OF COMMUTATIVE NORMED RINGS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1950

The author wishes to express his profound gratitude to Professors Karlin and Bohnenblust for the kind encouragement and generous assistance they have given during the preparation of this thesis.

ABSTRACT

In a complex commutative normed ring the unit sphere at the origin has a vertex at the unit element. If the ring is finite dimensional, the radical translated to the unit element intersects this sphere only at the unit element.

A finite dimensional ring containing an element of nilpotency degree equal to the dimension of the radical is a direct sum of a ring with a scalar product and a ring with a convolution product. Using this decomposition the conjugate space is made into a normed ring, and a duality theory is obtained.

General properties are given for completely continuous and weakly completely continuous elements of various types of rings.

In a star ring, if uniform convergence with respect to the maximal ideals implies weak convergence, then the square of a weakly completely continuous operator is completely continuous. Some of the consequences of this result are: (a) no infinite dimensional ring of this type is reflexive as a Banach space, (b) all weakly completely continuous elements of infinite dimensional indecomposable rings of this type lie in the radical, (c) a new proof of Dunford's theorem that the square of a weakly completely continuous operator from L into L is completely continuous is given.

PREFACE

This thesis contains an investigation of three general problems in the theory of complex commutative normed rings. Such rings will be denoted by the letter A.

The first problem deals with the structure of the unit sphere of A. In a Banach space it is well known that the unit sphere is a closed, convex, centrally symmetric body. In the case of a normed ring an additional restriction is placed on the norm, and hence on the unit sphere, by the inequality $\| xy \| \leq \| x \| \| y \|$. It is shown in Part II that this implies that the unit sphere centered at the origin has a vertex at the unit element u. (The term vertex is explained at the beginning of that Part.) This result is then interpreted in terms of the derivative of the norm as used by Ascoli. (1;53) ‡

The second problem considered is that of defining a normed ring structure in the space A^* conjugate to the Banach space of A. The possibility of accomplishing this was strongly suggested by several facts. For example, there exist numerous similarities between results in normed ring theory and in the theory of normed linear lattices [e.g. see (2)]. If E is such a lattice there is a natural way of defining a normed linear lattice in E^*. (2;3) As another example consider the set of pairs of complex numbers (x,y). This can be made into a Banach space, say A, by defining $\| (x,y) \| = \text{Max}(|x|,|y|)$. Its conjugate

‡ The first number within the parentheses refers to a numbered reference work at the end of this paper, and the second number refers to the page number of that work.

space $A^* = \{(u,v)\}$ is also a Banach space if $\|(u,v)\| = \sup_{\|(x,y)\|=1} \|ux+vy\| = |u|+|v|$ is taken as norm in A^*. Now if products are defined in A and A^* respectively as $(x,y)(x',y') = (xx',yy')$ (scalar product) and $(u,v)(u',v') = (uu',uv'+u'v)$ (convolution product), A and A^* become normed rings. This shows a duality between scalar and convolution products. Furthermore A, of dimension $n = 2$, has $k = 2$ maximal ideals, whereas A^*, also of dimension $n = 2$, has $n-k+1$ maximal ideals. This problem is discussed in Part III, where a complete duality theory for n- dimensional normed rings gives a partial answer.

Thirdly, an analysis of certain classical Banach space operators defined over normed rings is given in Part IV. In particular completely continuous and weakly completely continuous operators of the type $T_x : T_x y = xy$ are studied. The elements x which generate T_x are called completely continuous and weakly completely continuous, respectively. Partial results on completely continuous elements have been obtained independently by M. Freundlich (3); however, results presented here are believed to be more complete. A general theorem on weakly completely continuous transformations is proved, and completely continuous elements are discussed as a special case.

For purposes of reference some definitions and theorems from the theory of normed rings are given in Part I.

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PART I

INTRODUCTION

1.1 <u>Definition</u>. A complex commutative normed ring A is a set of elements x, y, ... having the following properties:

(1) A is a complex Banach space.

(2) For each $x, y \in A$, the <u>product</u> xy is uniquely defined, lies in A, and has the properties:

$xy = yx$

$x(yz) = (xy)z$

$x(y+z) = xy+xz$

$x(\mu y) = \mu xy$, where μ is a complex number.

(3) A <u>unit element</u> u exists in A such that $ux = x$ for all $x \in A$.

(4) The <u>norm</u> $\| \ \|$ in A has the properties:

$\| xy \| \leq \| x \| \| y \|$

$\| u \| = 1$.

1.2 <u>Remarks</u>. The concept of complex commutative normed ring was first introduced in 1932 by A. D. Michal and R. S. Martin, who also considered the case where the scalar field is the real numbers. (4;69) Henceforth complex commutative normed rings as defined above will be referred to simply as rings. Sets that are rings only in the algebraic sense will be explicitly called algebraic rings.

The zero element of A is denoted by θ to distinguish it from the scalar 0.

If, as a Banach space, A is n-dimensional, the ring is called an n-dimensional ring and is denoted by A_n.

A is a <u>star ring</u> if for every element $x \in A$ there is an element $x^* \in A$ such that $x(M) = \overline{x^*(M)}$ for all maximal ideals M (see section 1.4 below).

1.3 <u>Topologies in</u> A. Two topologies will be used in A, the <u>strong</u> (or <u>norm</u>) <u>topology</u> and the <u>weak topology</u>. In the first topology a neighborhood of an element x_0 is the set $N(x_0; \mathcal{E})$ of all x such that $\| x - x_0 \| < \mathcal{E}$, where $\mathcal{E}$ is an arbitrary positive number. A sequence $\{x_n\}$ is convergent, or more precisely strongly convergent, if $\lim_{n,m \to \infty} \| x_n - x_m \|$ exists. In this terminology condition (4) of Definition 1.1, $\| xy \| \leqslant \| x \| \, \| y \|$, expresses the continuity in the strong topology of xy with respect to (x, y). Unless specific mention is made to the contrary, it is the strong topology which will be used in A.

In the weak topology a neighborhood $N(x_0; f_1, \ldots, f_m; \mathcal{E})$ of x_0 is the set of all x such that $| f_i(x - x_0) | < \mathcal{E}$, $i = 1, \ldots, m$, where $f_1, \ldots, f_m$ are m arbitrary complex linear functionals over A, and $\mathcal{E}$ is an arbitrary positive number. Since all functionals which will occur are complex linear functionals, they will be referred to simply as functionals. A sequence $\{x_n\}$ is weakly convergent if $\lim_{n \to \infty} f(x_n)$ exists for every functional f over A.

1.4 <u>Maximal Ideals</u>. A subset I of A is called an ideal if

(1) $I \neq \{\theta\}$, A, and

(2) $x, y \in I$ and $x', y' \in A$ imply $xx' + yy' \in I$.

A maximal ideal is an ideal not contained in any other ideal. It has been shown (5;8) that every ideal is contained in a maximal ideal. As a consequence, the existence of the inverse x^{-1} of x is equivalent to the statement that x belongs to no maximal ideal. If M is a maximal ideal, the quotient space A/M is isomorphic to the field of complex numbers. This establishes a many-to-one mapping of A onto the complex numbers: to each x is associated a complex number $\lambda = x(M)$ depending only on M. For a fixed M, $x(M)$ is a multiplicative functional over A. However, if x is fixed and M varies over the set of maximal ideals of A, $x(M)$ describes the spectrum of x [i.e., the set of all complex λ's for which $(\lambda u - x)^{-1}$ does not exist]. The relation between functionals and hyperplanes of A is as follows: the kernel K of $f(x)$ [i.e. the set of zeros of $f(x)$] is a hyperplane of A passing through Θ, and every hyperplane K through Θ uniquely determines a functional $f(x)$ over A such that $f(K) = 0$ and $f(u) = 1$. In particular there is a one-to-one correspondence between multiplicative functionals over A and maximal ideals.

1.5 Radical. The radical R is defined as the intersection of all maximal ideals. R is an ideal. It has been shown (5;10) that R is the set of all generalized nilpotent elements of A, namely elements x such that $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = 0$. Indeed, for all $x \in A$, $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \sup_M |x(M)|$. In particular, R contains all the nilpotent elements.

1.6 Norms. A norm $\| \|'$ is called admissible in A if A is a normed ring with respect to this norm, that is if

(1) $\| \|'$ makes A a Banach space,

(2) $\| xy \|' \leqslant \|x\|' \|y\|'$, and

(3) $\| u \|' = 1$.

An example of a non-admissible norm is the pseudo-norm $\|x\|_1 = \sup_M |x(M)|$, since for example $\|x\|_1 = 0$ does not imply $x = \Theta$. Two norms $\| \|'$ and $\| \|''$, defined over A, are called isomorphic if positive constants a, b exist such that for all $x \in A$, $a\|x\|' \leq \|x\|'' \leq b\|x\|'$. If a normed vector space is complete with respect to two norms $\| \|'$ and $\| \|''$ and if $\|x\|'' \leq b\|x\|'$ for all x, then $\| \|'$ and $\| \|''$ are isomorphic.

The <u>conjugate space</u> A^* of A is the set of all functionals over A. The <u>natural norm</u> $\|f\| = \sup_{\|x\|=1} |f(x)|$ makes A^* a Banach space.

1.7 <u>Isomorphism</u>. Two rings are called:

(1) <u>isomorphic</u> if they are algebraically isomorphic;

(2) <u>equivalent</u> if they are isomorphic under a norm-preserving isomorphism;

(3) <u>homeomorphic</u> if they are homeomorphic topological spaces.

1.8 <u>Decomposability</u>. A is decomposable if it can be written as a direct sum of two of its ideals.

PART II

STRUCTURE OF THE UNIT SPHERE

2.1 **Definition.** A functional $f(x)$ is called a supporting plane of the unit sphere S at u if $\|f\| = 1$ and $f(u) = 1$.

2.2 **Definition.** $\mathcal{F} = \{f\}$ is called a total family of functionals if $f(x) = 0$ for all $f \in \mathcal{F}$ implies $x = \Theta$. (6;42)

2.3 **Example.** Any set of n linearly independent functionals over the complex n-dimensional Euclidean space C_n is a total family.

2.4 **Definition.** Let S denote the unit sphere centered at the origin. S is said to have a vertex at the point u if there exists a total family of functionals which are supporting planes of S at u.

2.5 **Definition.** Let $g(x)$ be the real-valued function defined by $g(x) = \sup\limits_{\lambda \neq 0} \frac{1}{|\lambda|} \log \|e^{\lambda x}\|$, λ complex. The set $G = \{x \mid g(x) = 1\}$ will be called the G - sphere.

2.6 **Theorem.** $g(x)$ has the following properties:

(1) $g(\Theta) = 0$, $g(u) = 1$

(2) $g(\alpha x) = |\alpha| g(x)$

(3) $g(x+y) \leqslant g(x) + g(y)$

(4) $g(x) \leqslant \|x\|$

(5) $x \neq \Theta$ implies $g(x) > 0$

(6) $\|x\| \leqslant e g(x)$

(7) If $x \in G$ [i.e., $g(x) = 1$], there exists a complex number ε of absolute value 1 such that the line segment from εu to x lies in G. (Flatness property)

Proof of (1)

By direct substitution $g(\theta) = \sup\limits_{\lambda \neq 0} \frac{1}{|\lambda|} \cdot 0 = 0$. Also,

$$g(u) = \sup_{\lambda \neq 0} \frac{1}{|\lambda|} \log \| e^{\lambda u} \| = \sup_{\lambda \neq 0} \frac{1}{|\lambda|} \log | e^{\lambda} | = 1 .$$

Proof of (2)

If $\alpha \neq 0$, $g(\alpha x) = \sup\limits_{\lambda \neq 0} \frac{1}{|\lambda|} \log \| e^{\lambda \alpha x} \| = |\alpha| \sup\limits_{\lambda \neq 0} \frac{1}{|\lambda \alpha|} \log \| e^{\lambda \alpha x} \|$ $= |\alpha| g(x)$. If $\alpha = 0$, $g(\alpha x) = 0$ by (1).

Proof of (3)

$$\begin{aligned} g(x+y) &= \sup_{\lambda \neq 0} \frac{1}{|\lambda|} \log \| e^{\lambda x} e^{\lambda y} \| \\ &\leq \sup_{\lambda \neq 0} \frac{1}{|\lambda|} \log \| e^{\lambda x} \| \, \| e^{\lambda y} \| \\ &= \sup_{\lambda \neq 0} \frac{1}{|\lambda|} (\log \| e^{\lambda x} \| + \log \| e^{\lambda y} \|) \\ &\leq \sup_{\lambda \neq 0} \frac{1}{|\lambda|} \log \| e^{\lambda x} \| + \sup_{\lambda \neq 0} \frac{1}{|\lambda|} \log \| e^{\lambda y} \| \\ &= g(x) + g(y) \end{aligned}$$

Proof of (4)

$$\| e^{\lambda x} \| = \quad u + \sum_{n=1}^{\infty} \frac{\lambda^n x^n}{n!} \qquad 1 + \sum_{n=1}^{\infty} \frac{|\lambda|^n \| x \|^n}{n!}$$ implies

$g(x) \leq \sup\limits_{\lambda \neq 0} \frac{1}{|\lambda|} |\lambda| \, \| x \| = \| x \|$. From this inequality it follows in particular that $g(x) < \infty$, for every x. Too, by (3) and (4) $g(x)$ is continuous.

Property (5) follows from the lemma:

Lemma. If for every complex ε of absolute value 1, x has the property $\| e^{\varepsilon x} \| \leq a$, then $\| x \| \leq a$.

Proof of the lemma

By contour integration around the circle $|\varepsilon| = 1$.

$$\begin{aligned} \| x \| &= \left\| \frac{1}{2\pi i} \cdot \oint_{|\varepsilon|=1} \left(u + \sum_{n=1}^{\infty} \frac{\varepsilon^n x^n}{n!}\right) \frac{d\varepsilon}{\varepsilon^2} \right\| \\ &= \left\| \frac{1}{2\pi i} \cdot \oint_{|\varepsilon|=1} e^{\varepsilon x} \frac{d\varepsilon}{\varepsilon^2} \right\| \leq a \end{aligned}$$

Proof of (5)

From the definition of $g(x)$, it follows that for all complex numbers $\lambda \neq 0$, $g(x) \geq \frac{1}{|\lambda|} \log \| e^{\lambda x} \|$. Hence for all λ,

$\| e^{\varepsilon\lambda x} \| \leqslant e^{|\lambda| g(x)}$, and therefore by the preceding lemma $\| \lambda x \| = |\lambda| \|x\| \leqslant e^{|\lambda| g(x)}$. Taking $x \neq \theta$, and assuming $g(x) \leqslant 0$, this inequality gives $|\lambda| \|x\| \leqslant 1$ for all λ [since $|\lambda| g(x) < 0$], which implies $\|x\| = 0$, and hence $x = \theta$; a contradiction. Therefore $g(x) > 0$. Properties (4) and (5) imply that $g(x)$ is finite for all x.

Proof of (6)

If $x = \theta$ the inequality is trivial. Assume therefore $x \neq \theta$. Since $g(x) > 0$, substituting $\lambda = \frac{1}{g(x)}$ in the inequality $|\lambda| \|x\| \leqslant e^{|\lambda| g(x)}$, used in proving (5), gives $\frac{1}{g(x)} \|x\| \leqslant e$ or $\|x\| \leqslant e g(x)$.

Proof of (7)

Since $g(x) = 1$, given $\delta > 0$ there exists a number λ_0 such that $\| e^{\lambda_0 x} \| \geqslant e^{|\lambda_0|(1-\delta)}$. If $\varepsilon\lambda_0 = |\lambda_0|$, then $\| e^{\lambda_0(x+\varepsilon u)} \| \geqslant e^{|\lambda_0|(1-\delta) + |\lambda_0|} = e^{|\lambda_0|(2-\delta)}$, so $g(x+\varepsilon u) \geqslant 2-\delta$, or $g\left(\frac{x+\varepsilon u}{2}\right) \geqslant 1 - \frac{\delta}{2}$; i.e., to each $\delta > 0$ there corresponds an ε, $|\varepsilon| = 1$, such that $g\left(\frac{x+\varepsilon u}{2}\right) \geqslant 1 - \frac{\delta}{2}$. Take now a sequence $\delta_n \to 0$, and the corresponding ε_n. By compactness of the circle $|\varepsilon| = 1$, a subsequence ε_n' can be selected that is convergent to ε, say. Thus, since $g(x)$ is continuous, $g\left(\frac{x+\varepsilon u}{2}\right) \geqslant 1$. But, because G is convex [by property (3) above], $g\left(\frac{x+\varepsilon u}{2}\right) \leqslant 1$. Hence $g\left(\frac{x+\varepsilon u}{2}\right) = 1$.

2.7 Remark. As defined above the function $g(x)$ is a Banach norm in A, and G is its unit sphere. Hence $g(x)$ induces a natural norm in A^*, the space of functionals f over A.

2.8 Definition. The g-norm of $f \in A^*$ is defined by

$$\|f\|_g = \sup_{g(x)=1} |f(x)|.$$

2.9 Theorem. The set of functionals $\mathfrak{F} = \{ f \mid \|f\|_g = 1, f(u) = 1 \}$ is a total family. Hence the sphere S has a vertex at u.

Proof. Let x' be such that $f(x') = 0$ for all $f \in \mathfrak{F}$ and $g(x') = 1$ (this last requirement can always be met by normalizing x'). If $x' \neq 0$, i.e., if $\mathfrak{F}$ is not a total family, then by property (7) of Theorem 2.6, an ε of absolute value 1 exists such that $g(u + \varepsilon x') = 2$. If $x = \varepsilon x'$, then $g(u + x) = 2$; and by property (2), $g(x) = g(\varepsilon x') = g(x') = 1$. Consider the two-dimensional complex space spanned by u and x. By a theorem of Banach (6;55), there exists a functional h defined over that space such that $\| h \|_g = 1$ and $h(u + x) = g(u + x) = 2$. By the Hahn-Banach Theorem (6;55) h has an extension f_1 to the full-space A. Therefore, $\| f_1 \|_g = 1$, and $f_1(u + x) = 2$. Hence, $f_1(u) + f_1(x) = 2$ and $| f_1(u) | \leq \| f_1 \|_g \cdot g(u) = 1$ $| f_1(x) | \leq \| f_1 \|_g \cdot g(x) = 1$. Consequently $f_1(u) = 1$ and $f_1(x) = 1$, i.e., $f_1 \in \mathfrak{F}$, and at the same time $f_1(x) = 1$, a contradiction. Hence x, and also x', must equal Θ, so $\mathfrak{F}$ is a total family. Therefore the G - sphere has a vertex at u. But S is contained in G [Theorem 2.6, property (4)] and has u in common with G. It follows that S also has a vertex at u.

2.10 Corollary. $\| u - x \| = \| u + x \| = \| u \|$ implies $x = \Theta$.

Proof. If f is an element of the total family $\mathfrak{F}$ of Theorem 2.9, then

$$| 1 - f(x) | = | f(u - x) | \leq 1$$

$$| 1 + f(x) | = | f(u + x) | \leq 1.$$

Together, these relations imply $f(x) = 0$. Hence $x = \Theta$.

Theorem 2.9 may be formulated analytically by means of the derivative of the norm:

2.11 Definition. The right and left derivatives of the norm at x_0 in the direction x are defined respectively as

$$\varphi_+(x_o, x) = \lim_{h\to 0^+} \frac{\|x_o + hx\| - \|x_o\|}{h}$$

$$\varphi_-(x_o, x) = \lim_{h\to 0^-} \frac{\|x_o + hx\| - \|x_o\|}{h} .$$

If for a fixed x, $\varphi_+(x_o, x) = \varphi_-(x_o, x)$, the derivative of the norm is said to be <u>defined at</u> x_o <u>in the direction</u> x; it is denoted by $\varphi(x_o, x)$. The derivative is <u>defined at</u> x_o if it is defined at x_o for every x.

2.12 <u>Theorem</u> [Ascoli (1;53)]. $\varphi_-(u, x) \leq \varphi_+(u, x)$.

<u>Proof.</u> When $h > 0$, the triangle inequality

$2 = \|u - hx + u + hx\| \leq \|u - hx\| + \|u + hx\|$ yields

$$\frac{\|u - hx\| - 1}{h} + \frac{\|u + hx\| - 1}{h} \geqslant 0.$$

Hence, letting $h \to 0^+$, $\varphi_+(u, -x) + \varphi_-(u, x) \geqslant 0$ or, $-\varphi_+(u, -x) \leqslant \varphi_+(u, x)$. But, clearly, $-\varphi_+(u, -x) = \varphi_-(u, x)$; so, $\varphi_-(u, x) \leq \varphi_+(u, x)$.

2.13 <u>Theorem</u> [Mazur (7;75)]. If a is a real number such that $\varphi_-(u, x_1) \leq a \leq \varphi_+(u, x_1)$, where x_1 denotes a fixed element of the space, there exists a functional F with the properties $F(u) = 1$, $F(x_1) = a$, and $F(x) \leq \|x\|$.

<u>Proof.</u> Consider the linear subspace $A_1 = \{x = su + tx_1\}$ of A, s and t being real numbers. The real-valued function defined over A_1 by $f(x) = s + ta$ is clearly distributive and continuous and hence is a functional. It has, for $x \in A_1$, the property $f(x) \leq \varphi_+(u, x)$. Since $tx_1 = x - su$, if h is positive and small $(1 + sh > 0)$, then

$$\frac{\|u + hx\| - 1}{h} = s + \frac{1+sh}{h}\left[\|u + \frac{h}{1+th} tx_1\| - 1\right].$$

Letting $h \to 0^+$, this gives $\varphi_+(u, x) = s + \varphi_+(u, tx_1)$. But $\varphi_+(u, tx) \geqslant ta$ in all cases; for if $t > 0$, $a \leq \varphi_+(u, x_1)$ gives

$ta \leq \varphi_+(u, tx_1)$; and if $t < 0$, $\varphi_-(u, x_1) \leq a$ gives $\varphi_-(u, -tx_1) \leq -ta$ or $\varphi_+(u, tx_1) \geq ta$. Hence, $\varphi_+(u, x) \geq s + ta = f(x)$. But Banach has shown (6;27) that there exists a functional $F(x)$ defined over all of A, such that $F(x) \leq \|x\|$ for $x \in A$, and $F(x) = f(x)$ for $x \in A_1$. In particular, when $x = u$ $(s = 1, t = 0)$, $F(u) = 1$, and when $x = x_1$ $(s = 0, t = 1)$, $F(x_1) = a$.

The conclusions of this theorem imply $\|F\| = 1$, and hence that F is a plane of support of S at u.

2.14 <u>Theorem</u>. A necessary and sufficient condition that S have a vertex at u is that $\varphi(u, x)$ be defined only in the direction $x = \alpha u$, where α is any complex number.

<u>Proof</u> (Necessity). Suppose $\varphi_+(u, x) = \varphi_-(u, x)$. Then for $f \in \mathfrak{F}$, and an arbitrary real number h,

$$f(u + hx) \leq \|f\|_g\, g(u + hx) \leq \|u + hx\|.$$

Hence, when $h > 0$,

$$f(x) \leq \frac{\|u + hx\| - 1}{h}$$

and when $h < 0$,

$$\frac{\|u + hx\| - 1}{h} \leq f(x).$$

Letting h tend to zero in these expressions gives $f(x) \leq \varphi_+(u, x)$ and $\varphi_-(u, x) \leq f(x)$. But $\varphi_+(u, x) = \varphi_-(u, x) = \varphi(u, x)$. Therefore $f(x) = \varphi(u, x)$ for all $f \in \mathfrak{F}$. Let α be the common value $\varphi(u, x)$. Then $f(x) = \alpha$ or $f(x - \alpha u) = 0$, for all $f \in \mathfrak{F}$. But this implies $x - \alpha u = \theta$, or $x = \alpha u$, which completes the proof of the necessity.

(Sufficiency). If $\varphi_-(u, x) \neq \varphi_+(u, x)$ for every x not of the form $x = \alpha u$, Theorem 2.12 implies $\varphi_-(u, x) < \varphi_+(u, x)$. So, if $x_1 \neq \alpha u$, there is a number a_1 such that $\varphi_-(u, x_1) < a_1 < \varphi_+(u, x_1)$. Then Theorem 2.13 guarantees the existence of a functional $F(x)$ of norm 1 such that $F(u) = 1$ and $F(x_1) = a_1$. For each $x_1 \neq \alpha u$ there is such an F. That the set $\{F\}$

of these functionals is a total family may be proved as follows. If x_0 is such that $F(x_0) = 0$ for all F in $\{F\}$, then x_0 cannot be of the form $x_0 = \alpha u$, $\alpha \neq 0$ as this would imply $F(u) = \frac{1}{\alpha} F(\alpha u) = 0$. But if $x_0 \neq \alpha u$, a real $a_0 \neq 0$ can be found such that $\varphi_-(u, x_0) < a_0 < \varphi_+(u, x_0)$ and hence also a functional $F_0 \in \{F\}$ for which $F_0(x_0) = a_0 \neq 0$, again a contradiction. So $x_0 \neq \Theta$, which proves that S has a vertex at u.

PART III

STRUCTURE OF FINITE DIMENSIONAL NORMED RINGS

3.1 <u>Theorem.</u> The radical R of the n-dimensional normed ring A_n is the set of all nilpotent elements of A_n.

<u>Proof.</u> R is an ideal and A_n satisfies the descending chain condition. Hence (8;64) R is nilpotent, i.e. there exists a positive integer p for which $R^p = \{\theta\}$. In particular, $x^p = \theta$ for every $x \in R$. Conversely, if x is nilpotent $\max_M |x(M)| = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = 0$, so x belongs to every maximal ideal M and hence to R.

3.2 <u>Theorem.</u> If $n > 1$, then A_n has at least one maximal ideal.

<u>Proof.</u> If A_n had no maximal ideals, then by a Theorem of Gelfand (5;8) A_n would be isomorphic to the space of complex numbers. But any two isomorphic vector spaces have the same dimension. Hence $n = 1$, a contradiction.

3.3 <u>Definition.</u> A set of maximal ideals $M_1, \ldots, M_\ell$ is said to be linearly independent if the multiplicative functionals $M_1(x), \ldots, M_\ell(x)$ are linearly independent.

3.4 <u>Lemma.</u> If $M_1, \ldots, M_\ell$ are linearly independent maximal ideals of the ring A (which need not be finite dimensional) then there exists an element y such that $M_1(y) \neq 0$ and $M_i(y) = 0$, $i = 2, 3, \ldots \ell$.

<u>Proof.</u> Consider $L_1 = \bigcap_{i=2}^{\ell} M_i$. Since each M_i is a hyperplane of A (i.e., a linear subspace of dimension $n - 1$), any hyperplane

containing L_1 can be expressed as a linear combination of $M_2, M_3, \ldots, M_\ell$. However, since $M_1, M_2, \ldots, M_\ell$ are linearly independent, the hyperplane M_1 is not expressible in that form, and therefore cannot contain L_1. But L_1 intersects M_1 as they have the point θ in common. Hence points y exist in L_1 which are not in M_1.

3.5 <u>Theorem.</u> In A, any finite set of maximal ideals is linearly independent.

<u>Proof.</u> In an arbitrary set of $k > 1$ maximal ideals let $M_1, \ldots, M_\ell$, be a maximal subset of linearly independent M's and suppose $\ell < k$. Then if $M_{\ell+1}$ denotes one of the M's in the set but not in this subset, there exist complex numbers α_i not all zero such that $M_{\ell+1} = \sum_{i=1}^{\ell} \alpha_i M_i$. Say $\alpha_1 \neq 0$; then by the preceding Lemma there is an element y for which $M_1(y) \neq 0$ and $M_i(y) = 0$, $i = 2, 3, \ldots, \ell$. Hence, for any x not in M_1,

$$\begin{aligned} M_{\ell+1}(x)\, M_{\ell+1}(y) = M_{\ell+1}(xy) &= \sum_{i=1}^{\ell} \alpha_i M_i(xy) \\ &= \sum_{i=1}^{\ell} \alpha_i M_i(x)\, M_i(y) \\ &= \alpha_1 M_1(x)\, M_1(y) \neq 0. \end{aligned}$$

Therefore $M_{\ell+1}(y) \neq 0$, and, for an arbitrary x,

$$M_{\ell+1}(x) = \alpha_1 \frac{M_1(y)}{M_{\ell+1}(y)} M_1(x) .$$

This states that $M_{\ell+1}$ and M_1 are the same maximal ideal, a contradiction. Thus $\ell = k$ and the Theorem follows.

3.6 <u>Corollary.</u> The maximal ideals of A_n are linearly independent.

3.7 <u>Lemma.</u> If A_n has k maximal ideals, then $\dim R = n - k$.

Proof. Since the M_i's are linearly independent and $\dim M_i = n - 1$,

$$\dim R = \dim \bigcap_{i=1}^{k} M_i$$
$$= n - 1 - (k - 1) = n - k$$

3.8 Remark. Since $\| u + x \| < 1$ implies the existence of x^{-1} (5;4), it follows that $x \in R$ implies $\| u + x \| \geqslant 1$. This states that the subspace $u + R$ obtained by translation of R to u cannot contain interior points of S. If A is finite dimensional, say $A = A_n$ then S and $u + R$ have only the point u in common as shown by the following Theorem.

3.9 Theorem. In A_n, if $x \in R$, $x \neq \theta$, then $\| u + x \| > 1$.

Proof. Let $p = p(x)$ be the smallest positive integer such that $x^p = \theta$ for each $x \in R$ (see Theorem 3.1). Then, if $x_1 \in R$, $x_1 \neq \theta$, $\| u + x_1 \| \leqslant 1$, and $p_1 = p(x)$,

$$1 \geqslant \| u + x_1 \|^n \geqslant \| (u + x_1)^n \|$$

$$= \left\| u + \binom{n}{1} x_1 + \cdots + \binom{n}{p_1 - 1} x_1^{p_1 - 1} + \binom{n}{p_1} x_1^{p_1} \right\|$$

$$\geqslant \binom{n}{p_1} \| x_1^{p_1} \| - \binom{n}{p_1 - 1} \left\| x_1^{p_1 - 1} + \frac{\binom{n}{p_1 - 2}}{\binom{n}{p_1 - 1}} x_1^{p_1 - 2} + \cdots + \frac{1}{\binom{n}{p_1 - 1}} u \right\|,$$

or,

$$1 \geqslant \frac{n!}{(n - p_1)!\,(p_1 - 1)!} \left[\frac{\| x_1^{p_1} \|}{p_1} - \frac{1}{n - p_1 + 1} \left\| x_1^{p_1 - 1} + \frac{\binom{n}{p_1 - 2}}{\binom{n}{p_1 - 1}} x_1^{p_1 - 2} + \cdots + \frac{1}{\binom{n}{p_1 - 1}} u \right\| \right].$$

As $n \to \infty$, the quantity in brackets tends to $\dfrac{\|x_1{}^{R}\|}{p}$ and $\dfrac{n!}{(n-p_1)!\,(p_1-1)!}$ tends to infinity. Hence, the right-hand side of the inequality tends to infinity, a contradiction. Therefore $\|u + x_1\| > 1$.

3.10 <u>Theorem.</u> If A_n has $k \geq 1$ maximal ideals, then A_n contains exactly k non-trivial idempotents. (Of course every ring contains the trivial idempotents Θ and u.)

<u>Proof.</u> By Corollary 3.6 and Lemma 3.4 elements $y_1, \ldots, y_k$ may be found for which

$$y_i(M_{i'}) = \begin{cases} 0 & \text{when } i \neq i' \\ \alpha_i \neq 0 & \text{when } i = i'. \end{cases}$$

The complex numbers α_i may be assumed all distinct (distinct values can always be obtained by multiplying the y_i's by appropriate non-zero constants). The element $y = \sum_{i=1}^{k} y_i$ then has the property $y(M_i) = \alpha_i$, i.e., its spectrum consists of the k distinct points α_i. It follows from a Theorem of Dunford and Hille (10,105) that A_n contains exactly k idempotents $j_1, \ldots, j_k$ with the properties

$$j_i\, j_{i'} = \delta_{i i'} \cdot j_i, \quad \sum_{i=1}^{k} j_i = u, \quad j_i \neq \Theta, u.$$

3.11 <u>First fundamental structure theorem.</u> Suppose the radical R of A_n contains an element z such that $z^r \neq \Theta$, $z^{r+1} = \Theta$, where $r = \dim R$. Then in terms of this z, every $x \in A_n$ has a unique representation

$$x = \sum_{i=1}^{k-1} x(M_i)\, j_i + \sum_{i=0}^{n-k} \alpha_i z^i$$

where z^0 is defined to mean j_k, and $j_1, \ldots, j_k$ are the idempotents of Theorem 3.10, and the α_i 's are complex numbers.

If y is another element of A_n with a representation

$$y = \sum_{i=1}^{k-1} y(M_i)\, j_i + \sum_{i=0}^{n-k} \beta_i z^i,$$

the sum $x + y$ is given by

$$x+y = \sum_{i=1}^{k-1} \left[x(M_i) + y(M_i) \right] j_i + \sum_{i=0}^{n-k} (\alpha_i + \beta_i) z^i,$$

and the product xy by

$$xy = \sum_{i=1}^{k-1} x(M_i)\, y(M_i)\, j_i + \sum_{i=0}^{n-k} \sum_{\ell=0}^{i} \alpha_\ell \beta_{i-\ell} z^i$$

which is of the mixed scalar and convolution type. Namely, in the representation of xy,

$$xy = \sum_{i=1}^{k-1} xy(M_i)\, j_i + \sum_{i=0}^{n-k} \gamma_i z^i,$$

the components are given by

$$xy(M_i) = x(M_i)\, y(M_i) \quad \text{(scalar product)},$$

and

$$\gamma_i = \sum_{\ell=0}^{i} \alpha_\ell \beta_{i-\ell} \quad \text{(convolution product)}.$$

Proof. Consider the ring decomposition $A_n = A_{k-1} \oplus A_{n-k+1}$ where $A_{k-1} = \left\{ \sum_{i=1}^{k-1} x(M_i)\, j_i \right\}$ and $A_{n-k+1} = j_k A_n$. ($j_k A_n$ is of dimension $n - k + 1$ since A_{k-1} is of dimension $k - 1$.) The ring A_{n-k+1} contains R, for if $x \in R$, then $x(M_i) = 0$ for $i = 1, \ldots, k$; and it also contains the element $j_k u = j_k$ which is not in R. Thus $A_{n-k+1} = \{\lambda j_k\} + R$, λ complex. Therefore, since the elements $z^0 = j_k$, $z, z^2, \ldots, z^{n-k}$ are obviously linearly independent, they constitute a basis in A_{n-k+1}, and every element of A_n has a unique representation as $x = x' \oplus x''$ where $x' = \sum_{i=1}^{k-1} x(M_i)\, j_i$, and $x'' = \sum_{i=0}^{n-k} \alpha_i z^i$. In this decomposition the sum and the product of two elements $x = x' \oplus x''$, $y = y' \oplus y''$ take the forms:

$$x + y = x' + y' \oplus x'' + y''$$

$$\text{where} \begin{cases} x' + y' = \sum_{i=1}^{k-1} \left[x(M_i) + y(M_i)\right] j_i \\ x'' + y'' = \sum_{i=0}^{n-k} (\alpha_i + \beta_i)\, z^i \end{cases}$$

and

$$xy = x'y' \oplus x''y''$$

$$\text{where} \begin{cases} x'y' = \sum_{i=1}^{k-1} xy(M_i)\, j_i = \sum_{i=1}^{k-1} x(M_i)\, y(M_i)\, j_i \quad \text{(\underline{scalar} \underline{product})} \\ x''y'' = \sum_{i=0}^{n-k} \gamma_i z^i = \sum_{i=0}^{n-k} \sum_{\ell=0}^{i} \alpha_\ell \beta_{i-\ell} z^i \quad \text{(\underline{convolution} \underline{product})} \end{cases}$$

3.12 __Remark.__ The coefficients α_i, $i = 0, 1, \ldots, n-k$ associated with each x by the representation above define $n-k+1$ functionals over A_n (6;111), say $f_i(x) = \alpha_i$. These functionals are linearly independent for if $f_{i'} = \sum_{\substack{i=0 \\ i \neq i'}}^{n-k} \mu_i f_i$, where i' is between 0 and $n-k$ and the μ_i's are constants, this would give $f_{i'}(z^{i'}) = \sum_{\substack{i=0 \\ i \neq i'}}^{n-k} \mu_i f_i(z^{i'}) = 0$, a contradiction, since by definition $f_{i'}(z^{i'}) = 1$.

It should also be observed that A_{k-1} and A_{n-k+1} in the preceding Theorem are normed rings. Their unit elements are $\sum_{i=1}^{k-1} j_i$ and j_k respectively, and the norm in these rings is the same as the norm in A_n. Indeed, any norm admissible in A_n is automatically admissible in both A_{k-1} and A_{n-k+1}. Conversely, if $\| \ \|'$ and $\| \ \|''$ are admissible norms in A_{k-1} and A_{n-k+1} respectively, then $\| x \| = \| x' \oplus x'' \| = \mathrm{Max}(\| x' \|', \| x'' \|'')$ defines an admissible norm in A_n. In particular, one may take the norms in A_{k-1} and A_{n-k+1} to be

$$\| x' \|' = \operatorname*{Max}_{1 \leq i \leq k-1} |x(M_i)|$$

$$\| x'' \|'' = \sum_{i=0}^{n-k} | \alpha_i | .$$

These will be referred to as the <u>natural flat norms</u> of A_{k-1} and A_{n-k+1}, and

$$\| x \| = \mathrm{Max} \left\{ \operatorname*{Max}_{1 \leq i \leq k-1} |x(M_i)| , \sum_{i=0}^{n-k} |\alpha_i| \right\}$$

will be called the <u>natural flat norm</u> of A_n. This terminology is justified by the following Theorem.

3.13 <u>Theorem</u>. The natural flat norm of A_n has the Flatness Property (7) of Theorem 2.6.

<u>Proof</u>. If $\| x' \|' = \operatorname*{Max}_{1 \leq i \leq k-1} |x(M_i)| = 1$, and the maximum occurs say for M_1, then taking ε to be $x(M_1)$ gives the desired property. Namely $\| x' + \bar{\varepsilon} u \| = 2$. Similarly, if $\| x'' \|''$ $= \sum_{i=0}^{n-k} | \alpha_i | = 1$, then $\varepsilon = \dfrac{\alpha_0}{|\alpha_0|}$ gives

$$\| x'' + \varepsilon u \|'' = \left| \alpha_0 + \frac{\alpha_0}{|\alpha_0|} \right| + \sum_{i=1}^{n-k} |\alpha_i| = |\alpha_0| + 1 + \sum_{i=1}^{n-k} |\alpha_i| = 2.$$

The Theorem follows immediately from these two cases.

3.14 <u>Theorem.</u> Any two normed rings A_n, A_n' both of dimension n with radicals of dimension r satisfying the condition of Theorem 3.11 are isomorphic.

<u>Proof.</u> According to Theorem 3.11 two general elements $x \in A_n$, $x' \in A_n'$ may be represented as

$$x = \sum_{i=1}^{k-1} x(M_i)\, j_i + \sum_{i=0}^{n-k} \alpha_i z^i$$

$$x' = \sum_{i=1}^{k-1} x'(M_i')\, j_i' + \sum_{i=0}^{n-k} \alpha_i' z'^i.$$

The isomorphism $A_n \cong A_n'$ now follows by defining $z^i \leftrightarrow z'^i$, and arbitrarily setting $j_i \leftrightarrow j_i'$. So $x \leftrightarrow x'$ whenever corresponding j_i- and z^i- components are equal. This isomorphism is obviously a homeomorphism. It is an equivalence if the norms in A_n and A_n' are replaced by the natural flat norms.

3.15 <u>Second fundamental theorem.</u> Let A_n satisfy the condition of Theorem 3.11, and let $A_n^* = \{f, g, \dots\}$ denote the conjugate space (as a Banach space) of A_n. Then A_n^* can be made into a normed ring by taking the functionals $M_1, \dots, M_k, f_1, \dots, f_{n-k}$, as a basis so that, for a given z, each $f \in A_n^*$ has a unique representation as

$$f = \sum_{i=1}^{k} f(j_i)\, M_i + \sum_{i=1}^{n-k} f(z^i)\, f_i.$$

The product and norm of elements of A_n^* are given by

$$f \circ g = \sum_{i=1}^{k} \left(\sum_{\ell=1}^{i} f(j_\ell)\, g(j_{i-\ell+1}) \right) M_i + \sum_{i=1}^{n-k} f(z^i)\, g(z^i)\, f_i \quad ,$$

$$\| f \| = \operatorname{Max} \left\{ \sum_{i=1}^{k} |f(j_i)| , \max_{1 \le i \le n-k} | f(z^i) | \right\}$$

<u>Proof.</u> The n functionals $M_1, \ldots, M_k, f_1, \ldots, f_{n-k}$ form a linearly independent set; for if

$f_{i'} = \sum_{i=1}^{k} \mu_i M_i + \sum_{\substack{i=1 \\ i \neq i'}}^{n-k} \lambda_i f_i$, then $1 = f_{i'}(z^{i'}) =$

$\sum_{i=1}^{n-k} \mu_i M_i(z^{i'}) + \sum_{\substack{i=1 \\ i \neq i'}}^{n-k} \lambda_i f(z^{i'}) = 0$, and if

$M_{i'} = \sum_{\substack{i=1 \\ i \neq i'}}^{k} \mu_i M_i + \sum_{i=1}^{n-k} \lambda_i f_i$, then $1 = M_{i'}(j_{i'}) =$

$\sum_{\substack{i=1 \\ i \neq i'}}^{k} \mu_i M_i(j_{i'}) + \sum_{i=1}^{n-k} \lambda_i f_i(j_{i'}) = 0$. Hence, the set $M_1, \ldots,$ $M_k, f_1, \ldots, f_{n-k}$ is a basis for A_n^* so that every $f \in A_n^*$ has a representation

$$f = \sum_{i=1}^{k} \mu_i M_i + \sum_{i=1}^{n-k} \lambda_i f_i .$$

To evaluate the μ's form

$$f(j_\ell) = \sum_{i=1}^{k} \mu_i M_i(j_\ell) + \sum_{i=1}^{n-k} \lambda_i f_i(j_\ell) = \mu_\ell .$$

Similarly,

$$f(z^\ell) = \sum_{i=1}^{k} \mu_i M_i(z^\ell) + \sum_{i=1}^{n-k} \lambda_i f_i(z^\ell) = \lambda_\ell .$$

Therefore,

$$f = \sum_{i=1}^{k} f(j_i) M_i + \sum_{i=1}^{n-k} f(z^i) f_i .$$

As defined in the statement of the Theorem, $f \circ g$ obviously satisfies the properties of a product in a normed ring. The unit element is

$$M_1 + \sum_{i=1}^{n-k} f_i .$$ Finally, the norm is admissible since it is the natural norm corresponding to the mixed product.

3.16 <u>Theorem.</u> In A_n^* the elements $f_1, \ldots, f_{n-k}$ are idempotents, and A_n^* has $n - k + 1$ maximal ideals.

<u>Proof.</u> The idempotency of the f_i's follows directly from the definition of $f \circ g$ in Theorem 3.15. Exhibiting in each case the components of their generic elements, the $n - k + 1$ maximal ideals are

$$\{(0, f(j_2), \ldots, f(j_k), f(z), f(z^2), f(z^3), \ldots, f(z^{n-k}))\}$$
$$\{(f(j_1), f(j_2), \ldots, f(j_k), 0, f(z^2), f(z^3), \ldots, f(z^{n-k}))\}$$
$$\{(f(j_1), f(j_2), \ldots, f(j_k), f(z), 0, f(z^3), \ldots, f(z^{n-k}))\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\{(f(j_1), f(j_2), \ldots, f(j_k), f(z), f(z^2), f(z^3), \ldots, 0)\}$$

3.17 <u>Corollary.</u> The normed ring A_n^* defined in Theorem 3.15 is the dual of A_n taken with the natural flat norm, that is $A_n^{**} = A_n$, so A_n is reflexive as a normed ring.

PART IV

OPERATORS

4.1 Definition. A set B is called (weakly) sequentially compact if every infinite sequence $\{x_n\}$ in B contains a subsequence converging (weakly) to a point in B.

4.2 Remark. The word operator will be used only in the sense of linear operator (i.e., distributive and continuous).

4.3 Definition. The operator T is (weakly) completely continuous if it transforms bounded sets into (weakly) sequentially compact sets.

4.4 Definition. The element $x \in A$ is said to be (weakly) completely continuous if the operator T_x defined by $T_x y = xy$ is (weakly) completely continuous.

4.5 Remarks. The standard abbreviations c. c and w. c. c. will be used to designate completely continuous and weakly completely continuous operators and elements, as defined above.

In A_n every x is trivially c. c. and w. c. c. As shown below, this is not true of infinite dimensional rings. Some of the rings used will be indecomposable. It is recalled that such rings cannot contain idempotents other than θ or u, for if A contains $j \neq \theta, u$, then A has the Peirce decomposition $A = (u - j) A \oplus jA$.

4.6 Theorem. If A is infinite dimensional and indecomposable, and if x is c. c., then $x \in P$.

Proof. In view of the preceding Remark, A contains no non-trivial idempotents. Hence, by a Theorem of Lorch (9;416) the spectrum

$\sigma(x)$ of x is connected. Therefore, if $\lambda_o = x(M_o)$ is an arbitrary point in $\sigma(x)$ and $\lambda_o \neq 0$ (if $\lambda_o = 0$ the result is proved), connectedness implies that a sequence of points λ_{o_n} can be found in $\sigma(x)$ which converges to λ_o, $\lambda_{o_n} \rightarrow \lambda_o$. But as T_x is c. c., it follows from a Theorem of Riesz (11;90) that the values μ for which $(u - \mu x)^{-1}$ does not exist are discrete, i.e., omitting the origin, $\sigma(x) = \left\{\frac{1}{\mu}\right\}$ is discrete, a contradiction. Hence $\sigma(x)$ reduces to the point $\lambda = 0$, and therefore $x \in R$.

4.7 <u>Theorem.</u> If

(1) A is infinite dimensional and indecomposable,

(2) $f(\lambda)$ is a function which is analytic on and inside the circle $|\lambda| = r$,

(3) $\|x\| \leq r$,

(4) the element $f(x)$ is c. c.,

then

(1) there exists a unique λ_o such that $x - \lambda_o u \in R$

(2) λ_o is a root of $f(\lambda)$.

<u>Proof.</u> The hypothesis implies that $f(\lambda)$ has an expansion $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ valid for $|\lambda| \leq r$. Therefore, since $\|x\| \leq r$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is an element of A. By the preceding Theorem $f(x) \in R$. Hence, for every maximal ideal M, $f(x)(M) = 0$, that is

$$f(x(M)) = \sum_{n=0}^{\infty} a_n [x(M)]^n = \sum_{n=0}^{\infty} a_n x^n(M)$$

$$= f(x)(M) = 0.$$

But $|x(M)| \leq \|x\| \leq r$, and $f(\lambda)$ has only a finite number of zeros in $|\lambda| \leq r$. Thus $\sigma(x)$ is finite and hence isolated, and that contradicts the fact that A is indecomposable (9;416) unless $\sigma(x)$ consists of only one point: $\sigma(x) = x(M) = \lambda_0$ for all M. It follows that λ_0 is the only value of λ for which $(x - \lambda u)(M) = 0$ for all M, that is for which $x - \lambda u \in R$. Furthermore $f(\lambda_0) = 0$.

4.8 Corollary. If A is infinite dimensional and indecomposable, and x^n is c. c., then $x \in R$.

Proof. Since $\lambda = 0$ is the only root of $f(\lambda) = \lambda^n$, this root is λ_0. Hence, $x - \lambda_0 u = x \in R$.

4.9 Remarks. The converse of Theorem 4.6 is not true. There exist indecomposable infinite dimensional rings whose radicals contain elements that are not c. c. For example, consider the ring of convergent power series $a = \sum_{n=0}^{\infty} a_n x^n$ for which $\sum_{n=0}^{\infty} |a_n| < \infty$. The sum and the product of two elements in this ring are defined as $a + b = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$, and $ab = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. The norm is given by $\| a \| = \sum_{n=0}^{\infty} |a_n|$. The only maximal ideal, hence R, consists of all a's for which $a_0 = 0$, that is $a = \sum_{n=1}^{\infty} a_n x^n$. The element $a = \sum_{n=1}^{\infty} \frac{1}{2^n} x^n$ which is in R is not c. c. For, consider in A the bounded set $\{b^{(k)}\}$ where $b^{(k)} = \sum_{n=1}^{\infty} \delta_{nk} x^n = x^k$. The transformed set $T_a\{b^{(k)}\}$

is the set of all elements $ab^{(k)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^{n+k}$, and every sequence of these elements diverges because, taking $k < \ell$,

$$\| ab^{(k)} - ab^{(\ell)} \| = \sum_{n=0}^{\ell-k-1} \frac{1}{2^{n+1}} + \sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{2^{\ell-k+n}} \right)$$
$$= 2 - \frac{3}{2^{\ell-k}} \geqslant \frac{1}{2} .$$

However, there are indecomposable infinite dimensional rings whose radicals are made up entirely of c. c. elements. For example, the ring of operators of the Volterra type $T_f\, g = \int_0^t f(t-s)\, g(s)\, ds$, $0 \leqslant t \leqslant 1$, from the space of real continuous functions $g(t)$ over the unit interval and into itself. This ring has been discussed by Michal and Elconin. (12) Each continuous function $f(t)$, $0 \leqslant t \leqslant 1$, gives rise to an operator T_f. The product $T_{f_3} = T_{f_1}\, T_{f_2}$ is defined by $f_3 = \int_0^t f_1(t-s)\, f_2(s)\, ds$, and the norm of T_f is taken to be the norm of f, $\| T_f \| = \| f \| = \sup_{0 \leqslant t \leqslant 1} | f(t) |$. This ring has no unit element, but may be imbedded in the space of couples (λ, T_f) with λ complex. Defining the product by $(\lambda_1, T_{f_1})(\lambda_2, T_{f_2}) = (\lambda_1 \lambda_2, \lambda_2 T_{f_1} + \lambda_1 T_{f_2} + T_{f_1} T_{f_2})$ and the norm by $\| (\lambda, T_f) \| = | \lambda | + \| T \|$ the space becomes a normed ring with the unit element (1,0). The radical is the only maximal ideal, $R = \{ (0, T_f) \}$. Now consider the bounded sequence of elements (λ_n, T_{g_n}), say $| \lambda_n |, \| g_n \| < K$ for all n. Then if $(0, T_f)$ is an arbitrary element of R, the elements $(0, \lambda_n T_f + T_f T_{g_n}) = (0, T_f)(\lambda_n, T_{g_n})$ form an equi-continuous family. For, since the interval $0 \leqslant s \leqslant 1$ is closed, $f(s)$ is uniformly continuous and hence given $\varepsilon > 0$ one can find $\delta > 0$ such that $| f(s_1) - f(s_2) | < \frac{\varepsilon}{3K}$ whenever

$|s_1 - s_2| < \delta$. Therefore, taking $0 \leqslant t_2 < t_1 \leqslant 1$, and $t_1 - t_2 < \delta_1 = \mathrm{Min}\left(\frac{\varepsilon}{3\|f\|K}, \frac{\varepsilon}{3}\right)$, and writing $T_{g_n'}$ for

$\lambda_n T_f + T_f T_{g_n} = T_{\lambda_n f} + \int_0^t f(t-s)\, g_n(s)\, ds$, gives for all n

$$|g_n'(t_1) - g_n'(t_2)| = \left| \lambda_n\left[f(t_1) - f(t_2)\right] + \int_0^{t_1} f(t_1 - s) g_n(s)ds - \int_0^{t_2} f(t_2 - s) g_n(s)ds \right|$$

$$= \left| \lambda_n\left[f(t_1) - f(t_2)\right] + \int_0^{t_2}\left[f(t_1 - s) - f(t_2 - s)\right] g_n(s)ds + \int_{t_1}^{t_2} f(t_1 - s)\, g_n(s)ds \right|$$

$$\leqslant |f(t_1) - f(t_2)|\, K + |f(t_1 - s) - f(t_2 - s)|\, K + (t_1 - t_2)\|f\| K$$

$$\leqslant \frac{\varepsilon}{3K} + \frac{\varepsilon}{3K} + \delta_1 \|f\| K \leqslant \varepsilon .$$

It follows by Ascoli's Theorem that the sequence g_n' contains a uniformly convergent subsequence g_{n_1}' . Hence $T_{g_{n_1}'}$ converges, so that T_f is c. c.

4.10 __Theorem.__ The set C of all c. c. elements of A is a closed ideal.

__Proof.__ Consider in A the bounded sequence $\{z_n\}$. Then if $x \in C$ a subsequence $\{z_{n_1}\}$ exists such that $\lim_{n_1, m_1 \to \infty} \|xz_{n_1} - xz_{m_1}\| = 0$. Similarly if $x' \in C$ there is a subsequence $\{z_{n_2}\}$ of $\{z_{n_1}\}$ such that $\lim_{n_2, m_2 \to \infty} \|xz_{n_2} - xz_{m_2}\| = 0$. Combining the two limits gives $\lim_{n_2, m_2 \to \infty} \|(x + x')z_{n_2} - (x + x')z_{m_2}\| = 0$. Therefore

$x + x' \in C$. Also, if y is an arbitrary element of A,

$$\lim_{n_1, m_1 \to \infty} \| xyz_{n_1} - xyz_{m_1} \| \leq \| y \| \cdot \lim_{n_1, m_1 \to \infty} \| xz_{n_1} - xz_{m_1} \| = 0 ,$$

i.e., $xy \in C$. Thus C is an ideal. To show that C is closed consider a convergent sequence of elements x_n in C, $x_n \to x$ say. The c. c. operators T_{x_n} defined over A by $T_{x_n} y = x_n y$ form a convergent sequence. Hence the limit operator T_x is c. c. (6;96), that is x is c. c.

4.11 <u>Lemma</u>. The limit of a convergent sequence of w. c. c. operators is a w. c. c. operator.

The proof of this Lemma is entirely similar to that given by Banach (6;96) in the corresponding case of c. c. operators, and will be omitted.

4.12 <u>Theorem</u>. The set W of all w. c. c. elements in A is a closed ideal.

<u>Proof</u>. Let $\{ z_n \}$ be bounded. If $x \in W$, a subsequence $\{ z_{n_1} \}$ can be found such that xz_{n_1} converges weakly. Similarly since $\{ z_{n_1} \}$ is bounded, if $x' \in W$, a subsequence $\{ z_{n_2} \}$ of $\{ z_{n_1} \}$ exists such that $\{ x'z_{n_2} \}$ converges weakly. It follows that $(x + x')z_{n_2}$ converges weakly, i.e., $x + x'$ W. Again since $\{ z_{n_1} \}$ is bounded, if $y \in A$, $\{ yz_{n_1} \}$ is also bounded. Hence for $x \in W$ there exists a subsequence $\{ yz_{n_2} \}$ for which $y(xz_{n_2}) = x(yz_{n_2})$ converges weakly. Thus W is an ideal. That W is strongly closed follows from Lemma 4.11. For if $x_n \in W$ and $\lim_{n \to \infty} \| x_n - x \| = 0$, then the w. c. c. operators T_{x_n} defined by $T_{x_n} y = x_n y$ converge to T_x , so that T_x is w. c. c., i.e., $x \in W$.

4.13 <u>Definition.</u> A ring that satisfies the two conditions:

(1) The ring is a star ring

(2) Uniform convergence of the sequence $x_n(M)$ with respect to the maximal ideals implies weak convergence of x_n

will be denoted by $\bar{A}$, and its conjugate space as a Banach space denoted by $\bar{A}^*$. The space of maximal ideals of $\bar{A}$ with the <u>star topology</u> of Gelfand and Šilov (13;30) will be called $\overline{\mathfrak{M}}$ [it is a compact Hausdorff space (13;31)]. $C(\overline{\mathfrak{M}})$ will designate the space of complex valued functions over $\overline{\mathfrak{M}}$, and $C(\overline{\mathfrak{M}})^*$ the conjugate space of $C(\overline{\mathfrak{M}})$.

4.14 <u>Theorem.</u> Every element $f \in \bar{A}^*$ can be extended to an element of $C(\overline{\mathfrak{M}})^*$.

<u>Proof.</u> By a Theorem of Gelfand (13;34) every $\varphi(M) \in C(\overline{\mathfrak{M}})$ is a uniform limit of functions $x_n(M)$. Condition (2) of Definition 4.13 implies the weak convergence of x_n, i.e., $\lim\limits_{n\to\infty} f(x_n)$ exists for every $f \in C(\overline{\mathfrak{M}})$. Define $f(\varphi) = \lim\limits_{n\to\infty} f(x_n)$. The function $f(\varphi)$ is clearly independent of the choice of the sequence $\{x_n\}$. It will now be verified that f is a continuous function over $C(\overline{\mathfrak{M}})$. Consider a fixed $\varphi_0 \in C(\overline{\mathfrak{M}})$, and a sequence $\varphi_n \to \varphi_0$, i.e., $\lim\limits_{n\to\infty} \|\varphi_n - \varphi_0\|_{C(\overline{\mathfrak{M}})} = 0$. Then for every n there exists a subsequence $\{x_{n_1}\}$ such that

$$\|x_{n_1} - \varphi_n\| < \frac{1}{n}$$

$$|f(x_{n_1}) - f(\varphi_n)| < \frac{1}{n} \quad , \quad f \in C(\overline{\mathfrak{M}})^* ,$$

whenever $n_1 \geq N_1$. It follows that $x_n(M)$ converges uniformly to φ_0. Hence, using Condition (2) of Definition 4.13,

$\lim\limits_{n\to\infty} |f(\varphi_n) - f(\varphi_0)| = 0$, or $f(\varphi_0) = \lim\limits_{n\to\infty} f(\varphi_n)$ for every $f \in C(\overline{\mathfrak{M}})^*$.

4.15 <u>Corollary.</u> $f(x(M)) = f(x)$ and the extension preserving this property is unique.

The following Lemma of Kakutani's (14;1012) is given without proof.

4.16 <u>Lemma.</u> Any functional f over $C(\overline{\mathfrak{M}})$ can be represented as a completely additive measure $\mu(E)$ with respect to the smallest Borel collection of sets E containing the open subsets of $\overline{\mathfrak{M}}$.

4.17 <u>Lemma.</u> $\bar{A}^*$ and $C(\overline{\mathfrak{M}})^*$ are isomorphic.

<u>Proof.</u> Indeed, the mapping defined by the identity mapping of $\bar{A}^*$ is by Corollary 4.15 a 1:1 onto mapping of $\bar{A}^*$ to $C(\overline{\mathfrak{M}})^*$. Moreover, as $f(x(M)) = f(x)$, it follows that

$$\| f \|_{\bar{A}^*} = \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|x\| \leq 1} |f(x(M))| .$$

But $\| x \| \leqslant 1$ implies $\sup_M |x(M)| \leqslant \| x \| \leqslant 1$, i.e., the set $\| x \| \leqslant 1$ is included in the set where $\sup_M |x(M)| \leqslant 1$. Hence,

$$\| f \|_{\bar{A}^*} \leqslant \| f \|_{C(\overline{\mathfrak{M}})^*} \cdot \sup_M |x(M)| = \| f \|_{C(\overline{\mathfrak{M}})^*} .$$

Whence, it follows from a result of Banach (6;41) that $\bar{A}^*$ is isomorphic to $C(\overline{\mathfrak{M}})^*$.

4.18 <u>Corollary.</u> Weak convergence in $\bar{A}^*$ is equivalent to weak convergence in $C(\overline{\mathfrak{M}})^*$.

4.19 <u>Theorem.</u> If T is a w. c. c. mapping of $\bar{A}$ into itself, then T^2 is c. c.

<u>Proof.</u> Consider over $\bar{A}^*$ the functional $f(T^2x)$. Using the notation (x, f) to denote $f(x)$ gives in that case $(T^2 x, f) = (Tx, T^* f)$. Let $\{x_n\}$ be a bounded sequence; then Tx_n is weakly convergent, $Tx_n \xrightarrow{w} x_0$ say. To complete the proof it must be shown that $T^2 x_n$ converges uniformly with respect to all functionals f of

norm $\|f\| \leq 1$. If the contrary is assumed, then there exists a sequence of f_n's of norm $\|f_n\| \leq 1$ such that

$$\left| (Tx_{n_\nu}, T^*f_{n_\nu}) - (Tx_{m_\nu}, T^*f_{m_\nu}) \right| \geq \varepsilon_0$$

where $\varepsilon_0 > 0$ has been preassigned arbitrarily. For convenience, suppose $n_\nu = n$, $n = 1, 2, \ldots$. As T^* is also w. c. c. [see (6;100) where it is proved that T c. c. implies T^* c. c.; the proof for T w. c. c. is similar], a subsequence of $\{Tf_n\}$ can be selected (which again is supposed to be the full sequence $\{Tf_n\}$) which is weakly convergent as elements, $Tf_n \xrightarrow{w} g_0 \in \bar{A}^*$ say. Thus the proof of the Theorem will be complete if the following Lemma furnishing the desired condition can be established.

4.20 <u>Lemma</u>. If

(1) for each M, $x_n(M)$ converges to $x_0(M)$

(2) $x_n(M)$ is uniformly bounded, i.e., $|x_n(M)| \leq K$ for every n and every M

(3) the functionals $f_n \in C(\overline{\mathfrak{M}})^*$ converge weakly as elements to f_0

then (f_n, x_n) converges.

<u>Proof</u>. By Lemma 4.16 and the weak convergence of f_n it follows that if μ_n is the measure function associated with f_n, then $\mu_n(E)$ converges to $\mu_0(E)$ for any measurable set $E \subset \overline{\mathfrak{M}}$.

Now,

$$\left|(f_n, x_n) - (f_0, x_0)\right| \leq \left|(f_n, x_0) - (f_0, x_0)\right| + \left|(f_n, x_0) - (f_n, x_n)\right|$$
$$= |I_1| + |I_2| .$$

Consider I_1 first. By Egoroff's Theorem (15;18), an E exists for which, for every M, $|x_n(M) - x_0(M)| \leq \varepsilon$ when $n \geq n_0$, such that

$\mu_o(E^c) \leq \varepsilon$, where E^c denotes the complement of E in $\mathfrak{M}$. The remark above shows that $\mu_n(E^c)$ tends to $\mu_o(E^c)$ as $n \to \infty$, so $\mu_n(E^c) \leq 2\varepsilon$ for $n \geq n_o$. Hence, as f_n tends to f_o weakly, $|I_1| \leq \varepsilon$ for $n \geq n_1$.

Secondly consider I_2.

$$\begin{aligned} I_2 &= \int_{\mathfrak{M}} \left[x_n(M) - x_o(M) \right] d\mu_n \\ &= \int_E \left[x_n(M) - x_o(M) \right] d\mu_n + \int_{E^c} \left[x_n(M) - x_o(M) \right] d\mu_n \\ &= I_{2,1} + I_{2,2} . \end{aligned}$$

The weak convergence of f_n implies that $\| f_n \| \leq K'$, where K' is a constant. Clearly, therefore,

$$| I_{2,1} | \leq \varepsilon \cdot (\text{Total Var. of } \mu_n) \leq \varepsilon \| f_n \| \leq \varepsilon K' .$$

Also,

$| I_{2,2} | \leq 4C\varepsilon$ for n sufficiently large. The Lemma now follows by combining the inequalities for $|I_1|$, $|I_{2,1}|$, and $|I_{2,2}|$.

4.21 <u>Theorem</u>. If $\bar{A}$ is infinite dimensional, it cannot be reflexive.

<u>Proof</u>. Assume $\bar{A}$ is reflexive. Then $\bar{A}$ is weakly complete (16;423). Hence the identity operator T_u is w. c. c., and so by Theorem 4.19 $(T_u = T_u^2)$ it is also c. c. Therefore the unit sphere is sequentially compact, and it follows that $\bar{A}$ is finite dimensional (6;84), a contradiction.

4.22 <u>Corollary</u>. If every element of $\bar{A}$ is c. c., then $\bar{A}$ is finite dimensional.

<u>Proof</u>. If $C = \bar{A}$, then $\bar{A}$ is reflexive. (3) The Corollary now follows from Theorem 4.21.

4.23 <u>Theorem</u> [Dunford and Pettis (17;385) and Phillips (18;536)]. If T is a w. c. c. operator mapping L into L, then T^2 is c. c.

Proof. By a fundamental result of Kakutani (14;1021) L^* is isomorphic to $C(\Omega)$, where Ω is a compact Hausdorff space. But T^* is w. c. c. [see (6;100); the proof in the w. c. c. case is similar], and $C(\Omega)$ is an $\bar{A}$-ring. Hence, by Theorem 4.19, $(T^*)^2$ is c. c., and, since $(T^*)^2 = (T^2)^*$, T^2 is c. c.

4.24 Theorem. If $\bar{A}$ is infinite dimensional and indecomposable, and if x is w. c. c., then $x \in R$.

Proof. Theorem 4.19 implies that x^2 is c. c. Hence, by Corollary 4.7, $x \in R$.

4.25 Theorem. If

(1) $\bar{A}$ is infinite dimensional and indecomposable

(2) $f(\lambda)$ is a function which is analytic on and inside the circle $|\lambda| = r$

(3) $\|x\| \leqslant r$

(4) the element $f(x)$ is w. c. c.,

then

(1) there exists a unique λ_0 such that $x - \lambda_0 u \in R$

(2) λ_0 is a root of $f(\lambda)$.

Proof. The result follows immediately from Theorem 4.7 upon observing that $[f(\lambda)]^2$ satisfies the conditions of that Theorem.

4.26 Corollary. If $\bar{A}$ is infinite dimensional and indecomposable, and x^n is w. c. c., then $x \in R$.

APPENDIX

Several problems related to topics in Part III are discussed in this Appendix.

1. Projection operators over A.

Definition. A projection P over A is an operator taking A into A such that

(1) $P(x+y) = Px + Py$

(2) $Pxy = PxPy$

(3) $P^2x = P(Px) = Px$.

Remark. An idempotent j of A generates over A the projection P defined by $Px = jx$. In this sense every ring admits two trivial projection operators: $Px = x$ and $Px = \Theta$. Every projection over A is not necessarily of the type $Px = jx$ as the following example shows. This example, it should be observed, involves not merely an algebraic ring, but a normed ring. Thus, even in the presence of a norm the representation of P in the form $Px = jx$ is not possible in general.

Example. Consider the ring A_3 of elements $x = (\xi, \eta, \zeta)$ (where ξ, η, ζ are complex) with product $xx' = (\xi\xi', \xi\eta' + \xi'\eta, \zeta\zeta')$, and norm $\|x\| = \mathrm{Max}(|\xi| + |\eta|, |\zeta|)$. This ring has two non-trivial idempotents, $j_1 = (1, 0, 0)$ and $j_2 = (0, 0, 1)$. Their products with the general element $x = (\xi, \eta, \zeta)$ are $j_1 x = (\xi, \eta, 0)$ and $j_2 x = (0, 0, \zeta)$. Now the operator U defined by $Ux = (\xi, 0, 0)$

is clearly a projection. However, for x arbitrary $Ux \neq \Theta, x, j_1 x, j_2 x$. Indeed, there is no fixed element $x' = (\xi', \eta', \zeta')$ such that $Ux = x'x$ for all x'. For this would require $\xi\xi' = \xi$, $\xi\eta' + \xi'\eta = 0$, and $\zeta\zeta' = 0$, whence $\xi' = 1$, $\eta' = -\dfrac{\eta}{\xi}$, $\zeta' = 0$ which means that x' is not constant.

If P is of the form $Px = jx$ the following result may be stated.

Theorem. If $Px = jx$, then PA is a closed ideal of A.

Proof. The case $j = \Theta$ is trivial, so assume $j \neq \Theta$. PA is obviously an ideal. In it consider a convergent sequence $jx_n \to y$. For n sufficiently large $\| jx_n - y \| < \dfrac{\varepsilon}{\|j\|}$, so $\| jx_n - jy \| = \| j^2 x_n - jy \| = \| j(jx_n - y) \| \leq \| j \| \dfrac{\varepsilon}{\|j\|} = \varepsilon$. Thus $jx_n \to jy = y$, i.e., PA is closed.

Remark. Even when a projection P is of the form $Px = jx$, in general PA will not be a maximal ideal of A. For example, in the ring A_3 above $P_2 A_3 = j_2 A_3 = \{(0, 0, \zeta)\}$ is not a maximal ideal. The maximal ideals in that case are $\{(0, \eta, \zeta)\}$ and $\{(\xi, \eta, 0)\}$.

2. Homomorphisms.

Theorem. If B is a normed ring with zero radical, and h is a homomorphism from A onto B whose kernel is a closed set, then h is continuous.

Proof. The kernel of h is defined by $K = \{ x \in A \mid h(x) = \Theta \}$. The mapping H of A/K onto B that is induced by h is an algebraic isomorphism. Since K, which is an ideal of A, is closed, it follows from a Theorem of Gelfand (5;17) that H is a homeomorphism. Hence H is continuous. Moreover, if X is the coset of A/K which corresponds to an arbitrary $x' \in A$, then $h(x) = H(X)$, and so $\| h(x) \| = \| H(X) \| \leq \| H \| \cdot \| X \|$. But, since X is an ideal of A, $\| X \| = \inf_{x' \in X} \| x' \| \leq \| x \|$. It follows that $\| f(x) \| \leq \| H \| \| x \|$, i.e., h is bounded and hence continuous.

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