ELECTRIC ANALOG COMPUTER TECHNIQUES APPLIED TO CERTAIN MECHANICAL VIBRATION PROBLEMS

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ABSTRACT

Solutions are given for two distinct types of vibration problems of importance in the field of applied mechanics. These solutions are of themselves useful, and in addition they represent the development of basic electric analog computer techniques which may be applied to the solution of a vast number of hitherto unsolved engineering problems.

Part One treats the problem of a linear system excited by one or more forces of varying frequency. An analytical solution is given for the case in which the exciting force is of constant amplitude and of a frequency which varies linearly with time. A device which generates electric forces of varying frequency is discussed, and the results of its application to the solution of time-varying frequency problems are shown. A detailed study is made of a low loss two degree of freedom system excited by a force of constant amplitude and a frequency varying linearly with time. This study demonstrates the tremendous saving in time which may be effected by electric analog computation methods. It is demonstrated how forces of time-varying frequency may be used to rapidly obtain a qualitative measure of a system's steady state frequency response.

The transient draft gear forces existing during the braking of a long train are treated in Part Two. An analytical solution and an Electric Analog Computer solution are given for a train of identical cars and draft gears assumed to behave as perfect springs. The electric analog for a nonlinear draft gear is developed. A computer solution is presented for a train containing nonlinear draft gears and, in addition, for certain distributions of cars of unequal weights. This problem, involving a system of fifty nonlinear elements and fifty separate excitation forces would be completely impracticable by even the best available digital computation methods.

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PART ONE

Response of Linear Physical Systems to Forces of Time-Varying Frequency

Mechanical vibration problems may be described as those problems wherein a mechanical system suffers the action of arbitrary forces. This is a broad definition and encompasses many types of systems; linear and nonlinear, lumped and distributed; and many varieties of forces. A great deal of mathematics has been developed to treat such problems. Since no one mathematical technique can adequately handle all mechanical vibration problems, these problems have been divided into various subclasses. For example, one very important type of vibration problem has to do with the behavior of lumped linear mechanical systems excited by a finite number of steady state sinusoidal forces of constant amplitude. Thus we have, in making such a distinction, limited both the type of system and the force used. And, for this particular type of problem we have available a highly developed system of mathematical analysis.

Another field of problems, with its associated mathematical techniques, deals with lumped linear mechanical systems excited by a finite number of arbitrary transient forces. The methods for handling such problems, or their electrical equivalents, have probably been advanced more by electrical engineers than by any other group, simply because electrical engineers have felt a greater need for such methods to cope with electrical problems of a corresponding nature. The operational methods of Heaviside and Laplace have received wide use in this field.

The operational method is not particularly useful for the solution of transient problems, nor is any mathematical technique useful for the solution of a problem which it describes, unless the mathematical result lends itself to convenient interpretation and calculation. An analytical result is not particularly useful in a given problem if an inordinate amount of time is involved in expressing this result in understandable terms, numerical or otherwise.

Some of the problems in the material to be presented are of this nature. The analytical solutions are extremely lengthy when reduced to numerical calculation. Nevertheless, the analytical solutions help to gain an understanding of the general nature of the problems. It will be shown how the electric analog method of computation lends itself nicely to more exact numerical work.

I. NATURE OF THE PROBLEM

To make the results worthwhile it is necessary to restrict the present study to the behavior of linear lumped systems excited by oscillatory forces of constant amplitude whose frequencies vary with time. The primary emphasis will be on mechanical systems, since they are the most common practical examples. The limitations of lumped linear systems and of constant amplitude forces are not necessary for solution by the electric analog computer; they make the presentation of concrete results more meaningful.

One type of problem involving time-varying frequencies is that occurring in frequency modulation broadcasting. Here the primary interest is in a frequency which is varying periodically and whose instantaneous deviation is small compared to the mean value. The treatment of this phenomenon has been accorded considerable discussion in engineering literature, and its treatment here will be brief.

The second type of problem, which is the more important mechanically, deals with a force whose frequency varies in a transient manner. Such forces are common in almost all types of reciprocating engines. As such machines are accelerated or decelerated the pulsating components of the forces involved change in frequency. For example, as a diesel engine is accelerated the oscillating forces of the pistons transmitted to the connecting rods and thence to the crankshaft increase in frequency. It is well known that if an oscillatory force is applied to a system containing inertia and elastic recovery, and with little damping, large stresses may be set up for particular values of frequency. One would expect, then, that if the frequency of the applied force(s) should pass through these critical system frequencies similar large stresses might be expected. This is indeed the case, and it is the exact nature of these phenomena which will be discussed. Since, in the case of machinery, the frequencies of the pulsating components of the forces are directly related to machine speed the critical frequencies may be referred to as "critical speeds".

II. SURVEY OF MATERIAL APPEARING IN LITERATURE

Steady State Frequency Modulation. Before proceeding farther it is perhaps advisable to define exactly what is meant by the term frequency. Given an oscillatory function of constant amplitude,

$$A \cos \left[g(t)\right] , \tag{1}$$

the instantaneous frequency, f, is defined by

$$f = \frac{1}{2\pi} g'(t) \quad . \tag{2}$$

This is the accepted definition appearing in all the literature which has come to the attention of the author. It will become apparent later that this definition is consistent with steady state notions of frequency.

In steady state frequency modulation an electrical signal, or perhaps a mechanical force, is made to vary such that its frequency is of the form

$$f = f_0 + a_1 \sin 2\pi p_1 t \quad . \qquad (3)$$

In broadcasting work f_0 is the carrier frequency, and p_1 is the frequency of the sinusoidal intelligence to be transmitted. Of course the intelligence usually consists of many frequencies, as for example in the transmission of speech or music. Rather complicated expressions have been developed (1) to express the exciting force in such cases as made up of each of a number of signals or "sidebands", each sideband being of a different frequency, amplitude, and phase. The complication of such expressions increases as the number of modulation frequencies, p_1, p_2, p_3, ... , increases. If the modulation should be a square wave, the frequency would be given by

$$f = f_0 + a_1 \Big[\sin 2\pi p_1 t - \tfrac{1}{3} \sin 6\pi p_1 t + \cdots + \frac{(-1)^{\frac{n-1}{2}}}{n} \sin 2\pi n p_1 t + \cdots \Big] \ n \text{ odd}, \qquad (4)$$

which represents an infinite number of modulation frequencies. In such cases the expressions mentioned above become useless, and a different method must be employed. The special cases of rectangular and triangular modulation waveshapes have been considered by others. (2) Beyond these special cases of periodic steady state frequency variation little has been treated in the literature.

<u>Acceleration Through Resonance.</u> Two noteworthy contributions have been made in treating the problem of a system excited by an oscillating force of constant amplitude whose frequency varies

linearly with time. The first of these, a mechanical engineering paper, treats in a fairly thorough manner the response of a single degree of freedom system excited by such a force. (3) The case of greatest interest is of course the one in which the exciting force's frequency passes through the system's resonance frequency, since it is in this case that large mechanical stresses occur. Hence the term acceleration through resonance. The second paper, (4) while it refers to systems of more than one degree of freedom, does not specifically treat these cases. The second paper, insofar as results are concerned, is largely a duplication of the first. However, it contains some worthwhile mathematical data.

III. ANALYTICAL EXPRESSIONS FOR SYSTEM RESPONSE

Further attention to the analytical treatment of steady state frequency modulation will not be given here, since this has already been carried as far as is deemed practical for the present. The response of a linear system to each of the sideband frequencies can be computed separately, and these results can then be combined into a final result. Further discussion of steady state frequency modulation will be postponed until after presentation of an applicable electric analog technique.

To consider acceleration through resonance and related phenomena the response of linear lumped systems to an excitation force

$$\cos\,(\omega_0 t + a^2t^2)$$

which is applied at time $t = 0$ will be discussed. By the definition of equation (2) this represents an angular frequency of

$$\omega = \omega_0 + 2a^2t \qquad (5)$$

which begins at an itial value ω_0 and varies linearly with time. There is no loss of generality in making the amplitude of the exciting force unity. The frequency may be either increasing or decreasing with time. In the latter case a of equation (5) is imaginary.

The direct Laplace transform method is not particularly useful since it leads to cumbersome expressions. For example the L transform of cos t^2 may be shown by direct integration to be *

$$L(\cos t^2) = \sqrt{\frac{\pi}{2}}\left[\frac{1}{2}\left(\cos\frac{s^2}{4} - \sin\frac{s^2}{4}\right) - \cos\frac{s^2}{4}\, S\left(\frac{s^2}{4}\right) + \sin\frac{s^2}{4}\, C\left(\frac{s^2}{4}\right)\right] \qquad (6)$$

where s is the complex variable of the Laplace transform, and C and S are Fresnel's integrals for a complex variable, defined by

$$C(z) = \frac{1}{\sqrt{2\pi}}\int_0^z \frac{\cos u}{\sqrt{u}}\, du \qquad (7)$$

and

$$S(z) = \frac{1}{\sqrt{2\pi}}\int_0^z \frac{\sin u}{\sqrt{u}}\, du \qquad (8)$$

Thus the Laplace transform method leads to expressions for which the direct inverse transforms are not known.

A more direct approach is to use the method of real convolution. It is known that the response of a system having a transfer function $F_1(s)$, to an exciting force $f_2(t)$, is given by (5)

$$R = \int_0^t f_1(t-\tau)\, f_2(\tau)\, d\tau \qquad (9)$$

where

$$L\left[f_1(t)\right] = F_1(s) \; .$$

* The Laplace transform notation used throughout is that of reference (5).

This method may be used to obtain the response corresponding to different system transfer functions.

Response of $\frac{1}{s+\delta}$. The response to $\cos(\omega_0 t + a^2t^2)$ of a system whose transfer function is given by

$$F_1(s) = \frac{1}{s+\delta} \quad , \ \delta \text{ real},$$

is obtained as follows.

Let $f_2(t) = \cos(\omega_0 t + a^2t^2) = e^{-j(\omega_0 t + a^2t^2)}$ Real Part.

From a table of transform pairs

$$f_1(t) = e^{-\delta t} \quad .$$

Then, from equation (9), the response is given by

$$R = \int_0^t e^{-\delta(t-\tau)} e^{-j(\omega_0\tau + a^2\tau^2)} d\tau \quad \text{Real Part}$$

$$= e^{-\delta t} \int_0^t e^{-j(\omega_0\tau + a^2\tau^2) + \delta\tau} d\tau \quad \text{Real Part} .$$

Completing the square of the exponent,

$$R = e^{-\delta t + j\frac{(\omega_0 + j\delta)^2}{4a^2}} \int_0^t e^{-j\left(a\tau + \frac{\omega_0 + j\delta}{2a}\right)^2} d\tau \quad \text{Real Part.} \tag{10}$$

Now, let

$$a\tau + \frac{\omega_0 + j\delta}{2a} = u \ ,$$

and equation (10) becomes

$$R = e^{-\delta t} \frac{e}{a}^{\,j\frac{(\omega_o+j\delta)^2}{4a^2}} \int_{\frac{\omega_o+j\delta}{2a}}^{at+\frac{\omega_o+j\delta}{2a}} e^{-ju^2}\,du \quad \text{Real Part.} \qquad (11)$$

But,

$$\int_{\frac{\omega_o+j\delta}{2a}}^{at+\frac{\omega_o+j\delta}{2a}} = \int_{0-j\infty}^{at+\frac{\omega_o+j\delta}{2a}} - \int_{0-j\infty}^{\frac{\omega_o+j\delta}{2a}} .$$

Thus,

$$R = \frac{e}{a}^{-\delta t + j\frac{(\omega_o+j\delta)^2}{4a^2}} \left\{ \int_{0-j\infty}^{at+\frac{\omega_o+j\delta}{2a}} e^{-ju^2}\,du \right.$$

$$\left. - \int_{0-j\infty}^{\frac{\omega_o+j\delta}{2a}} e^{-ju^2}\,du \right\} \quad \text{Real Part.} \qquad (12)$$

At this point there is introduced a new type of function which is given the name "Fresnel function" or "fresneloid" by Hok (4), because of its resemblance to Fresnel's integrals. This function is related to the error integral, but, as will be demonstrated later, the latter is not as convenient for use here. The fresneloid is denoted by Fr(z), where z is in general complex, and it is defined by

$$Fr(z) = -je^{jz^2} \int_{0-j\infty}^{z} e^{-jw^2}\,dw . \qquad (13)$$

Using this definition equation (12) becomes

$$R = \frac{j e^{-j(\omega_0 t + a^2 t^2)}}{a} Fr\left(at + \frac{\omega_0 + j\delta}{2a}\right)$$
$$-j \frac{e^{-\delta t}}{a} Fr\left(\frac{\omega_0 + j\delta}{2a}\right) \quad \text{Real Part.} \qquad (14)$$

The response may also be expressed in terms of the error integral, defined by

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-w^2} dw . \qquad (15)$$

For, comparing this with equation (13) there is obtained

$$Fr(z) = \frac{\sqrt{j\pi}}{2} e^{jz^2} \left[1 - \Phi(j^{1/2} z)\right], \qquad (16)$$

and equation (14) now becomes

$$R = \frac{\sqrt{\pi}}{2} \frac{e^{-\delta t - \frac{\delta \omega_0}{2a^2} + \frac{j}{4a^2}(\omega_0^2 - \delta^2)}}{a j^{1/2}} \left\{ \Phi\left[j^{1/2}\left(at + \frac{\omega_0 + j\delta}{2a}\right)\right]\right.$$
$$\left. - \Phi\left[j^{1/2}\left(\frac{\omega_0 + j\delta}{2a}\right)\right]\right\} \quad \text{Real Part.} \qquad (17)$$

<u>Response of $\frac{1}{(s+\alpha)^2 + \beta^2}$.</u> Again using the method of real convolution and a similar integration process the response of a system having a transfer function

$$F_1(s) = \frac{1}{(s+\alpha)^2 + \beta^2}, \quad \alpha, \beta \text{ real},$$

can be shown to be

$$R = -\frac{e^{-j(\omega_0 t + a^2 t^2)}}{2a\beta} Fr\left(\frac{-j\alpha - \beta - \omega_0}{2a} - at\right)$$
$$+ \frac{e^{-\alpha t + j\beta t}}{2a\beta} Fr\left(\frac{-j\alpha - \beta - \omega_0}{2a}\right) + \frac{e^{-j(\omega_0 t + a^2 t^2)}}{2a\beta} Fr\left(\right.$$
$$\left.\frac{-j\alpha + \beta - \omega_0}{2a} - at\right) - \frac{e^{-\alpha t - j\beta t}}{2a\beta} Fr\left(\frac{-j\alpha + \beta - \omega_0}{2a}\right)$$
$$\text{Real Part.} \qquad (18)$$

Response of $\frac{s}{(s+\alpha)^2+\beta^2}$. Applying the same method as before one obtains the response

$$R = \frac{(\alpha^2+\beta^2)^{1/2}}{2a\beta}\left\{-e^{-j(\omega_o t + a^2t^2+\psi)} Fr\left(\frac{-j\alpha-\beta-\omega_o}{2a} - at\right)\right.$$

$$+e^{-\alpha t+j(\beta t-\psi)} Fr\left(\frac{-j\alpha-\beta-\omega_o}{2a}\right) + e^{-j(\omega_o t+a^2t^2-\psi)} Fr\Bigg($$

$$\left.\frac{-j\alpha+\beta-\omega_o}{2a} - at\right) - e^{-\alpha t-j(\beta t-\psi)} Fr\left(\frac{-j\alpha+\beta-\omega_o}{2a}\right)\Bigg\} \text{ Real Part}$$

$$(19)$$

where $\psi = \tan^{-1}\frac{\beta}{\alpha}$.

The system responses given by equations (14), (18), and (19) are all that are needed for any linear system. The transfer function of any system can, by a partial fraction expansion, be considered as made up of terms of the type

$$\frac{1}{s+\delta}, \quad \frac{1}{s+\alpha+j\beta}, \quad \frac{1}{s+\alpha-j\beta},$$

since for any realizable system the complex poles occur in complex conjugate pairs. Then the last two terms above can be combined into terms of the form

$$\frac{1}{(s+\alpha)^2+\beta^2}, \quad \frac{s}{(s+\alpha)^2+\beta^2} .$$

Thus all the partial fraction expansions may be considered as made up of terms of the form

$$\frac{1}{s+\delta}, \quad \frac{1}{(s+\alpha)^2+\beta^2}, \quad \frac{s}{(s+\alpha)^2+\beta^2}$$

for which equations (14), (18), and (19) give the corresponding responses.

The convenience of the Fresnel function as contrasted with the more classical error integral may be demonstrated as follows.

Consider the response of a system exhibiting simple exponential damping. This system has a transfer function of the form

$$\frac{1}{s + \delta} \, .$$

(Of course a constant term may be added, depending upon the particular problem.) Suppose the excitation frequency begins at zero and increases linearly thereafter. Then, from equation (14) the response is

$$R = \frac{je^{-ja^2t^2}}{a} \operatorname{Fr}\left(at + \frac{j\delta}{2a}\right) - \frac{je^{-\delta t}}{a} \operatorname{Fr}\left(\frac{j\delta}{2a}\right) \text{ Real Part.} \quad (20)$$

This is, in form, very similar to the response of the same system to a sinusoidal excitation applied at t = 0. The first term of equation (20) is exactly like the steady state response in the case of the sinusoidal force, except that it is modified by the factor $\operatorname{Fr}(at + \frac{j\delta}{2a})$. The second, an exponential damping term multiplied by a constant, is analogous to the exponentially decaying term for sinusoidal excitation. However, expressing equation (20) in terms of the error integral

$$R = \frac{\sqrt{\pi}}{2aj^{1/2}} e^{-\delta t - j\frac{\delta^2}{4a^2}} \left\{ \Phi\left[j^{1/2}\left(at + \frac{j\delta}{2a}\right)\right] - \Phi\left[\frac{j^{3/2}\delta}{2a}\right] \right\}$$

$$\text{Real Part}$$

gives something which does not lend itself to such analogous interpretation.

<u>Response of $\frac{1}{s^2 + \beta^2}$ to $\cos a^2t^2$.</u> When a system has no damping, as indicated by a transfer function of the type

$$\frac{1}{s^2 + \beta^2} \, , \quad \beta \text{ real,}$$

the response may be more conveniently expressed in terms of Fresnel's integrals. In the case $\omega_0 = 0$ the response is shown by the method of real convolution to be

$$R = \frac{\sqrt{2\pi}}{4\beta}\left\{\cos\left(\beta t + \frac{\beta^2}{4a^2}\right)\left[-S\left[\left(at + \frac{\beta}{2a}\right)^2\right] + S\left[\left(at - \frac{\beta}{2a}\right)^2\right]\right]\right.$$
$$+\sin\left(\beta t + \frac{\beta^2}{4a^2}\right)\left[C\left[\left(at + \frac{\beta}{2a}\right)^2\right] - 2C\left(\frac{\beta^2}{4a^2}\right)\right.$$
$$\left.\left.+C\left[\left(at - \frac{\beta}{2a}\right)^2\right]\right]\right\}, \qquad (21)$$

where S and C are given by equations (7) and (8) respectively. Fresnel's integrals are tabulated in a number of references. (6)

The analytical form of a system's response may be considered for a force whose frequency is varying as an arbitrary function of time, i.e. other than linear, but the resulting expressions are of a form which is of little value here.

<u>Computation of Fr and Φ.</u> Unfortunately tabulated values of Fr or Φ for complex arguments are not readily available. Reference (4) gives a limited number of values of Fr in graphical form. Functions of this type have been used in other varieties of work, notably in the theory of probability and statistics, the theory of optics, the theory of heat transfer, the theory of electro-chemical diffusion, and in the mechanics of rocket flight. In connection with the latter a number of methods of computation have been developed. (7)

Series expansions for the error integral, Φ, are given as follows. (6)

$$|z|<1 \qquad \Phi(z) = \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{1!\,3} + \frac{z^5}{2!\,5} - \cdots\cdots\right)$$

$$|z|>1 \qquad \frac{\sqrt{\pi}}{2}\left[1-\Phi(z)\right] = \frac{e^{-z^2}}{2z}\left(1 - \frac{1}{2z^2} + \frac{1\cdot 3}{(2z^2)^2} - \frac{1\cdot 3\cdot 5}{(2z^2)^3} + \cdots\right)$$

Particular values are given by

$$\Phi(-z) = -\Phi(z)$$

$$\Phi(0) = 0$$

$$\Phi(\infty + j0) = 1 \quad .$$

For the fresneloid, Fr, useful expansions are

$$|z|<1 \qquad Fr(z) = e^{jz^2}\left[\frac{\sqrt{j\pi}}{2} - jz - \frac{z^3}{1!3} + j\frac{z^5}{2!5} + \frac{z^7}{3!7} - \cdots\right]$$

$$|z|>1 \qquad Fr(z) = \frac{1}{2z} + \frac{j}{4z^3} - \frac{3}{8z^5} - j\frac{3\cdot 5}{16z^7} + \frac{3\cdot 5\cdot 7}{32z^9} + \cdots$$

Or, if the argument is expressed in polar form,

$$z = re^{j\theta} \quad ,$$

there results

$$|z|<1 \qquad Fr(z) = e^{jz^2}\left\{\frac{1}{2}\sqrt{\frac{\pi}{2}} + r\sin\theta - \frac{r^3}{1!3}\cos 3\theta - \frac{r^5}{2!5}\sin 5\theta + \cdots + j\left[\frac{1}{2}\sqrt{\frac{\pi}{2}} - r\cos\theta - \frac{r^3}{1!3}\sin 3\theta + \frac{r^5}{2!5}\cos 5\theta + \cdots\right]\right\}$$

$$|z|>1 \qquad Fr(z) = \frac{1}{2r}\cos\theta + \frac{1}{4r^3}\sin 3\theta - \frac{3}{8r^5}\cos 5\theta - \frac{3\cdot 5}{16r^7}\sin 7\theta + \cdots + j\left[-\frac{1}{2r}\sin\theta + \frac{1}{4r^3}\cos 3\theta + \frac{3}{8r^5}\sin 5\theta - \frac{3\cdot 5}{16r^7}\cos 7\theta - \cdots\right] .$$

Also,

$$Fr(-z) = e^{jz^2}\sqrt{j\pi} - Fr(z)$$

$$\underset{b \text{ real} > 0}{Fr(jb)} = -j\sqrt{\frac{\pi}{2}}\, e^{-jb^2}\left[-j^{-1/2} - jC(b^2) + S(b^2)\right]$$

$$\underset{a \text{ real} > 0}{Fr(a)} = -j\sqrt{\frac{\pi}{2}}\, e^{ja^2}\left[-j^{-1/2} + C(a^2) - jS(a^2)\right]$$

$$Fr(0) = \frac{\sqrt{j\pi}}{2} \quad .$$

Complete System Response. The following outline is a summary of the steps necessary to obtain the response of a complete lumped linear system to an excitation by cos $(\omega_o t + a^2t^2)$.

1. Write the differential equations for the system and express them in transform form, including initial conditions as necessary.

2. Express the desired response in transform form as

$$R(s) = \frac{P(s)}{Q(s)} \quad .$$

3. Calculate the roots of $Q(s)$.

4. Corresponding to the roots in step 3 evaluate the response by use of equations (14), (18), and (19), and then add the contribution due to each root.

Step 3 alone may be extremely laborious for a practical problem if there are a number of roots (e.g. $Q(s)$ is a polynomial of high order) and it is desired to study the effect of a great many parameters each of which affects $Q(s)$. Even if complete tables of the Fr or Φ functions for complex arguments were available step 4 would be tedious, as examination of equations (14), (18), and (19) shows. In this connection it is well to emphasize that the single degree of freedom solutions available (3)(4) are solutions giving the envelope of the magnitude of the response; they do not, as such, give the phase angle. And, it is necessary to take the phase angles into consideration when performing the addition in step 4 above. This fact will be illustrated later by means of experimental results.

For these reasons it is desirable to resort to machine methods of computation, and the Electric Analog Computer fits well the needs of such computation.

IV. CONSIDERATION OF ELECTRIC ANALOG METHOD

Forcing Function Generator. Application of the electric analog method to this type of problem necessitates first a device which will supply electrically the forcing function or exciting force, and such a device will be described first. Following a brief description of its operation, analytical justification for this operation will be given.

(Fig. 1) shows a block diagram of the electronic device used. Two oscillators are used; one is fixed in frequency, f_o , and the other has a frequency which is made to vary from f_o, due to the reactance tube in its plate circuit, by an amount Δf. The frequency difference, Δf , is proportional to the input voltage to the reactance tube. The output of the variable frequency oscillator is passed through a high pass filter to eliminate any of the low frequency voltage components due to the modulating signal applied to the reactance tube. This output is then mixed in a multigrid tube with the fixed oscillator's signal, and the output, after suitable filtering, is a signal of constant amplitude and of frequency Δf. Thus the device produces an output signal of constant amplitude whose frequency is linearly proportional to the input voltage. The limitations of such a scheme will become apparent through the discussion to follow.

(Fig. 2) shows two views of the equipment used. The circuit techniques are standard. A type 6AC7 tube serves as the reactance tube since its transconductance is proportional to first grid voltage over a wide range.

Consider an oscillating tank circuit consisting of a fixed inductance, L , and a capacitance, C , which changes such that the capacitive reactance varies linearly with time. Thus,

$$C = \frac{C_o}{1 + bt} \quad .$$

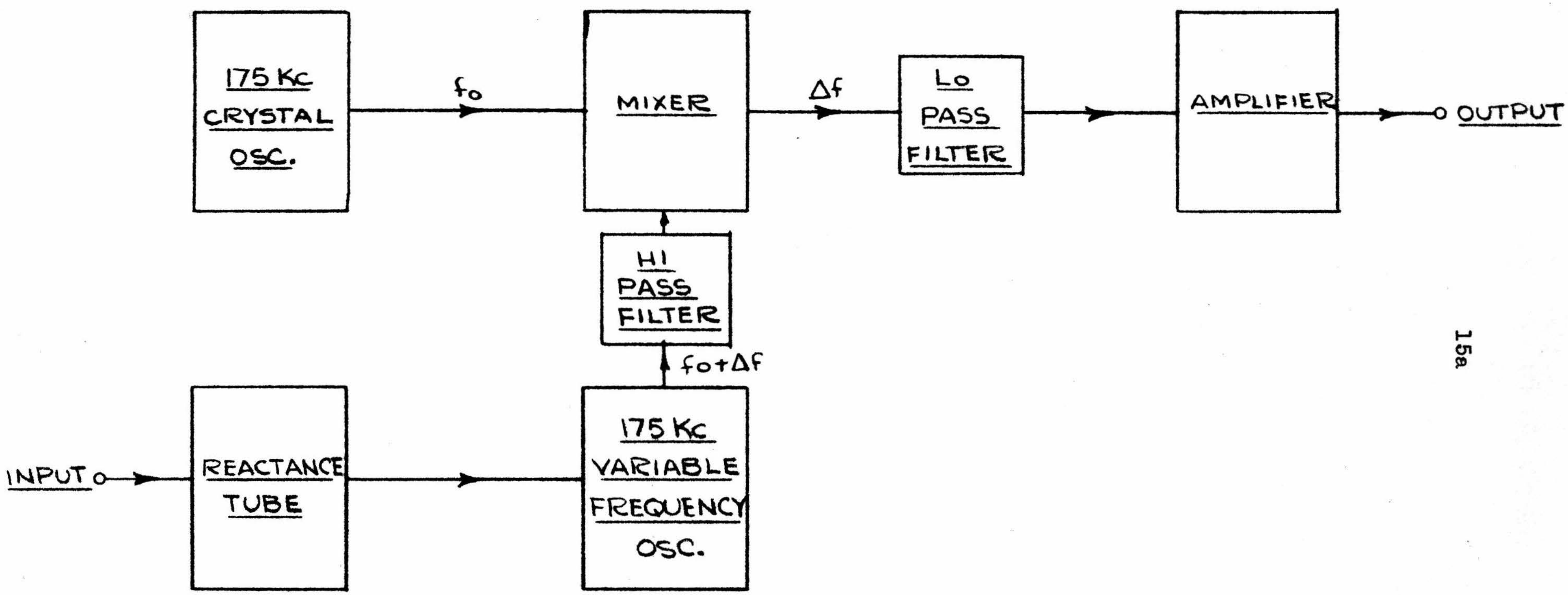
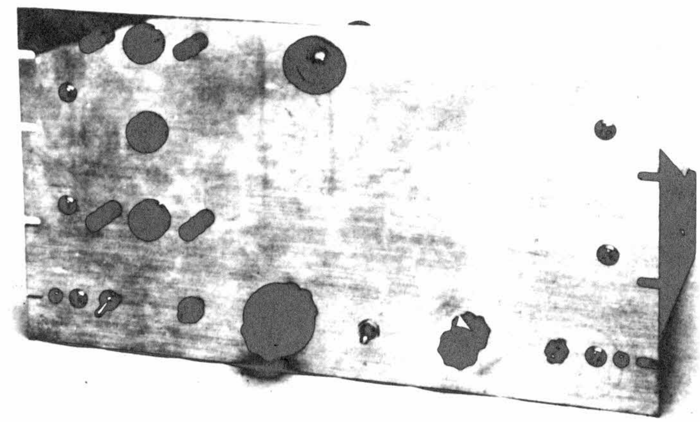


FIG. I BLOCK DIAGRAM OF FORCING FUNCTION GENERATOR



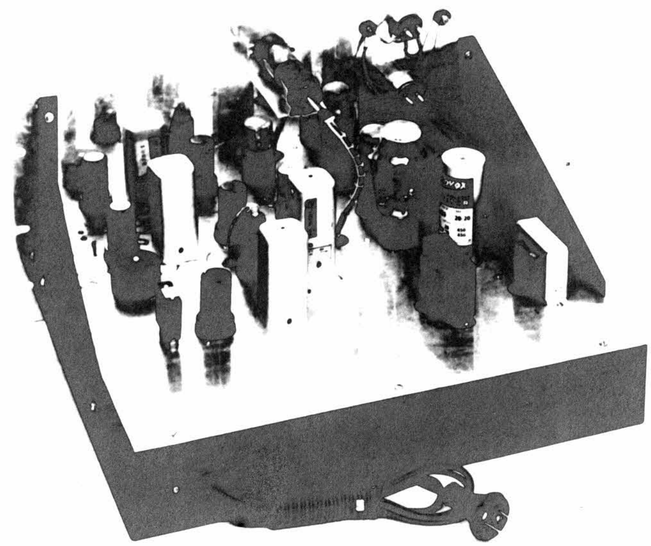


FIG. 2 FORCING FUNCTION GENERATOR

If q is the instantaneous charge on the capacitor,

$$L\ddot{q} + \frac{q}{C_0}(1+bt) = 0$$

describes the oscillation. This may be written

$$\ddot{q} + \omega_1^2(1+bt)q = 0, \tag{22}$$

where

$$\omega_1^2 = \frac{1}{LC_0} .$$

It has been shown by L. Knopoff that the general solution of equation (22) is

$$q = \omega_1\sqrt{1+bt}\left\{AJ_{\frac{1}{3}}\left[\frac{2\omega_1}{3b}(1+bt)^{3/2}\right] + BY_{\frac{1}{3}}\left[\frac{2\omega_1}{3b}(1+bt)^{3/2}\right]\right\}, \tag{23}$$

where A and B are arbitrary constants and $J_{1/3}$ and $Y_{1/3}$ are Bessel functions of one-third order and of the first and second kinds respectively.

The case of present interest is one in which the frequency deviation is small and the rate of percentage change of capacitive reactance is much less than ω_1. Thus,

$$bt \ll 1$$

$$b \ll \omega_1 ,$$

and using the asymptotic Bessel function relationships for large argument

$$J_{\frac{1}{3}}(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{5\pi}{12}\right) \tag{24}$$

$$Y_{\frac{1}{3}}(x) \approx \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{5\pi}{12}\right), \tag{25}$$

equation (23) may be expressed

$$q = \sqrt{\frac{3b\omega_1}{\pi}}(1+bt)^{-1/4}\left\{A\cos\left[\frac{2\omega_1}{3b}(1+bt)^{3/2} - \frac{5\pi}{12}\right]\right.$$
$$\left. + B\sin\left[\frac{2\omega_1}{3b}(1+bt)^{3/2} - \frac{5\pi}{12}\right]\right\}$$

or, choosing new arbitrary constants A' and φ ,

$$q = \frac{A'}{(1+bt)^{1/4}} \cos\left[\frac{2\omega_1}{3b}(1+bt)^{3/2} + \varphi\right] . \qquad (26)$$

But, since

$$(1+bt)^{3/2} = 1 + \tfrac{3}{2}bt + \tfrac{3}{8}(bt)^2 - \tfrac{3}{48}(bt)^3 + \tfrac{3}{128}(bt)^4 - \cdots ,$$

$$q = \frac{A'}{(1+bt)^{1/4}} \cos\left\{\frac{2\omega_1}{3b}\left[1 + \tfrac{3}{2}bt + \tfrac{3}{8}(bt)^2 - \tfrac{3}{48}(bt)^3 + \cdots\right] + \varphi\right\} .$$

As far as the frequency of oscillation is concerned there is no loss of generality in letting

$$\varphi = -\frac{2\omega_1}{3b} ,$$

in which case equation (26) becomes

$$q = \frac{A'}{(1+bt)^{1/4}} \cos\left\{\frac{2\omega_1}{3b}\left[\tfrac{3}{2}bt + \tfrac{3}{8}(bt)^2 - \tfrac{3}{48}(bt)^3 + \cdots\right]\right\} . \qquad (27)$$

This represents an oscillation of slowly damped amplitude and a frequency, referring to equation (2), of

$$f_1 = \frac{1}{2\pi} \cdot \frac{2\omega_1}{3b} \frac{d}{dt}\left[\tfrac{3}{2}bt + \tfrac{3}{8}(bt)^2 - \tfrac{3}{48}(bt)^3 + \tfrac{3}{128}(bt)^4 - \cdots\right]$$

$$= \frac{\omega_1}{2\pi}\left[1 + \tfrac{1}{2}bt - \tfrac{1}{8}(bt)^2 + \tfrac{1}{16}(bt)^3 - \cdots\right] . \qquad (28)$$

Now consider the steady state frequency of oscillation of the same tank circuit for a static setting of the capacitance which is equal in value to

$$\frac{C_0}{1+bt} .$$

This is given by

$$f_2 = \frac{\omega_1}{2\pi}\sqrt{1+bt}$$

$$= \frac{\omega_1}{2\pi}\left[1 + \tfrac{1}{2}bt - \tfrac{1}{8}(bt)^2 + \tfrac{1}{16}(bt)^3 - \cdots\right] . \qquad (29)$$

Comparison of equation (29) with equation (28) shows that f_1 and f_2 are exactly equal. Of course the same would be true if the inductive reactance varied. This leads to the following interesting conclusion:

For an LC oscillating tank circuit whose capacitive or inductive reactance varies linearly with time over a narrow range and at a fractional rate considerably less in value than the initial resonant frequency, the instantaneous frequency of oscillation is given by

$$\frac{1}{2\pi\sqrt{LC}} ,$$

where L and C are the instantaneous values of inductance and capacitance respectively.

Furthermore, intuition suggests that if the variation of reactance with time is other than linear, e.g. parabolic, the instantaneous frequency of oscillation is still given by the above expression as long as the total frequency deviation and rate of change are kept low.

Of course a practical tank circuit has associated with it certain losses in the inductive and capacitive elements. However, when such a circuit is employed in a vacuum tube oscillator the effect is very nearly that of the tube compensating for these losses.

Numerical considerations show how nearly linear the output frequency of an oscillator of this type may be expected to be. The value of ω_1 used in the forcing function generator corresponds to a frequency of (see Fig. 1) 175 kilocycles. The greatest excursion of frequency used in the computing work to be described later was 1 kc. And the greatest rate of change of capacitive reactance was that corresponding to varying the oscillator frequency 1 kc.

in 0.05 sec. Thus,

$$\text{max. } bt \approx \frac{2}{175} \tag{30}$$

$$\frac{\omega_1}{b} \approx 4.8 \times 10^6 \quad ,$$

and equations (24) and (25) are therefore justified. Hence, the frequency is correctly given by equations (28) and (29). An idea of the maximum departure of the frequency from linear is given by the ratio of the third to second terms in equation (28), evaluated when bt is a maximum. This gives

$$\frac{\frac{1}{8}\, b^2 t^2}{\frac{1}{2}\, bt} = 0.28\ \% \quad ,$$

which is acceptable for this work.

Attention will next be given to the mixer problem. A signal of fixed frequency $\frac{\omega_1}{2\pi}$ is applied to one grid of a multigrid mixer tube, and to the other grid is applied a signal given by equation (27). Thus the two signals are

$$e_1 = \cos \omega_1 t$$

$$e_2 = \frac{1}{(1+bt)^{1/4}} \cos \left\{ \frac{2\omega_1}{3b} \left[\frac{3}{2} bt + \frac{3}{8} (bt)^2 - \frac{3}{48} (bt)^3 + \cdot \cdot \cdot \right] \right\}_1 .$$

Assuming square law mixer action, the mixer output is given by

$$\begin{aligned} e &= (K_1 e_1 + K_2 e_2)^2 \\ &= K_1^2 \cos^2 \omega_1 t + \frac{K_2^2}{(1+bt)^{1/2}} \cos^2 \{ \quad \}_1 \\ &\quad + \frac{2 K_1 K_2}{(1+bt)^{1/4}} \cos \omega_1 t \cos \{ \quad \}_1 . \end{aligned} \tag{31}$$

Making use of the relationships

$$\cos x \cos y = \frac{1}{2} \left[\cos (x+y) + \cos (x-y) \right]$$

and

$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad ,$$

equation (31) becomes

$$e = \frac{K_1^2}{2}\left[1+\cos 2\omega_1 t\right] + \frac{K_2^2}{2(1+bt)^{1/2}}\left[1+\cos 2\{\quad\}_1\right]$$

$$+ \frac{K_1 K_2}{(1+bt)^{1/4}}\cos\left\{\frac{2\omega_1}{b}\left[bt+\frac{1}{8}(bt)^2-\frac{1}{48}(bt)^3+\cdot\cdot\cdot\right]\right\}$$

$$+ \frac{K_1 K_2}{(1+bt)^{1/4}}\cos\left\{\frac{2\omega_1}{b}\left[\frac{1}{8}(bt)^2-\frac{1}{48}(bt)^3+\frac{1}{128}(bt)^4-\cdot\cdot\cdot\right]\right\}. \tag{32}$$

If the stages following the mixer will pass neither the D. C. terms, (the amplifier of Fig. 1 meets this qualification), nor the terms of frequency $2\omega_1$ and higher, (see Lo Pass Filter of Fig. 1), the only term remaining in the output is

$$\frac{K_1 K_2}{(1+bt)^{1/4}}\cos\left\{\frac{2\omega_1}{b}\left[\frac{1}{8}(bt)^2-\frac{1}{48}(bt)^3+\frac{1}{128}(bt)^4-\cdot\cdot\cdot\right]\right\}.$$

This is a voltage whose amplitude decays as $(1+bt)^{-1/4}$ and whose frequency is

$$\frac{\omega_1}{2\pi}\left[\frac{1}{2}bt-\frac{1}{8}(bt)^2+\frac{1}{16}(bt)^3-\cdot\cdot\cdot\right],$$

which is the desired result. The small decay in amplitude is not troublesome and can be equalized without affecting the frequency.

With the foregoing considerations in mind the forcing function generator of (Fig. 2) was constructed. A plot of its static frequency output vs. D. C. input voltage is shown in (Fig. 3). The linearity for frequencies below 1000 cycles per sec., the maximum frequency used, is acceptable.

(Fig. 4) shows the forcing function generator output as a function of time when the input voltage varies linearly with time.

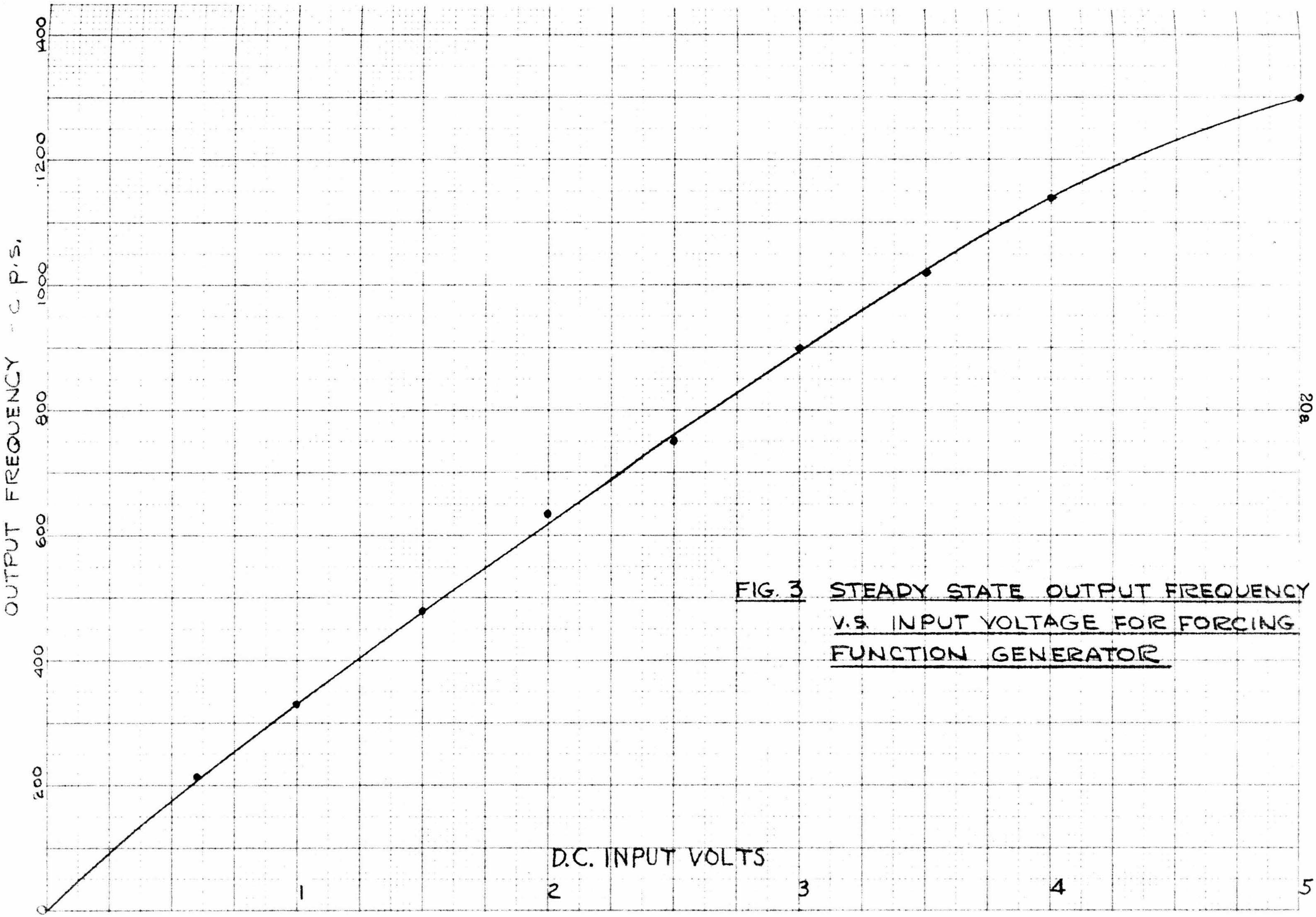


FIG. 3 STEADY STATE OUTPUT FREQUENCY V.S. INPUT VOLTAGE FOR FORCING FUNCTION GENERATOR

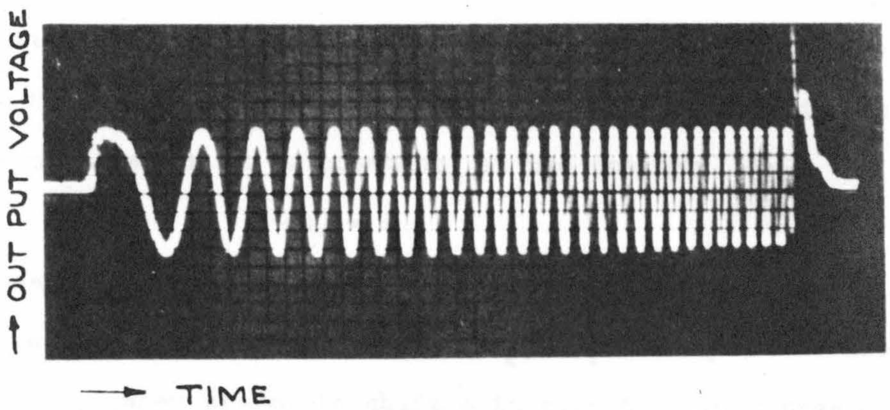


FIG. 4 FORCING FUNCTION GENERATOR OUTPUT.
FREQUENCY 0-1000 c.p.s. IN 0.05 SEC.

The input voltage is the voltage across a capacitor being charged by a constant current generator. Thus, the waveshape in (Fig. 4), since the frequency begins at zero, is cos a^2t^2. The instantaneous frequency at the right hand end of the trace is 1000 c.p.s.

<u>Measurement of Instantaneous Frequency.</u> In using the forcing function generator it is desirable to have some fairly direct method for measuring the instantaneous value of the time-varying frequency of the generator output.

One method of frequency measurement stems from the original definition of frequency. From equations (1) and (2) it is seen that a function cos $[g(t)]$ has the same instantaneous frequency as a sine wave sin mt (m constant) at the instant

$$g'(t) = m .$$

This means that on a graph of cos $[g(t)]$ and sin mt, plotted about the same zero line, the slope of cos $[g(t)]$ will equal the slope of sin mt , (when sin mt is shifted in time to have corresponding phase), at points on cos $[g(t)]$ which have instantaneous angular frequency m. Thus the two curves are tangent at these points. If cos $[g(t)]$ is now considered to be the output of the forcing function generator it may be presented on an oscilloscope screen, and sin mt may be superimposed on the same screen, (by using two beams or a rapid switching technique). A value of m may be chosen, and by matching the two patterns the instantaneous frequency of any portion of cos $[g(t)]$ is determined.

Resonant circuits, though useful for steady state frequency measurements, are not practical for measuring a time-varying frequency. For example it has been shown by Lewis (3) that if a linearly varying frequency is applied to a simple tuned resonant circuit the peak response, (or null for some circuits), of the circuit occurs later than the instant at which the instantaneous excitation frequency is equal to the circuit's resonant frequency. And the delay depends upon the rate of change of excitation frequency.

The most convenient method of frequency measurement found, and one giving sufficient accuracy, was that employing a Wien bridge. Such a bridge contains only resistances and capacitances and is not affected by the considerations of the previous paragraph.

(Fig. 5a) shows a Wien bridge of the type used. For steady state frequency measurements the supplied voltage is applied to the upper and lower diagonal points of the bridge, and at a frequency

$$f = \frac{1}{2\pi RC}$$

the voltage between the other pair of diagonal points is zero. By continuously varying R the bridge may be used to measure a range of frequencies.

Consider the right hand side of the bridge shown in (Fig. 5b). The source voltage is applied between a and d. In transform form

$$\frac{E_{db}}{E_{da}} = \frac{1}{RC}\left[\frac{a_1}{s+\frac{\delta_2}{RC}} + \frac{a_2}{s+\frac{\delta_1}{RC}}\right], \qquad (33)$$

where

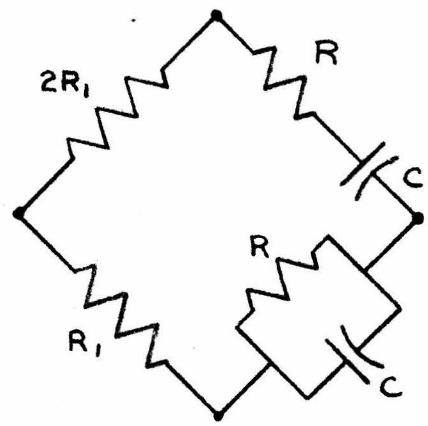
$$a_1 = \frac{1}{2} + \frac{3}{4}\sqrt{\frac{4}{5}}$$

$$a_2 = \frac{1}{2} - \frac{3}{4}\sqrt{\frac{4}{5}}$$

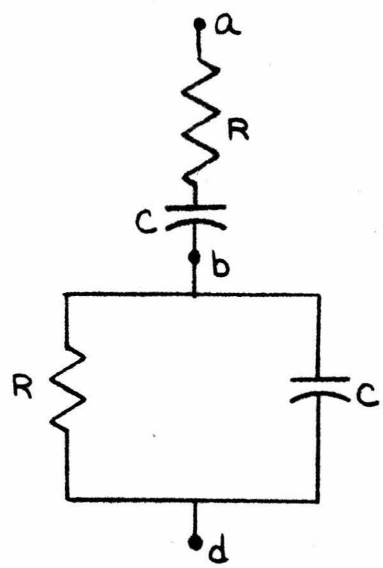
$$\delta_1 = \frac{3}{2} - \sqrt{\frac{5}{4}}$$

$$\delta_2 = \frac{3}{2} + \sqrt{\frac{5}{4}} \quad .$$

If $e_{da} = \cos a^2t^2$, (a frequency of $\frac{2a^2t}{2\pi}$), the response of this circuit is given by applying equation (14) to equation (33), wherein there results



(a) COMPLETE BRIDGE



(b) RIGHT SIDE OF BRIDGE

FIG. 5 WIEN BRIDGE FOR INSTANTANEOUS FREQUENCY MEASUREMENT

$$e_{db} = j\,\frac{e^{-ja^2t^2}}{aRC}\left\{a_1 Fr\left(at + j\frac{\delta_2}{2aRC}\right) + a_2 Fr\left(at + j\frac{\delta_1}{2aRC}\right)\right\}$$

Real Part. (34)

(The exponentially damped terms are omitted, as it may easily be shown that they are negligible for the values of time to be considered).

Let f_m be the frequency of $\cos a^2t^2$ at time t_m. Then

$$f_m = \frac{2a^2 t_m}{2\pi} .$$

Define

$$h = \frac{f_m}{t_m} \qquad (35)$$

as the frequency rate of change.

Now let one attempt to measure a frequency f_1 which occurs at a time t_1. To do this the bridge is adjusted such that

$$f_1 = \frac{1}{2\pi RC}$$

Then, at time t_1 the amplitude of the bridge response e_{db} is given by equation (34) to be

$$M = j\,\frac{2\pi f_1}{\sqrt{\pi h}}\left\{a_1 Fr\left[f_1\sqrt{\tfrac{\pi}{h}}(1+j\delta_2)\right] + a_2 Fr\left[f_1\sqrt{\tfrac{\pi}{h}}(1+j\delta_1)\right]\right\}. \qquad (36)$$

As a check, letting h approach zero, corresponding to a frequency which changes infinitely slowly, should give the steady state value of response. From the series expansions for $Fr(z)$,

$$\underset{z\to\infty}{Fr(z)} \longrightarrow \frac{1}{2z} .$$

Thus equation (36) becomes

$$\underset{h\to 0}{M} \approx j\sqrt{\frac{\pi}{h}}\left\{\frac{a_1}{\sqrt{\frac{\pi}{h}}(1+j\delta_2)} + \frac{a_2}{\sqrt{\frac{\pi}{h}}(1+j\delta_1)}\right\} . \qquad (37)$$

Substituting the values of a_1, a_2, δ_1, δ_2 into equation (37) yields

$$\underset{h\to 0}{M} \approx j\sqrt{\frac{\pi}{h}}\left\{\frac{\sqrt{\frac{\pi}{h}} + j0}{\frac{\pi}{h} + j\frac{3\pi}{h} - \frac{\pi}{h}}\right\} = \frac{1}{3} \quad ,$$

which is exactly the steady state result, since the left side of the bridge (Fig. 5a) is simply a 1 to 3 voltage divider.

At time t_1 the bridge does not give a null output but gives a voltage

$$\frac{1}{3}e^{-j\alpha^2 t_1^2}\text{Real Part} - j\frac{2\pi f_1}{\sqrt{\pi h}}\left\{a_1 Fr\left[f_1\sqrt{\frac{\pi}{h}}(1+j\delta_2)\right]\right.$$
$$\left. + a_2 Fr\left[f_1\sqrt{\frac{\pi}{h}}(1+j\delta_1)\right]\right\} e^{-j\alpha^2 t_1^2}\text{Real Part}. \qquad (38)$$

Since the difference of the real parts is the real part of the difference, equation (38) describes an oscillating voltage whose amplitude envelope has, at time t_1 , a magnitude

$$\left|\frac{1}{3} - j\frac{2\pi f_1}{\sqrt{\pi h}}\left\{a_1 Fr\left[f_1\sqrt{\frac{\pi}{h}}(1+j\delta_2)\right] + a_2 Fr\left[f_1\sqrt{\frac{\pi}{h}}(1+j\delta_1)\right]\right\}\right|. \qquad (39)$$

The nature of the envelope is shown by the bridge response of (Fig. 6).

The value given by expression (39), which determines the error of the bridge, is a function of f_1^2/h . For frequencies measured which are greater than 500 c.p.s. and rates of frequency change less than 20,000 c.p.s. per sec., which are the limits within which the bridge was used, calculations using series expansions for the Fr function show that the value of expression (39) is less than 1 per cent of the magnitude of the envelope at $t = 0$. The answer to the question: At what time does the envelope become zero?, involves finding the roots of expression (39), which is an extremely tedious calculation. However, it can be seen from (Fig. 6) that the envelope magnitude is varying fairly linearly with time. Consequently, one may conclude that the error in time, (or, since

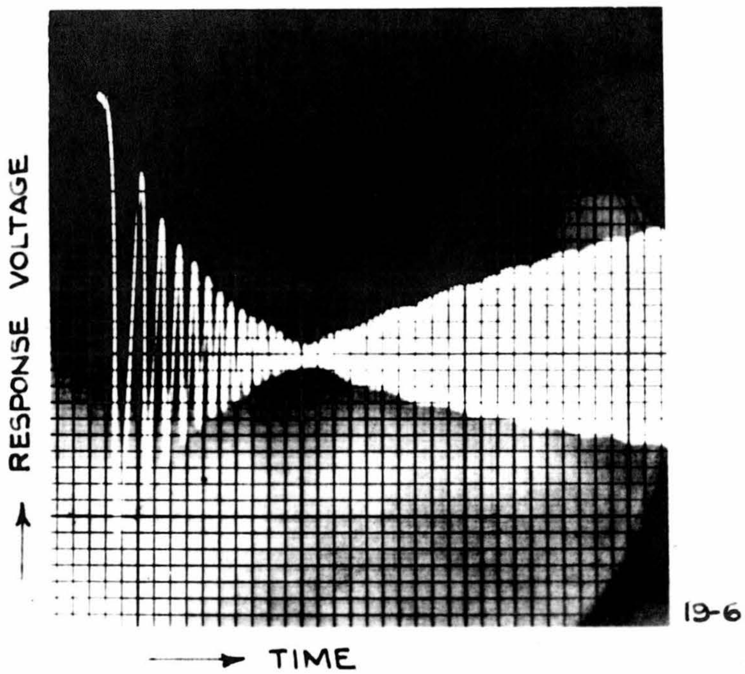


FIG. 6 RESPONSE OF WIEN BRIDGE TO A FREQUENCY VARYING LINEARLY WITH TIME

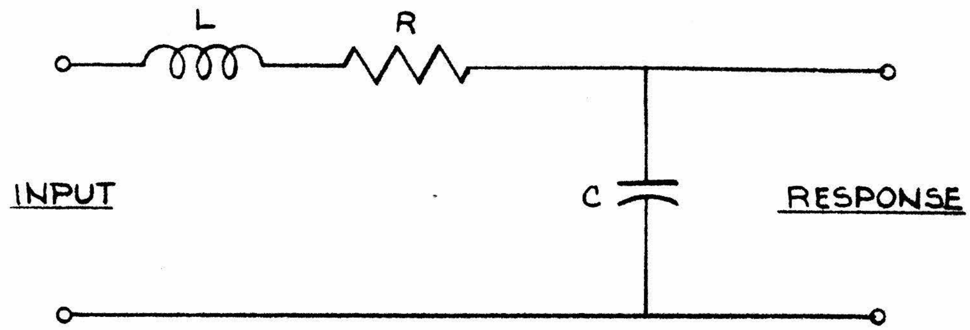
frequency is varying linearly with time, the error in instantaneous frequency measured), is also indicated quite accurately by expression (39). Thus, for the frequency measurements made the error, based upon theoretical limits, is less than one per cent. Of course the above reasoning does not indicate whether the error is plus or minus, but that is not important here.

Now that the forcing function has been developed it may be applied to various electric circuits which are the analogs of the physical systems to be studied. The basic concepts of the electric analog method of computation are treated in a number of references. (8)(9)

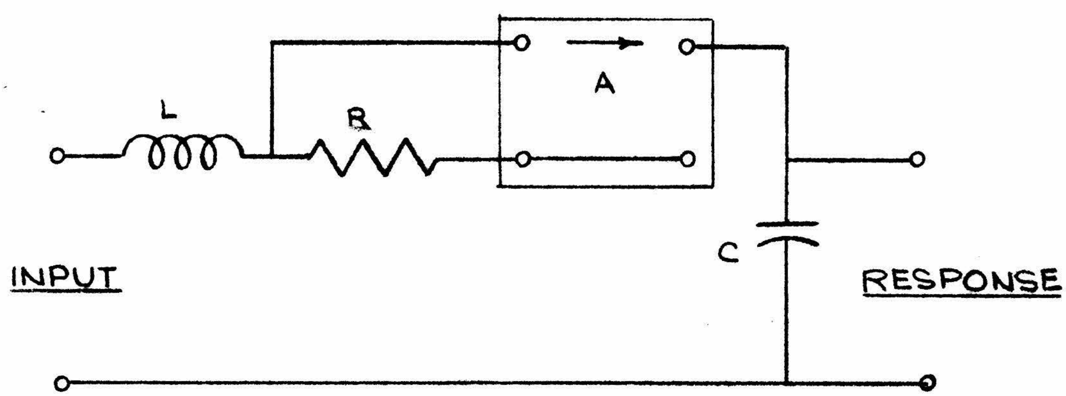
V. RESPONSE OF SIMPLE SYSTEMS TO LINEAR TIME-VARYING FREQUENCY

Single Degree of Freedom. The single degree of freedom case has been adequately described by Lewis (3) and Hok (4). However, as a prelude to the more difficult material to follow it is felt that to display here some electric analog computer results for the simple cases is perhaps wise.

In (Fig. 7a) is shown an electric circuit with a single degree of freedom. A voltage cos a^2t^2, when applied to this circuit, gives a response voltage shown as the upper trace in each photograph of (Fig. 8). The lower trace is the output of a null bridge circuit to which the same excitation is applied. The bridge is set such that the null frequency is the steady state resonant frequency of the circuit of (Fig. 7a). The three photographs are for three different rates of change of excitation frequency, the top photograph being for the slowest rate. It is noted that the peak response occurs later than the time at which the instantaneous frequency is equal to the steady state resonant frequency, the delay, measured



(a) DAMPED SYSTEM CIRCUIT



(b) CIRCUIT FOR ZERO DAMPING

FIG. 7 ELECTRIC CIRCUITS FOR SINGLE DEGREE OF FREEDOM

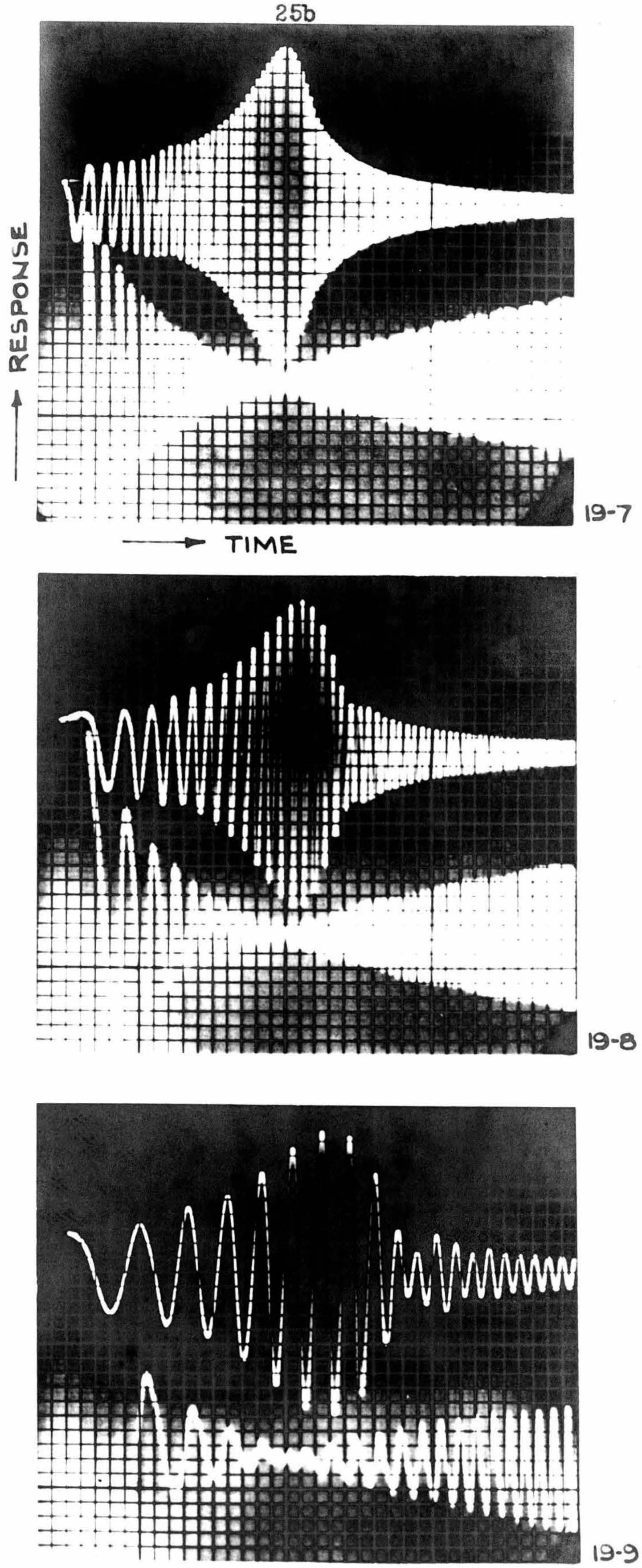


FIG. 8 SINGLE DEGREE OF FREEDOM DAMPED SYSTEM RESPONSE SHOWING SHIFT OF PEAK RESPONSE

in terms of frequency, being greater for higher rates of frequency change. In each photograph the total frequency range is the same. Since the vertical scale in each photograph is the same it is also noted that the magnitude of the response is higher for lower rates of change of frequency, approaching the steady state resonance curve form as the frequency is varied more and more slowly. The value of R (Fig. 7a) is adjusted to one-tenth the value for critical damping.

A single degree of freedom system having zero damping is obtained by using an amplifier (Fig. 7b), with positive gain, which just compensates for the loss in the resistance R and for the losses in the inductance L and capacitance C. A typical response for this system is shown in (Fig. 9).

More Than One Degree of Freedom. It will now be demonstrated that single degree of freedom solutions cannot be added together in amplitude only to get the solution for systems having more than one degree of freedom.

(Fig. 10) shows a circuit used to add two single degree of freedom solutions. The voltage e_1 is the response of a system function

$$\frac{1}{(s+\alpha_1)^2 + \beta_1^2}$$

to $\cos a^2t^2$, and e_2 is the response of

$$\frac{1}{(s+\alpha_2)^2 + \beta_2^2}$$

to the same excitation. These voltages are shown in (Fig. 11a) and (Fig. 11b) respectively. In (Fig. 11c) is shown the sum, $e_1 + e_2$. The envelope of the latter curve is clearly not the sum of the envelopes of e_1 and e_2 separately.

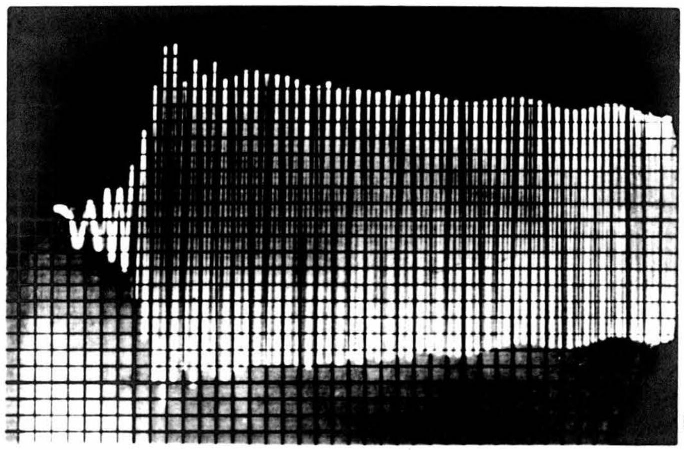


FIG. 9 RESPONSE OF SINGLE DEGREE OF FREEDOM SYSTEM HAVING ZERO DAMPING

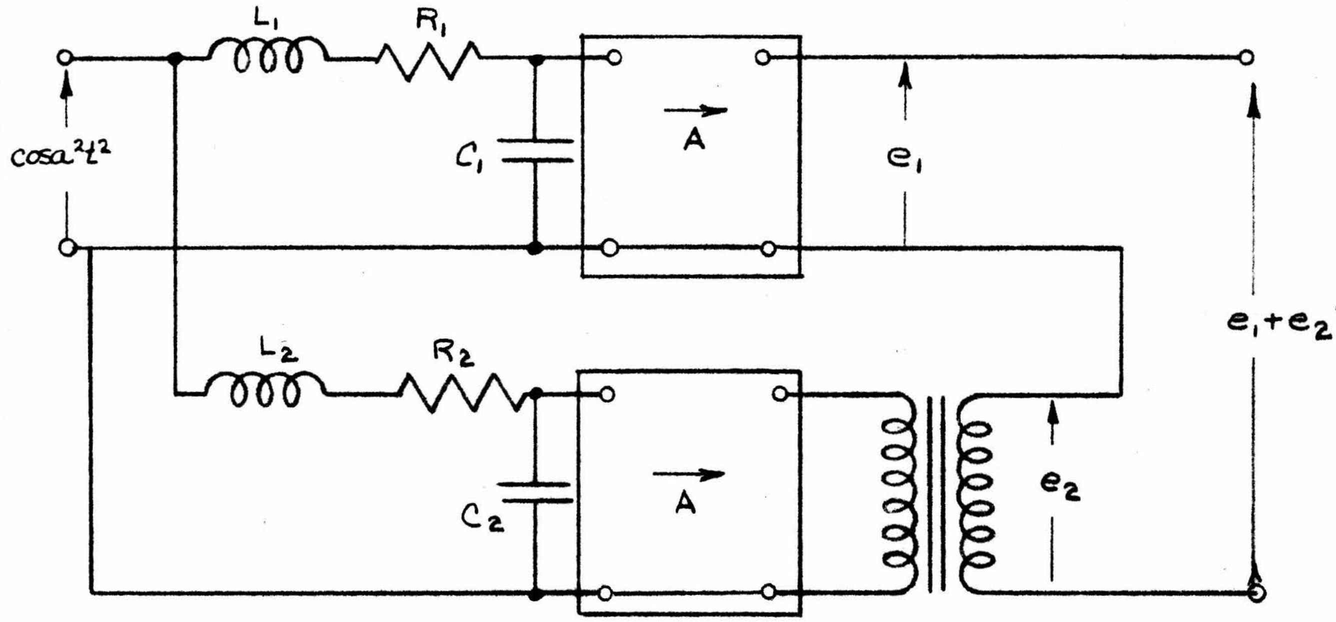
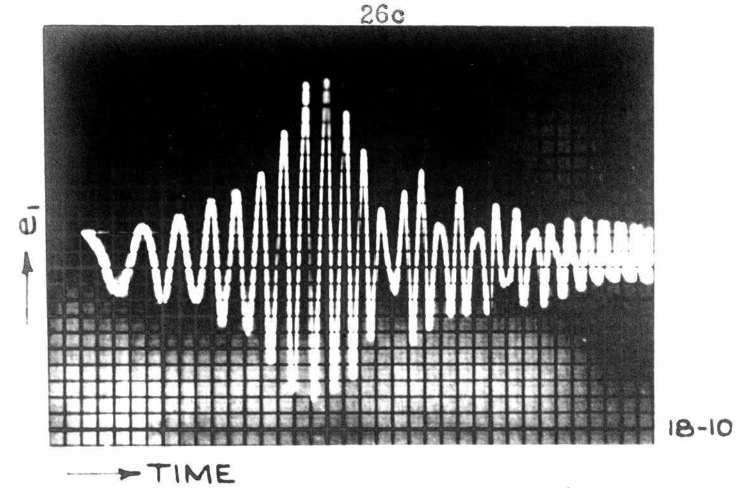
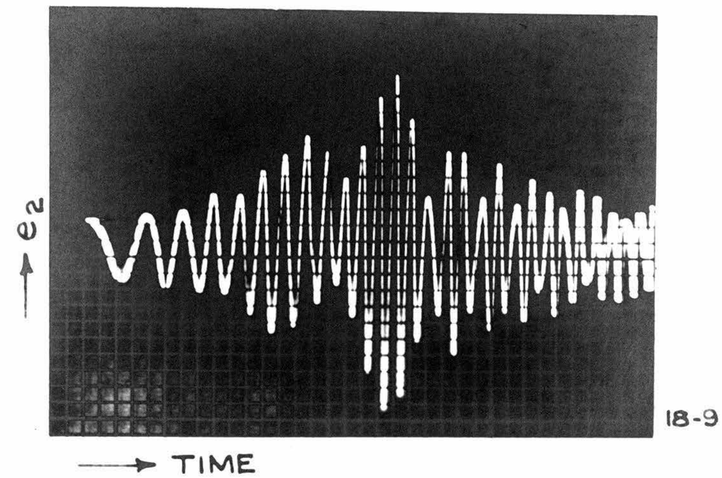


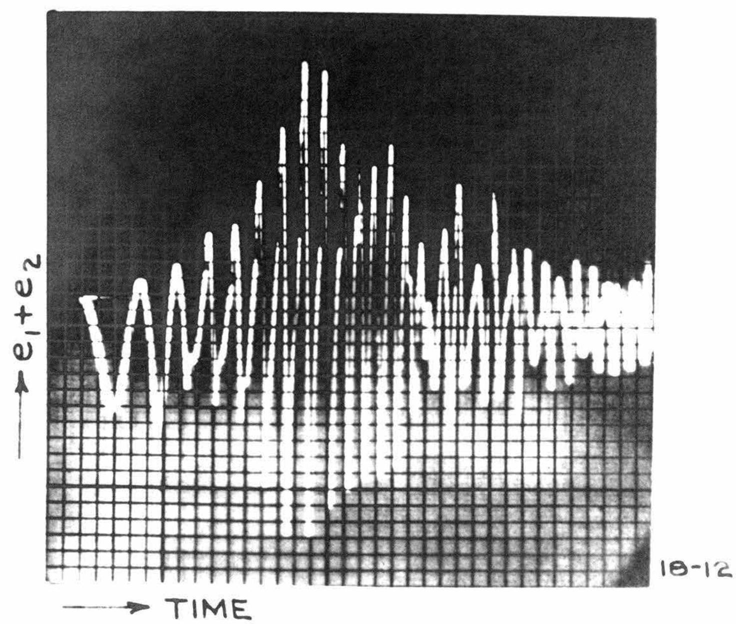
FIG. 10 CIRCUIT FOR COMBINING TWO SINGLE DEGREE OF FREEDOM SOLUTIONS



(a)



(b)



(c)

FIG. 11 RESPONSE OF CIRCUIT OF FIG. 10.

VI. COMPLETE SOLUTIONS FOR TWO DEGREE OF FREEDOM SYSTEM

The results of a detailed study of a two degree of freedom system made with the Cal Tech Electric Analog Computer will now be given

System Studied. The basic mechanical vibratory system considered is illustrated in (Fig. 12), together with the electric analog used on the computer. The differential equations for the mechanical and electrical systems are shown also, (in operational form where p = d/dt). The numerical relationships between the mechanical and electrical quantities are given in Table 1. This analogy and the general computer techniques are discussed in detail by Criner, McCann, and Warren (8).

Table 1 Electrical-Mechanical Analogy

$$L_1 = \frac{a}{n^2} I_1 \qquad \frac{1}{C_{12}} = aK_{12} \qquad R_1 = \frac{a}{n} G_1$$

where

a = arbitrary impedance base factor
n = computer time base change

$$\theta_1 = a q_1 \frac{T_0}{E_0}$$

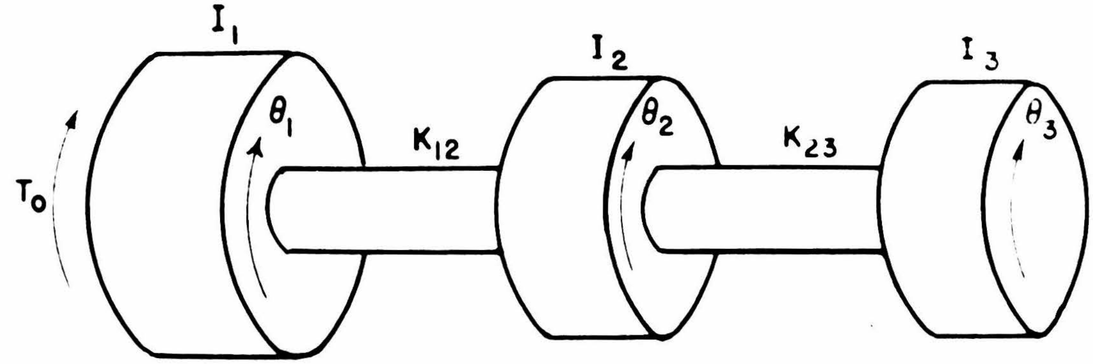
$$\text{Shaft stress} = K_{12}(\theta_1 - \theta_2) = \frac{1}{aC_{12}} a(q_1 - q_2)\frac{T_0}{E_0}$$

$$= \frac{q_1 - q_2}{C_{12}} \frac{T_0}{E_0} = \frac{E_1}{E_0} T_0$$

where E_1 is the capacitor voltage across C_{12}.

Dimensionless torque

$$\frac{T_1}{T_0} = \frac{K_{12}(\theta_1 - \theta_2)}{T_0} = \frac{E_1}{E_0}$$

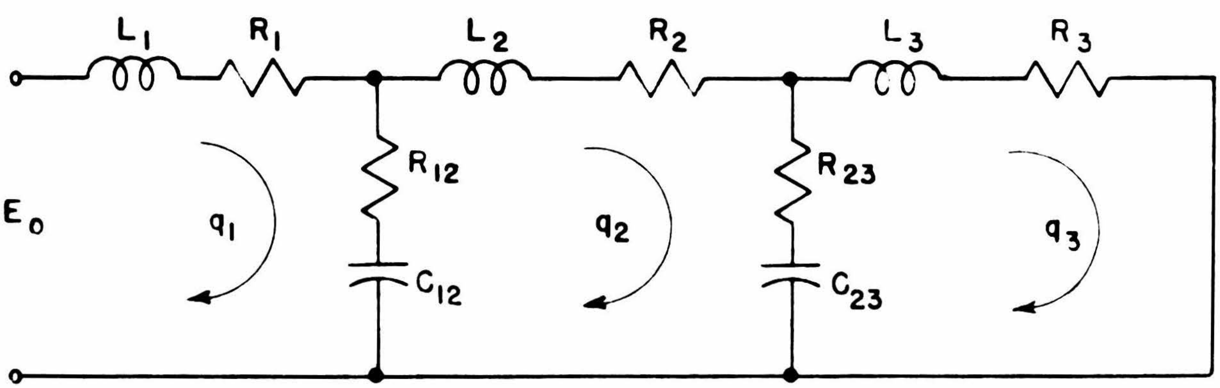


TORQUE EQUATIONS

(1a) $T_0 = I_1 p^2\theta_1 + G_1 p\theta_1 + G_{12}p(\theta_1-\theta_2) + K_{12}(\theta_1-\theta_2)$

(2a) $0 = I_2 p^2\theta_2 + G_2 p\theta_2 + G_{12}p(\theta_2-\theta_1) + G_{23}p(\theta_2-\theta_3) + K_{12}(\theta_2-\theta_1) + K_{23}(\theta_2-\theta_3)$

(3a) $0 = I_3 p\theta_3 + G_3 p\theta_3 + G_{23}p(\theta_3-\theta_2) + K_{23}(\theta_3-\theta_2)$



ELECTRIC CIRCUIT EQUATIONS

(1b) $E_0 = L_1 p^2 q_1 + R_1 p q_1 + R_{12}p(q_1-q_2) + \frac{1}{C_{12}}(q_1-q_2)$

(2b) $0 = L_2 p^2 q_2 + R_2 p q_2 + R_{12}p(q_2-q_1) + R_{23}p(q_2-q_3) + \frac{1}{C_{12}}(q_2-q_1) + \frac{1}{C_{23}}(q_2-q_3)$

(3b) $0 = L_3 p^2 q_3 + R_3 p q_3 + R_{23}p(q_3-q_2) + \frac{1}{C_{23}}(q_3-q_2)$

Fig. 12 Electrical Analogy for Linear Rotating System with Two Degrees of Freedom

The electric circuit of (Fig. 12) has also an analogous translational mechanical system in which the analog of electric charge is translational rather than rotational displacement. However, in attempting to make a somewhat general analysis of a two degree of freedom system it was felt that the most important type of mechanical system is the rotating system which has inherently low damping. This is representative of a great many machines. Of importance also is the effect of acceleration through resonance upon the mountings of such machines. Although damping may vary widely in translational vibratory systems, most rotating-equipment mountings can be analyzed as single degree of freedom systems.

As shown in (Fig. 12), two types of damping may be present, namely bearing frictions which are functions of absolute velocities, and internal shaft material frictions which are functions of relative velocities. However, it has been shown previously (8) that so long as the damping is low, (corresponding to only 10 or 20 per cent per cycle), it can be represented accurately by single damping factors acting upon the absolute motions of the respective inertia constants. Thus, in the analogy of (Fig. 12) only the resistors R_1, R_2, and R_3 were actually used.

The variable-frequency exciting torque, as shown in (Fig. 12), is simulated by the voltage E_o. If the excitation torque is applied to an end inertia constant, the location of the driving voltage for the analog is as shown. If it should be on the center inertia constant the voltage would be inserted in series in the central circuit loop.

The forcing function was restricted to a single frequency varying linearly with time, since this is most representative of acceleration through resonance. To consider the effects of variations in the rate of change of forcing function frequency or amplitude would result in too many parameters for a general study. Thus the excitation woltage is given by

$$E_o = E \cos (\omega_o t \pm \pi h t^2) \quad . \qquad (40)$$

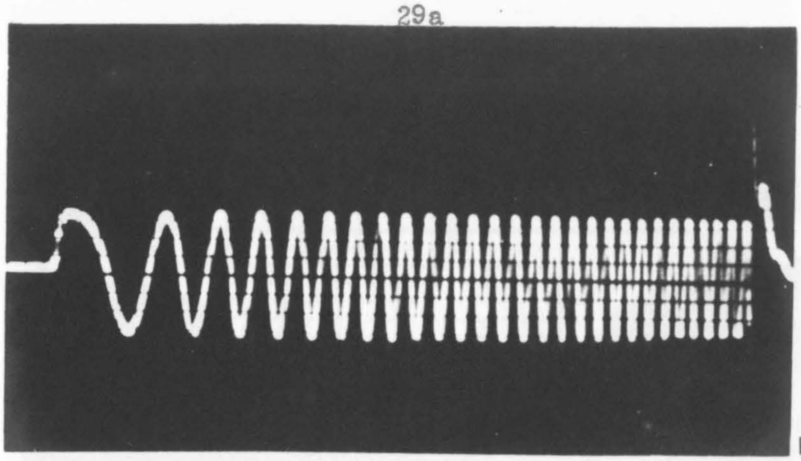
Certain types of systems, such as those driven by induction motors, produce sustained unidirectional torques upon which the variable-frequency oscillating torque is superimposed. These two components may be separated in the analysis of linear systems. In the solutions presented here no unidirectional component was present. For the cases of increasing frequency, ω_o of equation (40) was zero, and for the cases of decreasing frequency ω_o was made sufficiently high compared to the natural frequencies of the system to eliminate the effect of the starting frequency, ω_o .

The same basic principle as that of Lewis (3) was also used here for expressing the forcing function and the basic system parameters in dimensionless form. (See Table 2).

Typical Solutions. Typical solutions as obtained with the computer are shown in (Fig. 13) and (Fig. 14). The top oscillogram in (Fig. 13) shows the driving torque or voltage E_o of linearly increasing frequency. The lower oscillograms show the resulting torque or motion of the shaft K_{12} for two ratios of shaft spring constants. In the electrical analogy the shaft torques are obtained by measuring the capacitor voltages, and the solutions are expressed in the dimensionless form T_1/T_o, as obtained by recording the ratio E_1/E_o, or the ratio of the capacitor voltage to the applied voltage (see Table 1).

The solutions of (Fig. 14) apply to a system with natural frequencies which are rather far apart. Thus if the rate of change of frequency is relatively low two distinct resonant peaks are present.

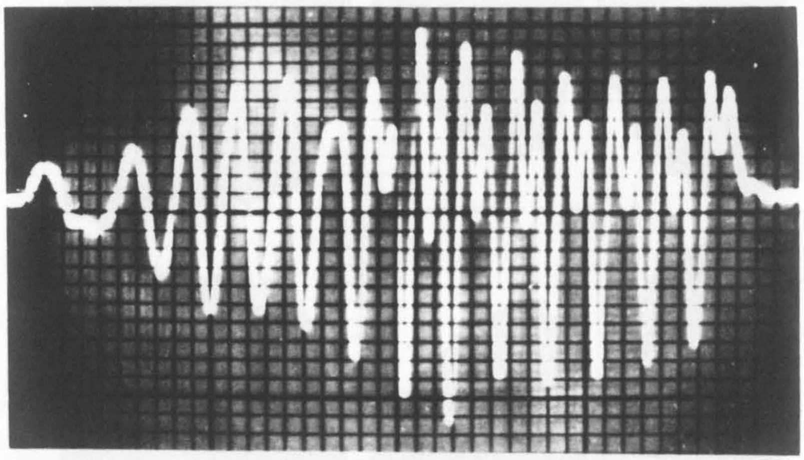
T_0



15-2

TIME

DRIVING TORQUE. ACTUAL FREQUENCY 0-1000 cps IN 0.05 SEC.

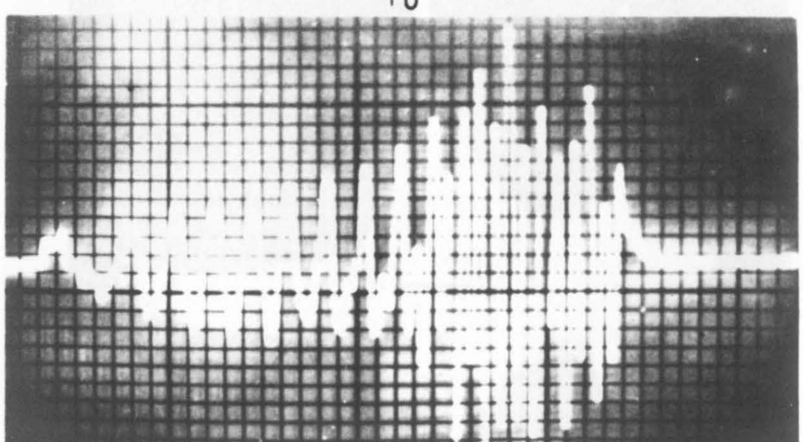


15-5

T_1

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 0.5, \quad \frac{K_{23}}{K_{12}} = 0.5, \quad N_2 = 306 \text{ cps}, \quad q = 4.35$

$\text{CREST } \frac{T_1}{T_0} = 5.2$



15-6

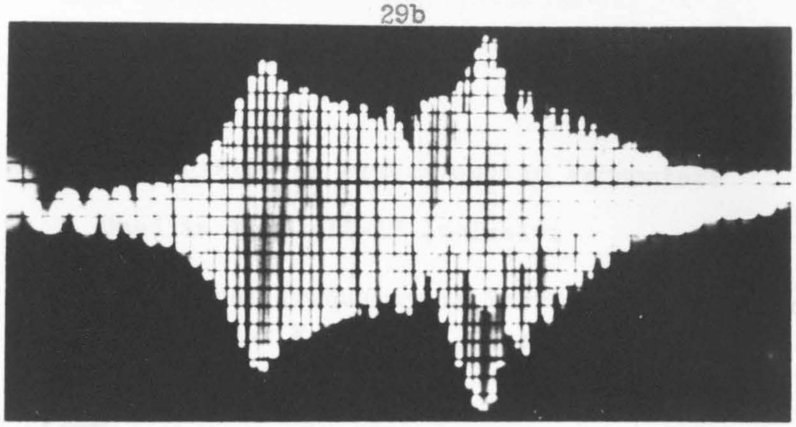
T_1

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 0.5, \quad \frac{K_{23}}{K_{12}} = 0.25, \quad N_2 = 433 \text{ cps}, \quad q = 8.7$

$\text{CREST } \frac{T_1}{T_0} = 5.5$

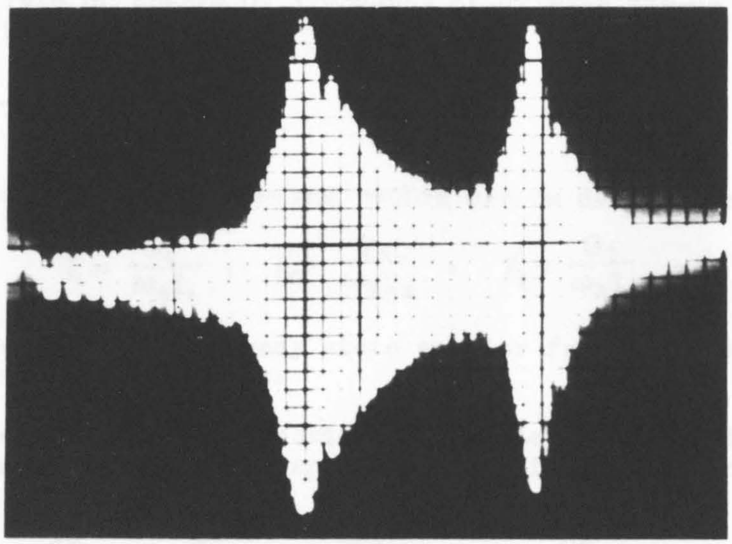
Fig. 13 Typical Solutions Showing Driving Torque and Resulting Shaft Torque T_1

(System of Fig. 12. $G_{12} = G_{23} = 0.$)



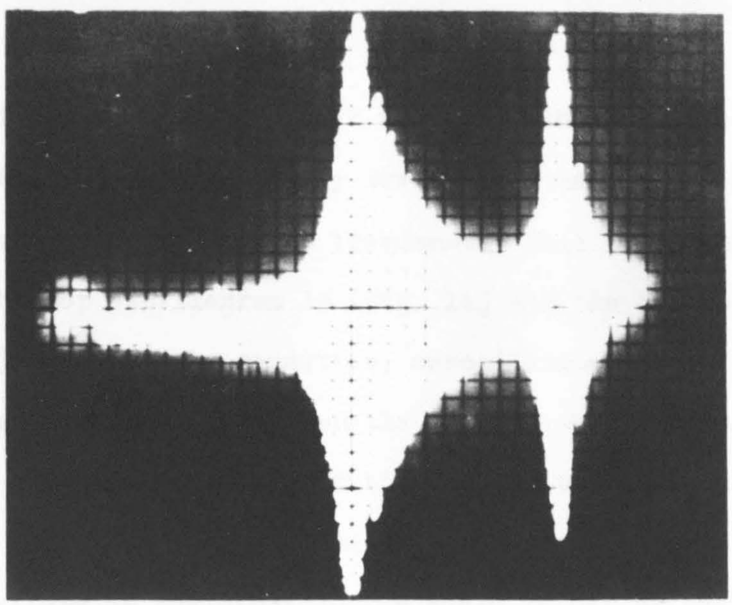
16-2

q = 26



16-3

q = 70



16-4

q = 104

Fig. 14 Solutions Showing Shift of Maximum Crest Torque From Region of Lower System Frequency to Higher System Frequency

$$\left(\frac{I_2}{I_1} = \frac{I_3}{I_1} = 0.5 \,, \quad \frac{K_{23}}{K_{12}} = 0.5. \right)$$

Table 2 Definition of Dimensionless Parameters

Used in Figs. 12-17 Inclusive

$N_2 = \frac{\omega_2}{2\pi}$ = natural frequency of inertia I_1 and spring constant K_{12} with I_2 locked in position

$$\omega_2 = \sqrt{\frac{K_{12}}{I_1}}$$

h = rate of change of frequency of driving torque T_o in cycles per sec. per sec.

$$q = \frac{N_2^2}{h}$$

If $G_{12} = G_{23} = 0$, damping ratios can be defined as follows

$$\gamma_1 = \frac{G_1}{\omega_3 I_1}, \quad \gamma_2 = \frac{G_2}{\omega_3 I_2}, \quad \gamma_3 = \frac{G_3}{\omega_3 I_3},$$

where ω_3 is the steady state angular frequency at which maximum torque occurs.

System and forcing function are completely defined in dimensionless form by specifying the following constants:

$$\frac{I_2}{I_1}; \quad \frac{I_3}{I_1}; \quad \frac{K_{23}}{K_{12}}; \quad q = \frac{N_2^2}{h}$$

This condition corresponds to high values of the parameter q (see Table 2). When q is sufficiently low the system may oscillate simultaneously at more than one frequency. This condition is illustrated by the top oscillogram in (Fig. 14) and the two solutions in (Fig. 13). Under this condition, cancellation or addition effects may take place in the period when the second resonant peak would normally be expected. Thus the ratios of the two peaks may vary appreciably as q is varied. It might be expected, therefore, that wide fluctuations in the crest torque would result as the system parameters are changed. This was not found to be the case however.

As Lewis (3) also observed for the single degree of freedom system, a delay always occurs in the instantaneous frequency of the forcing function at which the resonant peaks are reached, which becomes greater frequencywise as q is decreased.

To correlate with the Lewis data, some ten spot check measurements were made of a single degree of freedom system with the computer. These all check the Lewis data within 3 per cent.

Generalized Analysis. It is obvious that the complete transient solutions are of too complex a character to permit presentation in a generalized form. It was considered practical to plot only the maximum crest torque which results from a given solution as a function of the various system parameters. To further reduce the required data, a detailed study was made of the effect of damping expressed as the damping factor γ (see Table 2). For almost all practical rotating systems γ will be between 0.005 and 0.02. Solutions showing the effect of damping are shown in (Figs. 15 and 16). (Fig. 16) applies to the case where the driving torque is applied to an end inertia constant. (Fig. 15) applies to a special case with the driving torque on the center inertia constant, and with such symmetry that the system has only one mode of oscillation. The dotted curves for $\gamma = 0$ were taken from the Lewis data. (3) The correlation between this system and a single mass-spring system is discussed in the appendix.

The curves in (Figs. 15 and 16) show a rather narrow spread between $\gamma = 0$ and $\gamma = 0.02$ for all solutions resulting in shaft torques below about 10 times the driving torque, which is the practical range of interest. Thus it was considered sufficient to plot solutions for the single most typical value of γ of 0.01.

Another factor which is eliminated from the generalized analysis is the difference between the crest torque developed during acceleration

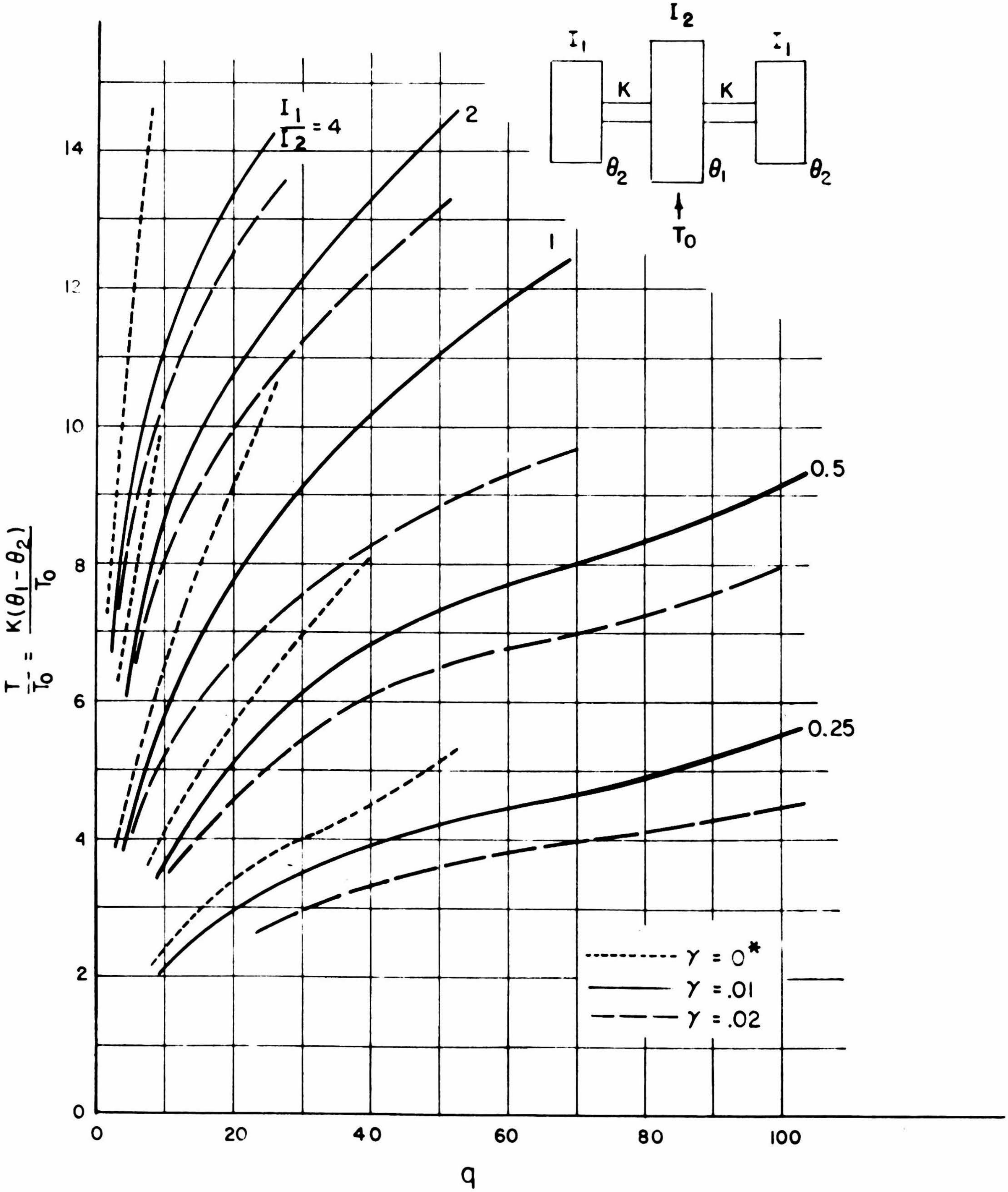


Fig. 15 Maximum Crest Shaft Torques for a Balanced System Having One Degree of Freedom and Driving Torque at the Center
($\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. See Table 2 for definition of parameters.
* Taken from reference 3. See appendix.)

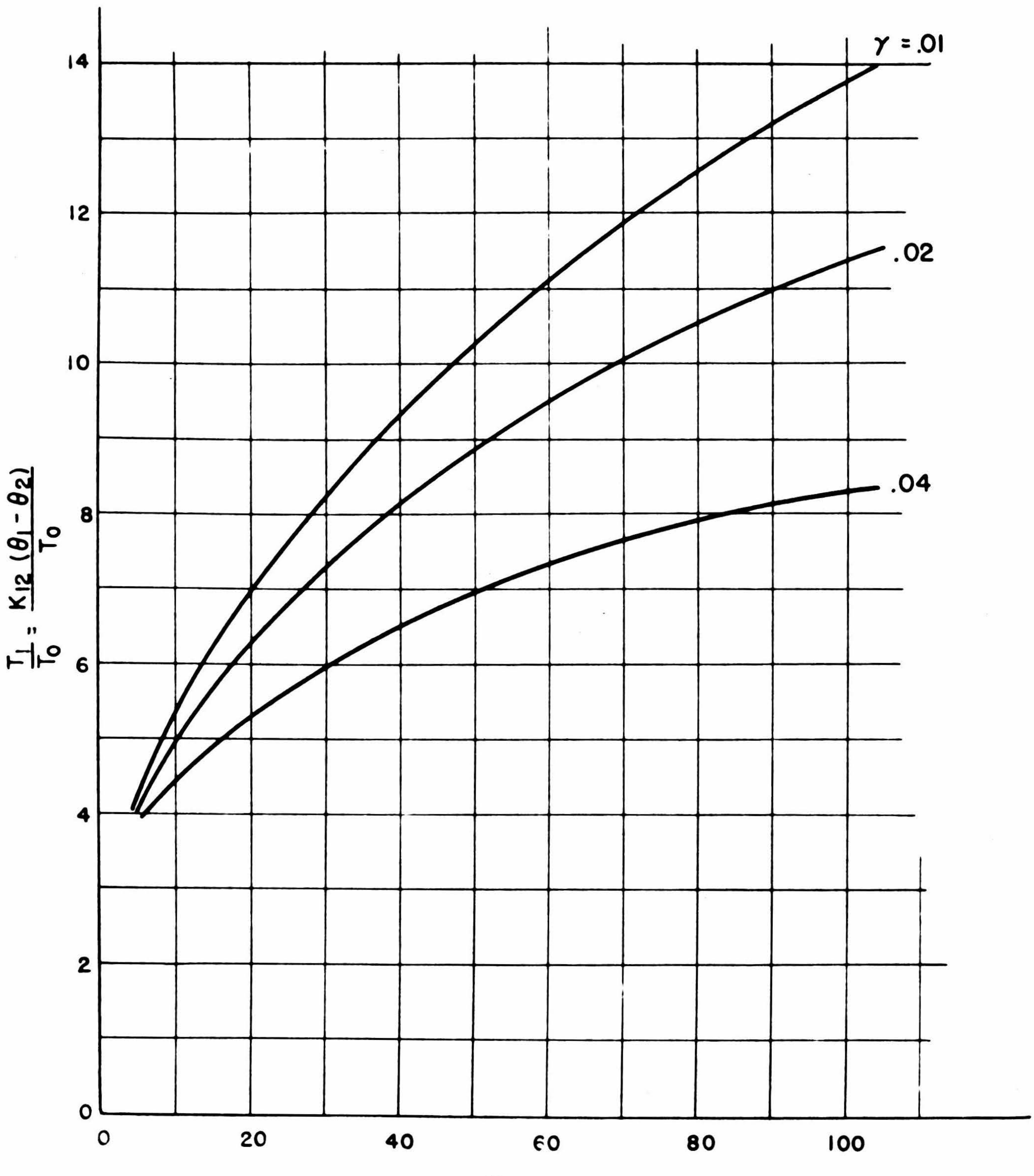


Fig. 16 Curves Showing Effect of Damping Upon Crest Shaft Torque

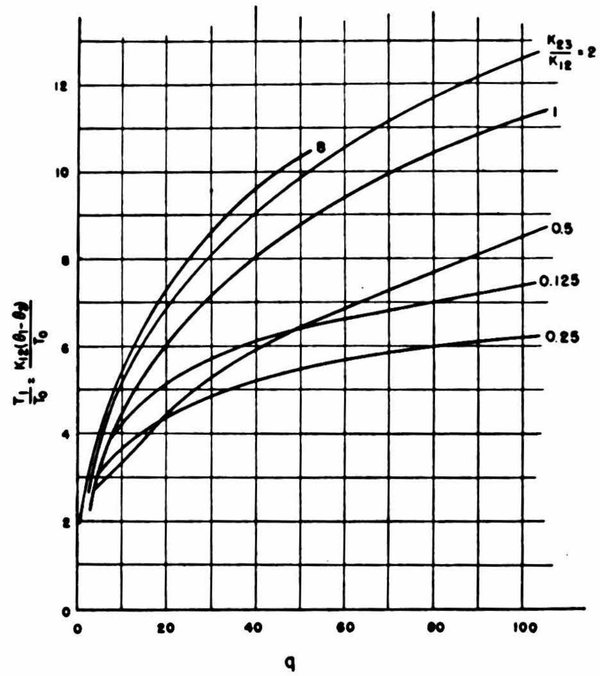
$$\left(\frac{I_2}{I_1} = \frac{I_3}{I_1} = 0.5,\ \frac{K_{23}}{K_{12}} = 1,\ \gamma_1 = \gamma_2 = \gamma_3 = \gamma \ . \right)$$

and deceleration of frequency. Solutions were obtained over a wide range of system parameters for both conditions, and it was found that the crest torque is always higher for the decelerating case. This was also found by Lewis to be true for the single degree of freedom system. The range of variation was found to be from 5 to 20 per cent higher than for accelerating frequency. However, in the vast majority of cases the difference lay between 10 and 15 per cent. Therefore it was considered sufficient to plot the generalized curves for only one of these two conditions. The accelerating case was chosen.

The generalized solutions are presented in (Fig. 15 and 17). (Fig. 17) applies to the case wherein the excitation torque is applied to an end inertia constant. In these curves the maximum crest shaft torques on both shafts are plotted in dimensionless form as a function of the parameter q. In all cases $I_2 = I_3$. Each set of curves applies to a different ratio of I_2/I_1, and on each set are curves for a range of ratios of K_{23}/K_{12}. The curves are so chosen and enough values of the ratios given that satisfactory interpolation can be made. The crossing over of the curves at lower values of q results from the multifrequency interaction discussed previously.

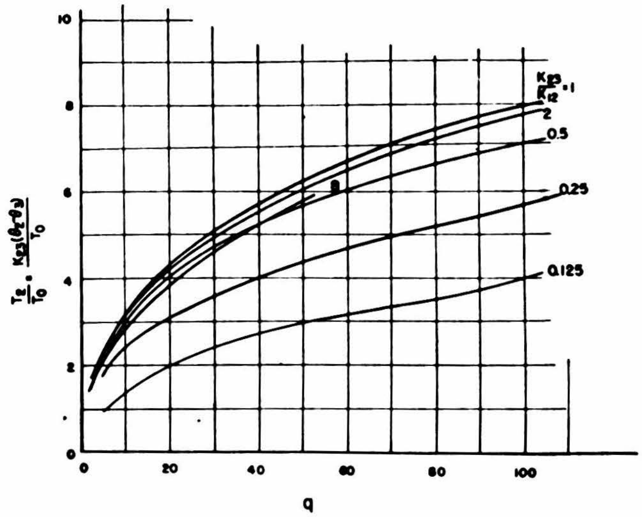
The data for the case in which the driving torque is applied to the center inertia constant are presented in (Fig. 15). This represents practically as general a treatment as is given in (Fig. 17) for the driving force on the end of the system, since in most practical cases the two shafts would have the same spring constant.

Space does not permit presenting a complete coverage for cases where all the inertia constants are different. However, it is a relatively simple matter to obtain such data with the electric analog computer.



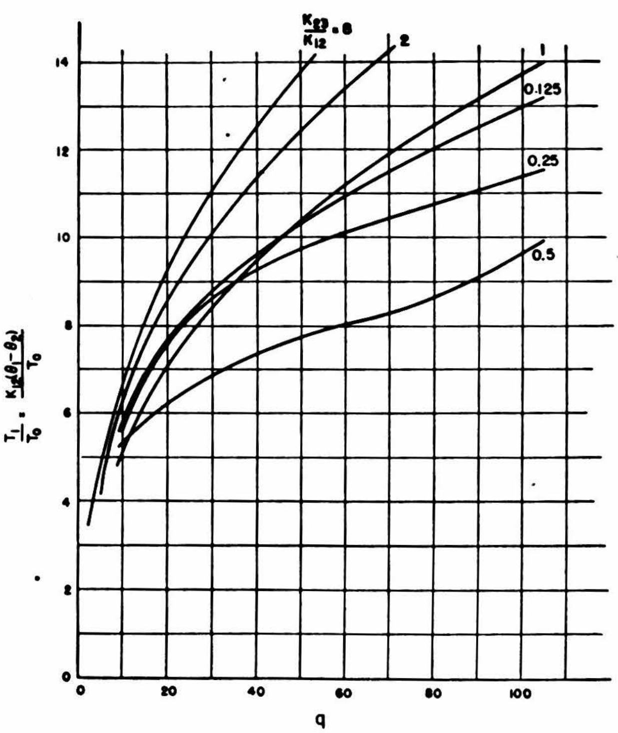
(a)

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 0.25$



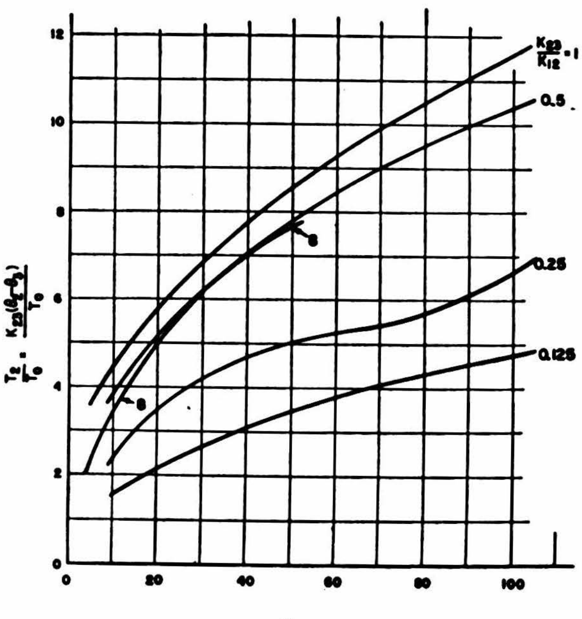
(a)

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 0.25$



(b)

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 0.5$

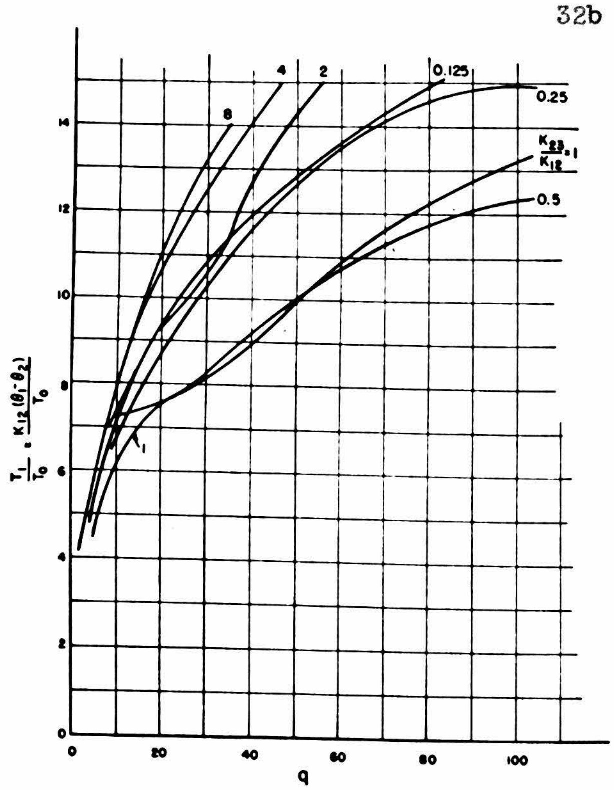


(b)

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 0.5$

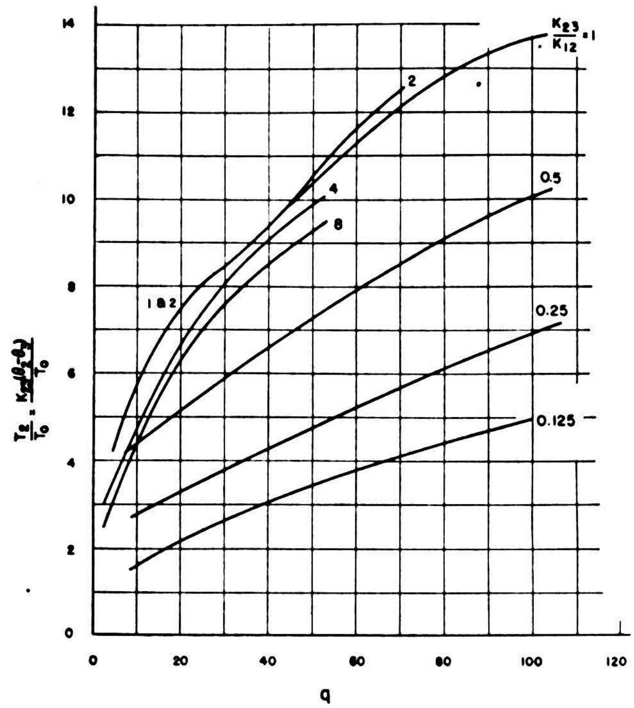
FIG. 17 GENERALIZED DIMENSIONLESS CURVES GIVING CREST MAGNITUDES OF SHAFT TORQUES WHICH OCCUR DURING ACCELERATION THROUGH RESONANCE OF TWO-DEGREE-OF-FREEDOM SYSTEM IN FIG. 12

($G_{12} = G_{23} = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = 0.01$.)



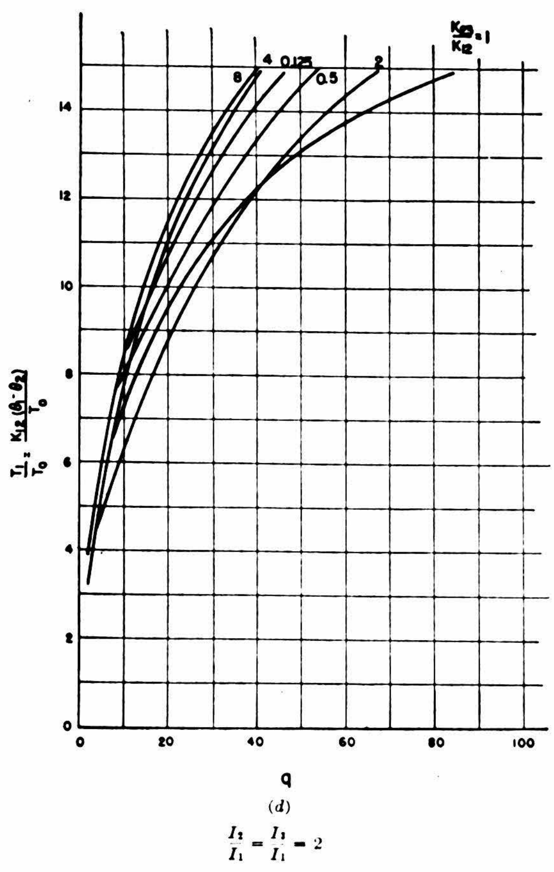
(c)

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 1$



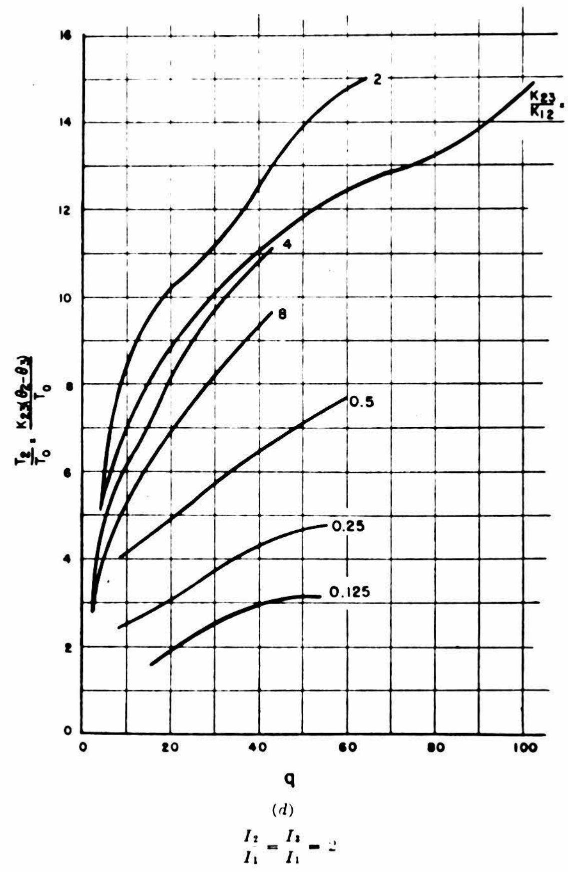
(c)

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 1$



(d)

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 2$



(d)

$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 2$

Fig. 17 (continued)

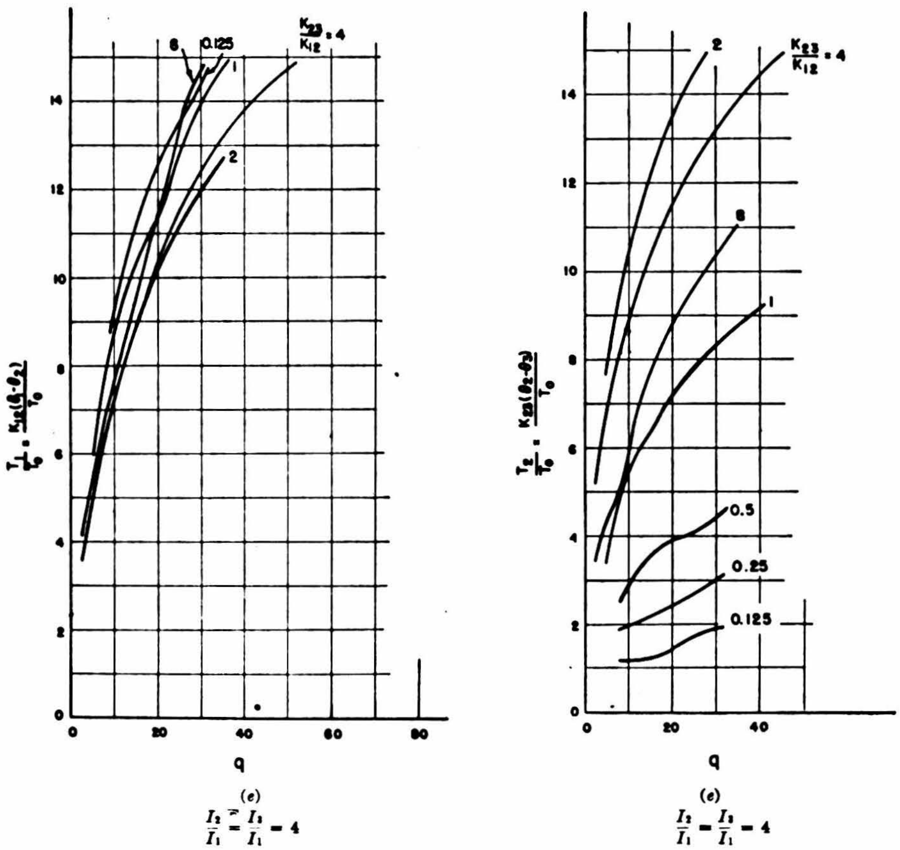


Fig. 17 (continued)

A hint at the time that would be involved in obtaining these solutions by analytical calculation may be had by noting that the system studied has a characteristic equation of 7th order, containing 3 real roots and 2 pairs of conjugate complex roots corresponding to the two natural frequencies.

Application of Curves. As an illustration of the use of the curves of (Figs. 15 and 17) for a specific problem, consider a 1200 rpm induction motor driving two generators. The induction motor rotor inertia constant is 3.3 slug ft., and each generator has an inertia constant of 6.6 slug ft. The shaft constants are $K_{12} = K_{23} = 0.5 \times 10^6$ lb. ft./radian. The value of γ for each machine is 0.01.

Consider first the case when the motor has the position I_1, (Fig. 12); thus

$$I_1 = 3.3 \text{ slug ft.}$$

$$I_2 = I_3 = 6.6 \text{ slug ft.}$$

$$K_{12} = K_{23} = 0.5 \times 10^6 \text{ lb. ft./radian}$$

The motor starting characteristic is such that it accelerates the system to 1200 rpm in 8 sec. During this period there exists, superimposed upon the unidirectional torque, an oscillating slip frequency torque which varies in frequency from 60 cycles per sec. to zero in the 8 seconds. It will be assumed that the acceleration is constant at least until the variable frequency has passed through the system natural frequencies, and that the rate of change of frequency during this time is given by the average value (see Table 1)

$$h = 60/8 = 7.5 \text{ cycles/sec.}^2 \; .$$

The crest magnitude of the pulsating torque is

$$T_o = 263 \text{ lb. ft.}$$

Now,

$$\omega_2 = \sqrt{\frac{K_{12}}{I_1}} = \sqrt{\frac{0.5 \times 10^6}{3.3}} = 123$$

$$N_2 = \frac{\omega_2}{2\pi} = \frac{123}{2\pi} = 19.5 \text{ c.p.s.}$$

$$q = \frac{N_2^2}{h} = \frac{(19.5)^2}{7.5} = 51$$

$$\frac{I_2}{I_1} = \frac{I_3}{I_1} = 2$$

$$\frac{K_{23}}{K_{12}} = 1$$

Referring to (Fig. 17d) it is seen that for a T_o having an increasing frequency, T_1/T_o = 13.2, T_2/T_o = 11.8. Since, however, the forcing function in this case is one of decreasing frequency, values of crest torque 15 per cent higher should be used. Thus T_1/T_o = 15.2, and T_2/T_o = 13.6.

The maximum pulsating components of shaft torques to be expected are therefore

$$T_1 = 263 \times 15.2 = 4000 \text{ lb. ft.}$$

$$T_2 = 263 \times 13.6 = 3580 \text{ lb. ft.}$$

Next, consider the motor in the position I_2 in (Fig. 15), with one generator symmetrically placed on each side. If the same shafts are used,

I_1 = 6.6 slug ft.

I_2 = 3.3 slug ft.

K = 0.5 x 10^6 lb. ft./radian.

Now,

$$\omega_2 = \sqrt{\frac{K}{I_1}} = \sqrt{\frac{0.5 \times 10^6}{6.6}} = 87$$

$$N_2 = \frac{\omega_2}{2\pi} = \frac{87}{2\pi} = 13.8 \text{ c.p.s.}$$

$$q = \frac{N_2^2}{h} = \frac{(13.8)^2}{7.5} = 25.5$$

Referring to (Fig. 15), for an increasing frequency, $I_1/I_2 = 2$, $\gamma = 0.01$, $q = 25.5$, T/T_0 is found to be 11.5. Again adding 15 per cent (for decreasing frequency), $T/T_0 = 13.2$. Hence the maximum pulsating component of shaft torque to be expected in this case is

$$T = 263 \times 13.2 = 3470 \text{ lb. ft.}$$

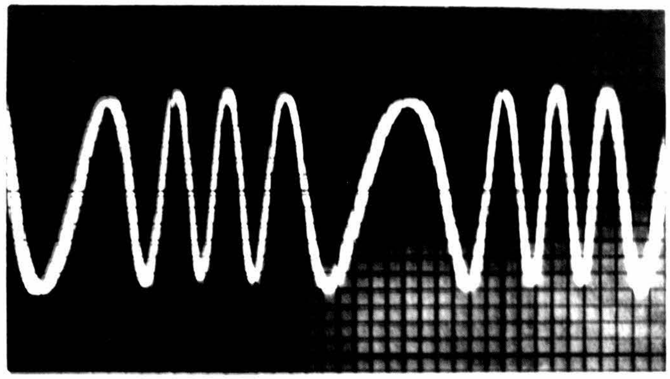
The stresses produced by the unidirectional component of the driving torque of the motor can be found by conventional methods of analysis.

VII. FURTHER CONSIDERATION OF ELECTRIC ANALOG METHOD

The foregoing type of treatment may be extended to more complicated systems and to one or more forcing functions of a general frequency vs. time characteristic. The present Cal Tech Electric Analog Computer is capable of representing systems with up to 100 degrees of freedom. In addition many nonlinear systems; and distributed systems, such as vibrating beams; may be handled by employing electrical analogies already developed.

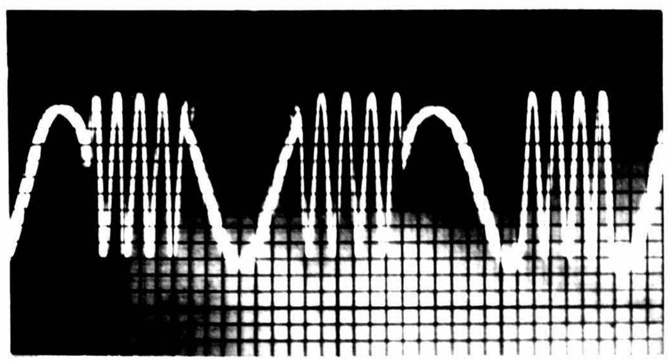
If the amplifier in (Fig. 1) is made to have a gain proportional to an auxiliary input voltage, (e.g. by varying the screen potential), some degree of amplitude modulation may be added to the frequency modulation of the output.

For treatment of periodic variation of frequency, i.e. steady state frequency modulation, applying the proper periodic voltage waveshape to the device of (Fig. 2) will result in the desired excitation function. This excitation may then be applied to the appropriate electric analog circuit. Waveshapes obtained in this manner for sinusoidal and square wave frequency modulations are shown in (Figs. 18a and 18b) respectively.



19-2

(a) SINUSOIDAL



19-5

(b) SQUARE WAVE

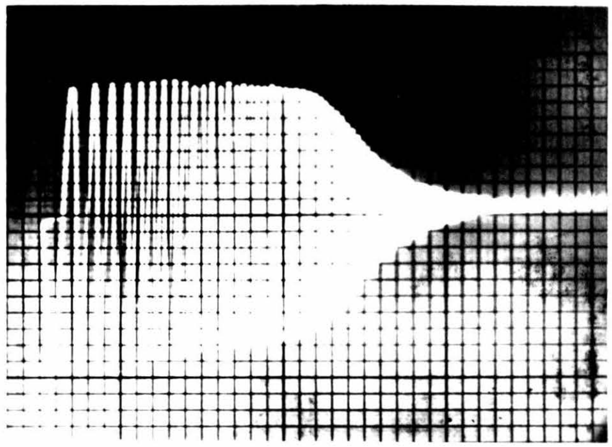
FIG. 18 SINUSOIDAL AND SQUARE WAVE STEADY STATE FREQUENCY MODULATION EXCITATIONS.

VIII. VARIABLE FREQUENCY TECHNIQUE USED TO OBTAIN STEADY STATE FREQUENCY RESPONSE

In many types of electrical work it has become fairly common practice to employ a time-varying frequency excitation in obtaining a measure of a system's steady state frequency response. For example, if a voltage which varies in frequency from zero to 20 kc. in periodic fashion is applied to an audio amplifier, the output of the amplifier, when viewed with an oscilloscope, will show a plot of the amplifier's frequency response in the audio range. This same method has been applied in obtaining rapidly the frequency response characteristics of amplifiers operating at higher frequencies, of loudspeakers, for aligning radio receivers, and in a number of other measurements. Such a technique gives a result which is of increasing accuracy as the rate of change of frequency is made less and less. From previous considerations it is seen that the accuracy is governed by the dimensionless quantities ω_p^2/h , where ω_p represents the various resonant frequencies and reciprocal time constants of the system, and h is the rate of change of frequency. Unless the values of ω_p^2/h are sufficiently large, which means that for a given system h must be sufficiently small, this technique is only a rough qualitative method.

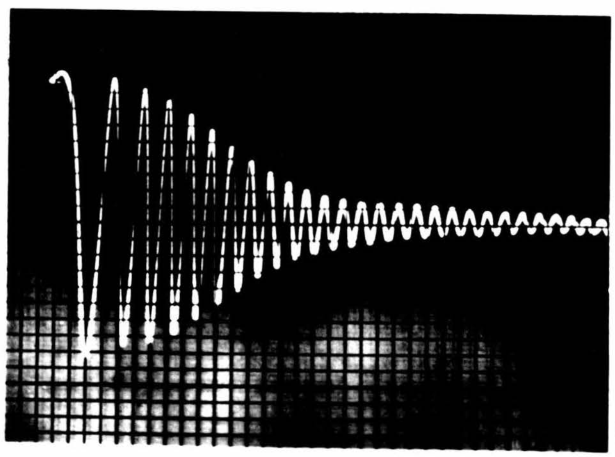
(Fig. 19) shows the response of a low pass filter to a frequency varying linearly from zero to 1000 cycles per sec. for two different rates of change of frequency. The slow frequency sweep, (Fig. 19a), gives a fairly accurate measure of the filter's frequency response. The faster sweep, (Fig. 19b), shows considerable error.

In (Fig. 20) are shown the response characteristics of two other filters. The upper photograph, (Fig. 20a), is for a high pass filter, and the lower photograph is for a band pass filter.



19-11

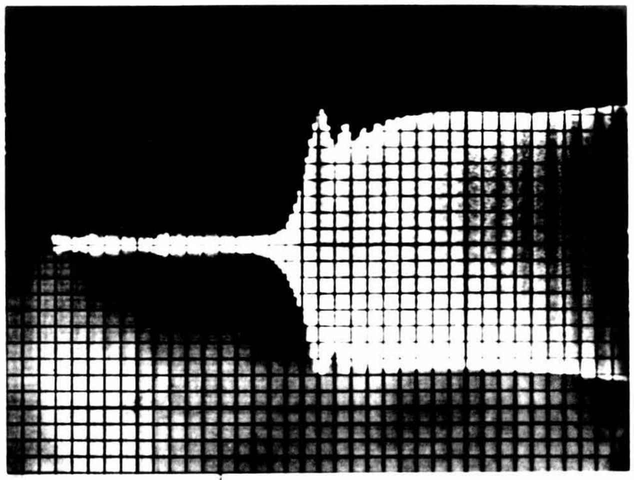
(a) SLOW SWEEP



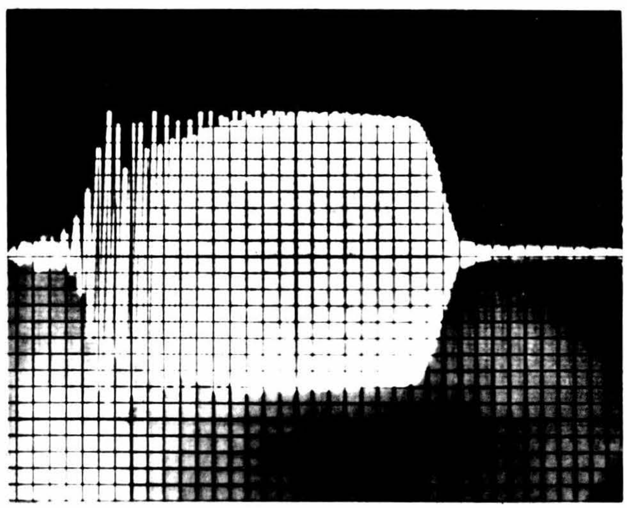
19-12

(b) FAST SWEEP

FIG. 19 RESPONSE OF LOW PASS FILTER TO LINEARLY VARYING FREQUENCY.


20-3

(a) HIGH PASS FILTER


20-2

(b) BAND PASS FILTER

FIG. 20 RESPONSE OF HIGH PASS AND BAND PASS FILTERS

APPENDIX

The zero damping curves in (Fig. 15) are obtained from the paper by Lewis. (3) Using operational notation, the equations satisfying the system in (Fig. 15) are

$$I_2 p^2 \theta_1 + 2K(\theta_1 - \theta_2) = T_0$$

$$I_1 p^2 \theta_2 + K(\theta_2 - \theta_1) = 0 \quad .$$

Multiplying the first equation by I_1, and the second by I_2, and subtracting,

$$I_1 I_2 p^2 (\theta_1 - \theta_2) + K(2I_1 + I_2)(\theta_1 - \theta_2) = T_0 I_1 \; .$$

Solving for the shaft torque, $K(\theta_1 - \theta_2)$,

$$K(\theta_1 - \theta_2) = \frac{T_0 I_1}{2I_1 + I_2} \; \frac{1}{\dfrac{I_1 I_2}{K(2I_1 + I_2)} p^2 + 1} \quad .$$

Similarly, for a torque T_0 applied to a single inertia-spring configuration (I_a, K_a), the shaft torque is

$$T_0 \; \frac{1}{\dfrac{I_a}{K_a} p^2 + 1} \quad .$$

Thus the response of the system in (Fig. 15) may be found in terms of that of the simple system provided that

$$K_a = K$$

$$I_a = \frac{I_1 I_2}{2I_1 + I_2} \quad .$$

In addition, the amplitude of response must be decreased by multiplying it by the factor

$$\frac{I_1}{2I_1 + I_2} \quad .$$

It remains to connect the parameter q of the Lewis paper, which will here be termed q_a , with the q defined in Table 2.

$$q_a = \frac{K_a}{4\pi^2 h I_a} = \frac{K(2I_1 + I_2)}{4\pi^2 h I_1 I_2}$$

$$q = \frac{K}{4\pi^2 h I_1}$$

Hence,

$$q_a = q\,\frac{2I_1 + I_2}{I_2} \quad .$$

Thus, to obtain the response of the balanced 3-body system in (Fig. 15), multiply the response of the Lewis paper by

$$\frac{I_1}{2I_1 + I_2}$$

for a q_a equal to

$$q \cdot \frac{2I_1 + I_2}{I_2} \quad .$$

The foregoing derivations may be made more general by including the effect of damping, thus enabling one to treat with Lewis' results a balanced system, such as that in (Fig. 15), which includes damping.

P A R T   T W O

B R A K I N G   O F   A   L O N G   T R A I N

PART TWO

Braking of a Long Train

In the stopping of a long railroad train, and particularly in the emergency braking of a long train, it is important to consider the longitudinal forces acting upon the cars and upon the coupling apparatus used between cars. The forces on the cars may, in some instances, be great enough to cause the cars to buckle lengthwise. The forces on a car determine the resulting acceleration, and the acceleration of the car in turn governs the forces to which the dunnage and lading are subjected.

The essential part of the coupling device used between cars is a spring, called a "draft gear". This spring is constructed such that it absorbs considerable energy during compression and subsequent expansion, a feature obtained by making the draft gear a combination of an elastic spring and sliding friction. This means that the draft gear exhibits a nonlinear characteristic. For this reason analytical solution of the braking problem, considering the draft gears as nonlinear, is extremely difficult.

There is a delay in the application of brakes along the train due to the acoustic delay of the air pressure wave used in applying the brakes. Thus the head end of the train is braked first, and the succeeding cars have their brakes applied sequentially.

An analysis of the long train braking problem for draft gears treated as perfect springs has been made by considering the train as a uniform elastic bar. (10)(11)(12) Though the uniform elastic bar does not accurately represent the actual system of lumped cars, it will be shown that for draft gears considered to be perfect springs the results are fairly accurate, particularly for longer trains.

I. ELECTRIC ANALOG OF TRAIN

If the train is considered to be composed of cars which act as rigid lumped masses, and if the only forces acting on each car are the braking force and the force of the draft gear(s), a section of the mechanical configuration for longitudinal motion is shown in (Fig. 21a). The analogous electric circuit is given in (Fig. 21b), together with a table of analogous electrical and mechanical quantities. This circuit is based upon the well known Mobility or Force-current analogy.

If each draft gear is considered to be a perfect spring with a spring constant K, addition of the forces on the nth mass, (Fig. 21a), yields

$$F_n = M_n \frac{dv_n}{dt} + K\int(v_n - v_{n-1})\,dt + K\int(v_n - v_{n+1})\,dt. \qquad (41)$$

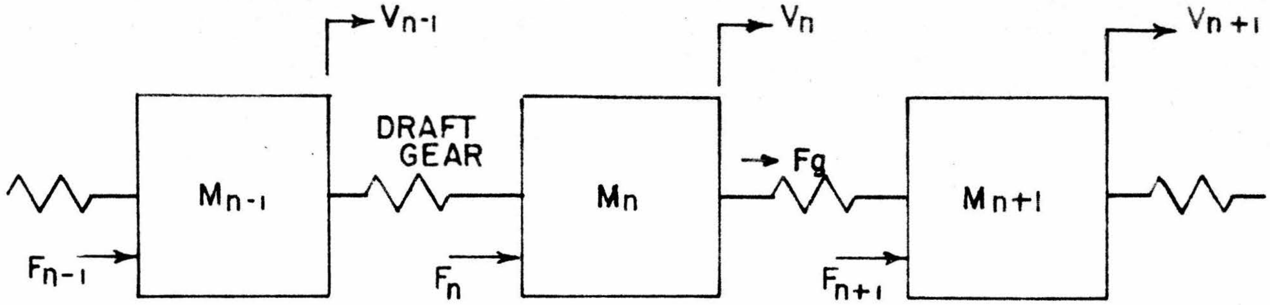
The corresponding electrical equation, obtained by adding the currents at the nth electrical node, is

$$I_n = C_n \frac{dE_n}{dt} + \frac{1}{L}\int(E_n - E_{n-1})\,dt + \frac{1}{L}\int(E_n - E_{n+1})\,dt. \quad (42)$$

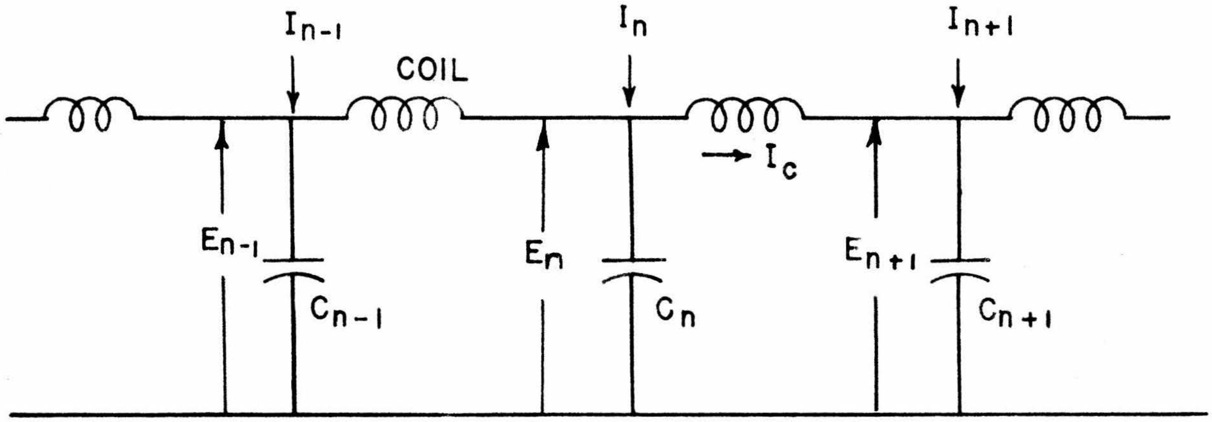
Comparing equations (42) and (41) shows that the mechanical spring constant is analogous to the inductance of each of the electric coils.

The detailed analogy for the case where the nonlinear behavior of the draft gears is considered will be given later.

Sequential application of car brakes is simulated in the electric circuit by sequential application of the nodal currents I_n.



a. MECHANICAL CONFIGURATION



b. ELECTRICAL CIRCUIT

MECHANICAL QUANTITY	ELECTRICAL EQUIVALENT
F_n BRAKING FORCE	I_n NODAL INPUT CURRENT
M_n CAR MASS	C_n CAPACITANCE
V_n CAR VELOCITY	E_n NODAL VOLTAGE
DRAFT GEAR FORCE F_g	COIL CURRENT I_c

FIG. 21 ELECTRICAL-MECHANICAL ANALOGY

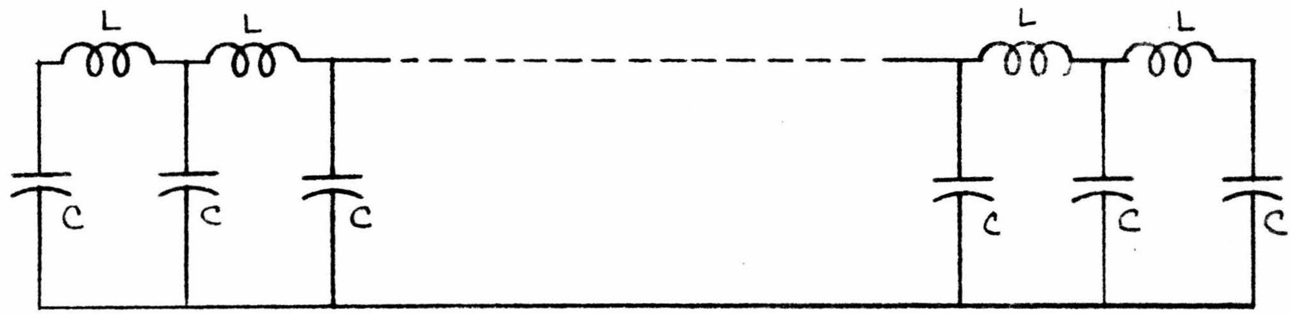
II. ANALYTICAL SOLUTION FOR IDEALIZED DRAFT GEARS

In considering application of the electric analog computer to the solution of any problem it is wise to first investigate the possibility of obtaining an analytical solution to the problem. If the time necessary to solve the problem by analytical means is less than would be required by using the analog computer the latter is of course not justified. Then too, the nature of the analytical solution often lends insight to the analog computer attack.

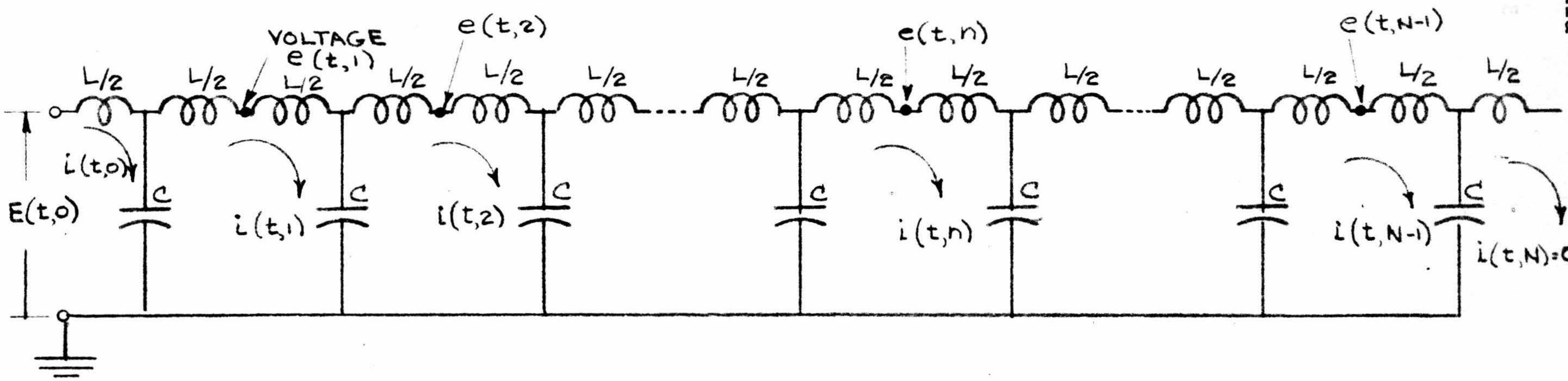
The analytical solution to be presented is for draft gears which are assumed to be perfect springs. The cars are assumed to be identical. In this case the electric circuit becomes repetitive in construction, as shown in (Fig. 22a).

The analytical solution will be given in terms of electrical quantities because of the possible electrical interest in this ladder network problem, the solution to which has not been found elsewhere in the literature.

<u>Force at One End of Train.</u> The first problem to be considered is that of a sudden braking force of constant amplitude applied at the front end of the train, such as would be due to application of the locomotive's brakes. In the electrical analogy this corresponds to a step current entering the front end of the circuit of (Fig. 22b). The train is composed of N cars, and the designation of voltages and currents is as shown in (Fig. 22b). The two end inductances, each L/2 in value, are added only to make the circuit end sections the same as the typical T cell of each section. Since the network is forced with a current, $i(t,0)$, the addition of L/2 at the front end of the circuit has no effect upon the solution of the problem.



(a) EQUIVALENT CIRCUIT FOR ENTIRE TRAIN



(b) CIRCUIT FOR N CARS WITH FORCE APPLIED AT LEFT END

FIG. 22 CIRCUITS FOR TRAIN WITH IDENTICAL CARS

The transient solution for a ladder network consisting of T cells of this type has been treated by Gardner and Barnes, (5) and the general solution in Laplace transform form is

$$E(s,n) = E(s,0)\cosh n\beta - Z\,I(s,0)\sinh n\beta \tag{43}$$

$$I(s,n) = I(s,0)\cosh n\beta - \frac{E(s,0)}{Z}\sinh n\beta . \tag{44}$$

where $E(s,n) = L_t\left[e(t,n)\right]$ and $I(s,n) = L_t\left[i(t,n)\right]$ are the Laplace transforms of voltage and current with respect to the time variable, and

$$\cosh\beta = \frac{LCs^2}{2} + 1 = \lambda \tag{45}$$

$$Z = \frac{1}{2}\sqrt{\frac{L}{C}(LCs^2+4)} . \tag{46}$$

Since, in this particular problem, $i(t,N) = 0$, equation (44) becomes, for $n = N$,

$$I(s,N) = I(s,0)\cosh N\beta - \frac{E(s,0)}{Z}\sinh N\beta = 0$$

from which

$$E(s,0) = Z\,I(s,0)\,\frac{\cosh N\beta}{\sinh N\beta} . \tag{47}$$

Substituting this into equations (43) and (44),

$$E(s,n) = Z\,I(s,0)\frac{\cosh N\beta}{\sinh N\beta}\cosh n\beta - Z\,I(s,0)\sinh n\beta$$

$$= Z\,I(s,0)\,\frac{\sinh(N-n)\beta}{\sinh N\beta}$$

$$I(s,n) = I(s,0)\cosh n\beta - I(s,0)\,\frac{\cosh N\beta}{\sinh N\beta}\sinh n\beta$$

$$= I(s,0)\,\frac{\sinh(N-n)\beta}{\sinh N\beta} . \tag{48}$$

For a unit step current input,

$$I(s,0) = \frac{1}{s} ,$$

$$I(s,n) = \frac{\sinh (N-n)\beta}{s \sinh N\beta} = \frac{A(s)}{s\,B(s)} . \quad (49)$$

A partial fraction expansion of equation (49) may be made by first finding the roots of the denominator. $\sinh N\beta = 0$ is satisfied by making

$$\beta_k = \pm j \frac{k\pi}{N} , \quad k = 0, 1, 2, \ldots$$

The values of s corresponding to these values of β_k may be found by using equation (45),

$$\cosh \pm j \frac{k\pi}{N} = \frac{LCs^2}{2} + 1$$

which gives the roots

$$s_k = \pm j \sqrt{\frac{2}{LC}} \left[1 - \cos \frac{k\pi}{N} \right]^{1/2} = \pm j\omega_k \quad (50)$$

$$k = 0, 1, 2, \ldots , N .$$

After $k = N$ the terms repeat, and no new roots are added. The ω_k's are the natural frequencies of the system.

Since $s_0 = 0$ is a zero of both A(s) and B(s) in equation (49), the function I(s,n) has only a first order pole at the origin, and the partial fraction expansion becomes (see Reference 5, Chap. VI)

$$I(s,n) = \frac{A(0)}{s\,B(0)} + \sum_{k=1}^{N} \frac{(K_k + \bar{K}_k)s + j\omega_k (K_k - \bar{K}_k)}{s^2 + \omega_k^2} \quad (51)$$

where

$$K_k = \left[\frac{A(s)}{s\,B'(s)} \right]_{s=j\omega_k} .$$

and $\overline{K}_k$ indicates the conjugate complex of K_k.

From equations (45), (49), and (50) it may be shown that

$$K_k = \frac{\sin\frac{k\pi}{N}\ \sin\frac{nk\pi}{N}}{2N\left(\cos\frac{k\pi}{N}-1\right)} \quad . \tag{52}$$

Since $\beta \to 0$ as $s \to 0$, and

$$\sinh z = z + \frac{z^3}{3!} + \cdots ,$$

then

$$\frac{A(0)}{B(0)} = 1 - \frac{n}{N} \quad .$$

As K_k is real, $\overline{K}_k = K_k$, and equation (51) now becomes

$$I(s,n) = \frac{1}{s}\left(1-\frac{n}{N}\right) + \sum_{k=1}^{N} \frac{2K_k s}{s^2+\omega_k^2} \quad .$$

Taking the inverse Laplace transform with respect to time,

$$i(t,n) = L_t^{-1}\left[I(s,n)\right] = 1 - \frac{n}{N} + \sum_{k=1}^{N} 2K_k \cos\omega_k t \tag{53}$$

where

$$2K_k = \frac{\sin\frac{k\pi}{N}\ \sin\frac{nk\pi}{N}}{N\left(\cos\frac{k\pi}{N}-1\right)} \tag{54}$$

and

$$\omega_k = \sqrt{\frac{2}{LC}}\left[1-\cos\frac{k\pi}{N}\right]^{1/2} . \tag{55}$$

Equations (53), (54), and (55) constitute the solution to the problem of a step current injected into the front end of the network.

<u>Force Applied to an Intermediate Car.</u> As the next step toward the complete solution of the problem, consider a step braking force applied to any one car along the train. This corresponds

to a unit step current applied to one capacitor-inductor junction in the electrical network, as shown in (Fig. 23a).

Let the total number of capacitors (cars) be M, and break the circuit into three separate networks, as shown in (Fig. 23b). It will be recognized that each of the end sections of (Fig. 23b) is identical to the circuit treated before. By inspection

$$M = N_1 + N_2 + 1 \; . \tag{56}$$

In joining the three networks together the following boundary conditions must be satisfied (see Fig. 23b):

$$I_1(s,0) + I_2(s,0) + sC\,E_o(s) = I_o(s) \tag{57}$$

$$E_1(s,0) + \frac{sL}{2} I_1(s,0) = E_o(s) \tag{58}$$

$$E_2(s,0) + \frac{sL}{2} I_2(s,0) = E_o(s) \tag{59}$$

From equation (47) there are obtained the relationships

$$E_1(s,0) = Z\,I_1(s,0)\coth N_1\beta$$

$$E_2(s,0) = Z\,I_2(s,0)\coth N_2\beta \; .$$

Substituting these into equations (58) and (59),

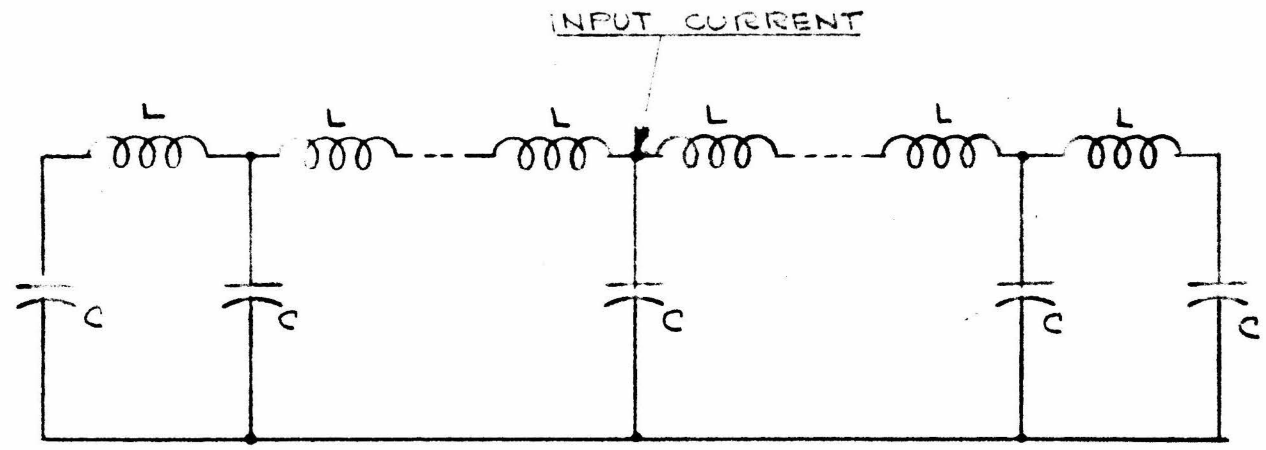
$$I_1(s,0) + I_2(s,0) + sC\,E_o(s) = I_o(s) \tag{57}$$

$$Z\,I_1(s,0)\coth N_1\beta + \frac{sL}{2} I_1(s,0) = E_o(s) \tag{60}$$

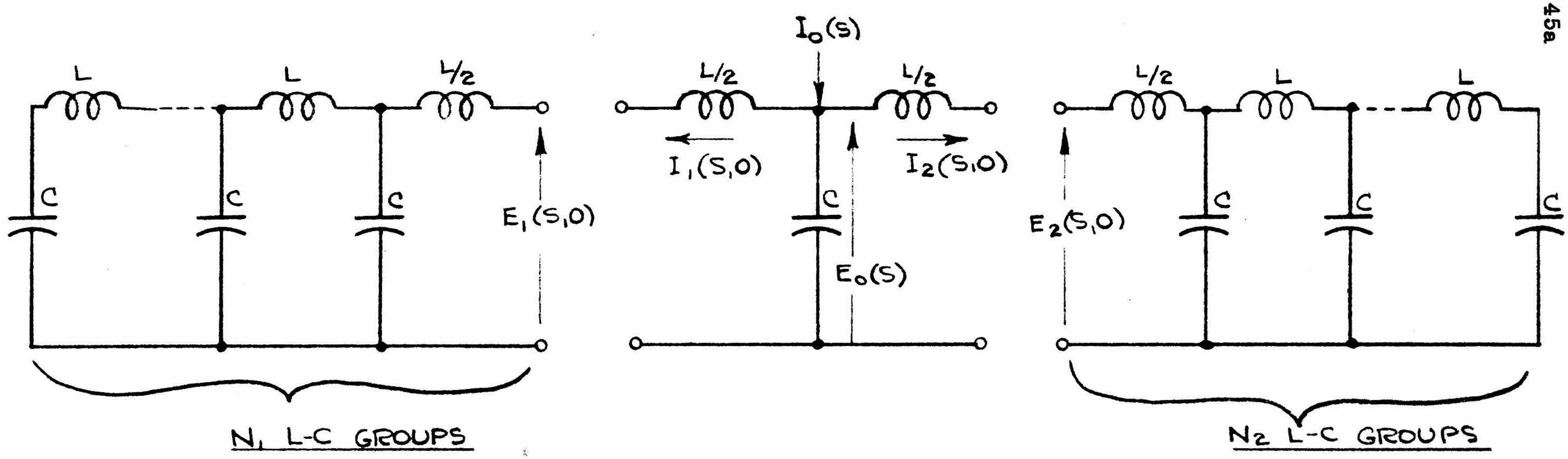
$$Z\,I_2(s,0)\coth N_2\beta + \frac{sL}{2} I_2(s,0) = E_o(s) \; . \tag{61}$$

Simultaneous solution of the above three equations yields

$$I_1(s,0) = \frac{I_o(s)\left[Z\coth N_2\beta + \frac{sL}{2}\right]}{\left\{\left[1 + ZsC\coth N_1\beta + \frac{s^2LC}{2}\right]\left[Z\coth N_2\beta + \frac{sL}{2}\right] + Z\coth N_1\beta + \frac{sL}{2}\right\}_1}$$



(a) TOTAL CIRCUIT



(b) CIRCUIT DIVIDED INTO THREE SECTIONS

FIG. 23 CIRCUITS FOR APPLICATION OF FORCE TO INTERMEDIATE CAR and

$$I_2(s,0) = \frac{I_0(s)\left[Z \coth N_1\beta + \frac{sL}{2}\right]}{\left\{ \quad \right\}_1} .$$

Designating successive loop currents in the N_1 circuit by n_1, from right to left, and in the N_2 circuit by n_2, from left to right, and using equation (48),

$$I_1(s,n_1) = \frac{I_0(s)\left[Z \coth N_2\beta + \frac{sL}{2}\right] \sinh (N_1-n_1)\beta}{\left\{ \quad \right\}_1 \sinh N_1\beta} \tag{62}$$

$$I_2(s,n_2) = \frac{I_0(s)\left[Z \coth N_1\beta + \frac{sL}{2}\right] \sinh (N_2-n_2)\beta}{\left\{ \quad \right\}_1 \sinh N_2\beta} . \tag{63}$$

By expanding the denominators of equations (62) and (63) and making repeated use of the relationship

$$\sinh x \cosh y + \cosh x \sinh y = \sinh (x+y)$$

there is obtained

$$I_1(s,n_1) = \frac{I_0(s)\left[Z \coth N_2\beta + \frac{sL}{2}\right] \sinh (N_1-n_1)\beta \sinh N_2\beta}{Z \sinh (N_1+N_2+1)\beta} \tag{64}$$

$$I_2(s,n_2) = \frac{I_0(s)\left[Z \coth N_1\beta + \frac{sL}{2}\right] \sinh (N_2-n_2)\beta \sinh N_1\beta}{Z \sinh (N_1+N_2+1)\beta} . \tag{65}$$

Referring to equation (48) it is observed that the $\sinh(N_1+N_2+1)$ term is exactly what is to be expected, since there are N_1+N_2+1 capacitors in the network, and since the characteristic equation of any linear system is independent of the location of the driving force.

For a unit step input current,

$$I_0(s) = \frac{1}{s} ,$$

and

$$I_1(s,n_1) = \frac{\left[Z \coth N_2\beta + \frac{sL}{2}\right] \sinh (N_1 - n_1)\beta \, \sinh N_2\beta}{s Z \sinh (N_1 + N_2 + 1)\beta} \tag{66}$$

$$I_2(s,n_2) = \frac{\left[Z \coth N_1\beta + \frac{sL}{2}\right] \sinh (N_2 - n_2)\beta \, \sinh N_1\beta}{s Z \sinh (N_1 + N_2 + 1)\beta} . \tag{67}$$

The solution of equations (66) and (67) is carried out the same as for equation (49), and the details will not be repeated here. The resulting solutions are

$$i_1(t,n_1) = \frac{N_1 - n_1}{N_1 + N_2 + 1} + \sum_{k=1}^{N_1+N_2+1} 2 K_{1K} \cos \omega_{1K} t \tag{68}$$

$$i_2(t,n_2) = \frac{N_2 - n_2}{N_1 + N_2 + 1} + \sum_{k=1}^{N_1+N_2+1} 2 K_{2K} \cos \omega_{1K} t , \tag{69}$$

where

$$\omega_{1K} = \sqrt{\frac{2}{LC}} \left[1 - \cos \frac{k\pi}{N_1 + N_2 + 1}\right]^{1/2} \tag{70}$$

$$K_{1K} = \frac{(-1)^k \left[\sin \frac{k\pi (N_2+1)}{N_1+N_2+1} - \sin \frac{k\pi N_2}{N_1+N_2+1}\right] \sin \frac{k\pi (N_1 - n_1)}{N_1+N_2+1}}{2(N_1+N_2+1)\left(1 - \cos \frac{k\pi}{N_1+N_2+1}\right)} \tag{71}$$

$$K_{2K} = \frac{(-1)^k \left[\sin \frac{k\pi (N_1+1)}{N_1+N_2+1} - \sin \frac{k\pi N_1}{N_1+N_2+1}\right] \sin \frac{k\pi (N_2 - n_2)}{N_1+N_2+1}}{2(N_1+N_2+1)\left(1 - \cos \frac{k\pi}{N_1+N_2+1}\right)} . \tag{72}$$

Next, let the loop currents and nodal points of the circuit be numbered from left to right, beginning at the extreme left end, as shown in (Fig. 24). Let the forcing current enter at nodal point L, and let successive loop currents bear the designation m.

If all currents are considered positive clockwise,

$$i_1(t,m) = -i_1(t,n_1) \tag{73}$$

$$i_2(t,m) = +i_2(t,n_2) \quad . \tag{74}$$

First, consider the currents to the left of point L. By inspection

$$N_1 = L - 1$$

$$N_2 = M - L$$

$$n_1 = L - m - 1 \quad .$$

Then,

$$i_1(t,m) = -\frac{m}{M} - \sum_{k=1}^{M} 2K_{1k} \cos \omega_{1k} t \tag{75}$$

$$K_{1k} = \frac{(-1)^k \left[\sin \frac{k\pi(M-L+1)}{M} - \sin \frac{k\pi(M-L)}{M} \right] \sin \frac{k\pi m}{M}}{2M\left(1 - \cos \frac{k\pi}{M}\right)} \tag{76}$$

$$\omega_{1k} = \sqrt{\frac{2}{LC}} \left[1 - \cos \frac{k\pi}{M} \right]^{1/2} . \tag{77}$$

To the right of point L,

$$n_2 = m - L \ ,$$

and the solution becomes

$$i_2(t,m) = 1 - \frac{m}{M} + \sum_{k=1}^{M} 2K_{2k} \cos \omega_{1k} t \tag{78}$$

where

$$K_{2k} = \frac{(-1)^k \left[\sin \frac{k\pi L}{M} - \sin \frac{k\pi(L-1)}{M} \right] \sin \frac{k\pi(M-m)}{M}}{2M\left(1 - \cos \frac{k\pi}{M}\right)} \tag{79}$$

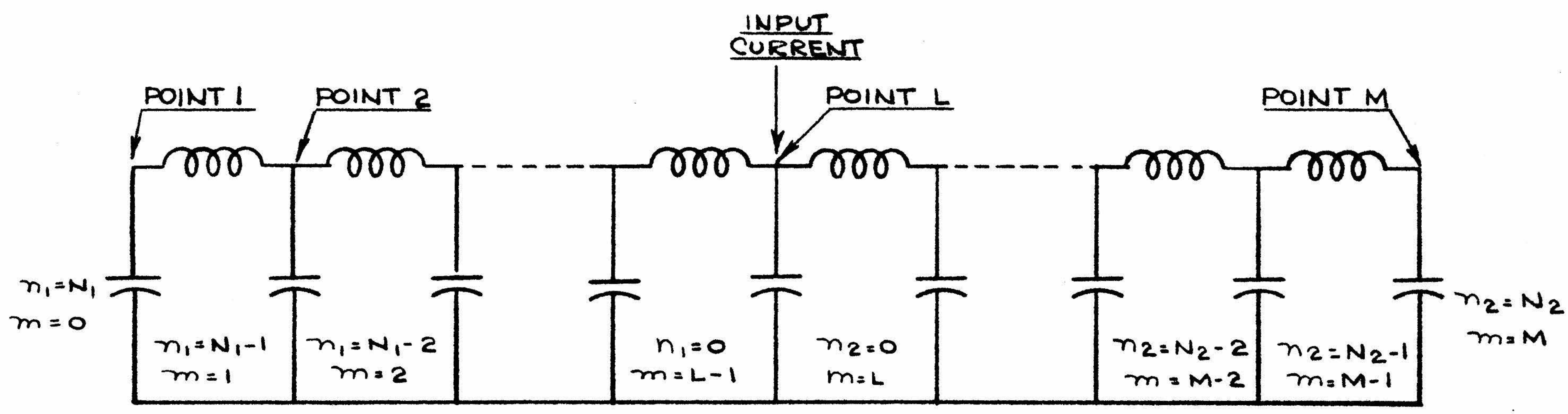


FIG. 24 CIRCUIT LOOP AND NODE DESIGNATION

and ω_{lK} is given by equation (77).

There appears to be no one simple expression which will present both i_1 and i_2 in the same equation. However, this is to be expected from the nature of the problem.

As a check on the correctness of the above solutions, letting $L = 1$, $M = N$, and $m = n$ for the i_2 solution; or letting $L = M = N$, and $m = N-n$ for the i_1 solution, gives the end drive case of the preceeding section.

Force Applied Sequentially to All Cars. To extend the problem, let a braking force be applied to each car in succession. This corresponds to step currents, (which will be assumed to be of unit magnitude), being applied successively to the different nodal points of the electric circuit, from left to right. The current at point 1, (Fig. 24), is applied at $t = 0$, and the time delay between application of successive currents is T_0. Extend the current notation to $i_1(t,m,L)$ and $i_2(t,m,L)$ to indicate by L the point at which current enters. The solution then consists of the solution for $L = 1$, multiplied by $u(t)$, ($u(x)$ is a function which is zero for x negative and unity for $x \geq 0$); the solution for $L = 2$, multiplied by $u(t-T_0)$; the solution for $L = 3$, multiplied by $u(t-2T_0)$; and so on up to the $L = M$ solution (M total cars). Of course, care must be taken to change from the i_2 to the i_1 solution at the proper point in the calculation. Thus, for any particular value of m the calculation may be indicated as follows:

$$\begin{aligned} i(t,m) = {} & i_2(t,m,1)\,u(t) + i_2(t,m,2)\,u(t-T_0) + i_2(t,m,3)\,u(t-2T_0) + \cdots \\ & + i_2(t,m,m)\,u[t-(m-1)T_0] + i_1(t,m,m+1)\,u[t-mT_0] \\ & + i_1(t,m,m+2)\,u[t-(m+1)T_0] + \cdots \\ & + i_1(t,m,M-1)\,u[t-(M-2)T_0] + i_1(t,m,M)\,u[t-(M-1)T_0] \end{aligned} \tag{80}$$

where i_1 and i_2 are given by equations (75) and (78) respectively.

It should perhaps be remarked in passing that if the tractive effort of the locomotive should be pulsating with a frequency near one of the system resonant frequencies given by the mechanical equivalent of equation (77), large spring forces could be expected.

It has been calculated that to use the foregoing equations to obtain a satisfactory transient solution for the force in only one draft gear and for one particular value of T_0 would require over 250,000 separate calculations of moderate complexity and over 7000 tabulations, plus the addition and possible plotting associated with equation (80). Furthermore, these equations apply only to the case in which all cars are identical. These difficulties, plus the desire to consider nonlinear draft gears, make the electric analog method attractive.

III. ELECTRIC ANALOG COMPUTER SOLUTION FOR 50 CAR TRAIN WITH IDEALIZED DRAFT GEARS

Fifty cars was chosen as representative of a long train.

Generation of Braking Currents. An electric circuit to supply one of the forcing currents to the analog network is shown in (Fig. 25). This circuit is essentially a high voltage source (450 volts) in series with a high resistance ($2x10^6$ ohms), which supplies a nearly constant current as long as the voltage of the capacitor in the train circuit remains low. The beginning of the current is controlled by the type 2D21 thyratron. The thyratron's firing is actuated by closing switch S, which reduces the grid voltage to zero. Switch S is actually one section of a commutating switch. Thus, when commutator action closes S momentarily the thyratron conducts and remains conducting. The commutator shorts the grid of each of 50 thyratrons in turn, corresponding to each successive

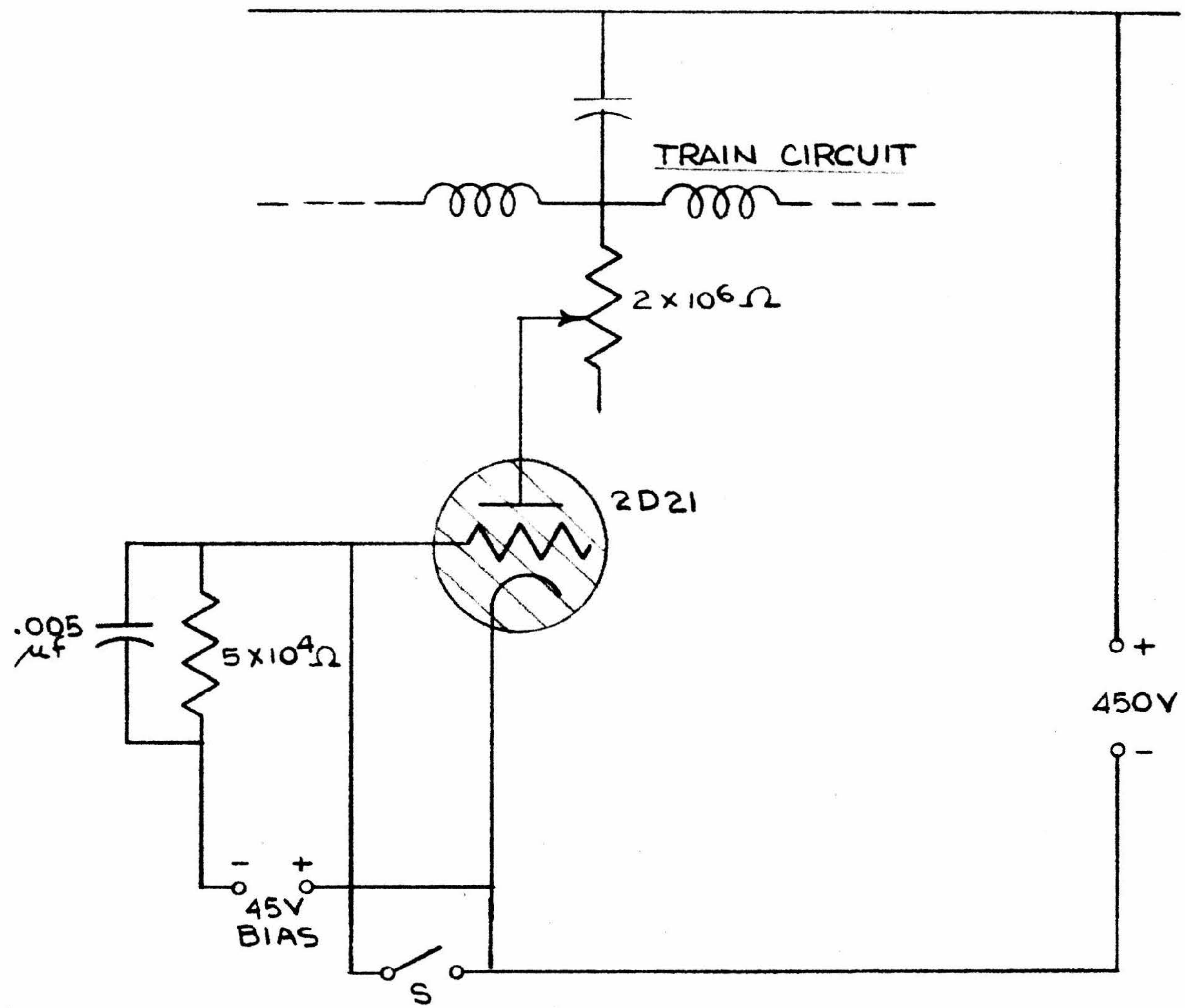


FIG. 25 CIRCUIT TO SIMULATE BRAKING FORCE APPLIED TO ONE CAR

current input to the analog circuit. The thyratrons' continuing to conduct corresponds to the mechanical brakes of each car remaining on once they are actuated. After the solution is no longer of interest the thyratrons are made nonconducting by opening the supply voltage circuit. After removal of energy from the train circuit the solution is repeated. In the electric analog this complete cycle requires 0.1 sec. Thus the solutions appear as stationary patterns on a cathode ray oscilloscope. The .005 μf condensers prevent firing of a thyratron due to spurious pulses, such as those from adjacent circuit elements. (Fig. 26) shows the 50 thyratron circuit chassis.

The braking force applied to each car is a step force of magnitude f_o which is applied at a time T_o later than the braking force of the preceding car. (Fig. 27) shows the total integrated braking force applied to the train, integrated in the sense that it represents the sum of all the braking forces applied to the train at any particular instant of time. On the computer T_o was fixed at 0.0005 sec. f_o was 1 milliampere.

Presentation of Solutions. It is known (12) that the draft gear forces vary according to the ratio of the natural velocity of wave propagation along the train to the velocity at which the braking forces are propagated along the train. Thus this dimensionless variable is chosen as a parameter in the presentation of solutions. (See Table 3.)

The cars are numbered consecutively from the head end of the train, the first mass being car number one. Actually, since all masses are assumed equal, the circuit represents a train of 49 cars headed by a locomotive of equal mass and braking force. This was done to keep the solution general and to afford a check against the solution for a uniform bar. Draft gear number one is located between

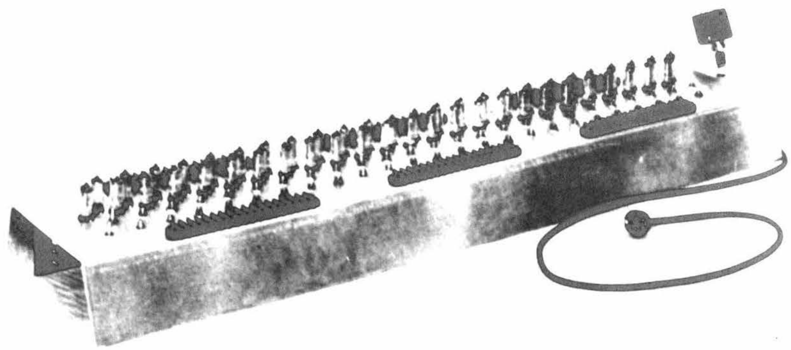
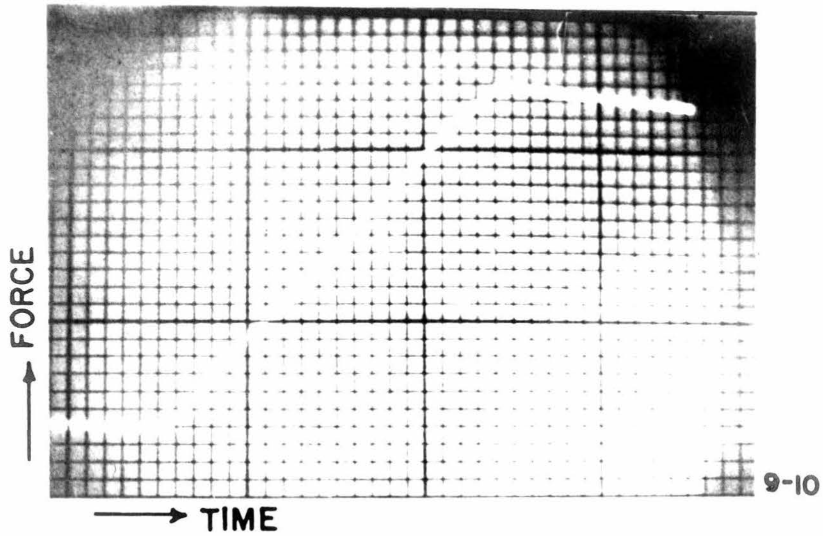
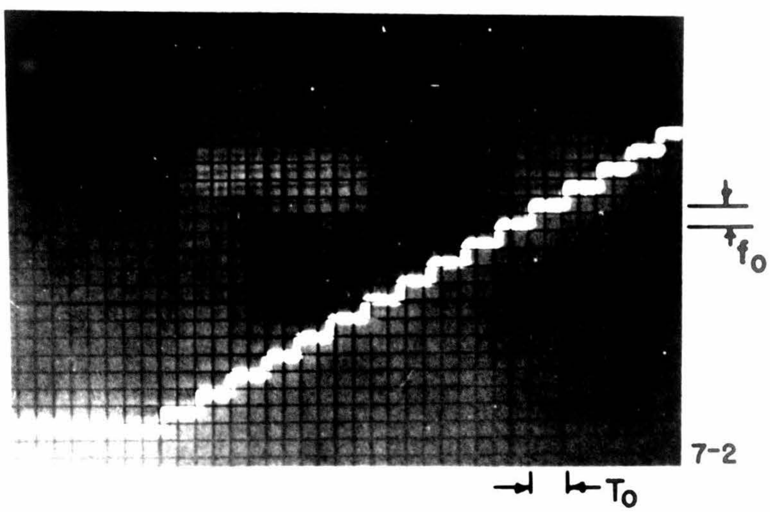


FIG. 26 50 THYRATRON FORCING FUNCTION UNIT



a. TOTAL INTEGRATED BRAKING FORCE



b. ENLARGED SECTION OF BEGINNING OF TOTAL INTEGRATED BRAKING FORCE.

FIG.27 TOTAL INTEGRATED BRAKING FORCE APPLIED TO TRAIN.

Table 3 Definition of Parameters
For Train With Idealized Draft Gears

K	Spring constant of each draft gear
M	Mass of each car
$V = \sqrt{\frac{K}{M}}$	Natural propagation velocity along train
V_1	Velocity of application of brakes along train
$p = \frac{V}{V_1}$	Ratio of natural velocity to braking velocity

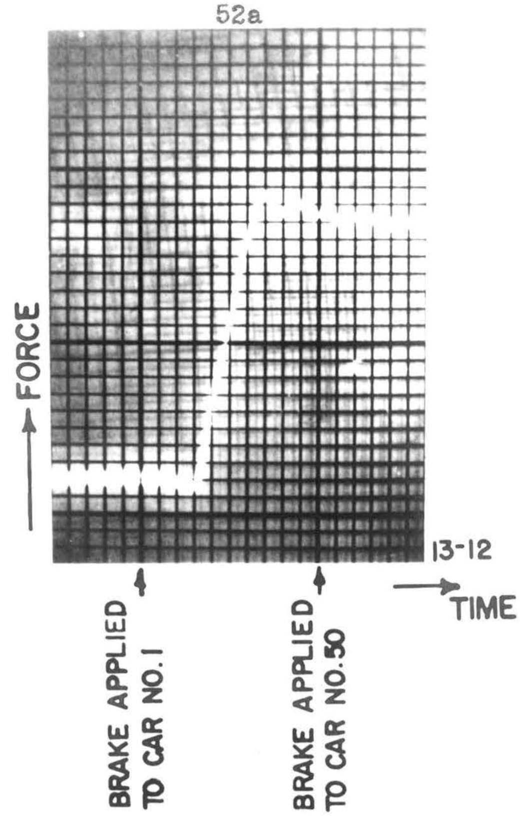
cars one and two, and the remaining draft gears are numbered consecutively.

Draft gear forces are expressed in units of braking force f_o .

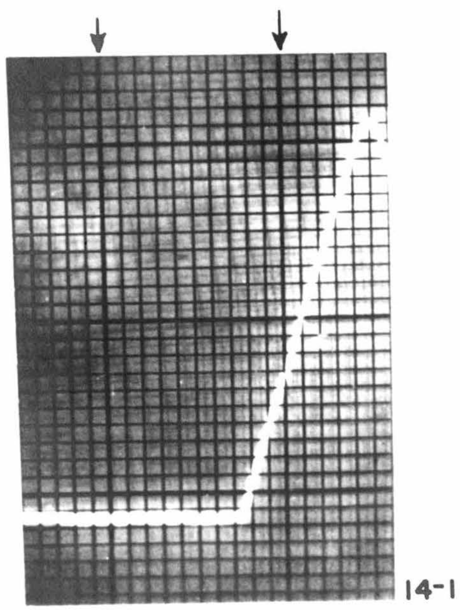
Results are presented in both photographic and graphical form. The horizontal time base is the same for all photographs except those of (Fig. 27). The time required to apply 50 brakes is represented by 10 horizontal divisions on the photographs. The application of the first brake, (car No. 1), occurs at the first heavy vertical line in all photographs except that of (Fig. 27a). On all photographs the vertical scale is arbitrarily adjusted for clearness and definition.

The parameter p (see Table 3) ranges from 0.5 to 2.0, the range of practical interest in freight service. On the computer p was varied by varying the value of the inductances in the train circuit.

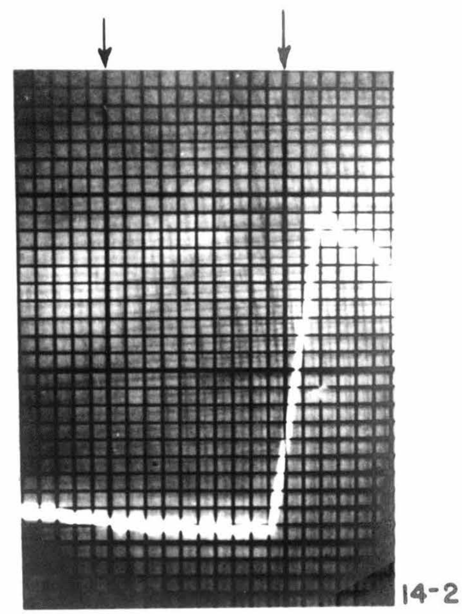
(Fig. 28) shows typical transient solutions of force vs. time for each of three different draft gears when $p = 0.5$. The delay effect is illustrated by the fact that the beginning of the force build up occurs later for the draft gears farthest from the head of the train. Furthermore, it may be observed that the time at which force build up begins in each draft gear is exactly the time at which the braking force is applied to the car adjacent. This is because the natural propagation velocity is low compared to braking velocity.



a. DRAFT GEAR NUMBER 15
PEAK FORCE 5.4 f_0



b. DRAFT GEAR NUMBER 36
PEAK FORCE 12.0 f_0



c. DRAFT GEAR NUMBER 45
PEAK FORCE 5.6 f_0

FIG. 28 TYPICAL SOLUTIONS FOR $p = 0.5$

Thus, for draft gear number 15 (Fig. 28a) the build up begins $\frac{15}{50} \times 10 = 3$ divisions to the right of the heavy vertical line which marks the application of brakes to car number 1. For draft gear number 36 it is $\frac{36}{50} \times 10 = 7.2$ divisions, and so on.

In (Fig. 29) are similar solutions for p = 1.075. This was the value of p which gave the maximum crest force, this force occurring in draft gear number 46.

(Fig. 30) presents solutions for p = 2.0. These solutions demonstrate the effects of reflections from the rear end of the train. For p = 2.0 the natural velocity of propagation along the train is just twice the brake application velocity, and it is quantitatively observed that the beginning of force build up occurs at a time which is, in general, one half the time for the case in which reflections did not affect the solution (p = 0.5, Fig. 28). Thus for draft gear number 25 (Fig. 30b) the force build up begins

$$\frac{25}{50} \times 10 \times \frac{1}{2} = 2.5 \text{ divisions}$$

after braking is begun. For draft gear number 33 it is

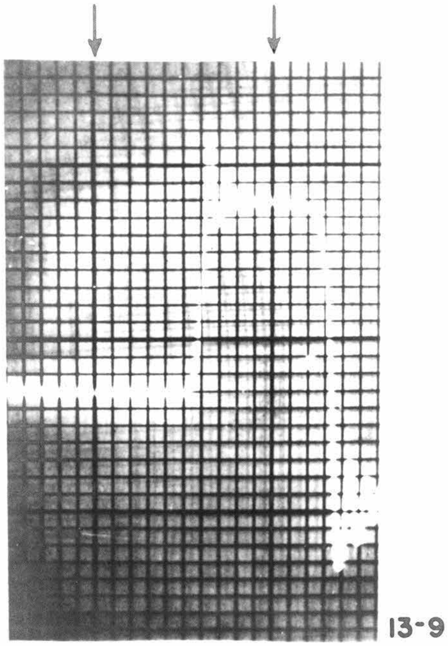
$$\frac{33}{50} \times 10 \times \frac{1}{2} = 3.3 \text{ divisions.}$$

For three different values of p the peak force in each draft gear of the train is plotted and compared to the theoretical force in a uniform elastic bar. (12) These plots are arranged as follows:

Fig. 31	p = 0.5
Fig. 32	p = 1.075
Fig. 33	p = 2.0

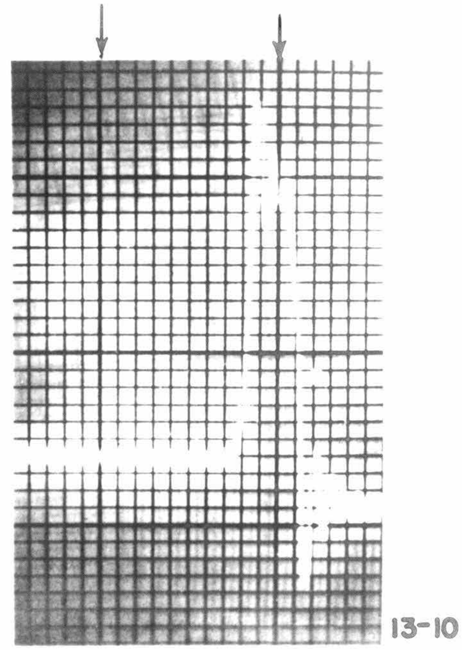
(Fig. 34) gives the number of the draft gear in which the maximum crest force occurs for a range of values of p from 0.5 to 2.0. These locations are compared with the location of the maximum longitudinal stress in a uniform bar (12).

(Fig. 35) presents the values of maximum crest draft gear force



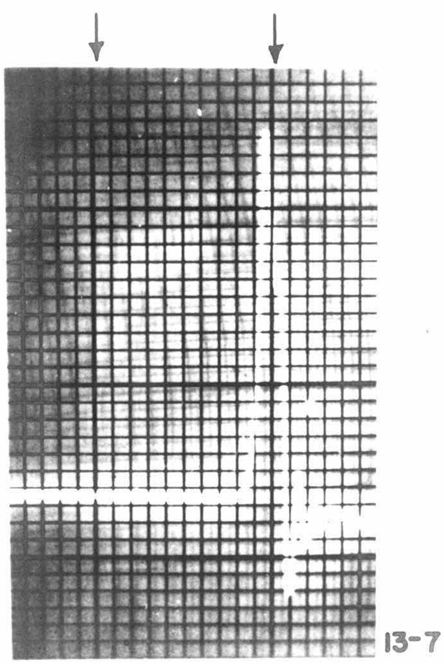
a. DRAFT GEAR NUMBER 30

PEAK FORCE 20.6 f_0



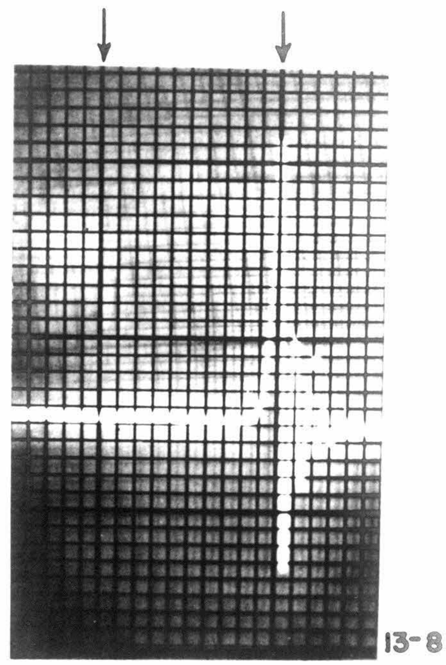
b. DRAFT GEAR NUMBER 42

PEAK FORCE 28.6 f_0



c. DRAFT GEAR NUMBER 46

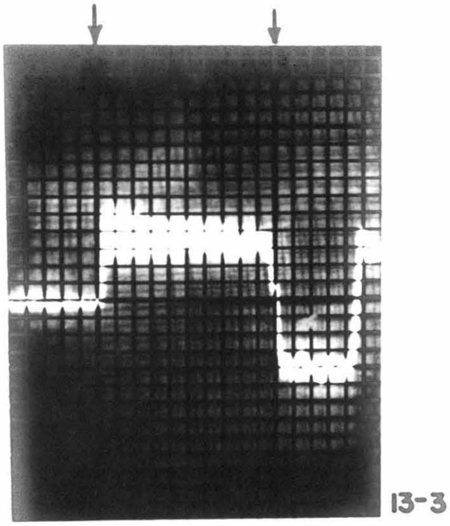
PEAK FORCE 31.5 f_0



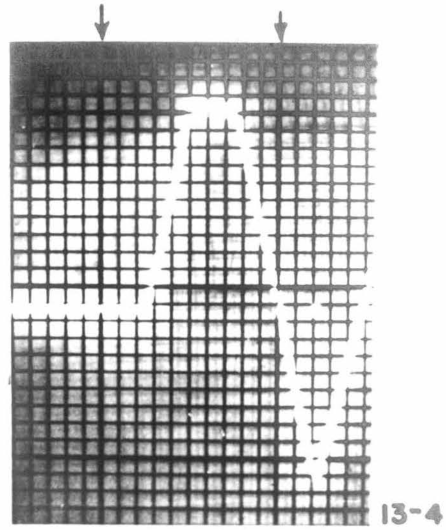
d. DRAFT GEAR NUMBER 49

PEAK FORCE 16.8 f_0

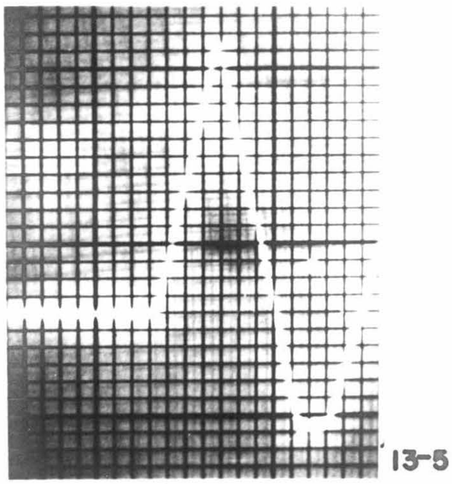
FIG. 29 TYPICAL SOLUTIONS FOR p = 1.075



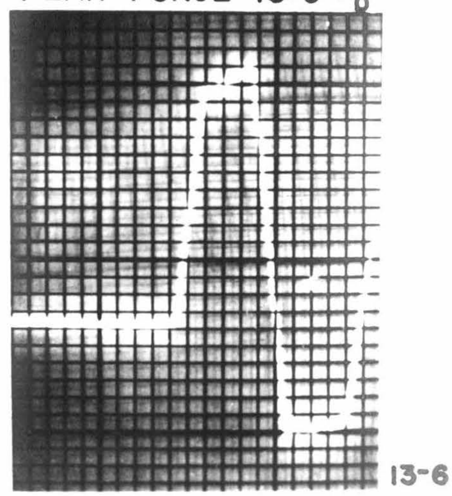
a. DRAFT GEAR NUMBER 12
PEAK FORCE 8.9 f_0



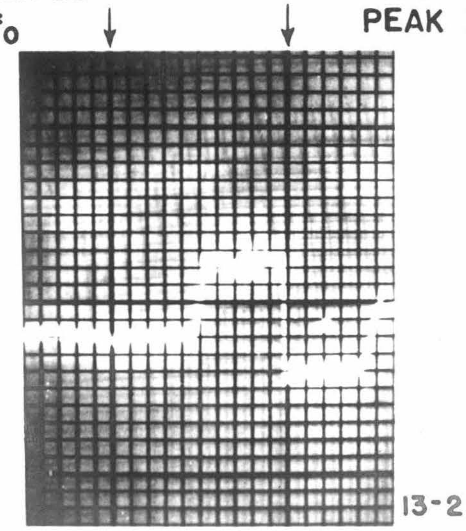
b. DRAFT GEAR NUMBER 25
PEAK FORCE 18.0 f_0



c. DRAFT GEAR NUMBER 33
PEAK FORCE 23.4 f_0



d. DRAFT GEAR NUMBER 43
PEAK FORCE 10.5 f_0



e. DRAFT GEAR NUMBER 49
PEAK FORCE 1.5 f_0

FIG.30 TYPICAL SOLUTIONS FOR $p = 2.0$

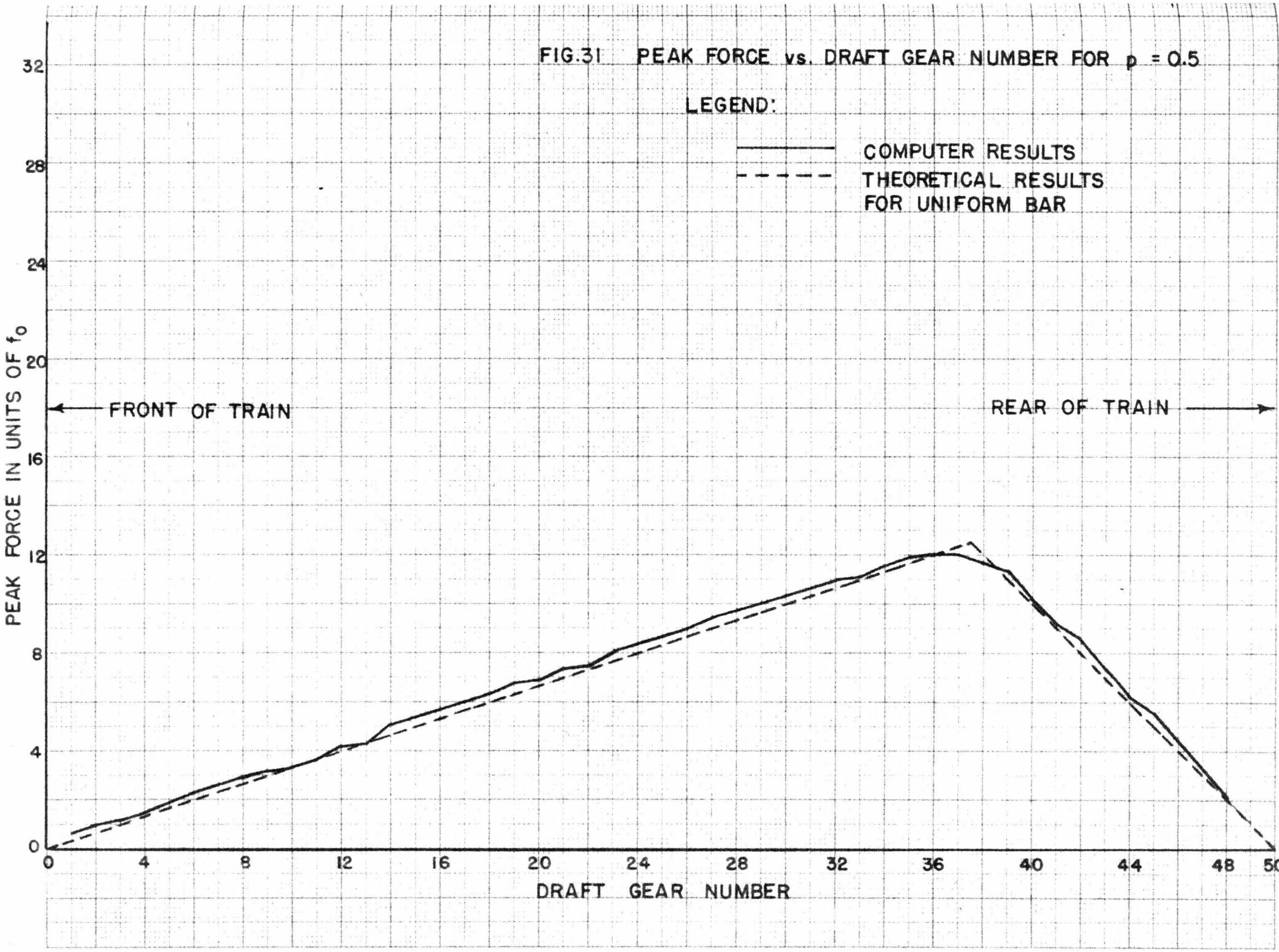


FIG.31 PEAK FORCE vs. DRAFT GEAR NUMBER FOR $p = 0.5$

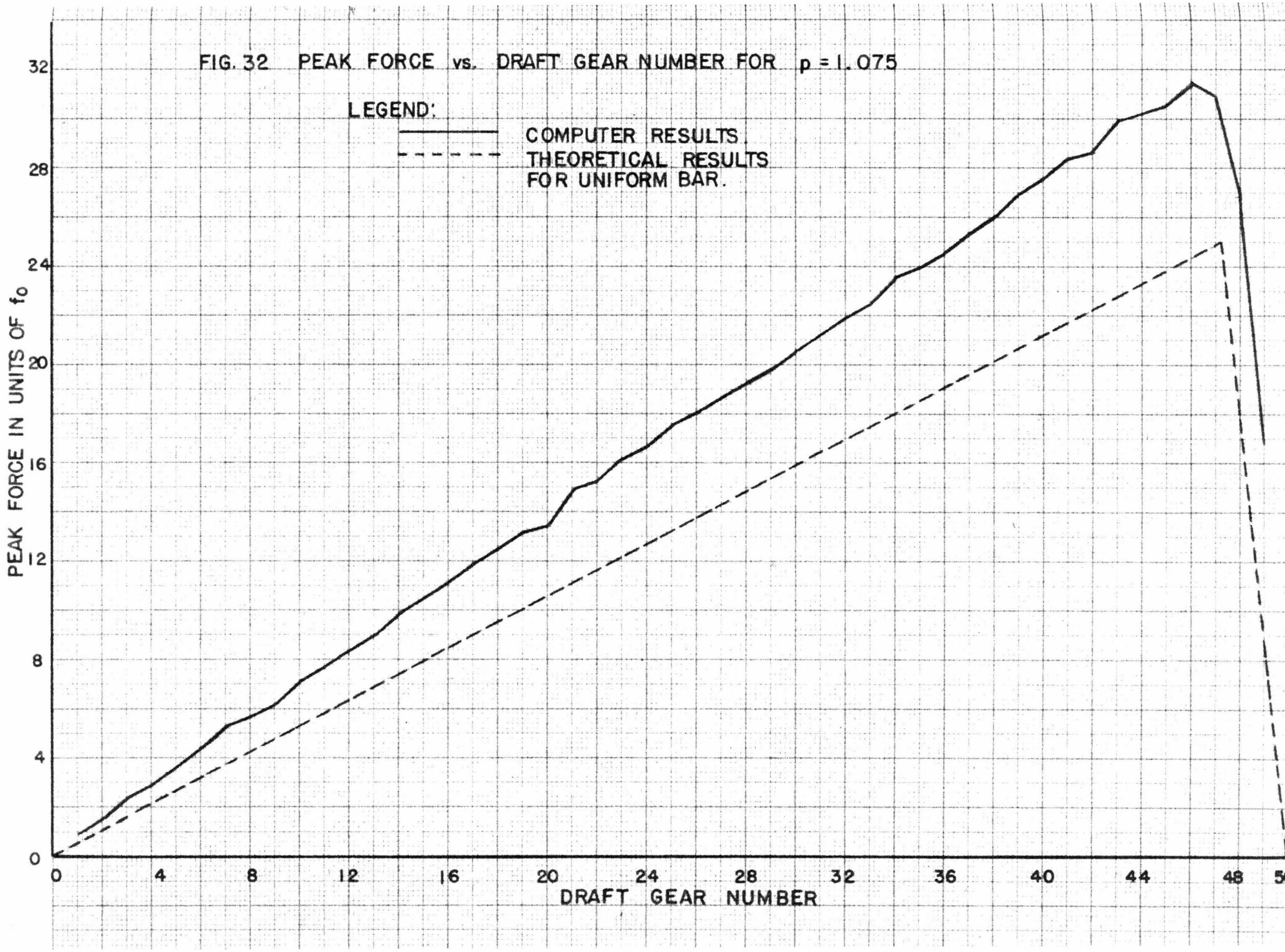


FIG. 32 PEAK FORCE vs. DRAFT GEAR NUMBER FOR $p = 1.075$

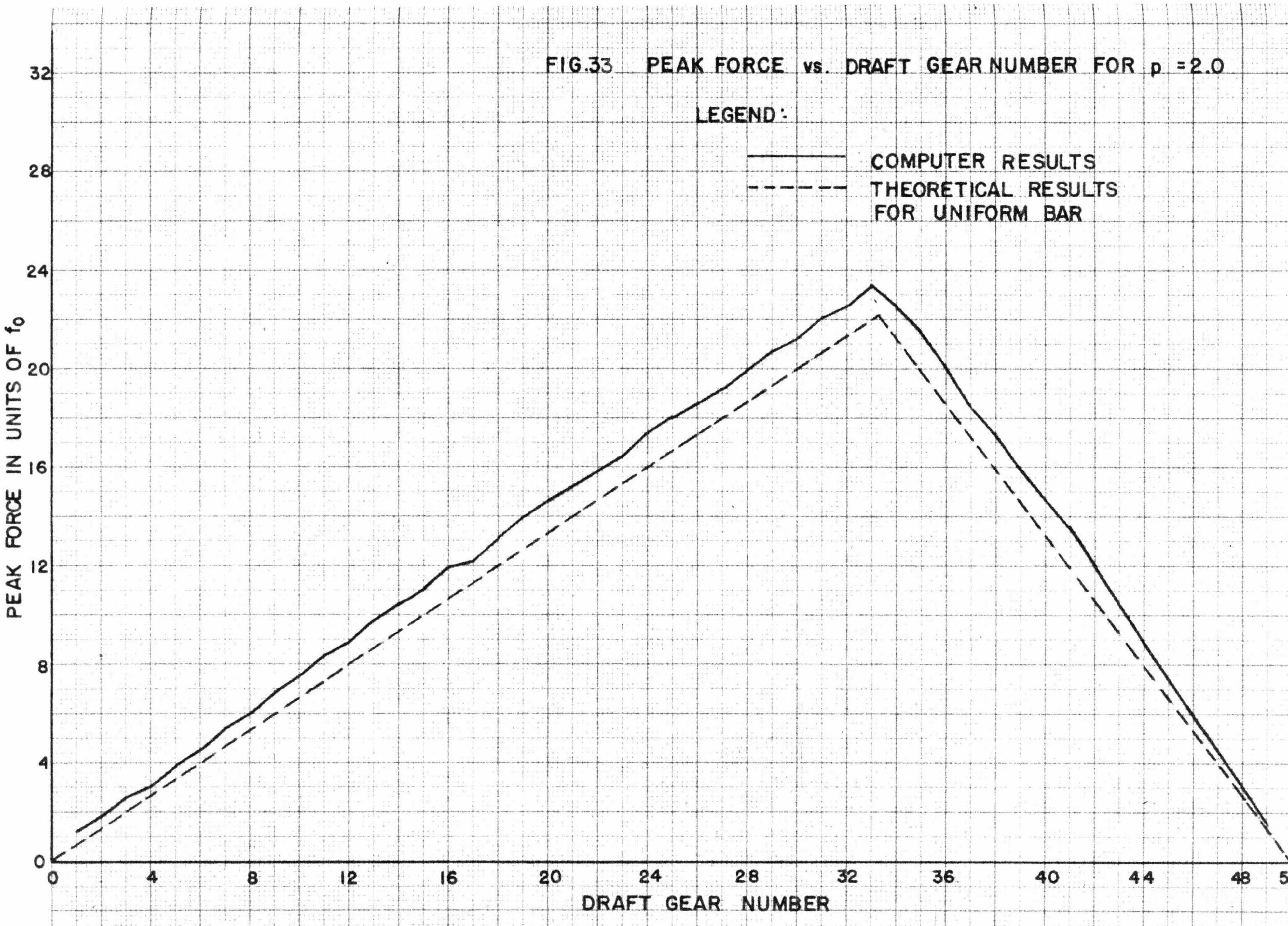


FIG.33 PEAK FORCE vs. DRAFT GEAR NUMBER FOR ρ = 2.0

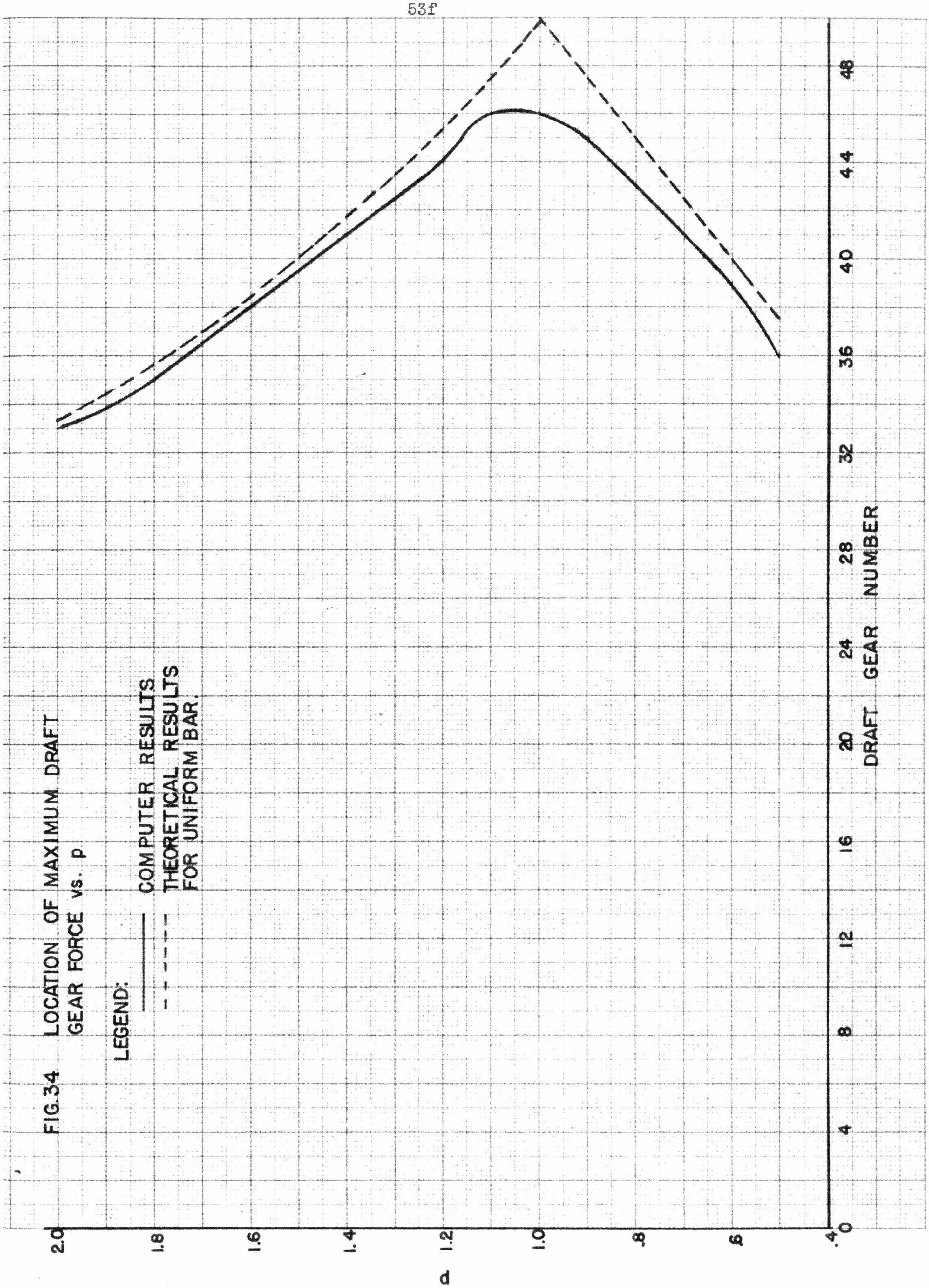


FIG.34 LOCATION OF MAXIMUM DRAFT GEAR FORCE vs. p

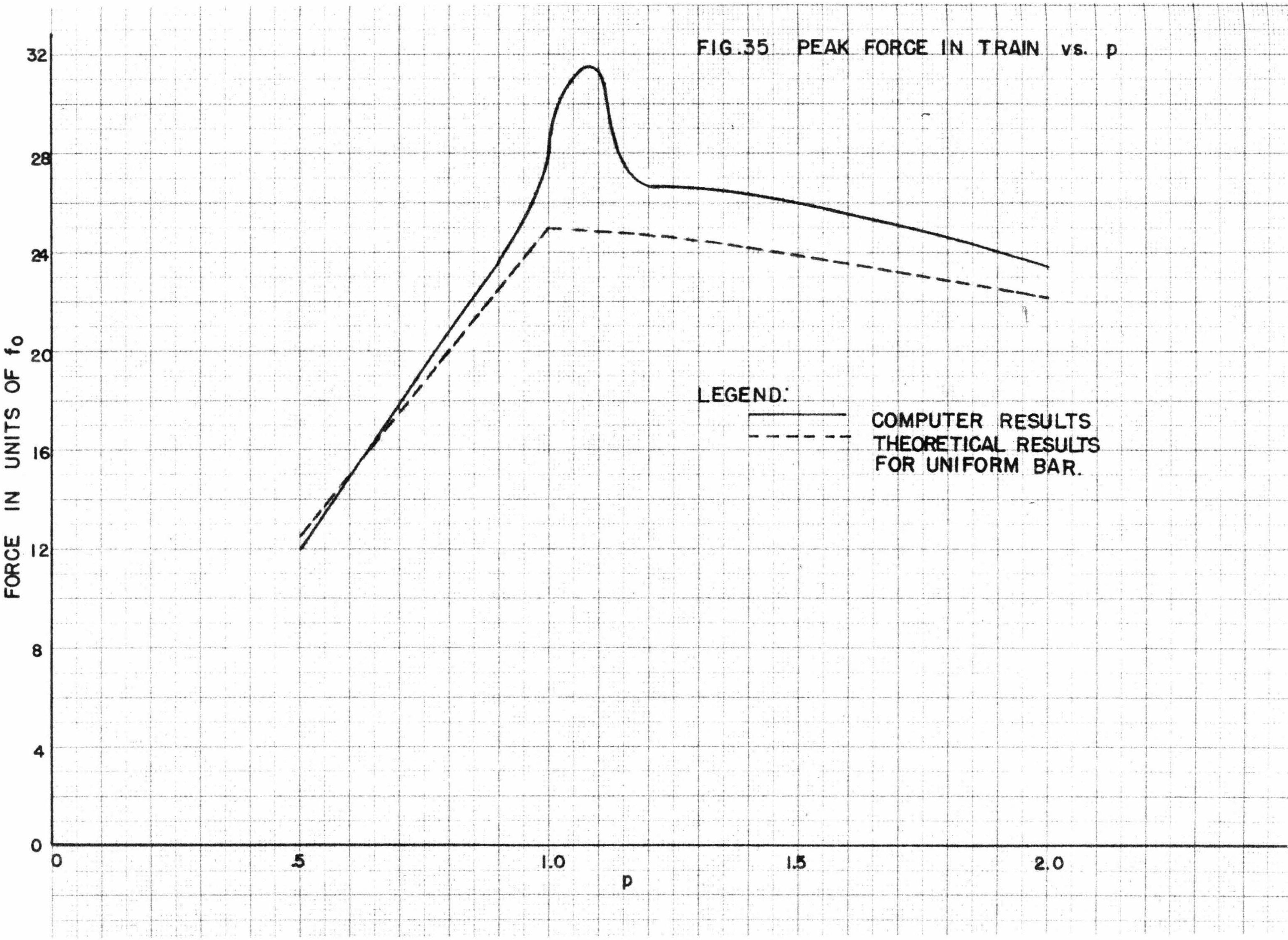


FIG.35 PEAK FORCE IN TRAIN vs. p

occurring in the train plotted against values of p, together with theoretical results for a uniform bar.

Referring to (Figs. 34 and 35) it is apparent that for a train of 50 identical cars connected by ideal springs the magnitude and location of maximum spring force for values of p between 0.5 and 2.0 agree fairly well with those for a uniform elastic bar. The agreement for values of p from 0.5 to 0.9 is remarkably close. However, it is to be noted that where disagreement does exist between the lumped mass and uniform bar results, the latter are optimistic. (See for example Fig. 32). That is, the forces acting are actually larger than would be expected from the uniform bar results.

It would be expected that for a train of more than 50 cars the results would resemble more closely those for a uniform bar, the agreement improving as more cars are added. Correspondingly, since significant differences between a 50 car train and a uniform bar are observed, the results for less than 50 cars could be expected to be considerably different from the uniform bar solutions.

IV. ELECTRIC ANALOG FOR NONLINEAR DRAFT GEAR

To extend the problem to include the effects of the nonlinearity of actual draft gears as used in freight service it is first necessary to develop an electric analog for the nonlinear draft gear. A force-travel characteristic for a typical draft gear is given by the solid curve of (Fig. 36). The upper curve is the closure curve of the draft gear, and the lower curve represents re-expansion. Thus, during closure and subsequent re-expansion considerable energy is dissipated by the draft gear, given quantitatively by the area of the loop. This curve is taken from "Edgewater Ring Spring Draft Gears", Circular No. 12, published by the Edgewater Steel Company.

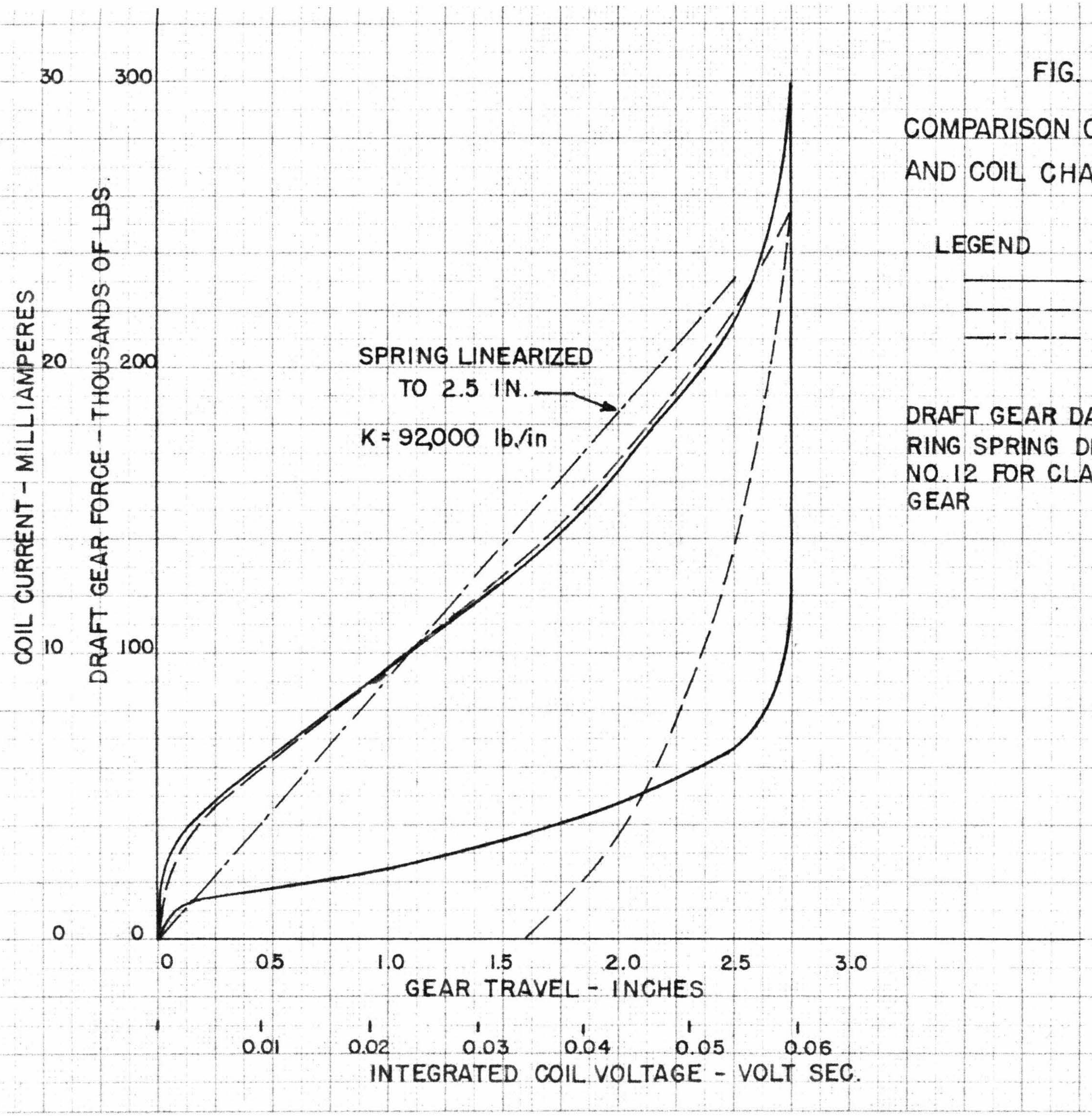


FIG. 36

COMPARISON OF DRAFT GEAR AND COIL CHARACTERISTICS

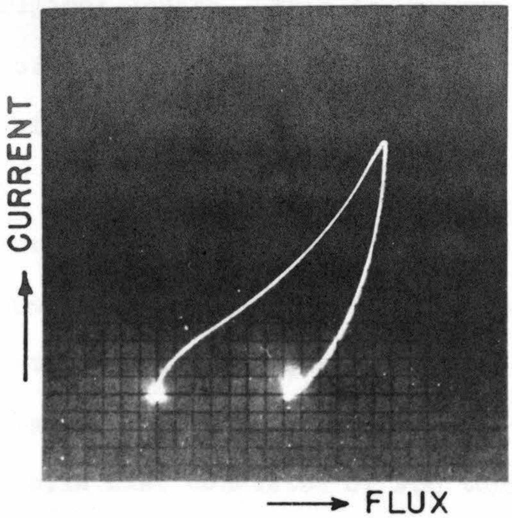
LEGEND

- DRAFT GEAR
- COIL
- LINEARIZED SPRING

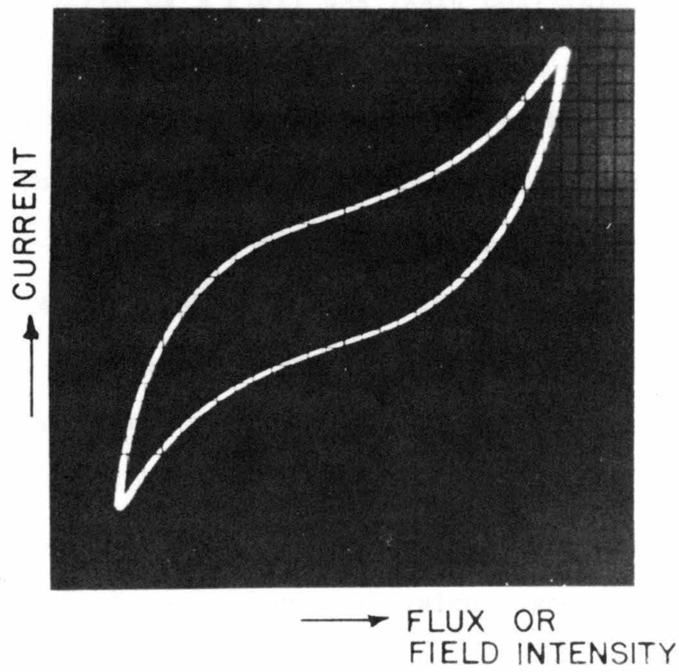
DRAFT GEAR DATA FROM EDGEWATER RING SPRING DRAFT GEAR CIRCULAR NO. 12 FOR CLASS B-32-KA DRAFT GEAR

The loop nature of the draft gear force-travel curve suggests magnetic hysterisis in an iron core coil as an electrical counterpart. Electric current is analogous to mechanical force, (see Fig. 21), and magnetic flux, (the time integral of coil voltage), is analogous to mechanical displacement. Using this analogy as a basis, a special coil was developed which has electrical properties sufficiently like the mechanical behavior of the draft gear. The dashed curve of (Fig. 36) shows the characteristic of the special coil in terms of the analogous electrical quantities. The data for the dashed curve of (Fig. 36) were taken from an oscillographic record of the type shown in (Fig. 37a), the coordinates of which will perhaps be most readily recognized by referring to the corresponding steady state curve of (Fig. 37b). (Fig. 37a) was obtained by passing a transient current through an initially demagnetized special coil and electrically integrating the coil voltage to obtain a voltage proportional to coil flux. The transient current was one which increased to a maximum and then returned to zero.

It is impossible to match the draft gear curve over its entire path, but the match shown in (Fig. 36) is considered sufficiently accurate for the following reasons. The close match on the upper curve, (the closure curve), is important because it is the closure characteristic which determines the initial travelling force wave which passes down the train. The total area inside the draft gear loop and that inside the corresponding electrical loop are approximately equal. The area inside each loop represents the energy extracted from the train during closure and subsequent re-expansion of the draft gear. Thus, while the re-expansion curves do not match closely the energy absorption per cycle is the same for the draft gear and its analog.



a. ELECTRICAL CHARACTERISTIC OF SPECIAL COILS USED TO MATCH DRAFT GEAR.



b. STEADY STATE DYNAMIC "B-H" CURVE OF SPECIAL COILS.

FIG 37 CHARACTERISTICS OF SPECIAL COILS USED.

It is noted that upon re-expansion the draft gear of (Fig. 36) returns to its original length. This is due to the fact that the sliding surfaces of the draft gear are graphite-lubricated. With many other types of draft gears the re-expansion does not result in complete recovery, and the draft gear retains a permanent "set" or deflection when force is removed. This effect corresponds more nearly to the magnetic analog exhibited as residual magnetism when coil current is returned to zero, and the re-expansion curves of such draft gears may be more nearly matched by a magnetic circuit.

The special coils used consisted of 950 turns of No. 28 wire on the center leg of a shell type core of 10 laminations of 24 gauge Allegheny EI-75 dynamo (Grade D) steel.

V. BRAKING OF A FIFTY CAR TRAIN CONTAINING NONLINEAR DRAFT GEARS

Solutions will now be given for a 50 car train containing the nonlinear draft gears described previously (Fig. 36). Since it is known that for a train with idealized draft gears, which act as perfect springs, the transient forces in the draft gears are directly dependent upon the weights of the cars, it is desirable to study the effect of different car weights in the train with nonlinear draft gears. The train studied is assumed to have no free slack. Results will be presented for the case in which all cars are identical and a few cases in which cars of unequal weights are distributed along the train. Two different values of braking force are used.

Due to the nonlinear nature of the problem it is not possible to express results in dimensionless form for a particular draft gear characteristic. For this particular study the following data were chosen as being representative of a typical freight train:

Length of cars	42 feet
Velocity of brake application along train	860 feet/sec.
Type of draft gear	Edgewater Class B-32-KA
Weight of empty car	25 tons
Weight of fully loaded car	100 tons
Braking force on each car	(1) 10,000 lbs. (2) 20,000 lbs.

Electrical-mechanical Analogy. The details of the electrical-mechanical analogy will be developed here.

Adding the forces on the nth mass of the mechanical configuration, (Fig. 21a), there results

$$F_n = M_n p_2 v_n + f\left(\frac{v_n}{p_2} - \frac{v_{n+1}}{p_2}\right) + f\left(\frac{v_n}{p_2} - \frac{v_{n-1}}{p_2}\right), \qquad (81)$$

where

$p_2 = d/dt_2$ is the differential operator in the mechanical system

$f\left(\frac{v}{p_2}\right)$ represents the force of the draft gear and is a double valued function.

The corresponding electrical equation is obtained by adding all the currents entering the nth node of the electrical circuit, (Fig. 21b). Thus,

$$I_n = C_n p_1 E_n + i\left(\frac{E_n}{p_1} - \frac{E_{n+1}}{p_1}\right) + i\left(\frac{E_n}{p_1} - \frac{E_{n-1}}{p_1}\right), \qquad (82)$$

where

$p_1 = d/dt_1$ is the differential operator in the electrical system

$i\left(\frac{E}{p_1}\right)$ gives the double valued functional relationship between coil current and coil flux, since flux is the integral of coil voltage.

Multiplying equation (82) by the constant A, there is obtained

$$AI_n = AC_n p_1 E_n + Ai\left(\frac{E_n}{p_1} - \frac{E_{n+1}}{p_1}\right) + Ai\left(\frac{E_n}{p_1} - \frac{E_{n-1}}{p_1}\right). \quad (83)$$

Also, time and the differential operators in the two systems are related by

$$d/dt_1 = p_1 = np_2 = n\ d/dt_2 \ .$$

This means that for an event which takes place after 1 second in the electrical system, the corresponding analogous event in the mechanical system takes place after n seconds.

Equation (83) now becomes

$$AI_n = AC_n p_2 n E_n + Ai\left(\frac{E_n}{np_2} - \frac{E_{n+1}}{np_2}\right) + Ai\left(\frac{E_n}{np_2} - \frac{E_{n-1}}{np_2}\right). \quad (84)$$

To correlate the mechanical and electrical systems, equations (81) and (84) are equated term by term. Then

$$F_n = AI_n \ , \quad (85)$$

and the constant A, called a "scale change", is in pounds per ampere. Further, the relationship between mechanical displacement and electrical coil flux may be independently chosen such that

$$\frac{V_n}{p_2} = AB\frac{E_n}{p_1} \ , \quad (86)$$

which defines the constant B. Then

$$M_n p_2 V_n = AC_n p_1 E_n = \frac{C_n p_2 n^2 V_n}{B}$$

or

$$C_n = M_n \frac{B}{n^2} \ . \quad (87)$$

Equating the last terms of equations (81) and (84),

$$f\left(\frac{v}{p_2}\right) = Ai\left(\frac{E_n}{p_1}\right).$$

Now that the correspondence between analogous mechanical and electrical quantities has been determined the numerical equivalences may be calculated. Data from a curve of the type shown in (Fig. 37a) was plotted such that it coincided as nearly as possible with the solid draft gear curve of (Fig. 36). Reading the corresponding vertical scales in (Fig. 36), one milliampere is equivalent to 10,000 lbs. force. Referring to equation (85),

$$A = 10^7 \text{ lb./amp.}$$

From the horizontal scales of (Fig. 36) one inch is equivalent to 0.0216 volt sec. Equation (86) yields, therefore,

$$B = 3.86 \times 10^{-7} \text{ ft.amp./lb. volt sec.}$$

In the electrical circuit the brakes are applied at the rate of 1000 cars per second. Mechanically the brakes are applied at a rate of

$$V_1 = \frac{\text{braking velocity}}{\text{car length}} \text{ cars per sec.} \qquad (88)$$

$$= \frac{860}{42} = 20.5 \text{ cars per sec.}$$

Thus, the time base change factor, n, is given by

$$n = \frac{1000}{20.5} = 48.8 \quad . \qquad (89)$$

Now, from equation (87)

$$C_n = M_n \frac{B}{n^2} = 1.62 \times 10^{-10} M_n \; .$$

A car weight of 100,000 lbs. calls for a capacitor of

$$1.62 \times 10^{-10} \times \frac{10^5}{32.2} = 0.503 \times 10^{-6} \text{ farads.}$$

From equations (88) and (89)

$$n = \frac{1000 \times \text{car length}}{\text{braking velocity}}$$

from which, using equation (87),

$$M_n = \frac{C_n}{B}\left(\frac{\text{car length}}{\text{braking velocity}}\right)^2 \times 10^6 \; .$$

Hence, for a fixed capacitance value as used in the computer solution the correct equivalent mechanical mass is proportional to

$$\left(\frac{\text{car length}}{\text{braking velocity}}\right)^2 ,$$

and since the original solutions are for 42 ft. cars and 860 ft. per sec. braking velocity the car weights must be multiplied by

$$\left[\frac{\text{car length (ft.)}}{42} \times \frac{860}{\text{braking velocity (fps)}}\right]^2$$

$$= \left[\frac{\text{car length (ft.)}}{\text{braking velocity (fps)}}\right]^2 \times 420 \qquad (90)$$

in case car lengths or braking velocities different from the original are used.

Operation of Computer. In obtaining results for the non-linear train a somewhat different computer technique was employed than for the idealized train. The coil characteristic shown by (Fig. 37a) is that of a coil whose core is initially demagnetized. However, after obtaining a solution the coils are left with a certain amount of residual magnetism. Consequently the next solution would not be correct. Thus it was necessary, after obtaining each solution, to completely demagnetize the coils by inserting at some point along the train circuit an alternating current source of sufficient magnitude and slowly decreasing its magnitude. Hence the solutions were obtained in "single shot" fashion by applying the braking current sequence only once and then, after observing the solution, demagnetizing the coils.

Presentation of Results. Results are presented in both photographic and graphical form in a number of figures.

The car weights used in this study are as follows:

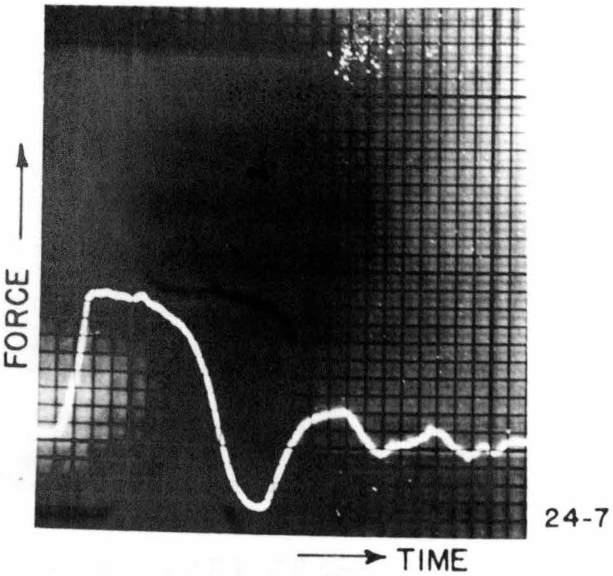
25 tons
50
70
100

These cover the range from empty to loaded cars.

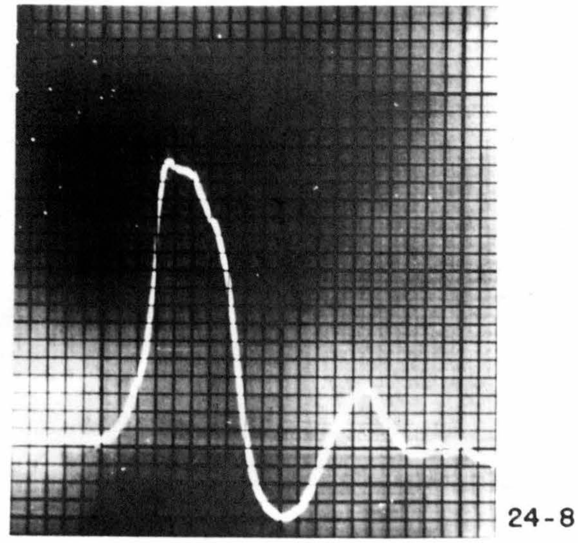
Typical transient solutions of draft gear force vs. time for a train composed of identical 25 ton cars and stopped with a braking force of 10,000 lbs. on each car are shown in (Fig. 38) for four different draft gears along the train. The forces of these same draft gears are also shown for car weights of 50 and 100 tons in (Figs. 39 and 40, respectively). In the latter two cases the braking force is the same as before, 10,000 lbs.

It would not be practical to present photographically the force in each draft gear for all cases studied. Of greatest interest is the peak force which occurs during the solution. For the uniform train of 25 ton cars the peak force occurring in each draft gear is plotted in (Fig. 42). Similar plots are made in (Figs. 43, 44, and 45) for uniform trains of 50, 70, and 100 ton car weights, respectively, and for a braking force of 10,000 lbs.

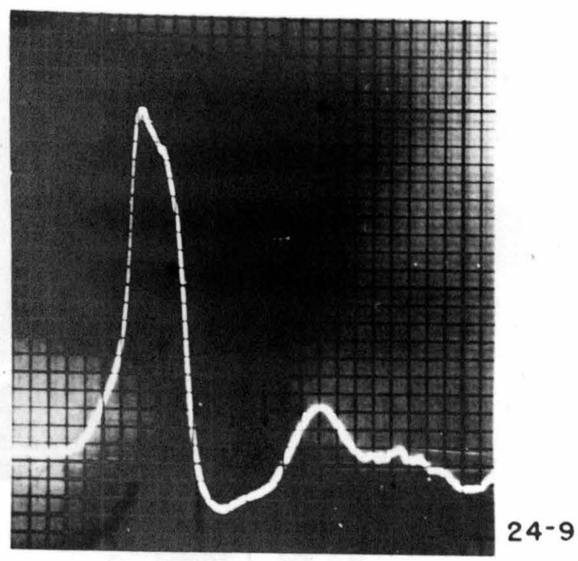
Comparison of the resulting forces for each of the uniform trains with the corresponding forces in a uniform elastic bar is made on (Figs. 42 through 47). To establish some criterion for comparison of the nonlinear draft gear with a perfect spring the draft gear is considered to be "linearized" by comparing it with a perfect spring which will store the same amount of energy when each is compressed a distance of 2.5 inches. The characteristic of this linearized equivalent spring is shown by the dash-dot curve of (Fig. 36) and has a spring constant of 92,000 lbs./inch. On the basis of this linearized spring constant the ratio of natural propagation velocity along the train to the velocity of brake application, the ratio defined as p in the treatment of the linearized train, (see Table 3), may be computed. Corresponding to the car weights used these velocity ratios are as follows:



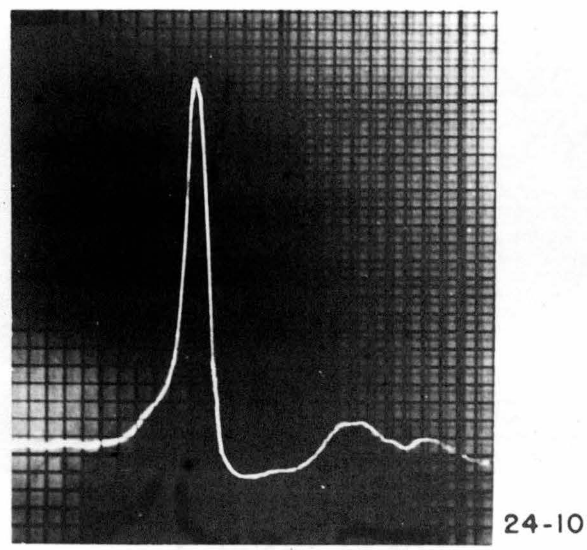
a. DRAFT GEAR
NUMBER 13



b. DRAFT GEAR
NUMBER 26



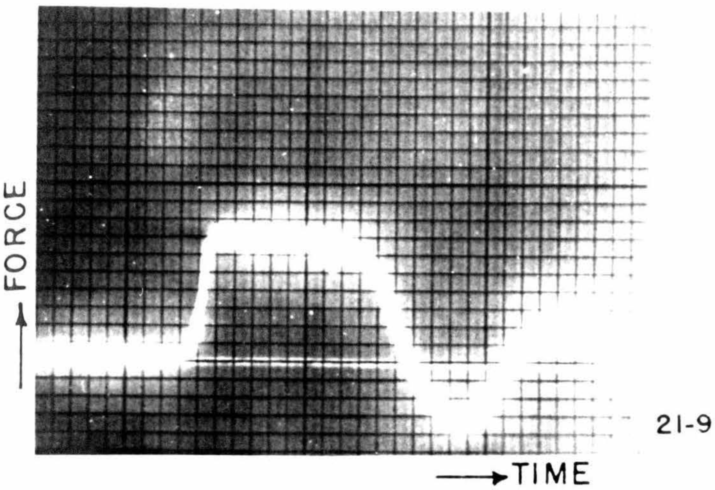
c. DRAFT GEAR
NUMBER 34



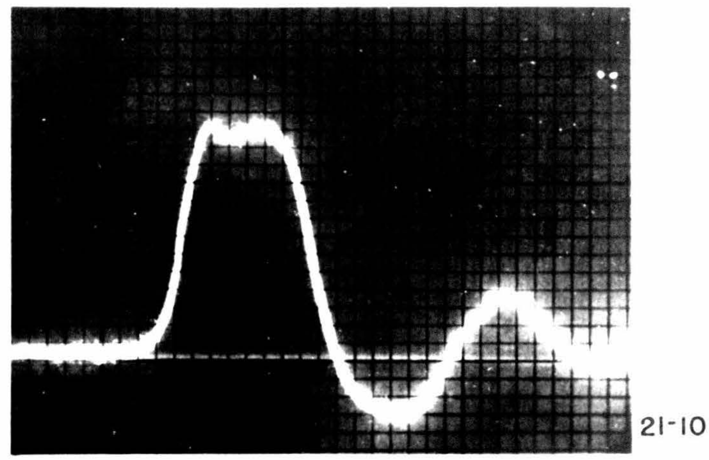
d. DRAFT GEAR
NUMBER 42

SCALE:
VERTICAL: 1 DIV. = 7300 LBS.
HORIZONTAL: 1 DIV. = 0.349 SEC.

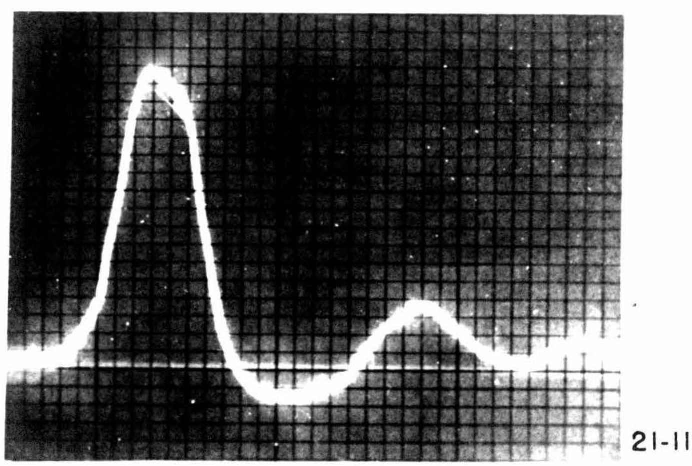
FIG. 38 TYPICAL SOLUTIONS FOR UNIFORM TRAIN OF 25 TON CARS. BRAKING FORCE 10,000 LBS.



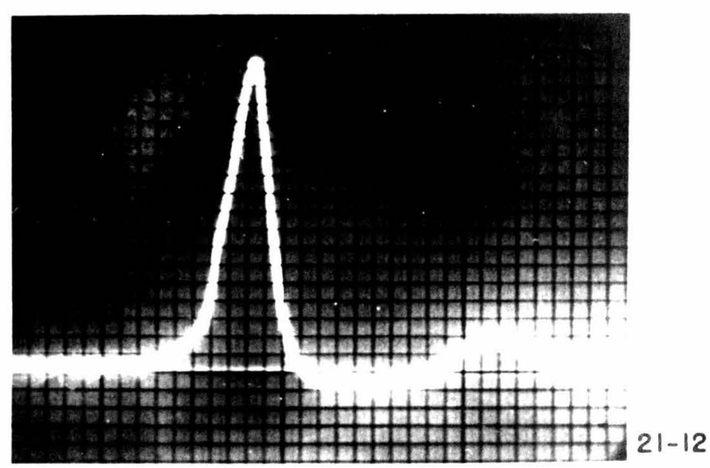
a. DRAFT GEAR
NUMBER 13



b. DRAFT GEAR
NUMBER 26



c. DRAFT GEAR
NUMBER 34



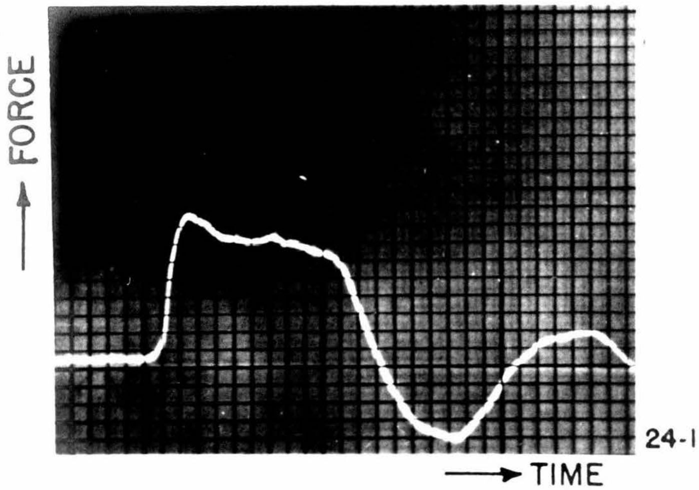
d. DRAFT GEAR
NUMBER 42

SCALE:

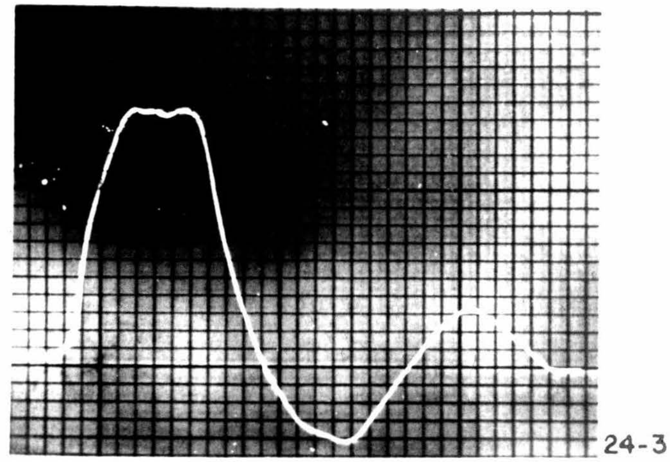
VERTICAL: 1 DIV. = 7300 LBS.

HORIZ: 1 DIV = 0.313 SEC.

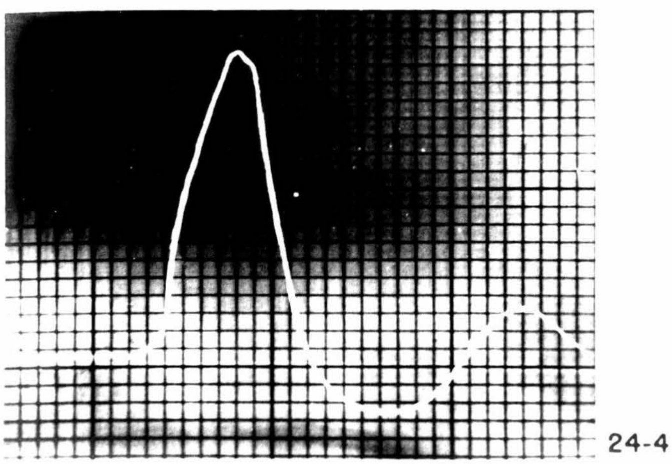
FIG.39 TYPICAL SOLUTIONS FOR UNIFORM TRAIN OF 50 TON CARS.
BRAKING FORCE 10,000 LBS.



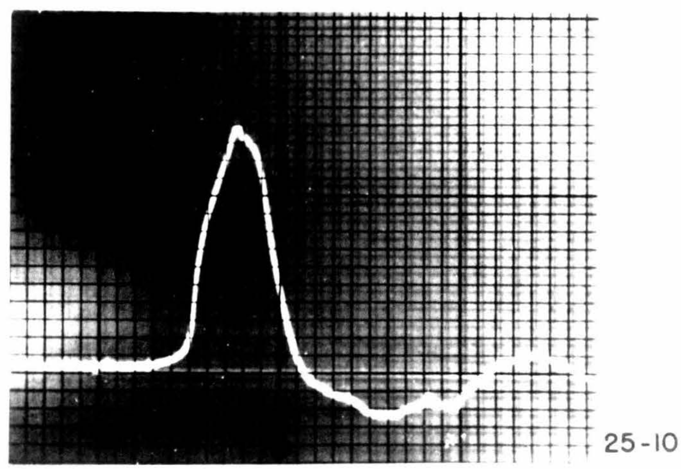
a. DRAFT GEAR
NUMBER 13



b. DRAFT GEAR
NUMBER 26



c. DRAFT GEAR
NUMBER 34

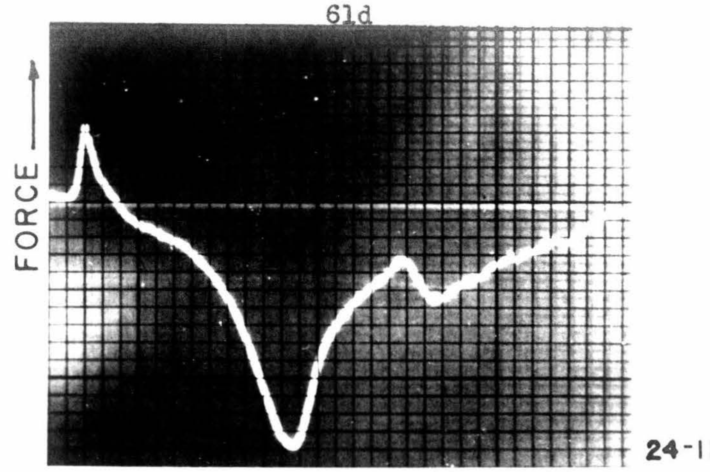


d. DRAFT GEAR
NUMBER 42

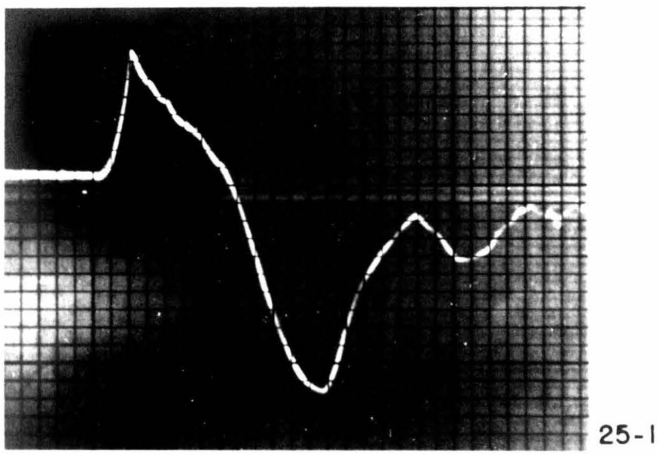
SCALE:

VERTICAL: 1 DIVISION = 4870 LBS.
HORIZONTAL: 1 DIV. = 0.349 SEC.

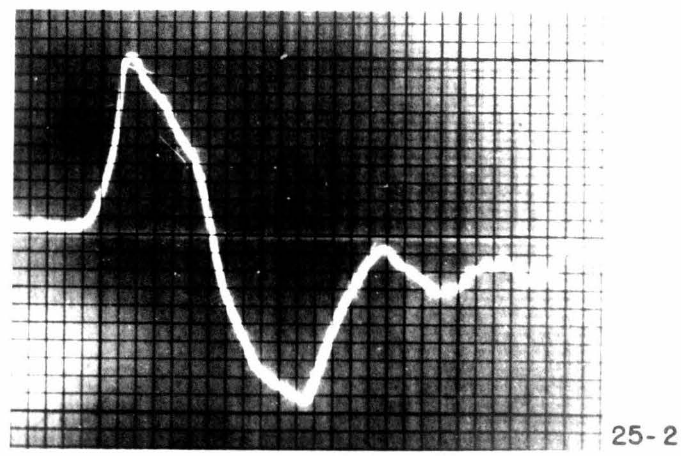
FIG. 40 TYPICAL SOLUTIONS FOR UNIFORM TRAIN OF 100 TON CARS.
BRAKING FORCE 10,000 LBS.



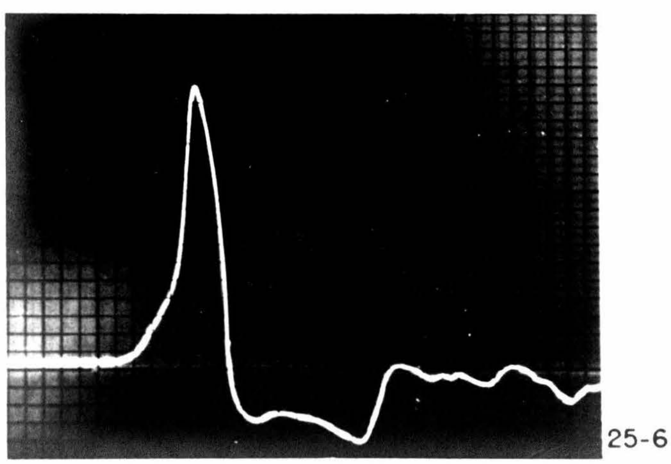
a. DRAFT GEAR
NUMBER 9



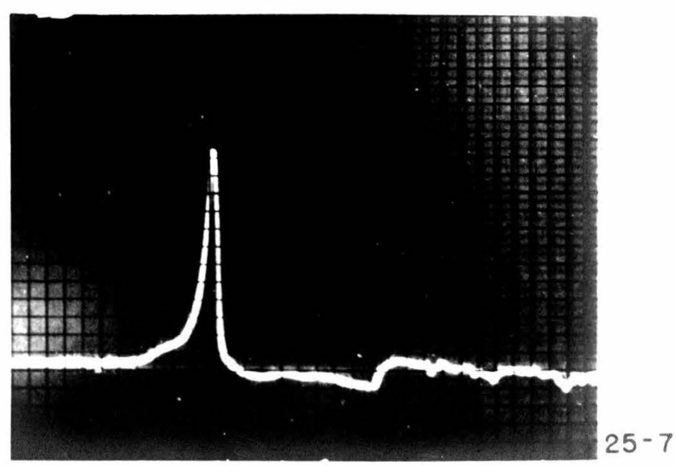
b. DRAFT GEAR
NUMBER 17



c. DRAFT GEAR
NUMBER 24



d. DRAFT GEAR
NUMBER 39



e. DRAFT GEAR
NUMBER 47

SCALE:

VERTICAL: 1 DIV. = 7300 LBS.

HORIZONTAL: 1 DIV. = 0.349 SEC.

FIG. 41 SOLUTIONS FOR TRAIN OF 25 TON CARS WITH FIVE HEAVY 100 TON CARS AT HEAD END. BRAKING FORCE 10,000 LBS.

FIG.42
PEAK FORCE vs. DRAFT GEAR NUMBER
FOR UNIFORM TRAIN OF 25 TON CARS

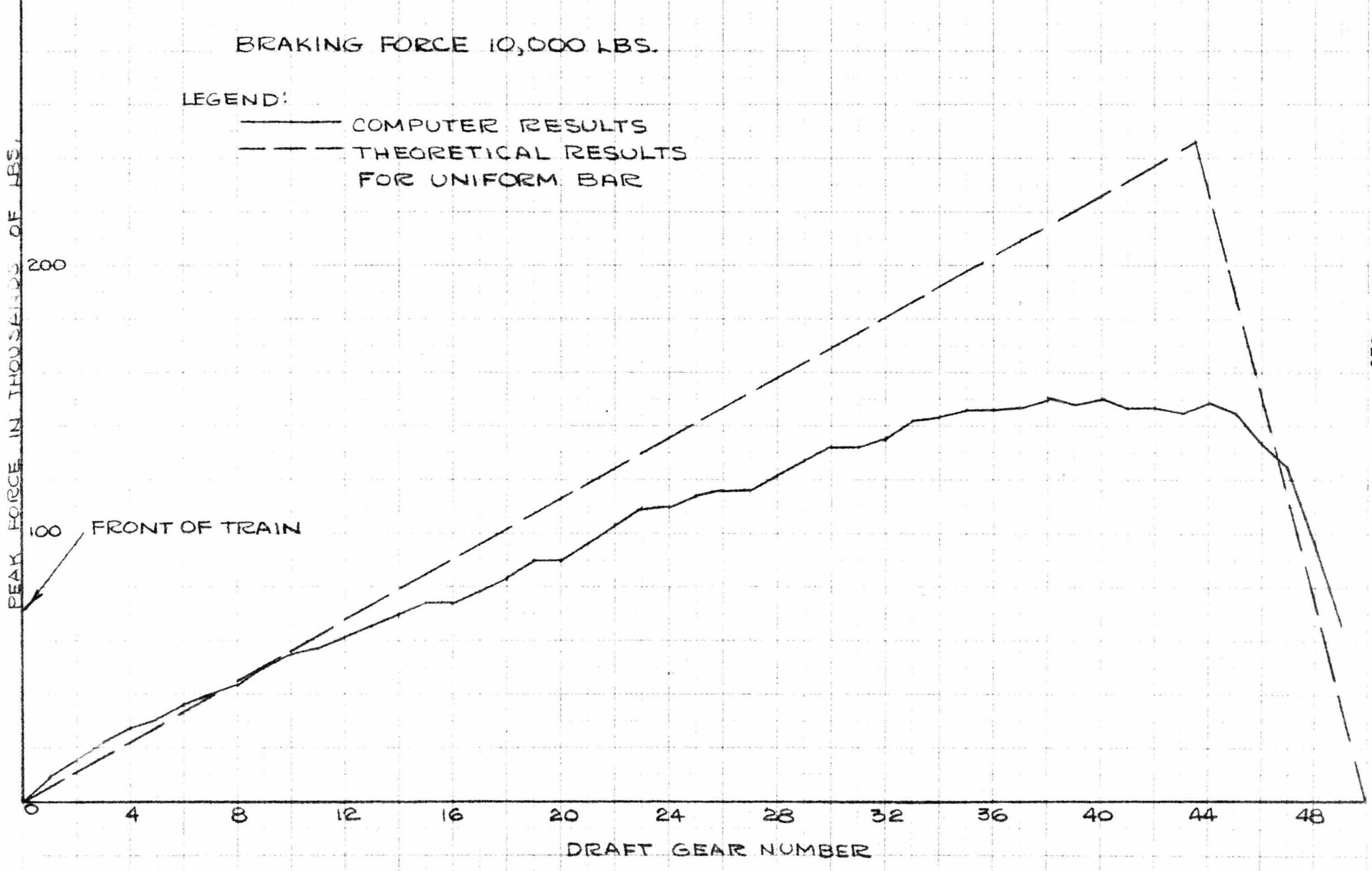
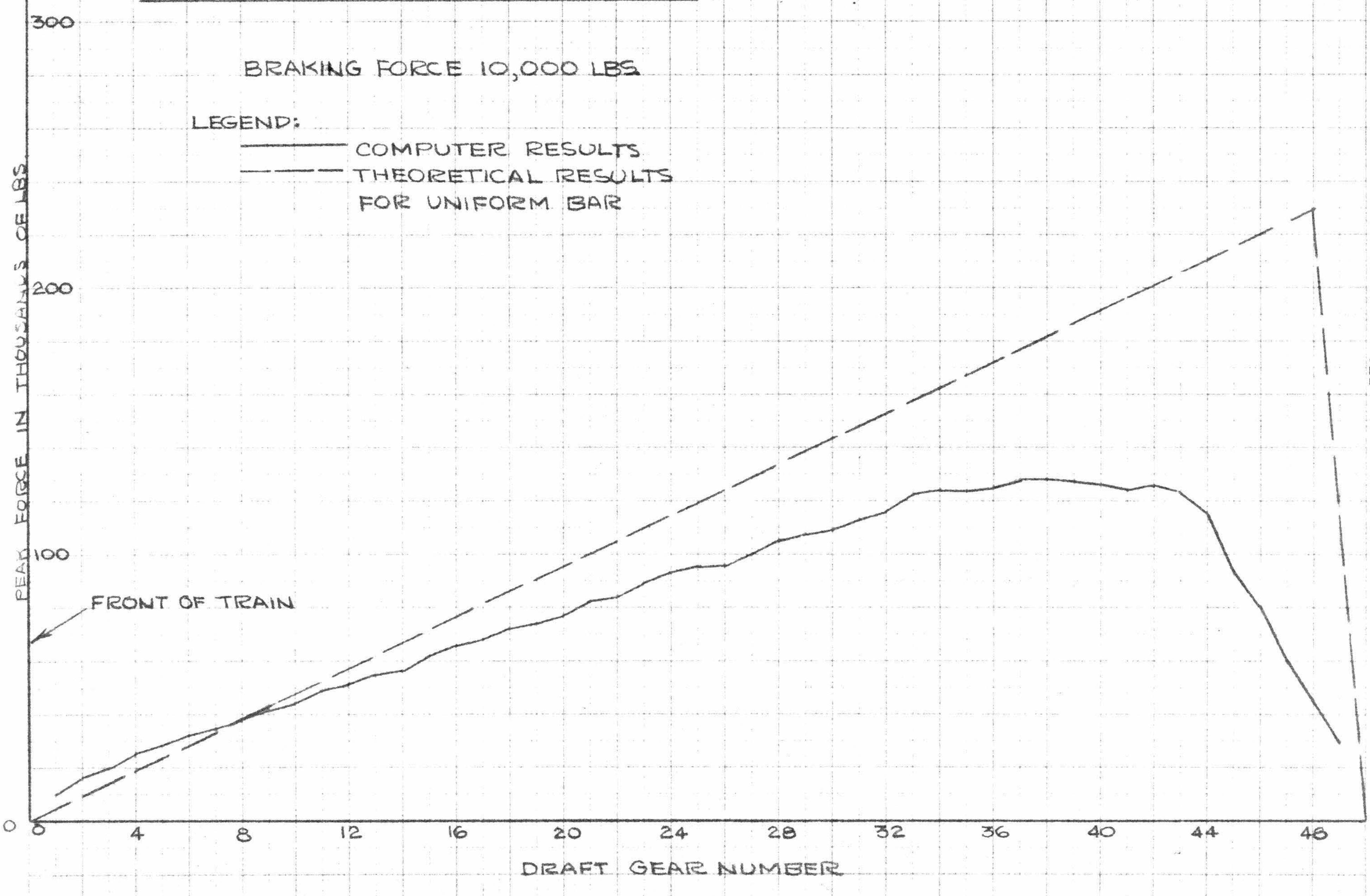


FIG. 43

PEAK FORCE vs DRAFT GEAR NUMBER FOR UNIFORM TRAIN OF 50 TON CARS



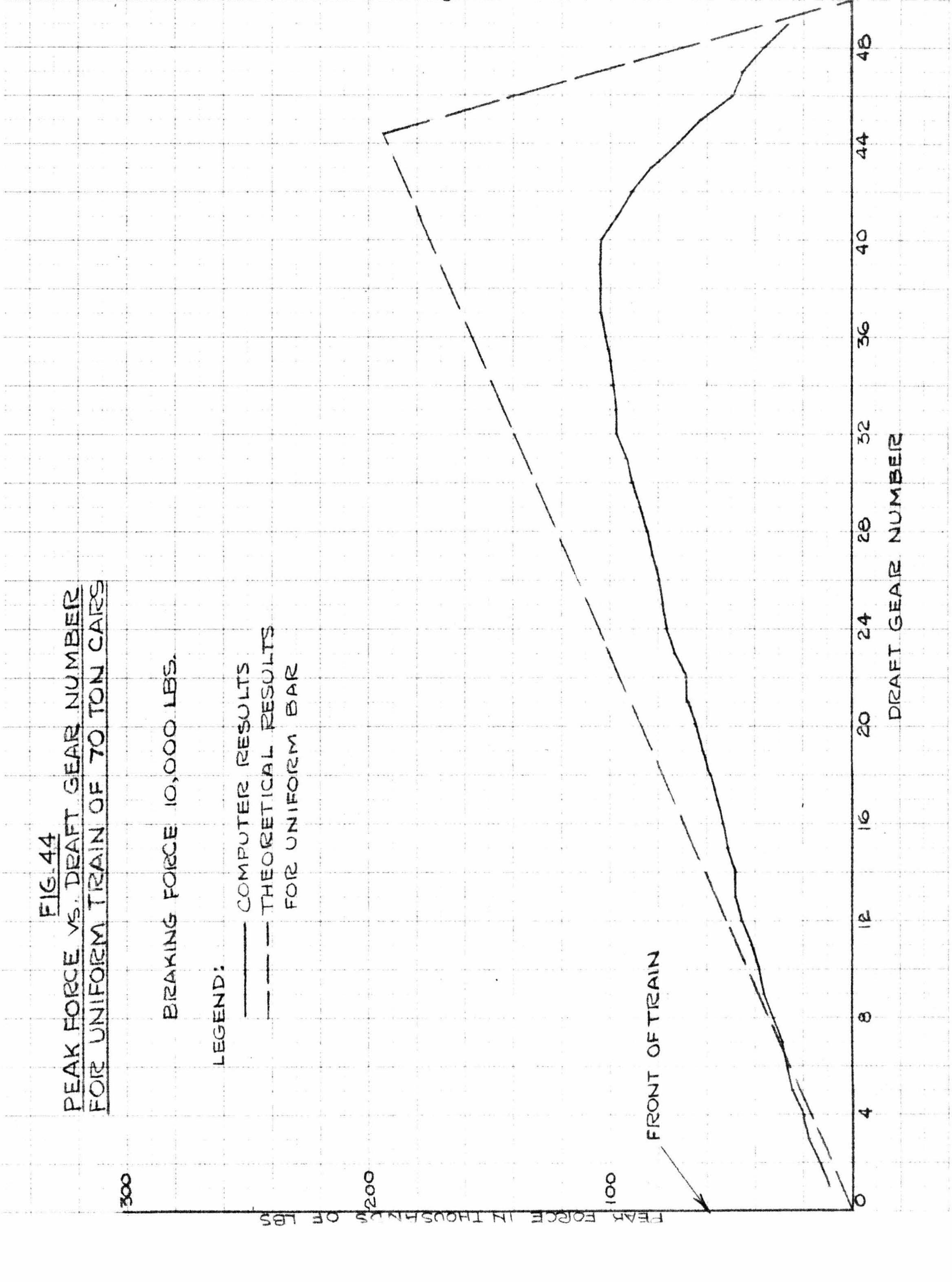
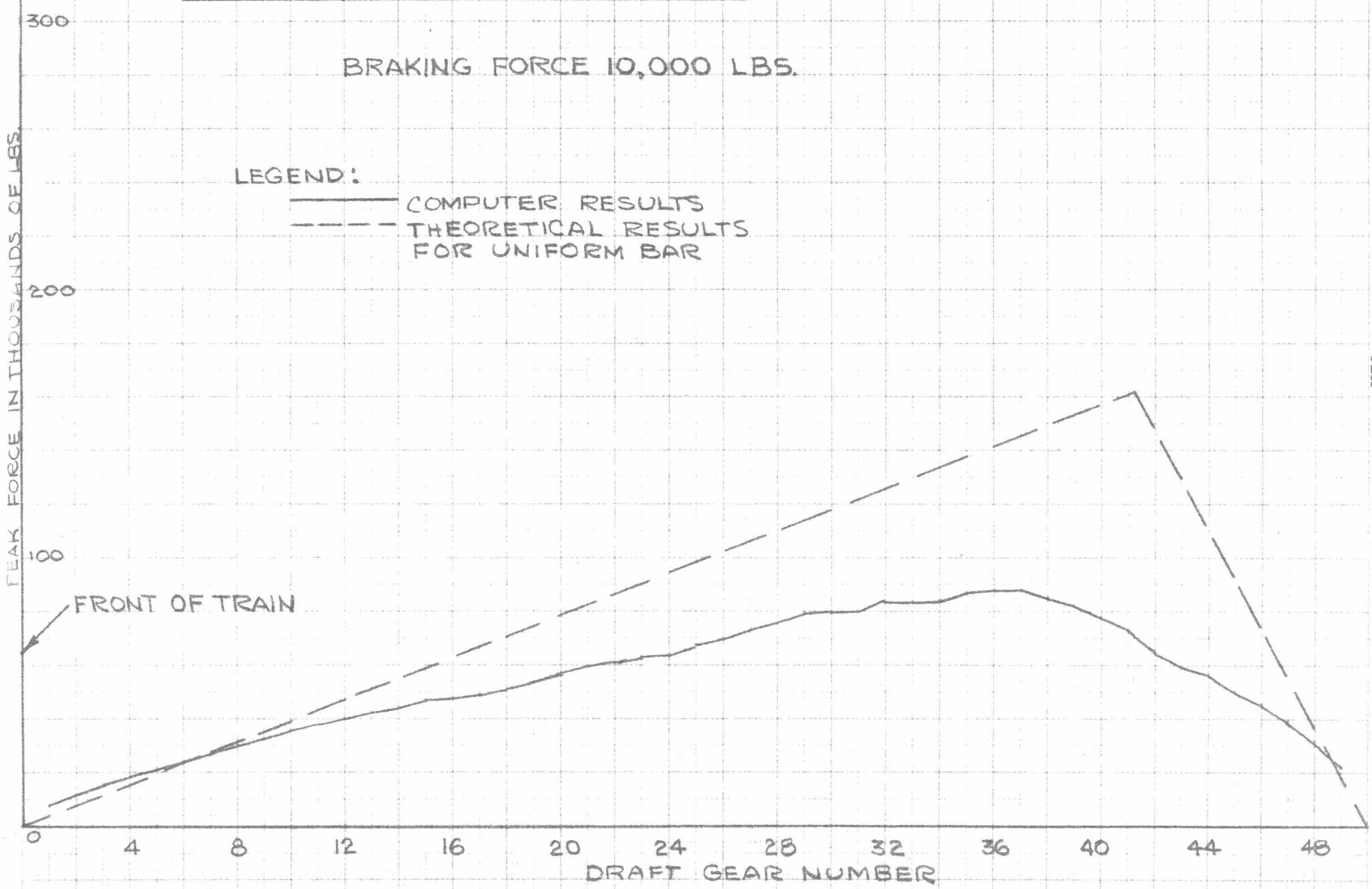


FIG. 44
PEAK FORCE VS. DRAFT GEAR NUMBER
FOR UNIFORM TRAIN OF 70 TON CARS
BRAKING FORCE 10,000 LBS.
LEGEND:
COMPUTER RESULTS
THEORETICAL RESULTS FOR UNIFORM BAR
FRONT OF TRAIN
PEAK FORCE IN THOUSANDS OF LBS
300
200
100
0
4
8
12
16
20
24
28
32
36
40
44
48
DRAFT GEAR NUMBER

FIG. 45
PEAK FORCE vs. DRAFT GEAR NUMBER
FOR UNIFORM TRAIN OF 100 TON CARS



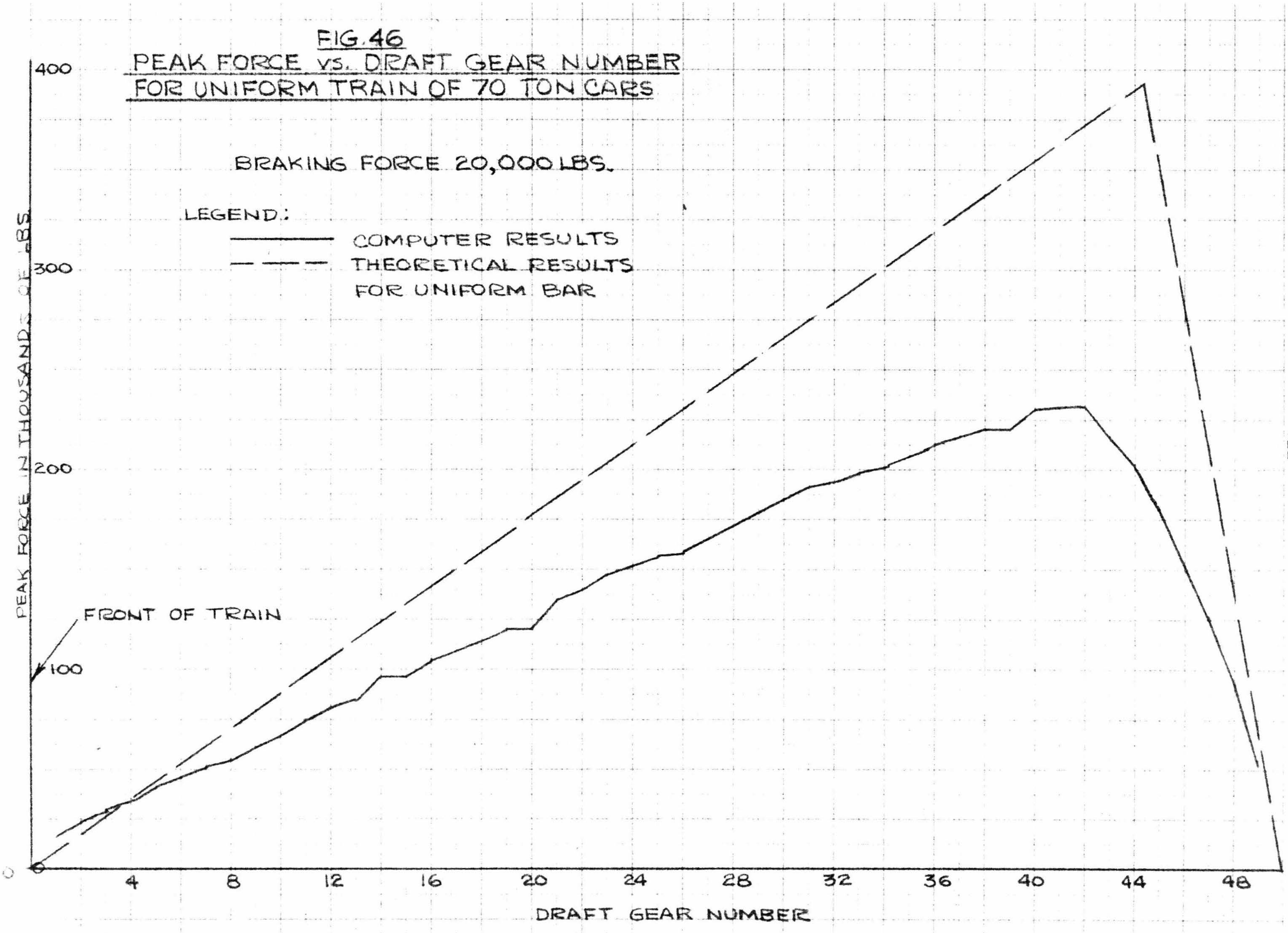


FIG. 46
PEAK FORCE vs. DRAFT GEAR NUMBER
FOR UNIFORM TRAIN OF 70 TON CARS
BRAKING FORCE 20,000 LBS.
LEGEND:
COMPUTER RESULTS
THEORETICAL RESULTS
FOR UNIFORM BAR
FRONT OF TRAIN
PEAK FORCE IN THOUSANDS OF LBS
0
100
200
300
400
4
8
12
16
20
24
28
32
36
40
44
48
DRAFT GEAR NUMBER

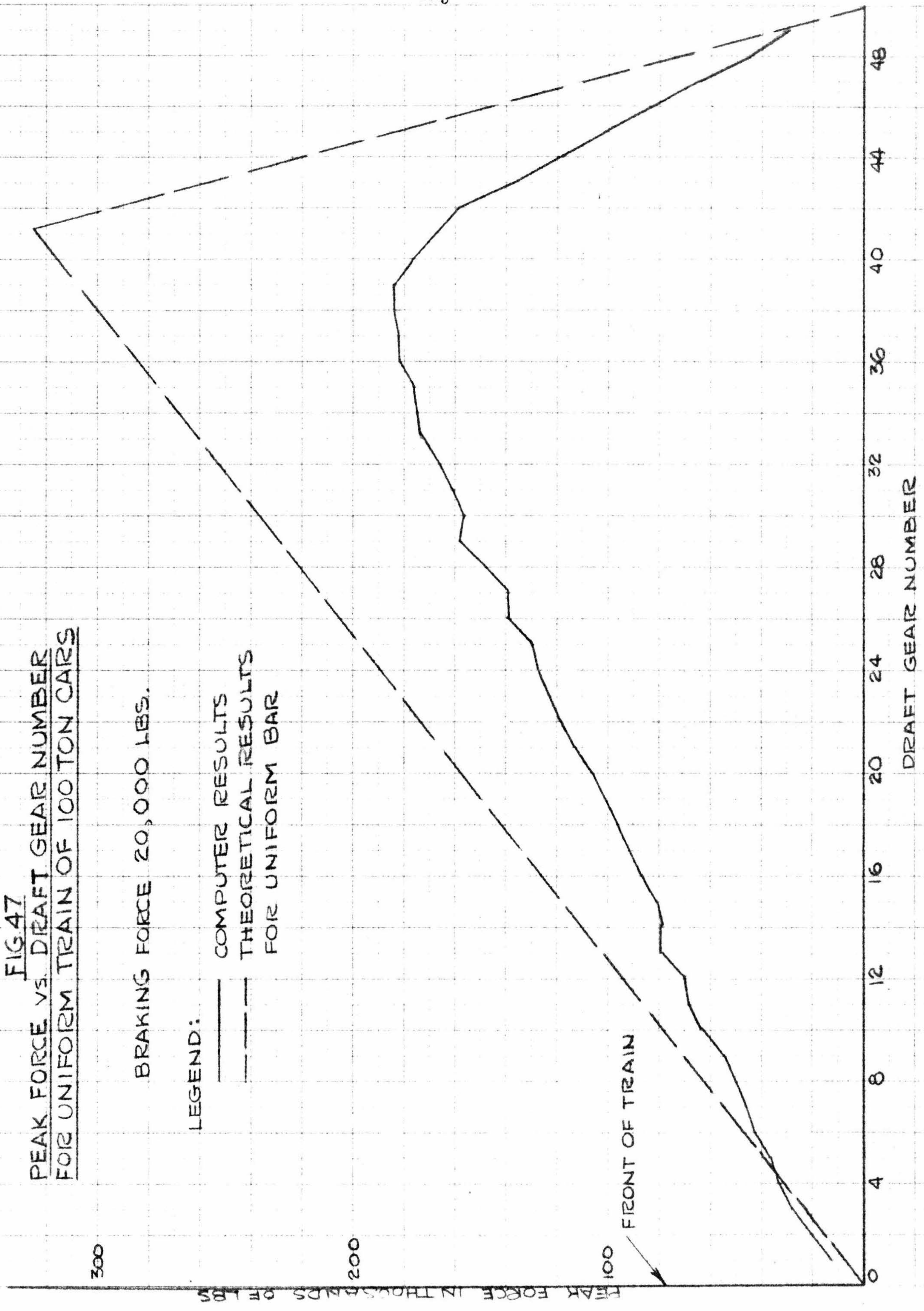


FIG. 47
PEAK FORCE vs. DRAFT GEAR NUMBER
FOR UNIFORM TRAIN OF 100 TON CARS
BRAKING FORCE 20,000 LBS.
LEGEND:
COMPUTER RESULTS
THEORETICAL RESULTS FOR UNIFORM BAR
FRONT OF TRAIN
300
200
100
PEAK FORCE IN THOUSANDS OF LBS
0
4
8
12
16
20
24
28
32
36
40
44
48
DRAFT GEAR NUMBER

Car Weight	Velocity Ratio p
25 tons	1.30
50	0.92
70	0.78
100	0.65

Now, the peak forces in a uniform bar for these velocity ratios can be calculated for each draft gear (12), and it is this information which is plotted as a dashed curve in (Figs. 42 through 47).

The agreement between solutions for a uniform bar and for a 50 car train containing perfect springs was discussed earlier, and by referring to that material for the above listed velocity ratios one may compare some of the results for a uniform train coupled by perfect springs with the results for a train containing practical draft gears. In general the forces encountered in the present investigation are considerably lower than for a uniform bar and lower still than for a 50 car train with perfect springs.

(Figs. 46 and 47) are for uniform trains of 70 and 100 ton cars, respectively, and for a higher value of braking force, 20,000 lbs.

In (Fig. 41) are shown solutions for a train consisting of five 100 ton cars at the head end and forty-five 25 ton cars following. Forces are positive upward, showing that in some draft gears the force goes considerably negative. These negative forces correspond to effective tension in the draft gear assembly. The draft gear proper experiences only compression and release, but the overall assembly containing the draft gear behaves as a spring which exhibits both tension and compression. Thus the negative forces represent elongation of the draft gear assembly.

Several other solutions are presented which show the effects of having unequal car weights along the train. In (Fig. 48) are plotted the peak forces in a train of forty-five 25 ton cars and

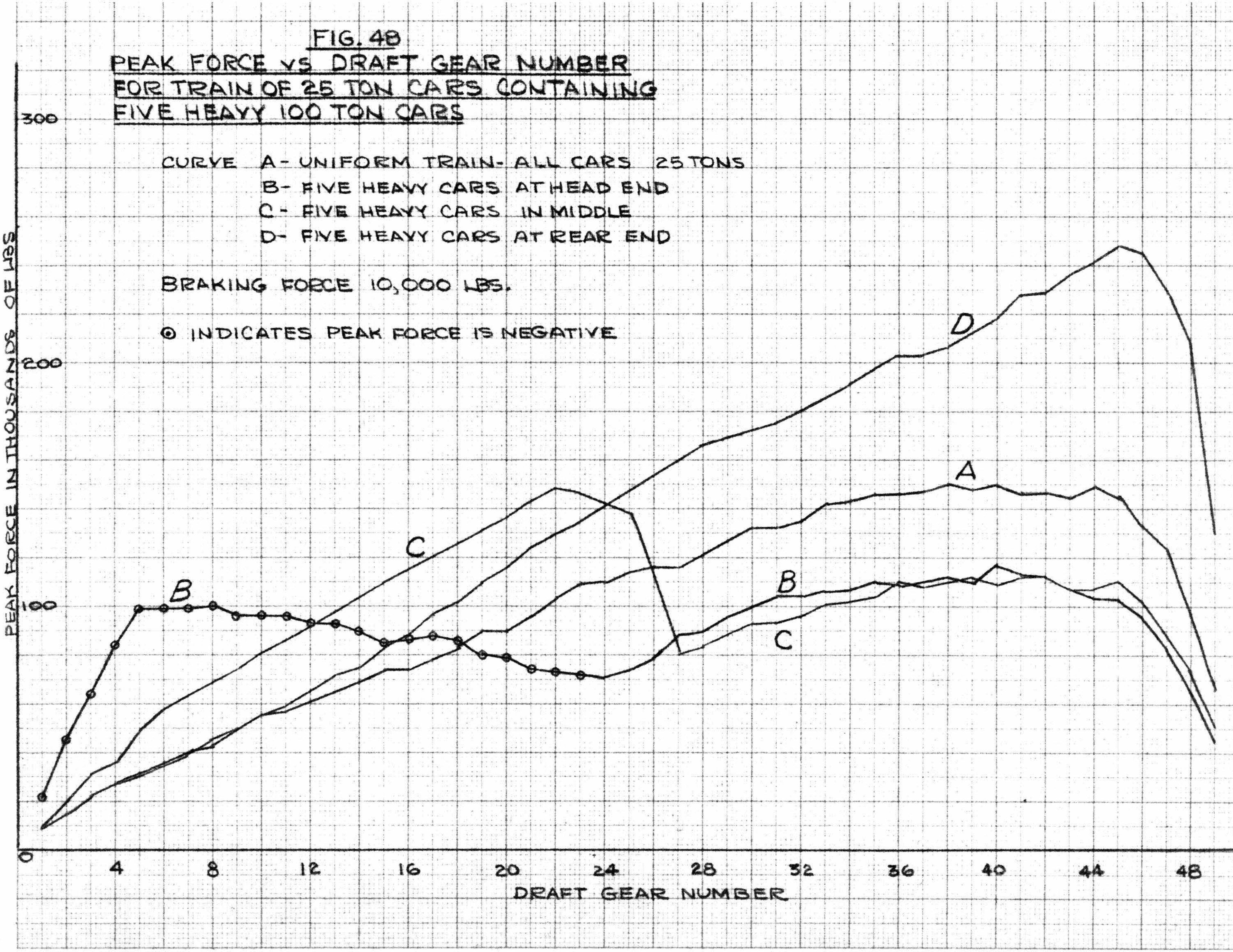


FIG. 48
PEAK FORCE vs DRAFT GEAR NUMBER
FOR TRAIN OF 25 TON CARS CONTAINING
FIVE HEAVY 100 TON CARS
CURVE A - UNIFORM TRAIN - ALL CARS 25 TONS
B - FIVE HEAVY CARS AT HEAD END
C - FIVE HEAVY CARS IN MIDDLE
D - FIVE HEAVY CARS AT REAR END
BRAKING FORCE 10,000 LBS.
⊙ INDICATES PEAK FORCE IS NEGATIVE
PEAK FORCE IN THOUSANDS OF LBS
300
200
100
0
4
8
12
16
20
24
28
32
36
40
44
48
DRAFT GEAR NUMBER
A
B
C
D

five 100 ton cars. The five heavy cars are placed in each of three different positions: at the head end, in the center, and at the rear of the train. For comparison purposes the results for a uniform train of 25 ton cars are again presented in the same figure. For a train of forty-five 100 ton cars and five 25 ton cars placed in each of the aforementioned places the results are given in (Fig. 49). Again the uniform train solution is presented for comparison.

It was shown previously that the results may be applied to cars of a different length and to braking velocities of different value from those listed in the original data of the problem. For example, if the cars were 50 feet long and the braking velocity were 900 feet per sec. the car weights would be multiplied, according to equation (90), by

$$\left(\frac{50}{900}\right)^2 \times 420 = 1.30 \quad ,$$

and (Fig. 42) would represent solutions for cars weighing

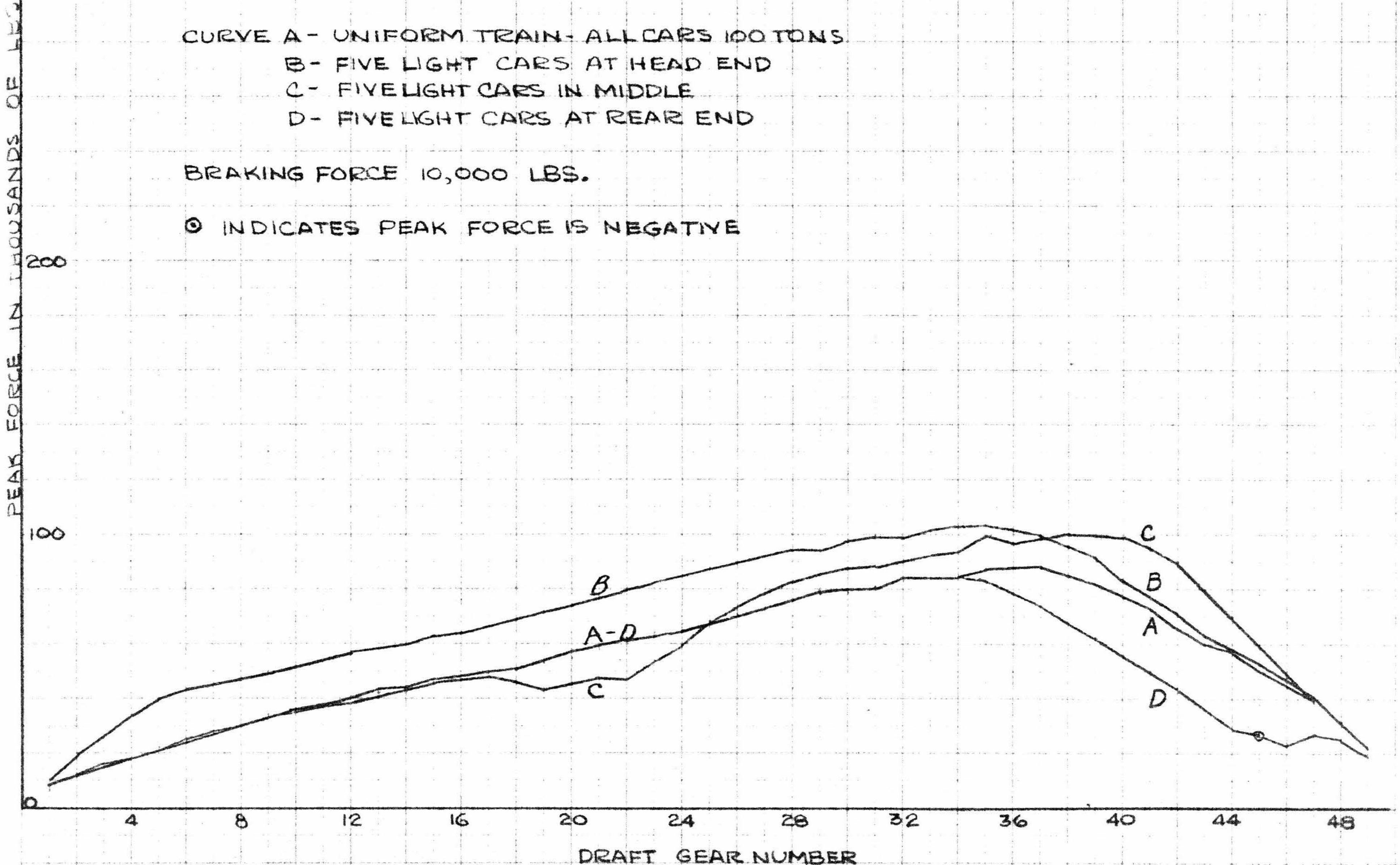
$$1.30 \times 25 = 32.5 \text{ tons.}$$

Other car weights would also be increased by the same factor.

Analysis of Results. The graphs which present peak forces for the uniform trains show that these forces are all less than for an equivalent linearized uniform bar. The reason for this may be thought of in either of two ways. Associated with the motion of each draft gear there is considerable energy absorption during any compression and subsequent re-expansion or partial re-expansion. This energy absorption represents an extraction of energy from a travelling wave which, were the draft gears perfect springs, would travel unimpeded along the train. Energy absorption at each draft gear subtracts from the force build up at points farther along the train, and consequently the peak forces are less. Looking at the phenomenon

FIG. 49

PEAK FORCE vs. DRAFT GEAR NUMBER
FOR TRAIN OF 100 TON CARS CONTAINING
FIVE LIGHT 25 TON CARS



in an alternative manner, it may be noted that each draft gear exerts less force upon re-expansion than it does for the same elongation or travel in compression. Hence, when a force wave reaches a particular draft gear the draft gear is compressed. But, upon subsequent re-expansion the draft gear does not transmit as much force to the succeeding car as would a perfect spring, and hence does not contribute as much to the forces farther along the train. The foregoing analysis assumes, however, that in the compression portion of the draft gear's cycle the draft gear does not rapidly stiffen or "go solid". Referring to (Fig. 36), the compression or upper curve of the draft gear characteristic exhibits no rapidly increasing slope below a force of 250,000 lbs. And, examination of the various solutions shows that none of them contains forces greater than this magnitude. Of course, if any of the draft gears should go solid the forces in those particular units could reach very large proportions.

It is interesting to note that for all the uniform trains considered the maximum value of peak draft gear force is nominally 40 per cent lower than for the equivalent uniform bar. The exact percentage ranges from 39 per cent to 46 per cent. It may also be observed that for the two cases in which the braking force is 20,000 lbs. the maximum peak force is almost exactly twice the value for 10,000 lbs. braking force.

In (Fig. 46) it is seen that the maximum force in draft gear number 42 is 230,000 lbs. This is approaching the value of force at which the draft gear goes solid. Hence it may be concluded that a braking force in excess of, say, 25,000 lbs. would cause some of the draft gears in a train of 70 ton cars to go solid.

It was pointed out earlier that for the train of 25 ton cars with five heavy 100 ton cars at the head end the peak force in some of the draft gears is negative. For draft gear number 9 (Fig. 41a) there is a small compression force followed by a much larger negative force. Farther along the train (draft gear 17, Fig. 41b) the positive force is larger, and the negative peak has diminished. At draft gear 24 (Fig. 41c) the two peaks are nearly equal, and for succeeding draft gears the positive or compression peak is the greater. These large negative peaks may be simply explained as follows. The first five cars, because of their greater inertia, tend to keep moving when brakes are applied. The lighter cars to the rear, of much lower inertia and with the same braking force, slow down considerably upon application of brakes. Thus the heavy cars tend to move away from the light cars, which explains the large tension forces. In (Fig. 48) it is shown that the greatest tension force occurs between the last heavy car and the first light car, which is consistent with the above explanation.

Another case of particular interest is the one involving a train of 25 ton cars with five 100 ton cars placed at the rear end. For the same braking force applied to all cars the light cars tend to decelerate rapidly while the heavy cars at the rear, because of their greater inertia tend to "run into" the lighter cars, thus producing large compression forces. This is borne out by examination of curve D in (Fig. 48). Also to be noted is the fact that the peak force in this case is 247,000 lbs., only slightly lower than the value of force at which the draft gear goes solid. Therefore, one may conclude that either the use of greater braking force or the placing of more than five heavy cars at the rear of the train will result in undesirably large forces in some of the draft gears at the rear of the train due to their going solid.

The other examples of trains of nonuniform makeup may be explained in the manner used in the previous two paragraphs.

Draft gears having force travel characteristics differing from that of the draft gear used here may be treated by suitable design of the special coils. If desired, other quantities such as car acceleration, time required to stop the train for a given initial velocity, car forces, or car velocities, can be studied for trains of any arbitrary makeup. Arbitrary train makeup may include variation of draft gear type along the train, number of cars, car weights, placement of cars, or braking force.

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