MULTIPATH INTERFERENCE OF MODULATED WAVES

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Abstract

Transmitter-Receiver systems are operationally defined in a fashion applicable to the treatment of multipath systems of reception. The systems of modulation now in general use are divided into two groups, the continuous wave and the pulse methods, and each is discussed in detail. A method is established for the solution of the continuous wave problem and the solutions considered for several types of continuous wave modulated waves. It is found that to a first approximation none of the methods considered has any degree of superiority in the small percentage modulation case but a second order approximations shows frequency modulation to be superior to amplitude and phase modulation reception under conditions of multipath transmission. The expected wave form of the received wave is evaluated for small percentage modulation conditions. Comparison of the theory with a practical case of reception of frequency modulated waves is made. Methods of solution of the problem of the reception of modulated waves by diffraction as in physical optics are established.

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I. Introduction

1.1 The Function of Transmitter-Receiver Systems

In radio communication, the intelligence or the information is transmitted by allowing some property of the radiated waves to vary with a function of the intelligence. This process is known as modulation. The modulation is usually carried out under conditions such that the average frequency of the spectrum belonging to the radiation is much higher than the frequencies usually associated with the intelligence or the information and is also much higher than the width of the frequency band corresponding to the principal lines in the actual spectrum transmitted. The reason for this procedure is a matter of the efficiency of radiation, propagation and reception.

The functions of the intelligence mentioned above are not necessarily point functions. Indeed we shall encounter such dependences as functions of the integral of the intelligence as a function of the time. Insofar as a mathematical expression of the dependence is concerned, the property of the radiation dependent on the intelligence also may not be simple.

However one can write operationally a simple expression for the radiated energy. One can state that an expression for the amplitude of the disturbance radiated is

$$O[t, I(t)] \qquad 1.1\text{-}1$$

where t is the time, I(t) is the intelligence or information function to be modulated and transmitted while O is an operator which represents the modulation process as a whole including the property of the radiated wave mentioned and its functional dependence on the intelligence. Typical examples of the nature of this operator will be discussed in sections 2.1 and 2.2.

The disturbance is radiated into space by antennae or is transmitted along wires. In due process a receiver recognizes the energy and attempts a decoding process on the incident energy so that the output from the receiver will be of the nature of the intelligence function before the modulation process took place at the transmitter. The process of decoding is called detection. The nature of the receiver must be such that the operation of the detector must correspond to the particular type of modulation used. If the received wave is exactly the same in every property except in amplitude and in time of arrival as the wave radiated by the transmitter, the received wave can be written in the form

$$kO\left[(t-t_o),\ I(t-t_o)\right] \qquad 1.1\text{-}2$$

where k is some constant while t_o is the time of transmission of the wave between the transmitter and the receiver. The output from the receiver is then

$$E(t) = P\, k0 \left[(t-t_o),\ I(t-t_o)\right]. \qquad 1.1\text{-}3$$

We choose operators, O, such that the constant k is transferable across the receiver operator symbol, P. Hence the detector output is

$$E(t) = k P\, 0 \left[(t-t_o),\ I(t-t_o)\right] \qquad 1.1\text{-}4$$

and the receiver output is proportional to

$$E(t) = I(t-t_o) \qquad 1.1\text{-}5$$

In other words, the ideal modulation-detection process should yield distortionless output from the receiver with exact reproduction of intelligence or information as measured before the process of modulation takes place.

When clear, undistorted reception is observed, the processes discussed above can be said to take place. However when the reception is distorted, the distortion can be attributed to the fact that the reception is no longer 1.1-2 but is something else. Under less ideal conditions the processes of the detection become

$$E(t) = P\, g(t) \qquad 1.1\text{-}6$$

where g(t) is the input wave function to the receiver corresponding to the particular circumstances surrounding the reception. The investigation of the nature of the distortion will depend therefore in part upon the nature of the reception and also in part upon the nature of the modulation and hence upon the nature of O for the system in use.

The nature of the receiver operator, P , for the particular system of modulation corresponding to the operator, O, is at times obscured by the mechanism of the detection process. In some cases the receiver operator, P , is complex and involves a number of mathematical operations. Some of these operations have no function and are the identity operator in the case of the reception of the idealized wave given by equation 1.1-2 and only appear to affect the wave when the reception is something other than the idealized wave. This property will necessitate a discussion in detail of the actual mechanism of the detection process for the individual cases of the inverse operator corresponding to particular methods of modulation.

Distortion in the reception can be attributed to the presence of any one or combinations of any of the following three causes when superimposed on the desired signal:

1. Random noise
2. Spurious signals occupying all or part of the same band of frequencies corresponding to that of the desired signal
3. Multipath interference.

Random noise such as that caused by the motion of electrons in resistors has the property that the energy is more or less uniformly distributed over the entire frequency spectrum. Calculations of the properties of the various systems of modulation under conditions of reception of a random noise superimposed on the desired signal have been made by several authors.[1,2] The results are available in terms of the characteristic variable corresponding to each system: the signal-to-noise ratio.

A treatment of the reception of spurious signals occupying the same or overlapping frequency channels is dependent upon the exact conditions corresponding to the particular disturbing spurious signals. The problem has not been discussed in all its generality in the literature for all the systems of modulation.

The treatment of the distortion in the reception due to the multipath interference of the transmitted wave to the exclusion of the other two possible causes already listed is the subject of the present paper.

It should be noted that to some extent, all these three effects are present in any reception of modulated radio waves. However, if the signal strength corresponding to the desired signal is sifficiently large compared with those of the disturbing influences, the observed distortion is very small indeed and usually is so small as not to be noticed. This property is a general property of all the methods of modulation to be discussed and section 1.2 will present a proof of the statement.

1.2 The Definition of Multipath Systems

When electromagnetic waves are propagated over two or more distinct optical paths, the interference phenomena resulting from the superposition of the waves at the receiver are termed multipath interference phenomena. That all of the paths of propagation involved in the interference have the same source is an essential part of the definition. This latter restriction enables us to distinguish between multipath distortion and adjacent channel distortion. The latter type of interference, already mentioned, arises because of the possibility of reception of electromagnetic waves from two or more sources which occupy all or part of the band of frequencies passed and detected by the receiver.

It is evident that there is little to choose between the definition of multipath interference and that of the phenomenon of interference in physical optics. Thus we can see that interference as an optical phenomenon can be considered as a special case of the more general field of multipath interference of modulated waves.

The transmission and reception of the waves are subject to the operational definitions of the processes set forth in the preceding section. It is seen that the output of the receiver can be given as the result of the receiver operating upon the superposition of all the received energy due to transmission over each of the paths and can be written as

$$E(t) = k\,P\left\{\sum_{n=1}^{N} A_n O\left[(t-t_n), I(t-t_n)\right]\right\} \qquad 1.2\text{-}1$$

where N is the number of the incoming signals, t_n is the time of transmission of the nth signal from the transmitter to the receiver, and A_n is the relative amplitude of the nth unmodulated wave received as observed by the receiver as though the other N-1 waves were absent. The unity of the source for all waves is evident from the expression. No other sources of energy except that from the one transmitter are permitted. The problem as stated in section 1.1 is reduced to the consideration of the expression 1.1-6 for the function g(t) given by the operation 1.2-1. Further investigation will depend upon the specific nature of the receiver system corresponding to the transmitter operator O. Specific properties of the operators, O, and P, in common use will be discussed in the next section.

In the case of modulated radio waves the above expression 1.2-1 represents a scalar quantity even though the problem is one of the transmission of electromagnetic waves. The problem has been reduced to one involving the receiver and the operation it performs and is not specifically dependent upon the space or the boundaries in which the receiver and transmitter are located. The problem is merely one of assumption of relative scalar wave intensities, A_n, and relative times of transmission, t_n, of the incident waves upon a receiver whose operation is $\mathcal{P}$. The electromotive forces induced in the circuits of the receiver are scalar quantities and problems which involved vector fields have been reduced to scalars and to scalar operations. In the case of the optical interference problem, the statement of the problem is already one which is in scalar form.

No extensive solution of the problem has been attempted heretofore. The cases of the interference of such waves which have been treated in the literature follow:

1) When they are Frequency Modulated, N = 2 and the waves are sinusoidally modulated[3,4,5,6,7]
2) When they are Amplitude Modulated, N = 2 and the waves are sinusoidally modulated[8]
3) When the problem is that of interference in physical optics.

In the following pages, the problem is broadened to include many specific types of modulation, general and specific forms of the intelligence I(t), and general and many specific conditions on the interfering waves as regards their relative amplitudes and phases.

The assumptions that will be made are summarized as follows:

1) That the receivers used are the ideal detectors

of the type mentioned in section 1.1

2) That the sources for all signals arriving at the receiver are one and the same and that the signals may be linearly superposed in the receiver so that the input to the receiver is

$$\sum_{n=1}^{N} A_n \, O[(t-t_n), I(t-t_n)] \qquad 1.2\text{-}2$$

3) That the functions I(t) and the results of the operations 1.1-1 and 1.1-3 are well-behaved.

We shall be able to express the output of the receiver in the form

$$E(t) = c\{I(t) + D[I(t); \rho_1, \rho_2, \ldots, \rho_N; (t_k-t_1), (t_k-t_2) \ldots (t_k-t_N)]\} \qquad 1.2\text{-}3$$

where the function D will be called the distortion. The distortion is the signal superimposed upon the desired intelligence I(t) in the detected wave. The distortion depends upon the nature of the intelligence, on the nature of the method of modulation, on the relative amplitudes of transmission ρ_i where $\rho_i = \frac{A_i}{A_k}$, and upon the relative times of transmission (t_k-t_i). A_k and t_k are the observed amplitude and time of transmission of some one of the incoming signals. From equations 1.1-4 and 1.1-5 we see that

D = 0 if N = 1 .

We can expand 1.2-3 by Taylor's theorem

$$E(t) = c\Big\{I(t-t_k) + D[I(t-t_k); \rho_1, \rho_2, \ldots, \rho_N; (t_k-t_1)(t_k-t_1) \ldots (t_k-t_1)] + \sum_{i=1}^{N} \frac{\partial D}{\partial(t_k-t_i)}\Big|_{t_i=t_1} (t_1-t_i) + \cdots \Big\} \qquad 1.2\text{-}4$$

but if all signals have the same time of transmission, t_1, the induced electromotive forces in the circuits of the receiver by each of the incoming waves are all in the same phase and the detected wave is exactly the same as if only one wave were observed. Hence

$$D\left[I(t-t_k);\rho_1,\rho_2,\ldots,\rho_{N-1},\rho_N;\ t_k-t_1,t_k-t_1,t_k-t_1,\ldots,t_k-t_1\right]=0 \qquad 1.2\text{-}5$$

Thus for small time differences, since D is in all cases well behaved, it is seen from 1.2-4 the distortion becomes vanishingly small with decreasing times (t_k-t_i).

Further if one of the incoming waves has an amplitude which is much larger than all the others, some estimate of the behavior of the distortion function can be made. Let the kth wave be the one which is larger in amplitude, A_k, than all the rest. Then since all the N-1 values of the ρ_i are small we can investigate the distortion by again expanding 1.2-3 by Taylor's theorem.

$$E(t)=c\left\{I(t)+D\left[I(t);1,0,0,\ldots,0;\ (t_k-t_1),\ t_k-t_2,\ldots\right]+\sum_{k=1}^{N}\left.\frac{\partial D}{\partial\rho_k}\right|_{\rho_k=0}\rho_k+\cdots\right. \qquad 1.2\text{-}6$$

But for only one wave arriving, the distortion is zero; or

$$D\left[I(t);1;t_1\right]=0 \qquad 1.2\text{-}7$$

Thus for small relative amplitudes, ρ_i, since D in all cases is a well behaved function, it is seen from 1.2-6 that the distortion becomes vanishingly small as one received wave becomes larger than all the rest of the amplitudes of the other received waves.

Physically, the conclusions of the preceding two paragraphs can be interpreted as follows: If the principal lines in the frequency spectrum are above the critical frequency of the Heaviside Layer the reception is dependent upon such phenomena as reflections from obstacles in the field of transmission or refraction from say an inversion layer in the atmosphere or is dependent upon line-of-sight propagation. In the case of line-of-sight reception, the principal or the line-of-sight wave is much larger in amplitude than any of the amplitudes of obscure reflected or other signals, the distortion present in the reception due to multipath interference is negligible or is very small. If one places reflectors near the receiving antenna, the interference due to the reception of a principal wave and the reflected waves is very small or negligible as the times of transmission of the various signals do not differ much from one another.

The distortion therefore depends, for all types of modulation and receivers upon a set of 2N -2 variables and the nature of the intelligence. The 2N-2 variables are the relative amplitudes of the N incoming waves and N-1 relative times of arrival of the N incoming waves. It is precisely this set of 2N-2 variables which will be an invariant for given positions of transmitter, receiver, and boundaries so that comparison of various methods of modulation as regards multipath interference can be made upon the basis of the nature of the receiver and not on the basis of the surroundings.

The question arises, "When do the differences in the times of transmission become significant and cause appreciable distortion?" It will be pointed out that the time differences required for appreciable distortion in modulated waves are of the order of 1/f where f is the highest frequency of the function I(t). For ordinary music or speech, this time is of the order of 50 microseconds. Electromagnetic waves in their propagation require corresponding path differences of approximately 9 miles or 15 kilometers for appreciable distortion of the intelligence. The distortion is evident with path differences of the order of one mile. In practical high frequency radio communication where line-of-sight propagation is not possible, path differences of one mile and above are not uncommon with suitable boundaries to the region of the transmission and distortion in the reception is noted.

II. Methods of Modulation and Detection

2.1 Continuous Wave Systems

The modern methods of modulation can be divided into two groups. They are the continuous wave methods and the pulse methods of modulation. The continuous wave methods of modulation are characterized by the fact that $O[I(t)]$ is a continuous function of t. The pulse wave methods of modulation are essentially sampling methods which transmit a few sine waves of a constant frequency, for a short time, the nature of this pulse and its relationship to the other pulses transmitted depending upon the value of I(t) at the time of initiation of the pulse. Most of the time, the pulse wave transmitter is inoperative as an intermittent or sampling technique for the transmission of information is utilized. The pulse methods of transmission of intelligence are further discussed in detail in section 2.3.

a) Amplitude Modulation

The nature of the transmitted wave in the case of amplitude modulation takes the form

$$O[I(t)] = c\left\{1 + kI(t)\right\} e^{i\omega_0 t}\Big|_{\text{Real Part}} \qquad 2.1\text{-}1$$

where c and k are constants. ω_0 is called the carrier frequency of the unmodulated wave or of the sine wave occuring when I(t) = 0. It is seen that this quantity represents a function whose envelope varies directly with the intelligence, I(t).

The detection process is defined such that if a wave of the form

$$g(t) = A(t)e^{if(t)}\Big/_{\text{Real Part}} \qquad 2.1\text{-}2$$

is received, then the output of the detection process is given by

$$E(t) = |A(t)| - \text{average}\,|A(t)| \qquad 2.1\text{-}3$$

It can be seen that in the case of ideal, or single signal reception, the detector output is kI(t) where I(t) is a function which is taken to average to zero over a long period of time. For if I(t) did not average to zero, it could be made to do so by the appropriate use of the function

$$I_1(t) = I(t) - \text{average}\; I(t)$$

in the expression 2.1-1. Further, steady state values of detected currents are usually filtered out by the receivers used so that the output of the receiver accounts for the second term of 2.1-3. Both I(t) and E(t) average to zero.

Any function of time can be put in the form 2.1-2 with ease in a number of ways. This will require that for any g(t), the form 2.1-2 be defined uniquely as regards the nature of the A(t) and the f(t). This will be done in section 3.1, enabling us to perform the operations of detection in this method of modulation and in the other methods of modulation classified as continuous wave methods.

b) <u>Frequency Modulation</u>

The transmitted wave in the case of frequency modulation takes the form

$$O[I(t)] = ce^{i[w_0 t + k\int I(t)dt]}\Big/_{\text{Real Part}} \qquad 2.1\text{-}4$$

The indefinite integral is written since the constant of integration which appears may be incorporated into the constant c which appears in the front of the expression. It is seen that the envelope in the present case is a constant while the intelligence is carried by the relative positions of the nulls in the function 2.1-4.

The detection process is defined for frequency modulation such that if a wave of the form 2.1-2 is received, where the form of 2.1-2 is uniquely expressed by the method of section 3.1, the output of the detection process is given by

$$E(t) = \frac{d}{dt} f(t) - \text{average} \frac{d\,f(t)}{dt} \qquad 2.1\text{-}5$$

Thus in the case of single signal or ideal reception, the output is given by kI(t) where, as before, steady state values of the output are filtered out by the receiver accounting for the second term in the expression 2.1-5. ω_0, the carrier frequency is a constant, and again the average value of I(t) over a long period of time is zero. As in the case of amplitude modulation, ω_0 is the angular frequency of the unmodulated wave obtained by letting I(t) = 0 and is called the carrier frequency. It should be noted that ω_0 is the carrier frequency only when

$$\text{Average } I(t) = 0 \qquad 2.1\text{-}6$$

In any other case, the non-zero steady state component of I(t) yields, upon integration, a constant times the time, which may be incorporated in the function $\omega_0 t$.

c) Phase Modulation

The transmitted wave in the case of phase modulation takes the form

$$O[I(t)] = ce^{i[\omega_0 t + kI(t)]} \Big/_{\text{Real Part}} \qquad 2.1\text{-}7$$

Again, the envelope of the transmitted wave is a constant. And again, the quantity, ω_0 is the angular frequency of the transmitted wave had I(t) been allowed to be zero and is called the carrier frequency. The average value of I(t) is usually taken as zero for any non-zero average value of I(t) merely results in a phase shift which may be incorporated in the constant c which appears in the front of the expression.

The detection process is defined for the phase modulated wave such that if a wave of the form 2.1-2 is received, the output from the receiver is given by

$$E(t) = [f(t) - \omega_0 t] - \text{average}[f(t) - \omega_0(t)] \qquad 2.1\text{-}8$$

Thus the average value of E(t) in this case is zero. In the case of single signal or ideal reception, the output is merely kI(t).

Some confusion has arisen in many quarters regarding the differences between frequency and phase modulation because of the similarity of the expressions 2.1-4 and 2.1-7 when I(t) is a pure sinusoid. And of course the expressions cited are identical in form and the results of the detection processes for ideal reception are the same for sinusoidal modulation. However this is as far as the similarity extends.

The dissimilarities between equations 2.1-4 and 2.1-7 and between 2.1-5 and 2.1-8 are evident. Further, even in the case of sinusoidal modulation, the reception is not the same when detection takes place under conditions of multipath transmission.

d) Single Sideband Modulation

If I(t) is a periodic function, the frequency spectrum of the transmitted wave for the amplitude modulated case consists of a number of lines equally spaced extending upon either side of the carrier frequency, ω_o. The totality of the information however may be carried by only one-half the lines in the frequency spectrum and thus may require much less energy to perform the transmission of the intelligence than in the case of amplitude modulation. Usually the upper half frequency spectrum is transmitted extending from ω_o upward. The line corresponding to the carrier frequency may or may not be transmitted. The nature of the transmitted wave is found by taking the spectral analysis of the expression 2.1-1 and eliminating the lines of the spectrum below the frequency ω_o. We thus obtain

$$O[I(t)] = k\left[\sum_{n=1}^{\infty} a_n \cos(\omega_o + n\omega)t + \sum_{b_n=1}^{\infty} b_n \sin(\omega_o + n\omega)t + c_o \cos\omega_o t\right] \qquad 2.1\text{-}9$$

where the a_n, and b_n, are the ordinary Fourier coefficients in the expansion of I(t)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} I(t)\cos n\omega t \, d(\omega t) \quad , \quad b_n = \frac{1}{\pi}\int_0^{2\pi} I(t)\sin n\omega t \, d(\omega t)$$

and c_o may be zero if the center frequency, ω_o is not transmitted. In the above expression, ω is the fundamental frequency of repetition of the intelligence, I(t). The extension when I(t) is a non-repetitious function, of the equation 2.1-9 to an expression involving Fourier Integrals is evident.

The detection process involves the artificial supplying of the carrier frequency in the proper phase in the case where the summation starts from n = 1 and the artificial supplying of the mirror lines necessary to complete the amplitude modulation spectrum and then performs a detection of the type already mentioned in the section on amplitude modulation.

If a wave of the form

$$g(t)=\sum_{n=1}^{\infty} a_n \cos\left[(\omega_0+n\omega)t+\alpha_n\right]+\sum_{n=1}^{\infty} b_n \sin\left[(\omega_0+n\omega)t+\beta_n\right] \qquad 2.1\text{-}10$$

is received, then the result of the detection process is the output

$$\mathcal{E}(t)=\sum_{n=1}^{\infty} a_n \cos(n\omega t+\alpha_n-u)+\sum_{n=1}^{\infty} b_n \sin(n\omega t+\beta_n-u) \qquad 2.1\text{-}11$$

where u is the phase of the carrier artificially supplied. The choice of the phase angle, u, is determined by setting u equal to the phase of the wave arriving at the receiver had an unmodulated carrier wave of frequency been transmitted rather than the modulated single sideband wave.

In the case of multipath propagation of a single sideband wave, no new lines are introduced in the frequency spectrum and a phase shift is merely introduced in each. It is this phase shift which produces the distortion observed. Under conditions of multipath transmission, the nature of the distortion in this and in other types of modulation here mentioned must be considered in detail and are discussed in section 3. Diagrams illustrating these systems of modulation are found in figure 1.

e) Generalized Amplitude Modulation

Indeed, section 2.1c) notwithstanding, amplitude modulation and phase modulation are closely akin, being members of a type of generalized amplitude modulation. They are special cases of a modulation in which the transmitter output is

$$O[I(t)] = c\,h[I(t)]e^{i\omega_0 t}\Big|_{\text{Real Part}} \qquad 2.1\text{-}12$$

where h is a point function of its argument. The output of the detection of this hypothetical modulation process, of which only two examples are in common usage, is such that if a wave of the form 2.1-2 were operated upon by the receiver, subject to the uniqueness conditions of section 3.1, the output from the receiver would be given by the solution for E(t) to the equation

$$c\,h[E(t)]e^{i\omega_0 t} = g(t) \qquad 2.1\text{-}13$$

Indeed in any type of modulation, the output from the detector is given by the solution to the equation

$$O[E(t)] = g(t) \qquad 2.1\text{-}14$$

In the case of phase modulation, the operator h takes the form $h[I] = e^{ikI}$ and for amplitude modulation it takes the form $h[I] = 1+kI$

Unfortunately the conditions of this definition for the expected output as a solution to the operational equation, 2.1-14, are not always followed in the construction of receivers. They are constructed so that the reception satisfies the sufficient condition that the output be proportional to the intelligence or the modulation in the case of single signal or ideal reception. This means that operators may be inserted in the equation 2.1-14 which take on the form of the identity operator for ideal reception but are not inoperative in the case of multipath reception. An example of such a case is the absolute value sign appearing in equation 2.1-3 for the amplitude modulation detector. We shall consider the generalized amplitude modulation case but we shall therefore avoid any possibility of difficulty by calculating the multipath distortion for the two subcases, phase and amplitude modulation as well. We can generalize amplitude modulation as a subcase of the modulation considered in 2.1-12 by considering only real operators, h, and then solving the equation characterizing the detection

$$h[u(t)] = |A(t)| - \text{ave}\,|A(t)|, \quad \varepsilon(t) = u(t) - \text{ave}\,u(t) \qquad 2.1\text{-}15$$

for $\varepsilon(t)$, where A(t) is defined in 2.1-2. Thus amplitude modulation but not phase modulation, occurs as a special case of the modulation and detection given by 2.1-15.

f). <u>Frequency Modulated Frequency Modulation</u>

The transmitted wave in Frequency Modulation of Frequency Modulation (F_M of FM) takes the form

$$O[I(t)] = C\epsilon^{i\left[\omega_0 t + K_1 \int \cos\left(\omega_1 t + k \int I(t)\,dt\right) dt\right]} \Big|_{\text{Real Part}} \qquad 2.1\text{-}16$$

ω_0 is called the angular frequency of the carrier or the carrier frequency. It is the frequency of the transmitted wave when K_1 is zero. ω_1 is called the subcarrier frequency. The subcarrier is modulated on the carrier by the process indicated in the equation.

Here, as frequency and phase modulation, the envelope of the transmitted wave is a constant. Also the envelope of the subcarrier before modulation is a constant. The argument for such a type of modulation is that the signal-to-noise ratio for this type of modulation is very small compared with the conventional types.

The method of detection is rather complicated from the standpoint of an operational definition. The detection process can be considered in two stages. The first is the operation by an ordinary frequency modulation detector upon the incoming wave which is of the form 2.1-2 according to the definitions of section 3.1. The output of the detection process at this stage is $\frac{df}{dt} - \text{ave}\,\frac{df}{dt}$. This output is then written in the form of 2.1-2 and the resultant operated upon with another frequency modulation detector. Writing

$$\frac{df}{dt} - ave\frac{df}{dt} = B(t)\,e^{ig(t)}\Big/_{Real\ Part} \qquad 2.1\text{-}17$$

the FM of FM detector output is $\frac{dg}{dt} - ave\frac{dg}{dt}$. Again this second writing of the wave in the form of 2.1-2 must be done uniquely. The detection of ideal reception as yielding I(t) is evident. The detection of Multipath transmitted waves will be considered in more detail.

2.2 The Mechanics of Detection[9]

Generally speaking a receiver for any type of modulation can be divided into a distinct number of sections. Each section contributes to the operator involved in the detection. The general classifications of these sections are as follows:

a. A linear amplifier and detector.

b. A series of limiters introducing a fixed or adjustable amplitude gate.

c. A converter or demodulator restoring the audio frequency characteristics of the original signal.

d. A series of audio frequency filters eliminating all frequencies not used in the desired signal followed by Audio frequency amplifiers which bring the signal to the desired level.

The linear amplifier is used to amplify the magnitude of the incoming signal and reproduce the wave shape of the currents induced in the antennae. The linear detector is possibly an heterodyne detector which usually shifts the position of the frequency spectrum of the incoming radiation downward in frequency but doesnot change the relative positions or magnitudes or phases of the lines in the spectrum.

If the signal transmitted from the source has the property that, during the time of actual operation the envelope of the transmitted function is either a constant not zero or is zero, then limiters ot amplitude may be introduced in the receiver. This has the property of clipping the peaks of the incoming waves. The distortion introduced by this operation is such as to add additional lines in the frequency spectrum but they are located at integral multiples of the existing lines in the spectrum. Modern transmission of modulated intelligence is established so that, as noted in section 1.1, the width of the frequency spectrum is much less than the carrier frequency. Receivers are designed so that a finite band-width corresponding to the bandwidth of the transmitted wave is used thus eliminating the higher order frequency lines introduced by the clipping process. The advantages of the clipping process are that it reduces the amount of the noise present in the detection.

With the pulse modulation techniques to be described in the next section more than one intelligence function can be transmitted upon the same carrier. The actual modulation due to a particular intelligence function occupies such a short period of time that other intelligence functions may be placed upon the same carrier. This procedure is called a time division multiplex system. At the present writing as many as ten intelligence functions have been placed upon one carrier. An oscillogram of the radiation from the transmitter antenna would show a succession of pulses from each of the intelligence functions in order, the entire procedure being repeated. At the receiver a gating process is necessary to separate the modulation waves for each of the intelligence functions involved in the multiplex. This process is merely a pass or no-pass system which lets the modulation corresponding to a particular intelligence function proceed along a particular channel but refuses passage to the other modulations of the other intelligence waves. Of course there must be as many gates and channels as there are intelligence functions in the multiplex.

The converter mentioned is the demodulator or inverse operator already mentioned and needs no further explanation.

There may be objectionable audio frequencies present in the resultant of the detected wave. Such situations are encountered in the pulse methods of modulation. Filters must be provided after the detector to eliminate these frequencies.

By means of review of the preceding section it may be said that there are no limiters in amplitude modulation and that the linear system is preserved not only in the relationship between the output of the detector and the input to the transmitter but also in the output of the transmitter and the input to the demodulator of c) mentioned in the cases of single signal transmission. Frequency modulation has a linear converter or demodulator called a discriminator which takes the value of the instantaneous frequency and supplies it to the A.F. amplifier. In pulse modulators, the limiters may either precede or follow the linear amplifiers and the demodulator is the subject of the sections following.

2.3 Pulse Wave Systems

a). Pulse Amplitude Modulation

A pulse amplitude modulation (PAM) transmitter radiates electromagnetic waves whose dependence upon time is given by

$$O[I(t)] = [1 + kI(t_0 + n\tau)] \quad t_0 + n\tau < t < t_0 + n\tau + \delta \qquad 2.3\text{-}1$$
$$= 0 \quad t_0 + n\tau + \delta < t < t_0 + (n+1)\delta$$

where t_0 is a constant, n is an integer, δ is the width of the pulse and $1/\tau$ is the repetition rate. Thus the radiation takes the form of a series of bursts of sine waves whose amplitudes depend upon the values of the intelligence at the times of initiation of the bursts or pulses. The spacing of the pulses is uniform and the sampling rate is $1/\tau$. During the dead time, other intelligence functions may be modulated and transmitted upon the same carrier, which procedure as already mentioned is the basis for the process of multiplex. (See Figure 2).

The problem can be looked upon as a special case of the continuous wave amplitude modulation where the function $1+kI(t)$ in 2.1-1 is zero most of the time and is $1+kI'(t_0+n\tau)$ at other times. However the distinction is made separating PAM from amplitude modulation because the former falls a little more naturally into a common classification along with the other methods of modulation of this group.

The detection of this type of modulation is similar to the mechanics of that of amplitude modulation. The (Fig. 3) mechanical process in amplitude modulation detection involves the amplification of the incoming signal, its rectification and passage through a low pass filter filtering out the high frequency components remaining after rectification. The cutoff of the amplitude modulation detection filter should be set somewhere below the carrier frequency, ω_0, and above the highest frequency of the intelligence, I(t), it is desired to reproduce. The rectification process and filtering in reality reproduces the envelope of the incoming function as implied by the method of detection given by equation 2.1-3.

The procedure of the PAM detection process is exactly the same as the above with the exception of the specification of the cut-off frequency of the low pass filter. It is made more restrictive in this case. The cut off of the receiver low-pass filter should be set at a frequency somewhere between $1/\tau$ and $2/\tau$.

Although some distortion is introduced by the procedure of finite sampling rates, this distortion can be made small by increasing the sampling rate. It has been shown,[10,11] however, that the transmitter sampling rate sufficient to reproduce the highest frequency component of the intelligence it is desired to transmit is twice this highest frequency.

The detection procedure is approximately equivalent to passing a smooth curve through the tops of the equally spaced pulses in the case of ideal or single signal reception. This reproduces approximately the intelligence wave to the limits of the sampling and of the filtering. The distortion due to multipath transmission will be compared with the ideal or single signal wave as detected although this wave is not exactly the intelligence due to the finite sampling and the filtering.

Writing 2.2-1 in the form of a spectral analysis

$$O[I(t)] = \sum_{n=0}^{T/k_1 - 1} \left[1 + kI(t_0 + n\tau)\right]\left[\frac{\delta}{T_f} + \sum_{m=1}^{\infty} \frac{2}{m\pi} \sin\frac{m\pi\delta}{T_f} \cos\frac{2\pi m}{T_f}\left(t - t_0 - n\tau - \frac{\delta}{2}\right)\right] \cos\omega_0\left[t - t_0 - n\tau - \frac{\delta}{2}\right] \qquad 2.3\text{-}2$$

where $T_f = \frac{T\tau}{k_1}$, and T is the fundamental frequency of the intelligence to be reproduced. k_1 is a dimensional constant chosen such that k_1 is unity in the system of units of time where T_f is the least common multiplier of $T\tau$.

The detection of the ideal wave by a PAM receiver yields the output

$$E(t) = \sum_{n=0}^{T/k_1 - 1} \left[1 + kI(t_0 + n\tau)\right] \sum_{m=1}^{T/k_1 - 1} \frac{2}{m\pi} \sin\frac{m\pi\delta}{T_f} \cos\frac{2\pi m}{T_f}\left(t - t_0 - n\tau - \frac{\delta}{2}\right) \approx I(t) \qquad 2.3\text{-}3$$

For any wave, g(t), arriving at a PAM receiver, if it is put into the form 2.1-2 according to the conditions of section 3.1, and if A(t) is periodic with frequency $1/T_f$ having a Fourier Analysis

$$A(t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_f} + \delta_n\right) \qquad 2.3\text{-}4$$

then the detector output is

$$E(t) = \sum_{n=1}^{T/t_1 - 1} a_n \cos\left(\frac{2n\pi t}{T_f} + \delta_n\right) \qquad 2.3\text{-}5$$

This will be more completely investigated in the case where the distortion arises due to multipath distortion.

b). Pulse Length Modulation

A pulse length modulated (PLM) transmitter radiates electromagnetic waves whose dependence upon time is given by

$$\begin{aligned} 0[I(t)] &= C_1 \cos w_0 t \quad & t_0 + n\tau < t < t_0 + n\tau + \delta_n \\ &= 0 & t_0 + n\tau + \delta_n < t < t_0 + (n+1)\tau \end{aligned} \qquad 2.3\text{-}6$$

where δ_n is explicitly given by the condition

$$\delta_n = K_2 \left(1 + k I(t_0 + n\tau)\right) \qquad 2.3\text{-}7$$

It can be seen that larger values of the intelligence cause longer pulses to be propagated. The leading edges of the pulses are equally spaced and are separated by the sampling rate, $1/\tau$. Multiplex in this case is dependent upon the maximum width of the pulse and hence is dependent upon the maximum value of the modulation, I_{max}. Thus the number of allowable intelligence functions having the same maximum values which can be inserted in a PLM multiplex with sampling rate, $1/\tau$, is equal to $\frac{\tau}{K_2(1+kI_{max})}$, or the next smaller integer.

It should be noted that another system of PLM has been proposed where the trailing edges of the pulses are equally spaced but the leading edges vary with the value of the intelligence at the time of initiation of the pulse. In this case the output of the transmitter is given by

$$\begin{aligned} O[I(t)] &= C_1 \cos \omega_0 t \quad t_0+n\tau-\delta_n < t < t_0+n\tau \\ &= 0 \quad t_0+n\tau < t < t_0+n\tau+\tau-\delta_{n+1} \end{aligned} \qquad 2.3\text{-}8$$

where δ_n is implicitly given by the condition

$$\delta_n = K_2(1+kI(t_0+\tau n-\delta_n)) \qquad 2.3\text{-}9$$

The two systems of modulation are similar and widths of the pulses transmitted in the two cases are the same in the limit when the width of the pulses are small compared with the sampling rate. The former system of modulation is the one which will be here considered however.

The mechanics of the detection of this type of modulation is similar in technique to that of the preceding section. The incoming signal is separated from the remainder of the multiplex components by a suitable gating system. The PLM system for a single intelligence function is then amplified and rectified. The resultant envelope of the incoming wave is passed through upper and lower level limiters to eliminate extraneous signals and the remainder is passed through a low pass filter to recover the original intelligence. The action of the low-pass filter upon pulses of varying length and constant height can be calculated and seen to be similar to the passage through the same filter of pulses of constant width but variable height (PAM) with the same sampling rate. This is so at least when the two cases consist of pulses which are small in width. This follows because the pulses may be looked upon as energy arriving at the filter; the rate of arrival of energy at the filter is the same for PLM as it is for PAM since the relative areas under pulses are proportional for the two systems.

The frequency analysis of the wave propagated by the transmitter takes the form

$$O[I(t)] = \sum_{n=0}^{T/k_1 - 1} C_1 \left[\frac{\delta_n}{T_f} + \sum_{m=1}^{\infty} \frac{2}{m\pi} \sin \frac{m\pi\delta}{T_f} \cos \frac{2\pi m}{T_f} \left(t - t_0 - n\tau - \frac{\delta_n}{2} \right) \right] \cos \omega_0 \left(t - t_0 - n\tau - \frac{\delta_n}{2} \right) \tag{2.3-10}$$

where the notation is similar to that used for PAM. After reception by a PLM receiver the ideal single signal wave becomes upon detection

$$E(t) = C_1 \sum_{n=0}^{T/k_1 - 1} \sum_{m=1}^{T/k_1 - 1} \frac{2}{m\pi} \sin \frac{m\pi\delta_n}{T_f} \cos \frac{2\pi m}{T_f} \left(t - t_0 - n\tau - \frac{\delta_n}{2} \right) \approx I(t) \tag{2.3-11}$$

The analysis for an arbitrary wave arriving at the receiver is more complex. Further analysis is reserved until the particular case of multipath transmission is considered.

c). Pulse Position Modulation

The concepts of Pulse Position Modulation (PPM) or Pulse Phase Modulation as it is sometimes called are very easy to grasp from the material of the preceding section on PLM. In PLM the length of the pulses transmitted varies directly as the modulation. In PPM the width of the pulses is a constant but the position of the pulses varies with the modulation. The output of the transmitter in the case of PPM is given by

$$O[I(t)] = \begin{cases} C_3 \cos \omega_0 t & t_0 + n\tau - \delta_n < t < t_0 + n\tau - \delta_n + p \\ 0 & t_0 + n\tau - \delta_n + p < t < t_0 + (n+1)\tau - \delta_{n+1} \end{cases} \qquad 2.3\text{-}12$$

where δ_n is given implicitly by the expression

$$\delta_n = K_3 \, I(t_0 + n\tau - \delta_n) \qquad 2.3\text{-}13$$

It can be seen that the width of each pulse is p and the position of the pulse with respect to its center position is a function of the modulation. The center position or un-modulated position of the pulse occurs when $I(t)=0$

In the unmodulated case all the pulses are equally spaced, occuring periodically with a period, τ. The sampling rate of the modulated case averages $1/\tau$. If PPM is used for multiplex transmission of intelligence, with a number of signals whose maximum values of intelligence are all $|I|_{max}$, the number of allowable channels is $\frac{\tau}{p + 2K_3 |I|_{max}}$, or the next smaller integer.

The mechanics of the detection process consists of the separation of the various channels by suitable gates. The single resulting intelligence PPM function is then amplified and rectified. The resultant envelope of the incoming wave is converted from a series of position pulses to a series of length pulses of the type used in modulation in equation 2.2-8. The length pulses are then detected in the fashion stated in the preceding section, using the upper and lower level limiters and the filter. Upper and lower level limiters may be inserted before the conversion to length pulses. Conversion from position pulses to length pulses is accomplished by allowing either the leading or the trailing edges of the position pulses to initiate a pulse in the receiver which is not terminated until a specified time. The trailing edges of the length pulses resulting are equally spaced and the length pulses are of the form of the pulses without carrier in equation 2.2-8. It is conceivable that pulses without carrier of the form of equation 2.2-6 could be generated in the receiver by having all leading edges of the length pulses equally spaced but letting the length pulses be terminated by either the leading or the trailing edges of the position pulses. Reference is made to plate 4.

e). Pulse Frequency Modulation

Pulse frequency modulation (PFM) transmitters radiate waves whose dependence upon time is given by

$$O[I(t)] = \begin{cases} C_4 \cos \omega_0 t & t_n < t < t_n + \delta \\ 0 & t_n + \delta < t < t_{n+1} \end{cases} \qquad 2.3\text{-}14$$

where t_n is given implicitly by the expression

$$\omega_1 t_n + \int_0^{t_n} k I(t) dt = 2n\pi \qquad 2.3\text{-}15$$

where ω_1 is the carrier frequency of the unsampled frequency modulation. The actual procedure used in PFM transmission is to frequency modulate the intelligence in the continuous wave sense and then perform a PPM sampling process on the resultant. It can be seen that the spacing of pulses in PFM bears a similar relationship to PPM as continuous wave FM does to continuous wave phase modulation. The number of pulses per unit time transmitted is directly proportional to the magnitude of I(t). The mean (or unmodulated) sampling rate is $2\pi/\omega_1$. Time division multiplex in this case would depend upon the rate of transmission of the pulses or upon the maximum value of the integral appearing in equation 2.2-15 where as has been customary, ave I(t) = 0 so that the integral is not divergent but oscillates with time. The number of time division multiplex signals which can be transmitted is given by $\dfrac{2\pi}{\omega_1 \delta + 2\left|\int_0^t k I(t) dt\right|_{max}}$ or the next smaller integer.

The detection process is similar to those previously considered insofar as preliminary stages are concerned. The various PFM multiplex systems are separated by a gating process. The individually channelled PFM waves are rectified, limited at upper and lower levels and the resultant envelope passed through a discriminator of the type described by equation 2.-15. The result is then passed through a low-pass filter, eliminating the higher frequency components. The behavior of the detector is uncomplicated in action when considering single signal reception.

e). Pulse Code Modulation

Pulse code modulation (PCM) is perhaps the most difficult type of pulse modulation thus far encountered as far as mathematical expression of the radiation is concerned. The method is based upon the expression of integers in the binary system of numbers. The present system of modulation has been proposed[12,13,14] because of its apparent superiority over all other systems of modulation here mentioned with regard to the signal-to-noise ratio despite the very large bandwidth of transmission required. The PCM transmitter radiates waves of the form

$$O[I(t)] = \delta_{nm} \cos \omega_0 t \quad t_0 + (n+m)\tau < t < t_0 + (n+m)\tau + p \qquad 2.3\text{-}16$$
$$= 0 \quad t_0 + (n+m)\tau + p < t < t_0 + (n+m+1)\tau$$

where δ_{nm} is the coefficient in the expansion of

$$1 + kI(t_0 + n\tau) = \sum_{m=0}^{N-1} \delta_{nm} 2^{N-m-1} + R_n \quad \text{(binary)} \qquad 2.3\text{-}17$$

where $\delta_{nm} = 0 \text{ or } 1$ and $R_n < 1$

such that $1 + k|I|_{max} < 2^N$ 2.3-18

in some arbitrary scale of magnitudes with N a position integer. The number of time division multiplex signals allowable is τ/p or the next smaller integer. Each group of N pulses, whether all are present or not is called a pulse code group. Synchronizing signals must be transmitted in order to ensure the start of a pulse group.

The receiver at the preliminary stages, behaves like the other pulse method receivers. It submits the multiplex PCM to separation by a suitable gating system followed by the usual amplification, rectification and upper and lower level limiting of the individually channelled PCM waves. Any given pulse code group is then multiplied by a wave generated inside the receiver initiated by the synchronizing signals as shown in figure 5. Multiplying a particular code group by this wave yields a series of N pulses whose heights are to each other, if present, in powers of 2. At this stage, the signal in the receiver has the form,

$$\mathcal{E}(t) = \delta_{nm} 2^{N-m-1} \quad t_0 + (n+m)\tau < t < t_0 + (n+m)\tau + p$$
$$= 0 \quad t_0 + (n+m)\tau + p < t < t_0 + (n+m+1)\tau \qquad 2.3\text{-}19$$

A given pulse code group, thus modified, is integrated by summing the amplitudes of the N pulses thus generated, the resultant integrated value being used to generate an amplitude pulse of amplitude equal to this integrated value. Thus each pulse code group is operated upon in such a fashion as to generate a single amplitude pulse. These equally spaced, equal width pulses are then passed through a low pass filter as in the final stages of PAM receivers, removing the intelligence. There is of course distortion introduced by both the finite sampling process and the reduction process for the sample pulses. The former distortions can be made small by letting the sampling rate, $1/N\tau$, become large and the latter distortion can be made smaller by letting the integer N become large. This system of modulation is advantageous because of the on-off character of the pulses transmitted. Thus the reception depends, in the ideal case, merely upon the presence or absence of a given pulse and not upon position, width or height of a pulse.

2.4 Optics as a Case of Modulation

The formulation of optical transmission as a case of modulated waves is not difficult. The transmitted wave is unmodulated in the cases of physical optics usually considered. The source output may be written

$$O[I(t)] = A\cos\omega_0 t \qquad 2.4\text{-}1$$

The receiver, (photographic plate, screen, etc.) which usually observes intensities so that if a wave of the form

$$f(t) = Ae^{ig(t)}\Big/_{\text{Real Part}} \qquad 2.4\text{-}2$$

were received, the receiver output would be

$$E(t) = A^2 \qquad 2.4\text{-}3$$

The amplitude function, A, appearing in 2.4-2 is assumed constant because, as will be seen in application of the method of 3.1, in multipath transmitted waves the amplitude function of the received wave is constant.

III. Solutions for Continuous Waves

3.1 The Vectorial Picture for the Resultant Wave

The concepts of representation of sinusoidally oscillating quantities by complex numbers are well known. If one attempts to express a quantity, $A\cos(\omega t+\theta)$, as a complex number where a, ω, θ are real constants, this can be done by writing

$$A\cos(\omega t+\theta) = Ae^{i[\omega t+\theta]}\Big/_{\text{Real Part}} \qquad 3.1\text{-}1$$

Thus the sinusoidally oscillating quantity can be pictorially represented by the motion of a point in the complex number plane; a complex number whose magnitude is A in the appropriate system of units and whose argument at some time, t, is $(\omega t+\theta)$. This complex number, or "vector" or "characteristic vector", moves with constant angular velocity, ω. The projection of the line connecting the origin and this point in the complex number plane on the line through the origin $\theta=0$ is the quantity considered.

Of course the convention, 3.1-1 is perfectly arbitrary. Indeed, remaining in the domain of complex representation, the projection on the coordinate axis of the motion of a point along any one of a group of restricted paths with appropriately varying velocity, such that the projection will be a sinusoidally varying oscillation, would have been satisfactory. However the simplicity of the representation 3.1-1 with its constant amplitude and constant angular velocity is appealing. It further has the advantage that the amplitude function, A, represents the envelope of the actual oscillating function as plotted in Cartesian coordinates while the zeros of the actual oscillating function are given by the zeros of the real part of the exponential function.

The fundamental arithmetic operations of addition and subtraction of sinusoidally oscillating quantities of the same frequency are shown to correspond to the analogous operations on their complex representations.

A more difficult question is the representation to be given to the sum of two sinusoidally oscillating quantities when the angular frequencies are different. It is well known that the resultant acoustic phenomenon, when the frequencies are close to one another, is that of beats. A convenient representation of the sum is

$$A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2) = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos[(\omega_2 - \omega_1)t + \theta_2 - \theta_1]} \cdot e^{i\left[\omega_1 t + \theta_1 + \tan^{-1} \frac{A_2 \sin[(\omega_2 - \omega_1)t + \theta_2 - \theta_1]}{A_1 + A_2 \cos[(\omega_2 - \omega_1)t + \theta_2 - \theta_1]}\right]} \quad \text{R.P.} \qquad 3.1\text{-}2$$

It can be seen that this representation is most convenient because the amplitude function represents the envelope which, if ω_1 and ω_2 are close to one another, represents the amplitude of the beat phenomenon. The exponential function leads to the zeros of the combination of the oscillatory functions. The result has a geometrical representation (fig. 6) which is the vector sum of the two vectors representing the simple oscillatory functions. Thus the concept of oscillatory functions of varying amplitude and phase (and frequency) has been introduced. When a function $A(t)e^{i(\omega t+\phi(t))}\Big|_{\text{Real Part}}$ is written, we shall reserve for A(t) the definition of amplitude, for ω_0 the mean value of $(\omega t+\phi)$ termed the carrier frequency, for $(\omega-\omega_0)t+\phi$ the definition of phase. It was upon this basis that the definitions of the various systems of continuous wave modulation (section 2.1) were made. Their complementary demodulation systems depend upon the generalization following.

When one is interested in writing the resultant of a number of oscillatory functions in the form of equation 2.1-2, the characteristic vector of the resultant is required to coincide with the resultant of the vector representing the separate functions thus fixing the amplitude and the argument of the exponential function. These demands fix the sum of two or more harmonic vibrations uniquely so that the equations of demodulation now have unique meaning.[15]

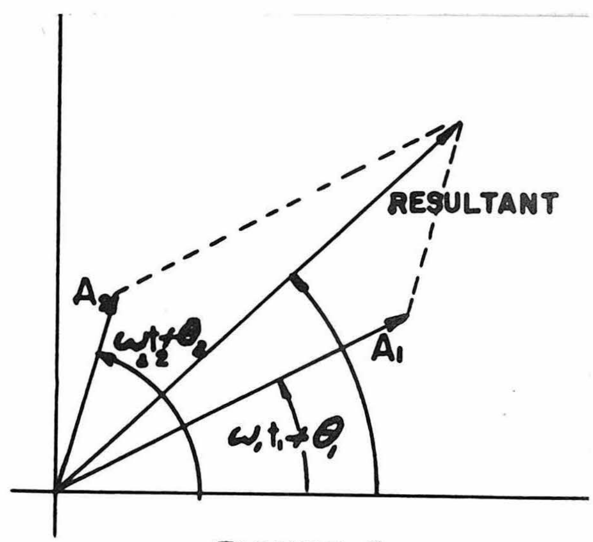

FIGURE 6

Examples follow in succeeding sections.

3.2 Small percentage modulation of Point functions of Intelligence.

In view of the definitions of the preceding section, the distortion function, D, defined in 1.2-3 or the output function E(t) of 1.1-3 can now be calculated for the conditions of multipath transmission and reception by the complementary receivers corresponding to the transmitters of continuous wave type modulation. The calculations are now made for amplitude, phase and generalized amplitude modulations. These types of continuous wave modulation methods fall into a common group as they are all types of modulation involving point functions of the intelligence.

a). Multipath Reception of Amplitude Modulated Waves

By 2.1-1 the transmitter output is

$$O[I(t)] = c\{1+kI(t)\}e^{i\omega_0 t}\Big|_{\text{Real Part}} \qquad 2.1\text{-}1$$

Thus the input to the receiver, by 1.2-2 is

$$g(t) = \sum_{n=1}^{N} A_n \{1+kI(t-t_n)\} e^{i(\omega_0)(t-t_n)}\Big|_{\text{Real Part}} \qquad 3.2\text{-}1$$

where the quantities A_n, t_n are those defined on page 9 (section 1.2). In order to determine the output of an amplitude modulation receiver receiving this wave, it is necessary to place 3.2-1 in the form 2.1-2 which can be accomplished by the method of the preceding section.

Performing the rearranging of terms in 3.2-1 indicated by the vector addition process, g(t) becomes

$$g(t)=\sqrt{\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} a_i a_j \cos\beta_{ij}}\; e^{i\left(w_0 t-\tan^{-1}\frac{\sum a_i \sin\beta_i}{\sum a_i \cos\beta_i}\right)} \qquad 3.2\text{-}2$$

where

$$a_i = A_i\left(1+kI(t-t_i)\right)$$

$$\beta_{ij} = \beta_i-\beta_j = w_0(t_i-t_j)$$

Thus by 2.1-3, the receiver output is

$$\mathcal{E}(t)=\sqrt{\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} a_i a_j \cos\beta_{ij}} - \text{ave}\sqrt{\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} a_i a_j \cos\beta_{ij}} \qquad 3.2\text{-}3$$

It is of interest to calculate the form of 3.2-3 when $k|I|_{max} \ll 1$.

Using the first two terms of the binomial expansion,

$$\mathcal{E}(t)=\sqrt{\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}} + \frac{\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}\, kI(t-t_i)}{\sqrt{\sum\sum A_i A_j \cos\beta_{ij}}}$$

$$-\text{ave}\left\{\sqrt{\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}} + \frac{\sum_{i=1}\sum_{j=1} A_i A_j \cos\beta_{ij}\, kI(t-t_i)}{\sqrt{\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}}}\right\}$$

and since (p.14)

$$\text{ave}\, I(t)=0,$$

$$\mathcal{E}(t)=\frac{k\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}\, I(t-t_i)}{\left(\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}\right)^{1/2}} \qquad 3.2\text{-}4$$

where $\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij} \neq 0$.

This is a form which lends itself nicely to numerical computation and the result of this calculation for various fundamental wave forms is seen in section 3.4.

The condition that

$$\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij} \neq 0$$

is essential to the expansion and the significance of the vanishing of this function is discussed in section 3.3.

b) <u>Phase Modulation</u>

By 2.1-7, the transmitter output is

$$O[I(t)] = Ce^{i(w_0 t + kI(t))}\Big|_{\text{Real Part}} \qquad 2.1\text{-}7$$

and the input to the receiver by 1.2-2 is

$$g(t) = \sum_{n=1}^{N} A_n e^{i[w_0(t-t_n) + kI(t-t_n)]} \qquad 3.2\text{-}5$$

The output of a phase modulation receiver detecting this wave is found by placing g(t) in the form 2.1-2 by the vector addition method of the previous section.

$$g(t) = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\Omega_{ij}}\; e^{i\left(w_0 t - \tan^{-1}\frac{\sum_{i=1}^{N} A_i \sin\Omega_i}{\sum_{i=1}^{N} A_i \cos\Omega_i}\right)} \qquad 3.2\text{-}6$$

where

$$\Omega_i = w_0 t_i - kI(t-t_i)$$

$$\Omega_{ij} = \Omega_i - \Omega_j = \beta_{ij} - k[I(t-t_i) - I(t-t_j)] \qquad 3.2\text{-}7$$

Then by 2.1-8 the receiver output is

$$\mathcal{E}(t) = \tan^{-1}\frac{\sum_{i=1}^{N} A_i \sin\Omega_i}{\sum_{i=1}^{N} A_i \cos\Omega_i} - \text{ave}\,\tan^{-1}\frac{\sum_{i=1}^{N} A_i \sin\Omega_i}{\sum_{i=1}^{N} A_i \cos\Omega_i}\,. \qquad 3.2\text{-}8$$

$\mathcal{E}(t)$ can be calculated when $|kI|_{max} \ll \pi$ by expanding $\tan^{-1}\frac{\sum_{i=1}^{N} A_i \sin\Omega_i}{\sum_{i=1}^{N} A_i \cos\Omega_i}$ in a Taylor's Series and considering the first two terms, where $\Omega_{ij} = \beta_{ij}$ as a first order calculation, with the effect of the intelligence acting as a perturbation.

$$\mathcal{E}(t) = \tan^{-1}\frac{\sum^{N} A_i \sin\beta_i}{\sum A_i \cos\beta_i} + k\frac{\sum^{N}\sum^{N} A_i A_j \cos\beta_{ij}\, I(t-t_i)}{\sum^{N}\sum^{N} A_i A_j \cos\beta_{ij}}$$

$$-\text{ave}\left\{\tan^{-1}\frac{\sum^{N} A_i \sin\beta_i}{\sum^{N} A_i \cos\beta_i} + k\frac{\sum^{N}\sum^{N} A_i A_j \cos\beta_{ij}\, I(t-t_i)}{\sum^{N}\sum^{N} A_i A_j \cos\beta_{ij}}\right\} \qquad 3.2\text{-}9$$

and by the discussion on page 14, ave $I(t) = 0$ so that

$$E(t) = k \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij}\, I(t-t_i)}{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij}} \qquad 3.2\text{-}10$$

The form of 3.2-10 is remarkably similar to 3.2-4, the result obtained for amplitude modulation of small percentage modulation. Indeed for the same type of I(t), and for similar relative characteristics (A_i, t_i) of the arriving waves, the detected wave is the same in both cases as the denominators of both expressions are constants, the time dependence appearing only in the numerators.

c) Generalized Amplitude Modulation

The transmitter output in generalized amplitude modulation is $O[I(t)] = c\, h[I(t)]\, e^{i w_0 t} \Big/_{\text{Real Part}}$ 2.1-12

and after transmission under multipath conditions, the input to the receiver is

$$g(t) = \sum_{n=1}^{N} A_n h[I(t-t_n)] e^{i w_0 (t-t_n)} \Big/_{\text{Real Part}} \qquad 3.2\text{-}11$$

where by the treatment of section 3.1,

$$g(t) = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \cos\beta_{ij}}\; e^{i\left(w_0 t - \tan^{-1}\frac{\sum a_i \sin\beta_i}{\sum a_i \cos\beta_i}\right)} \qquad 3.2\text{-}12$$

where $a_i = A_i h[I(t-t_i)]$ 3.2-13

The detector output, E(t), is, by 2.1-15

$$h[U(t)] = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij} h[I(t-t_i)] h[I(t-t_j)]} - \text{ave}\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij} h[I(t-t_i)] h[I(t-t_j)]} \qquad 3.2\text{-}14$$

To calculate the value of the function, E(t), for small percent modulation, the function is expanded in a Taylor's Series, calculating the change in $h[U(t)]$ because of a small variation in I(t), letting

$$I(t) = I_0 + \delta(t) \qquad 3.2\text{-}15$$

where $|\delta(t)|_{max} << I_0$, $ave\ \delta(t)=0$. The result will yield the fluctuations of U(t) about a steady state value, U_0.

$$U(t) = U_0 + \left.\frac{\partial U}{\partial I}\right|_{I_0} \delta(t) \qquad 3.2\text{-}16$$

The value U_0 arises by a calculation when I(t) is assumed a constant. The first two terms of the Taylor Series are

$$h(U_0) + \left.\frac{\partial h}{\partial U}\right|_{U_0} \left.\frac{\partial U}{\partial I}\right|_{I_0} \delta(t) = \sqrt{\sum_{i=1}^{\tilde N}\sum_{j=1}^{\tilde N} A_i A_j \cos\beta_{ij}}\; h(I_0) + \frac{\sum_{i=1}^{\tilde N}\sum_{j=1}^{\tilde N} A_i A_j \cos\beta_{ij}\, h'(I_0)\,\delta(t-t_i)}{\sqrt{\sum_{i=1}^{\tilde N}\sum_{j=1}^{\tilde N} A_i A_j \cos\beta_{ij}}}$$

$$- ave\left\{\sqrt{\sum_i^{\tilde N}\sum_j^{\tilde N} A_i A_j \cos\beta_{ij}}\; h(I_0) + \frac{\sum_i^{\tilde N}\sum_j^{\tilde N} A_i A_j \cos\beta_{ij}\, h'(I_0)\,\delta(t-t_i)}{\sqrt{\sum_i^{\tilde N}\sum_j^{\tilde N} A_i A_j \cos\beta_{ij}}}\right\} \qquad 3.2\text{-}17$$

and

$$h(U_0) = \sqrt{\sum_{i=1}^{\tilde N}\sum_{j=1}^{\tilde N} A_i A_j \cos\beta_{ij}}\; h(I_0) - ave\left[\sqrt{\sum_{i=1}^{\tilde N}\sum_{j=1}^{\tilde N} A_i A_j \cos\beta_{ij}}\; h(I_0)\right] \simeq 0 \qquad 3.2\text{-}18$$

so that 2.1-16 becomes

$$E(t) = \left.\frac{\partial U}{\partial I}\right|_{I_0} \delta(t) = \left.\frac{\partial E}{\partial I}\right|_{I_0} \frac{\sum_{i=1}^{\tilde N}\sum_{j=1}^{\tilde N} A_i A_j \cos\beta_{ij}\, \delta(t-\tau_i)}{\sum_{i=1}^{\tilde N}\sum_{j=1}^{\tilde N} A_i A_j \cos\beta_{ij}} \qquad 3.2\text{-}19$$

since, as usual, $ave\ \delta(t) = 0$

Again the similarity of the forms 3.2-19, 3.2-4, 3.2-10 is striking and is in fact significant. It will be shown that even for reception of frequency modulated waves, the result takes the same form for small percentage modulation (small fluctuations of intelligence about a zero value).

3.3 Small Percentage Modulation Case for Frequency Modulation Reception

By 2.1-4, the transmitted wave in the case of frequency modulated waves is

$$O[I(t)] = ce^{i[w_0 t + k\int I(t)dt]}\Big/_{\text{Real Part}} \qquad 2.1\text{-}4$$

where $\text{average } I(t) = 0$.

After multipath transmission, the received wave is

$$g(t) = \sum_{n=1}^{N} A_n e^{i[w_0(t-t_n) + k\int_0^t I(t-t_n)dt]} \qquad 3.3\text{-}1$$

which must be put into the form 2.1-2 by the method of section 3.1 so that the definition of the receiver operation in 2.1-5 may be applied. Letting

$$\Omega_{ij} = -w_0(t_i - t_j) + k\int_{t+t_j}^{t+t_i} I(t)dt = -\beta_{ij} + k\int_{t+t_j}^{t+t_i} I(t)dt \qquad 3.3\text{-}2$$

$$\Omega_i = -w_0 t_i + k\int_0^{t+t_i} I(t)dt \quad ,$$

$$g(t) = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\Omega_{ij}}\; e^{i\left(w_0 t + \tan^{-1}\frac{\sum_{i=1}^{N} A_i \sin\Omega_i}{\sum_{i=1}^{N} A_i \cos\Omega_i}\right)} \qquad 3.3\text{-}3$$

The detection process thus yields

$$E(t) = w_0 + \frac{d}{dt}\tan^{-1}\frac{\sum A_i \sin\Omega_i}{\sum A_i \cos\Omega_i} - \text{ave}\left[w_0 + \frac{d}{dt}\tan^{-1}\frac{\sum A_i \sin\Omega_i}{\sum A_i \cos\Omega_i}\right] \qquad 3.3\text{-}4$$

$$= \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\Omega_{ij}\, I(t-t_i)k}{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\Omega_{ij}} - \text{ave}\,\frac{\sum^{N}\sum^{N} A_i A_j \cos\Omega_{ij}\, I(t-t_i)k}{\sum^{N}\sum^{N} A_i A_j \cos\Omega_{ij}}$$

and for small frequency deviations,

$$\Omega_{ij} \approx \beta_{ij}$$

$$\text{for all } i, j \qquad \begin{matrix} i = 1, 2, \dots, N \\ j = 1, 2, \dots, N \end{matrix}$$

so by the usual assumption of ave I(t) = 0,

$$\mathcal{E}(t) = \frac{\sum^{N}\sum^{N} A_i A_j \cos\beta_{ij}\, I(t-t_i)}{\sum^{N}\sum^{N} A_i A_j \cos\beta_{ij}} \qquad 3.3\text{-}5$$

unless $\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij} = 0$. This result is the first term of a Taylor Expansion about the points $\Omega_{ij} = \beta_{ij}$.

The condition on the results of the two preceding sections that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij} \neq 0$$

is a stringent one. It implies that under unmodulated conditions, the receiver antenna would be placed at a point of no field. When modulated, the resultant wave is highly distorted. The detectors of phase and frequency modulation do not operate as ideal detectors under this condition. The expression for the output wave assumes a singular value under this condition but too the amplitude function of the resultant wave (3.3-3, 3.2-2) will have zeros at periodic intervals determined by the nature of the function I(t), where ave I(t) = 0. These zeros correspond to the "holes" in the envelope function of Corrington in his exposition of the two signal, sinusoidally modulated case of frequency modulation reception. The zeros in the envelope mean that insufficient signal is present to drive the limiters to saturation at periodic intervals resulting in the non-ideal detection. Experimentally it is recommended that the antenna of the receiver not be placed at such singular points in space (if they exist) to avoid this non-ideal detection.

The result of the last two sections is that, for small percentage modulation, the output waves for amplitude modulation, phase modulation, and frequency modulation, all take the same form. It may be suggested that the similarity of form is connected with the fact that under conditions of small percentage modulation each frequency spectrum has only three important lines- the carrier and two sidebands- although the side bands for different methods of modulation have different phase relationships to the carrier. This suggestion implies that perhaps the results of the multipath transmission of single-sideband modulated waves should not then take the same form as the result of the calculation above. The results for single sideband modulation are presented in section 3.6.

3.4 Small Percentage modulation Evaluated for Special Waveforms of Intelligence

It is of interest to calculate the reception to be observed under the conditions of small percent modulation discussed in the preceding sections. The common form of equations 3.3-5, 3.2-10 and 3.2-4 guarantee that the results of this section will hold for the three fundamental types of modulation.

a) Pulse Modulation

When the intelligence is a pulse we write the function I(t) in the form

$$I(t) = \begin{cases} 0 & t \neq n\tau_0 + c \\ 1 & t = n\tau_0 + c \end{cases}$$

where τ_0 is the repetition rate of the intelligence pulses. It will be shown that the assumption of zero width pulses is not essential to the results of the calculation.

By 3.3-5 the reception will be zero except at such times t_i when some one of incoming waves will bring the modulated pulse to the receiver. At these times the reception is

$$h_q = \mathcal{E}(t_q) = k\frac{\sum_i^N\sum_j^N A_iA_j\cos\beta_{ij}\,\delta_q^i}{\sum_i\sum_j A_iA_j\cos\beta_{ij}} = kA_q\frac{\sum^N A_i\cos\beta_{iq}}{\sum_{i=1}^N\sum_{j=1}^N A_iA_j\cos\beta_{ij}} \qquad \left(\delta_q^i = 0 \; i\neq q,\; = 1 \; i=q\right) \qquad 3.4\text{-}1$$

Thus the reception over a period of the modulation, τ_0, will consist of N pulses of various amplitudes given above. If the receiver output is placed upon an oscilloscope screen the result is a set of N pips whose amplitudes, h_q are given by

$$\frac{h_q}{\sum_{q=1}^N h_q} = \frac{A_q\sum_{i=1}^N A_i\cos\beta_{iq}}{\sum_{i=1}^N\sum_{j=1}^N A_iA_j\cos\beta_{ij}} \qquad q=1,2,\cdots,N \qquad 3.4\text{-}2$$

normalized in the appropriate system of units.

The system of equations, 3.4-1, homogeneous in the variables A_i, have a compatibility condition arrived at by adding all the equations. Whence

$$\sum_{q=1}^N h_q = k \qquad 3.4\text{-}3$$

Thus the sum of the amplitudes of the N pips is exactly equal to the change in amplitude, phase or frequency of the single pulse at the transmitter corresponding to the method of modulation.

The system of equations 3.4-1 holds for pulses of any shape as long as

1) the amplitude of the intelligence pulse is small enough to allow $k|I|_{max} \ll 1$ for phase and amplitude modulated pulses,

2) the area under the pulse integrated over one pulse is small enough so that $k\int_{\text{one pulse}} I(t)dt \ll 1$ for frequency modulated pulses and

3) the width of the pulse in time units is smaller than

the smallest of the times $(t_p - t_{p-1})$ where the incoming signals have been ordered in accordance with the time of their arrival. This avoids "overlap" of the incoming pulses in the various waves.

Thus the system of equations 3.4-1 holds for pulses of any shape as under the conditions above the condition of "no overlap" guarantees that the disturbance, if any exists, will be due to one incoming wave alone. So that the output will look as though N pulses were received at the times t_q, each of identical shape but whose relative amplitudes are in the ratio of the numbers, h_q.

A convenient notation for the system of equations 3.4-1 is given by plotting a set of N vectors in a plane whose amplitudes are given by the A_i and whose arguments measured from some arbitrary axis are given by the β_i . Let the vectors be called $\underset{\sim}{A}_i$. These are meant to be true vectors, not the artificial vectors introduced in section 3.1 to represent oscillating electrical quantities. Let $\underset{\sim}{R}$ be the vector sum of the $\underset{\sim}{A}_i$.

$$\underset{\sim}{R} = \sum_{i=1}^{N} \underset{\sim}{A}_i \qquad 3.4\text{-}4$$

Then

$$h_q = k \frac{\underset{\sim}{A}_q \cdot \underset{\sim}{R}}{\underset{\sim}{R} \cdot \underset{\sim}{R}} \qquad 3.4\text{-}5$$

b) <u>A Superposition Theorem</u>

From the results of the preceding sections exemplified by 3.3-5

$$E(t) = k \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij}\, I(t-t_i)}{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij}} \qquad 3.3\text{-}5$$

which becomes by 3.4-1

$$\mathcal{E}(t) = \sum_{i=1}^{\tilde{n}} h_i I(t-t_i) \qquad 3.4\text{-}5$$

Thus by this superposition theorem, the reception at a time, t, is given by the sum of the pip heights corresponding to <u>each</u> wave, had the transmitter been pulse modulated with deviation corresponding to $I(t-t_i)$.

By this theorem it is now possible to predict the reception for any wave shape of the intelligence function I(t) merely on the basis of the knowledge of the nature of the reception when pulse modulated. For a calculation of the result expected when the modulation is a square wave, triangular wave and pulse wave with possibility of overlap in the reception see figure 7.

From the figure the significance of the theorem 3.4-5 is readily seen. The calculations are made under the assumption that $k|I|_{max} \ll 1$ for amplitude modulation and phase modulation and that $k\int I(t)dt \ll 1$ for frequency modulation. No assumption need now be made about "overlap".

c) <u>Sinusoidal Modulation</u>

It is seen, from either 3.3-5 or 3.4-5 that when I(t) is a sinusoid the detected output is a sinusoid, undistorted with a phase shift and amplitude change introduced.

d) <u>Region of Validity of Superposition Theorem</u>

The superposition theorem holds for frequency modulated waves as long as $\cos\Omega_{ij} \approx \cos\beta_{ij}$ and $\sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij} \neq 0$.

I.e. the condition of validity is $\Omega_{ij} \approx \beta_{ij}$ 3.4-6

Now $\Omega_i(t) = \omega_0(t-t_i) + f(t-t_i)$

and $\Omega_{ij}(t) = \omega_0(t-t_j) + f(t-t_i) - f(t-t_j)$

$$= \beta_{ij} + f(t-t_i) - f(t-t_j)$$

Hence for 3.4-6 to hold, $|f(t-t_i) - f(t-t_j)| << 1$

By the Mean Value Theorem, for f(t) single valued, continuous first derivative, $|f(t-t_i) - f(t-t_j)| \leq |t_{ij}||f'(t_0)|$, $t_i < t_0 < t_j$

Hence a sufficient condition for the applicability of the superposition theorem is $|f'(t_0)||t_{ij}| \ll 1$ for all t_{ij}. For frequency modulation this result becomes $k|I|_{max} t_{N1} << 1$. For phase modulation the result of the sufficiency is $k|I'|_{max} t_{N1} << 1$. For amplitude modulation the condition of applicability of the superposition theorem is that $k|I|_{max} << 1$.

The order of magnitude of these limitations is given by the following figures. For signals starting simultaneously at the transmitter over the various paths with times of arrival between first and last signals of the order of ten microseconds, $k|I|_{max} \approx \frac{5 \times 10^{-2}}{10^{-5}} = 5 \times 10^3$ rad/sec , allowing maximum frequency deviations of the order of 10^3 cycles/sec. . The same order of magnitude of frequency deviation holds for the frequency and phase modulated cases, where the frequency deviation of a wave transmitted of the form

$$O[I(t)] = A e^{i(\omega_0 t + g(t))}$$

is defined as $\left|\frac{dg}{dt}\right|_{max}$. For amplitude modulation the superposition theorem limitation is a maximum modulation of the order of 10% where the percentage modulation of a wave of the form of 2.1-1 is given by 100kI(t).

When pulse modulated, the conditions for the validity of 3.3-5 et al. can be relaxed to, say for frequency modulation, the condition that the area under the pulse is small. I.e. $k \int_{\text{one cycle}} I(t)\,dt << 1$

If the pulses are 1/2 microsecond in width, the maximum allowable frequency deviation is of the order of 1.5×10^4 cycles/sec. under conditions of "no overlap".

3.5 <u>Large Percentage Modulation for Continuous Wave Systems</u>

The expansions 3.2-3, 3.2-8, and 3.3-4 summarize the expected detector output as functions of time for given I(t) for the three basic types of modulation: amplitude, phase and frequency modulation. Simple methods have been advanced for calculation of the expected output wave shape for given wave shape of the I(t) and for specific values of the A_i and t_i.

An attempt at a determination of the superiority of one system over another fails at the small percentage modulation stage as the results obtained, under suitable limitations, are identical for all three systems. Further comparison demands an investigation of the Taylor's Series for each type of modulation beyond the first term. These higher order calculations are now performed.

For amplitude modulation multipath reception the Taylor's Expansion of which 3.2-4 is a part is

$$\mathcal{E}(t) = \frac{k\sum_i^N\sum_j^N A_iA_j\cos\beta_{ij}I(t-t_i)}{\left(\sum_i^N\sum_j^N A_iA_j\cos\beta_{ij}\right)^{1/2}} + \frac{k^2}{2!}\frac{\sum_{i=1}^N\sum_{j=1}^N\sum_{k=1}^N\sum_{l=1}^N A_iA_jA_kA_l\sin\beta_{jl}\sin\beta_{ik}I(t-t_i)I(t-t_j)}{\left(\sum_i^N\sum_j^N A_iA_j\cos\beta_{ij}\right)^{3/2}} \quad 3.5\text{-}1$$

$$-\frac{3k^2}{3!}\frac{\sum^N\sum^N\sum^N\sum^N\sum^N\sum^N A_iA_jA_kA_lA_mA_n\sin\beta_{im}\sin\beta_{jn}\cos\beta_{kl}I(t-t_l)I(t-t_m)I(t-t_n)}{\left(\sum_i^N\sum_j^N A_iA_j\cos\beta_{ij}\right)^{5/2}} + \cdots$$

For phase modulation multipath reception the Taylor's Expansion to two terms is

$$\mathcal{E}(t) = k\frac{\sum_{i=1}^N\sum_{j=1}^N A_iA_j\cos\beta_{ij}I(t-t_i)}{\sum_{i=1}^N\sum_{j=1}^N A_iA_j\cos\beta_{ij}} + \frac{k^2}{2}\frac{\sum_{i=1}^N\sum_{j=1}^N\sum_{k=1}^N\sum_{l=1}^N A_iA_jA_kA_l\cos\beta_{ij}\sin\beta_{kl}\left[I^2(t-t_l)+2I(t-t_k)I(t-t_l)\right]}{\left(\sum_{i=1}^N\sum_{j=1}^N A_iA_j\cos\beta_{ij}\right)^2} + \cdots \quad 3.5\text{-}2$$

For frequency modulation multipath reception, the Taylor's Expansion again to two terms is

$$\mathcal{E}(t) = k\frac{\sum_{i=1}^N\sum_{j=1}^N A_iA_j\cos\beta_{ij}I(t-t_i)}{\sum_{i=1}^N\sum_{j=1}^N A_iA_j\cos\beta_{ij}} + \frac{k\sum_{i=1}^N\sum_{j=1}^N\sum_{k=1}^N\sum_{l=1}^N A_iA_jA_kA_l\cos\beta_{ij}\sin\beta_{kl}\left[I(t-t_k)+I(t-t_l)-2I(t-t_i)\right]\cdot\left[\int I(t-t_l)d\tau\right]}{\left(\sum_i^N\sum_j^N A_iA_j\cos\beta_{ij}\right)^2} + \cdots \quad 3.5\text{-}3$$

When sinusoidally modulated, the first order terms yield only the fundamental while the higher order terms in the expansion yield the observable distortion. The order of the highest harmonic produced in the detection is equal to the number of terms used in the expansion. Thus when only two terms in the expansions are considered, the fundamental and a second harmonic are produced. This result affords a criterion for the comparison of the various systems of continuous wave modulation thus far considered.

The criterion for superiority among the various systems of continuous wave modulation will be the relative amounts of second harmonic produced by the various systems of modulation-detection when sinusoidally modulated, two terms considered in each expansion when the number of interfering signals is limited to two (N = 2). For

$$I(t) = \sin w_s t$$

the output is of the form

$$\varepsilon(t) = b\cos(\omega_s t+\beta) + c\cos 2(\omega_s t+\gamma) \qquad 3.5\text{-}4$$

The criterion for the comparison of the various systems becomes a comparison of the magnitudes of the ratios c/b.

For amplitude modulation

$$c/b = \frac{c_0 k}{2}\sqrt{\sum_i^N\sum_j^N\sum_k^N\sum_l^N\sum_m^N\sum_n^N\sum_o^N\sum_p^N A_iA_jA_kA_lA_mA_nA_oA_p \sin\beta_{jl}\sin\beta_{np}\sin\beta_{ik}\sin\beta_{mo}\cos(\alpha_{im}+\alpha_{jn})} \qquad 3.5\text{-}5$$

For phase modulation,

$$c/b = \frac{c_0 k}{2}\sqrt{\sum_i^N\sum_j^N\sum_k^N\sum_l^N\sum_m^N\sum_n^N\sum_o^N\sum_p^N A_iA_jA_kA_lA_mA_nA_oA_p \cos\beta_{ij}\cos\beta_{mn}\sin\beta_{kl}\sin\beta_{op}\cdot\left\{\cos(\alpha_{im}+\alpha_{lp})+\tfrac{1}{4}\cos 2\alpha_{lp}-\tfrac{1}{2}\cos(\alpha_{ml}+\alpha_{pl})-\tfrac{1}{2}\cos(\alpha_{ip}+\alpha_{lp})\right\}} \qquad 3.5\text{-}6$$

For frequency modulation,

$$c/b = \frac{c_0 k}{\omega_s}\sqrt{\sum_{i=1}^N\sum_j^N\sum_k^N\sum_l^N\sum_m^N\sum_n^N\sum_o^N\sum_p^N A_iA_jA_kA_lA_mA_nA_oA_p \cos\beta_{ij}\cos\beta_{mn}\sin\beta_{kl}\sin\beta_{op}\cdot\left\{\cos(\alpha_{im}+\alpha_{lp})+\tfrac{1}{4}\cos 2\alpha_{lp}-\cos(\alpha_{ml}+\alpha_{pl})-\cos(\alpha_{km}+\alpha_{lp})+\tfrac{1}{2}\cos(\alpha_{ol}+\alpha_{pl})+\tfrac{1}{4}\cos(\alpha_{ko}+\alpha_{lp})\right\}} \qquad 3.5\text{-}7$$

where c_o is a constant common to all systems and where

$$\alpha_i = \omega_s t_i$$

$$\alpha_{ij} = \alpha_i - \alpha_j$$

For N = 2 these summations reduce to

$$c/b = \frac{c_0 k}{2}(1-\cos\alpha)\sin^2\beta A_1^2 A_2^2 = \frac{c_0 k}{2}A_1^4\rho^2\sin^2\beta(1-\cos\alpha)\,; \quad \rho = \frac{A_2}{A_1} < 1 \qquad 3.5\text{-}8$$

for amplitude modulation

$$c/b = \frac{c_0 k}{2}(1-\cos\alpha)\sin\beta A_1A_2(A_1^2-A_2^2) = \frac{c_0 k}{2}A_1^4\rho(1-\rho^2)\sin\beta(1-\cos\alpha) \qquad 3.5\text{-}9$$

for phase modulation

$$c/b = \frac{c_0 k}{\omega_s}(1-\cos\alpha)\sin\beta A_1A_2(A_1^2-A_2^2) = \frac{c_0 k}{\omega_s}A_1^4\rho(1-\rho^2)\sin\beta(1-\cos\alpha) \qquad 3.5\text{-}10$$

for frequency modulation. The dimensionality of the constants k used in these expressions prevents a conclusion that frequency modulation is far and above the best system of modulation considered. Indeed an examination of the conclusions on page

52 for the sinusoidal intelligence function show the allowable values of k for frequency and phase modulation differ precisely by a factor w_s to permit of the same order of approximation.

A comparison of the three distortion ratios, c/b, shows that in most of the cases of reception, transmission by amplitude modulation is superior to transmission between the same two points by either frequency or phase modulation. From 3.5-9 and 3.5-10 it is seen that the relative percentages of second harmonic are one-half smaller for phase modulation compared with frequency modulation and hence phase modulation, under this criterion, is always more free of distortion than frequency modulation than frequency modulation at the same carrier frequency. For most cases of transmission

$$(c/b)_{AM} < (c/b)_{PM} < (c/b)_{FM}$$

Indeed only for

$$\rho > \frac{\sqrt{\sin^2\beta + 4} - |\sin\beta|}{2} \geq .618$$

is phase modulation superior to amplitude modulation and only for

$$\rho > \frac{\sqrt{\sin^2\beta + 16} - |\sin\beta|}{4} \geq .781$$

is frequency modulation preferable to phase modulation.

If the point (ρ, β), plotted in the plane of figure 8, lies between the two curves, it is advantageous to transmit in the order of preference phase, amplitude, and frequency modulation. If the point lies above the upper curve it is advantageous to transmit, in the order of preference, phase, frequency, and amplitude modulation. If the point (ρ, β)

lies in the unshaded area, the order of preference is amplitude, phase, and frequency modulation. It is estimated that the allowable amplitude modulation for the criterion of negligible third harmonic for sinusoidal modulation is 25 percent. For frequency modulation transmission the maximum allowable frequency deviation for the comparison to hold is 1/4 the modulating frequency. The conclusions of this section compare favorably with the results of experiments performed with practical systems of modulation[16] that frequency modulation yields larger distortions due to multipath transmission than amplitude modulation.

The case of an intelligence function which is a pulse is also easily calculable in the case where several terms are considered in the Taylor's expansion. For amplitude modulation, the terms at the times t_q, when some one of the incoming waves brings a modulated pulse to the receiver become

$$\varepsilon(t_q) = \frac{k \sum_i \sum_j A_i A_j \cos\beta_{ij} \delta_q^i}{\sqrt{\sum_i \sum_j A_i A_j \cos\beta_{ij}}} + \frac{k^2}{2} \frac{\sum_i \sum_j \sum_k \sum_l A_i A_j A_k A_l \sin\beta_{jl} \sin\beta_{im} \delta_q^i \delta_q^j}{\left(\sum_{i=1}^{N} \sum_{j=1}^{N} A_i A_j \cos\beta_{ij}\right)^{3/2}} \qquad 3.5\text{-}11$$

$$- \frac{k^3}{2} \frac{\sum_i \sum_j \sum_k \sum_l \sum_m \sum_n A_i A_j A_k A_l A_m A_n \sin\beta_{im} \sin\beta_{jn} \cos\beta_{kl} \delta_q^l \delta_q^m \delta_q^n}{\left(\sum_i \sum_j A_i A_j \cos\beta_{ij}\right)^{5/2}} + \cdots$$

For phase modulation the result is

$$\varepsilon(t_q) = \frac{k \sum_i \sum_j A_i A_j \cos\beta_{ij} \delta_q^i}{\sum_i \sum_j A_i A_j \cos\beta_{ij}} + \frac{k^2}{2} \frac{\sum_i \sum_j \sum_k \sum_l A_i A_j A_k A_l \left\{\cos\beta_{ij} \sin\beta_{kl} \delta_q^l + 2\cos\beta_{il} \sin\beta_{kj} \delta_q^k\right\} \delta_q^l}{\left(\sum_i \sum_j A_i A_j \cos\beta_{ij}\right)^2} + \cdots \qquad 3.5\text{-}12$$

For frequency modulation the result is

$$\varepsilon(t_q) = \frac{k \sum_{i=1}^{N} \sum_{j=1}^{N} A_i A_j \cos\beta_{ij} \delta_q^i}{\sum_{i=1}^{N} \sum_{j=1}^{N} A_i A_j \cos\beta_{ij}} + \frac{k^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} A_i A_j A_k A_l \cos\beta_{ij} \sin\beta_{kl} \left(\delta_q^k + \delta_q^l - 2\delta_q^i\right) c}{\left(\sum_{i=1}^{N} \sum_{j=1}^{N} A_i A_j \cos\beta_{ij}\right)^2} + \cdots \qquad 3.5\text{-}13$$

where $\delta_j^i = \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} \begin{matrix} i \neq j \\ i = j \end{matrix}$

In the last expression the assumption is still made of small area, c, under the pulse. These expressions reduce to

$$\mathcal{E}(t_q)=\frac{kA_q\sum_{i=1}^{N}A_i\cos\beta_{iq}}{\left(\sum_{i=1}^{N}\sum_{j=1}^{N}A_iA_j\cos\beta_{ij}\right)^{1/2}}+\frac{k^2}{2}\,\frac{A_q^2\sum_{i=1}^{N}\sum_{j=1}^{N}A_iA_j\sin\beta_{iq}\sin\beta_{jq}}{\left(\sum_{i=1}^{N}\sum_{j=1}^{N}A_iA_j\cos\beta_{ij}\right)^{3/2}}-\frac{k^3}{2}\,\frac{A_q^2\sum_{i}\sum_{j}\sum_{k}A_iA_jA_k\sin\beta_{iq}\sin\beta_{jq}\cos\beta_{kq}}{\left(\sum_{i=1}^{N}\sum_{j=1}^{N}A_iA_j\cos\beta_{ij}\right)^{5/2}}+\cdots \qquad 3.5\text{-}14$$

for amplitude modulation

$$\mathcal{E}(t_q)=\frac{kA_q\sum_{i=1}^{N}A_i\cos\beta_{iq}}{\sum_{i=1}^{N}\sum_{j=1}^{N}A_iA_j\cos\beta_{ij}}+\frac{k^2}{2}\,\frac{2A_q^2\sum_{i=1}^{N}\sum_{j=1}^{N}A_iA_j\sin\beta_{jq}\sin\beta_{iq}+A_q\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}A_iA_jA_k\cos\beta_{ij}\sin\beta_{kq}}{\left(\sum_{i=1}^{N}\sum_{j=1}^{N}A_iA_j\cos\beta_{ij}\right)^2}+\cdots \qquad 3.5\text{-}15$$

for phase modulation while for frequency modulation the result is

$$\mathcal{E}(t_q)=\frac{kA_q\sum_{i=1}^{N}A_i\cos\beta_{iq}}{\sum_{i=1}^{N}\sum_{j=1}^{N}A_iA_j\cos\beta_{ij}}-2k^2cA_q\,\frac{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=6}^{N}A_iA_jA_k\cos\beta_{qk}\sin\beta_{ij}}{\left(\sum_{i=1}^{N}\sum_{j=1}^{N}A_iA_j\cos\beta_{ij}\right)^2}+\cdots \qquad 3.5\text{-}16$$

In terms of the vectors $\underset{\sim}{R}$, $\underset{\sim}{A}_i$, introduced in section 3.4 the pulses received become

$$\mathcal{E}(t_q)=k\,\frac{\underset{\sim}{A}_q\cdot\underset{\sim}{R}}{|R|}+\frac{k^2}{2}\,\frac{(\underset{\sim}{A}_q\times\underset{\sim}{R})\cdot(\underset{\sim}{A}_q\times\underset{\sim}{R})}{|R|^3}-\frac{k^3}{2}\,\frac{(\underset{\sim}{A}_q\times\underset{\sim}{R})\cdot(\underset{\sim}{A}_q\times\underset{\sim}{R})(\underset{\sim}{A}_q\cdot\underset{\sim}{R})}{|R|^5}+\cdots \qquad 3.5\text{-}17$$

$$\mathcal{E}(t_q)=k\,\frac{\underset{\sim}{A}_q\cdot\underset{\sim}{R}}{|R|^2}+\frac{k^2}{2}\,\frac{(2\underset{\sim}{A}_q\cdot\underset{\sim}{R}-R^2)}{|R|^4}\sqrt{|A_q|^2|R|^2-(\underset{\sim}{A}_q\cdot\underset{\sim}{R})^2}+\cdots \qquad 3.5\text{-}18$$

$$\mathcal{E}(t_q)=k\,\frac{\underset{\sim}{A}_q\cdot\underset{\sim}{R}}{|R|^2}+0k^2+\cdots \qquad 3.5\text{-}19$$

for amplitude, phase and frequency modulation respectively.

3.6 Multipath Transmission of Single Side Band Amplitude Modulated Systems (SSBAM)

The multipath reception of SSBAM is reducible to the condition that the detected output is in the form 2.1-11 if I(t) is in the form 2.1-9 as the input to the receiver can be placed in the form of equation 2.1-10. The input to the receiver is by 1.2-2

$$g(t)=k\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}A_i\left\{a_n\cos(w_0+nw)(t-t_i)+b_n\sin(w_0+nw)(t-t_i)\right\} \qquad 3.6\text{-}1$$

for the case where the lower half frequency spectrum and the carrier are suppressed. Writing 3.6-1 in the form 2.1-10 by the method of 3.1,

$$g(t) = k\sum_{n=1}^{\infty} \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos(\omega_0+n\omega)(t_i-t_j)} \begin{Bmatrix} a_n \cos[(\omega_0+n\omega)t - \phi_n] \\ + b_n \sin[(\omega_0+n\omega)t - \phi_n] \end{Bmatrix} \qquad 3.6\text{-}2$$

where

$$\phi_n = \tan^{-1} \frac{\sum_{i=1}^{N} A_i \sin(\omega_0+n\omega)t_i}{\sum_{i=1}^{N} A_i \cos(\omega_0+n\omega)t_i}$$

and the u of expression 2.1-11 is

$$u = \tan^{-1} \frac{\sum_{i=1}^{N} A_i \sin \omega_0 t_i}{\sum_{i=1}^{N} A_i \cos \omega_0 t_i} \qquad 3.6\text{-}3$$

Defining

$$\gamma_i = \omega t_i, \qquad \gamma_{ij} = \omega(t_i - t_j) = \gamma_i - \gamma_j$$

and in the usual fashion

$$\beta_i = \omega_0 t_i, \qquad \beta_{ij} = \omega(t_i - t_j) = \beta_i - \beta_j$$

the output from the detector is

$$\mathcal{E}(t) = k\sum_{n=1}^{\infty} \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos(\beta_{ij}+n\gamma_{ij})} \begin{Bmatrix} a_n \cos(n\omega t - v) \\ + b_n \sin(n\omega t - v) \end{Bmatrix} \qquad 3.6\text{-}4$$

since

$$v = \tan^{-1} \frac{\sum_{i=1}^{N} A_i \sin(\beta_i + n\gamma_i)}{\sum_{i=1}^{N} A_i \cos(\beta_i + n\gamma_i)} - \tan^{-1} \frac{\sum_{i=1}^{N} A_i \sin\beta_i}{\sum_{i=1}^{N} A_i \cos\beta_i} = \tan^{-1} \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \sin(\beta_{ij}+n\gamma_i)}{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos(\beta_{ij}+n\gamma_j)}$$

For N = 1, it is seen that the reception reduces to

$$\mathcal{E}(t) = kA\left\{\sum_{n=1}^{\infty} a_n \cos n(\omega t - \gamma) + \sum_{n=1}^{\infty} b_n \sin n(\omega t - \gamma)\right\}$$

or undistorted reception.

For a sinusoidal intelligence function, all the coefficients $a_n = b_n = 0$ for $n > 1$, and no distortion is present. A single line in the frequency spectrum is transmitted and although a phase shift may be present, no higher order harmonics are present.

In the more general case, it is seen that the various harmonics have suffered changes in their relative amplitudes, and relative phase shifts, causing the distortion although no new harmonics have been introduced. Suppose several signals arrive with negligible γ_{ij} but with not necessarily small β_{ij} (possible since the modulating frequency is in general very much smaller than the carrier frequency). Writing 3.6-4 in the form

$$E(t) = \sum_{n=1}^{\infty} \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos(\beta_{ij} + n\gamma_{ij})} \left\{ \begin{array}{l} \cos\theta_n [a_n \cos n\omega t + b_n \sin n\omega t] \\ + \sin\theta_n [a_n \sin n\omega t - b_n \cos n\omega t] \end{array} \right\}$$

where $\theta_n = \tan^{-1} \dfrac{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \sin(\beta_{ij} + n\gamma_{ij})}{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos(\beta_{ij} + n\gamma_{ij})}$

For $n\gamma_{ij} \ll 1$ and calling $\gamma_1 = 0$ and $n\gamma_i \ll 1$

then $\theta_n \approx 0$ and

$$E(t) = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij}} \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

an undistorted output.

3.7 Multipath Transmission of Frequency Modulated Frequency Modulated Systems

As an example of a case of frequency division multiplex, we consider frequency modulated frequency modulation. For the purposes of calculation consider a single channel modulation case. Rewriting the notation of 2.1-16 calling

$$p(t) = \cos[\omega_1 t + k\int I(t)dt]$$
$$X(t) = k\int I(t)dt \qquad 3.7\text{-}1$$
$$\Omega(t) = K\int p(t)dt + \omega_0 t$$

The received wave is

$$\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\Omega_{ij}}\; e^{i\left(\omega_0 t + \tan^{-1} \frac{\sum_{i=1}^{N} A_i \sin\Omega_i}{\sum_{i=1}^{N} A_i \cos\Omega_i}\right)} \qquad 3.7\text{-}2$$

which after a first detection yields output

$$\mathcal{E}_1(t) = \frac{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\Omega_{ij}\, p(t-t_j)}{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\Omega_{ij}} - \text{ave}\,\frac{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\Omega_{ij}\, p(t-t_j)}{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\Omega_{ij}} \qquad 3.7\text{-}3$$

An expression for the output of the second discriminator becomes now quite difficult to write.

Suppose the function I(t) is periodic with frequency ω_2. Then a frequency analysis of 3.7-3 would show various ordered spectra centered at frequencies $n\omega_1$ with lines separated by frequencies ω_2. The detection process by selective tuned circuits takes the first order spectrum, centered at ω_1, and performs a frequency modulation detection of the ordinary type upon it. This first order spectrum is difficult to calculate because of the interaction terms between the components of the function 3.7-3.

For both K, k, small, the results of a modulation is here calculated. The output of the first detector is

$$\mathcal{E}_1(t) = \frac{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}\, p(t-t_j)}{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}} - \text{ave}\,\frac{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}\, p(t-t_j)}{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}} \qquad 3.7\text{-}4$$

By the process described in section 3.1 this is written in the form

$$\mathcal{E}_1(t) = \sqrt{\left(\frac{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}\cos\chi_j}{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}}\right)^2 + \left(\frac{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}\sin\chi_j}{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}}\right)^2}\; e^{\,i\left(\omega_1 t + \tan^{-1}\frac{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}\sin\chi_j}{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}} A_i A_j \cos\beta_{ij}\cos\chi_j}\right)} \qquad 3.7\text{-}5$$

Thus the second detector output is

$$\mathcal{E}(t) = k\,\frac{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}}\sum\limits_{k=1}^{\tilde{n}}\sum\limits_{l=1}^{\tilde{n}} A_i A_j A_k A_l \cos\beta_{ij}\cos\beta_{kl}\cos\gamma_{jl}\, I(t-t_j)}{\sum\limits_{i=1}^{\tilde{n}}\sum\limits_{j=1}^{\tilde{n}}\sum\limits_{k=1}^{\tilde{n}}\sum\limits_{l=1}^{\tilde{n}} A_i A_j A_k A_l \cos\beta_{ij}\cos\beta_{kl}\cos\gamma_{jl}} \qquad 3.7\text{-}6$$

where

$$\beta_{ij} = \omega_0(t_i - t_j), \quad \gamma_{ij} = \omega_1(t_i - t_j)$$

When pulse modulated the amplitudes of the N pips observed are

$$h(t_g) = k\, A_g \frac{\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} A_i A_k A_l \cos\beta_{ig} \cos\beta_{kl} \cos\gamma_{gl}}{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} A_i A_j A_k A_l \cos\beta_{ij} \cos\beta_{kl} \cos\gamma_{jl}} \qquad 3.7\text{-}7$$

It is apparent that, as in 3.4-3,

$$\sum_{g=1}^{N} h(t_g) = k \qquad 3.7\text{-}8$$

and the superposition theorem 3.4-5 holds in this case as well.

$$\mathcal{E}(t) = \sum_{i=1}^{N} h(t_i)\, I(t-t_i) \qquad 3.7\text{-}9$$

where the h_i's are different in the present case from the values in the cases of amplitude, phase and frequency modulation.

If the analogy is carried a little farther, comparing the second order modulation system with first order systems, keeping K small, the first two terms in the Taylor's Expansion are

$$\mathcal{E}(t) = k \frac{\sum_i^N\sum_j^N\sum_k^N\sum_l^N A_i A_j A_k A_l \cos\beta_{ij} \cos\beta_{kl} \cos\gamma_{jl}\, I(t-t_j)}{\sum^N\sum^N\sum^N\sum^N A_i A_j A_k A_l \cos\beta_{ij} \cos\beta_{kl} \cos\gamma_{jl}}$$
$$+ k^2 \frac{\sum^N\sum^N\sum^N\sum^N\sum^N\sum^N\sum^N\sum^N A_i A_j A_k A_l A_m A_n A_o A_p \cos\beta_{ij} \cos\beta_{kl} \cos\beta_{mn} \cos\beta_{op} \cos\gamma_{jl} \cos\gamma_{np} \cdot \left[I(t-t_n) + I(t-t_p) - 2I(t-t_j) \right] \cdot \int I(t-t_p)\,dt}{\left[\sum^N\sum^N\sum^N\sum^N A_i A_j A_k A_l \cos\beta_{ij} \cos\beta_{kl} \cos\gamma_{jl}\right]^2} \qquad 3.7\text{-}10$$

It is noted that these equations are obtained from the analogous frequency modulated case by letting

$$A_i \longrightarrow A_i \sum_{j=1}^{N} A_j \cdot \cos\beta_{ij}$$

$$\beta_i \longrightarrow \gamma_i$$

A calculation of the amount of second harmonic present when sinusoidally modulated, for purposes of comparison with the systems already considered, yields, on the basis of the assumptions on page 54 and the substitution indicated above, from 3.5-10 3.7-11

$$c/b\Big|_{FM\,of\,FM} = \frac{c_o' k}{\omega_s}(1-\cos\alpha)\sin\gamma A_1A_2\left[A_1+A_2\cos\beta\right]\left[A_2+A_1\cos\beta\right]\left\{\left[A_1^2+A_1A_2\cos\beta\right]^2 + \left[A_2^2+A_1A_2\cos\beta\right]^2\right\}$$

where c_o in 3.5-10 was

$$c_o^{-1} = \sum_{i=1}^{2}\sum_{j=1}^{2} A_iA_j\cos\beta_{ij}\sqrt{\sum\sum\sum\sum A_iA_jA_kA_\ell\cos\beta_{ij}\cos\beta_{k\ell}\cos\alpha_{ik}} \qquad 3.7\text{-}12$$

and in the present case c_o' is

$$c_o'^{-1} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{\ell=1}^{2} A_iA_jA_kA_\ell\cos\beta_{ij}\cos\beta_{k\ell}\cos\gamma_{ik}\sqrt{\sum\sum\sum\sum\sum\sum\sum\sum A_iA_jA_kA_\ell A_mA_nA_oA_p\cdot\cos\beta_{ij}\cos\beta_{k\ell}\cos\beta_{mn}\cos\beta_{op}\cos\gamma_{ik}\cos\gamma_{mo}\cos\alpha_{im}} \qquad 3.7\text{-}13$$

No comparison with small signal amplitude, phase and frequency modulation is made as the criterion for comparison is difficult by virtue of the entrance of modulating frequency, carrier and subcarrier phase shifts (α, β, γ) into the expression 3.7-10 one term of which is not contained in 3.5-8. Hence the frequencies to be chosen for the ω_0, ω_1 in frequency modulation of frequency modulation for comparison with the ω_0 of frequency, phase or amplitude modulation is in doubt.

For K slightly larger than in the above considerations, the first two terms in the Taylor's Expansion must be considered, recombined according to the vector rule of section 3.1, and expanded as above, a calculation which will not be made here.

3.8 <u>100% Modulated Pulses</u>

When amplitude modulated pulses are 100% modulated, differing from the type of small percentage modulation considered previously, the calculation is not difficult.

These pulses are exemplified by their use in radar. If no "overlap" in the reception is observed, the reception evidently is a series of pulses at the times t_q whose amplitudes are

$$\mathcal{E}(t_q) = A_q \qquad 3.8-1$$

since, at the time t_q, only the energy due to the qth wave is arriving and no interference is observed with the other waves.

IV. Experimental Interpretations

4.1 The Inverse Problem

As noted in previous sections the calculation of the expected wave form at a particular point in space depends upon a knowledge of the transmission characteristics A_i and t_i for all the arriving waves for a given type of intelligence function and for a given type of modulation-detection system. Thus if the quantities A_i, t_i were known the formulas above would be useful. How can the transmission characteristics be measured?

Certainly the A_i are given by the results of section 3.8 where 100 percent modulated Amplitude Modulated Pulses were transmitted and detected. But can the process give the relative times of arrival t_i as well? Theoretically, yes, but the question of resolution in the receiver arises as a practical objection. It is noted that in all the formulas trigonometric functions of the β_{ij} are taken requiring a knowledge of the β_{ij} to an accuracy of high degree in the quantity δ_{ij} appearing in the expression

$$\beta_{ij} = 2\pi n_{ij} + \delta_{ij}$$

where n_{ij} is an integer such that $0 \leq \delta_{ij} < 2\pi$. For reasonable accuracy, the $\beta_{ij} - 2\pi n_{ij}$ should be calculated to say 5% accuracy or the δ_{ij} to $\pm .05$ radian. At a carrier frequency of $\omega_0 = 2\pi \times 10^8 \text{ sec}^{-1}$ the δ_{ij} imply knowledge of the path difference to an integral number of wave-lengths plus an excess known to about 2.5 cm. In practical transmission over many kilometers between transmitter and receiver it is perhaps too much to expect of the radar receiver to detect path differences of this order. Further, taking into account the finite pulse size, the resolution on the radar screen would be impractical, namely the n_{ij} could be estimated but the δ_{ij} could not whereas the δ_{ij} are the important quantities in these expressions.

However if pulses of small percentage modulation are transmitted by any of the three continuous wave methods considered to be fundamental in nature then the amplitudes of the pulses detected are the h_q not the A_q. But the A_q are known from the experiment of the preceding paragraph. Hence knowing the A_q and the h_q the system of equations

$$ch_q = A_q \sum_{i=1}^{N} A_i \cos\delta_{iq} \qquad q = 1, 2, \cdots, N \qquad 4.1\text{-}1$$

can be solved for the unknowns δ_{iq}. For N = 2 the solution becomes

$$\cos\delta = \frac{\left(\frac{h_1}{h_2}\right)\left(\frac{A_2}{A_1}\right) - \left(\frac{A_1}{A_2}\right)}{1 - h_1/h_2} \qquad 4.1\text{-}2$$

For $N > 2$ numerical approximation methods or graphical methods are necessary to evaluate the angles δ_{ij}. Unfortunately more than one solution may exist for $N > 2$ only one of which may correspond to the actual case under consideration. However the system has been reduced to a very small number of possible solutions; the particular one corresponding to the actual case under consideration being selected by a transmission under a condition where the percentage modulation is not small or where the intelligence is not a pulse.

With the information of the A_i, t_i the formulas of the preceding sections can be used as "predictors" for the expected wave shape of the detector output for a given type of modulation and intelligence function.

The geometrical interpretation of the inverse problem heretofore referred is that if the magnitudes of a number of vectors $\underset{\sim}{A}_i$ are given, then the problem is to orient the vectors in a plane such that the resultant of the vectors $\underset{\sim}{R}$ yields inner products $\underset{\sim}{A}_i \cdot \underset{\sim}{R}$ which are proportional to the observed h_i.

4.2 A Case of Multipath Frequency Modulation

As a check on the above concepts, an experimental test was made using sinusoidally modulated frequency modulation. The transmitter used was the commercial transmitter of KHJFM located atop Mount Lee in Griffith Park, Los Angeles, California operating at a carrier frequency of 99.8 megacycles. $\omega_0 = 2\pi \times 99.8 \times 10^6 \sec^{-1}$. The receiver was mounted in a mobile truck unit and consisted of various interchangeable antennas of several directivity patterns, a Hallicrafters model SX-28 receiver with an extra stage of limiting, and various monitoring equipment. The output was placed upon an oscilloscope and photographed. Fourier analyses of the detected waves were made and the results of the various data tabulated by percentage distortion present of the various harmonics under the possible conditions of operation.

The receiving unit was operated for the tests here reported at the intersection of Glenoaks Blvd. and Chevy Chase Dr. in northeastern Glendale, California, an airline distance of 5.4 miles from the transmitter. The reception was chosen such that line of sight conditions were not observed as under line of sight transmission the reception would be, as observed, almost free from distortion since the intensities of the reflections are in general small compared with the direct wave. Indeed the receiver was operated in a region where the observations indicated, by their distorted nature, that the received signal, of strength above the limiter level, was of a nature to be multipath distorted. Physically the location of the receiver was one inside a canyon with a rather narrow mouth, the canyon being pointed in a direction away from the transmitter (see Appendix 2), and having walls rising approximately 250 feet above the elevation of the receiver in the vicinity of the receiver. The transmitter was approximately 1050 feet above the elevation of the receiver.

By 3.5-3 when the higher (fourth and above) harmonics are small, the reception should yield a second harmonic distortion which increases in direct proportion to the maximum deviation frequency k, when the modulating frequency, ω_s, is kept a constant. The third harmonic should increase as the square of k under similar limitations. Curves of the observations are given in figures 9-11. The observed straggling of the second harmonic curves at the high deviation frequency end in a direction away from the predicted slope of unity in the logarithmic plot is probably due to the effect of the appreciable fourth harmonic and the fact that the fourth term in the Taylor Expansion, if present, introduces at the same time a second harmonic of comparable magnitude. The curves of second, third and fourth harmonic distortions should have slopes of 1, 2, and 3 for constant ω_s in the figures and within the accuracy of the measurements (to 0.1% in any percentage calculation) the predictions seem to be fulfilled.

If the deviation frequency is kept a constant but the modulation frequency is increased, the equations 3.5-3 state there should be a general decrease in the second harmonic distortion which is inversely proportional to the angular modulation frequency, ω_s, but that the relationship is obscured by the appearance of the functions $I(t-t_i)$ etc. Similarly the equations state the third harmonic distortion should vary inversely as ω_s^2 subject to the modification of the $I(t-t_i)$ functions and their integrals appearing in the numerators of the expressions in the Taylor expansion. From 3.5-10 the two signal case of reception implies that the second harmonic distortion is

$$\frac{c}{b} = \frac{k}{\omega_s} \frac{(1-\cos\alpha)\sin\beta \cdot \rho(1+\rho^2)}{(\rho^2+2\rho\cos\beta+1)\sqrt{\rho^4+4\rho^3\cos\beta\cos^2\frac{\alpha}{2}+2\rho^2(\cos^2\beta\cos\alpha+\cos^2\beta+\cos\alpha)+4\rho\cos\beta\cos^2\alpha/2+1}} \qquad 4.2\text{-}1$$

which becomes for α very small

$$c/b = \frac{k}{\omega_s} \frac{\alpha^2}{2} \frac{\sin\beta\, \rho(1+\rho^2)}{(1+2\rho\cos\beta+\rho^2)^2} \qquad 4.2\text{-}2$$

Hence the second harmonic distortion should increase linearly with ω_s and then start to decrease when the α is no longer small, oscillating in the process. The experimental results for the condition of reception cited are given in figures 12-14 with sample photographs of the received waves when the modulation is a sinusoid in appendix 4.

Further results of sinusoidally modulated frequency modulation transmission under multipath conditions are given in figures 15-18 for a second location of receiver. The second location of receiver was in the Arroyo Seco, Pasadena, California, a broader canyon than the first considered, in a region indicated in appendix 2, a distance of 8.9 airline miles from the transmitter.

4.3 The Antenna Problem

The results of the preceding section were taken for a particular location of receiver, and a particular type of antenna, whose beam was directed in a particular orientation. It is apparent the complexity of the results depends upon the number of interfering signals. Further in the cases of mountainous terrain considered, reflecting surfaces may exist on many sides of the receiving antenna. Thus a non-directional antenna (circular) would be highly susceptible to interference phenomena from multipath sources. In the case of practical radio reception where the interference is desired to be as low as possible, the advantages of reception with a directional antenna are apparent. A selective antenna of this type reduces the number of interfering signals and permits the fixing upon one strong signal and perhaps several weaker signals by virtue of the finite width of the antenna receiving beam. The ultimate in distortionless reception is obtained by a zero width beam or by a beam whose width is sufficiently small to resolve at least one signal, sufficiently strong to be above limiter level, and to make the presence of the remaining signals unknown to the receiver. The effect of substituting a directional antenna pointed in the direction of maximum signal strength for a circular antenna, directionless in the plane of polarization of the incoming radiation, is seen for the Glendale site in figures 19-20. The distortion is reduced under all parallel conditions of transmitter operation.

Upon deciding that for reception relatively more free of distortion, directional antennas should be used, two problems arise: where in a given vicinity should the antenna be located and how should the antenna be oriented? From the results of the preceding sections it is apparent that the reception depends to a very large extent upon trigonometric functions of the β_{ij} , implying that within a change in path difference of a half wave length the entire character of the distortion will change. At a carrier frequency of 100 megacycles the wave-length (3 meters) is sufficiently small to ensure that large variations in the reception will be observed even within a small area of possible location of the directional antenna. Too, as previously mentioned, there exists a prefered direction of the beam orientation of the directional antenna to bring the distortion to a minimum. Experimental data for these two problems taken at the Glendale site are given in table I and in figure 21, for the case of sinusoidally modulated frequency modulation.

4.4 <u>Sinusoidally Modulated Frequency Modulation</u>

The two signal theory for sinusoidally modulated frequency modulated waves as developed by Corrington and others can be extended to the case of sinusoidally modulated waves interfering when the multipath propagation is over more than two paths. The method is worthy of note here although of little practical use. The problem involves a prediction of the harmonic distortion to be observed at the receiver of frequency modulated waves in terms of the usual variables, the amplitudes of the various waves and the relative times of arrival.

As in 1.2-2 the input to the receiver is, by the usual hypotheses,

$$\sum_{i=1}^{\tilde{N}} A_i O[I(t-t_i)]$$

which for frequency modulated waves by 2.1-4 and 3.3-2 becomes

$$\sum_{j=1}^{\tilde{N}} A_j e^{i(\omega_0 t+\Omega_j)} \qquad 4.4\text{-}1$$

By the process of the vector addition indicated in section 3.1 the input to the receiver is placed in the form

$$\sum_{j=1}^{\tilde{N}} A_j e^{i(\omega_0 t+\Omega_j)} = \sqrt{\sum_{i=1}^{\tilde{N}}\sum_{j=1}^{\tilde{N}} A_i A_j \cos\Omega_{ij}}\; e^{i(\omega_0 t+\Omega_1(t)+\Phi(t))} \qquad 4.4\text{-}2$$

which yields by the hypothesis of frequency modulated receiver detection 2.1-5

$$\varepsilon(t) = \frac{d\Omega_1}{dt} + \frac{d\Phi}{dt} \qquad 4.4\text{-}3$$

The first term of the above expression is the output wave had we let N = 1 (for then $\Phi=0$) which reduces the problem to single signal reception and yields undistorted output. We may therefore interpret the second term of the above expression as the already noted distortion 1.2-3, which is superimposed upon the desired output $\frac{d\Omega_1}{dt}$. A detailed investigation of D is the subject of the subsequent analysis.

Let $\rho_i = \frac{A_i}{A_1}$; $\tau_i = \frac{A_{i+1}}{A_i} = \frac{\rho_{i+1}}{\rho_i}$

and we shall order the signals by letting

$$A_1 \geq A_2 \geq A_3 \cdots \geq A_{N-1} \geq A_N$$

so that $\rho_i \leq 1$, $\tau_i \leq 1$. If no two signals have the same amplitude the ordering is unique. Then

$$\Phi(t) = \tan^{-1} \frac{\sum_{i=1}^{N} \rho_i \sin \Omega_{i1}}{\sum_{i=1}^{N} \rho_i \cos \Omega_{i1}} = \arg\left(\sum_{j=1}^{N} \rho_j e^{i\Omega_{j1}}\right)$$

$$= \operatorname{Im} \ln \left(\sum_{j=1}^{N} \rho_j e^{i\Omega_{j1}}\right) = \operatorname{Im} \ln \left(1 + \sum_{j=2}^{N} \rho_j e^{i\Omega_{j1}}\right). \qquad 4.4\text{-}4$$

Now

$$\ln(1+x) = \sum_{p=1}^{\infty} (-1)^{p+1} \frac{x^p}{p}$$

so

$$\Phi(t) = \operatorname{Im} \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1+1}}{n_1} \left[\sum_{j=2}^{N} \rho_j e^{i\Omega_{j1}}\right]^{n_1}$$

$$= \operatorname{Im} \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1+1}}{n_1} \rho_2^{n_1} e^{in_1\Omega_{21}} \left[1 + \sum_{j=3}^{N} \frac{\rho_j}{\rho_2} e^{i\Omega_{j2}}\right]^{n_1} \qquad 4.4\text{-}5$$

By the binomial expansion

$$\Phi(t) = \operatorname{Im} \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{n_1} \frac{(-1)^{n_1+1}}{n_1} \rho_2^{n_1} e^{in_1\Omega_{21}} \binom{n_1}{n_2} \left[\sum_{j=3}^{N} \frac{\rho_j}{\rho_2} e^{i\Omega_{j2}}\right]^{n_2}$$

$$= \operatorname{Im} \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{n_1} \frac{(-1)^{n_1+1}}{n_1} \rho_2^{n_1} \left(\frac{\rho_3}{\rho_2}\right)^{n_2} e^{in_1\Omega_{21}} e^{in_2\Omega_{32}} \binom{n_1}{n_2} \left[1 + \sum_{j=4}^{N} \frac{\rho_j}{\rho_3} e^{i\Omega_{j3}}\right]^{n_2} \qquad 4.4\text{-}6$$

And continuing the expansion in a similar fashion

$$\Phi(t) = \operatorname{Im} \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{n_1} \sum_{n_3=0}^{n_2} \cdots \sum_{n_{N-1}=0}^{n_{N-2}} \frac{(-1)^{n_1+1}}{n_1} \rho_2^{n_1} \left(\frac{\rho_3}{\rho_2}\right)^{n_2} \cdots \left(\frac{\rho_N}{\rho_{N-1}}\right)^{n_{N-1}} \binom{n_1}{n_2}\binom{n_2}{n_3} \cdots \binom{n_{N-2}}{n_{N-1}} \cdot$$

$$\cdot\, e^{i[n_1\Omega_{21} + n_2\Omega_{32} + \cdots + n_{N-1}\Omega_{N,N-1}]}$$

or

$$\Phi(t) = \operatorname{Im} \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{n_1} \sum_{n_3=0}^{n_2} \cdots \sum_{n_{N-1}=0}^{n_{N-2}} \frac{(-1)^{n_1+1}}{n_1} \left[\tau_1^{n_1} \tau_2^{n_2} \cdots \tau_{N-1}^{n_{N-1}}\right] \binom{n_1}{n_2}\binom{n_2}{n_3} \cdots \binom{n_{N-2}}{n_{N-1}} e^{i\sum_{j=1}^{N-1} n_j \Omega_{j+1,j}} \qquad 4.4\text{-}7$$

or

$$\Phi(t) = \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{n_1} \sum_{n_3=0}^{n_2} \cdots \sum_{n_{N-1}=0}^{n_{N-2}} \frac{(-1)^{n_1+1}}{n_1} \sin\left(\sum_{m=1}^{N-1} n_m \Omega_{m+1,m}\right) \prod_{j=1}^{N-1} \tau_j^{n_j} \prod_{k=1}^{N-2} \binom{n_k}{n_{k+1}} \tag{4.4-8}$$

To this point no mention has been made of the specific nature of the modulation, the intelligence function being contained in the $\Omega_{m+1,m}$. For the special problem of sinusoidal frequency modulation with

$$\Omega_i(t) = -\beta_i + a \sin \omega_s (t - t_i)$$

the distortion function becomes

$$D = \frac{d\Phi}{dt} = 2\omega_s \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{n_1} \sum_{n_3=0}^{n_2} \cdots \sum_{n_{N-1}=0}^{n_{N-2}} \frac{(-1)^{n_1+1}}{n_1} \prod_{j=1}^{N-1} \tau_j^{n_j} \prod_{k=1}^{N-2} \binom{n_k}{n_{k+1}} \cdot$$
$$\cdot \left\{ \begin{array}{l} \cos B \cdot \sum_{m=0}^{\infty} (-1)^m (2m+1) J_{2m+1}(A) \cdot \sin (2m+1)(\omega_s t + D) \\ + \sin B \cdot \sum_{m=1}^{\infty} (-1)^m (2m\, J_{2m}(A)) \sin 2m(\omega_s t + D) \end{array} \right\} \tag{4.4-9}$$

where

$$B = \sum_{i=1}^{N-1} n_i \beta_{i+1,i}$$

$$A = a\sqrt{\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} n_i n_j \left(\cos \delta_{i+1,j+1} + \cos \delta_{i,j} - \cos \delta_{i,j+1} - \cos \delta_{j,i+1}\right)}$$

$$D = \tan^{-1} \frac{\sum_{i=1}^{N-1} n_i (\cos \delta_{i+1} - \cos \delta_i)}{\sum_{i=1}^{N-1} n_i (\sin \delta_{i+1} - \sin \delta_i)}$$

with

$$\beta_{ij} = \omega_0 (t_i - t_j)$$

$$\delta_{ij} = \omega_s (t_i - t_j)$$

$$\delta_i = \omega_s t_i$$

It is to be noted that if N = 1, $\Phi(t) = 0$ and no distortion is obtained, while if N = 2, 4.4-8 reduces to the expression of Corrington et. al., namely

$$\Phi(t) = \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1+1}}{n_1} \left[\sin(n_1 \Omega_{21})\right] \tau^{n_1}$$

4.4-10

and

$$\frac{d\Phi}{dt} = 2\omega_s \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1+1}}{n_1} \tau^{n_1} \Big[\cos(n_1\beta) \sum_{m=0}^{\infty} (2m+1)(-1)^m J_{2m+1}\left(2n_1 a \sin\frac{\delta}{2}\right) \cdot \sin(2m+1)(\omega_s t + \delta/2) + \sin(n_1\beta) \sum_{m=1}^{\infty} (-1)^m 2m J_{2m}(2n_1 a \sin\delta/2) \sin 2m(\omega_s t + \delta/2) \Big]$$

4.5 The Direction of Further Investigations

The problem of the transmission of intelligence between two points in space as free from multipath interference effects as possible seems to be dependent upon

1) the choice of a system of modulation-detection,

2) the choice of a suitable receiving antenna, and

3) the location of the antenna in a small region about the selected position of the receiver.

The answer to the second problem is perhaps the simplest of all; antennas should be chosen with a minimum beam width to exclude energy arriving at the antenna from directions which might interfere with the energy arriving through the principal lobe of the beam. Narrow beam antennas are possible, yet the idea of the desirability of the reception of intelligence from two or more sources makes a single zero beam (the limiting case) antenna, unless rotatable, undesirable. Even with narrow beam antennas, the possibility of multipath interference is not eliminated.

The location of the antenna in a small region about the position of the receiver is the next problem that can be attacked. The solution depends upon a knowledge of the N, and of the A_i, t_i, for each point in the vicinity of the receiver. The method of performing this calculation and measurement is that outlined in section 4.1. A map of the A_i, t_i is now possible for the region about the receiver. Further consideration of antenna location problems hinges upon the answer to the choice of a system of modulation-detection and the resulting equations of the reception.

It has not been possible to consider the problem of the selection of a system of modulation-detection in its entirety from an analytical viewpoint in this paper. Indeed experimental methods of determining the distortions to be observed for a given A_i, t_i are indicated for the various types of modulation in order to determine the superiority of the various systems. However controlled experiments in the field are complicated by the fact that the observer is not able to control the A_i, t_i at will. Laboratory controlled experiments are therefore indicated, however the problem is again complicated by the fact that for appreciable measurable distortion to result in the observations the delay between the interfering signals should at least be an appreciable portion of the wavelength of the intelligence. This may require delays of the order of a mile or so, a condition usually attainable in the field.

Thought however indicates that laboratory investigation of the multipath transmission of modulated waves involving the interference between many waves would best be accomplished by an acoustic analogy, performing the transmission and interference using sound waves rather than electromagnetic waves. To this end construction has been initiated upon a tank (dimensions 8'x8'x1") to be filled with water resembling a two dimensional medium for the propagation of sound. The medium is excited and the reception accomplished by hydrophones. The carrier frequency of the sound is to be of the order of 100 Kilocycles and the modulators and receivers are built to this carrier frequency. The modulation processes are scaled down from the analogous electromagnetic processes at higher carrier frequencies. The A_i, t_i are to be observed by suitably placed obstacles and reflectors.

No difficulty is encountered in using the sonic analgy despite the fundamental differences in the propagation, transmission and reception of sound as compared with the processes involving electromagnetic waves as the entire problem, from the approach of this paper, is reduced to a scalar problem in the receiver. The spatial configuration in the tank is not expected to resemble the spatial configuration in the electromagnetic case it is intended to reproduce. It is then possible to investigate various systems of modulation for various A_i, t_i, the conditions of the reception and for various forms and magnitudes of $I(t)$, the condition of transmission.

V. Multipath Interference of Pulse Modulated Systems

5.1 Comparison with Continuous Wave Systems

The number of variables involved in the interference of the various systems of pulse modulation considered in section 2.3 makes an analytical discussion of generalized (N-signal) interference prohibitive. However the appropriate use of gating systems and of limiters in the reception of pulse modulated waves makes the conclusion of smaller distortions than those present in the corresponding continuous wave cases evident.

The appropriate use of gating systems may convert a problem of multipath distortion into either a distortionless reception or into a common channel interference problem* in the case of multiplex systems. The former condition is of course ideal while the latter falls outside the range of multipath transmission of modulated waves and so neither case falls under consideration here. With gates of the minimum possible width allowed for the reception of the given type of pulse modulated system, the possibility of interference is reduced and if interference exists it is probably of the two signal type.

Further, if interference exists, it must exist between two signals whose relative times of arrival are nearly the same or are different by integer multiples of

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*2 or more signals on the same carrier frequency but with different intelligence functions modulating the carriers.

the average repetition rate of the pulses or close to these multiples. For pulse repetition rates of the order of 1/20,000 sec., the probability that interference exist, if at all, lies with the first possibility rather than with any other as path differences of 1/20,000 second or 9 miles or of multiples of 9 miles approaches the upper limit of possible interference in the practical case. But if two signals interfering arrive at almost the same times the distortion, by 1.2-5, will be small. Thus, in general, pulse modulated systems of transmission have smaller distortion possibilities than continuous wave systems as the conditions for distortion are removed by the gating systems.

The further use of upper and lower level limiters on all systems (except PAM) reduces the possible effect of interfering signals whose signal strength is too low compared with the strength of the strongest signal. Hence the pulse systems constitute a further improvement over the continuous wave systems (see figure 22). The appropriate choice of upper and lower limiter levels is an important factor in reducing distortion.

If distortion still arises after appropriate limiting and gating, then it must arise by means of an overlap of pulses between two interfering signals of approximately the same strength A_1, A_2 arriving at almost the same times. This situation can be removed by means of a directional antenna. If not so removable the distortion introduced must be investigated according to the individual systems of modulation.

5.2 Qualitative Analysis

a) Pulse Amplitude Modulation (PAM)

The use of a gate equal to the width of the pulse, or less, is indicated. No limiters are possible at the upper level but are possible at the lower level if the percentage modulation is kept small. In the two signal interference problem with overlap (figure 22), distorted pulses for the input to the receiver, and the train of distorted pulses is detected by the filter. First order effects are of course expected to be negligible.

It is evident that, at least theoretically, the gate in the case of PAM could be made much smaller than the width of the pulse and still reproduce with great accuracy the desired intelligence. In this case the distortion introduced by the multipath transmission is also reduced to a vanishingly small quantity as the width of the gate is reduced to zero.

b) Pulse Position and Pulse Length Modulation (PPM and PLM)

Each of the systems, pulse position and pulse length modulation, when properly limited and gated are more free from multipath distortion than PAM as the detection process is dependent only upon the time of initiation or of cession of a pulse sequence. The time of arrival of either a leading or a trailing edge of a pulse is the prime consideration. If multipath transmission exists, then all pulses may be initiated a fraction of a second earlier than in a one-signal case but they are all initiated at the same relative times.

Of course, if a number of weak signals, all of approximately the same strength form the input to the receiver, as limiting case, and energy is always coming into the receiver at all times, the PPM and PLM cases appear to breakdown in their operation and will yield highly distorted output waves.

c) Pulse Frequency Modulation (PFM)

PFM suitably gated and limited is more susceptible to interference than PPM because of the necessarily wide gate required compared with the width of the pulses. Limiting at both levels is possible. The repetition rate of the pulses is still fundamentally the same and to the first order no distortion will result in the detection of intelligence by the transmission of waves by this system of modulation. In the two signal-small overlap case of interference, the discriminators must act on the distorted pulses and it is expected that small distortions may arise from this case. Again if there are many signals arriving, all of apparently the same signal strength, high distortions may result as the entire gate width is filled with energy.

d) Pulse Code Modulation (PCM)

Pulse code modulation is known to be, at present, the ideal type of transmission-detection at hand as far as noise figures are concerned. It will be seen that similar conclusions can be drawn for the problem of Multipath interference. With the usual suitable limiting and gating the reception depends only upon the presence or absence of the pulses. If in the desired signal the pulse is present, it will also be present in the interfering signal. If the pulse is absent in the desired signal, it will also be absent in the interfering signal. Thus presence or absence in the two signal-small over-lap case is not dependent upon the interference at all and the reception is undistorted. The problem is further idealized by allowing, as in PAM the existence of gates, in the limit, of width much less than the width of the pulse.

Two problems, not here considered yield possibilities of distorted reception of multipath transmission of pulse modulated systems of waves. They are

1) pulse overlap by time delays of the order of the rate of repetition of the pulses and

2) complete overlap between two signals of equal carrier strength with carrier phase differences of an odd multiple of π, resulting in complete annihilation of the received waves (corresponding to the condition $\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \cos\beta_{ij} = 0$ in continuous wave interference problems).

5.3 The Synchronizing Problem

A further difficulty encountered in the detection of multipath transmitted pulse modulated waves is the problem of synchronization. In PAM, PLM, PFM, PPM the leading edges (or trailing edges) of the pulses act as "triggers" in the receiver, initiating processes such as the gating process in the detection. In PCM a separately transmitted sunchronizing pulse defines the code groups and initiates and times the internally generated processes in the receiver. In the case of multipath transmission of pulse modulated systems several synchronizing signals arrive at the receiver in the space of time one such signal would arrive had the transmission been of the ideal single-signal type.

In the cases where N, the number of interfering waves is small, it is possible to conceive of a means of avoiding the difficulty of the effect of multiple synchronizing signals. Let the receiver "fix" on some one of the synchronizing signals and then permit a mechanism to exist in the receiver which prevents further reception of synchronizing pulses until a short time before the anticipated reception of another synchronizing pulse on the single signal basis. Thus if τ is the period of repetition of synchronizing pulses, let the "dead time" or time of insensibility of the receiver to synchronizing pulses be $\tau - \delta$ where δ is a small fractioh of τ. This will in general permit, if N is small and overlap occurs, the fixing on the first of the group of arriving signals of synchronization, permitting the operation of the gate.

However the problem is extremely complicated if N is so large, and the time delays sufficient, so that there are no times when energy is not coming into the receiver. Since the synchronizing pulses are multipath transmitted as well as the modulated waves, it is difficult to find the criterion for synchronization in the receiver. One possible criterion, although susceptible to noise effects, applicable for PPM, PFM, PCM and PLM is the initiation of all the processes at the times of arrival of maximum energy at the receiver since all the maxima should have equal energy levels. It is evident that this criterion will not be adequate for PAM where the transmission depends upon the relative amplitudes of the maxima. PLM is complicated under this criterion by the fact that the gating width must be at least as long as the longest pulse to be received. If energy arrives at the receiver at all times, the PLM pulses which were to have been discrete, yet varying lengths,are now almost identical yielding highly distorted output.

Difficulty is also encountered in PCM since shorter delay times are sufficient to cause distortion because of the increased rate of transmission of pulses over the PAM sampling rate, even though the synchronizing pulses are transmitted at the same rate as PAM sampling. Shorter time delays are more probable in any practical physical configuration of transmitter, receiver and intervening space and boundaries.

One solution of the problem, admittedly inadequate, is to reduce the sampling rates at the transmitter sufficiently so that the time between pulses is always greater than the longest time delay present permitting the isolation of the effect of each pulse transmitted from all preceding or subsequent pulses transmitted. The distortions, although still present in some pulse systems (not in PCM) are reduced if arising because of multipath transmission, but are increased due to the loss of fidelity from lower sampling rates at the transmitter.

VI. Reception of Diffracted Modulated Waves

6.1 Diffraction Integrals

The preceding arguments may be considered to be calculations of the interference phenomena to be detected when a receiver observes radiation coming apparently from a number of distinct virtual sources, corresponding to the analogous interference problems in physical optics. However, when the incoming radiation appears to originate from a diffuse source, as in problems of diffraction in physical optics, the expressions must be modified to take this consideration into account.

In the preceding interference arguments, it will be noted, no consideration was made of the spatial conditions giving rise to the various incoming signals. The relative amplitudes and phases were assumed as phenomena of the reception and the calculations proceeded from these as bases. In order to avoid the entrance of the spatial configuration of slits, gratings, obstacles, etc., into the present problem, the arrival at the receiver of a given spatial distribution of intensities and phases is hypothesized independently of the conditions leading to these intensities and phases.

Designate spherical polar coordinates (r, θ, φ) (figure 23) about the antenna. Then let $A(\theta, \varphi)$ be the magnitude of the induced (scalar) e.m.f. per unit solid angle in the receiver due to energy arriving at the antenna from that portion of space designated by the coordinates (θ, φ)

encompassing a solid angle $\sin\theta\, d\theta\, d\varphi$. A is a relative quantity calculated proportional to the induced e.m.f. when the transmitter is unmodulated, sending merely a carrier. Let $t'(\theta,\varphi)$ be the relative time of arrival from the transmitter of the radiation arriving at angles (θ,φ). The problem of interference of N continuous wave signals is now extended to the present problem by letting N increase and the relative amplitudes, A, appearing in sec. 1.2 decrease until the limit of the present problem is attained. It is evident that for amplitude modulation, from 3.2-3 the receiver output is

$$\varepsilon(t) = \sqrt{\int_0^\pi\int_0^\pi\int_0^{2\pi}\int_0^{2\pi} a(\theta_1,\varphi_1)a(\theta_2,\varphi_2)\sin\theta_1\sin\theta_2\cos\omega_0\left[t'(\theta_1,\varphi_1)-t'(\theta_2,\varphi_2)\right]d\theta_1 d\theta_2 d\varphi_1 d\varphi_2}$$
$$-\text{ave}\sqrt{\int_0^\pi\int_0^\pi\int_0^{2\pi}\int_0^{2\pi} a(\theta_1,\varphi_1)a(\theta_2,\varphi_2)\sin\theta_1\sin\theta_2\cos\omega_0\left[t'(\theta_1,\varphi_1)-t'(\theta_2,\varphi_2)\right]d\theta_1 d\theta_2 d\varphi_1 d\varphi_2} \quad 6.1\text{-}1$$

where

$$a(\theta,\varphi) = A(\theta,\varphi)\left[1+kI(t-t'(\theta,\varphi))\right]$$

For phase modulation, from 3.2-8

$$\varepsilon(t) = \tan^{-1}\frac{\int_0^\pi\int_0^{2\pi} A(\theta,\varphi)\sin\Omega(\theta,\varphi)\sin\theta\, d\theta\, d\varphi}{\int_0^\pi\int_0^{2\pi} A(\theta,\varphi)\cos\Omega(\theta,\varphi)\sin\theta\, d\theta\, d\varphi} - \text{ave}\tan^{-1}\frac{\int_0^\pi\int_0^{2\pi} A(\theta,\varphi)\sin\Omega(\theta,\varphi)\sin\theta\, d\theta\, d\varphi}{\int_0^\pi\int_0^{2\pi} A(\theta,\varphi)\cos\Omega(\theta,\varphi)\sin\theta\, d\theta\, d\varphi} \quad 6.1\text{-}2$$

where

$$\Omega(\theta,\varphi) = \omega_0 t'(\theta,\varphi) - kI(t-t'(\theta,\varphi))$$

and for frequency modulation, from 3.3-4

$$\mathcal{E}(t)=\frac{\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}A(\vartheta_1,\varphi_1)A(\vartheta_2,\varphi_2)\cos\left\{\int_{t+t'(\vartheta_2,\varphi_2)}^{t+t'(\vartheta_1,\varphi_1)}I(t)dt-\omega_0\left[t'(\vartheta_1,\varphi_1)-t'(\vartheta_2,\varphi_2)\right]\right\}\cdot k I\left[t-t'(\vartheta_1,\varphi_1)\right]\sin\vartheta_1\sin\vartheta_2\,d\vartheta_1 d\vartheta_2 d\varphi_1 d\varphi_2}{\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}A(\vartheta_1,\varphi_1)A(\vartheta_2,\varphi_2)\cos\left\{\int_{t+t'(\vartheta_2,\varphi_2)}^{t+t'(\vartheta_1,\varphi_1)}I(t)dt-\omega_0\left[t'(\vartheta_1,\varphi_1)-t'(\vartheta_2,\varphi_2)\right]\right\}\cdot\sin\vartheta_1\sin\vartheta_2\,d\vartheta_1 d\vartheta_2 d\varphi_1 d\varphi_2}$$

$$-\text{ave}\,\frac{\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}A(\vartheta_1,\varphi_1)A(\vartheta_2,\varphi_2)\cos\left\{\int_{t+t'(\vartheta_2,\varphi_2)}^{t+t'(\vartheta_1,\varphi_1)}I(t)dt-\omega_0\left[t'(\vartheta_1,\varphi_1)-t'(\vartheta_2,\varphi_2)\right]\right\}\cdot k I\left[t-t'(\vartheta_1,\varphi_1)\right]\sin\vartheta_1\sin\vartheta_2\,d\vartheta_1 d\vartheta_2 d\varphi_1 d\varphi_2}{\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}A(\vartheta_1,\varphi_1)A(\vartheta_2,\varphi_2)\cos\left\{\int_{t+t'(\vartheta_2,\varphi_2)}^{t+t'(\vartheta_1,\varphi_1)}I(t)dt-\omega_0\left[t'(\vartheta_1,\varphi_1)-t'(\vartheta_2,\varphi_2)\right]\right\}\cdot\sin\vartheta_1\sin\vartheta_2\,d\vartheta_1 d\vartheta_2 d\varphi_1 d\varphi_2}\qquad 6.1\text{-}3$$

These formulas take the place of the functions written in sections 3.3, 3.4 for discrete multipath transmitted signals.

6.2 Small-percent Modulation

It is evident that in the case of small percent modulation being transmitted, the approximations of sections 3.3, 3.4 hold and the reception E(t), for any of the 3 systems is

$$H(t)=c_1 k\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}A(\vartheta_1,\varphi_1)A(\vartheta_2,\varphi_2)\cos\omega_0\left[t'(\vartheta_1,\varphi_1)-t'(\vartheta_2,\varphi_2)\right]\cdot\sin\vartheta_1\sin\vartheta_2\, I(t-t'(\vartheta_1,\varphi_1))\,d\vartheta_1 d\vartheta_2 d\varphi_1 d\varphi_2\qquad 6.2\text{-}1$$

where $E(t) = p[H(t)]$

Where p is an operator present to include the effects of any low pass filtering in the receiver. When I(t) is a sine wave it is evident E(t) is still a sine wave after the complex multipath diffraction. For more complex intelligence functions this integral must be evaluated for the given I(t), $A(\vartheta, \varphi)$, $t'(\vartheta, \varphi)$.

6.3 Scalar Diffraction through a Slit

As an illustrative example of the problem of diffraction consider the problem of Fraunhofer diffraction of modulated plane waves through a slit of width a, when the receiver and transmitter are located far from the slit, the walls of the slit are opaque, the dimensions of the slit are large compared to the carrier wavelength so that there is no interaction of the radiation with the walls of the slit and a directional receiving antenna is used. If the direction of propagation of the incident radiation makes an angle α with the normal to the slit and the receiver is located at an angle β with the normal to the slit, then 6.2 becomes

$$H(t) = c_1 k \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \cos\left[\frac{x_1 - x_2}{c} \omega_0 (\sin\alpha - \sin\beta)\right] I\left[t - \frac{x_1}{c}(\sin\alpha - \sin\beta)\right] dx_1 dx_2 \qquad 6.3\text{-}1$$

Let $V = \sin\alpha - \sin\beta$

Integrating over x_2

$$H(t) = \frac{2cc_1 k}{\omega_0 V} \int_{-a/2}^{a/2} \sin\frac{a\omega_0 V}{2c} \cos\frac{\omega_0 V x_1}{c} I\left(t - \frac{x_1 V}{c}\right) dx_1 \qquad 6.3\text{-}2$$

For $V = 0$ there is no distortion since

$$\mathcal{E}(t) = c_1 k a^2 I(t) \qquad 6.3\text{-}3$$

For I(t) a sinusoid, $\mathcal{E}(t)$ is also a sinusoid of the same frequency (no apparent distortion), but for I(t) other than sinusoidal, $V \neq 0$, distortions are present given by 6.3-2 Further, nulls in the reception exist where $\frac{a\omega_0 V}{2c} = p\pi$ (p = any integer) corresponding to the usual nulls in the reception of diffracted unmodulated waves through a slit in physical optics. Reception of a square wave modulated case of small percentage modulation of a continuous wave by diffraction through a slit is given graphically in figure 24.

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Appendix I. <u>The Envelope of a Single Sideband Wave, Square Wave Modulated</u>

A square wave of amplitude ± 1, period $\frac{2\pi}{\omega}$, has a Fourier analysis

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\omega t}{2n+1} \qquad \text{I-1}$$

Thus by 2.1-9, the transmitted wave of a single-sideband transmitter is

$$O[I(t)] = \frac{4k}{2\pi}\sum_{n=0}^{\infty}\frac{\sin(\omega_0+(2n+1)\omega)t}{2n+1} \qquad \text{I-2}$$

with the carrier suppressed, and where ω_0 is the carrier angular frequency. Rewriting I-2 by the method of section 3.1,

$$O[I(t)] = \frac{4k}{2\pi}\sqrt{\left[\sum_{n=0}^{\infty}\frac{\sin(2n+1)\omega t}{2n+1}\right]^2+\left[\sum_{n=0}^{\infty}\frac{\cos(2n+1)\omega t}{2n+1}\right]^2}\;\cdot$$

$$\cdot\,\sin\left[\omega_0 t+\tan^{-1}\frac{\sum_{n=0}^{\infty}\frac{\sin(2n+1)\omega t}{2n+1}}{\sum_{n=0}^{\infty}\frac{\cos(2n+1)\omega t}{2n+1}}\right]$$

Assuming $\omega \ll \omega_0$, the envelope of the SSBAM wave is

$$\frac{4k}{2\pi}\sqrt{\left[\sum_{n=0}^{\infty}\frac{\sin(2n+1)\omega t}{2n+1}\right]^2+\left[\sum_{n=0}^{\infty}\frac{\cos(2n+1)\omega t}{2n+1}\right]^2}$$

$$= \frac{4k}{2\pi}\sqrt{\left(\pi/4\right)^2+f^2(\omega t)} \qquad \text{I-3}$$

by I-1. It is seen that

$$f(\omega t) = \sum_{n=0}^{\infty}\frac{\cos(2n+1)\omega t}{2n+1} \qquad \text{I-4}$$

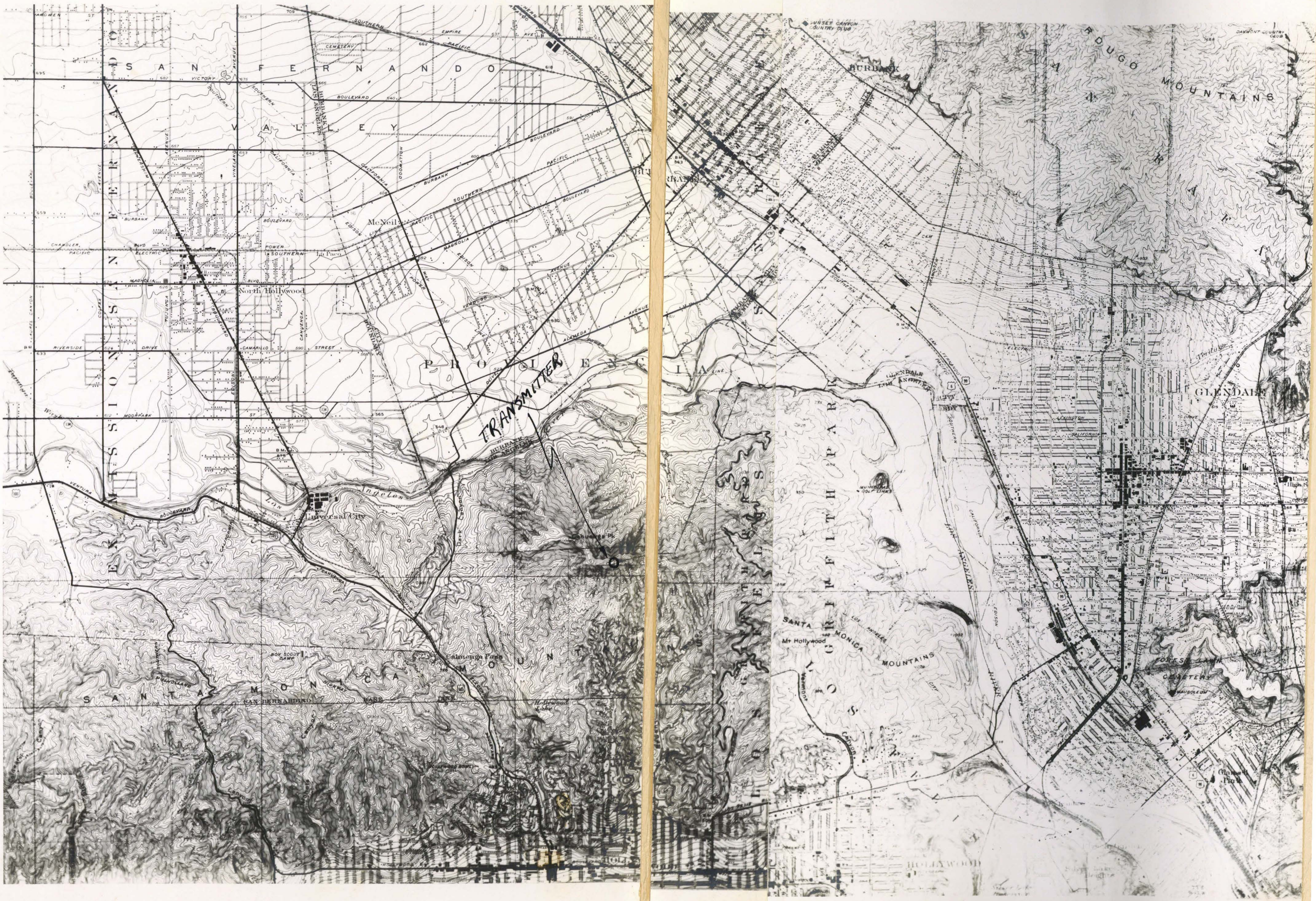
has a discontinuity at $\omega t = m\pi$, m an integer, but the divergence is of the order of $\frac{1}{\sin \omega t}$ permitting a numerical solution to the summation of the series I-4. By the identity

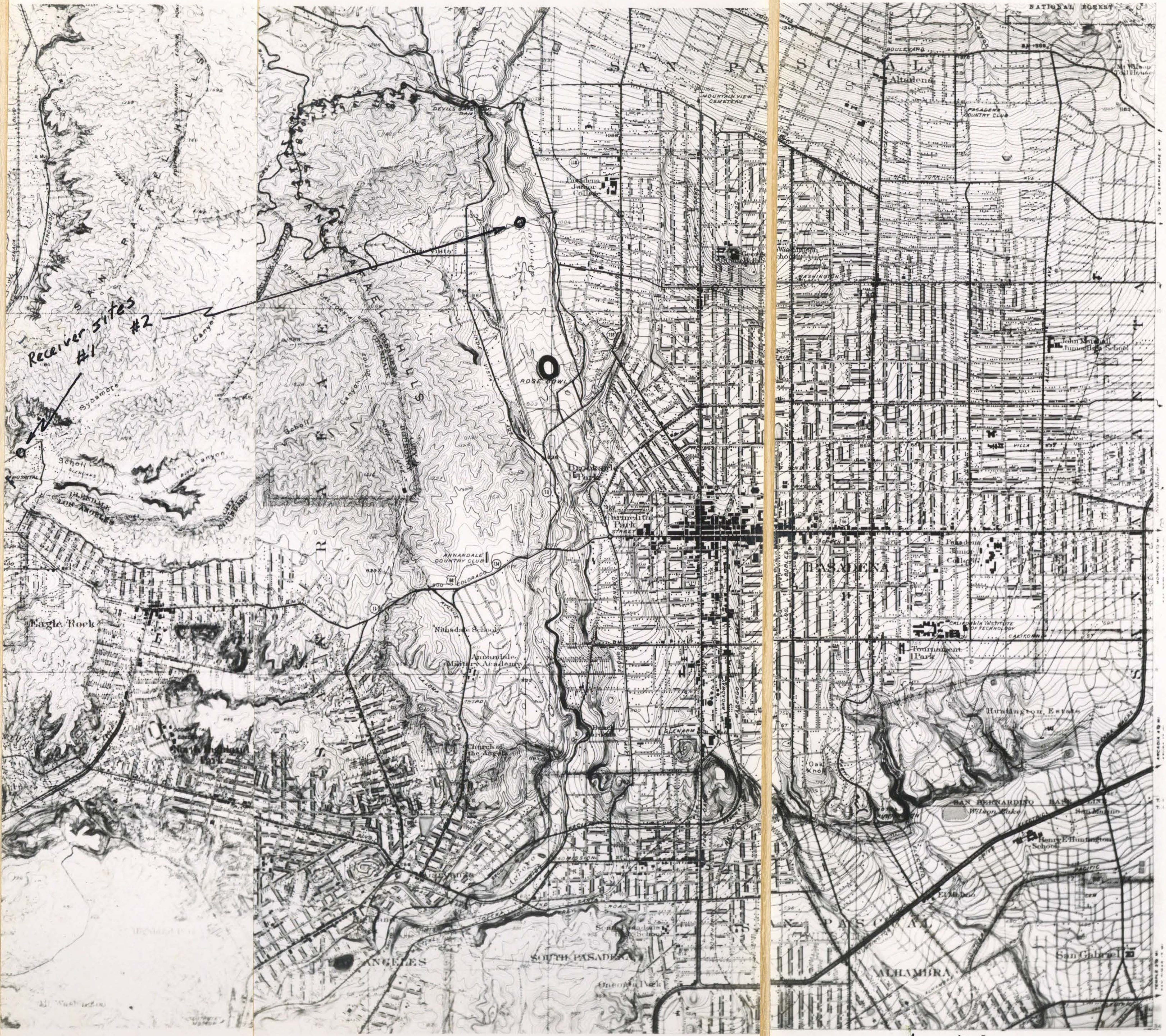
$$\sin\theta\cos(2n+1)\theta = \frac{\sin(2n+2)\theta-\sin 2n\theta}{2}$$

$$f(\omega t) = \frac{1}{2\sin\omega t} \sum_{n=0}^{\infty} \frac{\sin(2n+2)\omega t - \sin 2n\omega t}{2n+1}$$

$$= \frac{1}{2\sin\omega t} \sum_{n=1}^{\infty} \sin 2n\omega t \left[\frac{1}{2n-1} - \frac{1}{2n+1}\right] = \frac{1}{\sin\omega t} \sum_{n=1}^{\infty} \frac{\sin 2n\omega t}{4n^2-1}$$

This last summation converges sufficiently rapidly to be summed, and the result substituted into I-3. The result of the calculation is the envelope in figure 1.


SAN FERNANDO
VALLEY
MISSION
North Hollywood
McNeil
Universal City
PROVIDENCIA
TRANSMITTER
BURBANK
MOUNTAINS
SANTA MONICA
Mt Hollywood
GLENDALE
HOLLYWOOD


Receiver sites
#1
#2
ROSE BOWL
PASADENA
Eagle Rock
SOUTH PASADENA
ALHAMBRA
Altadena
Huntington Estate
Tournament Park
Annandale Military Academy
Church of the Angels
San Gabriel
NATIONAL FOREST

Appendix 2

Appendix III. <u>The Reception of a Square Wave Modulated Continuous Wave Signal Through a Slit.</u>

It is evident that for a slit of width a, the result of the calculation 6.3-2 for I(t) the unit step (or impulse) function

$$I(t) = 0 \quad t<0$$
$$= 1 \quad t>0$$

is the second curve of figure 24. The result of the operation of the low pass filter upon this wave is the third curve of figure 24.

The calculation of the effect of the filter can be made in several ways. Assuming I(t) a square wave of amplitude 1, period $\frac{2\pi}{\omega}$

$$I(t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)(\omega t)}{2n+1}$$

where upon performing the integration indicated in 6.3-2

$$H(t) = \frac{2c^2c_1k}{\omega_0 v^2}\sin\frac{a\omega_0 v}{2c}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\omega t}{2n+1}\left\{\frac{\sin[\omega_0-(2n+1)\omega]\frac{av}{2c}}{\omega_0-(2n+1)\omega}+\frac{\sin[\omega_0+(2n+1)\omega]\frac{av}{2c}}{\omega_0+(2n+1)\omega}\right\}$$

The action of the filter is to eliminate the high frequency terms. Thus with $\omega_0 \gg \omega$, the terms in the summation of large amplitude for $n \approx \frac{\omega_0}{2\omega}$ are filtered out by the receiver and

$$\mathcal{E}(t) \approx \frac{2c^2c_1k}{\omega_0^2v^2}\sin\frac{a\omega_0 v}{2c}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\omega t}{2n+1}\left\{\sin[\omega_0-(2n+1)\omega]\frac{av}{2c}+\sin[\omega_0+(2n+1)\omega]\frac{av}{2c}\right\}$$

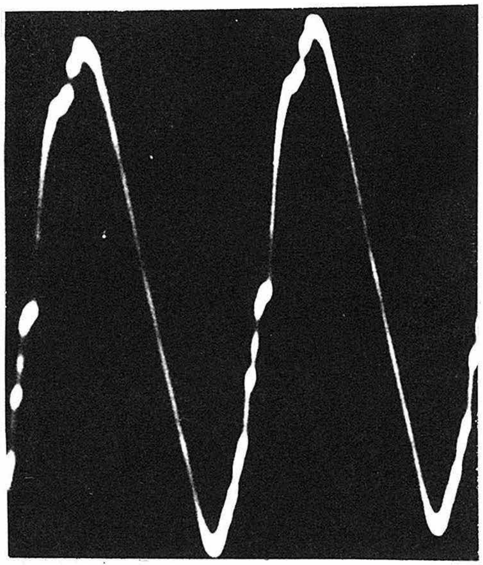
$$= \frac{2c^2c_1k}{\omega_0^2v^2}\sin\frac{2a\omega_0 v}{2c}\left\{\sum_{n=0}^{\infty}\frac{\sin(2n+1)(\omega t-\frac{\omega a v}{2c})}{2n+1}+\sum_{n=0}^{\infty}\frac{\sin(2n+1)(\omega t+\frac{\omega a v}{2c})}{2n+1}\right\}$$

Hence the result of the detection of the wave transmitted through the slit is a superposition of two square waves of equal amplitude differing in phase by the amount $\frac{\omega a v}{c}$ where a is the width of the slit. For $\frac{a\omega_0 v}{2c} = 0$ the reception is zero.

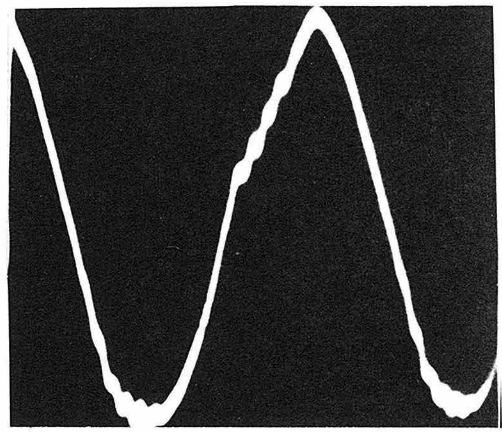
Appendix IV.

Samples of Oscillographic Data Used.

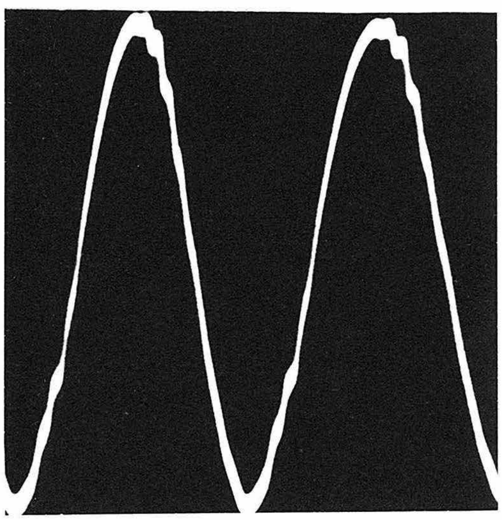
Directional Antenna



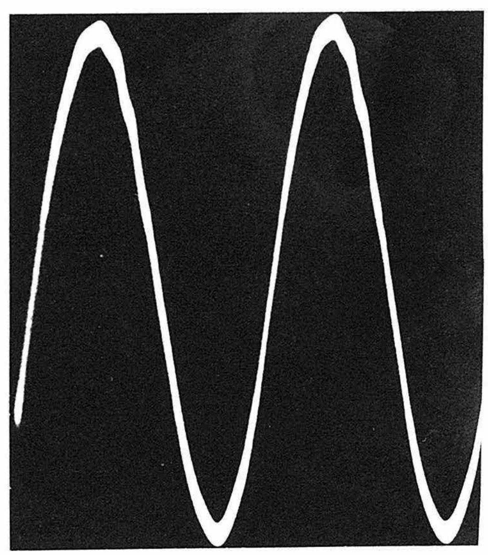
70 Kc. Deviation
Frequency



50 Kc. Deviation
Frequency



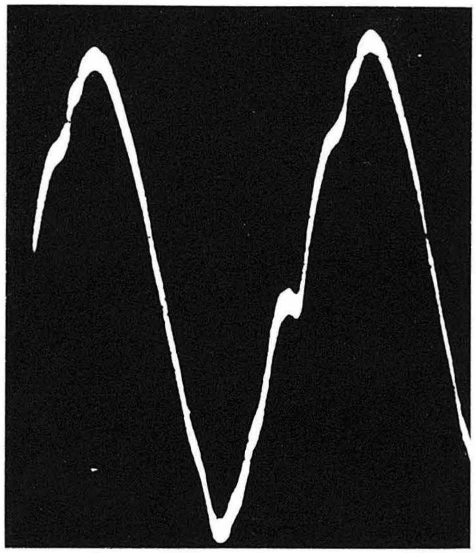
40 Kc. Deviation
Frequency



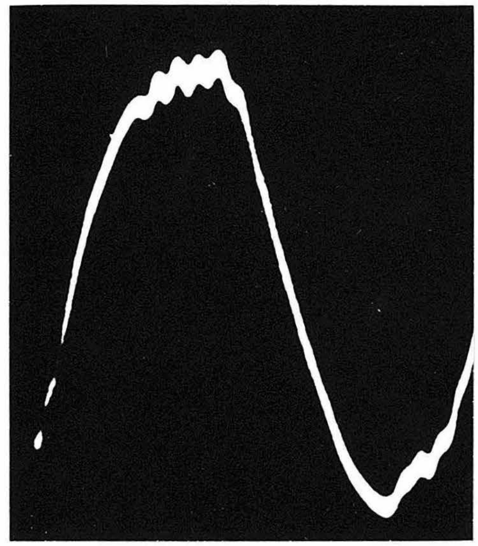
30 Kc. Deviation
Frequency

5000 c.p.s. Modulating Frequency

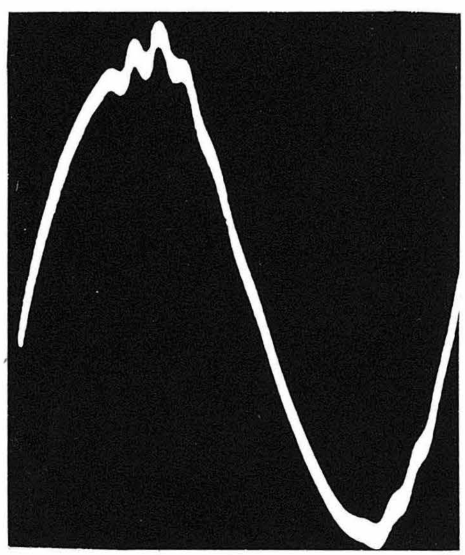
Plate I: Glendale Site. Frequency Modulation Reception.



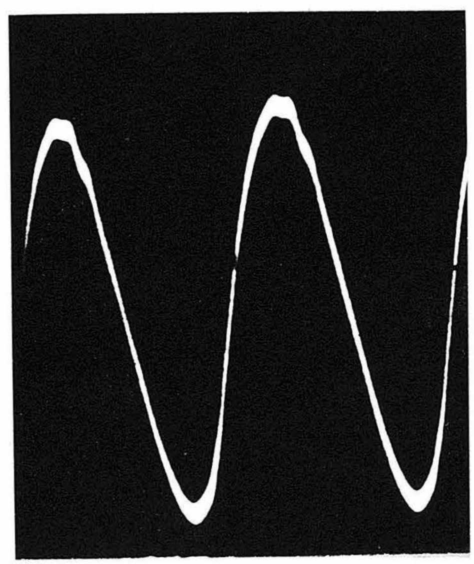
75 Kc. Deviation
Frequency



50 Kc. Deviation
Frequency

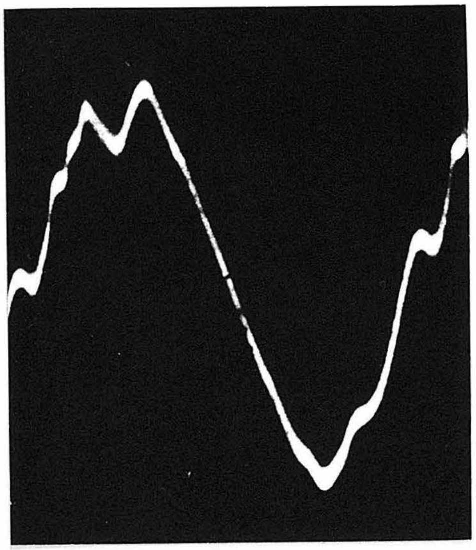


40 Kc. Deviation
Frequency

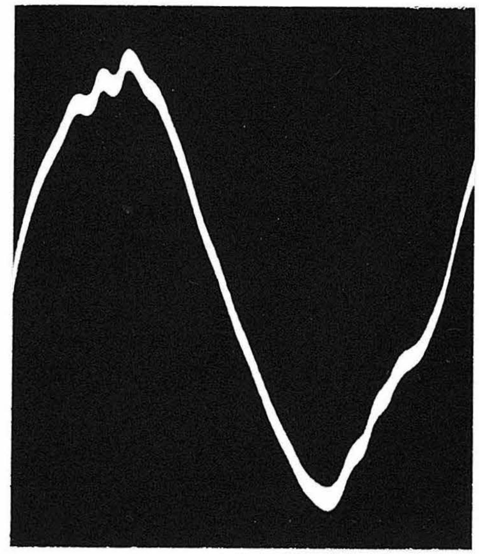


30 Kc. Deviation
Frequency

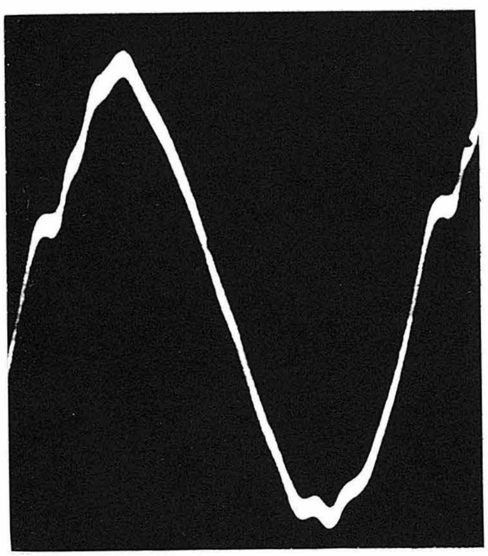
6000 c.p.s. Modulating Frequency
Plate II: Glendale Site. Frequency Modulation Detection.



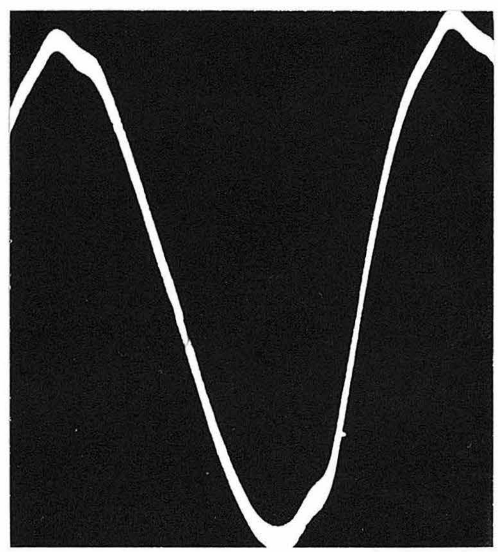
75 Kc. Deviation
Frequency



50 Kc. Deviation
Frequency

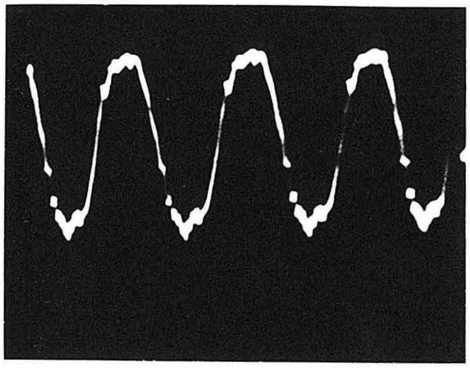


40 Kc. Deviation
Frequency

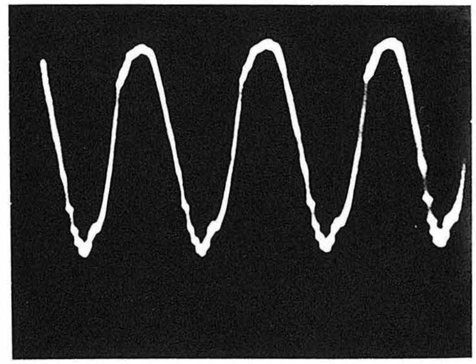


30 Kc. Deviation
Frequency

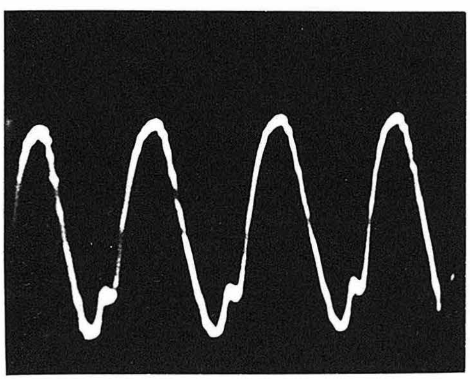
7500 c.p.s. Modulating Frequency
Plate III. Glendale Site. Frequency Modulation Reception.



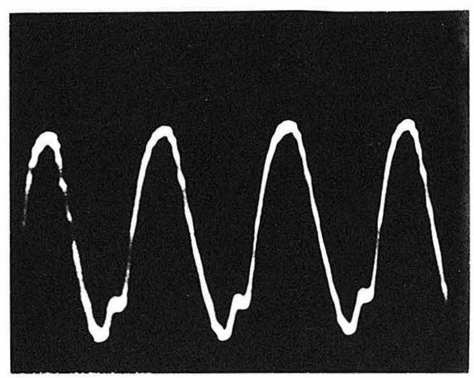
-30° (Below Limiter Level)



0° (Broadside to Transmitter)



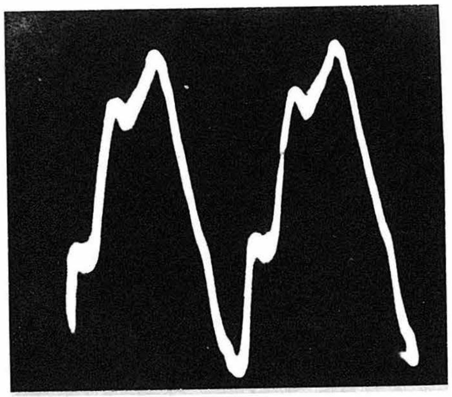
60°



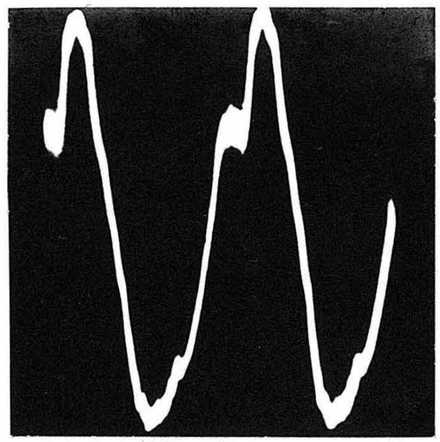
90°

Plate IV: Distortion as a function of Polar Angle of Antenna Glendale Site. Frequency Modulation Reception. 5000 c.p.s. modulating frequency. 75 Kc. Deviation Frequency.

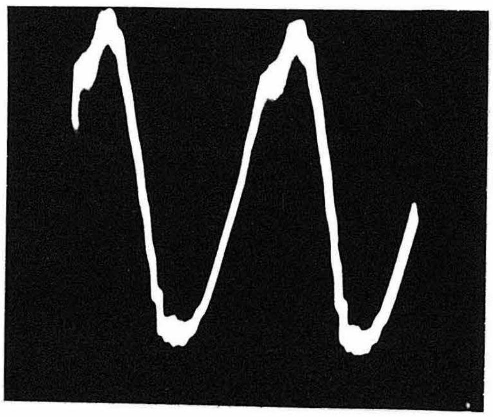
Circular Antenna



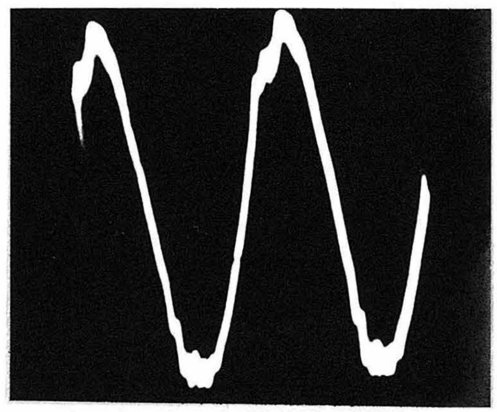
75 Kc. Deviation
Frequency



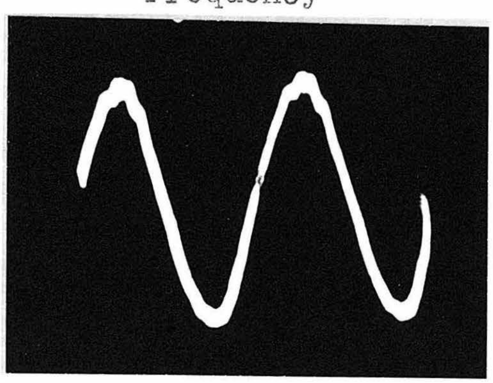
70Kc. Deviation
Frequency



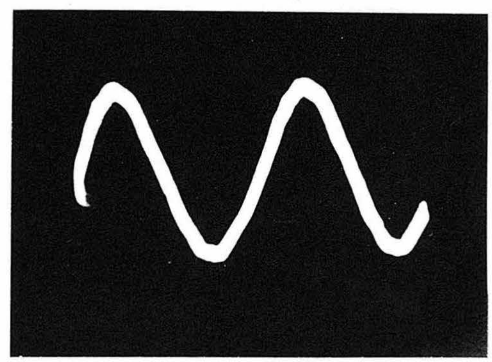
50 Kc. Deviation
Frequency



40 Kc. Deviation
Frequency



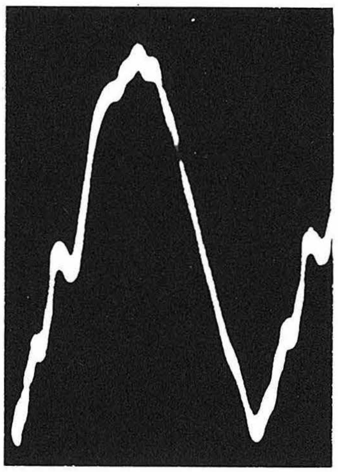
30 Kc. Deviation
Frequency



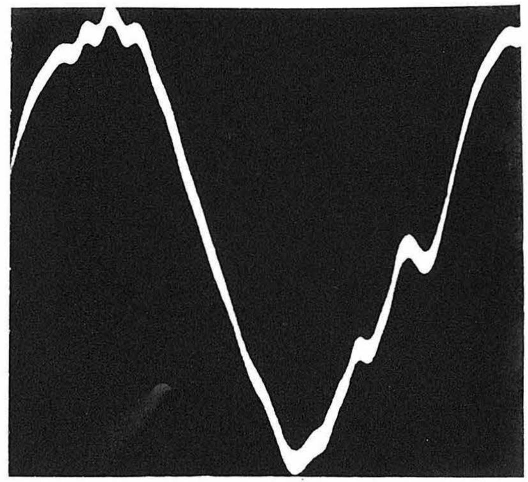
20 Kc. Deviation
Frequency

Plate V. Effect of Circular Antenna.
Glendale Site. Frequency Modulation Reception.
5000 cp.s. Modulating Frequency

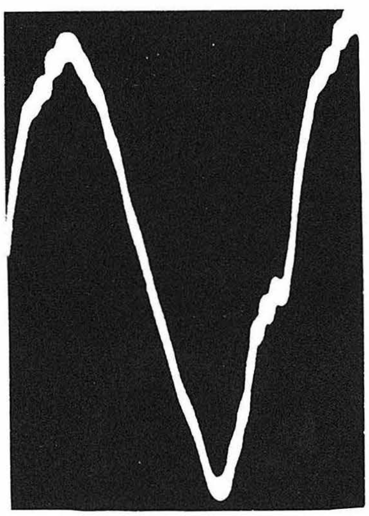
Directional Antenna

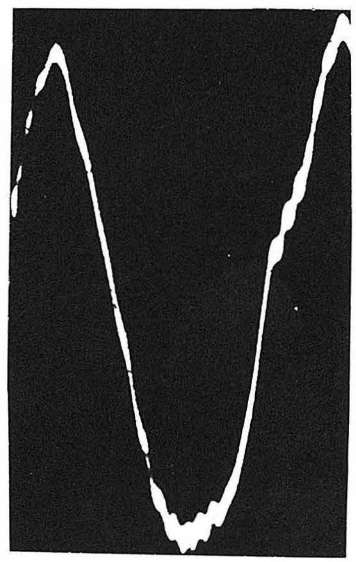


26.5 meters
200°
75 Kc. Deviation Frequency



17.5 Meters
-60°
75 Kc. Deviation Freq.





50°, 22.0 Meters

70 Kc. Deviation Freq.

75 Kc. Deviation Freq.

Plate VI: Effect of Displacement of Antenna.
Glendale Site. Frequency Modulation Reception.
5000 c.p.s. Modulating Frequency.

SQUARE WAVE MODULATION OF CONTINUOUS WAVE SYSTEMS

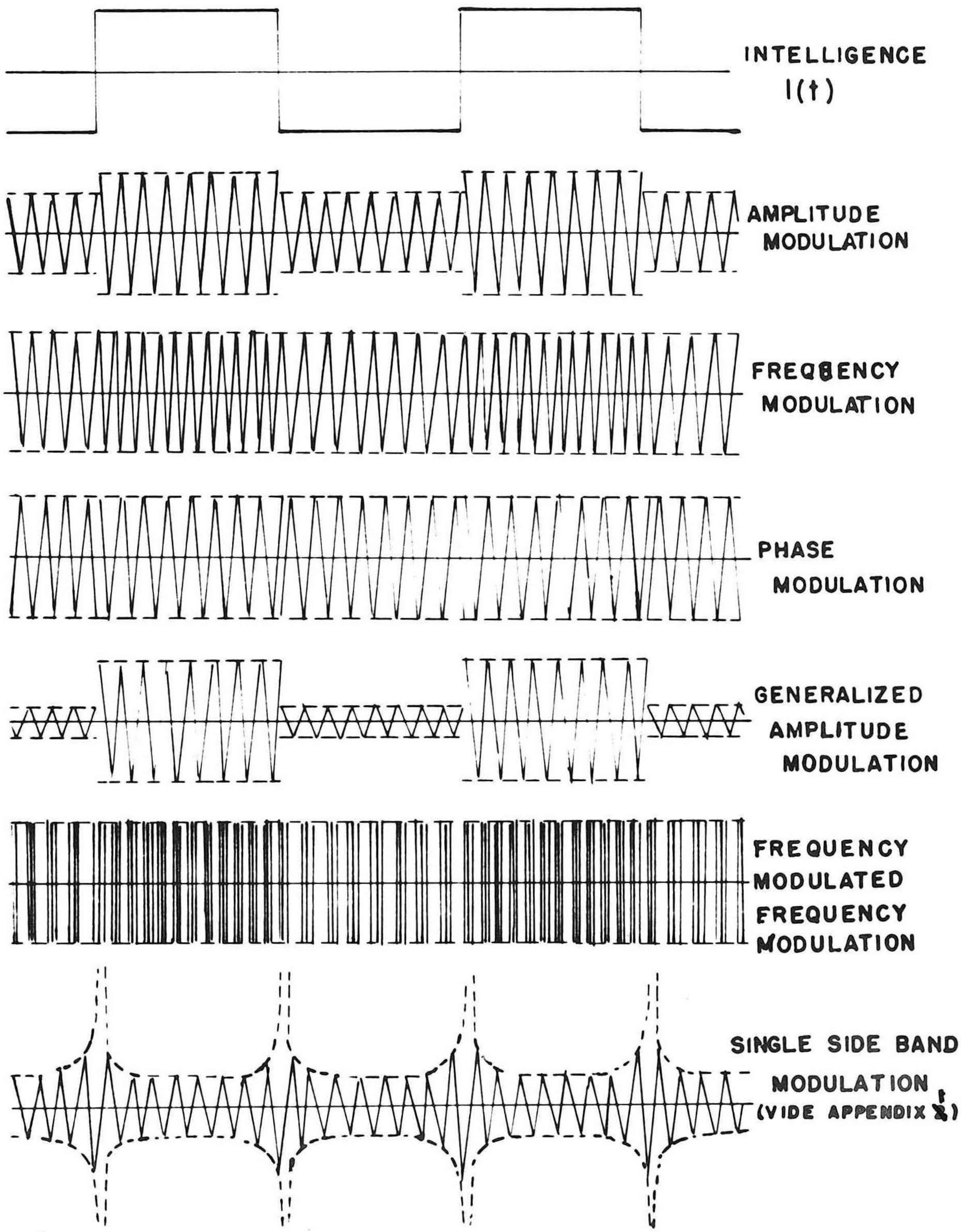


FIGURE I

SQUARE WAVE MODULATION OF PULSE WAVE SYSTEMS

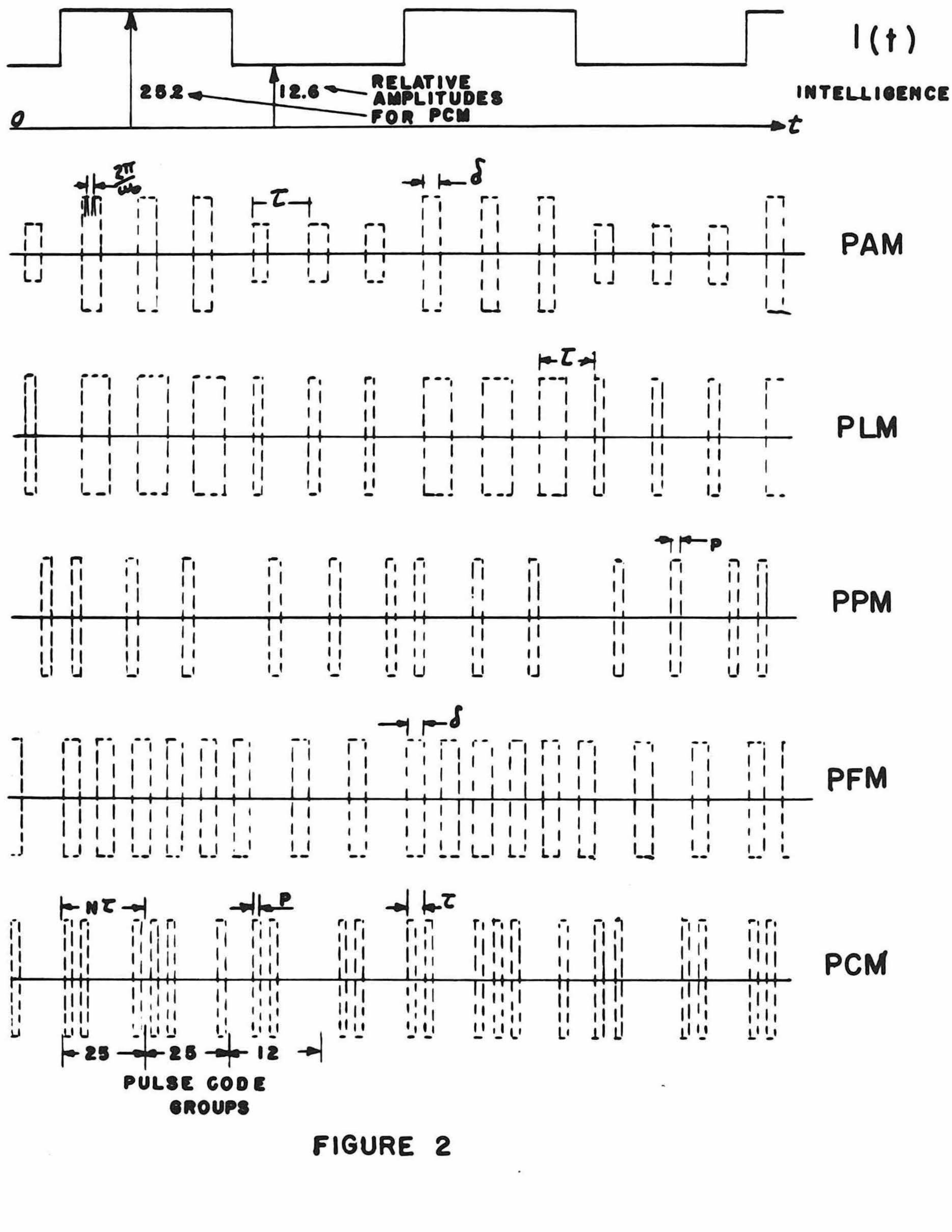


FIGURE 2

DETECTION OF PAM

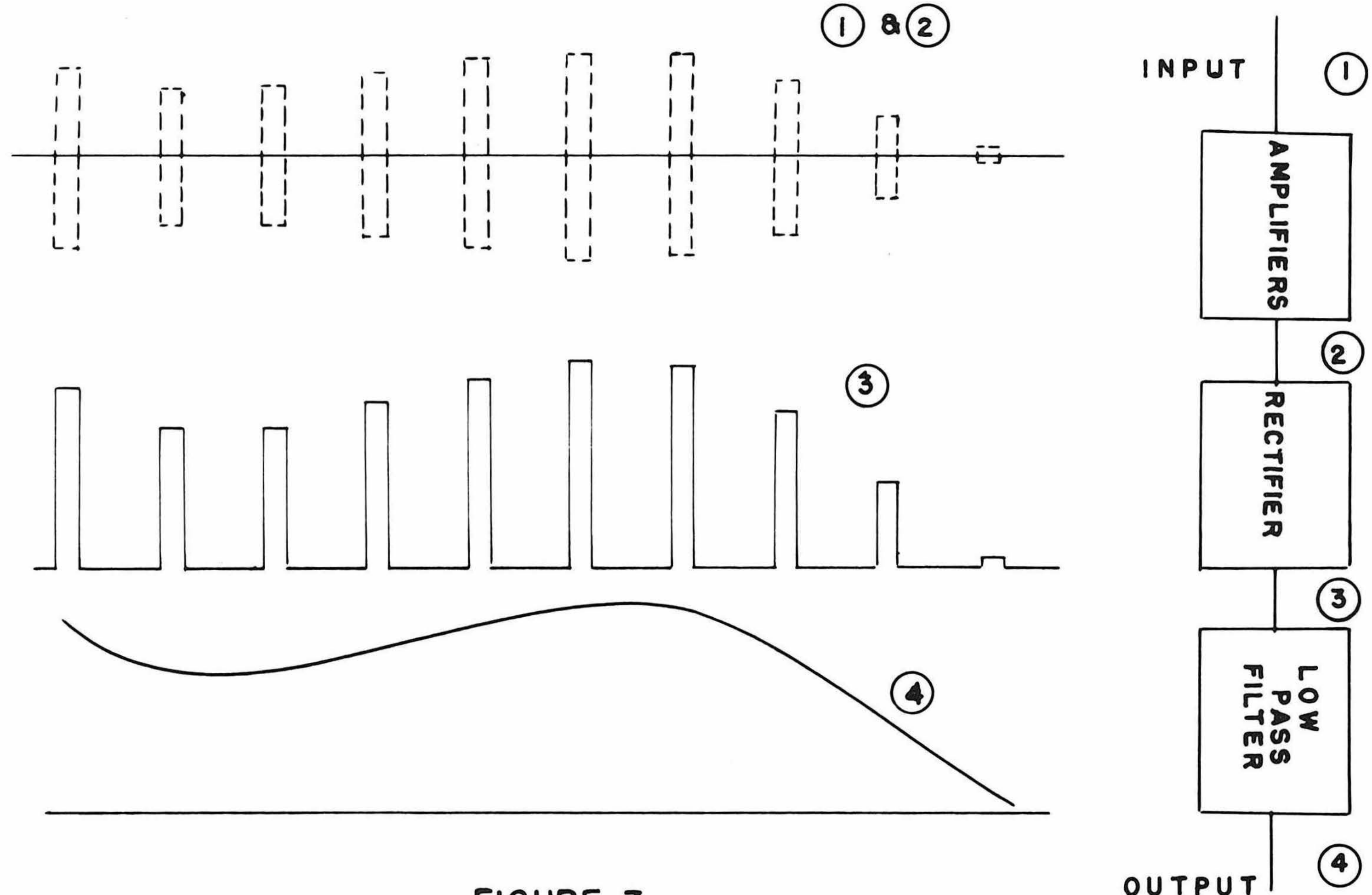


FIGURE 3

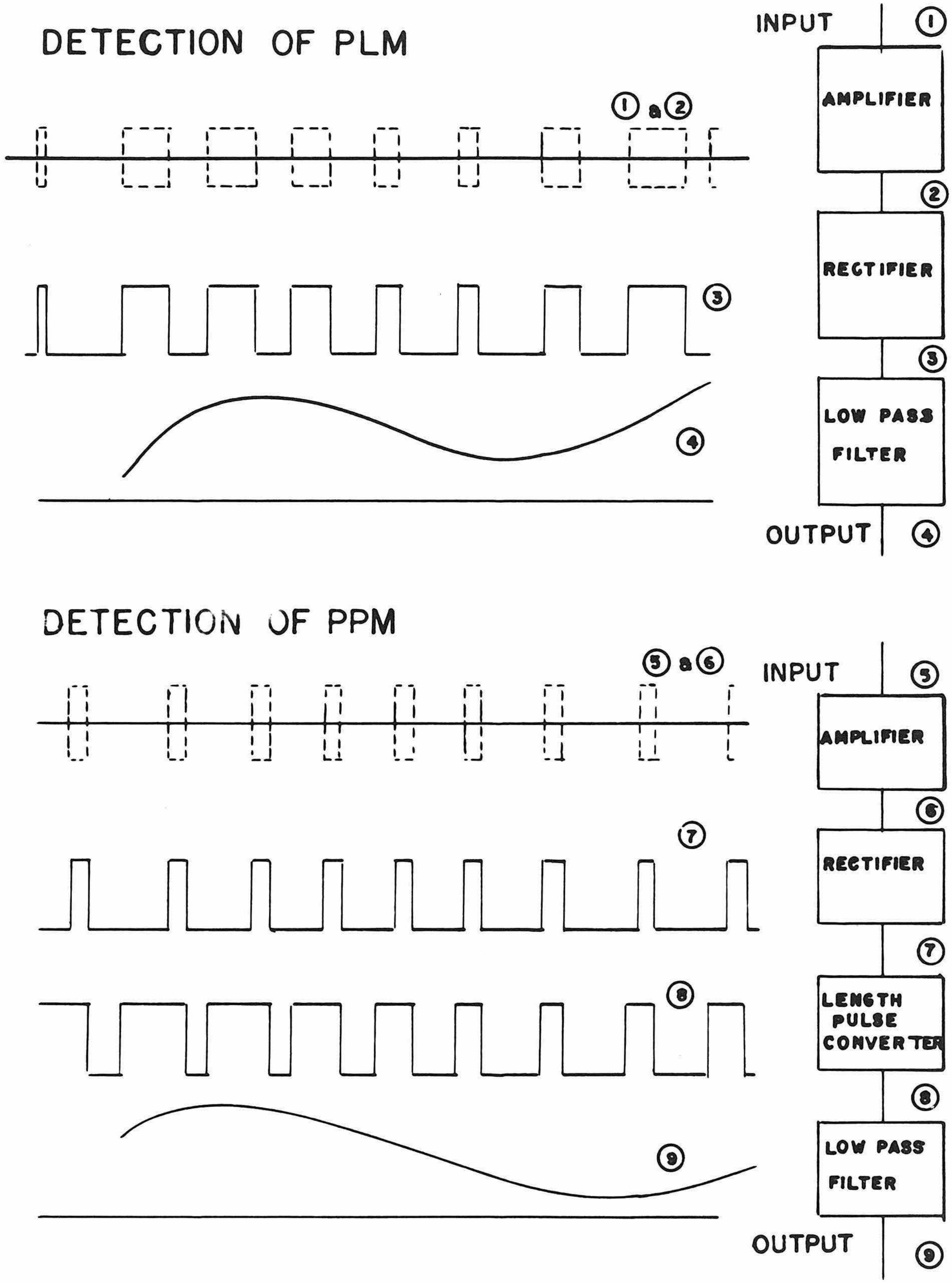

DETECTION OF PLM
INPUT
①
AMPLIFIER
① a ②
②
RECTIFIER
③
③
LOW PASS
FILTER
④
OUTPUT
④
DETECTION OF PPM
⑤ a ⑥
INPUT
⑤
AMPLIFIER
⑥
⑦
RECTIFIER
⑦
⑧
LENGTH
PULSE
CONVERTER
⑧
LOW PASS
FILTER
⑨
OUTPUT
⑨

FIGURE .4

DETECTION OF PCM

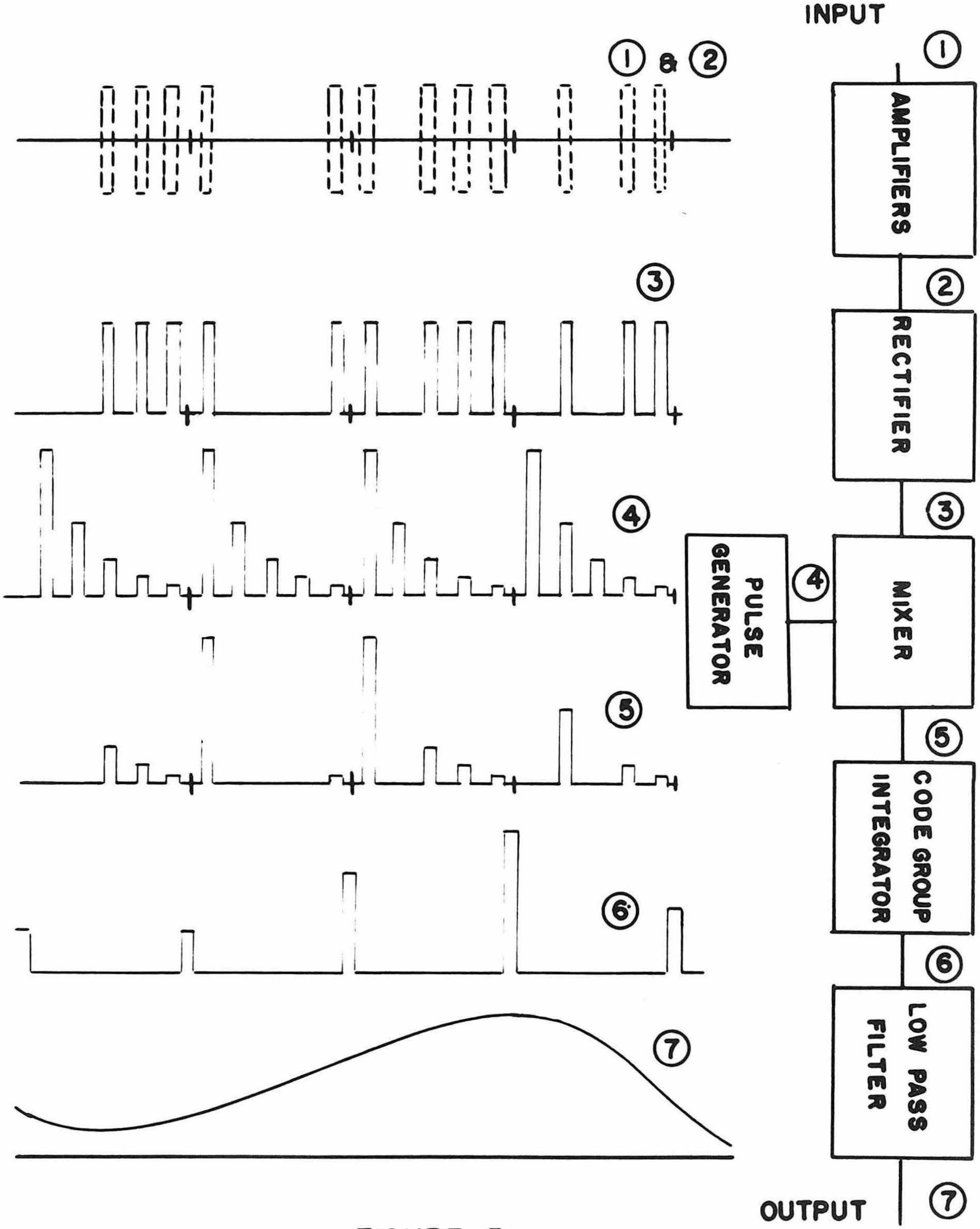


FIGURE 5

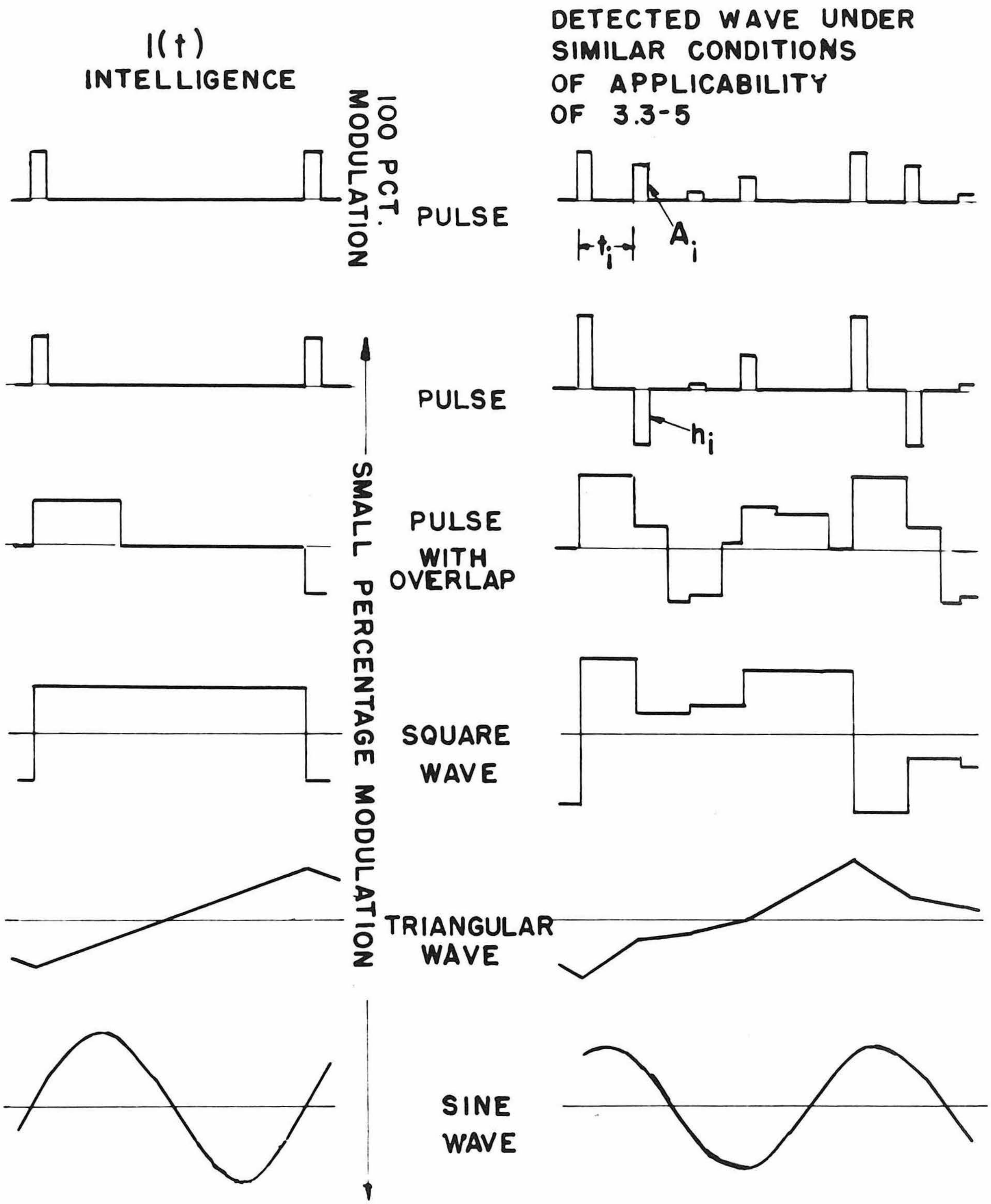

I(t)
INTELLIGENCE
DETECTED WAVE UNDER SIMILAR CONDITIONS OF APPLICABILITY OF 3.3-5
100 PCT. MODULATION
PULSE
A
t
PULSE
h
SMALL PERCENTAGE MODULATION
PULSE WITH OVERLAP
SQUARE WAVE
TRIANGULAR WAVE
SINE WAVE

FIGURE 7

TWO SIGNAL COMPARISON
OF
AMPLITUDE (AM), PHASE (PM), AND FREQUENCY (FM) MODULATION

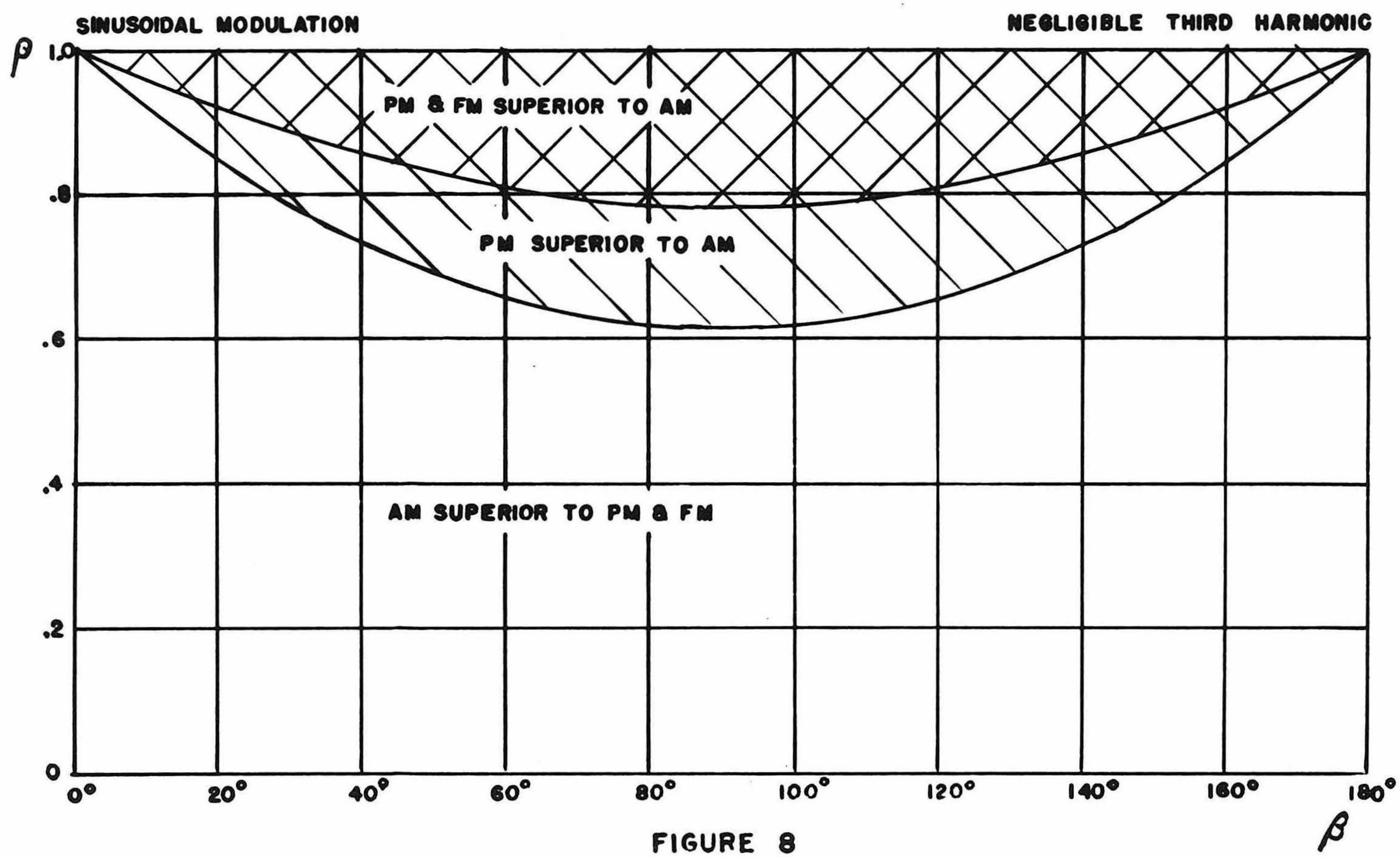


FIGURE 8

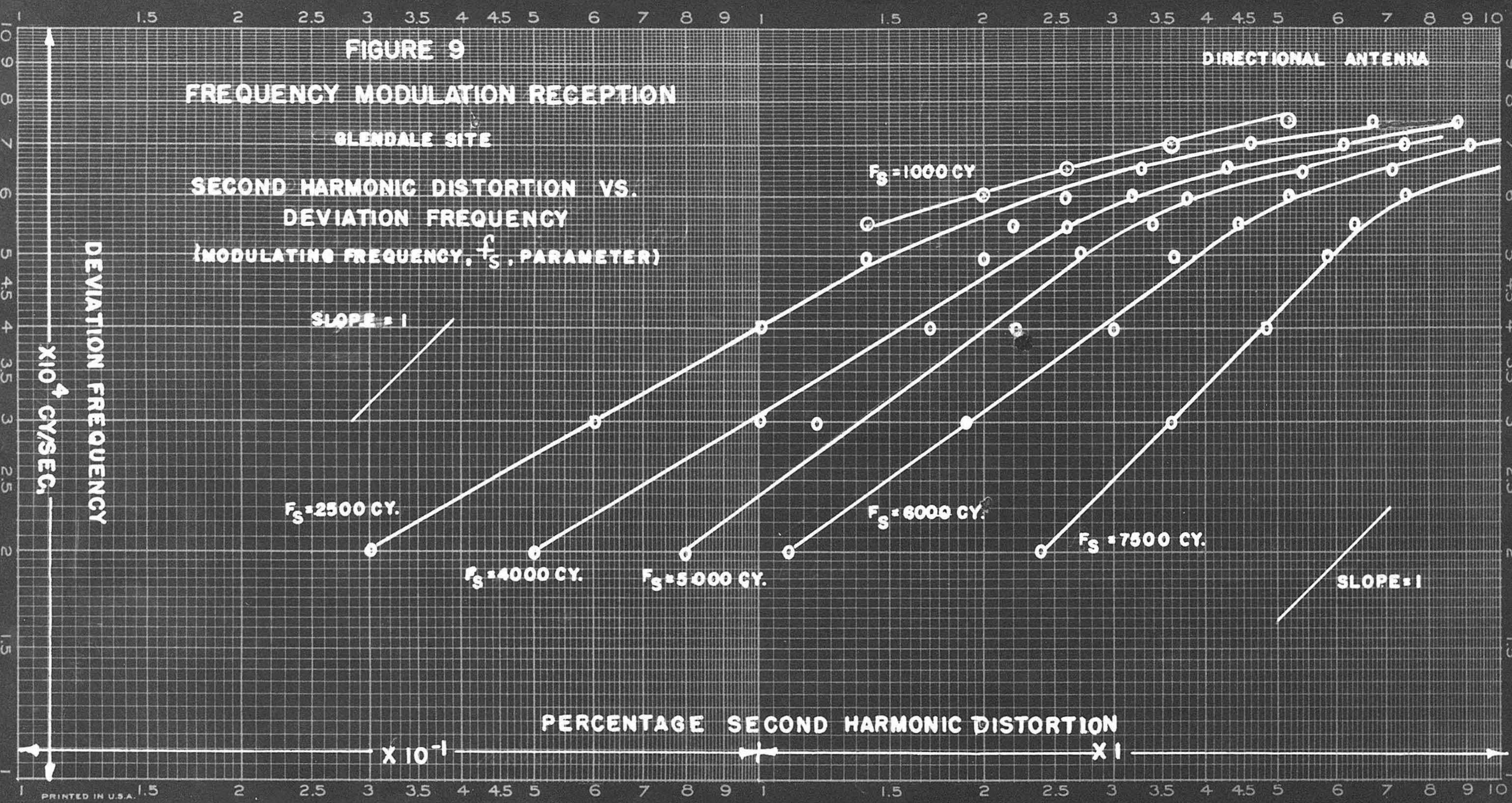


FIGURE 9
FREQUENCY MODULATION RECEPTION
GLENDALE SITE
SECOND HARMONIC DISTORTION VS.
DEVIATION FREQUENCY
(MODULATING FREQUENCY, f_S, PARAMETER)
DIRECTIONAL ANTENNA
SLOPE = 1
SLOPE = 1
F_S = 1000 CY
F_S = 2500 CY.
F_S = 4000 CY.
F_S = 5000 CY.
F_S = 6000 CY.
F_S = 7500 CY.
DEVIATION FREQUENCY
X10^4 CY/SEC.
PERCENTAGE SECOND HARMONIC DISTORTION
X 10^{-1}
X 1
PRINTED IN U.S.A.

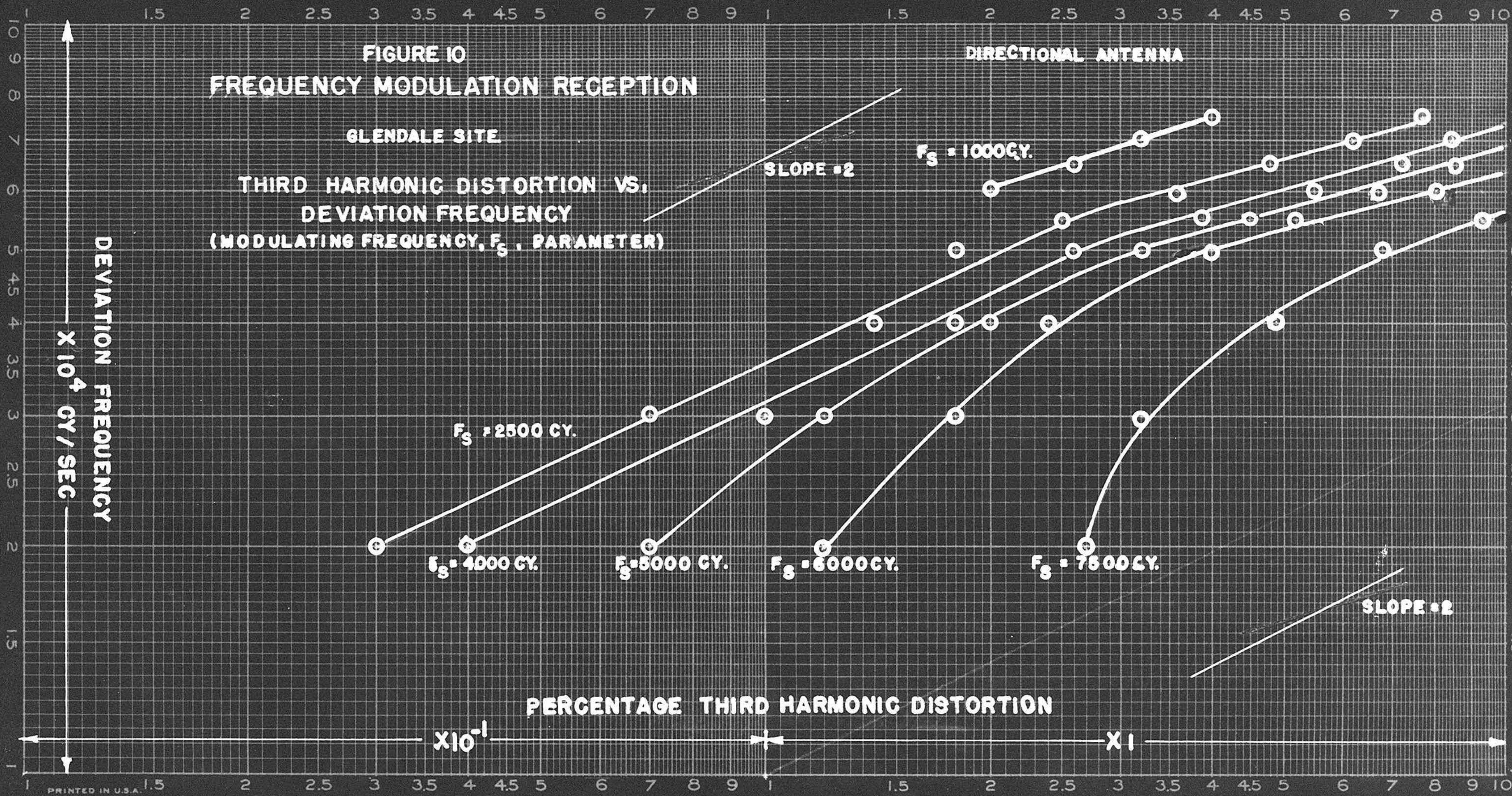


FIGURE 10
FREQUENCY MODULATION RECEPTION
GLENDALE SITE
THIRD HARMONIC DISTORTION VS.
DEVIATION FREQUENCY
(MODULATING FREQUENCY, F_S, PARAMETER)
DIRECTIONAL ANTENNA
SLOPE = 2
F_S = 1000 CY.
F_S = 2500 CY.
F_S = 4000 CY.
F_S = 5000 CY.
F_S = 6000 CY.
F_S = 7500 CY.
SLOPE = 2
PERCENTAGE THIRD HARMONIC DISTORTION
X10^{-1}
X1
DEVIATION FREQUENCY
X 10^4 CY/SEC
PRINTED IN U.S.A.

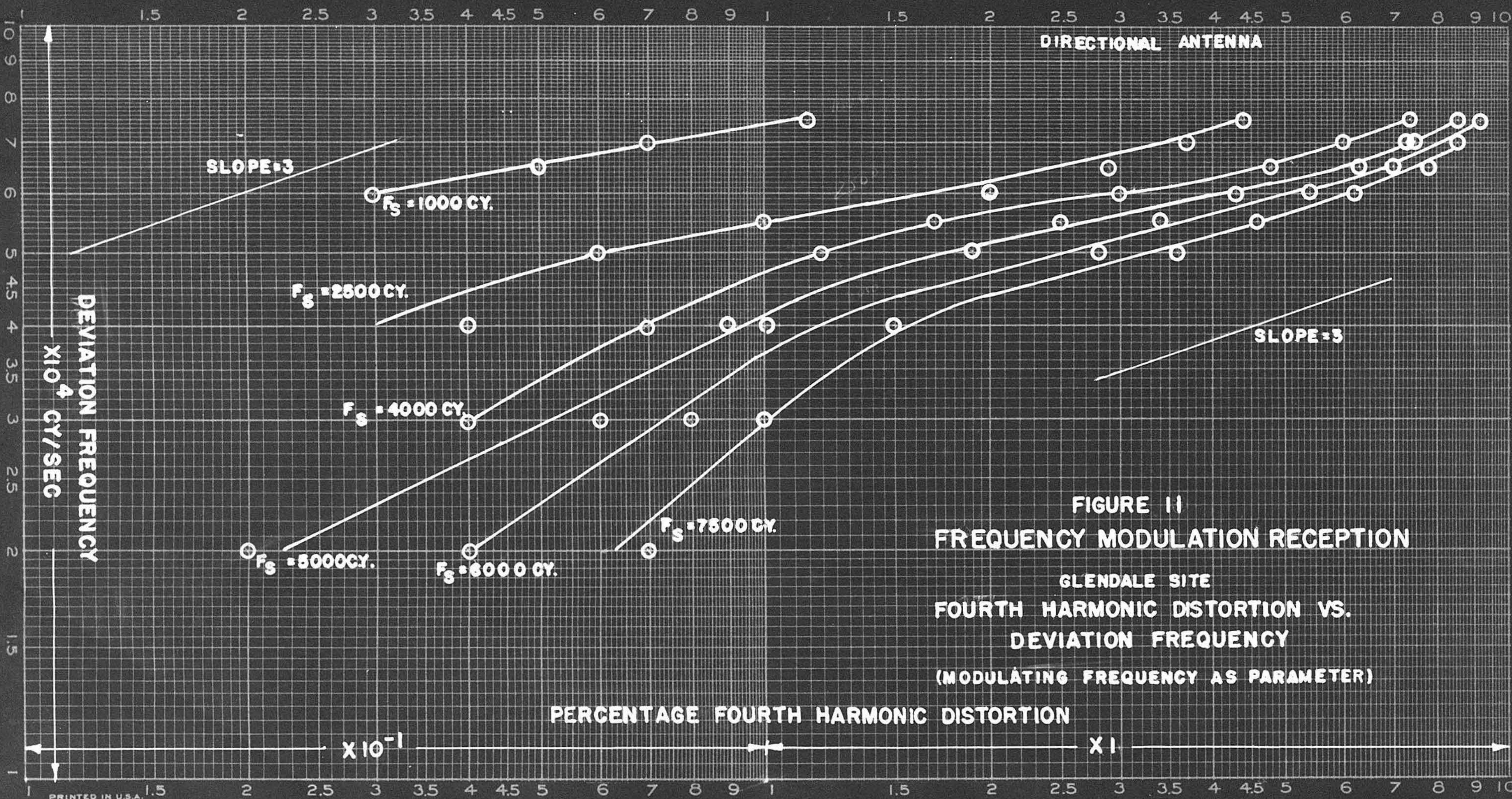


FIGURE 11
FREQUENCY MODULATION RECEPTION
GLENDALE SITE
FOURTH HARMONIC DISTORTION VS. DEVIATION FREQUENCY
(MODULATING FREQUENCY AS PARAMETER)

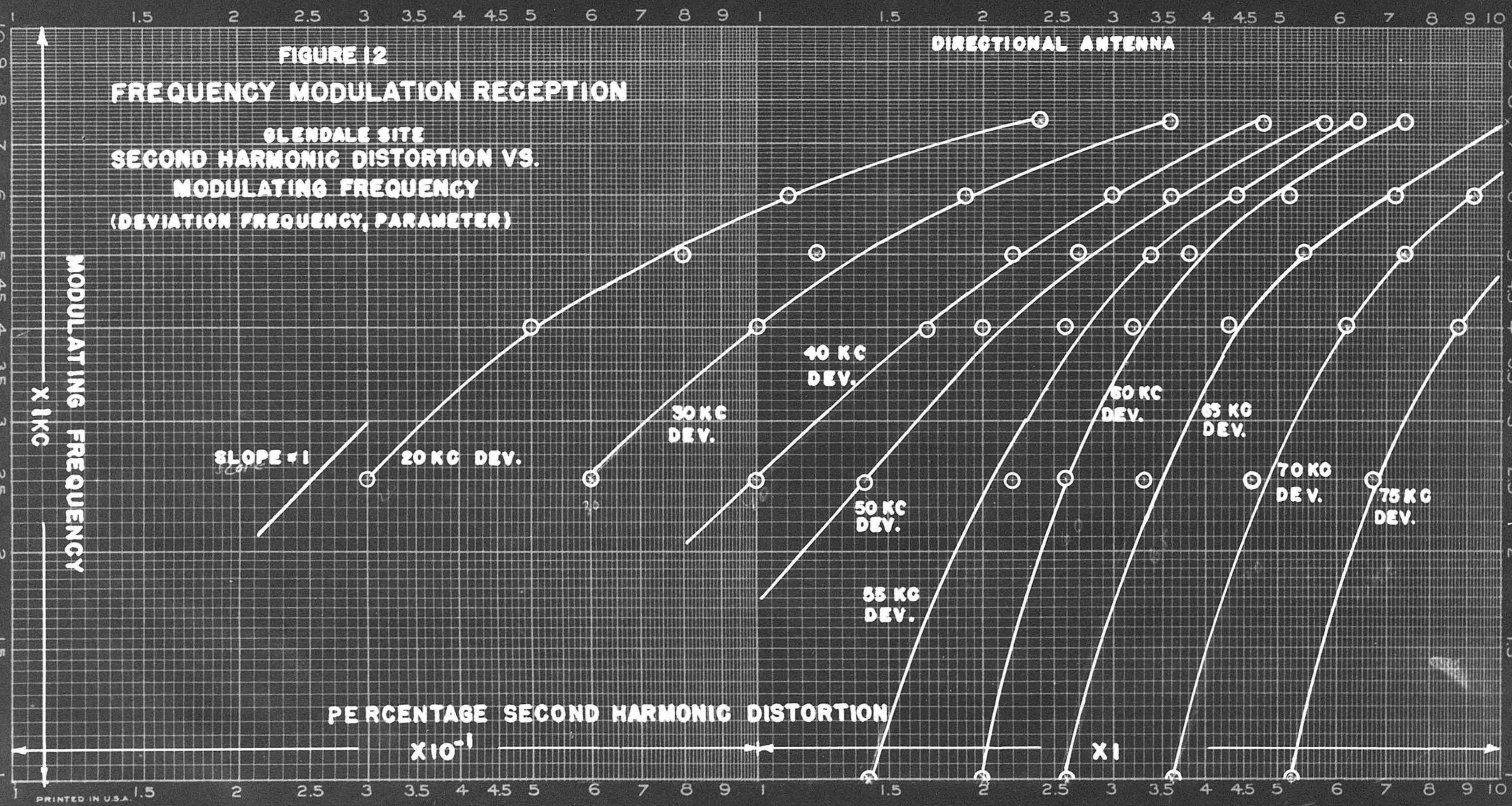


FIGURE 12
FREQUENCY MODULATION RECEPTION
GLENDALE SITE
SECOND HARMONIC DISTORTION VS.
MODULATING FREQUENCY
(DEVIATION FREQUENCY, PARAMETER)
DIRECTIONAL ANTENNA
SLOPE = 1
20 KC DEV.
30 KC DEV.
40 KC DEV.
50 KC DEV.
55 KC DEV.
60 KC DEV.
65 KC DEV.
70 KC DEV.
75 KC DEV.
MODULATING FREQUENCY
X 1KC
PERCENTAGE SECOND HARMONIC DISTORTION
X10^{-1}
X1
PRINTED IN U.S.A.

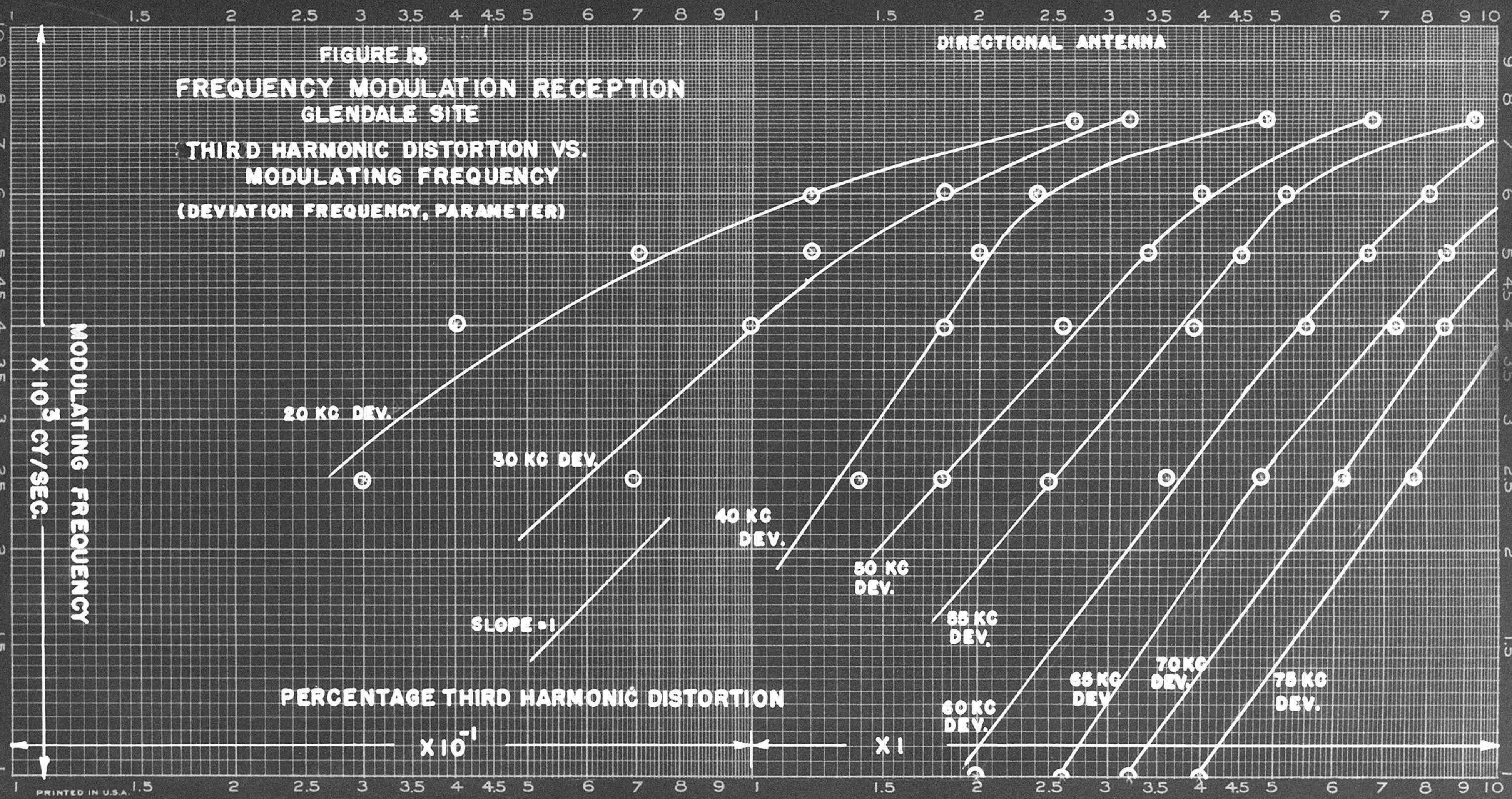


FIGURE 13
FREQUENCY MODULATION RECEPTION
GLENDALE SITE
THIRD HARMONIC DISTORTION VS.
MODULATING FREQUENCY
(DEVIATION FREQUENCY, PARAMETER)
DIRECTIONAL ANTENNA
20 KC DEV.
30 KC DEV.
40 KC DEV.
50 KC DEV.
55 KC DEV.
60 KC DEV.
65 KC DEV.
70 KC DEV.
75 KC DEV.
SLOPE = 1
PERCENTAGE THIRD HARMONIC DISTORTION
X 10⁻¹
X 1
MODULATING FREQUENCY
X 10³ CY/SEC.
PRINTED IN U.S.A.

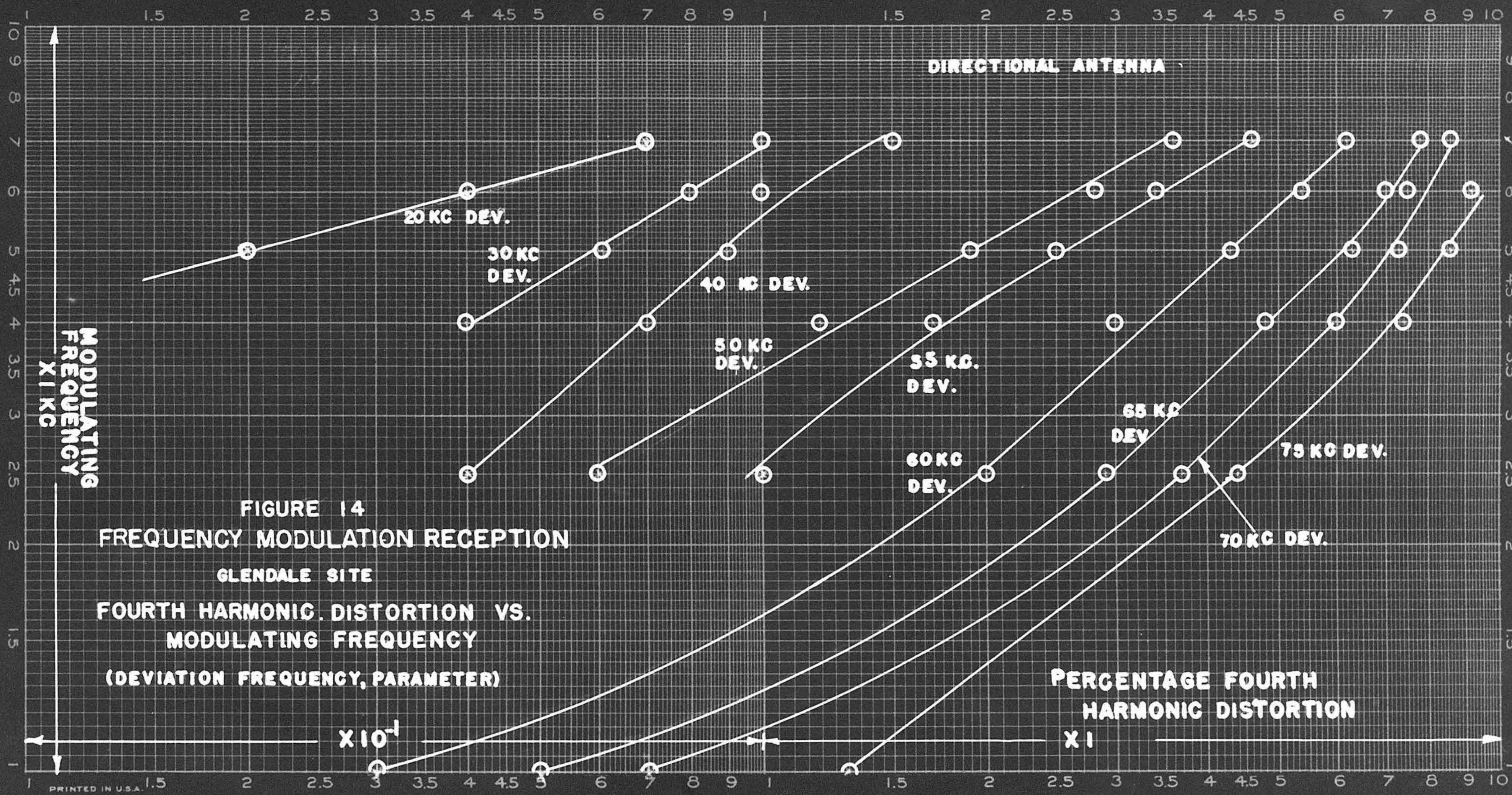
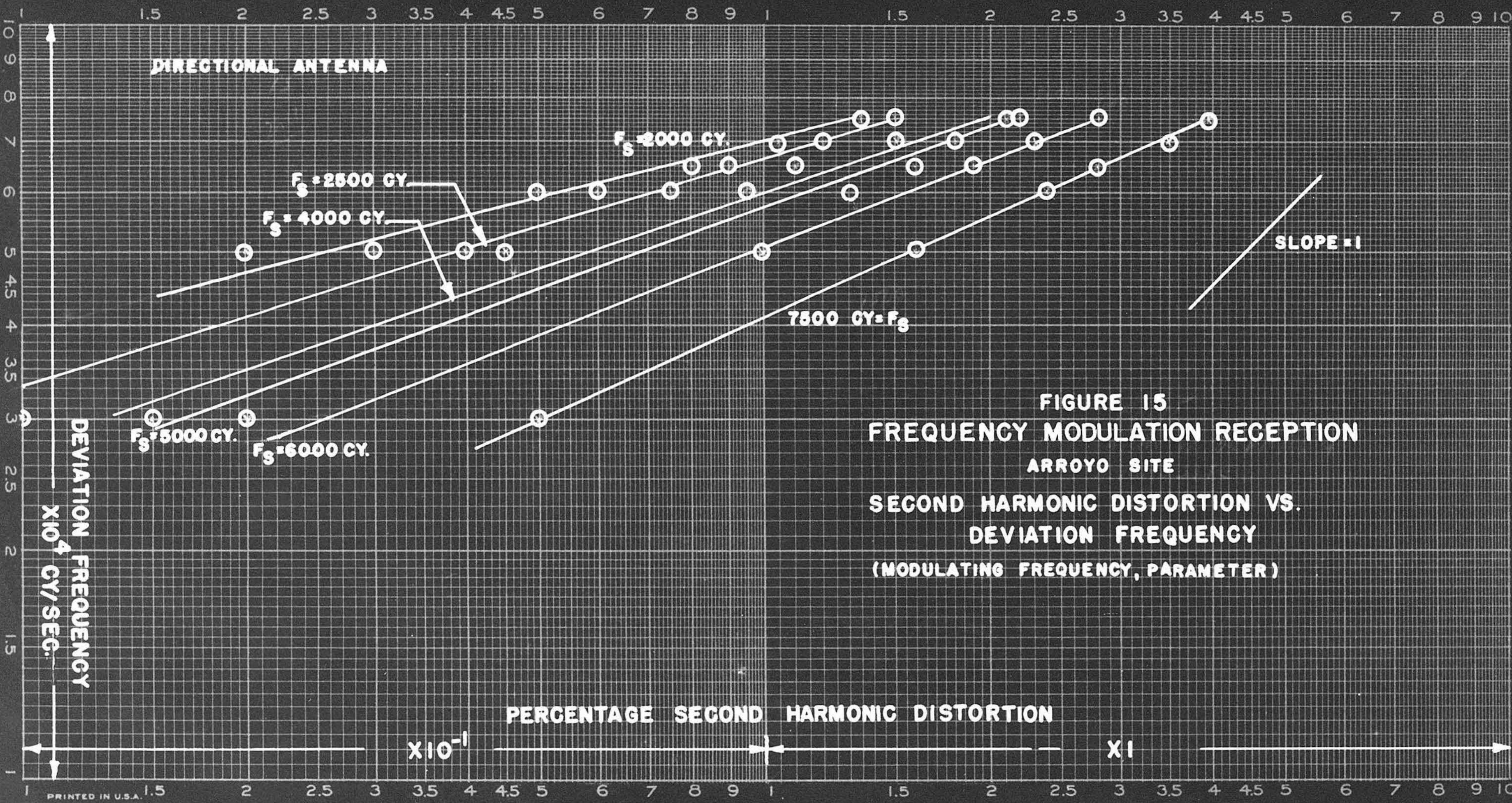


FIGURE 14
FREQUENCY MODULATION RECEPTION
GLENDALE SITE
FOURTH HARMONIC DISTORTION VS. MODULATING FREQUENCY
(DEVIATION FREQUENCY, PARAMETER)
DIRECTIONAL ANTENNA
MODULATING FREQUENCY X I KC
PERCENTAGE FOURTH HARMONIC DISTORTION
X10⁻¹
X I
20 KC DEV.
30 KC DEV.
40 KC DEV.
50 KC DEV.
55 KC. DEV.
60 KC DEV.
65 KC DEV
70 KC DEV.
75 KC DEV.
PRINTED IN U.S.A.


DIRECTIONAL ANTENNA
F_S=2000 CY.
F_S=2500 CY.
F_S=4000 CY.
F_S=5000 CY.
F_S=6000 CY.
7500 CY=F_S
SLOPE=1
FIGURE 15
FREQUENCY MODULATION RECEPTION
ARROYO SITE
SECOND HARMONIC DISTORTION VS.
DEVIATION FREQUENCY
(MODULATING FREQUENCY, PARAMETER)
DEVIATION FREQUENCY
X10[4] CY/SEC.
PERCENTAGE SECOND HARMONIC DISTORTION
X10[-1]
X1
PRINTED IN U.S.A.

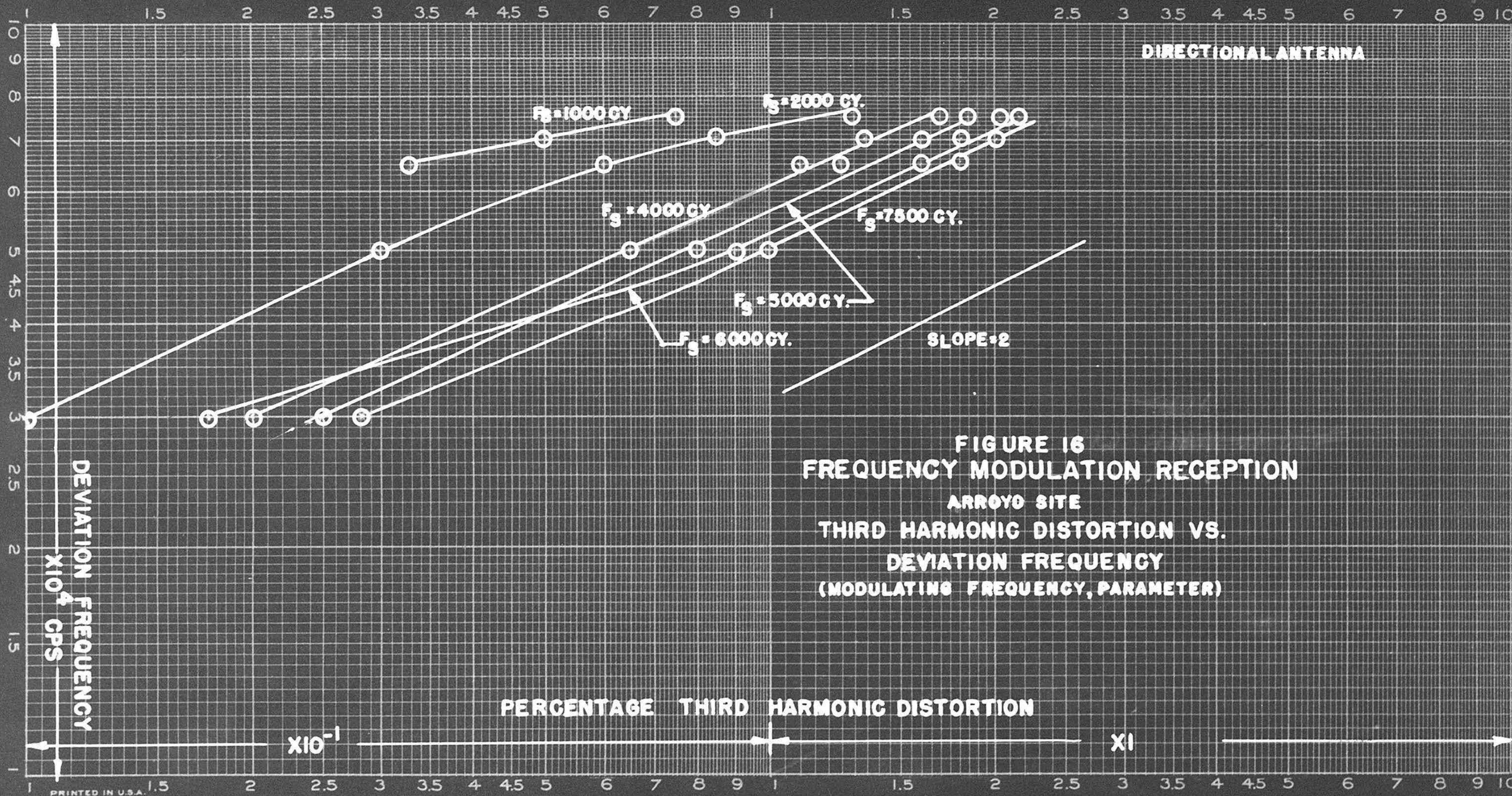


FIGURE 16
FREQUENCY MODULATION RECEPTION
ARROYO SITE
THIRD HARMONIC DISTORTION VS.
DEVIATION FREQUENCY
(MODULATING FREQUENCY, PARAMETER)

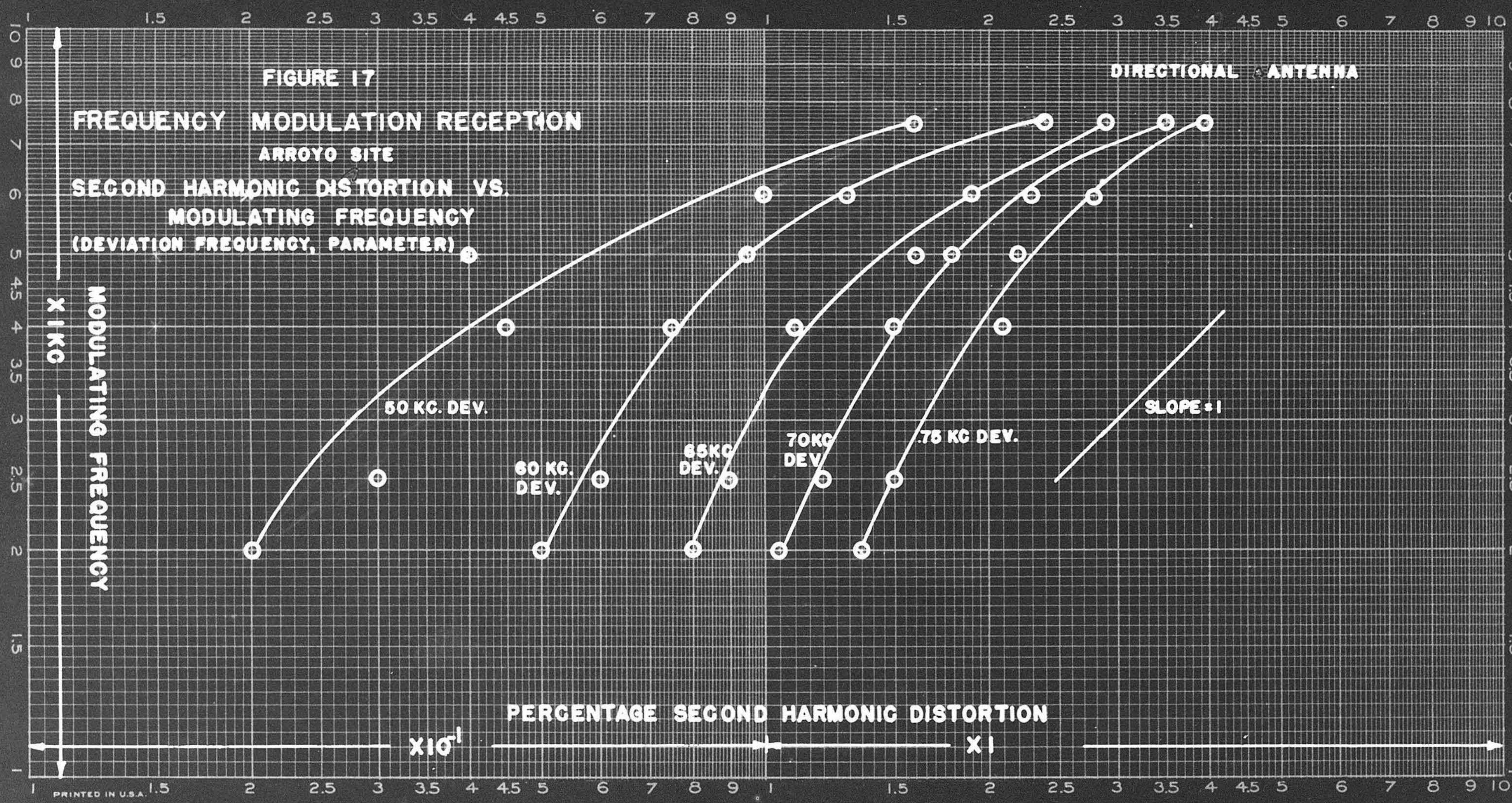


FIGURE 17
FREQUENCY MODULATION RECEPTION
ARROYO SITE
SECOND HARMONIC DISTORTION VS.
MODULATING FREQUENCY
(DEVIATION FREQUENCY, PARAMETER)
DIRECTIONAL ANTENNA
50 KC. DEV.
60 KC. DEV.
65KC DEV.
70KC DEV.
75 KC DEV.
SLOPE = 1
MODULATING FREQUENCY
X 1 KC
PERCENTAGE SECOND HARMONIC DISTORTION
X10⁻¹
X 1
PRINTED IN U.S.A.

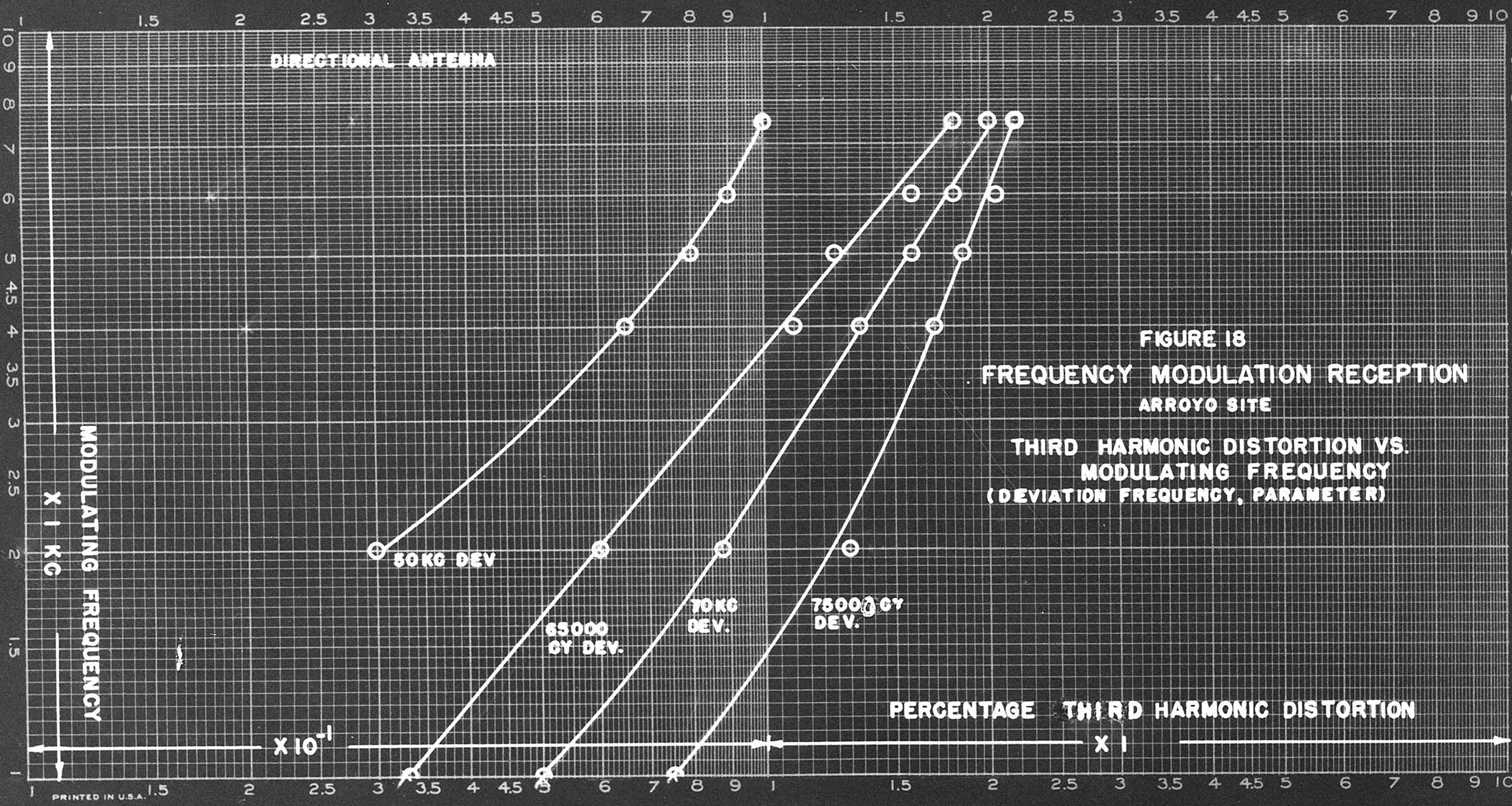


FIGURE 18
FREQUENCY MODULATION RECEPTION
ARROYO SITE
THIRD HARMONIC DISTORTION VS. MODULATING FREQUENCY
(DEVIATION FREQUENCY, PARAMETER)
DIRECTIONAL ANTENNA
50KC DEV
65000 CY DEV.
70KC DEV.
75000 CY DEV.
MODULATING FREQUENCY
X 1 KC
PERCENTAGE THIRD HARMONIC DISTORTION
X 10^-1
X 1
PRINTED IN U.S.A.

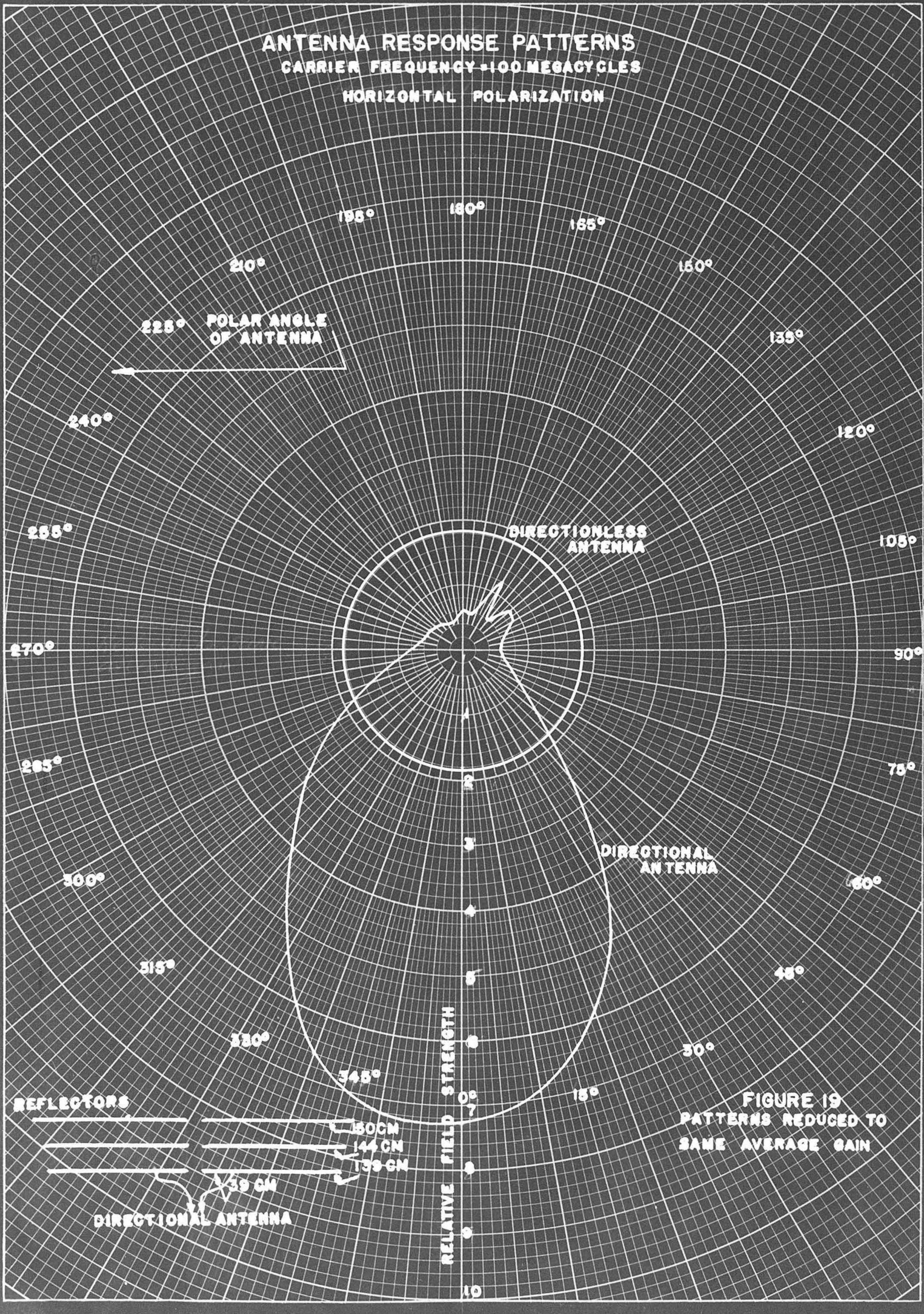


FIGURE 19
PATTERNS REDUCED TO SAME AVERAGE GAIN

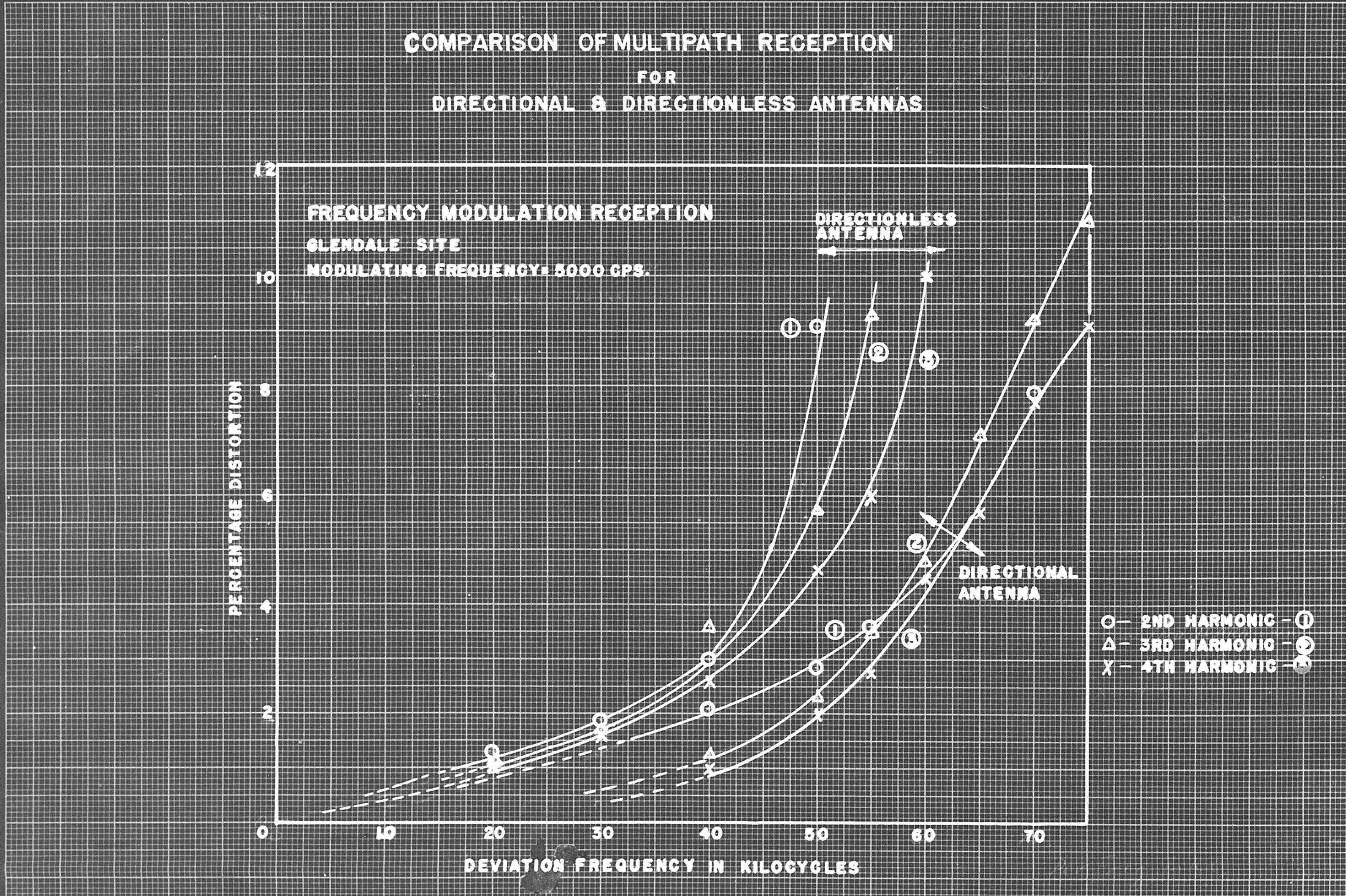
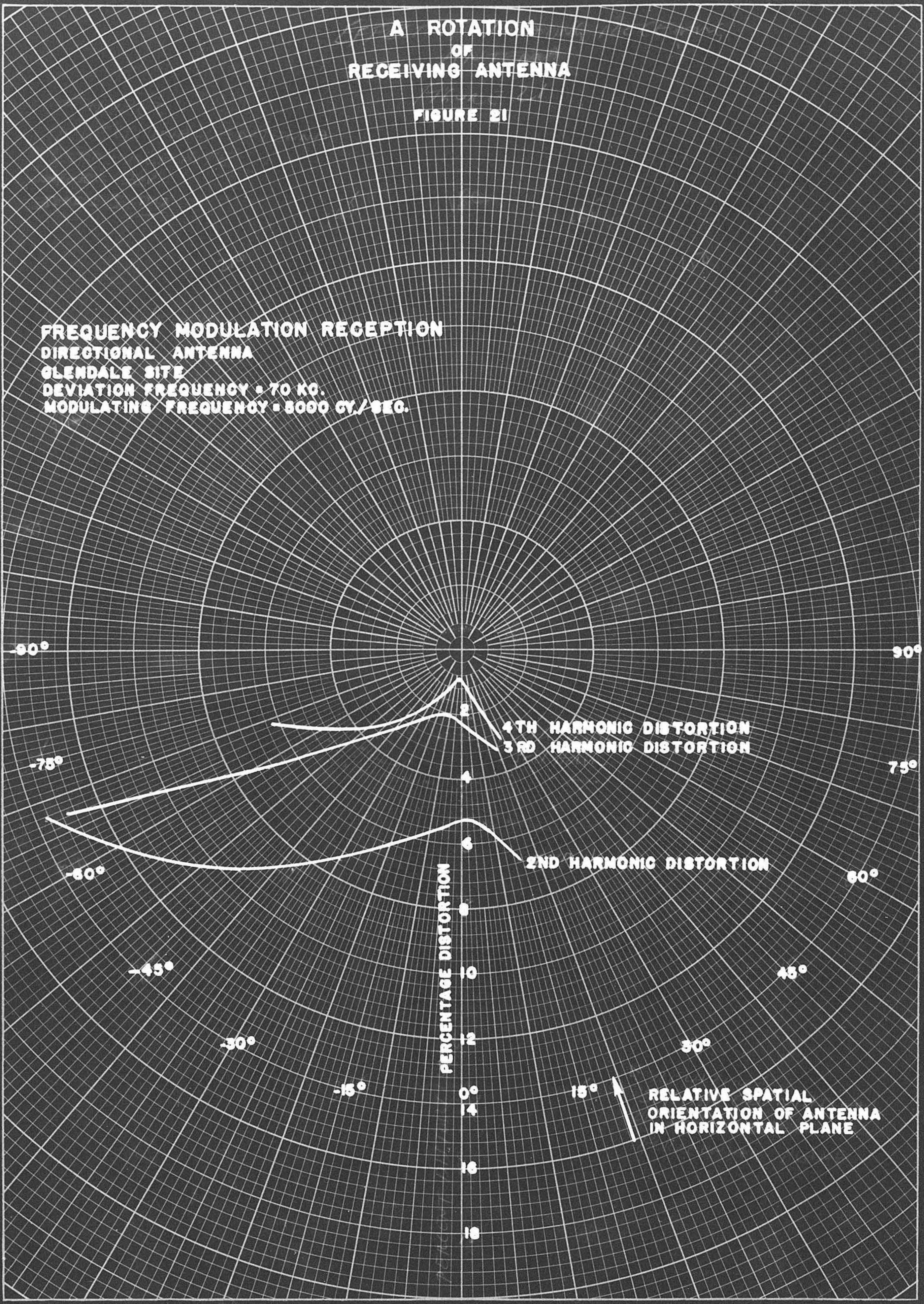
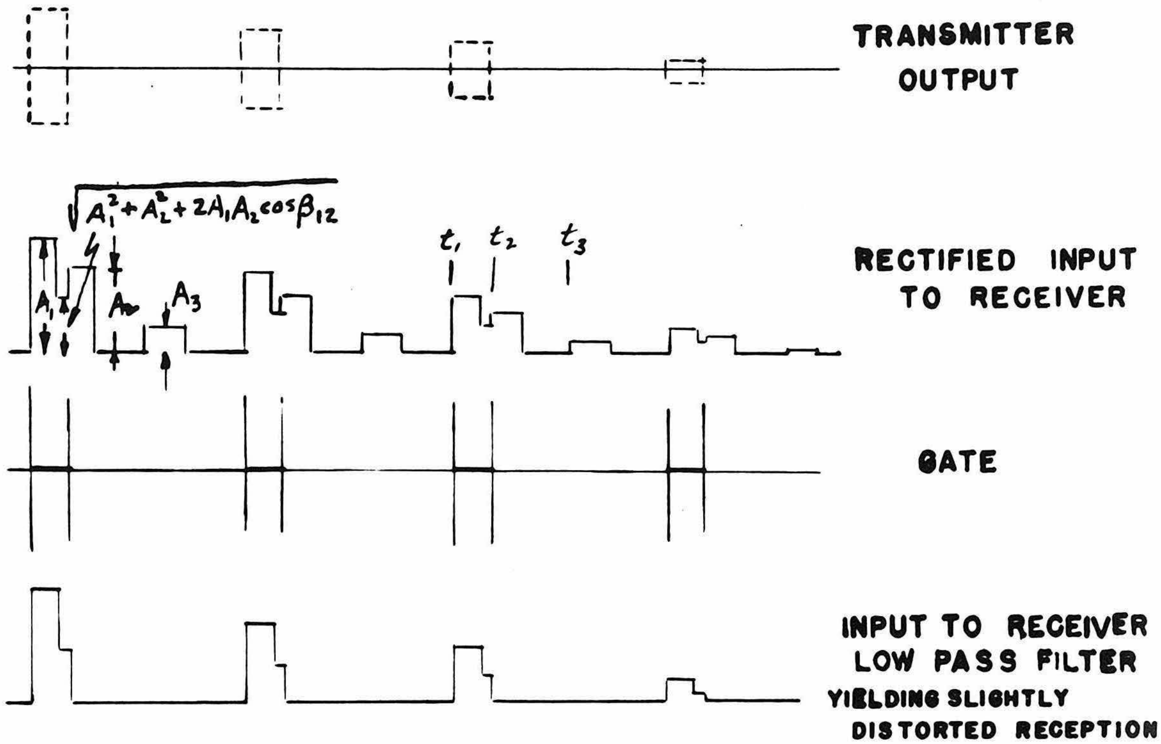

COMPARISON OF MULTIPATH RECEPTION
FOR
DIRECTIONAL & DIRECTIONLESS ANTENNAS
FREQUENCY MODULATION RECEPTION
GLENDALE SITE
MODULATING FREQUENCY= 5000 CPS.
DIRECTIONLESS ANTENNA
DIRECTIONAL ANTENNA
PERCENTAGE DISTORTION
DEVIATION FREQUENCY IN KILOCYCLES
0
2
4
6
8
10
12
10
20
30
40
50
60
70
○ — 2ND HARMONIC — ①
△ — 3RD HARMONIC — ②
X — 4TH HARMONIC — ③

FIGURE 20


A ROTATION
OF
RECEIVING ANTENNA
FIGURE 21
FREQUENCY MODULATION RECEPTION
DIRECTIONAL ANTENNA
GLENDALE SITE
DEVIATION FREQUENCY = 70 KC.
MODULATING FREQUENCY = 5000 CY./SEC.
-90°
90°
-75°
75°
-60°
60°
-45°
45°
-30°
30°
-15°
15°
0°
2
4
6
8
10
12
14
16
18
PERCENTAGE DISTORTION
4TH HARMONIC DISTORTION
3RD HARMONIC DISTORTION
2ND HARMONIC DISTORTION
RELATIVE SPATIAL
ORIENTATION OF ANTENNA
IN HORIZONTAL PLANE

MULTIPATH RECEPTION OF PAM



MULTIPATH RECEPTION OF PCM

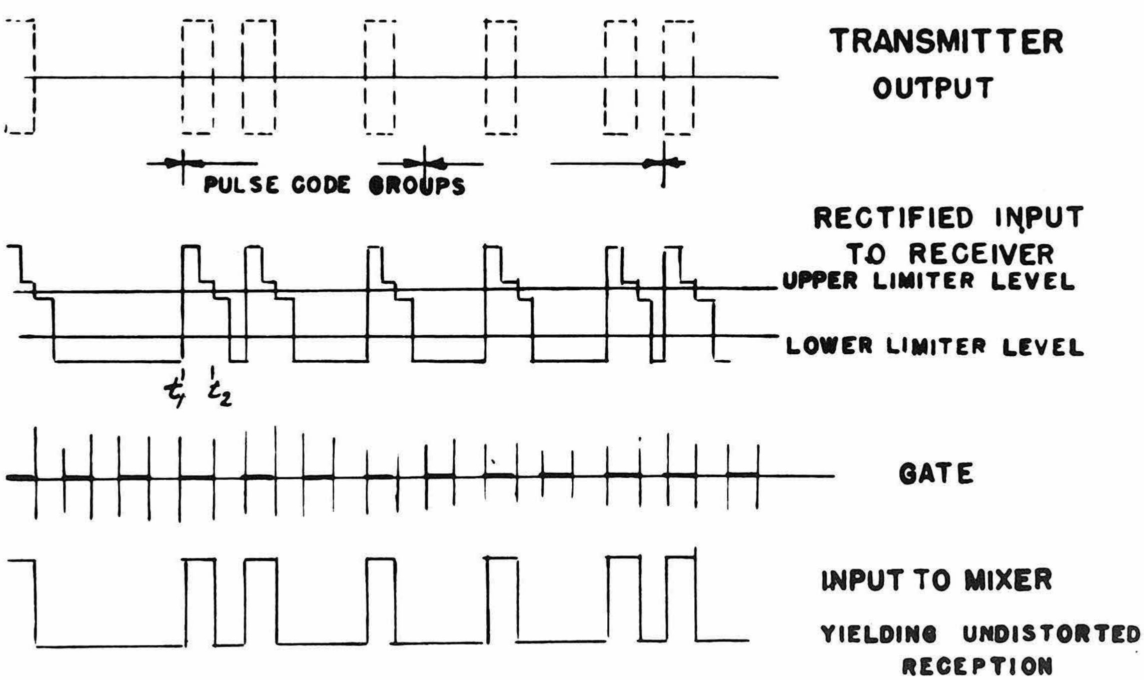


FIGURE 22

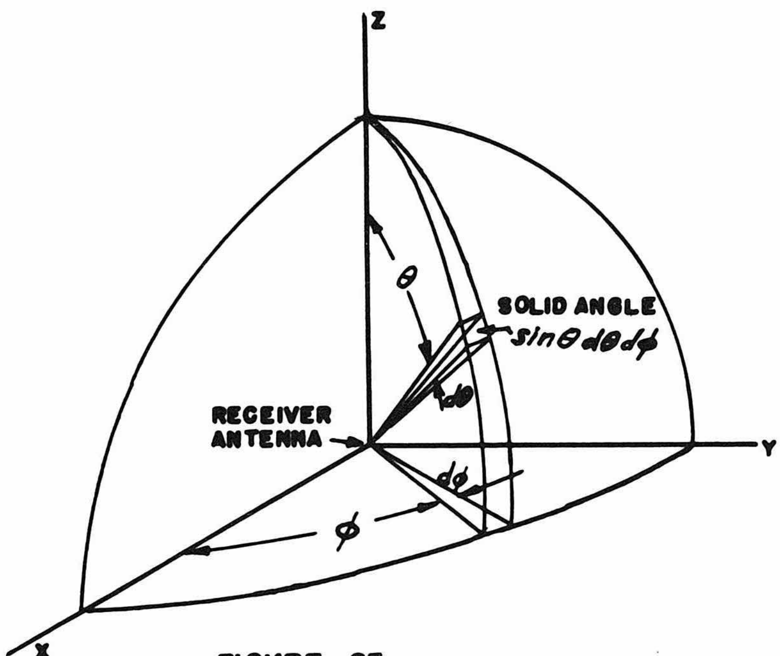


FIGURE 23

FIGURE 24

DIFFRACTION THROUGH SLIT

SQUARE WAVE MODULATION OF CONTINUOUS WAVE TRANSMITTER

SMALL PERCENTAGE MODULATION

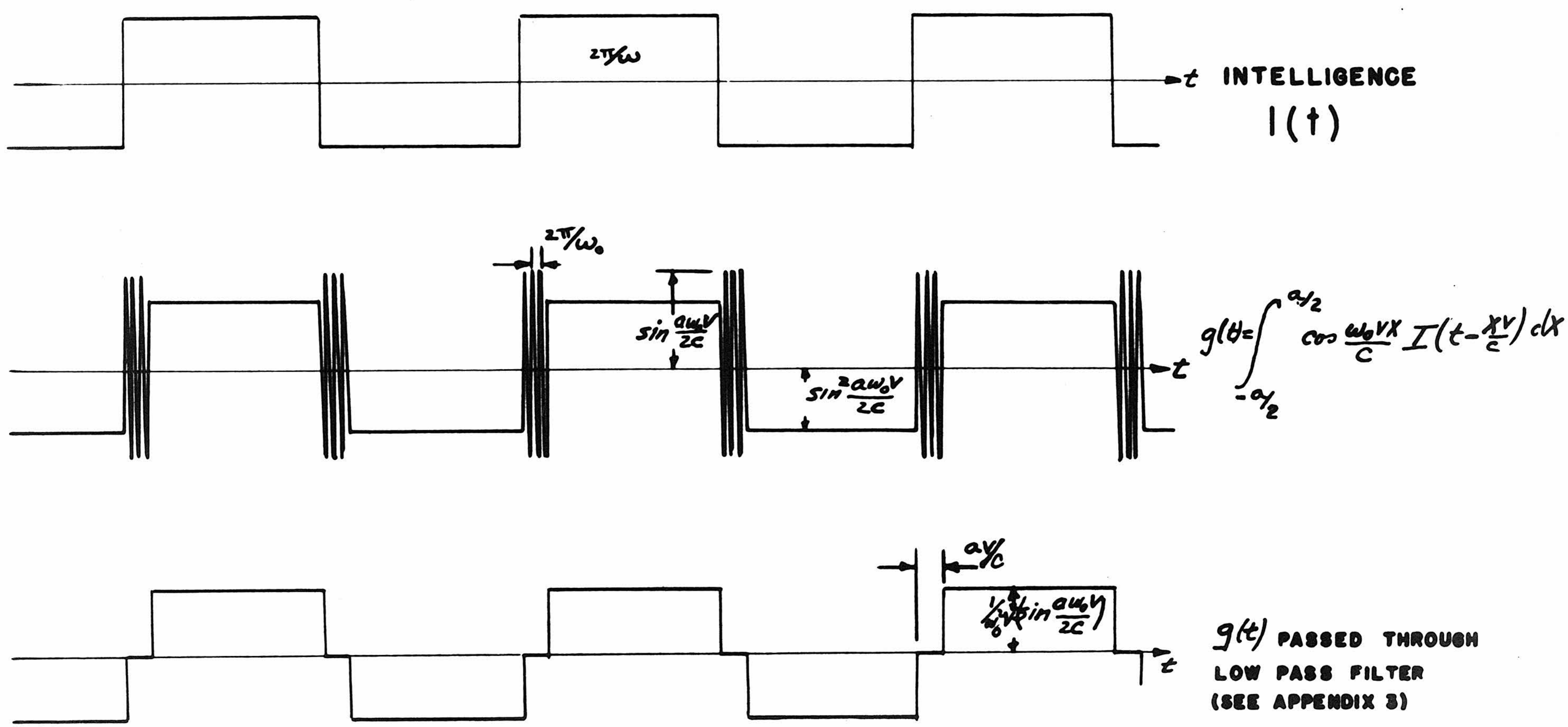
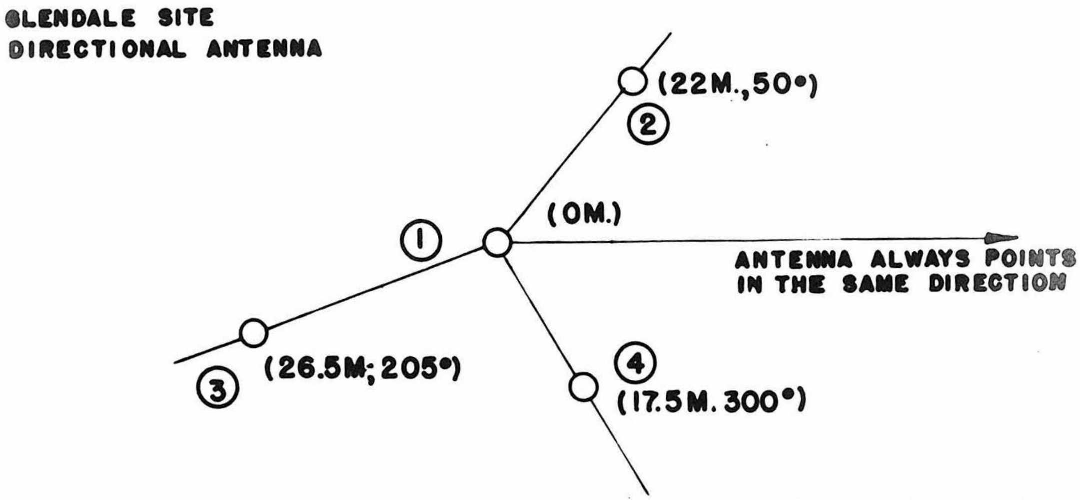


TABLE I

SPATIAL DISPLACEMENT OF ANTENNA



ALL PRECEDING DATA
TAKEN AT POSITION 1.

FREQUENCY MODULATION RECEPTION

POSITION	CARRIER FREQUENCY	MODULATING FREQUENCY	DEVIATION FREQUENCY	PERCENTAGE SECOND HARMONIC	PERCENTAGE THIRD HARMONIC
1	99.8 MC.	5000 CPS.	75 KC.	10.6	12.2
2	99.8	5000	75	10.6	9.6
2	99.8	5000	70	8.8	8.3
3	99.8	5000	75	10.4	11.4
4	99.8	5000	70	9.8	10.4