Invariant Combinatorics on Borel Equivalence Relations

Thesis by

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To Esther

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ABSTRACT

This thesis comprises four independent parts and an appendix.

1. We define and study expansion problems on countable structures in the setting of descriptive combinatorics. We consider both expansions on countable Borel equivalence relations and on countable groups, in the Borel, measure, and category settings, and establish some basic correspondences between the two notions. We then explore in detail many examples, including finding spanning trees in graphs, finding monochromatic sets in Ramsey's Theorem, and linearizing partial orders.

2. Standard results in descriptive set theory provide sufficient conditions for a set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ to admit a Borel uniformization, namely, when P has small or large sections. We consider an invariant analogue of these results with respect to a Borel equivalence relation E. Given E, we show that every such P admits an E-invariant Borel uniformization if and only if E is smooth. We also compute the definable complexity of counterexamples in the case where E is not smooth, using category, measure, and Ramsey-theoretic methods. We also show that the set of pairs (E, P) such that P has large sections and admits an E-invariant Borel uniformization is $\mathbf{\Sigma}^1_2$-complete.

3. Let E, F be Borel equivalence relations on X, Y, and P be an E-invariant Borel set whose sections contain countably many F-classes. We explore obstructions to the existence of Borel E-invariant uniformizing sets for P, i.e., sets choosing one F-class from every section. We survey known results, and prove new dichotomies for the case where P has σ-bounded finite sections. On the way, we prove a dichotomy characterizing the essential values of Borel cocycles into residually finite Polish groups.

4. We show that the Kechris–Solecki–Todorčević dichotomy implies the Harrington–Kechris–Louveau dichotomy. We also give a simple proof of a graph-theoretic dichotomy of Miller for doubly-indexed sequences of analytic graphs, and show that this dichotomy generalizes to finite-dimensional hypergraphs but not to $\aleph_0$-dimensional hypergraphs.

5. An effective version of Nadkarni's Theorem was proved in Ditzen's unpublished Ph.D. thesis. The appendix contains a streamlined exposition of the proof and provides an alternative proof of the Effective Ergodic Decomposition Theorem for invariant measures (also originally proved by Ditzen). In addition, we show that the existence of an invariant Borel probability measure is not effective.

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CONTENTS

Chapter 1

INTRODUCTION

Broadly speaking, descriptive set theory is the study of definable sets in Polish spaces, i.e., separable and completely metrizable topological spaces. Examples of Polish spaces include the space $\mathbb{R}$ of real numbers, the Cantor space $2^{\mathbb{N}}$ of infinite binary sequences, and the Baire space $\mathbb{N}^{\mathbb{N}}$ of infinite sequences of natural numbers. What it means to be definable is intentionally flexible; it is beneficial to consider various notions of definability depending on the context. Most commonly we consider the Borel sets, that is, the sets generated by the basic open sets via the operations of countable union, countable intersection and complementation. In this thesis we also consider various other classes of sets, such as the larger classes of analytic, measurable, or Baire measurable sets, or the more restricted classes of G_δ sets or "effectively Borel" sets.

Restricting ourselves to definable sets has many benefits. First, we avoid pathologies that may exist in the general context, such as non-measurable sets or sets of reals of cardinality strictly between $\aleph_0$ and the continuum. Second, using the structure afforded to us by the definitions of these sets, we get more refined and broadly applicable versions of general results. For example, the Perfect Set Theorem asserts that not only does the continuum hypothesis hold for analytic sets, but in this case there is a concrete witness to uncountability—every uncountable analytic set contains a homeomorphic copy of the Cantor space. As another example, by the Lusin–Novikov Uniformization Theorem not only do Borel families of countable sets have choice functions (a consequence of the axiom of choice), such families admit choice functions that are Borel. Finally, by considering the definable complexity of sets and functions between them, we get a much finer analysis of and comparison between the complexity of various sets, structures, and combinatorial problems; one can view this as analogous to the study of complexity and reductions in computer science.

One concept that has become especially important, both in theory and in application, is that of a *Borel equivalence relation*. A Borel equivalence relation on a Polish space X is an equivalence relation E that is Borel, considered as a set of pairs in the product $X \times X$. The main notion of relative complexity between various Borel equivalence relations is that of *Borel reduction*: if E is a Borel equivalence relation on X and F is a Borel equivalence relation on Y, a Borel reduction from E to F is a Borel function $f : X \to Y$ so that $x_0 E x_1 \iff f(x_0) F f(x_1)$. We write $E \leq_B F$ when there is a Borel reduction from E to F, and in this case we view F as being "at least as complicated" as E.

We give a brief overview of the "base" of the poset of Borel equivalence relations with respect to Borel reducibility: The simplest Borel equivalence relations are the *smooth* ones, i.e., those that admit Borel reductions to equality on $\mathbb{R}$. Above these there is $\mathbb{E}_0$, the *eventual equality* relation on $2^{\mathbb{N}}$:

$$x \mathbb{E}_0 y \iff \exists n \forall k \geq n (x_k = y_k).$$

By the Harrington–Kechris–Louveau Theorem, if E is a Borel equivalence relation that is not smooth, then $\mathbb{E}_0 \leq_B E$. The structure of the Borel equivalence relations above $\mathbb{E}_0$ is much more complicated; see for example [Gao08; Kan08; Kec25].

The central focus of this thesis is on finding Borel solutions to combinatorial problems that are "compatible with" or "invariant with respect to" various Borel equivalence relations. We explain in detail below what this means in various contexts, and give an overview of our results.

1.1 Definable expansions

A *countable* Borel equivalence relation is an equivalence relation whose equivalence classes are countable. Given a countable Borel equivalence relation E, a *Borel structuring of E* is a Borel assignment of a first-order structure on each E-class C. When each of these structures comes from a class $\mathcal{K}$ of countable structures, we call this a *Borel $\mathcal{K}$-structuring of E*.

Broadly speaking, given a combinatorial problem on countable structures, we are interested in solving it in a "uniformly Borel" way on a countable Borel equivalence relation, possibly after throwing away a meagre set or a null set. There are various examples of interest coming from graph theory, such as finding (edge) colourings, perfect matchings or spanning trees in graphs. Other examples include finding infinite monochromatic sets as in Ramsey's Theorem, or linearizing partial orders. Such problems have been studied extensively in this context; see for example [KM20; Pik21; CK18; GX24].

In Chapter 2, we consider these problems within the framework of *expansions*. Given first-order languages $\mathcal{L} \subseteq \mathcal{L}^*$ and an $\mathcal{L}$-structure $\boldsymbol{A}$, we call an $\mathcal{L}^*$-structure $\boldsymbol{A}^*$ an *expansion* of $\boldsymbol{A}$ if $\boldsymbol{A} = \boldsymbol{A}^* \restriction \mathcal{L}$, where $\boldsymbol{A}^* \restriction \mathcal{L}$ denotes the *reduct* of $\boldsymbol{A}^*$ to $\mathcal{L}$. If $\mathcal{K}$ is a class of $\mathcal{L}$ structures and $\mathcal{K}^*$ is a class of $\mathcal{L}^*$-structures, the *expansion problem for* $(\mathcal{K}, \mathcal{K}^*)$

is the problem of determining whether every structure in $\mathcal{K}$ admits an expansion in $\mathcal{K}^*$. All of the combinatorial problems above, such as graph colourings and Ramsey's Theorem, can be phrased as expansion problems.

We study here the "uniformly Borel" analogue of expansion problems. That is, we consider expansion problems $(\mathcal{K}, \mathcal{K}^*)$ for which every element of $\mathcal{K}$ admits an expansion to an element of $\mathcal{K}^*$. Given such a problem, a countable Borel equivalence relation E and a Borel $\mathcal{K}$-structuring $\mathbb{A}$ of E, we study the following problem: is there a Borel $\mathcal{K}^*$-structuring $\mathbb{A}^*$ of E whose restriction to every E-class is an expansion of the original structuring $\mathbb{A}$?

We also study an "equivariant" version of the Borel expansion problem. Let Γ be a group and $(\mathcal{K}, \mathcal{K}^*)$ is an expansion problem with $\mathcal{K}, \mathcal{K}^*$ Borel. We let $\mathcal{K}(\Gamma)$ (resp. $\mathcal{K}^*(\Gamma)$) denote the set of structures in $\mathcal{K}$ (resp. $\mathcal{K}^*$) whose universe is Γ. The action of Γ on itself by multiplication on the left induces a natural action of Γ on $\mathcal{K}(\Gamma), \mathcal{K}^*(\Gamma)$. The Γ-*equivariant expansion problem* is then the following: is there a Borel map $f : \mathcal{K}(\Gamma) \to \mathcal{K}^*(\Gamma)$, taking $\boldsymbol{A} \in \mathcal{K}(\Gamma)$ to an expansion $f(\boldsymbol{A})$, which is equivariant with respect to the induced action of Γ on $\mathcal{K}(\Gamma), \mathcal{K}^*(\Gamma)$?

Our primary objective is to study this correspondence between the Borel expansion problem on CBER and the Borel equivariant expansion for countable groups Γ. More generally, we study also the connection between these problems in the settings of *measure* and *category*, i.e., when we are allowed to solve these problems after possibly removing a null or meagre set. We show that there is natural correspondence between Γ-equivariant expansions for countable groups Γ and countable Borel equivalence relations which arise via free Borel actions of Γ. We then apply our results to various examples.

We include below a representative sample of our results; see Chapter 2 for more precise definitions and further results.

In terms of measure, we consider *random expansions* for countable groups, where we say an invariant measure ν on $\mathcal{K}^*(\Gamma)$ is a random expansion of an invariant measure μ on $\mathcal{K}(\Gamma)$ when the reduct of ν is equal to μ. We show that the existence of random expansions on Γ depends only on its orbit equivalence class, where we say groups Γ, Λ are *orbit equivalent* if there is a countable Borel equivalence relation E induced by free probability-measure-preserving actions of both Γ and Λ.

Theorem 1.1.1. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem and Γ, Λ be countably infinite groups. If Γ, Λ are orbit equivalent, then Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ if* *and only if Λ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$.*

For category, we consider *generic* equivariant expansions on G_δ classes of structures, i.e., equivariant expansions on comeagre subsets of $\mathcal{K}(\Gamma)$. Given an expansion problem $(\mathcal{K}, \mathcal{K}^*)$ and a countably infinite group Γ, we say $\mathcal{K}$ admits Γ-equivariant expansions *generically* if there is a comeagre invariant Borel set $X \subseteq \mathcal{K}(\Gamma)$ such that there is a Borel Γ-equivariant expansion map $X \to \mathcal{K}^*(\Gamma)$.

We show that when $\mathcal{K}$ consists of structures with *trivial algebraic closure* that are not definable from equality, whether or not $\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically is independent of the group Γ. (A structure is said to have trivial algebraic closure if its automorphism group has infinite orbits, even after fixing finitely many points, and is definable from equality when relations between tuples of points depend only on their equality types.)

Theorem 1.1.2. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem. Suppose that $\mathcal{K}$ is G_δ and the generic element of $\mathcal{K}$ has trivial algebraic closure and is not definable from equality. Then the following are equivalent:*

1. *For every countably infinite group Γ, $\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically.*
2. *There exists a countably infinite group Γ for which $\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically.*

As one concrete example, we consider the problem of choosing from a countable linear order without endpoints a subset that is order-isomorphic to $\mathbb{Z}$, in a Borel way.

Theorem 1.1.3.

1. *Let $\mathcal{K}$ be the class of linear orders without endpoints, and $\mathcal{K}^*$ be the class of linear orders without endpoints along with a subset of order-type $\mathbb{Z}$. For any countably infinite group Γ, $\mathcal{K}$ does not admit Γ-equivariant expansions to $\mathcal{K}^*$ generically. In particular, every non-smooth CBER E admits a Borel assignment of linear orders to every E-class so that there is no Borel way to choose an infinite subset of each E-class that has order-type $\mathbb{Z}$.*
2. *There is a Borel Γ-invariant set $X \subseteq \mathcal{K}(\Gamma)$ and a Borel equivariant expansion map $f : X \to \mathcal{K}^*(\Gamma)$ such that, for all invariant random $\mathcal{K}$-structures μ on Γ, μ*

admits a random expansion to $\mathcal{K}^*$ if and only if $\mu(X) = 1$, in which case $f_*\mu$ gives such an expansion. Moreover, we can choose f so that for all $L \in X$, $f(L)$ picks out an interval in L.

1.2 Invariant uniformization

Let X, Y be sets and let $P \subseteq X \times Y$ satisfy $\forall x \in X \exists y \in Y (x, y) \in P$. A *uniformization* of P is a function $f : X \to Y$ so that $\forall x \in X((x, f(x)) \in P)$. If E is an equivalence relation on X, we say P is *E-invariant* if $x_0 E x_1 \implies P_{x_0} = P_{x_1}$, where $P_x = \{y \in Y : (x, y) \in P\}$ is the *x-section of* P. In this case, an *E-invariant uniformization* is a uniformization f such that $x_0 E x_1 \implies f(x_0) = f(x_1)$.

A uniformization for P can be viewed as a *choice function* for the family $\{P_x : x \in X\}$. When X, Y are Polish spaces and P is Borel, standard results in descriptive set theory give sufficient conditions for the existence of Borel uniformizations of P, such as when P has *small* or *large* sections; see e.g. [Kec95, Section 18].

Suppose now that P has small or large sections. In Chapter 3, we study the existence of Borel E-invariant uniformizations of P, when E is a Borel equivalence relation on X and P is E-invariant.

Given a Borel equivalence relation E, we consider the property that every E-invariant P with countable (resp. K_σ, non-meagre, non-null) sections admits a Borel E-invariant uniformization. We show that in every case, E has this property if and only if E is smooth.

Theorem 1.2.1 (Kechris–Wolman)**.** *Let E be a Borel equivalence relation on a Polish space X. Then the following are equivalent:*

- *(i) E is smooth;*
- *(ii) every E-invariant Borel set P with non-null sections admits a Borel E-invariant uniformization;*
- *(iii) every E-invariant Borel set P with non-meagre sections admits a Borel E-invariant uniformization;*
- *(iv) every E-invariant Borel set P with K_σ sections admits a Borel E-invariant uniformization;*
- *(v) every E-invariant Borel set P with countable sections admits a Borel E-invariant uniformization.*

We also characterize the minimal definable complexity of counterexamples. Below we state the largest class for which there are always Borel invariant uniformizations; for each of these cases, we also give counterexamples at the next level of the Borel hierarchy.

Theorem 1.2.2 (Kechris–Wolman)**.** *Let X, Y be Polish spaces, E a Borel equivalence relation on X, and $P \subseteq X \times Y$ an E-invariant Borel relation. Suppose one of the following holds:*

(i) $P_x \in \mathbf{\Delta^0_2}$ *and* $\mu_x(P_x) > 0$, *for all* $x \in X$, *and some Borel assignment* $x \mapsto \mu_x$ *of probability Borel measures* μ_x *on* Y*;*

(ii) $P_x \in F_\sigma$ *and* P_x *non-meager, for all* $x \in X$*;*

(iii) $P_x \in G_\delta$ *and* P_x *non-empty and* K_σ *(in particular countable), for all* $x \in X$*.*

Then there is a Borel E-invariant uniformization.

We next consider "local" dichotomies. Miller recently proved a dichotomy showing that $\mathbb{E}_0$ is essentially the only obstruction to the existence of invariant uniformizations, in the case of countable sections [Mild, Theorem 2]. We provide a different proof of this dichotomy, using Miller's $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy [Mil12] and Lecomte's $\aleph_0$-dimensional hypergraph dichotomy [Lec09]. We also prove an $\aleph_0$-dimensional $(\mathbb{G}_0, \mathbb{H}_0)$-type dichotomy, which generalizes Lecomte's dichotomy in the same way that the $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy generalizes the $\mathbb{G}_0$ dichotomy, and use this to give still another proof of this theorem.

Informally, dichotomies such as [Mild, Theorem 2] provide upper bounds on the complexity of the collection of Borel sets satisfying certain combinatorial properties; for example, (the effective version of) Miller's dichotomy gives a bound of $\mathbf{\Pi^1_1}$ for the set of pairs (E, P) admitting Borel E-invariant uniformizations, when P has countable sections. Thus, one method of showing that there is no analogous dichotomy in other cases is to provide lower bounds on the complexity of such sets. We show that this is the case for the large section problem, namely, we show that the set of such pairs (E, P), where P has large sections, is $\mathbf{\Sigma^1_2}$-complete.

Theorem 1.2.3 (Kechris–Wolman)**.** *Let $\mathcal{P}$ be the class of pairs (E, P) such that E is a Borel equivalence relation on $\mathbb{N}^\mathbb{N}$ and $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ is Borel and E-invariant, has large sections, and admits an E-invariant Borel uniformization. Then $\mathcal{P}$ is $\mathbf{\Sigma^1_2}$-complete.*

The following is still open:

Problem 1.2.4. Is there an analogous dichotomy or anti-dichotomy result for the case where P has K_σ sections?

We end the chapter with some partial results concerning the more general problem of *invariant countable uniformization*, i.e., invariant uniformization where we choose a countable set in each section instead of a single point.

1.3 Invariant uniformization over quotients

Suppose now that E is a Borel equivalence relation on a Polish space X and F is a Borel equivalence relation on a Polish space Y. We say $P \subseteq X \times Y$ is $E \times F$-invariant if

$$x_0 E x_1 \ \ \& \ \ y_0 F y_1 \ \ \& \ \ (x_0, y_0) \in P \implies (x_1, y_1) \in P.$$

Suppose now that P is Borel and $E \times F$-invariant. We say P has *countable sections over* F if the sections of P each contain countably many F-classes. A *Borel E-invariant uniformization of P over F* is a Borel set $U \subseteq P$ that is $E \times F$-invariant, and whose sections each contain exactly one F-class.

In this chapter, we look at dichotomies characterizing the existence of Borel E-invariant uniformizations of P over F, when P is Borel, $E \times F$-invariant and has countable sections over F.

We begin by considering the case where the sections of P contain finitely many F-classes. The following has been shown independently by the author and Miller (personal communication), who pointed out that it follows from their work on essential values of Borel cocycles.

Theorem 1.3.1. *Let E, F be Borel equivalence relations on Polish spaces X, Y, and $P \subseteq X \times Y$ be a Borel $E \times F$-invariant set whose sections contain exactly n F-classes, $n \geq 2$. There is a finite basis of minimal obstructions to the existence of Borel E-invariant uniformizations of P over F, corresponding to the set of minimal fixed-point-free subgroups of S_n (up to conjugacy).*

If E is an equivalence relation and Γ is a group, a map $\rho : E \to \Gamma$ is a *cocycle* if it satisfies the cocycle identity

$$\rho(x, y)\rho(y, z) = \rho(x, z)$$

for $xEyEx$. If Γ is a countable discrete group, E is a Borel equivalence relation on a Polish space X, $\rho : E \to \Gamma$ is a Borel cocycle and $\Lambda \subseteq \Gamma$ is a non-empty set, we say Λ is an *essential value* of ρ if for every partition $(B_n)_{n\in\mathbb{N}}$ of X into Borel sets, there is some n so that for all $\lambda \in \Lambda$ there are $x \neq y \in B_n$ with $\rho(x, y) = \lambda$.

If Γ is a group, E is a Borel equivalence relation on X, F is a Borel equivalence relation on Y, and $\rho : E \to \Gamma$, $\pi : F \to \Gamma$ are cocycles, a *continuous embedding* of ρ into π is a continuous injection $f : X \to Y$ so that $x_0 E x_1 \iff f(x_0) F f(x_1)$, and $\rho(x_0, x_1) = \pi(f(x_0), f(x_1))$ for $x_0 E x_1$.

Miller has proved the following dichotomy characterizing the essential values of Borel cocycles into countable groups.

Theorem 1.3.2 (Miller [Mila, Theorem 1]). *Suppose $\Lambda \leq \Gamma$ are countable discrete non-trivial groups. There is a canonical cocycle $\wp_\Lambda : \mathbb{E}_0 \to \Lambda$ so that for every Borel equivalence relation E and every Borel cocycle $\rho : E \to \Gamma$, the following are equivalent:*

1. *Λ is an essential value of ρ.*
2. *There is a continuous embedding of $\wp_\Lambda$ into ρ.*

We include a simple proof of Theorem 1.3.1 from Miller's dichotomy for essential values of Borel cocycles.

We then consider the case where P has countable sections over F, but can be partitioned into countably many $E \times F$-invariant Borel sets that have bounded finite sections over F. We prove a dichotomy characterizing the essential values of Borel cocycles into pro-finite Polish groups, and use this to generalize Theorem 1.3.1 to this setting.

Theorem 1.3.3. *Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of finite groups, and $\mathcal{F}_n$ be a family of non-trivial subsets of Γ_n that is closed under conjugation. For every Borel equivalence relation E on X and every Borel cocycle $\rho : E \to \prod_n \Gamma_n$, the following are equivalent:*

1. *The family $(\mathcal{F}_n)_n$ is an essential value for ρ, meaning that for every cover of X by Borel sets $B_{i,k}$, there are i, k such that $\mathrm{proj}_{\Gamma_i}(\rho(E{\restriction}B_{i,k} \setminus \Delta(B_{i,k})))$ contains an element of $\mathcal{F}_i$.*
2. *There is a subgroup $\Lambda \leq \prod_n \Gamma_n$ so that $\mathrm{proj}_{\Gamma_n}(\Lambda)$ contains an element of $\mathcal{F}_n$ for all $n \in \mathbb{N}$, a (somewhat canonical) cocycle $\wp : \mathbb{E}_0 \to \prod_n \Gamma_n$ that has Λ as an essential value, and a continuous embedding of $\wp$ into ρ.*

Theorem 1.3.4. *Let E, F be Borel equivalence relations on Polish spaces X, Y, and $P_n \subseteq X \times Y$ be Borel $E \times F$-invariant sets whose sections contain exactly $\alpha(n) \geq 2$ F-classes. There is a (somewhat canonical) basis of minimal obstructions to the existence of Borel E-invariant uniformizations of $\bigcup_n P_n$ over F, corresponding to fixed-point-free subgroups of $\prod_n S_{\alpha(n)}$.*

1.4 Descriptive dichotomy theorems

We have discussed so far various dichotomy theorems in descriptive set theory concerning Borel equivalence relations. We recall now some of the most important, namely Silver's Theorem and the Harrington–Kechris–Louveau Theorem. If E is a Borel equivalence relation on X and F is a Borel equivalence relation on Y, a *continuous embedding* of E into F is a reduction $f : X \to Y$ from E to F that is continuous and injective.

Theorem 1.4.1 (Silver)**.** *Let E be a co-analytic equivalence relation on a Polish space. Exactly one of the following hold:*

1. *E has countably many equivalence classes.*
2. *There is a continuous embedding of equality on $2^{\mathbb{N}}$ into E.*

Theorem 1.4.2 (Harrington–Kechris–Louveau)**.** *Let E be a Borel equivalence relation on a Polish space. Exactly one of the following hold:*

1. *E is smooth, i.e., there is a Borel reduction of E to equality on $\mathbb{R}$.*
2. *There is a continuous embedding of $\mathbb{E}_0$ into E.*

We note that the Harrington–Kechris–Louveau Theorem extends prior results of Glimm and Effros, who proved this in the case that E is induced by a continuous action of a locally compact Polish group (and in particular when E is countable).

We also recall the graph-theoretic dichotomy of Kechris–Solecki–Todorčević. We say a (directed) graph G on a Polish space X is *analytic* if it is analytic as a subset of $X \times X$, and we say G has *countable Borel chromatic number* if there is a cover of X by countably many G-independent Borel sets. If G is a graph on X and H is a graph on Y, a *homomorphism* from G to H is a map $f : X \to Y$ so that

$$x_0 G x_1 \implies f(x_0) H f(x_1).$$

Fix a set $S \subseteq 2^{<\mathbb{N}}$ that contains exactly one sequence of every length, and which is *dense*, meaning that for all $t \in 2^{<\mathbb{N}}$ there is some $s \in S$ with $t \subseteq s$. Define the directed graph $\mathbb{G}_0$ on $2^{\mathbb{N}}$ by

$$\mathbb{G}_0 = \{(s^\frown(0)^\frown x, s^\frown(1)^\frown x) : s \in S, x \in 2^{\mathbb{N}}\},$$

where $^\frown$ denotes concatenation of sequences.

Theorem 1.4.3 (Kechris–Solecki–Todorčević)**.** *Let G be a directed analytic graph on a Polish space. Exactly one of the following hold:*

1. *G has countable Borel chromatic number.*
2. *There is a continuous homomorphism of $\mathbb{G}_0$ into G.*

The original proofs of all of these dichotomies used techniques of effective descriptive set theory. In [Mil12], Miller found a classical proof of the Kechris–Solecki–Todorčević Theorem, and used this to give a classical proof of Silver's Theorem and the Glimm–Effros dichotomy for countable Borel equivalence relations. Miller also proved a generalization of the Kechris–Solecki–Todorčević Theorem, and used this to give a classical proof of the full Harrington–Kechris–Louveau Theorem.

Since then, it has remained an open question whether there is a proof of the Harrington–Kechris–Louveau Theorem directly from the Kechris–Solecki–Todorčević Theorem. In Chapter 5, we show that this is indeed the case.

Theorem 1.4.4. *There is a proof of the Harrington–Kechris–Louveau Theorem directly from the Kechris–Solecki–Todorčević Theorem.*

We then give new proofs of a dichotomy of Miller for doubly-indexed sequences of analytic graphs and show that this dichotomy generalizes to finite-dimensional hypergraphs but not infinite-dimensional hypergraphs, in contrast to most other graph-theoretic dichotomies. We also prove that if $\mathbb{G}_0$ is written as a finite union of Baire measurable graphs, then there is a continuous embedding of $\mathbb{G}_0$ into one of these graphs, without applying the Kechris–Solecki–Todorčević Theorem.

1.5 Ditzen's effective version of Nadkarni's Theorem

An effective version of Nadkarni's Theorem was proved in Ditzen's unpublished Ph.D. thesis. Appendix A contains a streamlined exposition of the proof, and provides an alternative proof of the Effective Ergodic Decomposition Theorem for invariant measures (also originally proved by Ditzen). In addition, we show that the existence of an invariant Borel probability measure is not effective. We use this example to construct an effectively Borel non-smooth equivalence relation that does not effectively admit a compact action realization, giving a concrete witness to [FKSV23, Proposition 4.3.17].

Chapter 2

DEFINABLE EXPANSIONS ON COUNTABLE GROUPS AND COUNTABLE BOREL EQUIVALENCE RELATIONS

Michael S. Wolman

2.1 Introduction

A *countable Borel equivalence relation (CBER)* on a Polish space X is a Borel equivalence relation $E \subseteq X^2$ whose equivalence classes are countable. Given a CBER E, a *(Borel) structuring of* E is a Borel assignment of a first-order structure on each E-class C (see Sections 2.2.5 and 2.3.1 for precise definitions).

In this paper, we are primarily concerned with the *descriptive combinatorics* of *locally countable structures*. Broadly speaking, given a combinatorial problem on countable structures, we are interested in solving it in a "uniformly Borel" way, possibly after throwing away a meagre set or a null set. For instance, given a Borel structuring of a CBER E by countable graphs, we may be interested in characterizing exactly when one can find a Borel colouring of these graphs with countably many colours, i.e., a colouring so that the assignment of the colour classes to the vertices in each E-class is a Borel structuring of E. Other examples of combinatorial problems include finding proper edge colourings, perfect matchings or spanning trees in graphs, finding infinite monochromatic sets (as in Ramsey's Theorem), and extending a given partial order into a linear order; see Section 2.2.2 for more. We refer the reader to [KM20; Pik21] for a survey of results in descriptive combinatorics, and to [CK18; BC24] for more on the structurability of CBER.

For "locally finite" structures, many of these combinatorial problems can be expressed in terms of *constrain satisfaction* or *locally checkable labelling* problems on graphs. In this setting, there has been a lot of recent progress towards finding solutions to various expansion problems, using tools from theoretical computer science and finite combinatorics such as the Lovász Local Lemma and connections with LOCAL algorithms in distributed computing [Ber23; BCGGRV22; GR23]. However, we note that many problems are not locally finite and hence do not fit within this framework; for example, linearizations of partial orders, or Ramsey's Theorem (see e.g. [GX24]).

Here, we consider these problems in the more general framework of *expansions*. Given first-order languages $\mathcal{L} \subseteq \mathcal{L}^*$ and an $\mathcal{L}$-structure $\boldsymbol{A}$, we call an $\mathcal{L}^*$-structure $\boldsymbol{A}^*$ an *expansion* of $\boldsymbol{A}$ if $\boldsymbol{A} = \boldsymbol{A}^* \restriction \mathcal{L}$, where $\boldsymbol{A}^* \restriction \mathcal{L}$ denotes the *reduct* of $\boldsymbol{A}^*$ to $\mathcal{L}$. If $\mathcal{K}$ is a class of $\mathcal{L}$ structures and $\mathcal{K}^*$ is a class of $\mathcal{L}^*$-structures, the *expansion problem for* $(\mathcal{K}, \mathcal{K}^*)$ is the problem of determining whether every structure in $\mathcal{K}$ admits an expansion in $\mathcal{K}^*$. For a countably infinite set X, let $\mathcal{K}(X)$ denote the set of structures in $\mathcal{K}$ whose universe is X, and call $\mathcal{K}$ a *Borel class of structures* if $\mathcal{K}(X)$ is Borel for all countably infinite sets X.

Given an expansion problem $(\mathcal{K}, \mathcal{K}^*)$ for which every element of $\mathcal{K}$ admits an expansion to an element of $\mathcal{K}^*$, we get a corresponding "uniformly Borel" expansion problem: For every CBER E and any structuring of E with elements of $\mathcal{K}$, is there a structuring of E with elements of $\mathcal{K}^*$ which is an expansion of the original structuring on every E-class? In general, one can view this Borel expansion problem as asking if there is a "canonical" assignment of an expansion in $\mathcal{K}^*$ to every element of $\mathcal{K}$; this is made precise in [CK18, arXiv version, Appendix B; BC24].

One can also interpret the Borel expansion problem in terms of definable equivariant maps. If Γ is a group acting on a countably infinite set X and $(\mathcal{K}, \mathcal{K}^*)$ is an expansion problem with $\mathcal{K}, \mathcal{K}^*$ Borel, we may consider the Γ-*equivariant expansion problem*: Is there a Borel map $f : \mathcal{K}(X) \to \mathcal{K}^*(X)$, taking $\boldsymbol{A} \in \mathcal{K}(X)$ to an expansion $f(\boldsymbol{A})$, which is equivariant with respect to the induced action of Γ on $\mathcal{K}(X), \mathcal{K}^*(X)$?

There is a natural correspondence between Γ-equivariant expansions for countable groups Γ, in the special case where Γ acts on $X = \Gamma$ by multiplication on the left, and CBER which arise via free Borel actions of Γ (see Section 2.3.2). By the Feldman–Moore Theorem, every CBER is induced by a Borel action of a countable group, though in general we cannot expect this action to be free [Kec25, Section 11]. Nevertheless, CBER induced by free Borel actions of countable groups are a great source of (counter-)examples in the study of Borel expansions on CBER (especially with respect to the Schreier graphs of their actions), and remain very relevant in the study of the descriptive combinatorics of locally countable structures.

The primary objective of this paper is to study this correspondence between the Borel expansion problem on CBER and the Borel equivariant expansion for countable groups Γ. More generally, we study also the connection between these problems in the settings of *measure* and *category*, i.e., when we are allowed to solve these problems after possibly removing a null or meagre set. We shall see that by exploiting this connection, we can apply results and techniques from the theory of CBER to prove theorems about equivariant expansions on countable groups (see e.g. Sections 2.3.4, 2.3.5, and 2.4). Conversely, we apply tools from symbolic dynamics and probability theory (such as the mass transport principle and random walks on groups) to the study of equivariant expansions, which gives in some cases precise characterizations of exactly when certain structurings of CBER admit definable expansions (c.f. Sections 2.3.3, 2.3.6, and 2.4).

The connection between expansions on CBER and equivariant expansions in the purely Borel setting has been studied independently in [BC24], with the goal of making precise the relation between the existence of Borel expansions on CBER and "canonical" expansions from $\mathcal{K}$ to $\mathcal{K}^*$. There, Banerjee and Chen [BC24, Corollary 3.29, Remark 2.25] show that every Borel structuring of a CBER admits a Borel expansion if and only if there is a Borel $S_{\mathbb{N}}$-equivariant expansion map, for classes $\mathcal{K}$ of structures that interpret the theories of *Lusin–Novikov functions* and *countable separating families* (c.f. [BC24, Definitions 3.18, 3.23]). (We note however that the classes we study in this paper do not interpret these theories.) Expansion problems have also been studied in the context of *invariant random structures* on groups, i.e., invariant probability measures on $\mathcal{K}(\Gamma)$, and one can view equivariant expansions as a natural strengthening of this notion; see e.g. [KM20, Sections 6, 15] for some examples in graph combinatorics, or [GLM24; Alp22] for linearizations of partial orders.

Organization. The structure of this paper is as follows. In Section 2.2 we give precise definitions of expansions on CBER and equivariant expansions on groups for expansion problems, in the Borel, Baire category, and measurable settings. We also give examples of various expansion problems of interest, that we study in detail in Section 2.4.

In Section 2.3, we prove several general theorems relating equivariant expansions on groups Γ with Borel expansions on CBER induced by free Borel actions of Γ. We describe in Section 2.3.2 a weak duality between the two notions, which can be viewed as an analogue of [BC24, Corollary 3.29] for this setting. We then consider *random expansions* for countable groups, where we say an invariant measure ν on $\mathcal{K}^*(\Gamma)$ is a random expansion of an invariant measure μ on $\mathcal{K}(\Gamma)$ when the reduct of ν is equal to μ (c.f. Section 2.2.3). We show that the existence of random expansions on Γ depends only on its orbit equivalence class, where we say groups Γ, Λ are *orbit equivalent* if there is a CBER E induced by free probability-measure-preserving actions of both Γ and Λ.

Theorem 2.1.1 (Theorem 2.3.7)**.** *Let* $(\mathcal{K}, \mathcal{K}^*)$ *be an expansion problem and* Γ, Λ *be* *countably infinite groups. If* Γ, Λ *are orbit equivalent, then* Γ *admits random expansions from* $\mathcal{K}$ *to* $\mathcal{K}^*$ *if and only if* Λ *admits random expansions from* $\mathcal{K}$ *to* $\mathcal{K}^*$.

We note that this has already been observed in some special cases, for example with linearizations in [Alp22], though we show here that it holds more generally for all expansion problems. We also give a sort of converse in Proposition 2.3.8.

Next, we consider *generic* equivariant expansions on G_δ classes of structures, i.e., equivariant expansions on comeagre subsets of $\mathcal{K}(\Gamma)$ (c.f. Section 2.2.3). Given an expansion problem $(\mathcal{K}, \mathcal{K}^*)$ and a countably infinite group Γ, we say $\mathcal{K}$ admits Γ-equivariant expansions *generically* if there is a comeagre invariant Borel set $X \subseteq \mathcal{K}(\Gamma)$ such that there is a Borel Γ-equivariant expansion map $X \to \mathcal{K}^*(\Gamma)$. We show that when $\mathcal{K}$ consists of structures with *trivial algebraic closure* that are not definable from equality, whether or not $\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically is independent of the group Γ. (A structure is said to have trivial algebraic closure if its automorphism group has infinite orbits, even after fixing finitely many points, and is definable from equality when relations between tuples of points depend only on their equality types; see Definition 2.3.11 for precise definitions of these terms.)

Theorem 2.1.2 (Theorem 2.3.13)**.** *Let* $(\mathcal{K}, \mathcal{K}^*)$ *be an expansion problem. Suppose that* $\mathcal{K}$ *is* G_δ *and the generic element of* $\mathcal{K}$ *has trivial algebraic closure and is not definable from equality. Then the following are equivalent:*

1. *For every countably infinite group* Γ*,* $\mathcal{K}$ *admits* Γ*-equivariant expansions to* $\mathcal{K}^*$ *generically.*
2. *There exists a countably infinite group* Γ *for which* $\mathcal{K}$ *admits* Γ*-equivariant expansions to* $\mathcal{K}^*$ *generically.*

A CBER E is *smooth* if there is a Borel set that contains exactly one point from every E-class. We give in Section 2.3.6 sufficient conditions for an expansion problem to satisfy (a) that every structuring of a smooth CBER admits a Borel expansion (Proposition 2.3.22 and Remark 2.3.23), or (b) that every non-smooth CBER admits a structuring with no Borel expansion (Proposition 2.3.25 and Corollary 2.3.26).

In Section 2.4 we analyze in detail the expansion problem for the examples described in Section 2.2.2, using in particular the tools we developed in Section 2.3. We summarize our results in Table 2.1; we highlight a few of these below.

Call a CBER *aperiodic* if it has only infinite equivalence classes. In [KST99] it is shown that for every non-smooth aperiodic CBER E, there are Borel sets A, B which have infinite intersection with every E-class, but for which there is no Borel bijection $f : A \to B$ whose graph is contained in E. By contrast, we have the following:

Theorem 2.1.3 (Generic bijections (Theorem 2.4.3))**.** *Let E be an aperiodic CBER on X and $A, B \subseteq X$ be sets that have infinite intersection with every E-class. Then there is a comeagre E-invariant set $Y \subseteq X$ and a Borel bijection $f : A \cap Y \to B \cap Y$ whose graph is contained in E, i.e., such that $xEf(x)$ for all $x \in A \cap Y$.*

The problem of whether an invariant random partial order on a countably infinite group Γ can be linearized was studied in [GLM24; Alp22]. Alpeev [Alp22] has shown that this random expansion property holds for Γ if and only if Γ is amenable. By contrast, for equivariant maps and CBER we have the following:

Theorem 2.1.4 (Linearizations (Theorems 2.4.11 and 2.4.12))**.**

1. *Let $\mathcal{K}$ be the class of partial orders and $\mathcal{K}^*$ be the class of linear orders extending a given partial order. For every countably infinite group Γ, $\mathcal{K}$ does not admit Γ-equivariant expansions to $\mathcal{K}^*$ generically.*

2. *For every non-smooth CBER E, there is a Borel assignment of a partial order to every E-class so that there is no Borel way of extending these partial orders to linear orders on every E-class. Moreover, if E is aperiodic then one can ensure that for every E-invariant probability Borel measure μ, there is no Borel extension of the partial orders to linear orders μ-a.e.*

A CBER E is *treeable* if there is a Borel assignment of a connected acyclic graph to every E-class. The class of treeable CBER has been studied extensively; see e.g. [Kec25, Section 10]. In Section 2.4.5, we consider CBER E that admit Borel spanning trees for *every* Borel assignment of a connected graph to every E-class. Clearly every such CBER is treeable. We show that the *hyperfinite* CBER have this property, where a CBER is said to be hyperfinite if it can be written as an increasing union of CBER with finite equivalence classes.

Theorem 2.1.5 (Spanning trees (Theorem 2.4.16))**.** *Let E be a CBER. If E is hyperfinite, then for every Borel assignment of a connected graph to each E-class, there is a Borel assignment of a spanning tree to each E-class.*

It is unknown whether the class of CBER with this spanning tree property coincides with the treeable CBER or the hyperfinite CBER, or if it lies somewhere in between.

As a final example, we consider the problem of choosing from a linear order without endpoints a subset that is order-isomorphic to $\mathbb{Z}$, in a Borel way. We give a complete classification of the invariant random structures on countably infinite groups that admit random expansions for this problem, and show moreover that these expansions can always be taken to come from equivariant Borel maps. In particular, this characterizes exactly when a Borel structuring of a CBER admits a Borel expansion for this problem μ-a.e., for any invariant measure μ.

Theorem 2.1.6 ($\mathbb{Z}$-lines (Theorems 2.4.18 and 2.4.20))**.**

1. *Let $\mathcal{K}$ be the class of linear orders without endpoints, and $\mathcal{K}^*$ be the class of linear orders without endpoints along with a subset of order-type $\mathbb{Z}$. For any countably infinite group Γ, $\mathcal{K}$ does not admit Γ-equivariant expansions to $\mathcal{K}^*$ generically. In particular, every non-smooth CBER E admits a Borel assignment of linear orders to every E-class so that there is no Borel way to choose an infinite subset of each E-class that has order-type $\mathbb{Z}$.*

2. *There is a Borel Γ-invariant set $X \subseteq \mathcal{K}(\Gamma)$ and a Borel equivariant expansion map $f : X \to \mathcal{K}^*(\Gamma)$ such that, for all invariant random $\mathcal{K}$-structures μ on Γ, μ admits a random expansion to $\mathcal{K}^*$ if and only if $\mu(X) = 1$, in which case $f_*\mu$ gives such an expansion. Moreover, we can choose f so that for all $L \in X$, $f(L)$ picks out an interval in L.*

We also give in Section 2.4.7 a survey of recent results regarding the existence of Borel proper edge colourings of bounded-degree graphs (i.e. definable Vizing's Theorem), and in Section 2.4.8 a survey of the current landscape regarding the existence of Borel perfect matchings in bipartite graphs (i.e. definable Hall's Theorem).

We end with a list of open problems in Section 2.5.

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2.2 Preliminaries

For a background on general descriptive set theory, see [Kec95]. For a survey of the theory of CBER, see [Kec25]. For the basics of structurability of CBER, see [CK18].

2.2.1 Languages and structures

By a **language**, we will always mean a countable relational first-order language, i.e., a countable set $\mathcal{L} = \{R_i : i \in I\}$, where each R_i is a relation symbol with associated arity $n_i \geq 1$.

Fix now a language $\mathcal{L}$ and let X be a set. An **$\mathcal{L}$-structure on X** is a tuple $\boldsymbol{A} = (X, R^{\boldsymbol{A}})_{R\in\mathcal{L}}$ where $R^{\boldsymbol{A}} \subseteq X^n$ for each n-ary relation symbol $R \in \mathcal{L}$. We call X the **universe of $\boldsymbol{A}$**, and let $\mathrm{Mod}_{\mathcal{L}}(X)$ denote the **space of $\mathcal{L}$-structures on X**.

For $\boldsymbol{A} \in \mathrm{Mod}_{\mathcal{L}}(X)$ and $Y \subseteq X$, let $\boldsymbol{A}\restriction Y \in \mathrm{Mod}_{\mathcal{L}}(Y)$ denote the **restriction of $\boldsymbol{A}$ to Y**, given by

$$R^{\boldsymbol{A}\restriction Y}(y_1, \dots, y_n) \iff R^{\boldsymbol{A}}(y_1, \dots, y_n)$$

for all n-ary $R \in \mathcal{L}$ and $y_1, \dots, y_n \in Y$. For $\mathcal{L}' \subseteq \mathcal{L}$, we let $\boldsymbol{A}\restriction\mathcal{L}' = (X, R^{\boldsymbol{A}})_{R\in\mathcal{L}'}$ denote the **reduct of $\boldsymbol{A}$ to $\mathcal{L}'$**, i.e., the structure we get when we "forget" the relations in $\mathcal{L} \setminus \mathcal{L}'$. (Note that the notation $\boldsymbol{A}\restriction(-)$ is used both for restrictions and reducts; which one we are referring to throughout this paper will be clear from context.)

If $\boldsymbol{A}$ is an $\mathcal{L}$-structure on X and $f : X \to Y$ is a bijection, we write $f(\boldsymbol{A})$ for the **push-forward structure on Y**, i.e., the structure on Y given by

$$R^{\boldsymbol{A}}(x_0, \dots, x_{n-1}) \iff R^{f(\boldsymbol{A})}(f(y_0), \dots, f(y_{n-1}))$$

for all n-ary relations $R \in \mathcal{L}$. When $X = Y$ this defines the **logic action** of S_X on $\mathrm{Mod}_{\mathcal{L}}(X)$, where S_X is the group of bijections of X.

We are primarily interested in the cases where X is a countably infinite set, or when X is a Polish space. We will reserve the symbols $\boldsymbol{A}, \boldsymbol{B}, \dots$ for $\mathcal{L}$-structures on countable sets, and the symbols $\mathbb{A}, \mathbb{B}, \dots$ for $\mathcal{L}$-structures on Polish spaces.

When X is a countable set, one can view $\mathrm{Mod}_{\mathcal{L}}(X)$ as a compact Polish space, namely,

$$\mathrm{Mod}_{\mathcal{L}}(X) = \prod_{i\in I} 2^{X^{n_i}}.$$

The logic action of S_X on $\mathrm{Mod}_{\mathcal{L}}(X)$ is continuous for this topology.

By a **class of $\mathcal{L}$-structures**, we mean a class $\mathcal{K}$ of countably infinite $\mathcal{L}$-structures closed under isomorphism. Given such a class $\mathcal{K}$ and a countably infinite set X, we let $\mathcal{K}(X) = \mathcal{K} \cap \mathrm{Mod}_{\mathcal{L}}(X)$ denote the space of $\mathcal{L}$-structures in $\mathcal{K}$ with universe X. We call $\mathcal{K}$ a **Borel class of $\mathcal{L}$-structures** (resp. a G_δ **class of $\mathcal{L}$-structures**, a **closed class of $\mathcal{L}$-structures**) if $\mathcal{K}(X)$ is Borel (resp. G_δ, closed) as a subset of $\mathrm{Mod}_{\mathcal{L}}(X)$ for some (equivalently any) countably infinite set X.

For a countably infinite set X and an $\mathcal{L}$-structure $\boldsymbol{A}$ let $\mathrm{Age}_X(\boldsymbol{A})$ denote the **age of $\boldsymbol{A}$ (on X)**, that is, the set finite $\mathcal{L}'$-structures that that embed into $\boldsymbol{A}$ and whose universe is contained in X, for finite $\mathcal{L}' \subseteq \mathcal{L}$. Let

$$\mathrm{Age}_X(\mathcal{K}) = \bigcup \{\mathrm{Age}_X(\boldsymbol{A}) : \boldsymbol{A} \in \mathcal{K}(X)\}$$

for any class $\mathcal{K}$ of $\mathcal{L}$-structures. For $\boldsymbol{A}_0 \in \mathrm{Age}_X(\mathrm{Mod}_{\mathcal{L}})$ and $\boldsymbol{A} \in \mathrm{Mod}_{\mathcal{L}}(X)$, we write $\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$ if $(\boldsymbol{A} \restriction \mathcal{L}') \restriction F = \boldsymbol{A}_0$, where $\boldsymbol{A}_0$ is an $\mathcal{L}'$-structure with universe $F \subseteq X$. The topology of $\mathcal{K}(X)$ is generated by basic clopen sets of the form

$$N(\boldsymbol{A}_0) = \{\boldsymbol{A} \in \mathcal{K}(X) : \boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}\}$$

for $\boldsymbol{A}_0 \in \mathrm{Age}_X(\mathcal{K})$. We note that there is an analogous logic action of S_X on $\mathrm{Age}_X(\mathcal{K})$.

We note that our definition of the age of $\boldsymbol{A}$ differs from the usual one (see e.g. [Hod93, Section 7]), which considers all finite $\mathcal{L}$-structures that embed into $\boldsymbol{A}$ *without* restricting to finite sublanguages. We choose here to restrict to finite sublanguages so that $\mathrm{Age}_X(\mathcal{K})$ corresponds naturally to a basis for the topology on $\mathcal{K}(X)$. (We specify the universe X in $\mathrm{Age}_X(\mathcal{K})$ for the same reason.)

2.2.2 Expansions

Let $\mathcal{L} \subseteq \mathcal{L}^*$ be languages, $\boldsymbol{A}$ be an $\mathcal{L}$-structure and $\boldsymbol{A}^*$ be an $\mathcal{L}^*$-structure. We say $\boldsymbol{A}^*$ is an **expansion** of $\boldsymbol{A}$ if $\boldsymbol{A}^* \restriction \mathcal{L} = \boldsymbol{A}$.

Given a class of $\mathcal{L}$-structures $\mathcal{K}$ and a class of $\mathcal{L}^*$-structures $\mathcal{K}^*$ with $\mathcal{L} \subseteq \mathcal{L}^*$, the **expansion problem for** $(\mathcal{K}, \mathcal{K}^*)$ is the question of whether or not every element of $\mathcal{K}$ admits an expansion in $\mathcal{K}^*$. We call such pairs $(\mathcal{K}, \mathcal{K}^*)$ **expansion problems**.

Below we give examples of expansion problems $(\mathcal{K}, \mathcal{K}^*)$ we will consider in this paper. In all of these examples the expansion problem will have a positive solution, i.e., every element of $\mathcal{K}$ admits an expansion in $\mathcal{K}^*$; we will be interested in finding "definable" expansions, for various notions of definability that we make precise below.

We omit $\mathcal{L}, \mathcal{L}^*$ from these examples, as they will be clear from context.

Example 2.2.1 (Bijections)**.**

$$\begin{aligned}\mathcal{K} &= \{(X,R,S) \mid R,S \subseteq X \;\;\&\;\; X,R,S \text{ are all countably infinite}\},\\ \mathcal{K}^* &= \{(X,R,S,T) \mid (X,R,S) \in \mathcal{K} \;\;\&\;\; T \text{ is the graph of a bijection } R \to S\}.\end{aligned}$$

In this case, $\mathcal{K}, \mathcal{K}^*$ are G_δ.

Example 2.2.2 (Ramsey's Theorem)**.**

$$\begin{aligned}\mathcal{K} &= \{(X,R,S) \mid R,S \text{ partition } [X]^2\},\\ \mathcal{K}^* &= \{(X,R,S,T) \mid (X,R,S) \in \mathcal{K} \;\;\&\;\; T \subseteq X \text{ is infinite}\\ &\qquad\qquad \text{and homogeneous for the partition } R,S\},\end{aligned}$$

where $[X]^2$ is the set of two-element subsets of X. Here $\mathcal{K}$ is closed and $\mathcal{K}^*$ is G_δ. Ramsey's Theorem is exactly the statement that every element of $\mathcal{K}$ admits an expansion in $\mathcal{K}^*$.

Example 2.2.3 (Linearizations)**.**

$$\begin{aligned}\mathcal{K} &= \{(X,P) \mid P \text{ is a partial order on } X\},\\ \mathcal{K}^* &= \{(X,P,L) \mid (X,P) \in \mathcal{K} \;\;\&\;\; P \subseteq L \;\;\&\;\; L \text{ is a linear order on } X\}.\end{aligned}$$

$\mathcal{K}, \mathcal{K}^*$ are both closed classes of structures.

Example 2.2.4 (Vertex colourings)**.** Fix $d \geq 2$, and let

$$\begin{aligned}\mathcal{K} &= \{(X,E) \mid (X,E) \text{ is a connected graph of max degree } \leq d\},\\ \mathcal{K}^* &= \{(X,E,S_0,\ldots,S_d) \mid (X,E) \in \mathcal{K} \;\;\&\;\; S_0,\ldots,S_d \text{ is a vertex colouring of } (X,E)\}.\end{aligned}$$

Here $\mathcal{K}, \mathcal{K}^*$ are G_δ.

Example 2.2.5 (Spanning trees)**.**

$$\begin{aligned}\mathcal{K} &= \{(X,E) \mid (X,E) \text{ is a connected graph}\},\\ \mathcal{K}^* &= \{(X,E,T) \mid (X,E) \in \mathcal{K} \;\;\&\;\; (X,T) \text{ is a spanning subtree of } (X,E)\}.\end{aligned}$$

$\mathcal{K}, \mathcal{K}^*$ are both G_δ.

Example 2.2.6 ($\mathbb{Z}$-lines)**.**

$$\begin{aligned}\mathcal{K} &= \{(X,L) \mid (X,L) \text{ is a linear order without endpoints}\},\\ \mathcal{K}^* &= \{(X,L,Z) \mid (X,L) \in \mathcal{K} \;\;\&\;\; Z \subseteq X \;\;\&\;\; (Z, L{\restriction}Z) \cong (\mathbb{Z},<)\},\end{aligned}$$

where $<$ is the usual order on $\mathbb{Z}$. Here $\mathcal{K}$ is G_δ, and $\mathcal{K}^*$ is Borel.

Example 2.2.7 (Vizing's Theorem)**.** Fix $d \geq 2$, and let

$$\mathcal{K} = \{(X, E) \mid (X, E) \text{ is a connected graph of max degree } \leq d\},$$
$$\mathcal{K}^* = \{(X, E, S_0, \ldots, S_d) \mid (X, E) \in \mathcal{K} \ \& \ S_0, \ldots, S_d \text{ is an edge colouring of } (X, E)\}.$$

Here $\mathcal{K}, \mathcal{K}^*$ are G_δ. Vizing's Theorem states that every element of $\mathcal{K}$ admits an expansion in $\mathcal{K}^*$.

Example 2.2.8 (Matchings)**.** A bipartite graph is said to satisfy **Hall's Condition** if $|A| \leq |N(A)|$ for every finite set of vertices A contained in one part of the graph, where $N(A)$ denotes the set of neighbours of A. We say a graph is **locally-finite** if every vertex has finitely-many neighbours. Let now

$$\begin{aligned}\mathcal{K} = \{(X, E) \mid (X, E) &\text{ is a connected, bipartite,} \\ &\text{locally finite graph satisfying Hall's Condition}\}, \\ \mathcal{K}^* = \{(X, E, M) \mid (X, E) &\in \mathcal{K} \ \& \ M \subseteq E \text{ is a perfect matching}\}.\end{aligned}$$

Here $\mathcal{K}, \mathcal{K}^*$ are Borel, and by Hall's Theorem every element of $\mathcal{K}$ admits an expansion in $\mathcal{K}^*$.

2.2.3 Equivariant and random expansions

Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem, X be a countably infinite set, $\Gamma \leq S_X$ be a subgroup of S_X and $Z \subseteq \mathcal{K}(X)$ be Γ-invariant. We say a map $f : Z \to \mathcal{K}^*(X)$ is **Γ-equivariant** if it commutes with the Γ action, i.e., $\gamma \cdot f(\boldsymbol{A}) = f(\gamma \cdot \boldsymbol{A})$ for all $\gamma \in \Gamma, \boldsymbol{A} \in \mathcal{K}(X)$. We call a function $f : Z \to \mathcal{K}^*(X)$ an **expansion map** if $f(\boldsymbol{A})$ is an expansion of $\boldsymbol{A}$ for all $\boldsymbol{A} \in \mathcal{K}(X)$.

In this paper, we will always consider the case where Γ is a countably infinite group acting on $X = \Gamma$ by multiplication on the left. It may also be interesting to consider the more general setting where the action of Γ on X is not free, though we do not explore this here.

Let Γ be a countably infinite group. Given a Γ-invariant Borel set $Z \subseteq \mathcal{K}(\Gamma)$, we say Z **admits Γ-equivariant expansions to** $\mathcal{K}^*$ if there is a Borel Γ-equivariant expansion map $Z \to \mathcal{K}^*(\Gamma)$. If $Z = \mathcal{K}(\Gamma)$, we say $\mathcal{K}$ **admits Γ-equivariant expansions to** $\mathcal{K}^*$. If $\mathcal{K}$ is a G_δ class of structures, we say $\mathcal{K}$ **admits Γ-equivariant expansions to** $\mathcal{K}^*$ **generically** if Z admits a Γ-equivariant expansion to $\mathcal{K}^*$ for some Γ-invariant dense G_δ set $Z \subseteq \mathcal{K}(\Gamma)$.

We let $P(\mathcal{K}(\Gamma))$ denote the space of probability Borel measures on $\mathcal{K}(\Gamma)$. Note that the action of Γ on $\mathcal{K}(\Gamma)$ gives rise to an action of Γ on $P(\mathcal{K}(\Gamma))$, where $\gamma \cdot \mu = \gamma_* \mu$ is the push-forward of μ under $\gamma : \mathcal{K}(\Gamma) \to \mathcal{K}(\Gamma)$. We say μ is **Γ-invariant** if it is fixed by the Γ-action, in which case we say μ is an **invariant random $\mathcal{K}$-structure on Γ**.

If μ is an invariant random $\mathcal{K}$-structure on Γ, we say $\mathcal{K}$ **admits Γ-equivariant expansions to $\mathcal{K}^*$ μ-a.e.** if Z admits a Γ-equivariant expansion to $\mathcal{K}^*$ for some Γ-invariant μ-conull set $Z \subseteq \mathcal{K}(\Gamma)$.

Note that the reduct $\pi : \mathcal{K}^*(\Gamma) \to \mathcal{K}(\Gamma)$ induces a map $\pi_* : P(\mathcal{K}^*(\Gamma)) \to P(\mathcal{K}(\Gamma))$. If μ (resp. ν) is an invariant random $\mathcal{K}$-structure (resp. $\mathcal{K}^*$-structure) on Γ, we say ν is a **Γ-invariant random expansion of μ to $\mathcal{K}^*$** if $\pi_*\nu = \mu$. We say μ **admits a Γ-invariant random expansion to** $\mathcal{K}^*$ if such a ν exists, and that Γ **admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$** if this holds for all such μ. Note that if $\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ μ-a.e., then μ admits a Γ-invariant random expansion to $\mathcal{K}^*$.

We may omit Γ from these definitions when it is clear from context.

2.2.4 Countable Borel equivalence relations

A **countable Borel equivalence relation (CBER)** is an equivalence relation E on a standard Borel space X which is Borel as a subset of X^2, and whose equivalence classes $[x]_E$ are countable for all $x \in X$.

If Γ is a group acting on a set X, we let $E_\Gamma^X \subseteq X^2$ denote the **orbit equivalence relation**

$$xE_\Gamma^X y \iff \exists \gamma \in \Gamma(\gamma \cdot x = y)$$

induced by the action of Γ on X. When Γ is countable, X is standard Borel and the action of Γ is Borel, then E_Γ^X is a CBER. Conversely, by the Feldman–Moore Theorem [FM77], every CBER E on a standard Borel space X is the orbit equivalence relation induced by a Borel action of some countable group Γ on X.

By the **free part** of an action of Γ on X we mean the set

$$Fr(X) = \{x \in X : \forall \gamma \neq 1_\Gamma(\gamma \cdot x \neq x)\}.$$

Note that when X is standard Borel and the action is Borel, $Fr(X)$ is Borel in X. Moreover, if X is Polish and the action of Γ is continuous, then $Fr(X)$ is G_δ in X, hence Polish in the subspace topology.

Given a CBER E on X and $A \subseteq X$, we let $[A]_E = \{x \in X : \exists y \in A(xEy)\}$ denote the **(E-)saturation** of A. We say A is **E-invariant** if $[A]_E = A$. Note that by the Feldman–Moore Theorem, if A is Borel then so is $[A]_E$. We call A a **complete section** for E if $X = [A]_E$.

Let E, F be CBER on X, Y respectively. We say that a Borel map $f : X \to Y$ is a **homomorphism** from E to F, denoted $f : E \to_B F$, if $xEy \implies f(x)Ff(y)$. It is a **reduction** $f : E \leq_B F$ if the converse holds as well, i.e., $xEy \iff f(x)Ff(y)$; an **embedding** $f : E \sqsubseteq_B F$ if it is an injective reduction; an **isomorphism** $f : E \cong_B F$ if it is a surjective embedding; a **class-bijective homomorphism** $f : E \to_B^{cb} F$ if it is a homomorphism for which $f : [x]_E \to [f(x)]_F$ is a bijection for all $x \in X$; and an a **invariant embedding** $f : E \sqsubseteq_B^i F$ if it is a class-bijective reduction.

A CBER E is **finite** if all of its classes are finite, and **aperiodic** if all of its classes are infinite. Given a CBER E on X, we can always partition X into Borel pieces Y, Z on which E is respectively finite and aperiodic.

A CBER E is **smooth** if $E \leq_B \Delta_Y$, where Y is equality on a standard Borel space Y. Equivalently, E is smooth iff it admits a Borel **selector**, i.e., a Borel map $s : X \to X$ which is E**-invariant** (a homomorphism $s : E \to \Delta_X$) and so that $s(x)Ex$ for all $x \in X$. The **Glimm–Effros Dichotomy** for CBER states that for every CBER E, either E is smooth or $\mathbb{E}_0 \sqsubseteq_B E$, where $\mathbb{E}_0$ is the eventual equality relation

$$x \mathbb{E}_0 y \iff \exists n \forall k (x_{n+k} = y_{n+k})$$

on $2^{\mathbb{N}}$, c.f. [Kec25, Theorem 6.5].

We say E is **hyperfinite** if E can be written as an increasing union of finite CBER; see [Kec25, Theorem 8.2] for alternate characterizations of hyperfiniteness. In particular, $\mathbb{E}_0$ and all smooth CBER are hyperfinite.

An **invariant probability measure** for a Borel action of a countable group Γ on X is a probability Borel measure μ on X such that $\gamma_* \mu = \mu$ for all $\gamma \in \Gamma$, where $\gamma_* \mu$ is the push-forward of μ along γ. An **invariant probability measure** for a CBER E on X is a probability Borel measure μ on X such that $f_* \mu = \mu$ for every Borel bijection $f : X \to X$ whose graph is contained in E. If E is the orbit equivalence relation induced by a Borel action of a countable group Γ, then these two notions coincide; see [KM04, Proposition 2.1]. A probability Borel measure μ on X is E**-ergodic** if $\mu(A) \in \{0, 1\}$ for all E-invariant Borel sets A.

If Γ is a countable group and X is standard Borel, the **shift action** of Γ on X^Γ is given by $(\gamma \cdot y)(\delta) = y(\gamma^{-1}\delta)$ for $\gamma, \delta \in \Gamma$ and $y \in X^\Gamma$. If μ is any probability Borel measure on X, then the product measure μ^Γ is an invariant probability measure on X^Γ which concentrates on the free part, i.e., $\mu^\Gamma(Fr(X^\Gamma)) = 1$.

We say E is **generically ergodic** if A is either meagre or comeagre for all E-invariant Borel sets A. For example, $\mathbb{E}_0$ is generically ergodic, as are the orbit equivalence relations induced by the shift actions of countably infinite groups.

A CBER E on X is **compressible** if there is a Borel map $f : X \to X$ whose graph is contained in E, and so that $f(C) \subsetneq C$ for every E-class $C \in X/E$. By **Nadkarni's Theorem**, E is compressible iff it does not admit an invariant probability measure [Nad90; BK96].

2.2.5 Structures and expansions on CBER

Fix a language $\mathcal{L}$ and a class $\mathcal{K}$ of $\mathcal{L}$-structures.

Let E be an aperiodic CBER on a standard Borel space X. A **(Borel) $\mathcal{L}$-structure on** E is an $\mathcal{L}$-structure $\mathbb{A} = (X, R^{\mathbb{A}})_{R\in\mathcal{L}}$ with universe X so that (a) $R^{\mathbb{A}} \subseteq X^n$ is Borel for each n-ary $R \in \mathcal{L}$, and (b) each $R^{\mathbb{A}}$ only relates elements in the same E-class, i.e.,

$$R^{\mathbb{A}}(x_0, \ldots, x_{n-1}) \implies x_0 E \cdots E x_{n-1}.$$

Given such a structure $\mathbb{A}$ and an E-class $C \in X/E$, we let $\mathbb{A}\restriction C$ denote the **restriction of $\mathbb{A}$ to C**, which is a countable $\mathcal{L}$-structure. We call $\mathbb{A}$ a **Borel $\mathcal{K}$-structuring of** E if $\mathbb{A}\restriction C \in \mathcal{K}$ for every $C \in X/E$.

Consider now $\mathcal{L}^* \supseteq \mathcal{L}$ and a class $\mathcal{K}^*$ of $\mathcal{L}^*$-structures. If $\mathbb{A}^*$ is an $\mathcal{L}^*$-structure on E, the reduct $\mathbb{A}^*\restriction\mathcal{L}$ of $\mathbb{A}$ is an $\mathcal{L}$-structure on E. We say that $\mathbb{A}^*$ is an **expansion** of an $\mathcal{L}$-structure $\mathbb{A}$ on E if $\mathbb{A}^*\restriction\mathcal{L} = \mathbb{A}$.

Let $\mathbb{A}$ be a Borel $\mathcal{K}$-structuring of E. We say $\mathbb{A}$ is **Borel expandable to** $\mathcal{K}^*$ if it admits an expansion which is a Borel $\mathcal{K}^*$-structuring of E; we say E is **Borel expandable for** $(\mathcal{K}, \mathcal{K}^*)$ if this holds for all such $\mathbb{A}$. When E lives on a Polish space X, we say $\mathbb{A}$ is **generically expandable to** $\mathcal{K}^*$ if its restriction to a comeagre invariant Borel set is Borel expandable to $\mathcal{K}^*$; we say E is **generically expandable for** $(\mathcal{K}, \mathcal{K}^*)$ if this holds for all such $\mathbb{A}$. If μ is a probability Borel measure on X, we say $\mathbb{A}$ is **μ-a.e. expandable to** $\mathcal{K}^*$ if its restriction to a μ-conull invariant Borel set is Borel expandable to $\mathcal{K}^*$; we say (E, μ) is **a.e. expandable for** $(\mathcal{K}, \mathcal{K}^*)$ this holds for all such $\mathbb{A}$.

2.3 General results

In this section, we assume that all classes of structures are Borel.

2.3.1 Universal structurings of CBER

We begin by describing an alternate characterization of Borel structures and expansions on CBER which will be useful later; see also [BC24, Definition 3.1].

Let E be an aperiodic CBER on a standard Borel space X. A **Borel family of enumerations of** E is a Borel map $g : X \to X^N$, where N is some countably infinite set, so that $g_x = g(x) : N \to X$ is a bijection of N with $[x]_E$ for all $x \in X$; such maps always exist by the Lusin–Novikov Theorem (c.f. [Kec95, 18.15]). A map $\rho : E \to G$ from E to a group G is a **cocycle** if

$$\rho(y, z)\rho(x, y) = \rho(x, z)$$

for all $xEyEz$. If $g : X \to X^N$ is a Borel enumeration of X, there is an associated Borel cocycle $\rho_g : E \to S_N$ given by $\rho_g(x, y) = g_y^{-1} \circ g_x$.

Let $\mathcal{L}$ be a language, $\mathcal{K}$ be a class of $\mathcal{L}$-structures and $g : X \to X^N$ be a Borel enumeration of E. Given an $\mathcal{L}$-structure $\mathbb{A}$ on E, one gets a map $F = F_g^{\mathbb{A}} : X \to \mathrm{Mod}_{\mathcal{L}}(N)$ given by setting

$$R^{F(x)}(n_1, \dots, n_k) \iff R^{\mathbb{A}}(g_x(n_1), \dots, g_x(n_k))$$

for k-ary $R \in \mathcal{L}$ and $n_1, \dots, n_k \in N$. Note that if xEy then

$$\begin{aligned} R^{\rho_g(x,y)\cdot F(x)}(n_1, \dots, n_k) &\iff R^{F(x)}(\rho_g(x, y)^{-1}(n_1), \dots, \rho_g(x, y)^{-1}(n_k)) \\ &\iff R^{\mathbb{A}}(g_x(\rho_g(x, y)^{-1}(n_1)), \dots, g_x(\rho_g(x, y)^{-1}(n_k))) \\ &\iff R^{\mathbb{A}}(g_y(n_1), \dots, g_y(n_k)) \\ &\iff R^{F(y)}(n_1, \dots, n_k), \end{aligned}$$

that is, $\rho_g(x, y) \cdot F(x) = F(y)$. Call such a map ρ_g**-equivariant**. Note that $g_x : F(x) \cong \mathbb{A}\restriction[x]_E$ for $x \in X$.

Conversely, given a ρ_g-equivariant map $F : X \to \mathrm{Mod}_{\mathcal{L}}(N)$, one can define a Borel $\mathcal{L}$-structure $\mathbb{A}$ on E by setting

$$R^{\mathbb{A}}(x_1, \dots, x_k) \iff R^{F(x)}(g_x(x_1)^{-1}, \dots, g_x^{-1}(x_k))$$

for any k-ary $R \in \mathcal{L}$ and $xEx_1E \dots Ex_k$. It is easy to verify, using the fact that $\rho_g(x, y)F(x) = F(y)$ for xEy, that this definition does not depend on the choice of x and that $g_x : F(x) \cong \mathbb{A}\restriction[x]_E$ for $x \in X$.

We therefore have that the map $\mathbb{A} \mapsto F_g^{\mathbb{A}}$ is a bijective correspondence between $\mathcal{L}$-structures on E and ρ_g-equivariant maps $F : X \to \mathrm{Mod}_{\mathcal{L}}(N)$. It is easy to see that

$\mathbb{A}$ is Borel iff $F_g^{\mathbb{A}}$ is Borel, and that $\mathbb{A}$ is a $\mathcal{K}$-structuring of E iff the image of $F_g^{\mathbb{A}}$ is contained in $\mathcal{K}(N)$, whenever $\mathcal{K}$ is a class of $\mathcal{L}$-structures.

We remark that if $\mathcal{L} \subseteq \mathcal{L}^*$ are languages and $\mathbb{A}, \mathbb{A}^*$ are $\mathcal{L}, \mathcal{L}^*$-structures on E, then $\mathbb{A}^*$ is an expansion of $\mathbb{A}$ iff $F_g^{\mathbb{A}} = \pi \circ F_g^{\mathbb{A}^*}$, where $\pi : \mathrm{Mod}_{\mathcal{L}^*}(N) \to \mathrm{Mod}_{\mathcal{L}}(N)$ is the reduct. That is, $\mathbb{A}$ admits a Borel expansion iff there is a ρ_g-equivariant Borel lift F of $F^{\mathbb{A}}$ to $\mathcal{K}^*(X)$:

$$\begin{tikzcd} & K^*(X) \arrow[d, "\pi"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "F^{\mathbb{A}}"] & K(X) \end{tikzcd}$$

Fix now a class of $\mathcal{L}$-structures $\mathcal{K}$. In [CK18, Theorem 4.1, Remark 4.3], a *universal* $\mathcal{K}$-structurable CBER lying over E is constructed, which we denote $E \ltimes_g \mathcal{K}$. Explicitly, $E \ltimes_g \mathcal{K}$ lives on $X \times \mathcal{K}(N)$, and is given by

$$(x, \boldsymbol{A})(E \ltimes_g \mathcal{K})(y, \boldsymbol{B}) \iff xEy \ \ \& \ \ \rho_g(x, y) \cdot \boldsymbol{A} = \boldsymbol{B}.$$

The projection $\pi_0 : X \times \mathcal{K}(N) \to X$ is a class-bijective homomorphism $E \ltimes_g \mathcal{K} \to E$, along which g can be pulled back to a Borel enumeration $\tilde{g}$ of $E \ltimes_g \mathcal{K}$. If $\rho_{\tilde{g}}$ is the associated Borel cocycle, then $\rho_{\tilde{g}} = \rho_g \circ (\pi_0 \times \pi_0)$, and it follows that

$$\rho_{\tilde{g}}((x, \boldsymbol{A}), (y, \boldsymbol{B})) \cdot \boldsymbol{A} = \rho_g(x, y) \cdot \boldsymbol{A} = \boldsymbol{B}.$$

The canonical $\mathcal{K}$-structure $\mathbb{A}$ on $E \ltimes_g \mathcal{K}$ is then the one induced by the projection $X \times \mathcal{K}(N) \to \mathcal{K}(N)$, which we have observed is $\rho_{\tilde{g}}$-equivariant. Note that while $E \ltimes_g \mathcal{K}$ depends only on the cocycle ρ_g, $\mathbb{A}$ depends on g.

The above constructions do not depend on the choice of g, up to canonical Borel isomorphism. That is, given Borel enumerations $g : X \to X^M$, $h : X \to X^N$ of E, let $\tau(x) = \tau_{g,h}(x) = h_x^{-1} \circ g_x$. Then $\tau : X \to N^M$ is Borel and each $\tau(x) : M \to N$ is a bijection. Moreover, we have

$$\tau(y) \circ \rho_g(x, y) = \rho_h(x, y) \circ \tau(x)$$

for xEy (i.e., ρ_g, ρ_h are cohomologous, as witnessed by τ). In particular, if $\mathbb{A}$ is a Borel $\mathcal{L}$-structure on E and $F_g^{\mathbb{A}}, F_h^{\mathbb{A}}$ the corresponding maps, then $F_h^{\mathbb{A}}(x) = \tau(x) \cdot F_g^{\mathbb{A}}(x)$ for $x \in X$. Additionally, the map $(x, \boldsymbol{A}) \mapsto (x, \tau(x) \cdot \boldsymbol{A})$ is a Borel isomorphism $E \ltimes_g \mathcal{K} \cong_B E \ltimes_h \mathcal{K}$.

2.3.2 Expansions on CBER induced by free actions

Fix an expansion problem $(\mathcal{K}, \mathcal{K}^*)$ and a countably infinite group Γ. We wish to relate the existence of Γ-equivariant expansions with Borel expansions on CBER induced by free actions of Γ.

Suppose $E = E_\Gamma^X$ is a CBER induced by a free Borel action of Γ on a standard Borel space X. This gives rise to a Borel enumeration $g : X \to X^\Gamma$ of E, namely $g_x(\gamma) = \gamma^{-1} \cdot x$. In this case, $\rho_g(x, \gamma x) = \gamma \in S_\Gamma$, so $\rho_g : E \to \Gamma$. In particular, we have that $E \ltimes_g \mathcal{K} = E_\Gamma^{X \times \mathcal{K}(\Gamma)}$ is the orbit equivalence relation induced by the diagonal action of Γ on $X \times \mathcal{K}(\Gamma)$.

Thus, the characterization of Borel $\mathcal{K}$-structurings of E described in Section 2.3.1 gives:

Proposition 2.3.1. *Let $\mathcal{L}$ be a language, $\mathcal{K}$ be a Borel class of $\mathcal{L}$-structures and Γ be a countably infinite group. Fix a free Borel action of Γ on a standard Borel space X. There is a canonical bijection $\mathbb{A} \mapsto F^{\mathbb{A}}$ between the set of Borel $\mathcal{K}$-structurings of E_Γ^X and the set of Γ-equivariant Borel maps $X \to \mathcal{K}(\Gamma)$, defined by setting*

$$R^{F^{\mathbb{A}}(x)}(\gamma_1, \ldots, \gamma_n) \iff R^{\mathbb{A}}(\gamma_1^{-1} \cdot x, \ldots, \gamma_n^{-1} \cdot x)$$

for $x \in X$, Borel $\mathcal{K}$-structurings $\mathbb{A}$ on E_Γ^X, n-ary relation symbols $R \in \mathcal{L}$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$.

Proposition 2.3.2. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem and Γ be a countably infinite group. Fix a free Borel action of Γ on a standard Borel space X and a Borel $\mathcal{K}$-structuring $\mathbb{A}$ of E_Γ^X, and let $f : X \to \mathcal{K}(\Gamma)$ be the corresponding equivariant Borel map. There is a canonical bijection between Borel expansions $\mathbb{A}^*$ of $\mathbb{A}$ and equivariant Borel maps $g : X \to \mathcal{K}^*(\Gamma)$ satisfying $\pi \circ g = f$, where $\pi : \mathcal{K}^*(\Gamma) \to \mathcal{K}(\Gamma)$ is the reduct from $\mathcal{L}^*$ to $\mathcal{L}$.*

Remark 2.3.3. If $X \subseteq Fr(\mathcal{K}(\Gamma))$ is invariant and Borel then there is a canonical Borel $\mathcal{K}$-structuring of E_Γ^X corresponding to the inclusion $X \to \mathcal{K}(\Gamma)$. More generally, if Z is a standard Borel space on which Γ acts and $X \subseteq Z \times \mathcal{K}(\Gamma)$ is Borel, invariant and free for the diagonal Γ action on the product, then there is a canonical Borel $\mathcal{K}$-structuring of E_Γ^X corresponding to the projection $Z \times \mathcal{K}(\Gamma) \supseteq X \to \mathcal{K}(\Gamma)$.

In particular, this gives a weak correspondence between equivariant expansions on Γ and expansions of CBER induced by free actions of Γ.

Proposition 2.3.4. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem and Γ be a countably infinite group.*

(1) If $\mathcal{K}(\Gamma)$ admits a Borel equivariant expansion, then every CBER induced by a free Borel action of Γ is Borel expandable.

(2) An invariant Borel set $X \subseteq Fr(\mathcal{K}(\Gamma))$ admits a Borel equivariant expansion iff the canonical $\mathcal{K}$-structuring of E_Γ^X admits a Borel expansion.

(3) Suppose there is a free Borel action of Γ on a standard Borel space Z admitting an invariant measure μ, and so that the canonical $\mathcal{K}$-structure $\mathbb{A}$ on $Z \times \mathcal{K}(\Gamma)$ is λ-a.e. expandable for every Γ-invariant probability Borel measure λ on $Z \times \mathcal{K}(\Gamma)$ whose push-forward to Z is μ. Then Γ admits random expansions.

Proof. (1) Let h be such an expansion. For any Borel equivariant $f : X \to \mathcal{K}(\Gamma)$, $h \circ f : X \to \mathcal{K}^*(\Gamma)$ is Borel, equivariant, and satisfies $\pi \circ h \circ f = f$, so this follows by Proposition 2.3.2.

(2) This follows immediately from Proposition 2.3.2.

(3) For any invariant random $\mathcal{K}$-structure ν on Γ, take $\lambda = \nu \times \mu$. Then λ is Γ-invariant, so there is a λ-conull invariant Borel set $X \subseteq \mathcal{K}(\Gamma) \times Z$ and an expansion $\mathbb{A}^*$ of $\mathbb{A}\restriction X$. Let $f : X \to \mathcal{K}^*(\Gamma)$ be the corresponding equivariant map, and let $\kappa = f_*(\lambda)$. Then κ is an invariant random $\mathcal{K}^*$-structure on Γ, and $\pi_*\kappa = (\pi \circ f)_*\lambda = (\mathrm{proj}_{\mathcal{K}(\Gamma)})_*\lambda = \nu$. $\square$

2.3.3 Uniform random expansions

Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem, Γ be a countably infinite group, and μ be an invariant random $\mathcal{K}$-structure on Γ. In general, it is possible that μ admits an invariant random expansion to $\mathcal{K}^*$, but $\mathcal{K}$ does not admit equivariant expansions to $\mathcal{K}^*$ μ-a.e. That is, there may be some invariant random expansion ν of μ to $\mathcal{K}^*$, but no Borel equivariant expansion map $f : \mathcal{K}(\Gamma) \supseteq Z \to \mathcal{K}^*(\Gamma)$, defined on a μ-conull set Z, so that $\nu = f_*\mu$ (see e.g. Remark 2.4.10).

On the other hand, we will see that for Examples 2.2.1, 2.2.4, and 2.2.6, every invariant random $\mathcal{K}$-structure on Γ which admits an invariant random expansion to $\mathcal{K}^*$ admits such an expansion of the form $f_*\mu$, where $f : \mathcal{K}(\Gamma) \supseteq Z \to \mathcal{K}^*(\Gamma)$ is a Borel equivariant expansion map defined on a μ-conull set Z. Moreover, in these cases, we shall see that this holds *uniformly in μ*: there is a single function f that works for all such μ. That is, there is an invariant Borel set $Z \subseteq \mathcal{K}(\Gamma)$ and Borel equivariant expansion map

$f : Z \to \mathcal{K}^*(\Gamma)$ so that μ admits an invariant random expansion to $\mathcal{K}^*$ iff $\mu(Z) = 1$, in which case $f_*\mu$ gives such an expansion.

One can view such a function f as both classifying exactly when invariant random expansions exist, as well as giving uniformly all possible invariant random expansions. When such an f exists, one can further characterize exactly when a Borel $\mathcal{K}$-structuring of a CBER is a.e. expandable, for any CBER induced by a free Borel action of Γ:

Proposition 2.3.5. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem and Γ be a countably infinite group. Suppose that there is a Borel Γ-invariant set $Z \subseteq \mathcal{K}(\Gamma)$ which admits a Γ-equivariant expansion to $\mathcal{K}^*$, and such that the following holds: For every invariant random $\mathcal{K}$-structure μ on Γ, μ admits an invariant random expansion to $\mathcal{K}^*$ iff $\mu(Z) = 1$.*

Then for any CBER E on X induced by a free Borel action of Γ, any E-invariant probability Borel measure μ on X and any Borel $\mathcal{K}$-structuring $\mathbb{A}$ of E, $\mathbb{A}$ is μ-a.e. expandable to $\mathcal{K}^*$ if and only if $F^{\mathbb{A}}(x) \in Z$ for μ-almost every $x \in X$, where $F^{\mathbb{A}} : X \to \mathcal{K}(\Gamma)$ is the equivariant map from Proposition 2.3.1.

Proof. If $F^{\mathbb{A}}(x) \in Z$ for μ-almost every $x \in X$, then by Proposition 2.3.2, the composition of $F^{\mathbb{A}}$ with the expansion map $Z \to \mathcal{K}^*(\Gamma)$ gives a Borel expansion of $\mathbb{A}$ to $\mathcal{K}^*$ on an invariant μ-conull set.

Conversely, suppose that $Y \subseteq X$ is an E-invariant Borel μ-conull set and $\mathbb{A}^*$ is a Borel expansion of $\mathbb{A}\restriction Y$ to $\mathcal{K}^*$. Let $\pi : \mathcal{K}^*(\Gamma) \to \mathcal{K}(\Gamma)$ denote the reduct, and note that $F^{\mathbb{A}} = \pi \circ F^{\mathbb{A}^*}$, so that $F^{\mathbb{A}^*}_*\mu$ is an invariant random expansion of $F^{\mathbb{A}}_*\mu$. By assumption, this implies that $F^{\mathbb{A}}_*\mu(Z) = 1$, i.e., $F^{\mathbb{A}}(x) \in Z$ for μ-a.e. $x \in X$. $\square$

2.3.4 Invariant random expansions on CBER

We consider now a notion of invariant random structures and random expansions on CBER. When E is a CBER arising from a free Borel action of Γ with an invariant measure, this will correspond to a weakening of the hypotheses of Proposition 2.3.4(3), and we will show in this case that the existence of random expansions on E corresponds exactly to the existence of random expansions on Γ. Crucially, this notion of random expansion will be purely in terms of the CBER with no reference to Γ, allowing us to compare the existence of random expansion between various groups.

Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem, fix an aperiodic CBER E on a standard Borel space X and let $g : X \to X^N$ be a Borel enumeration of E.

An **invariant random $\mathcal{K}$-structuring of E (with respect to g)** is an invariant probability Borel measure for $E \ltimes_g \mathcal{K}$. If μ is an invariant probability Borel measure for E, an **invariant random $\mathcal{K}$-structuring of (E, μ) (with respect to g)** is an invariant random $\mathcal{K}$-structuring ν of E whose push-forward along the projection $X \times \mathcal{K}(N) \to X$ is μ. We say an invariant random $\mathcal{K}^*$-structuring κ of E is an **expansion** of an invariant random $\mathcal{K}$-structuring ν of E if the push-forward of κ along the reduct $X \times \mathcal{K}^*(N) \to X \times \mathcal{K}(N)$ is ν.

If $h : X \to X^M$ is another Borel enumeration of E and $\tau = \tau_{g,h} : X \to M^N$ the induced Borel map described in Section 2.3.1, we recall that $(x, \boldsymbol{A}) \mapsto (x, \tau(x) \cdot \boldsymbol{A})$ is a Borel isomorphism $E \ltimes_g \mathcal{K} \cong_B E \ltimes_h \mathcal{K}$. It is easy to see that the following diagram commutes, where π denotes the reduct from $\mathcal{K}^*$ to $\mathcal{K}$.

$$\begin{tikzcd}
X \times \mathcal{K}^*(N) \arrow[r, "(\mathrm{id}_X, \pi)"] \arrow[d, "(\mathrm{id}_X, \tau)"] & X \times \mathcal{K}(N) \arrow[r, "\mathrm{proj}_X"] \arrow[d, "(\mathrm{id}_X, \tau)"] & X \arrow[d, "\mathrm{id}_X"] \\
X \times \mathcal{K}^*(M) \arrow[r, "(\mathrm{id}_X, \pi)"] & X \times \mathcal{K}(M) \arrow[r, "\mathrm{proj}_X"] & X
\end{tikzcd}$$

If μ is an invariant probability Borel measure on E, we say (E, μ) **admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$** if for every invariant random $\mathcal{K}$-structure ν on (E, μ) there is an invariant random $\mathcal{K}^*$-structure κ on (E, μ) that is an expansion of ν. We say E **admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$** if (E, μ) admits invariant random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ for every invariant probability Borel measure μ on E. By the prior remarks, these definitions do not depend on the choice of Borel enumeration of E.

The following key fact relates invariant random expansions between CBER and groups. Special cases of this have been shown, for example with linearizations in [Alp22]; we note here that it holds more generally for all expansion problems.

Proposition 2.3.6. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem and Γ be a countably infinite group. The following are equivalent:*

1. *Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$;*
2. *Every CBER E induced by a free Borel action of Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$;*
3. *There is a CBER E induced by a free Borel action of Γ and an E-invariant probability Borel measure μ, such that (E, μ) admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$.*

Proof. (1) $\implies$ (2): Let E be a CBER on X induced by a free Borel action of Γ. By the remarks at the start of Section 2.3.2, we may assume that $E \ltimes \mathcal{K}$ is the orbit equivalence relation on $X \times \mathcal{K}(\Gamma)$ arising from the diagonal Γ action. If ν is an invariant probability Borel measure on $X \times \mathcal{K}(\Gamma)$, then the push-forward of ν along the projection $X \times \mathcal{K}(\Gamma) \to \mathcal{K}(\Gamma)$ gives an invariant random $\mathcal{K}$-structure ν' on Γ. By (1), there is an invariant random $\mathcal{K}^*$-structure κ on $\mathcal{K}^*(\Gamma)$ which is an expansion of ν'.

Let now $\{\kappa_{\boldsymbol{A}}\}_{\boldsymbol{A}\in\mathcal{K}(\Gamma)}$ be the measure disintegration of κ with respect to ν' over the reduct $\mathcal{K}^*(\Gamma) \to \mathcal{K}(\Gamma)$, and $\{\nu_{\boldsymbol{A}} \times \delta_{\boldsymbol{A}}\}_{\boldsymbol{A}\in\mathcal{K}(\Gamma)}$ be a measure disintegration of ν with respect to ν' over the projection $X \times \mathcal{K}(\Gamma) \to \mathcal{K}(\Gamma)$ (see for example [Kec95, 17.35]). Then it is straightforward to check that $\lambda = \int(\nu_{\boldsymbol{A}} \times \kappa_{\boldsymbol{A}})d\nu'(\boldsymbol{A})$ is a Γ-invariant probability Borel measure on $X \times \mathcal{K}^*(\Gamma)$ which is an extension of ν.

(2) $\implies$ (3): Consider e.g. the free part of the shift of Γ on 2^Γ.

(3) $\implies$ (1): Suppose (E, μ) admits random expansions form $\mathcal{K}$ to $\mathcal{K}^*$, where E is induced by a free Borel action of Γ on X preserving the probability Borel measure μ. Let ν be an invariant random $\mathcal{K}$-structure on Γ, and consider $\mu \times \nu$ on $X \times \mathcal{K}(\Gamma)$. This is Γ-invariant; hence it admits an expansion κ, and the push-forward of κ along the projection $X \times \mathcal{K}^*(\Gamma) \to \mathcal{K}^*(\Gamma)$ is an invariant random expansion of ν. $\square$

Recall now that two countably infinite groups Γ, Λ are **orbit equivalent** if there is a CBER E on a standard Borel space X admitting an invariant probability Borel measure and free Borel actions of Γ, Λ on X with $E = E^X_\Gamma = E^X_\Lambda$. A CBER is said to be **measure-hyperfinite** if, for every invariant probability Borel measure μ, it is hyperfinite when restricted to a Borel invariant μ-conull set (see also [Kec25, Sections 8.5, 9]). For example, all countable infinite amenable groups are orbit equivalent and their actions generate measure-hyperfinite CBER [Dye59; OW80].

Theorem 2.3.7. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem and Γ, Λ be countably infinite groups. If Γ, Λ are orbit equivalent, then Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ iff Λ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$.*

In particular, the following are equivalent:

1. *Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ for all amenable groups Γ;*

2. *Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ for some amenable group Γ; and*

3. *every aperiodic measure-hyperfinite CBER admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$.*

Proof. Let E be a CBER on a standard Borel space X admitting an invariant probability Borel measure μ, and fix free Borel actions of Γ, Λ on X inducing E. Suppose Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$. By (1) $\implies$ (2) of Proposition 2.3.6 (E, μ) admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$, and hence by (3) $\implies$ (1) of Proposition 2.3.6 so does Λ.

By Proposition 2.3.6, it is clear that (1) $\iff$ (2) $\impliedby$ (3). If E is an aperiodic measure-hyperfinite CBER and μ is an E-invariant probability Borel measure, then by restricting to a μ-conull set we see that E is hyperfinite, hence generated by a free $\mathbb{Z}$ action. Thus (1) $\implies$ (3) by Proposition 2.3.6. □

We note that converse holds as well:

Proposition 2.3.8. *Let Γ, Λ be countably infinite groups and suppose that for every expansion problem $(\mathcal{K}, \mathcal{K}^*)$, Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ iff Λ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$. Then Γ, Λ are orbit equivalent.*

Proof. Let $\mathcal{K}$ be the class of countable sets (in the empty language), and take $\mathcal{K}^*$ to be a class of structures so that $\mathcal{K}^*$-structurability of a CBER E corresponds exactly to E arising from a free Borel action of Γ (see e.g. [CK18, Section 3.1]).

We claim that Γ admits an invariant random $\mathcal{K}^*$-structure. To see this, fix a free Borel action of Γ on a standard Borel space X preserving a probability Borel measure μ (e.g. the free part of the shift on 2^Γ). By definition, E_Γ^X admits a Borel $\mathcal{K}^*$-structure $\mathbb{A}^*$. By Proposition 2.3.1 this corresponds to a Γ-equivariant Borel map $F^{\mathbb{A}} : X \to \mathcal{K}^*(\Gamma)$, so $F^{\mathbb{A}}_* \mu$ is an invariant random $\mathcal{K}^*$-structure on Γ. As $\mathcal{K}(\Gamma)$ is a singleton, Γ admits invariant random expansions from $\mathcal{K}$ to $\mathcal{K}^*$.

Thus, by our assumption, Λ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$. Fix a free Borel action of Λ on a standard Borel space Y preserving a probability Borel measure ν. By (1) $\implies$ (2) of Proposition 2.3.6 there is an $(E \ltimes \mathcal{K}^*)$-invariant probability Borel measure κ on $Y \times \mathcal{K}^*(\Lambda)$. Note that $E \ltimes \mathcal{K}^*$ admits a Borel $\mathcal{K}^*$-structuring, and therefore arises from a free Borel action of Γ. Thus the actions of Γ, Λ on $Y \times \mathcal{K}^*(\Lambda)$ witness the orbit equivalence of Γ, Λ. □

Remark 2.3.9. This is really an observation about invariant random structures on groups: Two countably infinite groups Γ, Λ are orbit equivalent if and only if, for every class $\mathcal{K}$ of structures, there is an invariant random $\mathcal{K}$-structure on Γ exactly when there is an invariant random $\mathcal{K}$-structure on Λ.

Finally, we remark that random expansions always exist for closed classes of structures on amenable groups.

Proposition 2.3.10. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem. If $\mathcal{K}^*$ is closed, then Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ for every countably infinite amenable group Γ.*

Proof. Let μ be an invariant random $\mathcal{K}$-structure on Γ, and let $\pi : \mathcal{K}^*(\Gamma) \to \mathcal{K}(\Gamma)$ denote the reduct. Then $\pi_* : P(\mathcal{K}^*(\Gamma)) \to P(\mathcal{K}(\Gamma))$, where $P(X)$ denotes the space of probability Borel measures on a space X. Note that π_* is continuous and $P(\mathcal{K}^*(\Gamma))$ is compact metrizable [Kec95, 17.22, 17.28], so in particular $A = (\pi_*)^{-1}(\mu) \subseteq P(\mathcal{K}^*(\Gamma))$ is compact metrizable. By the invariance of μ and the equivariance of π, we see that A is Γ-invariant. It is also easy to see that it is non-empty: as π is continuous every fibre is compact, so we may choose in a Borel way some $\nu_x \in P(\pi^{-1}(x))$ for $x \in \mathcal{K}(\Gamma)$ and let $\nu = \int \nu_x d\mu \in A$ (c.f. [Kec95, 28.8]). As Γ is amenable there is a fixed point in A, which is an invariant random expansion of μ. □

2.3.5 Generic equivariant expansions

If $\mathcal{K}$ is G_δ classes of structures of $\mathcal{L}$-structures and Φ is an isomorphism-invariant property of $\mathcal{L}$-structures, we say **the generic element of $\mathcal{K}$ satisfies** Φ if for some countably infinite set X, $\mathcal{K}_\Phi(X) = \{\boldsymbol{A} \in \mathcal{K}(X) : \Phi(\boldsymbol{A})\}$ is comeagre in $\mathcal{K}(X)$. We note that this does not depend on the choice of X: If $f : X \to Y$ is a bijection, this induces a homeomorphism $\mathcal{K}(X) \to \mathcal{K}(Y)$ taking $\mathcal{K}_\Phi(X)$ to $\mathcal{K}_\Phi(Y)$.

In this section, we show that if $\mathcal{K}$ is a G_δ class of structures and the generic element of $\mathcal{K}$ has trivial algebraic closure, then the question of whether $\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically does not depend on the group Γ.

Definition 2.3.11. Let $\mathcal{L}$ be a language and $\boldsymbol{A}$ be a countable $\mathcal{L}$-structure with universe X. We say $\boldsymbol{A}$ has the **weak duplication property (WDP)** if, for every $\boldsymbol{A}_0 \in \mathrm{Age}_X(\boldsymbol{A})$ and any finite $F \subseteq X$, there is an embedding of $\boldsymbol{A}_0$ into $\boldsymbol{A}$ whose image is disjoint from F.

We say that $\boldsymbol{A}$ is **definable from equality** if for all n-ary $R \in \mathcal{L}$ and n-tuples $\bar{x}, \bar{y}$ in X with the same equality type, $R^{\boldsymbol{A}}(\bar{x}) \iff R^{\boldsymbol{A}}(\bar{y})$. (The **equality type** of $\bar{x}$ is the set of pairs (i, j) with $\bar{x}_i = \bar{x}_j$.)

For $F \subseteq X$, let $\mathrm{Aut}_F(\boldsymbol{A})$ denote the group of automorphisms of $\boldsymbol{A}$ that fix F pointwise, i.e., such that $f(x) = x$ for $x \in F$. We say $\boldsymbol{A}$ has **trivial algebraic closure (TAC)** if, for every finite $F \subseteq X$, the action of $\mathrm{Aut}_F(\boldsymbol{A})$ on $X \setminus F$ has no finite orbits. See e.g. [Hod93, Section 4.2] or [CK18, 50] for alternative characterizations.

Example 2.3.12. Let $\mathcal{K}$ be the class described in one of Examples 2.2.1 to 2.2.3, 2.2.5, and 2.2.6. The generic element of $\mathcal{K}$ has TAC and is not definable from equality. On the other hand, if $\mathcal{K}$ is the class of connected graphs of maximum degree d (c.f. Examples 2.2.4 and 2.2.7) then no element of $\mathcal{K}$ has TAC: If $G \in \mathcal{K}$ and (u, v) is an edge of G, then the orbit of v under the action of $\mathrm{Aut}_{\{u\}}(G)$ has size at most d.

Theorem 2.3.13. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem. Suppose that $\mathcal{K}$ is G_δ and the generic element of $\mathcal{K}$ has TAC and is not definable from equality. Then the following are equivalent:*

1. *For every countably infinite group Γ, $\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically.*
2. *There exists a countably infinite group Γ for which $\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically.*

In particular, this applies to Examples 2.2.1 to 2.2.3, 2.2.5, and 2.2.6 by Example 2.3.12.

Lemma 2.3.14. *Let $\boldsymbol{A}$ be an $\mathcal{L}$-structure on a countably infinite set X. Then $\boldsymbol{A}$ has TAC if and only if, for every $\boldsymbol{A}_0 \in \mathrm{Age}_X(\boldsymbol{A})$ with universe F, every $F_0 \subseteq F$, every embedding $f : F_0 \to X$ of $\boldsymbol{A}_0 \restriction F_0$ into $\boldsymbol{A}$ and every finite $G \subseteq X$ there is an embedding g of $\boldsymbol{A}_0$ into $\boldsymbol{A}$ which extends f and such that $g(F \setminus F_0) \cap G = \emptyset$.*

Proof. ($\implies$) Let $\boldsymbol{A}_0, F, F_0, G$ be as in the lemma and suppose that $\boldsymbol{A}$ has TAC. By Fraïssé's Theorem and [Hod93, Theorem 7.1.8] $\boldsymbol{A}$ is homogeneous, so we may assume wlog that f is the identity. By Neumann's Separation Lemma, there is some $g \in \mathrm{Aut}_{F_0}(\boldsymbol{A})$ such that $g(F \setminus F_0) \cap G = \emptyset$, in which case we may take $g \restriction F$.

($\impliedby$) Note first that $\boldsymbol{A}$ is homogeneous, i.e., every isomorphism between finite substructures of $\boldsymbol{A}$ extends to an automorphism of $\boldsymbol{A}$. To see this, by [Hod93, Lemma 7.1.4] it suffices to show that if $\boldsymbol{A}_0 \in \mathrm{Age}_X(\boldsymbol{A})$ has universe F, $F_0 \subseteq F$ and $f : F_0 \to X$ is an embedding of $\boldsymbol{A}_0 \restriction F_0$ into $\boldsymbol{A}$, then f can be extended to an embedding of $\boldsymbol{A}_0$ into $\boldsymbol{A}$, which follows from our assumption (taking $G = \emptyset$).

Let now F be a finite set and $x \in X \setminus F$ in order to show that x has infinite orbit under $\mathrm{Aut}_F(\boldsymbol{A})$. Let $C \subseteq X$ be finite and let $\boldsymbol{A}_0 = \boldsymbol{A} \restriction (F \cup \{x\})$. By assumption, there is an embedding f of $\boldsymbol{A}_0$ into $\boldsymbol{A}$ that is the identity on F and such that $f(x) \notin C$. By homogeneity, this can be extended to an automorphism of $\boldsymbol{A}$, so that in particular the orbit of x is not contained in C. As C, x were arbitrary, we conclude that $\boldsymbol{A}$ has TAC. $\square$

In particular, if $\boldsymbol{A}$ has TAC then $\boldsymbol{A}$ is homogeneous and has the WDP (to see the latter, take $F_0 = \emptyset$ and $G = F$ in the lemma).

Remark 2.3.15. If $\mathcal{K}$ is a class of structures and X a countably infinite set, the set of structures in $\mathcal{K}(X)$ which have the WDP (resp. are definable from equality, have TAC) is G_δ (resp. closed, G_δ) in $\mathcal{K}(X)$.

Lemma 2.3.16. *Let $\mathcal{K}$ be a class of structures with the WDP that are not definable from equality, and Γ be a countably infinite group. For any $\boldsymbol{A} \in \mathcal{K}(\Gamma)$ there is some $\boldsymbol{B} \in Fr(\mathcal{K}(\Gamma))$ isomorphic to $\boldsymbol{A}$. If $\mathcal{K}$ is G_δ, then $Fr(\mathcal{K}(\Gamma))$ is a dense G_δ set in $\mathcal{K}(\Gamma)$.*

Proof. Let $\boldsymbol{A} \in \mathcal{K}(\Gamma)$ and fix an enumeration $\{\gamma_n\}_{n\in\mathbb{N}}$ of Γ. Let $R \in \boldsymbol{\mathcal{L}}$ be a relation such that $R^{\boldsymbol{A}}$ is not definable from equality. We will construct an increasing sequence of finite partial bijections $f_n : \Gamma \to \Gamma$ so that γ_n is in the domain of f_{2n} and in the range of f_{2n+1}, and moreover so that for all n there is a tuple $\bar{x}$ such that $\bar{x}, \gamma_n\bar{x}$ are contained in the domain of f_{2n} and $R^{\boldsymbol{A}}(f_{2n}(\bar{x}) \iff \neg R^{\boldsymbol{A}}(f_{2n}(\gamma_n\bar{x}))$. Assuming this has been done, we let $f = \bigcup_n f_n$ and $\boldsymbol{B} = f^{-1}(\boldsymbol{A})$. Then $\boldsymbol{B} \cong \boldsymbol{A} \in \mathcal{K}(\Gamma)$, and for all $\gamma \in \Gamma$ there is some $\bar{x}$ such that

$$R^{\boldsymbol{B}}(\bar{x}) \iff R^{\boldsymbol{A}}(f(\bar{x})) \iff \neg R^{\boldsymbol{A}}(f(\gamma\bar{x})) \iff \neg R^{\boldsymbol{B}}(\gamma\bar{x}),$$

so that $\gamma\boldsymbol{B} \neq \boldsymbol{B}$ for all $\gamma \in \Gamma$.

We construct these maps as follows. Set $f_{-1} = \emptyset$ for convenience. Given f_{2n}, we let f_{2n+1} be an arbitrary extension with γ_n in its range. Suppose now we have constructed $f_{2n-1} : F \to G$. Because $\boldsymbol{A}$ has the WDP and $R^{\boldsymbol{A}}$ is not definable from equality, there are tuples $\bar{y}, \bar{z}$ with the same equality type so that $R^{\boldsymbol{A}}(\bar{y}) \iff \neg R^{\boldsymbol{A}}(\bar{z})$ and the sets $\bar{y}, \bar{z}, G$ are pairwise disjoint. Since Γ is infinite, we can find a tuple $\bar{x}$ with the same equality type as $\bar{y}$ so that $\bar{x}, \gamma_n\bar{x}, F$ are pairwise disjoint. We then define f_{2n} to extend f_{2n-1} by sending $\bar{x} \mapsto \bar{y}$, $\gamma_n\bar{x} \mapsto \bar{z}$ and, if $f_{2n}(\gamma_n)$ has not already been defined, setting it to any element of Γ not already in the range of f_{2n}.

If $\mathcal{K}$ is G_δ, then $Fr(\mathcal{K}(\Gamma))$ is a G_δ set as the action of Γ is continuous. Moreover, if $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ has universe F and $\boldsymbol{A} \in N(\boldsymbol{A}_0)$, then the same construction starting instead with the f_{-1} as the identity on F gives $\boldsymbol{B} \in N(\boldsymbol{A}_0) \cap Fr(\mathcal{K}(\Gamma))$, so the free part is dense. $\square$

Lemma 2.3.17. *Let $\mathcal{K}$ be a G_δ class of structures with TAC and Γ be a countably infinite group. The set of $\boldsymbol{A} \in \mathcal{K}(\Gamma)$ with the following extension property is a dense G_δ set in $\mathcal{K}(\Gamma)$:*

($*$) *Let $\boldsymbol{A}_0, \boldsymbol{A}_1 \in \mathrm{Age}_\Gamma(\mathcal{K})$ be $\mathcal{L}'$-structures with disjoint universes F, G respectively. Let $F_0 \subseteq F$ and $f : (F \setminus F_0) \cup G \to \Gamma$ be an injection. If $\boldsymbol{A}_0 \restriction F_0 \sqsubseteq \boldsymbol{A}$ and $\boldsymbol{A}_0, \boldsymbol{A}_1$ embed into $\boldsymbol{A}$, then there is some $\gamma \in \Gamma$ such that the map $F_0 \ni x \mapsto x$, $(F \setminus F_0) \cup G \ni x \mapsto \gamma f(x)$ embeds both $\boldsymbol{A}_0, \boldsymbol{A}_1$ into $\boldsymbol{A}$.*

Proof. Fix $\boldsymbol{A}_0, \boldsymbol{A}_1, F, G, F_0, f$ as in ($*$) and let $a : F \to \Gamma, b : G \to \Gamma$ be injections. Let U be the set of all $\boldsymbol{A} \in \mathcal{K}(\Gamma)$ such that, if $\boldsymbol{A}_0 \restriction F_0 \sqsubseteq \boldsymbol{A}$ and a, b are embeddings of $\boldsymbol{A}_0, \boldsymbol{A}_1$ into $\boldsymbol{A}$, then the conclusion of ($*$) holds for $\boldsymbol{A}$. We will show that U is open and dense. As there are only countably many choices for $\boldsymbol{A}_0, \boldsymbol{A}_1, F, G, F_0, f, a, b$, the intersection of all such U is a dense G_δ set whose elements satisfy ($*$).

It is clear that U is open. To see that it is dense, fix $\boldsymbol{B}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ and let $\boldsymbol{A} \in N(\boldsymbol{B}_0)$. If $\boldsymbol{A}_0 \restriction F_0 \not\sqsubseteq \boldsymbol{A}$ or some a, b is not an embedding of $\boldsymbol{A}_0, \boldsymbol{A}_1$ into $\boldsymbol{A}$ then $\boldsymbol{A} \in U$. Otherwise, let $\boldsymbol{B}_0$ have universe H, and assume wlog that $F_0 \subseteq H$. Because $\boldsymbol{A}$ has TAC and by Lemma 2.3.14, there is an embedding $g_0 : F \to \Gamma$ of $\boldsymbol{A}_0$ into $\boldsymbol{A}$ extending the identity on F_0 so that $g_0(F \setminus F_0) \cap H = \emptyset$. By the WDP, there is an embedding $g_1 : \boldsymbol{A}_1 \to \boldsymbol{A}$ whose image is disjoint from $H \cup g_0(F)$. Fix γ so that $H \cap \gamma f((F \setminus F_0) \cup G) = \emptyset$, and let $h : \Gamma \to \Gamma$ be a bijection so that the following hold: h is the identity on H, $h(\gamma f(x)) = g_0(x)$ for $x \in F \setminus F_0$ and $h(\gamma f(x)) = g_1(x)$ for $x \in G$. Let $\boldsymbol{B} = h^{-1}(\boldsymbol{A})$. Then $\boldsymbol{B} \in N(\boldsymbol{B}_0)$, $\boldsymbol{B} \cong \boldsymbol{A}$ and $\boldsymbol{B}$ satisfies ($*$) as witnessed by γ, so $\boldsymbol{B} \in U$. $\square$

Proof of Theorem 2.3.13. Clearly (1) $\implies$ (2).

(2) $\implies$ (1): As the class $\mathcal{K}'$ of elements of $\mathcal{K}$ with TAC that are not definable from equality is a comeagre G_δ set in $\mathcal{K}$, we may assume wlog that $\mathcal{K} = \mathcal{K}'$.

Let Γ, Λ be countably infinite groups and $Z \subseteq \mathcal{K}(\Lambda)$ be a Borel comeagre Λ-invariant set that admits a Λ-equivariant expansion to $\mathcal{K}^*$.

Let $f_n \in S_\Lambda$ be a dense sequence of bijections, i.e., a sequence such that for every finite partial bijection $\Lambda \to \Lambda$ there is some f_n extending it. Since S_Λ acts on $\mathcal{K}(\Lambda)$

by homeomorphisms, we can assume wlog that Z is G_δ and that $f_n(\boldsymbol{A}) \in Z$ for all $\boldsymbol{A} \in Z, n \in \mathbb{N}$. By Lemma 2.3.16, we may also assume that $Z \subseteq Fr(\mathcal{K}(\Lambda))$.

Let $X = Fr(\mathcal{K}(\Gamma))$, which by Lemma 2.3.16 is a dense G_δ set in $\mathcal{K}(\Gamma)$, and let $E = E_\Gamma^X$. Let $\mathbb{A}$ be the canonical $\mathcal{K}$-structuring of E. We will show that there is a Borel expansion of $\mathbb{A}$ restricted to a comeagre Γ-invariant set $Y \subseteq X$, and hence by Proposition 2.3.4(2) there is a Borel Γ-equivariant expansion map $Y \to \mathcal{K}^*(\Gamma)$. As Γ was arbitrary, this proves (1).

Our proof strategy is as follows: We find a Borel Γ-invariant comeagre set $Y \subseteq X$ and a free Λ-action on Y so that $E_\Lambda^Y = E_\Gamma^Y$. By Proposition 2.3.1 applied to $\mathbb{A}\restriction Y$, this gives a Λ-equivariant Borel map $F : Y \to \mathcal{K}(\Lambda)$. We will ensure that the image of F is contained in Z. By Proposition 2.3.2, this implies the existence of a Borel expansion of $\mathbb{A}\restriction Y$, completing the proof.

Below, we let $\forall^*$ denote the category quantifier: If W is a topological space and $A \subseteq W$ is Baire-measurable, $\forall^* w A(x)$ means that A is comeagre (see [Kec95, 8.J]).

Let $\mathcal{G}$ denote the **intersection graph** of E. That is, the vertices of $\mathcal{G}$ are finite subsets of X which are contained in a single E-class, and

$$a\mathcal{G}b \iff a \neq b \;\;\&\;\; (a \cap b \neq \emptyset).$$

By the proof of [KM04, Lemma 7.3], we may fix a countable Borel colouring c of $\mathcal{G}$.

Let $(R_n, \gamma_n, \bar{\delta}_n)$ be a sequence of triples so that (1) the tuples in $\{\bar{\delta}_n, \gamma_n\bar{\delta}_n : n \in \mathbb{N}\}$ are pairwise disjoint, (2) for every n, $R_n \in \mathcal{L}$, and if R_n has arity k then $\bar{\delta} \in \Gamma^k$, and (3) for every $R \in \mathcal{L}$ of arity k and every equality type of tuples of length k, there are infinitely many n with $R_n = R$ such that $\bar{\delta}_n$ has this equality type. Let

$$O_n = \{\boldsymbol{A} \in X : R_n^{\boldsymbol{A}}(\bar{\delta}_n) \;\;\&\;\; \neg R_n^{\boldsymbol{A}}(\gamma_n\bar{\delta}_n)\}$$

and $B_{n+1} = O_n \setminus \bigcup_{i<n} O_i$, $n \in \mathbb{N}$. We also set $B_0 = X \setminus \bigcup_n O_n$.

Claim 2.3.18. Suppose $\boldsymbol{A} \in \mathcal{K}(\Gamma)$ satisfies $(*)$ from Lemma 2.3.17. Then $|\Gamma\cdot\boldsymbol{A} \cap B_n| = \infty$ for infinitely many n.

Proof. Let $N \in \mathbb{N}$ be arbitrary. We may find some $n \geq N$ so that $R_n^{\boldsymbol{A}}$ is not definable from equality, and there are tuples $\bar{x}, \bar{y}$ with the same equality type as $\bar{\delta}_n$ so that $R_n^{\boldsymbol{A}}(\bar{x}) \;\;\&\;\; \neg R_n^{\boldsymbol{A}}(\bar{y})$. By the WDP, we may assume $\bar{x}, \bar{y}$ are disjoint. For $i < n$, find $\bar{x}_i$ with the same equality type as $\bar{\delta}_i$ which are disjoint from each other and from $\bar{x}, \bar{y}$. By $(*)$, there is some γ so that $R_n^{\boldsymbol{A}}(\gamma\bar{\delta}_n)$, $\neg R_n^{\boldsymbol{A}}(\gamma\gamma_n\bar{\delta}_n)$ and for $i < n$, if $R_i^{\boldsymbol{A}}(\bar{x}_i)$ then $R_i^{\boldsymbol{A}}(\gamma\gamma_i\bar{\delta}_i)$ and otherwise $\neg R^{\boldsymbol{A}}(\gamma\bar{\delta}_i)$. It follows that $\gamma^{-1}\boldsymbol{A} \in B_n$. $\triangleleft$

For $\alpha \in \mathbb{N}^{\mathbb{N}}$, we define an equivalence relation E^α on X as follows: We set E_0^α to be equality. Given E_n^α, we set $xE_{n+1}^\alpha y$ if either $xE_n^\alpha y$ or $x\not{E}_n^\alpha y$, $c([x]_{E_n^\alpha} \cup [y]_{E_n^\alpha}) = \alpha(n)$, and $x, y \in \bigcup_{i<n} B_i$. We then set $E^\alpha = \bigcup_n E_n^\alpha$.

Note that if $[x]_{E_n^\alpha}$ is not a singleton for some α, n, then $x \in \bigcup_{i<n} B_i$. We note also that we may analogously construct E_i^s for $s \in \mathbb{N}^{<\mathbb{N}}, i \leq |s|$, and that $E_i^\alpha = E_i^s$ for all such s, i and $\alpha \supseteq s$. We set $E^s = E_{|s|}^s$.

Claim 2.3.19. $\forall x \in X \forall^* \alpha([x]_E = [x]_{E^\alpha})$.

Proof. Fix $x \in X$ and let yEx. We show that the set of α with $y \in [x]_{E^\alpha}$ is open and dense. It is clearly open, as if $yE^\alpha x$ then $yE_n^\alpha x$ for some n. To see it is dense, fix $s \in \mathbb{N}^n$. We may assume wlog that $x, y \in \bigcup_{i<n} B_i$. But then any $\alpha \supseteq s$ with $\alpha(n) = c([x]_{E^s} \cup [y]_{E^s})$ satisfies $yE^\alpha x$. As there are only countably many yEx, the set of all α for which $[x]_E = [x]_{E^\alpha}$ is dense G_δ. $\triangleleft$

Let now $\mathcal{L}_0 = (f_\lambda)_{\lambda \in \Lambda}$, where each f_λ is a binary relation, and let $\mathbf{\Lambda} = (\Lambda, f_\lambda^{\mathbf{\Lambda}})_{\lambda \in \Lambda}$ be the $\mathcal{L}_0$-structure where $f_\lambda^{\mathbf{\Lambda}}(\delta) = \lambda\delta$ is interpreted as (the graph of) multiplication on the left by λ.

Let $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. We define an $\mathcal{L}_0$-structure $\mathbb{A}^{\alpha,\beta}$ on X as follows. We will define an increasing sequence of structures $\mathbb{A}_n^{\alpha,\beta}$ on E_n^α and then take $\mathbb{A}^{\alpha,\beta} = \bigcup_n \mathbb{A}_n^{\alpha,\beta}$. We will ensure at every stage n of this process that, if C is an E_n^α-class, then $\mathbb{A}_n^{\alpha,\beta} \upharpoonright C$ will be isomorphic to a substructure of $\mathbf{\Lambda}$.

Fix a Borel linear order $<$ on X and an enumeration of Λ. We define $\mathbb{A}_0^{\alpha,\beta}$ by setting $f_{id}^{\mathbb{A}_0^{\alpha,\beta}}(x) = x$ for all $x \in X$, and leaving the other relations undefined. Suppose now that we have defined $\mathbb{A}_n^{\alpha,\beta}$, in order to define $\mathbb{A}_{n+1}^{\alpha,\beta}$. Let C be an E_{n+1}^α-class. If C is an E_n^α-class, then we define $\mathbb{A}_{n+1}^{\alpha,\beta}$ to be equal to $\mathbb{A}_n^{\alpha,\beta}$ on C. Otherwise, C is the union of two E_n^α-classes C_0, C_1. Order them so that the $<$-least element of C_0 is $<$-below the $<$-least element of C_1. For each C_i, as $\mathbb{A}_n^{\alpha,\beta} \upharpoonright C_i$ is isomorphic to a substructure of $\mathbf{\Lambda}$, there is a unique embedding $f_i : C_i \to F_i \subseteq \Lambda$ taking the $<$-least element of C_i to the identity in Λ. We then take λ to be the $\beta(n)$-th element of Λ, in our fixed enumeration, satisfying $F_0 \cap F_1 \cdot \lambda = \emptyset$. We define now $\mathbb{A}_{n+1}^{\alpha,\beta} \upharpoonright C$ to be the pullback of $\mathbf{\Lambda} \upharpoonright (F_0 \cup F_1 \cdot \lambda)$ via the injection $(f_0 \cup f_1 \cdot \lambda) : (C_0 \cup C_1) \to (F_0 \cup F_1 \cdot \lambda)$, where $f_1 \cdot \lambda$ denotes the map $x \mapsto f_1(x) \cdot \lambda$. (Note that we are multiplying F_1, f_1 by λ on the right. This is because $\mathbf{\Lambda}$ is defined in terms of multiplication on the left, and this commutes with multiplication on the right.)

As with the E^α, we may define $\mathbb{A}_k^{s,t}$ for $s,t \in \mathbb{N}^n$ and $k \leq n$, and we let $\mathbb{A}^{s,t} = \mathbb{A}_n^{s,t}$. Note that $\mathbb{A}_n^{\alpha,\beta} = \mathbb{A}^{\alpha\restriction n, \beta\restriction n}$ for all α, β.

Claim 2.3.20. $\forall^* x \in X \forall^* \alpha, \beta (\mathbb{A}^{\alpha,\beta}\restriction[x]_{E^\alpha} \cong \mathbf{\Lambda})$.

Proof. Fix $x \in X$ satisfying $(*)$ of Lemma 2.3.17. Note that by construction, $\mathbb{A}^{\alpha,\beta}\restriction[x]_{E^\alpha}$ embeds into $\mathbf{\Lambda}$ for all α, β, and hence there is a unique embedding $f^{\alpha,\beta} : \mathbb{A}^{\alpha,\beta}\restriction[x]_{E^\alpha} \to \mathbf{\Lambda}$ taking x to the identity. Thus it suffices to show that the set of all α, β for which $f^{\alpha,\beta}$ is surjective is a dense G_δ set. To see this, fix $\lambda \in \Lambda$. We will show that the set of all α, β so that $\lambda \in f^{\alpha,\beta}([x]_{E^\alpha})$ is a dense open set. As there are only countably many such λ, this completes the proof.

Note that we may define similarly $f^{s,t} : [x]_{E^s} \to \Lambda$ for $s,t \in \mathbb{N}^n$. Then $f^{\alpha,\beta} = \bigcup_n f^{\alpha\restriction n,\beta\restriction n}$, so it is clear that the set of α, β with λ in its image is an open set. To see that it is dense, fix $s,t \in \mathbb{N}^n$ and consider $f^{s,t}$. If λ is in the image of $f^{s,t}$ then any α, β extending s,t satisfies that λ is in the image of $f^{\alpha,\beta}$. So suppose otherwise. By Claim 2.3.18, there is some yEx so that $[y]_{E^s}$ is a singleton. It is easy to see that if u is an extension of s whose new values are all $c([x]_{E^s} \cup \{y\})$, then for sufficiently long u we have $yE^u x$. Pick such a u of minimal length, so that at stage $|u|$ of the construction of E^u we merge $[x]^{E^s}$ with $\{y\}$. Let C_0, C_1 denote these two sets, ordered as in the construction of $\mathbb{A}^{\alpha,\beta}$, and let $f_i : C_i \to F_i \subseteq \Lambda$ be the corresponding embeddings.

If v is any extension of t of length $|u|$, $m = v(|u|-1)$ and λ_m is the m-th element of Λ such that $F_0 \cap F_1 \cdot \lambda_m = \emptyset$, then $\mathbb{A}^{u,v}\restriction C \cong \mathbf{\Lambda}\restriction(F_0 \cup F_1 \cdot \lambda_m)$ via the map $f = f_0 \cup f_1 \cdot \lambda_m$. In particular, if $x \in C_0$ and $f_0(x) = \delta$, then $f^{u,v} = f \cdot \delta^{-1}$. On the other hand, if $x \in C_1$ and $f_1(x) = \delta$, then $f^{u,v} = f \cdot \lambda_m^{-1}\delta^{-1}$. We will show that we can choose m so that $\lambda = f^{u,v}(y)$, and hence such that λ is in the image of $f^{\alpha,\beta}$ for all α, β extending u, v. As s, t were arbitrary, the set of all such α, β is dense and we are done.

Consider now two cases. If $x \in C_0$ and $f_0(x) = \delta$, then by assumption $\lambda\delta \notin F_0$. Since C_1 is a singleton, F_1 contains only the identity. Pick m so that $\lambda_m = \lambda\delta$. For such an m we have $f^{u,v}(y) = f_1(y)\lambda_m\delta^{-1} = \lambda\delta\delta^{-1} = \lambda$. On the other hand, if $x \in C_1$ and $f_1(x) = \delta$, then by assumption $\lambda\delta \notin F_1$. In this case again F_0 contains only the identity. Pick m with $\lambda_m = \delta^{-1}\lambda^{-1}$. For such an m we have $f^{u,v}(y) = f_0(y)\lambda_m^{-1}\delta^{-1} = \lambda\delta\delta^{-1} = \lambda$. $\triangleleft$

For all α, β, let $X^{\alpha,\beta} \subseteq X$ denote the set of all x for which $[x]_{E^\alpha} = [x]_E$ and $\mathbb{A}^{\alpha,\beta}\restriction[x]_E \cong \mathbf{\Lambda}$. There is a free Borel action of Λ on this set, namely the action where

$\lambda \cdot x = f_\lambda^{\mathbb{A}^{\alpha,\beta}}(x)$. By Proposition 2.3.1, the structure $\mathbb{A}$ on $E \restriction X^{\alpha,\beta} = E_\Lambda^{X^{\alpha,\beta}}$ gives rise to a Λ-equivariant Borel map $F^{\alpha,\beta} : X^{\alpha,\beta} \to \mathcal{K}(\Lambda)$.

Claim 2.3.21. $\forall^* x \in X \forall^* \alpha, \beta (x \in X^{\alpha,\beta} \implies F^{\alpha,\beta}(x) \in Z)$.

Proof. Fix any $\boldsymbol{A} \in X$ that is isomorphic to an element of Z and satisfies $(*)$ of Lemma 2.3.17. Note that the set of all such $\boldsymbol{A}$ is comeagre (for the first condition, note that any bijection $\Lambda \to \Gamma$ gives a homeomorphism $\mathcal{K}(\Lambda) \to \mathcal{K}(\Gamma)$ and consider the image of Z). If $\boldsymbol{A} \in X^{\alpha,\beta}$, let $\boldsymbol{B}^{\alpha,\beta} = F^{\alpha,\beta}(\boldsymbol{A})$. We will show that the set of all α, β for which $\boldsymbol{A} \in X^{\alpha,\beta} \implies \boldsymbol{B}^{\alpha,\beta} \in Z$ is a dense G_δ set.

Fix $s, t \in \mathbb{N}^n$ and let $C^{s,t} = [\boldsymbol{A}]_{E^s}$. Let $f^{s,t} : C^{s,t} \to I^{s,t} \subseteq \Lambda$ be the unique embedding of $\mathbb{A}^{s,t} \restriction C^{s,t}$ into $\boldsymbol{\Lambda}$ which takes $\boldsymbol{A}$ to the identity (note that this $f^{s,t}$ is the same as the one described in the proof of the previous claim). Let also $D^{s,t} = \{\gamma : \gamma^{-1}\boldsymbol{A} \in C^{s,t}\}$, let $g^{s,t} : D^{s,t} \to C^{s,t}$ be the map $\gamma \mapsto \gamma^{-1}\boldsymbol{A}$ and let $h^{s,t} = f^{s,t} \circ g^{s,t}$. Let $\boldsymbol{B}^{s,t} = h^{s,t}(\boldsymbol{A} \restriction D^{s,t})$. It is easy to see, by Proposition 2.3.1, that $\boldsymbol{B}^{s,t} \sqsubseteq \boldsymbol{B}^{\alpha,\beta}$ whenever $\alpha \supseteq s, \beta \supseteq t$ and $\boldsymbol{A} \in X^{\alpha,\beta}$.

Let now U_n be a sequence of dense open sets in $\mathcal{K}(\Lambda)$ so that $Z = \bigcap_n U_n$. We will show that for all N, the set of α, β so that $\boldsymbol{A} \in X^{\alpha,\beta} \implies \boldsymbol{B}^{\alpha,\beta} \in U_N$ is dense and open. Since U_N is open,

$$\boldsymbol{A} \in X^{\alpha,\beta} \implies [\boldsymbol{B}^{\alpha,\beta} \in U_N \iff \exists \mathcal{L}' \subseteq \mathcal{L} \exists n (N(\boldsymbol{B}^{\alpha\restriction n, \beta\restriction n} \restriction \mathcal{L}') \subseteq U_N)].$$

Thus the set of all such α, β is exactly the set of α, β satisfying

$$\exists \mathcal{L}' \subseteq \mathcal{L} \exists n (N(\boldsymbol{B}^{\alpha\restriction n, \beta\restriction n} \restriction \mathcal{L}') \subseteq U_N),$$

which is clearly open, so it remains to show that it is dense. That is, we need to show that for all s, t, there are u, v extending s, t so that $N(\boldsymbol{B}^{u,v} \restriction \mathcal{L}') \subseteq U_N$ for some finite $\mathcal{L}' \subseteq \mathcal{L}$.

Fix $s, t \in \mathbb{N}^n$. By assumption, there is some $\boldsymbol{A}' \in Z$ that is isomorphic to $\boldsymbol{A}$. As Z is closed under the functions f_n described at the start of the proof, we may assume that $\boldsymbol{B}^{s,t} \sqsubseteq \boldsymbol{A}'$. Since $\boldsymbol{A}' \in Z \subseteq U_N$, there is some $\mathcal{L}'$-structure $\boldsymbol{B}_0 \in \mathrm{Age}_\Lambda(\mathcal{K})$ so that $\boldsymbol{A}' \in N(\boldsymbol{B}_0) \subseteq U_N \cap N(\boldsymbol{B}^{s,t} \restriction \mathcal{L}')$. Let F be the universe of $\boldsymbol{B}_0$, so that $\boldsymbol{B}_0 = (\boldsymbol{A}' \restriction \mathcal{L}') \restriction F$ and wlog $F \supseteq I^{s,t}$. We will show that there are u, v extending s, t so that $\boldsymbol{B}^{u,v} \restriction \mathcal{L}' = \boldsymbol{B}_0$, which would complete the proof.

We will show how to do this assuming that $F = I^{s,t} \sqcup \{\lambda\}$. By repeating this argument recursively we can handle all finite F.

By the proof of Claim 2.3.18, there is some $m > n$, a finite structure $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\boldsymbol{A})$ and an injection f from the universe of $\boldsymbol{A}_0$ to $\Gamma \setminus \{1_\Gamma\}$ so that if $x \mapsto \gamma f(x)$ is an embedding of $\boldsymbol{A}_0$ into $\boldsymbol{A}$, then $\gamma^{-1}\boldsymbol{A} \in B_m$. It is clear from the proof that we can also assume that the universe of $\boldsymbol{A}_0$ is disjoint from $D^{s,t}$. Now $\boldsymbol{B}_0$ embeds into $\boldsymbol{A}$, as it embeds into $\boldsymbol{A}' \cong \boldsymbol{A}$, so by $(*)$ there is some γ so that (a) the map $x \mapsto \gamma f(x)$ is an embedding of $\boldsymbol{A}_0$ into $\boldsymbol{A}$ and (b) the map $(h^{s,t})^{-1} \cup \{(\lambda, \gamma)\}$ is an embedding $F \to \Gamma$ of $\boldsymbol{B}_0$ into $\boldsymbol{A}$.

By (a) and our choice of $\boldsymbol{A}_0$, $\gamma^{-1}\boldsymbol{A} \in B_m$ so the E^s-class of $\gamma^{-1}\boldsymbol{A}$ is a singleton. Extend s to a sequence u of length $m+1$ by setting the new values to be $c([\boldsymbol{A}]_{E^s} \cup \{\gamma^{-1}\boldsymbol{A}\})$. Thus $[\boldsymbol{A}]_{E^u_m} = [\boldsymbol{A}]_{E^s}$ and $[\boldsymbol{A}]_{E^u} = [\boldsymbol{A}]_{E^s} \cup \{\gamma^{-1}\boldsymbol{A}\}$. By the proof of the previous claim, there is an extension v of t of length $m+1$ so that $f^{u,v}(\gamma^{-1}\boldsymbol{A}) = \lambda$.

We claim now that $\boldsymbol{B}^{u,v} \restriction \mathcal{L}' = \boldsymbol{B}_0$. To see this, note that $D^{u,v} = D^{s,t} \cup \{\gamma\}$, $I^{u,v} = I^{s,t} \cup \{\lambda\} = F$ and $h^{u,v} = (h^{s,t} \cup \{(\gamma, \lambda)\})$. Now $\boldsymbol{B}^{u,v} = h^{u,v}(\boldsymbol{A} \restriction D^{u,v})$, so this follows immediately from (b). $\triangleleft$

By the Kuratowski–Ulam Theorem [Kec95, 8.41] and the claims above, we may fix some α, β so that $X^{\alpha,\beta}$ is comeagre and $x \in X^{\alpha,\beta} \implies F^{\alpha,\beta}(x) \in Z$ for the generic $x \in X$. In particular, there is a comeagre Γ-invariant Borel set $Y \subseteq X^{\alpha,\beta}$ such that $F^{\alpha,\beta}(Y) \subseteq Z$, which proves (1) by the remarks at the start of the proof. $\square$

2.3.6 Enforcing smoothness

Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem. We are broadly interested in relating the class of Borel expandable CBER with natural classes of CBER such as being smooth or compressible. In this section, we give some sufficient conditions for an expansion problem $(\mathcal{K}, \mathcal{K}^*)$ to be Borel expandable for exactly the class of smooth CBER.

Proposition 2.3.22. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem. If there is a Borel expansion map $f : \mathcal{K}(\mathbb{N}) \to \mathcal{K}^*(\mathbb{N})$, then every smooth aperiodic CBER is Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$.*

Note that we do not require f to satisfy any additional properties (such as equivariance).

Proof. Let E be a smooth aperiodic CBER on X. Since E is smooth, we can identify every E-class with $\mathbb{N}$ in a Borel way, i.e., there is a Borel enumeration $g : X \to X^{\mathbb{N}}$ so that if xEy then $g(x) = g(y)$ (for example take any Borel enumeration h of E and a selector s for E and let $g = h \circ s$). If $F : X \to \mathcal{K}(\mathbb{N})$ is Borel and E-invariant, then composing this with f gives a Borel E-invariant map $F^* = f \circ F : X \to \mathcal{K}^*(\mathbb{N})$ so that $F^*(x)$ is an expansion of $F(x)$ for all x. By the correspondence described in Section 2.3.1, it follows that E is Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$. $\square$

Remark 2.3.23. In many cases of interest (including all of the examples in Section 2.2.2), a Borel expansion map $\mathcal{K}(\mathbb{N}) \to \mathcal{K}^*(\mathbb{N})$ can easily be shown to exist, for example by recursively constructing an expansion for a given $\boldsymbol{A} \in \mathcal{K}(\mathbb{N})$, or via an application of the Compactness Theorem (see e.g. [Kec95, 28.8]). Note that such constructions depend crucially on the given enumeration of the universe of $\boldsymbol{A}$, and hence are not in general equivariant.

Definition 2.3.24. Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem and $\mathcal{E}$ be a class of aperiodic CBER. We say $(\mathcal{K}, \mathcal{K}^*)$ **enforces** $\mathcal{E}$ if an aperiodic CBER E belongs to $\mathcal{E}$ whenever E is $\mathcal{K}$-structurable and Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$.

When $\mathcal{E}$ is the class of aperiodic smooth CBER we say that such $(\mathcal{K}, \mathcal{K}^*)$ **enforces smoothness**.

Proposition 2.3.25. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem. If some hyperfinite, compressible, aperiodic CBER is not Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$, then $(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness. In particular, this holds if some aperiodic CBER is not generically expandable for $(\mathcal{K}, \mathcal{K}^*)$.*

Proof. Let E be a hyperfinite, compressible, aperiodic CBER and let F be any non-smooth aperiodic CBER. By the Glimm-Effros Dichotomy and compressibility we have $E \sqsubseteq_B^i F$ (c.f. [DJK94]). It is now easy to see that if F is Borel expandable then so is E, hence if E is not Borel expandable then neither is F. The second part follows from the first by [KM04, Theorem 12.1, Corollary 13.3]. $\square$

Corollary 2.3.26. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem. Suppose $\mathcal{K}$ is G_δ and there is a countably infinite group Γ with $Fr(\mathcal{K}(\Gamma)) \neq \emptyset$ so that there is no Borel equivariant expansion map $X \to \mathcal{K}^*(\Gamma)$ for any comeagre invariant Borel set $X \subseteq Fr(\mathcal{K}(\Gamma))$. Then $(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness.*

Proof. Let $\mathbb{A}$ be the canonical Borel $\mathcal{K}$-structuring of $E = E_\Gamma^{Fr(\mathcal{K}(\Gamma))}$. By our assumption on Γ and Proposition 2.3.4(2), $\mathbb{A}$ does not admit a Borel expansion when restricted to any comeagre E-invariant Borel set, and in particular E is not generically expandable. The conclusion follows by Proposition 2.3.25. $\square$

Remark 2.3.27. By Theorem 2.3.13, if the generic element of $\mathcal{K}$ has TAC and not definable from equality then the hypotheses of Corollary 2.3.26 hold for some group Γ iff they hold for all groups Γ.

We note the following weak converse:

Proposition 2.3.28. *Let $(\mathcal{K}, \mathcal{K}^*)$ be an expansion problem. If some non-smooth CBER admits a Borel $\mathcal{K}$-structuring and $\mathcal{K}$ admits a Borel Γ-equivariant expansion to $\mathcal{K}^*$ for some countably infinite group Γ then $(\mathcal{K}, \mathcal{K}^*)$ does not enforce smoothness.*

Proof. Let E be a hyperfinite compressible CBER. If some non-smooth CBER admits a Borel $\mathcal{K}$-structuring, then so does E (as E invariantly embeds into any non-smooth CBER). Now consider the orbit equivalence relation E of the shift of Γ on 2^Γ. This action is generically ergodic, hence by [KM04, Theorem 12.1, Theorem 13.3] it is hyperfinite, compressible and non-smooth on an invariant dense G_δ set $Y \subseteq Fr(2^\Gamma)$. Thus E_Γ^Y admits a Borel $\mathcal{K}$-structuring, and by Proposition 2.3.4(1) it is Borel expandable, so $(\mathcal{K}, \mathcal{K}^*)$ does not enforce smoothness. $\square$

2.4 Examples

In this section, we will consider in detail definable expansion problems for Examples 2.2.1 to 2.2.8. We summarize what is known for these problems in Table 2.1.

Expansions on CBER

	Borel expandable	Generically expandable	Expandable a.e.
Bijections	Smooth[1]	All	Classified
Ramsey's Theorem	Smooth [GX24]	? (CE)	? (CE)
Linearizations	Smooth	? (CE)	? (CE)
Vertex colouring	All[1]	All[1]	All[1]
Spanning trees	?[3]	All	?[3]
$\mathbb{Z}$-lines	Smooth	? (CE)	Classified
Vizing's Theorem	Smooth[5]	? (PP[6])	All [GP20]
Matchings	Smooth [CJMST20]	? (CE[7], PP[8])	? (CE[7], PP[8])

Equivariant expansions on groups

	Borel expansions	Generic expansions	Expansions a.e.	Random expansions
Bijections	None[1]	All	Classified	Classified
Ramsey's Theorem	None	None	? (CE$_\Gamma$)	? (CE$_\Gamma$)
Linearizations	None	None	? (CE$_\Gamma$)	Amen. [Alp22]
Vertex colouring	All[1,2]	All[1]	Classified[1]	All[1]
Spanning trees	?[4]	All	?[4]	?[4]
$\mathbb{Z}$-lines	None	None	Classified	Classified
Vizing's Theorem	None	? (PP[6])	All[2] [GP20]	All [GP20]
Matchings	None [CJMST20]	All[9]	? (CE$_\Gamma$[7], PP[8])	? (CE$_\Gamma$[7], PP[8])

Classified: In the sense of Section 2.3.3 and Proposition 2.3.5.
CE: There are counterexamples coming from free continuous actions of Γ, for all Γ.
CE$_\Gamma$ There are counterexamples for all countably infinite groups Γ.
PP: There are partial positive results (see the corresponding section for details and references).

[1] Essentially [KST99].
[2] On the free part $Fr(\mathcal{K}(\Gamma))$.
[3] This lies somewhere between hyperfinite and treeable.
[4] This lies somewhere between amenable and treeable.
[5] Smooth for $d \geq 3$ [CJMST20], All for $d = 2$ [KST99].
[6] Subexponential growth [BD25] and bipartite [BW23].
[7] See [Lac88; CK13; Kun24; BKS22].
[8] See [LN11; MU16; CM17; BKS22; BCW24; BPZ24; KL23].
[9] All for graphs of bounded degree $d > 2$ (None for $d = 2$).

Table 2.1: A summary of known results for Examples 2.2.1 to 2.2.8

2.4.1 Bijections

Fix $\mathcal{K}, \mathcal{K}^*$ as in Example 2.2.1, that is,

$$\mathcal{K} = \{(X, R, S) \mid R, S \subseteq X \ \ \& \ \ X, R, S \text{ are all countably infinite}\},$$
$$\mathcal{K}^* = \{(X, R, S, T) \mid (X, R, S) \in \mathcal{K} \ \ \& \ \ T \text{ is the graph of a bijection } R \to S\}.$$

Theorem 2.4.1 (Essentially [KST99]). *$(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness, and (E, μ) is not a.e. expandable for any CBER E and any E-invariant probability Borel measure μ.*

Proof. Let E be a non-smooth aperiodic CBER. By the Glimm–Effros Dichotomy, $\mathbb{E}_0 \sqsubseteq_B E$. By [KST99, Theorem 1.1] $\mathbb{E}_0$ is not Borel expandable, and it follows that E is not Borel expandable either. If μ is an E-invariant probability Borel measure, then (E, μ) is not a.e. expandable by the same argument as in [KST99, Section 1] for the shift on $2^{\mathbb{Z}}$. $\square$

More generally, if μ is an ergodic invariant probability Borel measure for a CBER E on a standard Borel space X and $A, B \subseteq X$ are Borel, then $\mu(A) = \mu(B)$ iff $\mathbb{A} = (X, A, B)$ admits a Borel expansion μ-a.e. (see e.g. [KM04, Lemma 7.10]). A similar proof gives a characterization of the invariant random $\mathcal{K}$-structures that admit invariant random expansions.

Theorem 2.4.2. *Let Γ be a countably infinite group. There is a Borel Γ-invariant set $X \subseteq \mathcal{K}(\Gamma)$ and a Borel equivariant expansion map $f : X \to \mathcal{K}^*(\Gamma)$ such that, for all invariant random $\mathcal{K}$-structures μ on Γ, μ admits a random expansion to $\mathcal{K}^*$ if and only if $\mu(X) = 1$, in which case $f_*\mu$ gives such an expansion.*

Moreover, let $\boldsymbol{A} = (\Gamma, A, B) \sim \mu$ for an invariant random $\mathcal{K}$-structure μ on Γ. If μ admits an invariant random expansion then $\mathbb{P}[1_\Gamma \in A] = \mathbb{P}[1_\Gamma \in B]$, and the converse holds when μ is ergodic.

In particular, if E is a CBER on Z induced by a free Borel action of Γ, μ is an E-invariant probability Borel measure and $\mathbb{A}$ is a Borel $\mathcal{K}$-structuring of E, then $\mathbb{A}$ is μ-a.e. expandable to $\mathcal{K}^*$ iff $F^{\mathbb{A}}(z) \in X$ for μ-a.e. $z \in Z$.

Proof. The "in particular" part follows immediately from Proposition 2.3.5.

Let $\Gamma = \{\gamma_n\}$ be an enumeration of Γ. For $A, B \subseteq \Gamma$, define sets $X_n^{A,B}$ recursively by

$$X_n^{A,B} = \left(A \setminus \bigcup_{m<n} X_m^{A,B}\right) \cap \left(B \setminus \bigcup_{m<n} X_m^{A,B} \cdot \gamma_m\right) \cdot \gamma_n^{-1}.$$

The collections $\{X_n^{A,B}\}, \{X_n^{A,B} \cdot \gamma_n\}$ consist of pairwise disjoint sets, and the map taking $\gamma \in X_n^{A,B}$ to $\phi^{A,B}(\gamma) = \gamma \cdot \gamma_n$ is a bijection from $\bigcup_n X_n^{A,B} \subseteq A$ to $\bigcup_n X_n^{A,B} \cdot \gamma_n \subseteq B$.

It is easy to see by induction that $X_n^{\gamma \cdot A, \gamma \cdot B} = \gamma \cdot X_n^{A,B}$ for $\gamma \in \Gamma$, so the map $(A, B) \mapsto \phi^{A,B}$ is Γ-equivariant. Additionally, either $\mathrm{dom}(\phi^{A,B}) = A$ or $\mathrm{ran}(\phi^{A,B}) = B$: If $\gamma \in A \backslash \mathrm{dom}(\phi^{A,B}), \gamma' \in B \backslash \mathrm{ran}(\phi^{A,B})$ and $\gamma_n = \gamma^{-1}\gamma'$ then $\gamma \in X_n^{A,B}$, a contradiction.

We let $X \subseteq \mathcal{K}(\Gamma)$ be the set of all $\boldsymbol{A} = (\Gamma, A, B)$ such that $\phi^{A,B}$ is a bijection $A \to B$. It is clear that X is invariant and Borel, and that $\boldsymbol{A} \mapsto (\boldsymbol{A}, \phi^{A,B})$ defines a Borel equivariant expansion $f : X \to \mathcal{K}^*(\Gamma)$.

Now let μ be an invariant random $\mathcal{K}$-structure on Γ. If $\mu(X) = 1$ then $f_*\mu$ is an invariant random expansion of μ. If ν is an invariant random expansion of μ and $(\Gamma, A, B, T) \sim \nu$ then

$$\begin{aligned} \mathbb{P}[1_\Gamma \in A] &= \mathbb{P}[\exists \gamma((1_\Gamma, \gamma) \in T)] = \sum_\gamma \mathbb{P}[(1_\Gamma, \gamma) \in T] = \sum_\gamma \mathbb{P}[(\gamma^{-1}, 1_\Gamma) \in T] \\ &= \mathbb{P}[\exists \gamma((\gamma, 1_\Gamma) \in T)] = \mathbb{P}[1_\Gamma \in B], \end{aligned} \tag{2.1}$$

and since ν is a random expansion of μ we have that $\mathbb{P}[1_\Gamma \in A] = \mathbb{P}[1_\Gamma \in B]$ for $(\Gamma, A, B) \sim \mu$.

Suppose now μ is ergodic and $\mathbb{P}[1_\Gamma \in A] = \mathbb{P}[1_\Gamma \in B]$ for $(\Gamma, A, B) \sim \mu$. As in (2.1), we find that $\mathbb{P}[1_\Gamma \in \mathrm{dom}(\phi^{A,B})] = \mathbb{P}[1_\Gamma \in \mathrm{ran}(\phi^{A,B})]$. If $\mathbb{P}[A = \mathrm{dom}(\phi^{A,B})] = 1$ then

$$\mathbb{P}[1_\Gamma \in B] = \mathbb{P}[1_\Gamma \in A] = \mathbb{P}[1_\Gamma \in \mathrm{dom}(\phi^{A,B})] = \mathbb{P}[1_\Gamma \in \mathrm{ran}(\phi^{A,B})],$$

and it follows that $\mathbb{P}[B = \mathrm{ran}(\phi^{A,B})] = 1$. Similarly, if $\mathbb{P}[B = \mathrm{ran}(\phi^{A,B})] = 1$ then $\mathbb{P}[A = \mathrm{dom}(\phi^{A,B})] = 1$. By ergodicity, one of these must hold, and so

$$\mathbb{P}[A = \mathrm{dom}(\phi^{A,B})] = \mathbb{P}[B = \mathrm{ran}(\phi^{A,B})] = 1$$

and hence $\mu(X) = 1$.

It remains to show that if μ admits an invariant random expansion then $\mu(X) = 1$, and for this it suffices to prove that if ν is an invariant random $\mathcal{K}^*$-structure on Γ and $(\Gamma, A, B, T) \sim \nu$ then $\mathbb{P}[(A, B) \in X] = 1$. By considering an ergodic decomposition of $\mathcal{K}^*(\Gamma)$ (cf. [Kec25, Theorem 5.12]) we may assume ν is ergodic, in which case this follows by the same argument as in the previous paragraph. $\square$

It is not hard to verify that the set X constructed in the proof of Theorem 2.4.2 is a dense G_δ set in $\mathcal{K}(\Gamma)$, so that the canonical $\mathcal{K}$-structuring of $E_\Gamma^{Fr(\mathcal{K}(\Gamma))}$ admits a Borel expansion on a comeagre invariant Borel set. More generally, we have the following:

Theorem 2.4.3. *Every aperiodic CBER is generically expandable. In particular, $\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically for every countably infinite group Γ.*

Proof. The second part follows from the first by Proposition 2.3.4. In order to prove the first part, let E be an aperiodic CBER on a Polish space X and let $R, S \subseteq X$ be Borel sets that have infinite intersection with every E-class. Let

$$xFy \iff xEy \ \& \ (x \in R \iff y \in R) \ \& \ (x \in S \iff y \in S).$$

By (the proof of) [KM04, Corollary 13.3], there is a comeagre E-invariant Borel set for which the aperiodic part of $F{\upharpoonright}C$ is compressible. As the finite part A of $F{\upharpoonright}C$ is smooth, and hence so is $E{\upharpoonright}[A]_E$, it suffices to prove the following:

Claim 2.4.4. Suppose F is compressible. Then there is a Borel bijection $R \to S$ whose graph is contained in E.

Proof. We note first that we may assume that $R \setminus S, S \setminus R$ have infinite intersection with every E-class. Indeed, E is smooth on the set of points for which this is false, and one can easily construct expansions on smooth CBER. By taking our bijection to be the identity on $R \cap S$, we may therefore assume that R, S are disjoint.

Fix a Borel action of a countable group Γ on X so that $xEy \iff \exists \gamma \in \Gamma(\gamma x = y)$. Fix an enumeration $(\gamma_n)_{n\in\mathbb{N}}$ of Γ and for $x \in R$ let $n(x)$ be the least n such that $\gamma_n x \in S$. Let $f(x) = (\gamma_{n(x)}, n(x))$, so that $f : R \to S \times \mathbb{N}$ is a Borel embedding of $F{\upharpoonright}R$ into $F{\upharpoonright}S \times I_{\mathbb{N}}$, where $I_{\mathbb{N}}$ is the equivalence relation on $\mathbb{N}$ with a single equivalence class. Since $F{\upharpoonright}S$ is compressible, there is a Borel isomorphism $g : F{\upharpoonright}S \times I_{\mathbb{N}} \to F{\upharpoonright}S$ such that $xFg(x, n)$ for all $x \in S, n \in \mathbb{N}$ (see e.g. the proof of [DJK94, Proposition 2.5]). Thus $g \circ f : R \to S$ is a Borel injection whose graph is contained in E. Since $F{\upharpoonright}R$ is compressible, the proof of [DJK94, Proposition 2.3] applied to $g \circ f$ gives a Borel bijection $R \to S$ whose graph is also contained in E. $\triangleleft$

$\square$

2.4.2 Ramsey's Theorem

Fix $\mathcal{K}, \mathcal{K}^*$ as in Example 2.2.2, that is,

$$\begin{aligned} \mathcal{K} &= \{(X, R, S) \mid R, S \text{ partition } [X]^2\}, \\ \mathcal{K}^* &= \{(X, R, S, T) \mid (X, R, S) \in \mathcal{K} \ \& \ T \subseteq X \text{ is infinite} \\ &\qquad\qquad \text{and homogeneous for the partition } R, S\}. \end{aligned}$$

Theorem 2.4.5. *Let Γ be a countably infinite group. Then $\mathcal{K}$ does not admit Γ-equivariant expansions to $\mathcal{K}^*$ generically.*

In particular, $(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness.

Proof. The proof of the second part follows from the first by Corollary 2.3.26.

Suppose now by way of contradiction that $f : X \to \mathcal{K}^*(\Gamma)$ was a Borel equivariant expansion on an invariant dense G_δ set $X \subseteq \mathcal{K}(\Gamma)$. Since an expansion $f(x)$ in this case is just a choice of a subset of Γ that is homogeneous for the partition given by $x \in X$, we may view f as an equivariant Borel map $X \to 2^\Gamma$.

Note that the generic element of $\mathcal{K}$ has TAC and is not definable from equality, so by Lemma 2.3.17 we may assume that for any $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ and any $\boldsymbol{A} \in X$ there is some $\gamma \in \Gamma$ such that $\gamma \boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$. By shrinking X further still we may assume that f is continuous.

Now fix an arbitrary $\boldsymbol{A} \in X$ and let $T = f(\boldsymbol{A}) \subseteq \Gamma$. Fix $\gamma_0 \neq \gamma_1 \in T$. By continuity there is some $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ so that $\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$ and $\gamma_0, \gamma_1 \in f(\boldsymbol{B})$ whenever $\boldsymbol{B} \in X \cap N(\boldsymbol{A}_0)$. By equivariance, $\gamma\gamma_0, \gamma\gamma_1 \in f(\boldsymbol{B})$ whenever $\boldsymbol{B} \in X \cap N(\gamma \boldsymbol{A}_0)$.

Let now F be the universe of $\boldsymbol{A}_0$, and assume wlog that $\gamma_0, \gamma_1 \in F$. We consider the case where $[T]^2 \subseteq R^{\boldsymbol{A}}$; the case where $[T]^2 \subseteq S^{\boldsymbol{A}}$ is handled identically. Fix $\gamma \in \Gamma$ so that $F \cap \gamma F = \emptyset$, and let $\boldsymbol{A}_1 = (F \cup \gamma F, R^{\boldsymbol{A}_1}, S^{\boldsymbol{A}_1}) \in \mathrm{Age}_\Gamma(\mathcal{K})$ satisfy $\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}_1, \gamma \boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}_1$ and $\{\gamma_0, \gamma\gamma_0\} \in S^{\boldsymbol{A}_1}$. Let δ be such that $\delta \boldsymbol{A}_1 \sqsubseteq \boldsymbol{A}$. Then $\delta \boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}, \delta\gamma \boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$ so $\delta\gamma_0, \delta\gamma_1, \delta\gamma\gamma_0 \in T$. Also, $\{\delta\gamma_0, \delta\gamma_1\} \in R^{\boldsymbol{A}}$ and $\{\delta\gamma_0, \delta\gamma\gamma_0\} \in S^{\boldsymbol{A}}$. This contradicts the fact that T is homogeneous for the partition $(R^{\boldsymbol{A}}, S^{\boldsymbol{A}})$. ☐

Remark 2.4.6. In [GX24], it is shown that an aperiodic CBER is Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$ iff it is smooth, and in particular that $(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness. Theorem 2.4.5, along with Proposition 2.3.22 and Remark 2.3.23, give an alternative proof of this result. Gao and Xiao consider more generally the case of k-colourings of sets of size n (with the appropriate modifications made to $\mathcal{K}, \mathcal{K}^*$) for $k, n \geq 2$ [GX24, Theorem 1.3]; we note that our proof of Theorem 2.4.5 holds in this more general setting as well (where one takes in this case $\boldsymbol{A}_1$ to contain n copies of $\boldsymbol{A}_0$).

In [GX24, Section 3], a variation of the Ramsey extension property is introduced that enforces (and is actually equivalent to) hyperfiniteness. This extension property involves choosing in an "almost invariant" way (see [GX24, Definition 3.1]) an expansion on each E-class, and in particular does not fit into our framework of expansion problems.

Example 2.4.7. We showed above that there is a Borel $\mathcal{K}$-structuring of a CBER X which does not admit an expansion on any comeagre set. Our proof gave a $\mathcal{K}$-structure (X, R, S) such that, when viewing (X, R) as a graph, each connected component is isomorphic to the Rado graph.

Below are two more such examples. In the first, (X, R) is acyclic, and in the second it is bipartite.

1. By [KST99, Proposition 6.2] there is a Borel acyclic graph $\mathbb{G}_0 \subseteq \mathbb{E}_0$ on $2^{\mathbb{N}}$ for which every Borel independent set is meagre. Let $\mathbb{A} = (2^{\mathbb{N}}, \mathbb{G}_0, \mathbb{E}_0 \setminus \mathbb{G}_0)$, and note that this is a Borel $\mathcal{K}$-structuring of $\mathbb{E}_0$. If $X \subseteq 2^{\mathbb{N}}$ is Borel and $\mathbb{E}_0$-invariant and $\mathbb{A}^* = (\mathbb{A}{\upharpoonright}X, T)$ is a Borel expansion of $\mathbb{A}{\upharpoonright}X$, then $T \cap C$ is $\mathbb{G}_0$-independent for every $\mathbb{E}_0$-class $C \subseteq X$ (as $\mathbb{G}_0$ is acyclic), so T is meagre. Since $\mathbb{E}_0$ is generated by the action of a countable group of automorphisms of $2^{\mathbb{N}}$, $X = [T]_{\mathbb{E}_0}$ is meagre as well. In particular, X is not comeagre, and hence $\mathbb{E}_0$ is not generically expandable.

2. (Kechris) Consider an irrational rotation R on the circle $\mathbb{T}$ and let $E = E_R^{\mathbb{T}}$. Define $f : E \setminus \Delta_{\mathbb{T}} \to 2$ by $f(x, y) = 1$ iff $R^n(x) = y$ for some even $n \in \mathbb{Z}$, where $\Delta_{\mathbb{T}} \subseteq \mathbb{T}^2$ denotes the diagonal. Let $\mathbb{A} = (\mathbb{T}, f^{-1}(0), f^{-1}(1))$ and note that $\mathbb{A}$ is a Borel $\mathcal{K}$-structuring of E. If $X \subseteq \mathbb{T}$ is Borel and E-invariant and $\mathbb{A}^* = (\mathbb{A}{\upharpoonright}X, T)$ is a Borel expansion of $\mathbb{A}{\upharpoonright}X$. then clearly f takes the value 1 on T. One can easily extend T to a Borel set $T \subseteq A \subseteq X$ so that $R^2(A) = A$. It follows, as R^2 is generically ergodic, that A, and hence $A \cup R(A) = X$, are meagre. In particular, X is not comeagre, and hence E is not generically expandable.

By Proposition 2.3.25, these examples give alternative proofs that $(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness.

Additionally, both of these examples are not a.e. expandable for the Haar measure, by a similar ergodicity argument. Proposition 2.4.8 provides another example that is not expandable a.e.

Proposition 2.4.8. *Let Γ be a countably infinite group and let μ be the law of the partition (R, S) of $[\Gamma]^2$ obtained by including every pair $\{\gamma, \delta\}$ in R independently with probability p, $0 < p < 1$. Then μ is an invariant random $\mathcal{K}$-structure on Γ that does not admit an invariant random expansion.*

Proof. It is clear that μ is Γ-invariant. Suppose there was an invariant random expansion ν of μ, and let $(R, S, T) \sim \nu$. We may view R as a graph on Γ, which is almost surely isomorphic to the Rado graph, and view T as either an infinite clique or an infinite independent set.

For any finite $F \subseteq \Gamma$, $R{\upharpoonright}F$ is the random graph on F whose edges are included independently with probability p. The expected size of the largest clique in $R{\upharpoonright}F$ is $\Theta(\log(|F|))$, and hence so is the expected size of the largest independent set [Bol01, Theorem 11.4]. Thus $\mathbb{E}[|F \cap T|] \in O(\log(|F|))$. On the other hand,

$$\mathbb{E}[|F \cap T|] = \mathbb{E}[\sum_{\gamma \in F} \mathbb{1}_{\gamma \in T}] = \sum_{\gamma \in F} \mathbb{P}[\gamma \in T] = |F| \cdot \mathbb{P}[1_\Gamma \in T]$$

by invariance of ν. Taking $|F| \to \infty$ we find that $\mathbb{P}[1_\Gamma \in T] = 0$, and hence that $T = \emptyset$ almost surely, a contradiction. $\square$

2.4.3 Linearizations

Fix

$$\begin{aligned} \mathcal{K} &= \{(X, P) \mid P \text{ is a partial order on } X\}, \\ \mathcal{K}^* &= \{(X, P, L) \mid (X, P) \in \mathcal{K} \ \ \& \ \ P \subseteq L \ \ \& \ \ L \text{ is a linear order on } X\}, \end{aligned}$$

as in Example 2.2.3.

The expansion problem for invariant random $\mathcal{K}$-structures on groups has been studied in [GLM24; Alp22]. In particular, Alpeev has shown that:

Theorem 2.4.9 ([Alp22, Corollary 1.1])**.** *A countable group Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ if and only if Γ is amenable.*

Remark 2.4.10. Note that even in the case when Γ is amenable and all invariant random partial orders admit invariant random extensions to linear orders, we cannot expect to find an equivariant map $f : \mathcal{K}(\Gamma) \to \mathcal{K}^*(\Gamma)$ for which an extension is given by the pushfoward measure along f. This is unlike the case of bijections, or (as we will see below) for vertex colourings or $\mathbb{Z}$-lines. As a trivial example, note that the empty partial order is a fixed point in $\mathcal{K}(\Gamma)$, and there is no equivariant expansion of this partial order when Γ is not left-orderable. We give a more interesting example in Theorem 2.4.11.

Theorem 2.4.11. *Let E be an aperiodic CBER. There is a Borel $\mathcal{K}$-structuring of E that is not μ-a.e. expandable for any E-invariant probability Borel measure μ.*

Proof. We may assume that E is not smooth, as no smooth CBER admits an invariant probability Borel measure.

It suffices to show this for the CBER $F = \Delta_{2^{\mathbb{N}}} \times \mathbb{E}_0$. Indeed, suppose $\mathbb{A}$ were a Borel $\mathcal{K}$-structuring of F with this property. By [Kec25, Corollary 8.14], we can assume that E lives on $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ and that $F \subseteq E$. Then every E-invariant measure is also F-invariant, $\mathbb{A}$ is also a Borel $\mathcal{K}$-structuring of E, and if $\mathbb{A}^*$ is an expansion of $\mathbb{A}$ on E (restricted to some invariant Borel set) then its restriction to each F-class is an expansion of $\mathbb{A}$ on F. It follows immediately that $\mathbb{A}$ witnesses that this holds for E as well.

Let $\mathbb{F}_0$ be the index 2 subequivalence relation of $\mathbb{E}_0$ given by

$$x\mathbb{F}_0 y \iff x\mathbb{E}_0 y \;\&\; |\{i : x(i) \neq y(i)\}| = 0 \mod 2.$$

We define a Borel $\mathcal{K}^*$-structuring L of $\mathbb{F}_0$ as follows: For $x\mathbb{F}_0 y$, we say xLy if either $x = y$ or $\sum_{i<n} x(i) = 0 \mod 2$, where n is maximal with $x(n) \neq y(n)$. This is clearly reflexive and anti-symmetric. To see that it is transitive, let $xLyLz$ and let l, m, n be maximal such that $x(l) \neq y(l)$, $y(m) \neq z(m)$ and $x(n) \neq z(n)$. It is clear that $l \neq m$. If $l < m$, then $m = n$ and $\sum_{i<m} x(i) = \sum_{i<m} y(i) = 0 \mod 2$, so xLz. Otherwise $l > m$, so $l = n$ and $\sum_{i<l} x(i) = 0 \mod 2$, and so xLz.

Note that in particular, for all xLz there are only finitely many y with $xLyLz$. Indeed, if n is maximal with $x(n) \neq z(n)$, then y must agree with either x or z at all coordinates $m > n$. Additionally, it is clear that the restriction of L to every $\mathbb{F}_0$-class is total.

Below, we identify i^k for $i \in 2, k \leq \infty$ with the constant sequence of length k with value i, and let $^\frown$ denote concatenation of sequences. (We also abuse notation and write i for i^1.)

We view L as a Borel $\mathcal{K}$-structuring of $\mathbb{E}_0$. Suppose that $X \subseteq 2^{\mathbb{N}}$ is Borel and $\mathbb{E}_0$-invariant, and that L' is a Borel expansion of $L{\restriction}X$ to $\mathcal{K}^*$. We claim that $\mu(X) = 0$, where μ is the Haar measure on $2^{\mathbb{N}}$. To see this, define $f : 2^{\mathbb{N}} \setminus \{1^\frown 0^\infty\} \to 2^{\mathbb{N}} \setminus \{0^\infty\}$ by

$$f(0^\frown i^\frown x) = 1^\frown (1-i)^\frown x, \qquad f(1^\frown 0^n{}^\frown 1^\frown i^\frown x) = 0^{n+1}{}^\frown 1^\frown (1-i)^\frown x.$$

It is clear that this is a Borel function whose graph is contained in $\mathbb{F}_0$. Moreover, it is not hard to verify that $f(x)$ is the immediate successor of x in L, whenever this is defined. It follows in particular that L orders every $\mathbb{F}_0$-class with order-type $\mathbb{Z}$, except for $[0^\infty]_{\mathbb{F}_0}, [1^\frown 0^\infty]_{\mathbb{F}_0}$, and that f generates L (i.e., $xLy \iff \exists n f^n(x) = y$).

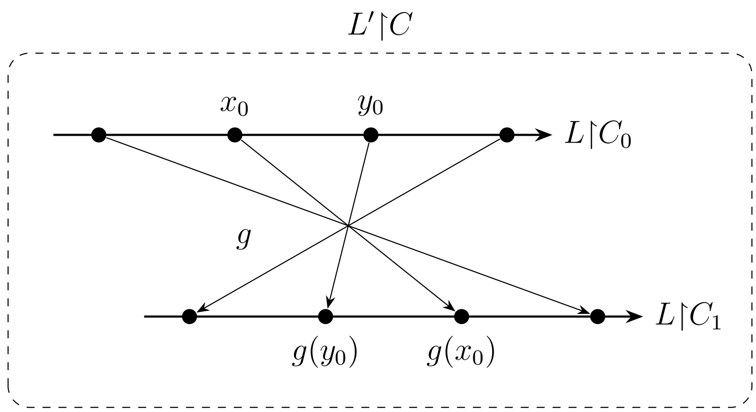

Figure 2.1: There is some $L{\restriction}C_0$-maximal x_0 for which $x_0 L' g(x_0)$.

Let $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the map which flips the first coordinate of every sequence. Then g is a Borel involution, its graph is contained in $\mathbb{E}_0$, and $x \not\mathbb{F}_0 g(x)$ for all x. It is also easy to verify that g is an isomorphism $\mathbb{F}_0 \cong \mathbb{F}_0$, and that it is order-reversing for L.

We will use L' to find a Borel $\mathbb{F}_0$-invariant set $Y \subseteq X$ so that every $\mathbb{E}_0{\restriction}X$-class contains exactly one $\mathbb{F}_0$-class in Y. It is easy to see that μ is $\mathbb{F}_0$-invariant and $\mathbb{F}_0$-ergodic, so this implies that Y is either μ-null or μ-conull. It clearly cannot be μ-conull, so $\mu(Y) = 0$ and hence $\mu(X) = 0$ as well.

Let now C be an $\mathbb{E}_0$-class in X. We show how to choose an $\mathbb{F}_0$-class from C in a uniformly Borel way, and then take Y to be the set of all such choices. If $0^\infty \in C$, then we choose $[0^\infty]_{\mathbb{F}_0}$, so suppose this is not the case. Let C_0, C_1 be the two $\mathbb{F}_0$-classes in C. If some C_i is an initial segment of $L'{\restriction}C$, then we choose C_i. Otherwise, we claim that for $i \in 2$ there is a unique pair $(x_i, y_i) \in C_i$ so that $x_i L y_i$, $x_i L' g(x_i)$ and $g(y_i) L' y_i$ (see Fig. 2.1). We then take x to be the lexicographically least element of $\{x_0, x_1, y_0, y_1\}$ and choose $[x]_{\mathbb{F}_0}$.

We show this for $i = 0$, the case $i = 1$ being symmetric. As C_1 is not an initial segment of $L'{\restriction}C$, there are $x \in C_0, y \in C_1$ with $x L' y$. If $y L g(x)$, then $x L' g(x)$. Otherwise, $g(x) L y$ so $g(y) L x$ (as g is order-reversing) and so $g(y) L' y$. Thus, by possibly setting $x = g(y)$, we may assume that $x L' g(x)$. Clearly there is an L-maximal such x, as we have assumed that C_0 is not an initial segment of $L'{\restriction}C$. We may then take $x_0 = x$, $y_0 = f(x)$.

Let now $\mathbb{A}$ be the $\mathcal{K}$-structuring of $\Delta_{2^{\mathbb{N}}} \times \mathbb{E}_0$ given by pulling back L along the projection $\mathrm{proj}_2 : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ to the second coordinate (note that this is a class-bijective map $\Delta_{2^{\mathbb{N}}} \times \mathbb{E}_0 \to_B^{cb} \mathbb{E}_0$). We claim that for any $(\Delta_{2^{\mathbb{N}}} \times \mathbb{E}_0)$-invariant probability Borel

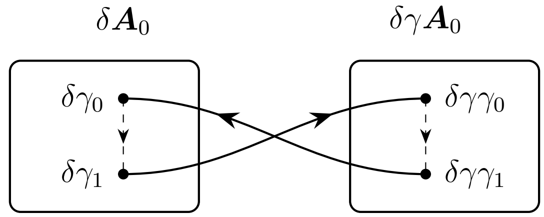

Figure 2.2: A copy of $\delta\boldsymbol{A}_1$ in $\boldsymbol{A}$. The solid arrows are relations in $\boldsymbol{A}_1$, and the dashed arrows are relations that are forced to exist in L.

measure μ and any invariant Borel set $X \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, if there is a Borel expansion of $\mathbb{A}\restriction X$ to $\mathcal{K}^*$ then $\mu(X) = 0$. By considering an ergodic decomposition we may assume that μ is ergodic, in which case it is equal to the Haar measure on $\{x\} \times 2^{\mathbb{N}}$ for some $x \in 2^{\mathbb{N}}$, so this follows by the analogous fact for L on $\mathbb{E}_0$. $\square$

With respect to category, we have the following:

Theorem 2.4.12. *Let Γ be a countably infinite group. Then $\mathcal{K}$ does not admit Γ-equivariant expansions to $\mathcal{K}^*$ generically.*

In particular, $(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness.

Proof. The second part follows from the first by Corollary 2.3.26.

Suppose by way of contradiction that there was such an expansion $f : X \to \mathcal{K}^*(\Gamma)$. By shrinking X, we may assume that X is G_δ, f is continuous, and by Lemma 2.3.17 (and the fact that there is a unique generic partial order) that for every $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ and every $\boldsymbol{A} \in X$ there is some $\gamma \in \Gamma$ with $\gamma\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$.

Fix an arbitrary $\boldsymbol{A} = (\Gamma, P) \in X$ and let $(\Gamma, L) = f(\boldsymbol{A})$, so that $P \subseteq L$. Since P contains a copy of every element of $\mathrm{Age}_\Gamma(\mathcal{K})$ we may in particular find $\gamma_0, \gamma_1 \in \Gamma$ that are P-incomparable. Suppose wlog that $\gamma_0 \mathrel{L} \gamma_1$. By continuity and equivariance, there is some $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ so that $\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$ and for all $\gamma \in \Gamma$ and $\boldsymbol{B} \in X$, if $\gamma\boldsymbol{A}_0 \sqsubseteq \boldsymbol{B}$ then $\gamma\gamma_0$ is less than $\gamma\gamma_1$ in $f(\boldsymbol{B})$.

Let now F be the universe of $\boldsymbol{A}_0$ and assume wlog that $\gamma_0, \gamma_1 \in F$. Find some $\gamma \in \Gamma$ so that $F \cap \gamma F = \emptyset$, and let $\boldsymbol{A}_1 = (F \cup \gamma F, P_1) \in \mathrm{Age}_\Gamma(\mathcal{K})$ be a structure with universe $F \cup \gamma F$ so that $\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}_1, \gamma\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}_1$ and such that $\gamma_1 \mathrel{P_1} \gamma\gamma_0$ and $\gamma\gamma_1 \mathrel{P_1} \gamma_0$. Let δ be such that $\delta\boldsymbol{A}_1 \sqsubseteq \boldsymbol{A}$. Then $\delta\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}, \delta\gamma\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$ so $\delta\gamma_0 \mathrel{L} \delta\gamma_1$ and $\delta\gamma\gamma_0 \mathrel{L} \delta\gamma\gamma_1$. On the other hand, as $\delta P_1 \subseteq P \subseteq L$, we have $\delta\gamma_1 \mathrel{L} \delta\gamma\gamma_0$ and $\delta\gamma\gamma_1 \mathrel{L} \delta\gamma_0$ (see Fig. 2.2.) It follows that $\delta\gamma_0 \mathrel{L} \delta\gamma_1 \mathrel{L} \delta\gamma_0$, a contradiction.

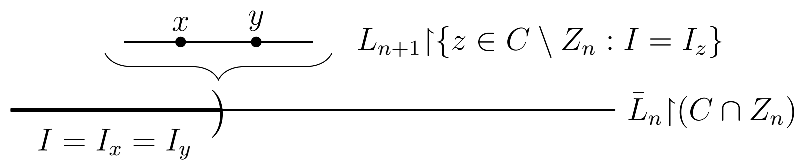

Figure 2.3: Extending $\bar{L}_n$ to $\bar{L}_{n+1}$.

□

We consider now expansions on CBER.

Proposition 2.4.13. *Let E be a CBER on a standard Borel space X and let P be a Borel partial order on E. Then the set of Borel subsets $Y \subseteq X$ for which $P\restriction Y$ admits a Borel linearization on $E\restriction Y$ forms a σ-ideal.*

Proof. This class is clearly closed under taking Borel subsets. Suppose now that $Y = \bigcup_n Y_n$ and for every n there is a Borel linearization L_n of $P\restriction Y_n$ on $E\restriction Y_n$. Let $Z_n = \bigcup_{i<n} Y_n$. We will recursively construct an increasing sequence $\bar{L}_n$ of Borel linearizations of $P\restriction Z_n$ on $E\restriction Z_n$ and then take $L = \bigcup_n \bar{L}_n$.

We begin by setting $\bar{L}_0 = \emptyset$. Suppose now that we have constructed $\bar{L}_n$, and let C be an $E\restriction Z_{n+1}$-class in order to define $\bar{L}_{n+1}\restriction C$. For $x \in C \setminus Z_n$, let

$$I_x = \{y \in C \cap Z_n : \exists z \in C \cap Z_n (z \, P \, x \;\;\&\;\; y \, \bar{L}_n \, z)\}.$$

Note that I_x is an initial segment of $\bar{L}_n\restriction(C \cap Z_n)$. Now for $x, y \in C$, we say $x \, \bar{L}_{n+1} \, y$ iff one of the following hold (c.f. Fig. 2.3):

1. $x, y \in Z_n$ and $x \, \bar{L}_n \, y$,
2. $x \in Z_n, y \notin Z_n$ and $x \in I_y$,
3. $x \notin Z_n, y \in Z_n$ and $y \notin I_x$,
4. $x, y \notin Z_n$ and $I_x \subsetneq I_y$, or
5. $x, y \notin Z_n$, $I_x = I_y$ and $x \, L_{n+1} \, y$.

One easily checks that $\bar{L}_{n+1}$ is a linear order on Z_{n+1}.

□

In particular, if a partial order P on a CBER E can be decomposed into a countable union of chains and antichains, then P admits a Borel linearization on E.

Proposition 2.4.14. *Let T be a Borel locally countable directed tree on a standard Borel space X whose (undirected) connected components form a CBER E, and let P be the smallest partial order containing T. Then (E, P) admits a Borel linearization.*

Proof. For all xEy, define $d(x, y)$ as follows: Consider the unique (undirected) path from x to y in T. Weigh each edge in this path by 1 if it occurs in T, and by -1 if its reverse appears in T, and let $d(x, y)$ be the sum of the edge weights along this path. Fix a Borel linear order $\leq$ on X and define $x \, L \, y$ if either $d(x, y) > 0$, or $d(x, y) = 0$ and $x \leq y$. It is straightforward to verify that L is a linear order on E extending P (for transitivity, note that $d(x, z) = d(x, y) + d(y, z)$ whenever these are defined). □

2.4.4 Vertex colourings

In this section, we fix $d \geq 2$ and let

$$\mathcal{K} = \{(X, E) \mid (X, E) \text{ is a connected graph of max degree } \leq d\},$$
$$\mathcal{K}^* = \{(X, E, S_0, \dots, S_d) \mid (X, E) \in \mathcal{K} \;\; \& \;\; S_0, \dots, S_d \text{ is a vertex colouring of } (X, E)\},$$

as in Example 2.2.4.

It was shown in [KST99, Proposition 4.6] that every CBER is Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$, and therefore that $(\mathcal{K}, \mathcal{K}^*)$ admits random expansions by Proposition 2.3.4. Their proof essentially establishes the following (see also [BC24, Proposition 4.29]).

Theorem 2.4.15 (Essentially [KST99])**.** *Let Γ be a countably infinite group. There is an invariant dense G_δ set $Fr(\mathcal{K}(\Gamma)) \subseteq X \subseteq \mathcal{K}(\Gamma)$ which admits a Borel equivariant expansion, and such that X is maximal with this property: For any invariant $X \supsetneq Y$, there is no equivariant expansion $Y \to \mathcal{K}^*(\Gamma)$.*

Interpreted in the language of [BC24], X consists of exactly the set of orbits which satisfy an appropriate separation axiom (as used for example in the proof of [BC24, Proposition 4.29]).

Proof. We say $G \in \mathcal{K}(\Gamma)$ is **bad** if there are $\gamma, \delta \in \Gamma$ so that $\delta \, G \, \gamma\delta$ and $\gamma G = \gamma$, and G is **good** otherwise.

If G is bad, then so is every graph in its orbit $\Gamma \cdot G$, and in this case there is no equivariant expansion map $\Gamma \cdot G \to \mathcal{K}^*(\Gamma)$. Indeed, suppose c were such an expansion map, and view $c(\gamma G)$ as a map $\Gamma \to d+1$ that is a colouring of γG for $\gamma \in \Gamma$. Fix some γ, δ so that $\delta \mathrel{G} \gamma\delta$ and $\gamma G = G$. Then

$$c(G)(\gamma\delta) = c(\gamma G)(\gamma\delta) = (\gamma c(G))(\gamma\delta) = c(G)(\delta)$$

by equivariance of c, a contradiction.

We take X to be the set of good graphs. Clearly X is a G_δ set containing the free part of $\mathcal{K}(\Gamma)$. It is also not hard to see that it is dense (for example, the free part is dense by Lemma 2.3.16 as the generic element of $\mathcal{K}$ satisfies the WDP and is not definable from equality). It thus remains only to show that X admits a Borel equivariant expansion.

To see this, it suffices to construct a Borel map $f : X \to d+1$ so that $f(G) \neq f(\gamma^{-1}G)$ whenever $G \in X, \gamma \in \Gamma$ and $1_\Gamma \mathrel{G} \gamma$. Indeed, thinking of $f(G)$ as the colour of 1_Γ in G, this extends uniquely to an equivariant map $g : X \to (d+1)^\Gamma$ sending G to the colouring

$$g(G)(\gamma) = f(\gamma^{-1}G).$$

It is clear that g is equivariant and Borel, and $g(G)$ is a proper colouring of G because if $\gamma \mathrel{G} \gamma\delta$ then $1_\Gamma \mathrel{\gamma^{-1}G} \delta$ so

$$g(G)(\gamma) = f(\gamma^{-1}G) \neq f(\delta^{-1}\gamma^{-1}G) = g(G)(\gamma\delta).$$

Let H_n be an enumeration of $\mathrm{Age}_\Gamma(\mathcal{K})$ and let X_n be the set of all $G \in X \cap N(H_n)$ such that $\gamma^{-1}G \notin N(H_n)$ for all neighbours γ of 1_Γ in G. It is clear that at most one of $G, \gamma^{-1}G \in X_n$ whenever $G \in X$ and $1_\Gamma \mathrel{G} \gamma$. We claim that $X = \bigcup_n X_n$. Indeed, as $G \in X$ is good and has bounded degree, there is some finite $F \subseteq \Gamma$ so that $G{\restriction}F \neq \gamma^{-1}G{\restriction}F$ for all neighbours γ of 1_Γ in G, in which case $G \in X_n$ for n such that $H_n = G{\restriction}F$.

Let $Y_n = X_n \setminus \bigcup_{i<n} X_i$. The sets Y_n partition X and if $G \in Y_n$ and $1_\Gamma \mathrel{G} \gamma$ then $\gamma^{-1}G \notin Y_n$. We now define $f : X \to d+1$ recursively on each Y_n as follows: supposing f has already been defined on $\bigcup_{i<n} Y_i$, we define $f(G)$ for $G \in Y_n$ to be the least element of $d+1$ which is not equal to $f(\gamma^{-1}G)$ for all neighbours γ of 1_Γ in G. □

Note that if we replace $\mathcal{K}$ with the class of d-regular graphs, then the sets X_n in the previous proof are clopen, so the construction yields a continuous equivariant expansion $X \to \mathcal{K}^*(\Gamma)$.

One can also consider vertex colourings with fewer colours. This has been studied extensively in the case of expansions on CBER; we refer the reader to [KM20, Part I] for a survey of this topic.

2.4.5 Spanning trees

In this section, consider

$$\mathcal{K} = \{(X, E) \mid (X, E) \text{ is a connected graph}\},$$
$$\mathcal{K}^* = \{(X, E, T) \mid (X, E) \in \mathcal{K} \;\; \& \;\; (X, T) \text{ is a spanning subtree of } (X, E)\},$$

as in Example 2.2.5.

We say a CBER E is **treeable** if there is a Borel $\mathcal{K}'$-structuring of E, where $\mathcal{K}'$ is the class of connected trees. Let $\mathcal{T}$ denote the class of treeable CBER, and say an expansion problem **enforces treeability** if it enforces $\mathcal{T}$.

A countably infinite group Γ is **antitreeable** if for every free Borel action of Γ on a standard Borel space X admitting an invariant probability Borel measure, the CBER E_Γ^X is not treeable (c.f. [Kec25, 115]).

Theorem 2.4.16. *Let E be a CBER. If E is hyperfinite then it is Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$, and if it is Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$ then it is treeable.*

In particular,

1. *$(\mathcal{K}, \mathcal{K}^*)$ enforces treeability;*
2. *$\mathcal{K}$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically for all countably infinite groups Γ;*
3. *Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ for all amenable groups Γ; and*
4. *Γ does not admit random expansions from $\mathcal{K}$ to $\mathcal{K}^*$ for all antitreeable groups Γ.*

Proof. It is clear that if E is Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$ then it is treeable (consider the complete graph on each E-class), and in particular that (1) holds.

To see that hyperfinite CBER are Borel expandable for $\mathcal{K}, \mathcal{K}^*$, let E be a hyperfinite CBER and $\mathbb{A}$ be a Borel $\mathcal{K}$-structuring of E. Write $E = \bigcup_n E_n$ for an increasing union of finite CBER. We recursively construct an increasing sequence of Borel sets $\mathbb{A}_n^*$ so that $\mathbb{A}_n^* \subseteq E_n$ and for every E_n-class C, $\mathbb{A}_n^* \restriction C$ is a spanning subforest of $\mathbb{A} \restriction C$. We describe the construction of $\mathbb{A}_{n+1}^*$, given $\mathbb{A}_n^*$.

Let C be an E_{n+1}-class, $G = \mathbb{A} \restriction C$, $T = \mathbb{A}_n^* \restriction C$. Then $T \subseteq G$ is a forest of trees, so we can easily find a spanning forest $T \subseteq T' \subseteq G$. We set $\mathbb{A}_{n+1}^* \restriction C = T'$. As every E_{n+1}-class is finite, it is clear that this can be done in a uniformly Borel way.

It follows that $Fr(\mathcal{K}(\mathbb{Z}))$ admits an equivariant random expansion to $\mathcal{K}^*$, and by Theorem 2.3.13 this holds for all countably infinite groups Γ. (3) follows by Proposition 2.3.4(3), as every CBER generated by a Borel action of an amenable group is measure-hyperfinite, and (4) is an immediate consequence of Proposition 2.3.6. $\square$

Thus the class of CBER that are Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$ lies somewhere between the hyperfinite and the treeable CBER.

Problem 2.4.17. Is every treeable CBER expandable for $(\mathcal{K}, \mathcal{K}^*)$? Does $(\mathcal{K}, \mathcal{K}^*)$ enforce hyperfiniteness?

2.4.6 $\mathbb{Z}$-lines

Let

$$\mathcal{K} = \{(X, L) \mid (X, L) \text{ is a linear order without endpoints}\},$$
$$\mathcal{K}^* = \{(X, L, Z) \mid (X, L) \in \mathcal{K} \ \& \ Z \subseteq X \ \& \ (Z, L\restriction Z) \cong (\mathbb{Z}, <)\},$$

as in Example 2.2.6.

Theorem 2.4.18. *Let Γ be a countably infinite group. Then $\mathcal{K}$ does not admit Γ-equivariant expansions to $\mathcal{K}^*$ generically.*

In particular, $(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness.

Proof. The second part follows from the first and Corollary 2.3.26.

For $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ and $\boldsymbol{A} \in \mathcal{K}(\Gamma)$, let $C(\boldsymbol{A}_0, \boldsymbol{A}) = \{\gamma : \gamma\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}\}$. We say $\boldsymbol{A}$ **contains $\boldsymbol{A}_0$ densely often** if $\boldsymbol{A}\restriction C(\boldsymbol{A}_0, \boldsymbol{A})$ is a dense linear order with at least two points.

We claim that the generic element of $\mathcal{K}(\Gamma)$ contains every $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ densely often. As there are only countably many such $\boldsymbol{A}_0$, it suffices to show this for some $\boldsymbol{A}_0$. It is easy to see that the set of all $\boldsymbol{A}$ for which $C(\boldsymbol{A}_0, \boldsymbol{A})$ contains at least two points is open and dense, so we show that the set of $\boldsymbol{A}$ for which $\boldsymbol{A}\restriction C(\boldsymbol{A}_0, \boldsymbol{A})$ is dense is a dense G_δ set.

To see this, we show that for any fixed $\gamma_0, \gamma_1 \in \Gamma$, the set of $\boldsymbol{A}$ satisfying

$$\gamma_0, \gamma_1 \in C(\boldsymbol{A}_0, \boldsymbol{A}) \ \& \ \gamma_0 \, L^{\boldsymbol{A}} \, \gamma_1 \implies \exists\delta(\delta \in C(\boldsymbol{A}_0, \boldsymbol{A}) \ \& \ \gamma_0 \, L^{\boldsymbol{A}} \, \delta \, L^{\boldsymbol{A}} \, \gamma_1)$$

is dense and open. This set is clearly open. To see that it is dense, fix $\boldsymbol{B}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ and let $\boldsymbol{A} \in N(\boldsymbol{B}_0)$. If $\gamma_i \notin C(\boldsymbol{A}_0, \boldsymbol{A})$ for some $i \in 2$ or $\gamma_1 \, L^{\boldsymbol{A}} \, \gamma_0$, we are done.

Otherwise, we may assume that γ_0, γ_1 are in the universe of $\boldsymbol{B}_0$ and that $\gamma_i \boldsymbol{A}_0 \sqsubseteq \boldsymbol{B}_0$ for $i \in 2$. Let F be the universe of $\boldsymbol{B}_0$ and fix δ so that $F \cap \delta\gamma_0^{-1} F = \emptyset$. Let $\boldsymbol{B}_1$ be a linear order with universe $F \cup \delta\gamma_0^{-1} F$ so that $\boldsymbol{B}_0 \sqsubseteq \boldsymbol{B}_1, \delta\gamma_0^{-1}\boldsymbol{B}_0 \sqsubseteq \boldsymbol{B}_1$ and $\gamma_0 \, L^{\boldsymbol{B}_1} \, \delta \, L^{\boldsymbol{B}_1} \, \gamma_1$. Then for any $\boldsymbol{B} \in N(\boldsymbol{B}_1) \subseteq N(\boldsymbol{B}_0)$ we have $\gamma_0 \boldsymbol{A}_0, \gamma_1 \boldsymbol{A}_0, \delta \boldsymbol{A}_0 \sqsubseteq \boldsymbol{B}$ and $\gamma_0 \, L^{\boldsymbol{B}} \, \delta \, L^{\boldsymbol{B}} \, \gamma_1$.

Suppose now that there is a comeagre Borel equivariant set $X \subseteq \mathcal{K}(\Gamma)$ and a Borel equivariant expansion $f : X \to \mathcal{K}^*(\Gamma)$. By shrinking X, we may assume that it is G_δ and that f is continuous. We view f as a function $X \to 2^\Gamma$ taking $\boldsymbol{A} \in X$ to a subset of Γ so that $\boldsymbol{A} \restriction f(\boldsymbol{A}) \cong \mathbb{Z}$.

Fix now some $\boldsymbol{A} \in X$ in which every element of $\mathrm{Age}_\Gamma(\mathcal{K})$ appears densely often and let $\gamma_0 \in f(\boldsymbol{A})$ be arbitrary. By continuity and equivariance, there is some $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K})$ so that $\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$ and whenever $\gamma \boldsymbol{A}_0 \sqsubseteq \boldsymbol{B} \in X$ we have $\gamma\gamma_0 \in f(\boldsymbol{B})$. In particular, $C(\boldsymbol{A}_0, \boldsymbol{A}) \subseteq f(\boldsymbol{A})$. But $\boldsymbol{A}_0$ appears densely often in $\boldsymbol{A}$, contradicting the fact that $\boldsymbol{A} \restriction f(\boldsymbol{A}) \cong \mathbb{Z}$. $\square$

We consider now the measurable case. For this, we will need the following lemma, due to Lyons and Schramm [LS99], on the existence of "densities" of infinite random subsets of a group Γ (see also [HP24, Section 4.2]).

Let Γ be a countably infinite group and let $(Z_n)_{n \in \mathbb{N}}$ be a random walk on Γ with symmetric step distribution μ whose support generates Γ. (Note that we do not assume μ to be finitely supported.) For $\gamma \in \Gamma$, let $\mathbb{P}_\gamma$ denote the law of the random walk $(Z_n)_{n \in \mathbb{N}}$ starting at γ.

Define $\Omega(\Gamma, \mu)$ to be the set of all $W \subseteq \Gamma$ for which there exists $r \in [0, 1]$ so that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}(Z_i \in W) = r, \ \mathbb{P}_\gamma\text{-a.s. for all } \gamma \in \Gamma,$$

and for $W \in \Omega(\Gamma, \mu)$ we let $\mathrm{Freq}_\mu(W)$ be the unique such r. We note that $\Omega(\Gamma, \mu) \subseteq 2^\Gamma, \mathrm{Freq}_\mu : \Omega(\Gamma, \mu) \to [0, 1]$ are Borel and Γ-invariant. (Here, $\mathbb{1}(Z_n \in W)$ is equal to 1 when $Z_n \in W$ and 0 otherwise.)

Lemma 2.4.19 (Existence of frequencies [LS99, Lemma 4.2]; c.f. [HP24, Lemma 4.4])**.** *Let Γ be a countably infinite group and μ be a symmetric probability measure on Γ whose support generates Γ. Let ν be an invariant random equivalence relation on Γ and let $E \sim \nu$. Then ν-almost surely, every equivalence class of E is contained in $\Omega(\Gamma, \mu)$.*

We include a proof of Lemma 2.4.19 in Appendix 2.A, for the reader's convenience.

Theorem 2.4.20. *Let Γ be a countably infinite group. There is a Borel Γ-invariant set $X \subseteq \mathcal{K}(\Gamma)$ and a Borel equivariant expansion map $f : X \to \mathcal{K}^*(\Gamma)$ such that, for all invariant random $\mathcal{K}$-structures μ on Γ, μ admits a random expansion to $\mathcal{K}^*$ if and only if $\mu(X) = 1$, in which case $f_*\mu$ gives such an expansion.*

Moreover, we can choose f so that for all $\boldsymbol{A} \in X$, $f(\boldsymbol{A})$ picks out an interval I in $\boldsymbol{A}$ with $\boldsymbol{A}{\restriction}I \cong \mathbb{Z}$.

In particular, if E is a CBER on Z induced by a free Borel action of Γ, μ is an E-invariant probability Borel measure and $\mathbb{A}$ is a Borel $\mathcal{K}$-structuring of E, then $\mathbb{A}$ is μ-a.e. expandable to $\mathcal{K}^*$ iff $F^{\mathbb{A}}(z) \in X$ for μ-a.e. $z \in Z$.

Proof. The "in particular" part follows immediately from Proposition 2.3.5.

For notational convenience, we identify $\boldsymbol{A} \in \mathcal{K}(\Gamma)$ with $L^{\boldsymbol{A}}$. Let κ be a fixed symmetric probability measure on Γ whose support generates Γ.

For a given $L \in \mathcal{K}(\Gamma)$, let Z_L denote the set of all intervals I in L for which $L{\restriction}I \cong \mathbb{Z}$. We define $X \subseteq \mathcal{K}(\Gamma)$ to be the set of all L for which $\sup_{I \in Z_L} \mathrm{Freq}_\kappa(I)$ exists, is non-zero and is attained by finitely many $I \in Z_L$. For such L, we define $f(L)$ to be the L-least interval $I \in Z_L$ maximizing $\mathrm{Freq}_\kappa(I)$. It is clear that f gives a Borel equivariant expansion $X \to \mathcal{K}^*(\Gamma)$.

Suppose now that μ is an invariant random $\mathcal{K}$-structure on Γ admitting a random expansion ν. Let $(L, S) \sim \nu$ be a random variable with law ν. We claim that ν-almost surely, for all $x, y \in S$, there are finitely many points between x and y in L. To see this, define $g(x, y, L, S)$, for $x, y \in \Gamma$, by setting $g(x, y, L, S) = 1$ if y is the L-least element of S with $x \mathrel{L} y$, and 0 otherwise. Note that $\sum_y g(x, y, L, S)$ is equal to 1 if x lies between two elements of S, and 0 otherwise. On the other hand, $\sum_x g(x, y, L, S)$ is 0 when $y \notin S$, and when $y \in S$ it is equal to the size of the interval $(z, y]$ in L, where z is the $L{\restriction}S$-predecessor of y.

Let now $G(x, y) = \mathbb{E}[g(x, y, L, S)]$. Then by the mass transport principle (which in this case follows simply from the invariance of ν),

$$\sum_x G(x, y) = \sum_x G(y, x) = \mathbb{E}[\sum_x g(y, x, L, S)] \leq 1$$

for all $y \in \Gamma$. It follows that the size of the interval $[y, z]$ in L is almost surely finite for all $y, z \in S$. In particular, if we take $I(L, S)$ to be the smallest interval in L containing

S, then $I(L,S) \in Z_L$ almost surely. Thus, by replacing ν with the law of $(L, I(L,S))$, we may assume that $S \in Z_L$.

We now show that $L \in X$ almost surely, i.e., $\mu(X) = 1$. By Lemma 2.4.19 we may assume that $I \in \Omega(\Gamma, \kappa)$ for all $I \in Z_L$, i.e., that $\mathrm{Freq}_\kappa(I)$ is defined for all such intervals. By considering an ergodic decomposition of ν (cf. [Kec25, Theorem 5.12]), we may assume that ν is ergodic. Let $\mathbb{P}[1_\Gamma \in S] = r > 0$. We claim that $\mathrm{Freq}_\kappa(S) = r$ almost surely. Indeed, by ergodicity $\mathrm{Freq}_\kappa(S)$ is constant a.s., and by the Dominated Convergence Theorem and invariance we have

$$\mathbb{E}[\mathrm{Freq}_\kappa(S)] = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{P}[Z_i \in S] = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} r = r.$$

It follows that $\sup_{I \in Z_L} \mathrm{Freq}_\kappa(I) > 0$, and by Fatou's Lemma

$$\sum_{I \in Z_L} \mathrm{Freq}_\kappa(I) \leq \mathrm{Freq}_\kappa(\bigcup Z_L) \leq 1,$$

so the max is attained by finitely many $I \in Z_L$. $\square$

2.4.7 Vizing's Theorem

Fix $d \geq 2$ and let

$$\begin{aligned} \mathcal{K} &= \{(X,E) \mid (X,E) \text{ is a connected graph of max degree } \leq d\}, \\ \mathcal{K}^* &= \{(X,E,S_0,\dots,S_d) \mid (X,E) \in \mathcal{K} \ \& \ S_0,\dots,S_d \text{ is an edge colouring of } (X,E)\}, \end{aligned}$$

as in Example 2.2.7.

By Vizing's Theorem, every element of $\mathcal{K}$ admits an expansion in $\mathcal{K}^*$. This is false in the Borel context. In particular, Marks has shown that there is a d-regular acyclic Borel bipartite graph with Borel edge-chromatic number $2d-1$ [Mar16], and in [CJMST20] it is shown that there are counter-examples even for hyperfinite graphs.

On the other hand, Vizing's Theorem holds in the Borel setting for $d = 2$ [KST99] and for graphs of subexponential growth [BD25], in the measurable setting [GP20; Gre25], and in the Baire-measurable setting for bipartite graphs [BW23].

To summarize, we have the following:

Theorem 2.4.21.

1. *[CJMST20] For $d \geq 3$, $(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness.*
2. *[KST99] For $d = 2$, every CBER is Borel expandable for $(\mathcal{K}, \mathcal{K}^*)$.*

3. *[BD25] Let $\mathcal{K}_0 \subseteq \mathcal{K}$ be the subclass of graphs of subexponential growth. Then every CBER is Borel expandable for $(\mathcal{K}_0, \mathcal{K}^*)$.*

4. *[BW23] Let $\mathcal{K}_1 \subseteq \mathcal{K}$ be the subclass of bipartite graphs. Then every CBER is generically expandable for $(\mathcal{K}_1, \mathcal{K}^*)$. In particular, $\mathcal{K}_1$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically for every countably infinite group Γ.*

5. *[GP20; Gre25] Every CBER is a.e. expandable for $(\mathcal{K}, \mathcal{K}^*)$ for every (not necessarily invariant) probability Borel measure. In particular, for every countably infinite group Γ:*

   a) there is a Borel invariant set $Z \subseteq Fr(\mathcal{K}(\Gamma))$ which admits a Borel Γ-equivariant expansion to $\mathcal{K}^*$ and such that every invariant random $\mathcal{K}$-structure μ on Γ which concentrates on $Fr(\mathcal{K}(\Gamma))$ satisfies $\mu(Z) = 1$; and

   b) Γ admits random expansions from $\mathcal{K}$ to $\mathcal{K}^*$.

Proof. By [CJMST20, Theorem 1.4], we may fix an aperiodic hyperfinite CBER E and Borel d-regular acyclic graph G on E which does not admit a Borel edge colouring with $d+1$ colours. By [JKL02, Lemma 3.23], E does not admit an invariant probability Borel measure (as G is a treeing of E for which every component has infinitely-many ends), so by Nadkarni's Theorem E is compressible. (1) follows by Proposition 2.3.25.

For the "in particular" parts of (4), (5), note that $Fr(\mathcal{K}_1(\Gamma))$ is dense G_δ in $\mathcal{K}_1(\Gamma)$ and apply Proposition 2.3.4. For (5a), apply the proof of [GP20, Theorem 4.3] to the canonical $\mathcal{K}$-structuring of $Fr(\mathcal{K}(\Gamma))$, as in the proof of (1) above. □

Remark 2.4.22. When $d = 2$, the same construction as in the proof of Theorem 2.4.15 gives an analogous characterization of exactly when there is a Borel equivariant expansion map from $Z \subseteq \mathcal{K}(\Gamma)$ to $\mathcal{K}^*$.

We note also that $(\mathcal{K}, \mathcal{K}^*)$ enforces smoothness even if we restrict $\mathcal{K}$ to the class of n-regular acyclic bipartite graphs (with a given bipartition), for $n > d/2 + 1$, by [CJMST20, Theorem 1.4].

Finally, we remark that the proof of the main result of [GP20] (along with Nadkarni's Theorem) gives the stronger fact that for every CBER E on X and Borel $\mathcal{K}$-structuring $\mathbb{A}$ of E, there is a Borel E-invariant set C so that $\mathbb{A}\restriction C$ admits a Borel expansion to $\mathcal{K}^*$ and $E\restriction(X \setminus C)$ is compressible.

In the Baire-measurable setting, Qian and Weilacher have shown that if we replace $\mathcal{K}^*$ with $(d+2)$-edge colourings, then every CBER is generically expandable [QW22]. It is open whether every CBER is generically expandable for $(\mathcal{K}, \mathcal{K}^*)$.

2.4.8 Matchings

As in Example 2.2.8, let

$$\begin{aligned}\mathcal{K} = \{(X,E) \mid (X,E) &\text{ is a connected, bipartite,} \\ &\text{locally finite graph satisfying Hall's Condition}\}, \\ \mathcal{K}^* = \{(X,E,M) \mid (X,E) &\in \mathcal{K} \;\;\&\;\; M \subseteq E \text{ is a perfect matching}\}.\end{aligned}$$

All graphs below are assumed to be in $\mathcal{K}$, unless specified otherwise.

By Hall's Theorem, every element of $\mathcal{K}$ admits an expansion in $\mathcal{K}^*$. This is false in the Borel context. Laczkovich and Conley and Kechris have given examples of d-regular hyperfinite graphs with Borel chromatic number 2 which do not admit Borel perfect matchings, even generically or a.e., for d even [Lac88; CK13]. Marks later showed that there are d-regular, acyclic graphs with Borel chromatic number 2 that do not have Borel perfect matchings for all $d \geq 2$ [Mar16], and in [CJMST20, Theorem 1.4] this was extended to hyperfinite graphs. Kun has given examples of such graphs that are not hyperfinite and do not admit Borel perfect matchings a.e. [Kun24], and in [BKS22] a hyperfinite one-ended bounded-degree graph with Borel chromatic number 2 is constructed which does not admit a Borel perfect matching a.e.

On the other hand, if we strengthen our structural assumptions on the graphs one can guarantee the existence of Borel perfect matchings generically or a.e. For instance, Marks and Unger have shown that if we strengthen Hall's Condition to assume that $|N(A)| \geq (1+\varepsilon)|A|$ for some fixed $\varepsilon > 0$ then there is always a Borel perfect matching generically [MU16], and Lyons and Nazarov have shown that Borel perfect matchings exist a.e. for graphs that instead satisfy an analogous expansion property for measure [LN11]. Conley and Miller have shown that acyclic graphs of minimum degree at least 2 which do not have infinite injective rays of degree 2 on even vertices have Borel perfect matchings generically, and a.e. when the graph is hyperfinite [CM17] (they showed this even for locally countable graphs in the measurable setting). Bowen, Kun, and Sabok have shown that Borel perfect matchings exist a.e. for hyperfinite measure-preserving regular graphs that are one-ended or have odd degree [BKS22], and in [BCW24] the odd-degree case is shown to hold even when the measure is not preserved. Borel perfect matchings also exist generically for regular graphs that are one-ended [BPZ24] or have odd degree [BCW24], and for bounded-degree non-amenable vertex-transitive graphs [KL23] (note that this last result applies to all graphs, not just those in $\mathcal{K}$).

We note also that Borel perfect matchings have been shown to exist a.e. for some Schreier graphs of free actions of groups; see e.g. [LN11; MU16; CL17; GMP17; BKS22; GJKS24; Wei24] and [KM20, Sections 14, 15].

Remark 2.4.23. Let G be any graph (not necessarily in $\mathcal{K}$). A **fractional perfect matching** on G is an assignment to each edge of G a weight in $[0, 1]$ so that for every vertex v in G, the sum of the weights of the edges incident to v is equal to 1. Perfect matchings are then the same as $\{0, 1\}$-valued fractional perfect matchings. We say a fractional perfect matching is **non-integral** if it takes values in $(0, 1)$.

The general strategy employed by [BKS22; BPZ24; BCW24] to find Borel perfect matchings in a Borel locally finite graph G is to start with a Borel non-integral fractional perfect matching on G, and then to attempt to round this Borel fractional perfect matching to be $\{0, 1\}$-valued (off of a meagre or null set).

When G is d-regular there is always a Borel non-integral fractional perfect matching on G, namely the one giving weight to $1/d$ to every edge. However, Borel non-integral fractional perfect matchings can also be shown to exist (possibly off of a meagre or null set) in other contexts; see [Tim23] for an example of this in the measurable setting. The results of these papers can therefore be applied to a larger class of graphs than e.g. the regular ones.

It may therefore be interesting to consider separately the expansion problems for $(\mathcal{K}, \mathcal{K}')$ and $(\mathcal{K}', \mathcal{K}^*)$, where $\mathcal{K}'$ is the class of graphs equipped with a (non-integral) fractional perfect matching, though we do not explore this here.

We summarize a few of the aforementioned results below, in the language and setting of expansions.

Let $\mathcal{K}_d$ (resp. $\mathcal{K}_{d,ac}$) denote the subclass of $\mathcal{K}$ consisting of d-regular (resp. d-regular acyclic) graphs. Note that these are G_δ classes of structures. Let $\mathcal{K}_0 \subseteq \mathcal{K}$ denote the class of graphs that are either acyclic with no infinite injective rays of degree 2 on even vertices, are regular and one-ended, or are regular of odd degree. Let $\mathcal{K}_1 \subseteq \mathcal{K}$ denote the class of graphs that satisfy the strengthening of Hall's Condition for ε-expansion for some $\varepsilon > 0$, or are vertex-transitive, non-amenable and have bounded degree. These are Borel classes of structures.

Theorem 2.4.24.

1. *[CJMST20] $(\mathcal{K}_{d,ac}, \mathcal{K}^*)$ enforces smoothness for $d \geq 2$. [Lac88; CK13] In particular, $(\mathcal{K}_d, \mathcal{K}^*)$ and $(\mathcal{K}, \mathcal{K}^*)$ enforce smoothness.*

2. *$\mathcal{K}_2$ does not admit Γ-equivariant expansions to $\mathcal{K}^*$ generically, for any countably infinite group Γ.*

3. *For every countably infinite group Γ and $d > 2$, $\mathcal{K}_d$ admits Γ-equivariant expansions to $\mathcal{K}^*$ generically. [MU16] So does $\mathcal{K}_{d,ac}$.*

4. *[MU16; CM17; BPZ24; BCW24; KL23] Every CBER is generically expandable for $(\mathcal{K}_0 \cup \mathcal{K}_1, \mathcal{K}^*)$.*

5. *[CM17; BKS22] Every hyperfinite CBER is a.e. expandable for $(\mathcal{K}_0, \mathcal{K}^*)$ for every invariant probability Borel measure. In particular, for every countably infinite amenable group Γ:*

    a) *there is a Borel invariant set $Z \subseteq Fr(\mathcal{K}_0(\Gamma))$ which admits a Borel Γ-equivariant expansion to $\mathcal{K}^*$ and such that every invariant random $\mathcal{K}_0$-structure on Γ which concentrates on $Fr(\mathcal{K}_0(\Gamma))$ satisfies $\mu(Z) = 1$; and*

    b) *Γ admits random expansions from $\mathcal{K}_0$ to $\mathcal{K}^*$.*

Proof. (1) By [CJMST20, Theorem 1.4], there is an aperiodic hyperfinite CBER and a Borel d-regular acyclic graph G on E which does not admit a Borel perfect matching. By [JKL02, Lemma 3.23], E does not admit an invariant probability Borel measure (as G is a treeing of E for which every component has infinitely-many ends), so by Nadkarni's Theorem E is compressible. We then apply Proposition 2.3.25.

(2) Suppose otherwise, and let $X \subseteq \mathcal{K}_2(\Gamma)$ be Borel, comeagre and invariant, and let $f : X \to \mathcal{K}^*(\Gamma)$ be a Borel equivariant expansion. It is not hard to see that the set of all $\boldsymbol{A}$ for which

$$\text{for all } \boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K}_2) \text{ there is some } \gamma \in \Gamma \text{ with } \gamma\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$$

is a dense G_δ set in $\mathcal{K}_2(\Gamma)$, and we may therefore assume that every element of X has this property. By further shrinking X, we may assume that f is continuous.

Fix now some $\boldsymbol{A} \in X$ and $\gamma_0, \gamma_1 \in \Gamma$ so that γ_0, γ_1 are matched in $f(\boldsymbol{A})$. By continuity and equivariance, there is some $\boldsymbol{A}_0 \in \mathrm{Age}_\Gamma(\mathcal{K}_2)$ so that $\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}$, and whenever $\gamma\boldsymbol{A}_0 \sqsubseteq \boldsymbol{B} \in X$ we have that $\gamma\gamma_0, \gamma\gamma_1$ are matched in $f(\boldsymbol{B})$.

Let $\boldsymbol{A}_1 \in \mathrm{Age}_\Gamma(\mathcal{K}_2)$ and $\gamma \in \Gamma$ be such that $\boldsymbol{A}_0, \gamma\boldsymbol{A}_0 \sqsubseteq \boldsymbol{A}_1$, $\boldsymbol{A}_1$ is connected, and the unique path in $\boldsymbol{A}_1$ whose first edge is $\{\gamma_0, \gamma_1\}$ and whose last edge is $\{\gamma\gamma_0, \gamma\gamma_1\}$ has even length. Let δ be such that $\delta \boldsymbol{A}_1 \sqsubseteq \boldsymbol{A}$. Then $\{\delta\gamma_0, \delta\gamma_1\}, \{\delta\gamma\gamma_0, \delta\gamma\gamma_1\} \in f(\boldsymbol{A})$, but the unique path in $\boldsymbol{A}$ containing these edges at either end has even length, which is impossible as $\boldsymbol{A}$ is a bi-infinite line and $f(\boldsymbol{A})$ is a perfect matching.

(3) By [BPZ24, Theorem 1.2], it suffices by Proposition 2.3.4(2) to show that the generic element of $\mathcal{K}_d$ is one-ended (note that $Fr(\mathcal{K}_d(\Gamma))$ is comeagre in $\mathcal{K}_d(\Gamma)$). To see this, note that a d-regular graph G is one-ended if and only if for every finite set F of vertices and all vertices u, v, one of the following holds:

- There is some finite set of vertices F' such that at least one of u, v is contained in F', and the boundary of F' is contained in F.
- There is a path in G from u to v which does not include any vertices in F.

It is easy to see that the set of graphs satisfying one of these conditions for any fixed F, u, v is open and dense in $\mathcal{K}_d$, and hence the set of graphs satisfying these conditions for all F, u, v is comeagre.

For $\mathcal{K}_{d,ac}$, this follows by Proposition 2.3.4(2) and [MU16, Theorem 1.3].

(4) is an immediate consequence of the (proofs in) the cited papers, and (5b) follows similarly by Proposition 2.3.4(3).

(5a) We split $\mathcal{K}_0$ into three parts: The acyclic graphs with no infinite injective rays of degree 2 on even vertices, the regular one-ended graphs, and the regular odd-degree graphs. We will give some detail for the last case, and then sketch the first two.

For the regular odd-degree graphs, we argue as follows: We consider each degree $d \geq 3$ separately. Let X be the free part of $\mathcal{K}_0(\Gamma)$ restricted to the regular d-degree graphs and let $\mathbb{A}$ be the canonical structuring of X. By the proof of [BKS22, Theorem 1.3] one can associate to each $t \in 2^{\mathbb{N}}$ a Borel fractional perfect matching on $\mathbb{A}$ so that for every invariant probability Borel measure μ on X, for almost every t the corresponding fractional perfect matching is $\{0,1\}$-valued for μ-a.e. component of $\mathbb{A}$. By [Kec95, 18.6], we can choose in a uniformly Borel way a Borel fractional perfect matching f_μ for every ergodic invariant measure μ on X, so that f_μ is $\{0,1\}$-valued for μ-a.e. component of $\mathbb{A}$. Let X_μ denote the set of components for which f_μ is $\{0,1\}$-valued. By considering an ergodic decomposition of X (cf. [Kec25, Theorem 5.12]), the set $Z = \bigcup_\mu X_\mu$ is Borel, and $f = \bigcup_\mu f_\mu \restriction X_\mu$ gives a Borel perfect matching of $\mathbb{A} \restriction Z$. Moreover, $\mu(Z) = 1$ for every invariant probability Borel measure on X. By Proposition 2.3.4(2), we are done.

For the acyclic graphs with no infinite injective rays of degree 2 on even vertices, the argument is similar: We consider an ergodic decomposition, and note that the proof of [CM17, Theorem B] is effective enough that the union of the solutions (and their domains) for all ergodic invariant probability Borel measures is still Borel.

For the regular one-ended graphs, we again consider an ergodic decomposition and argue that the proof of [BKS22, Theorem 1.1] is sufficiently uniform. For a fixed measure μ, the proof proceeds by constructing a transfinite sequence of fractional perfect matchings, and arguing that this must stabilize at some countable ordinal. We claim that the construction of this sequence is effective (in μ). Then, by the Boundedness Theorem for analytic well-founded relations [Kec95, 31.1] there is a uniform bound on how long these sequences take to stabilize for all (ergodic) invariant probability Borel measures, so we are done by the same argument as in the previous two cases.

The verification that the construction is effective is tedious but straightforward. The most subtle step is in the use of the Choquet–Bishop–de Leeuw Theorem, which is sufficiently effective for separable metrizable spaces as this essentially boils down to an application of compact uniformization; see e.g. [Phe01, Section 3; Sim09, IV.9; Kec95, 28.8]. □

2.5 Problems

Problem 2.5.1. If $(\mathcal{K}, \mathcal{K}^*)$ satisfies the hypotheses of Theorem 2.3.13 and admits generic equivariant expansions, is every CBER generically expandable for $(\mathcal{K}, \mathcal{K}^*)$?

Problem 2.5.2. Does the conclusion of Theorem 2.3.13 hold for classes of structures without TAC?

In [CK18], it is shown that for many natural classes of aperiodic CBER $\mathcal{E}$, there is a Borel class of structures $\mathcal{K}$ so that $E \in \mathcal{E}$ if and only if E admits a Borel $\mathcal{K}$-structuring. Nonetheless, it is interesting whether there is any “natural” class of problems (e.g. problems that are studied in finite combinatorics) that carve out interesting classes $\mathcal{E}$ of CBER. Example 2.2.6 was an attempt to characterize hyperfiniteness, though we have seen that it actually enforces smoothness.

Problem 2.5.3. Let $\mathcal{E}$ be a class of aperiodic CBER such as those that are hyperfinite, (non)-compressible or treeable. Is there a “natural” expansion problem $(\mathcal{K}, \mathcal{K}^*)$ for which an aperiodic CBER E is Borel expandable if and only if $E \in \mathcal{E}$?

In [GX24] a problem is described for which E admits solutions exactly when E is hyperfinite. However, this does not fit the framework of expansion problems, as it involves finding "approximate" solutions.

We note that the spanning tree example (Example 2.2.5) corresponds to a class of CBER that lies somewhere between hyperfinite and treeable.

Problem 2.5.4. What is the class of CBER that are Borel expandable for the spanning tree problem?

In general, it would be interesting to answer the problems remaining in Table 2.1. We highlight a few of these below.

Problem 2.5.5. For the Ramsey expansion problem (Example 2.2.2), when an invariant random structure admits an invariant random expansion? Can we characterize a.e. expansions (in the sense of Section 2.3.3)? Under what assumptions to generic expansions exist on CBER?

Problem 2.5.6. Can we say more about when a Borel structuring of a CBER by partial orders is expandable to a Borel structuring by linear orders? In particular, is there a characterization of exactly which invariant random expansions come from push-forwards along a.e. equivariant expansion maps?

Problem 2.5.7. Does Vizing's Theorem hold generically?

There are many open problems regarding the existence of perfect matchings; see Section 2.4.8 for details. As noted in Remark 2.4.23, one can often find perfect matchings by first finding non-integral fractional perfect matchings, and then rounding them.

Problem 2.5.8. What can be said about the expansion problem of finding a non-integral fractional perfect matching on a Borel graph? When can we round non-integral fractional perfect matchings to perfect matchings?

See e.g. [BKS22; Tim23; BPZ24] for some partial results and examples.

2.A Existence of frequencies

The purpose of this appendix is to prove Lemma 2.4.19 on the existence of frequencies. The proof is essentially the same as that of [LS99, Lemma 4.2], though we work here in a more general setting; we also thank Minghao Pan for sharing with us his notes about this proof.

We begin by recalling some definitions.

Let Γ be a countably infinite group and let $(Z_n)_{n\in\mathbb{N}}$ be a random walk on Γ with symmetric step distribution μ whose support generates Γ. (Note that we do not assume μ to be finitely supported.) For $\gamma \in \Gamma$, let $\mathbb{P}_\gamma$ denote the law of the random walk $(Z_n)_{n\in\mathbb{N}}$ starting at γ.

Define $\Omega(\Gamma,\mu)$ to be the set of all $W \subseteq \Gamma$ for which there exists $r \in [0,1]$ so that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}(Z_i \in W) = r, \ \mathbb{P}_\gamma\text{-a.s. for all } \gamma \in \Gamma,$$

and for $W \in \Omega(\Gamma,\mu)$ we let $\mathrm{Freq}_\mu(W)$ be the unique such r. We note that $\Omega(\Gamma,\mu) \subseteq 2^\Gamma, \mathrm{Freq}_\mu : \Omega(\Gamma,\mu) \to [0,1]$ are Borel and Γ-invariant. (Here, $\mathbb{1}(Z_n \in W)$ is equal to 1 when $Z_n \in W$ and 0 otherwise.)

Note that if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}(Z_i \in W)$ converges $\mathbb{P}_\gamma$-almost surely for some γ, then it does for all γ. To see this, note that the support of μ generates Γ, so if the sequence diverges with positive probability for a random walk starting at some γ, then this happens with positive probability for a random walk starting at any γ. Similarly, we see that the value of the limit (should it exist) does not depend on the choice of γ.

Let now $\mathcal{K}$ denote the class of equivalence relations. Let ν be an invariant random equivalence relation on Γ, i.e. an invariant random $\mathcal{K}$-structure on Γ, and let $E \sim \nu$. Let e denote the identity in Γ. We will show that ν-almost surely, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}(Z_i \in C)$ converges to a constant value $\mathbb{P}_e$-a.s. for every E-class C. By the previous remark, this proves Lemma 2.4.19.

A *two-sided* random walk starting at γ is a sequence of random variables $(Z_n)_{n\in\mathbb{Z}}$ so that $(Z_n)_{n\in\mathbb{N}}$ and $(Z_{-n})_{n\in\mathbb{N}}$ are random walks starting at γ. Let $\hat{\mathbb{P}}_\gamma$ denote the law of the two-sided random walk starting at γ.

Note that Γ acts on $\Gamma^\mathbb{Z}$ by coordinate-wise multiplication, so that we may consider $\mathcal{K}(\Gamma) \times \Gamma^\mathbb{Z}$ with the diagonal action of Γ:

$$\gamma \cdot (E, (Z_n)_{n\in\mathbb{Z}}) = (\gamma \cdot E, (\gamma \cdot Z_n)_{n\in\mathbb{Z}}).$$

We also define the *shift map* $S : \mathcal{K}(\Gamma)\times\Gamma^\mathbb{Z} \to \mathcal{K}(\Gamma)\times\Gamma^\mathbb{Z}$ to be the map $S(E,(Z_n)_{n\in\mathbb{Z}}) = (E,(Z_{n+1})_{n\in\mathbb{Z}})$. Note that the actions of Γ, S on $\mathcal{K}(\Gamma)\times\Gamma^\mathbb{Z}$ commute.

Let $\mathcal{I}$ denote the σ-algebra of Γ-invariant Borel sets in $\mathcal{K}(\Gamma) \times \Gamma^{\mathbb{Z}}$, and set $\lambda = \nu \times \hat{\mathbb{P}}_e$. Also, for $E \in \mathcal{K}(\Gamma)$, let Γ/E denote the set of equivalence classes of E.

Claim 2.A.1. If $A \in \mathcal{I}$, then $\lambda(A) = \lambda(S \cdot A)$.

Proof. Let $W^n_\gamma = \{Z = (Z_n)_{n\in\mathbb{Z}} : Z_n = \gamma\}$. Note that by the symmetry of μ,

$$\sum_{\gamma\in\Gamma} \hat{\mathbb{P}}_\gamma[W^j_{\gamma_j} \cap \cdots \cap W^k_{\gamma_k}] = \prod_{i=j}^{k-1} \mu(\gamma_i^{-1}\gamma_{i+1})$$

for all $j < 0 < k \in \mathbb{Z}$ and $\gamma_j, \ldots, \gamma_k \in \Gamma$. It follows that $\sum_{\gamma\in\Gamma} \hat{\mathbb{P}}_\gamma$ is shift-invariant.

Let now $\kappa = \nu \times \sum_{\gamma\in\Gamma} \hat{\mathbb{P}}_\gamma$. Note that κ is Γ-invariant. If $A \in \mathcal{I}$, then by Γ-invariance we have

$$\kappa(A \cap W^0_e) = \sum_\gamma \kappa(A \cap W^0_e \cap W^{-1}_\gamma) = \sum_\gamma \kappa(A \cap W^0_\gamma \cap W^{-1}_e) = \kappa(A \cap W^{-1}_e).$$

It follows that

$$\lambda(SA) = \kappa(SA \cap W^0_e) = \kappa(SA \cap W^{-1}_e) = \kappa(S(A \cap W^0_e)) = \kappa(A \cap W^0_e) = \lambda(A).$$

$\triangleleft$

For $C \subseteq \Gamma$, $Z = (Z_n)_{n\in\mathbb{Z}} \in \Gamma^{\mathbb{Z}}$, $m < n \in \mathbb{Z}$, let

$$\alpha^n_m(C, Z) = \frac{1}{n-m} \sum_{i=m}^{n-1} \mathbb{1}(Z_i \in C),$$

and for $(E, Z) \in \mathcal{K}(\Gamma) \times \Gamma^{\mathbb{Z}}$, $n, k \in \mathbb{N}$ let

$$F^n_k(E, Z) = n \cdot \max\{\alpha^n_0(C_0, Z) + \cdots + \alpha^n_0(C_{k-1}, Z) : C_0, \ldots, C_{k-1} \text{ are distinct } E\text{-classes}\}.$$

It is easy to see that $F^{i+j}_k(E, Z) \leq F^i_k(E, Z) + F^j_k(S^i \cdot (E, Z))$ and that each F^n_k is Γ-invariant. By Claim 2.A.1 and Kingman's Subadditive Ergodic Theorem (see e.g. [Ste89]) there are Γ, S-invariant maps $F_k, k \in \mathbb{N}$ so that $F_k(E, Z) = \lim_{n\to\infty} \frac{F^n_k(E,Z)}{n}$ λ-a.s.

Let now

$$A^n(E, Z) = \{|\alpha^m_0(C, Z) - \alpha^k_0(C, Z)| : k, m \geq n \;\; \& \;\; C \in \Gamma/E\}.$$

Claim 2.A.2. $\lim_{n\to\infty} \max(A^n(E, Z)) = 0$ for almost every (E, Z).

Proof. Fix (E,Z) for which $F_k(E,Z) = \lim_{n\to\infty} \frac{F_k^n(E,Z)}{n}$ for all $k \in \mathbb{N}$.

For any E-class C and $n \in \mathbb{N}$, there is some k so that $\alpha_0^n(C,Z) = \frac{F_k^n(C,Z)}{n} - \frac{F_{k-1}^n(C,Z)}{n}$, namely the k for which C is the k-th most frequently visited E-class in the first n steps of Z. It follows that

$$\alpha_0^m(C,Z) \in S_n = \left\{ \frac{F_k^m(E,Z)}{m} - \frac{F_{k-1}^m(E,Z)}{m} : k \geq 1, m \geq n \right\}$$

for $m \geq n$.

Let $S_n(\delta)$ denote the δ-neighbourhood of S_n in $[0,1]$. Since $|\alpha_0^{m+1}(C,Z) - \alpha_0^m(C,Z)| \leq \frac{1}{m}$, the set $\{\alpha_0^m(C,Z) : m \geq n\}$ is contained in a single connected component of $S_n(\frac{1}{n})$, for every E-class C. It therefore suffices to show that for all $\varepsilon > 0$, there is some n sufficiently large that $S_n(\frac{1}{n})$ has length at most ε.

Fix now $\varepsilon > 0$, and fix k sufficiently large that $F_j(E,Z) - F_{j-1}(E,Z) \leq \varepsilon$ for $j \geq k$. Fix n so that $|F_j(E,Z) - \frac{F_j^m(E,Z)}{m}| \leq \frac{\varepsilon}{k+1}$ for all $j \leq k$ and $m \geq n$. It follows that every point in S_n is within distance ε from 0, or $\frac{\varepsilon}{k+1}$ from $F_j(E,Z) - F_{j-1}(E,Z)$ for some $j \leq k$, so that $S_n(\frac{1}{n})$ has length at most $3(\varepsilon + \frac{1}{n})$. Since ε was arbitrary, this proves the claim. ◁

It follows that for almost every (E,Z), $(\alpha_0^n(C,Z))_{n\in\mathbb{N}}$ is Cauchy for every E-class C, and hence this sequence converges. Symmetrically, $(\alpha_{-n}^0(C,Z))_{n\in\mathbb{N}}$ converges a.s. for every E-class C.

Note that with probability 1

$$\max_{C\in\Gamma/E} |\alpha_n^{2n}(C,Z) - \alpha_0^n(C,Z)| = 2 \cdot \max_{C\in\Gamma/E} |\alpha_0^{2n}(C,Z) - \alpha_0^n(C,Z)| \xrightarrow{n\to\infty} 0,$$

so we may fix a sequence n_k such that

$$\mathbb{P}\Big[\max_{C\in\Gamma/E} |\alpha_{n_k}^{2n_k}(C,Z) - \alpha_0^{n_k}(C,Z)| \geq 2^{-k}\Big] \leq 2^{-k}.$$

By Claim 2.A.1,

$$\mathbb{P}\Big[\max_{C} |\alpha_{n_k}^{2n_k}(C,Z) - \alpha_0^{n_k}(C,Z)| \geq 2^{-k}\Big] = \mathbb{P}\Big[\max_{C} |\alpha_0^{n_k}(C,Z) - \alpha_{-n_k}^{0}(C,Z)| \geq 2^{-k}\Big],$$

so by the Borel–Cantelli Lemma we have that for almost every (E,Z),

$$\max_{C\in\Gamma/E} |\alpha_0^{n_k}(C,Z) - \alpha_{-n_k}^{0}(C,Z)| < 2^{-k}$$

for all but finitely many k. It follows that

$$\lim_{n\to\infty} \alpha_0^n(C,Z) = \lim_{n\to\infty} \alpha_{-n}^0(C,Z)$$

almost surely for all $C \in \Gamma/E$. But for any fixed $C \subseteq \Gamma$, $\alpha_0^n(C,Z), \alpha_{-n}^0(C,Z)$ are independent, so the limits are independent and hence must be constant a.s.

Chapter 3

INVARIANT UNIFORMIZATION

Alexander S. Kechris and Michael S. Wolman

3.1 Introduction

3.1.1 Invariant uniformization and smoothness

Given sets X, Y and $P \subseteq X \times Y$ with $\mathrm{proj}_X(P) = X$, a **uniformization** of P is a function $f\colon X \to Y$ such that $\forall x \in X((x, f(x)) \in P)$. If now E is an equivalence relation on X, we say that P is ***E*-invariant** if $x_1 E x_2 \implies P_{x_1} = P_{x_2}$, where $P_x = \{y\colon (x, y) \in P\}$ is the x-**section** of P. Equivalently this means that P is invariant under the equivalence relation $E \times \Delta_Y$ on $X \times Y$, where Δ_Y is the equality relation on Y. In this case an ***E*-invariant uniformization** is a uniformization f such that $x_1 E x_2 \implies f(x_1) = f(x_2)$.

Also if E, F are equivalence relations on sets X, Y, resp., a **homomorphism** of E to F is a function $f\colon X \to Y$ such that $x_1 E x_2 \implies f(x_1) F f(x_2)$. Thus an invariant uniformization is a uniformization that is a homomorphism of E to Δ_Y.

Consider now the situation where X, Y are Polish spaces and P is a Borel subset of $X \times Y$. In this case standard results in descriptive set theory provide conditions which imply the existence of Borel uniformizations. These fall mainly into two categories, see [Kec95, Section 18]: "small section" and "large section" uniformization results. We will concentrate here on the following standard instances of these results:

Theorem 3.1.1 (Measure uniformization)**.** *Let X, Y be Polish spaces, μ a probability Borel measure on Y and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(\mu(P_x) > 0)$. Then P admits a Borel uniformization.*

Theorem 3.1.2 (Category uniformization)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(P_x$ is non-meager$)$. Then P admits a Borel uniformization.*

Theorem 3.1.3 (K_σ uniformization)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(P_x$ is non-empty and $K_\sigma)$. Then P admits a Borel uniformization.*

A special case of Theorem 3.1.3 is the following:

Theorem 3.1.4 (Countable uniformization)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(P_x$ is non empty and countable$)$. Then P admits a Borel uniformization.*

Suppose now that E is a Borel equivalence relation on X and P in any one of these results is E-invariant. When does there exist a **Borel $\boldsymbol{E}$-invariant uniformization**, i.e., a Borel uniformization that is also a homomorphism of E to Δ_Y? We say that E satisfies **measure (resp., category, $\boldsymbol{K_\sigma}$, countable) invariant uniformization** if for every Y, μ, P as in the corresponding uniformization theorem above, if P is moreover E-invariant, then it admits a Borel E-invariant uniformization.

The following gives a complete answer to this question. Recall that a Borel equivalence relation E on X is **smooth** if there is a Polish space Z and a Borel function $S\colon X \to Z$ such that $x_1 E x_2 \iff S(x_1) = S(x_2)$.

Theorem 3.1.5. *Let E be a Borel equivalence relation on a Polish space X. Then the following are equivalent:*

(i) E is smooth;

(ii) E satisfies measure invariant uniformization;

(iii) E satisfies category invariant uniformization;

(iv) E satisfies K_σ invariant uniformization;

(v) E satisfies countable invariant uniformization.

One can compute the exact definable complexity of counterexamples to invariant uniformization. Let $\mathbb{E}_0$ denote the non-smooth Borel equivalence relation on $2^{\mathbb{N}}$ given by $x \mathbb{E}_0 y \iff \exists m \forall n \geq m(x_n = y_n)$. In the proof of Theorem 3.1.5, it is shown that for $E = \mathbb{E}_0$ on $X = 2^{\mathbb{N}}$ we have the following:

(1) Failure of measure invariant uniformization: There are Y, μ, E-invariant $P \in F_\sigma$ with $\mu(P_x) = 1$, for all $x \in X$, which has no Borel E-invariant uniformization.

(2) Failure of category invariant uniformization: There is Y and an E-invariant $Q \in G_\delta$ with Q_x comeager, for all $x \in X$, which has no Borel E-invariant uniformization.

(3) Failure of countable invariant uniformization: There is Y and an E-invariant $P \in F_\sigma$ such that P_x is non-empty and countable, for all $x \in X$, which has no Borel E-invariant uniformization.

The definable complexity of Q, P in (2), (3) is optimal. In the case of measure invariant uniformization, however, there are counterexamples which are G_δ, and this together with (1) gives the optimal definable complexity of counterexamples to measure invariant uniformization. These results are the contents of Theorems 3.1.6 and 3.1.7.

Theorem 3.1.6. *Let $X \subseteq 2^{\mathbb{N}}$ be the sequences with infinitely many ones. There is a Polish space Y, a probability Borel measure μ on Y and an $\mathbb{E}_0$-invariant G_δ set $P \subseteq X \times Y$ with P_x comeager and $\mu(P_x) = 1$, for all $x \in X$, which has no Borel $\mathbb{E}_0$-invariant uniformization.*

Theorem 3.1.7. *Let X, Y be Polish spaces, E a Borel equivalence relation on X and $P \subseteq X \times Y$ an E-invariant Borel relation. Suppose one of the following holds:*

(i) $P_x \in \mathbf{\Delta^0_2}$ *and* $\mu_x(P_x) > 0$, *for all* $x \in X$, *and some Borel assignment* $x \mapsto \mu_x$ *of probability Borel measures* μ_x *on* Y;

(ii) $P_x \in F_\sigma$ *and* P_x *non-meager, for all* $x \in X$;

(iii) $P_x \in G_\delta$ *and* P_x *non-empty and* K_σ *(in particular countable), for all* $x \in X$.

Then there is a Borel E-invariant uniformization.

The proof of Theorem 3.1.6 uses the Ramsey property.

3.1.2 Local dichotomies

The equivalence of (i) and (v) in Theorem 3.1.5 essentially reduces to the fact that if E is a countable Borel equivalence relation (i.e., one for which all of its equivalence classes are countable) which is not smooth, then the relation

$$(x, y) \in P \iff xEy,$$

is clearly E-invariant with countable nonempty sections but has no E-invariant uniformization. Considering the problem of invariant uniformization "locally", Miller [Mild] recently proved the following dichotomy that shows that this is essentially the only obstruction to (v). Below $\mathbb{E}_0 \times I_{\mathbb{N}}$ is the equivalence relation on $2^{\mathbb{N}} \times \mathbb{N}$ given by $(x, m)\mathbb{E}_0 \times I_{\mathbb{N}}(y, n) \iff x\mathbb{E}_0 y$. Also if E, F are equivalence relations on spaces X, Y, resp., an **embedding** of E into F is an injection $\pi\colon X \to Y$ such that $x_2 E x_2 \iff \pi(x_1) F \pi(x_2)$.

Theorem 3.1.8 ([Mild, Theorem 2]). *Let X, Y be Polish spaces, E a Borel equivalence relation on X and $P \subseteq X \times Y$ an E-invariant Borel relation with countable non-empty sections. Then exactly of the following holds:*

(1) There is a Borel E-invariant uniformization,

(2) There is a continuous embedding $\pi_X\colon 2^{\mathbb{N}} \times \mathbb{N} \to X$ of $\mathbb{E}_0 \times I_{\mathbb{N}}$ into E and a continuous injection $\pi_Y\colon 2^{\mathbb{N}} \times \mathbb{N} \to Y$ such that for all $x, x' \in 2^{\mathbb{N}} \times \mathbb{N}$,

$$\neg(x \; \mathbb{E}_0 \times I_{\mathbb{N}} \; x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset$$

and

$$P_{\pi_X(x)} = \pi_Y([x]_{\mathbb{E}_0 \times I_{\mathbb{N}}}).$$

We provide a different proof of this dichotomy, using Miller's $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy [Mil12] and Lecomte's $\aleph_0$-dimensional hypergraph dichotomy [Lec09]. Our proof relies on the following strengthening of $(i) \implies (v)$ of Theorem 3.1.5, which is interesting in its own right:

Theorem 3.1.9. *Let F be a smooth Borel equivalence relation on a Polish space X, Y be a Polish space, and $P \subseteq X \times Y$ be a Borel set with countable sections. Suppose that*

$$\bigcap_{x \in C} P_x \neq \emptyset$$

for every F-class C. Then P admits a Borel F-invariant uniformization.

We also prove an $\aleph_0$-dimensional $(\mathbb{G}_0, \mathbb{H}_0)$-type dichotomy, which generalizes Lecomte's dichotomy in the same way that the $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy generalizes the $\mathbb{G}_0$ dichotomy, and use this to give still another proof of Theorem 3.1.8.

In the case of countable uniformization, the Lusin-Novikov theorem asserts that P can be covered by the graphs of countably-many Borel functions. When E is smooth, the proof of Theorem 3.1.5 gives an invariant analogue of this fact (cf. Theorem 3.2.3). De Rancourt and Miller [dRM] have shown that $\mathbb{E}_0$ is essentially the only obstruction to invariant Lusin-Novikov:

Theorem 3.1.10 ([dRM, Theorem 4.11]). *Let X, Y be Polish spaces, E a Borel equivalence relation on X and $P \subseteq X \times Y$ an E-invariant Borel relation with countable non-empty sections. Then exactly one of the following holds:*

(1) There is a sequence $g_n : X \to Y$ of Borel E-invariant uniformizations with $P = \bigcup_n \operatorname{graph}(g_n)$.

(2) There is a continuous embedding $\pi_X : 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E and a continuous injection $\pi_Y : 2^{\mathbb{N}} \to Y$ such that for all $x \in 2^{\mathbb{N}}$, $P(\pi_X(x), \pi_Y(x))$.

We provide a different proof of this theorem in Section 3.4.4, directly from Miller's $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy.

3.1.3 Anti-dichotomy results

Our next result can be viewed as a sort of anti-dichotomy theorem for large-section invariant uniformizations (see also the discussion in [TV21, Section 1]). Informally, dichotomies such as Theorem 3.1.8 provide upper bounds on the complexity of the collection of Borel sets satisfying certain combinatorial properties. Thus, one method of showing that there is no analogous dichotomy is to provide lower bounds on the complexity of such sets.

In order to state this precisely, we first fix a "nice" parametrization of the Borel relations on $\mathbb{N}^{\mathbb{N}}$, i.e., a $\mathbf{\Pi}^1_1$ set $D \subseteq 2^{\mathbb{N}}$ and a map $D \ni d \mapsto D_d$ such that each $D_d \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}, d \in D$ is Borel, each Borel set in $\mathbb{N}^{\mathbb{N}}$ appears as some D_d, and so that these satisfy some natural definability properties (cf. [AK00, Section 5]).

Define now

$$\mathcal{P} = \{(d, e) : D_d \text{ is an equivalence relation on } \mathbb{N}^{\mathbb{N}} \text{ and } D_e \text{ is } D_d\text{-invariant}\},$$

and let $\mathcal{P}^{unif}$ denote the set of pairs $(d, e) \in \mathcal{P}$ for which D_e admits a D_d-invariant uniformization. More generally, for any set A of properties of sets $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, let $\mathcal{P}_A$ (resp. $\mathcal{P}^{unif}_A$) denote the set of pairs (d, e) in $\mathcal{P}$ (resp. $\mathcal{P}^{unif}$) such that D_e satisfies all of the properties in A. Let $\mathcal{P}_{ctble}$ (resp. $\mathcal{P}^{unif}_{ctble}$) denote $\mathcal{P}_A$ (resp. $\mathcal{P}^{unif}_A$) for A consisting of the property that P has countable sections.

One can easily check that $\mathcal{P}$ is $\mathbf{\Pi}^1_1$ and that $\mathcal{P}^{unif}$ is $\mathbf{\Sigma}^1_2$. The same is true for $\mathcal{P}_{ctble}$ and $\mathcal{P}^{unif}_{ctble}$. In the latter case, however, the effective version of Theorem 3.1.8 (see Theorem 3.4.14) gives a better bound on the complexity:

Proposition 3.1.11. *The set $\mathcal{P}^{unif}_{ctble}$ is $\mathbf{\Pi}^1_1$.*

By contrast, in the case of large sections, we prove the following, where a set B in a Polish space X is called $\mathbf{\Sigma}^1_2$-complete if it is $\mathbf{\Sigma}^1_2$, and for all zero-dimensional Polish spaces Y and $\mathbf{\Sigma}^1_2$ sets $C \subseteq Y$ there is a continuous function $f : Y \to X$ such that $C = f^{-1}(B)$.

Theorem 3.1.12. *The set $\mathcal{P}_A^{unif}$ is $\mathbf{\Sigma^1_2}$-complete, where A is one of the following sets of properties of $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$:*

1. *P has non-meager sections;*
2. *P has non-meager G_δ sections;*
3. *P has non-meager sections and is G_δ;*
4. *P has μ-positive sections for some probability Borel measure μ on $\mathbb{N}^{\mathbb{N}}$;*
5. *P has μ-positive F_σ sections for some probability Borel measure μ on $\mathbb{N}^{\mathbb{N}}$;*
6. *P has μ-positive sections for some probability Borel measure μ on $\mathbb{N}^{\mathbb{N}}$ and is F_σ.*

The same holds for comeager instead of non-meager, and μ-conull instead of μ-positive.

In fact, there is a hyperfinite Borel equivalence relation E with code $d \in D$ such that for all such A above, the set of $e \in D$ such that $(d, e) \in \mathcal{P}_A^{unif}$ is $\mathbf{\Sigma^1_2}$-complete.

Problem 3.1.13. Is there an analogous dichotomy or anti-dichotomy result for the case where P has K_σ sections?

While we do not know the answer to this problem, we note that Theorem 3.1.9 is false when the sections are only assumed to be K_σ:

Proposition 3.1.14. *There is a smooth countable Borel equivalence relation F on $\mathbb{N}^{\mathbb{N}}$ and an open set $P \subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that*

$$\bigcap_{x \in C} P_x \neq \emptyset$$

for every F-class C, but which does not admit a Borel F-invariant uniformization.

3.1.4 Invariant countable uniformization

We next consider a somewhat less strict notion of invariant uniformization, where instead of selecting a single point in each section we select a countable nonempty subset. More precisely, given Polish spaces X, Y, a Borel equivalence relation E on X and an E-invariant Borel set $P \subseteq X \times Y$, with $\mathrm{proj}_X(P) = X$, a Borel ***E*-invariant countable uniformization** is a Borel function $f\colon X \to Y^{\mathbb{N}}$ such that $\forall x \in X \forall n \in \mathbb{N}((x, f(x)_n) \in P)$ and $x_1 E x_2 \implies \{f(x_1)_n \colon n \in \mathbb{N}\} = \{f(x_2)_n \colon n \in \mathbb{N}\}$.

Equivalently, if for each Polish space Y, we denote by $\mathbb{E}^Y_{ctble}$ the equivalence relation on $Y^{\mathbb{N}}$ given by

$$(x_n)\mathbb{E}^Y_{ctble}(y_n) \iff \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\},$$

then an E-invariant countable uniformization is a Borel homomorphism f of E to $\mathbb{E}^Y_{ctble}$ such that for each x, n, we have that $(x, f(x)_n) \in P$.

We say that E satisfies **measure (resp., category, $\boldsymbol{K_\sigma}$) countable invariant uniformization** if for every Y, μ, P as in the corresponding uniformization theorem above, if P is moreover E-invariant, then it admits a Borel E-invariant countable uniformization.

Recall that a Borel equivalence relation E on X is **reducible to countable** if there is a Polish space Z, a countable Borel equivalence relation F on Z and a Borel function $S\colon X \to Z$ such that $x_1 E x_2 \iff S(x_1) F S(x_2)$.

As in the proof below of Theorem 3.1.5, part **(A)**, one can see that if a Borel equivalence relation E on X is reducible to countable, then E satisfies measure (resp. category, K_σ) countable invariant uniformization. We conjecture the following:

Conjecture 3.1.15. *Let E be a Borel equivalence relation on a Polish space X. Then the following are equivalent:*

(a) E is reducible to countable;

(b) E satisfies measure countable invariant uniformization;

(c) E satisfies category countable invariant uniformization;

(d) E satisfies K_σ countable invariant uniformization.

We discuss some partial results in Section 3.5.

3.1.5 Further invariant uniformization results and smoothness

We have so far considered the existence of Borel invariant uniformizations, generalizing the standard "small section" and "large section" uniformization theorems. One can also consider invariant analogues of uniformization theorems for more general pointclasses, such as the following:

Theorem 3.1.16 (Jankov, von Neumann uniformization [Kec95, 18.1])**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ be a $\boldsymbol{\Sigma^1_1}$ set such that P_x is non-empty, for all $x \in X$. Then P has a uniformization function which is $\sigma(\boldsymbol{\Sigma^1_1})$-measurable.*

Theorem 3.1.17 (Novikov-Kondô uniformization [Kec95, 36.14])**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ be a $\mathbf{\Pi^1_1}$ set such that P_x is non-empty, for all $x \in X$. Then P has a uniformizatoin function whose graph is $\mathbf{\Pi^1_1}$.*

Let E be a Borel equivalence relation on X. We say E satisfies **Jankov-von Neumann (resp. Novikov-Kondô) invariant uniformization** if for every Y, P as in the corresponding uniformization theorem above, if P is moreover E-invariant, then it admits an E-invariant uniformization which is definable in the same sense as in the corresponding uniformization theorem.

The following characterization of those Borel equivalence relations that satisfy these properties essentially follows from the proof of Theorem 3.1.5.

Theorem 3.1.18. *Let E be a Borel equivalence relation on a Polish space X. Then the following are equivalent:*

(i) E is smooth;

(ii) E satisfies Jankov-von Neumann invariant uniformization;

(iii) E satisfies Novikov-Kondô invariant uniformization.

3.1.6 Remarks on invariant uniformization over products

One can consider more generally the question of invariant uniformization over products. Let X, Y be Polish spaces, E a Borel equivalence on X, F a Borel equivalence on Y, and $P \subseteq X \times Y$ an $E \times F$-invariant set. In this case, one can ask whether there is an $E \times F$-invariant Borel set $U \subseteq P$ so that each section U_x intersects one, or even finitely-many, F-classes. This paper then considers the special case where $F = \Delta_Y$ is equality.

In the case where P has countable sections and F is smooth, one can reduce this to the case where F is equality to get analogues of Theorems 3.1.8 and 3.1.10.

Miller [Mild, Theorem 2.1] has proved a generalization of Theorem 3.1.8 where P has countable sections and the equivalence classes of F are countable, and de Rancourt and Miller [dRM, Theorem 4.11] have proved a generalization of Theorem 3.1.10 where the sections of P are contained in countably many F-classes (but are not necessarily countable).

The problem of invariant uniformization is also discussed in [Mye76; BM75] where they consider the question of invariant uniformization over products when E, F come from Polish group actions, and specifically when E, F are the isomorphism relation on a class of structures. Myers [Mye76, Theorem 10] gives an example in which there is no Baire-measurable invariant uniformization, so that in particular the invariant Jankov-von Neumann and invariant Novikov-Kondô uniformization don't hold.

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3.2 Proof of Theorem 3.1.5

(A) We first show that (i) implies (ii), the proof that (i) implies (iii) being similar. Fix a Polish space Z and a Borel function $S\colon X \to Z$ such that $x_1 E x_2 \iff S(x_1) = S(x_2)$. Fix also Y, μ, P as in the definition of measure invariant uniformization. Define $P^* \subseteq Z \times Y$ as follows:

$$(z, y) \in P^* \iff \forall x \in X\Big(S(x) = z \implies (x, y) \in P\Big).$$

Then P^* is $\mathbf{\Pi^1_1}$ and we have that

$$S(x) = z \implies P^*_z = P_x,$$
$$z \notin S(X) \implies P^*_z = Y.$$

Thus $\forall z \in Z(\mu(P^*_z) > 0)$. Then, by [Kec95, 36.24], there is a Borel function $f^*\colon Z \to Y$ such that $\forall z \in Z((z, f^*(z)) \in P^*)$. Put

$$f(x) = f^*(S(x)).$$

Then f is an E-invariant uniformization of P.

We next prove that (i) implies (iv) (and therefore (v)). Fix Z, S as in the previous case and Y, P as in the definition of K_σ invariant uniformization. Define P^* as before. Then $A = \{(z, y)\colon \exists x \in X(S(x) = z \;\; \& \;\; P(x, y))\}$ is a $\mathbf{\Sigma^1_1}$ subset of P^*, so by the Lusin separation theorem there is a Borel subset P^{**} of P^* such that $A \subseteq P^{**}$. By [Kec95, 35.47], the set C of all $z \in Z$ such that P^{**}_z is K_σ is $\mathbf{\Pi^1_1}$ and contains the $\mathbf{\Sigma^1_1}$ set $S(X)$, so by separation there is a Borel set B with $A \subseteq B \subseteq C$. Then if $Q \subseteq Z \times Y$ is defined by

$$(z, y) \in Q \iff z \in B \;\&\; (z, y) \in P^{**},$$

we have that

$$S(x) = z \implies Q_z = P_x,$$

and every Q_z is K_σ. It follows, by [Kec95, 35.46], that $D = \mathrm{proj}_Z(Q)$ is Borel and there is a Borel function $g\colon D \to Y$ such that $\forall z \in D(z, g(z)) \in Q$. Since $f(X) \subseteq D$, the function

$$f(x) = g(S(x))$$

is an E-invariant uniformization of P.

(B) We will next show that ¬(i) implies ¬(ii), ¬(iii), and ¬(v) (and thus also ¬(iv)). We will use the following lemma. Below for Borel equivalence relations E, E' on Polish spaces X, X', resp., we write $E \leq_B E'$ iff there is a Borel map $f\colon X \to X'$ such that $x_1 E x_2 \iff f(x_1) E' f(x_2)$, i.e., E can be **Borel reduced** to E' (via the reduction f).

Lemma 3.2.1. *Let E, E' be Borel equivalence relations on Polish spaces X, X', resp., such that $E \leq_B E'$. If E fails (ii) (resp., (iii), (iv), (v)), so does E'.*

Proof. Let $f\colon X \to X'$ be a Borel reduction of E into E'. Assume first that E fails (ii) with witness Y, μ, P. Define $P' \subseteq X' \times Y$ by

$$(x', y) \in P' \iff \forall x \in X\Big(f(x) E' x' \implies (x, y) \in P\Big).$$

Then note that

$$f(x) E' x' \implies P'_{x'} = P_x,$$
$$x' \notin [f(X)]_{E'} \implies P'_{x'} = Y.$$

Now clearly P' is $\mathbf{\Pi^1_1}$ and invariant under the Borel equivalence relation $E' \times \Delta_Y$. Then by a result of Solovay (see [Kec95, 34.6]), there is a $\mathbf{\Pi^1_1}$-rank $\varphi\colon P' \to \omega_1$ which is $E' \times \Delta_Y$-invariant. Consider then the $\mathbf{\Sigma^1_1}$ subset P'' of P' defined by

$$(x', y) \in P'' \iff \exists x \in X\Big(f(x) E' x' \ \&\ (x, y) \in P\Big).$$

By boundedness there is a Borel $E' \times \Delta_Y$-invariant set P''' with $P'' \subseteq P''' \subseteq P'$. Let now $Z \subseteq X'$ be defined by

$$x' \in Z \iff \mu(P'''_{x'}) > 0.$$

Then Z is Borel and E'-invariant and contains $[f(X)]_{E'}$. Finally define $Q \subseteq X' \times Y$ by

$$(x', y) \in Q \iff \Big(x' \in Z \ \&\ (x', y) \in P'''\Big) \text{ or } x' \notin Z.$$

Then $f(x) = x' \implies Q_{x'} = P_x$, so Y, μ, Q witnesses the failure of (ii) for E'.

The case of (iii) is similar and we next consider the case of (iv). Repeat then the previous argument for case (ii) until the definition of P'''. Then define $Z' \subseteq X'$ by

$$x' \in Z' \iff P'''_{x'} \text{ is } K_\sigma \text{ and nonempty.}$$

Then Z' is $\mathbf{\Pi^1_1}$, by [Kec95, 35.47] and the relativization of the fact that every nonempty Δ^1_1 K_σ set contains a Δ^1_1 member, see [Mos09, 4F.15]. It is also E'-invariant and contains $[f(X)]_{E'}$. Let then Z be E'-invariant Borel with $[f(X)]_{E'} \subseteq Z \subseteq Z'$ and define Q as before but replacing "$x' \notin Z$" by "($x' \notin Z$ and $y = y_0$)", for some fixed $y_0 \in Y$. Then Y, Q witnesses the failure of (iv) for E'.

Finally, the case of (v) is similar to (iv) by now defining

$$x' \in Z' \iff P'''_{x'} \text{ is countable and nonempty,}$$

and using that Z' is $\mathbf{\Pi^1_1}$ by [Kec95, 35.38] (and [Mos09, 4F.15] again). □

Assume now that E is not smooth. Then by [HKL90] we have $\mathbb{E}_0 \leq_B E$. Thus by Lemma 3.2.1 it is enough to show that $\mathbb{E}_0$ fails (ii), (iii), and (v) (thus also (iv)).

We first prove that $\mathbb{E}_0$ fails (ii). We view here $2^{\mathbb{N}}$ as the Cantor group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ with pointwise addition $+$ and we let μ be the Haar measure, i.e., the usual product measure. Let then $A \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ be an F_σ set which has μ-measure 1 but is meager. Let $X = Y = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ and define $P \subseteq X \times Y$ as follows:

$$(x, y) \in P \iff \exists x' \mathbb{E}_0 x (x' + y \in A).$$

Clearly P is F_σ and, since $P_x = \bigcup_{x' \mathbb{E}_0 x}(A - x')$, clearly $\mu(P_x) = 1$. Moreover P is $\mathbb{E}_0$-invariant. Assume then, towards a contradiction that f is a Borel $\mathbb{E}_0$-invariant uniformization. Since $x \mathbb{E}_0 x' \implies f(x) = f(x')$, by generic ergodicity of $\mathbb{E}_0$ there is a comeager Borel $\mathbb{E}_0$-invariant set $C \subseteq X$ and y_0 such that $\forall x \in C (f(x) = y_0)$; thus $\forall x \in C (x, y_0) \in P$, so $\forall x \in C \exists x' \mathbb{E}_0 x (x' \in A - y_0)$. If $G \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is the subgroup consisting of the eventually 0 sequences, then $x \mathbb{E}_0 y \iff \exists g \in G (g + x = y)$; thus $C = \bigcup_{g \in G}(g + (A - y_0))$, so C is meager, a contradiction.

To show that $\mathbb{E}_0$ fails (v), define

$$(x, y) \in P \iff x \mathbb{E}_0 y.$$

Then any Borel $\mathbb{E}_0$-invariant uniformization of P gives a Borel selector for $\mathbb{E}_0$, a contradiction.

Finally to see that $\mathbb{E}_0$ fails (iii), use above $B = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \setminus A$, instead of A, to produce a G_δ set Q as follows:

$$(x, y) \in Q \iff \forall x' \mathbb{E}_0 x (x' + y \in B).$$

Then Q is $\mathbb{E}_0$-invariant and has comeager sections. If g is a Borel $\mathbb{E}_0$-invariant uniformization, then by the ergodicity of $\mathbb{E}_0$, there is a μ-measure 1 set D and y_0 such that $\forall x \in D \forall x' \mathbb{E}_0 x (x' \in B - y_0)$, so $D \subseteq B - y_0$, and thus $\mu(D) = 0$, a contradiction.

This completes the proof of Theorem 3.1.5.

Remark 3.2.2. Andrew Marks and Dino Rossegger have pointed out that the construction in Proposition 3.5.8 actually gives a strengthening of Theorem 3.1.5: If E is not smooth then E fails co-countable invariant uniformization, i.e., there is an E-invariant Borel set P whose sections are co-countable which does not admit a Borel E-invariant uniformization.

To see this, define a hyperfinite Borel equivalence relation E on $(2^{\mathbb{N}})^{\mathbb{N}}$ by xEy iff there is a permutation σ of $\mathbb{N}$ fixing all but finitely many numbers, so that $x_n = y_{\sigma(n)}$ for $n \in \mathbb{N}$. H. Friedman has shown the following strengthening of Theorem 3.5.6 for this equivalence relation [Fri81, Proposition C]: If $F : (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ is Borel and E-invariant, then there is some $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $F(x) = x_0$.

Let now P be as in the proof of Proposition 3.5.8. Then P is Borel, E-invariant, and has co-countable sections. If F were a Borel E-invariant uniformization of P, then there would be some x with $F(x) = x_0$. But by definition $\neg P(x, x_0)$, a contradiction.

Let now E' be a non-smooth Borel equivalence relation on X. By [HKL90; DJK94], if E' is not smooth then there is a Borel reduction f from E to E'. Define P' as in the proof of Lemma 3.2.1. Then P' has co-countable sections and does not admit a Borel E'-invariant uniformization, so it remains to check that P' is Borel. To see this, write

$$Q(x', y, x) \iff f(x) E' x' \ \ \& \ \neg P(x, y).$$

Then Q is Borel and its sections $Q_{(x',y)}$ are either an E-class or empty, hence countable. Thus by Lusin–Novikov, $P' = X \times 2^{\mathbb{N}} \setminus \mathrm{proj}_{X \times 2^{\mathbb{N}}}(Q)$ is Borel.

(C) We note the following strengthening of Theorem 3.1.5 in the case that E is smooth, where $K(Y)$ denotes the Polish space of compact subsets of Y [Kec95, 4.F]:

Theorem 3.2.3. *Let X, Y be Polish spaces, E be a smooth Borel equivalence relation on X, and $P \subseteq X \times Y$ be a Borel E-invariant set with non-empty sections.*

1. *If P has countable sections, then $P = \bigcup_n \operatorname{graph}(g_n)$ for a sequence of E-invariant Borel maps $g_n : X \to Y$.*

2. *If P has K_σ sections, then $P_x = \bigcup_n K_n(x)$ for a sequence of E-invariant Borel maps $K_n : X \to K(Y)$.*

3. *If P has comeager sections, then $P \supseteq \bigcap_n U_n$ for a sequence of E-invariant Borel sets $U_n \subseteq X \times Y$ with dense open sections. Moreover, if P has dense G_δ sections, we can find such U_n with $P = \bigcap_n U_n$.*

Proof. The first two assertions follow from [Kec95, 18.10, 35.46] applied to Q from the proof of (i) $\implies$ (iv) of Theorem 3.1.5.

For the third, let Z, S, P^*, P^{**} be as in the proof of (i) $\implies$ (iv). By [Kec95, 16.1] the set C of all z for which P^{**}_z is comeager is Borel, so $Q(z, y) \iff [C(z) \implies P^{**}(z, y)]$ is Borel with comeager sections and $S(x) = z \implies P_x = Q_z$.

If moreover P has G_δ sections, we instead let A be the set of all z for which P^{**}_z is comeager and G_δ, which is $\mathbf{\Pi^1_1}$ by [Kec95, 35.47]. Then $S(X) \subseteq A$ is $\mathbf{\Sigma^1_1}$, so by the Lusin separation theorem there is a Borel set $S(X) \subseteq C \subseteq A$. We then define Q as above, so that Q moreover has G_δ sections.

The result then follows by [Kec95, 35.43]. □

(D) Theorems 3.1.1 to 3.1.3 are effective, meaning that whenever P is (lightface) Δ^1_1 and satisfies the hypotheses of one of these theorems, then P admits a Δ^1_1 uniformization (cf. [Mos09, 4F.16, 4F.20] and the discussion afterwards). Similarly, [HKL90] implies that if E is smooth and Δ^1_1 then it has a Δ^1_1 reduction to $\Delta(2^{\mathbb{N}})$. The proof of Theorem 3.1.5 therefore gives the following effective refinement:

Theorem 3.2.4. *Let E be a smooth Δ^1_1 equivalence relation on $\mathbb{N}^{\mathbb{N}}$ and $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be Δ^1_1 and E-invariant. Then P admits a Δ^1_1 E-invariant uniformization whenever one of the following holds:*

(i) P has μ-positive sections, for some Δ^1_1 probability measure μ on $\mathbb{N}^{\mathbb{N}}$;

(ii) P has non-meager sections;

(iii) P has non-empty K_σ sections;

(iv) P has non-empty countable sections.

In (i) above, we identify probability Borel measures on $\mathbb{N}^{\mathbb{N}}$ with points in $[0,1]^{\mathbb{N}^{<\mathbb{N}}}$ [Kec95, 17.7].

It is also interesting to consider whether the converse holds. For example, let E be a Δ^1_1 equivalence relation on $\mathbb{N}^{\mathbb{N}}$, and suppose that for every Δ^1_1 E-invariant set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ which satisfies one of (i)-(iv) above, P admits a Δ^1_1 E-invariant uniformization. Must it be the case that E is smooth?

If we replace Δ^1_1 by Borel, then E must indeed be smooth by Theorem 3.1.5. However, to prove this we use the fact that every non-smooth Δ^1_1 equivalence relation embeds $\mathbb{E}_0$ [HKL90], and this is not effective: There are non-smooth Δ^1_1 equivalence relations on $\mathbb{N}^{\mathbb{N}}$ which do not admit Δ^1_1 embeddings of $\mathbb{E}_0$.

Restricting our attention to those P which have countable sections, it turns out that the converse to Theorem 3.2.4 is false. In fact, using the theory of turbulence, one can construct the following very strong counterexample:

Theorem 3.2.5. *There is a Π^0_1 set $N \subseteq \mathbb{N}^{\mathbb{N}}$ and a Δ^1_1 equivalence relation E on N which is not smooth, and such that every Δ^1_1 E-invariant set $P \subseteq N \times \mathbb{N}^{\mathbb{N}}$ with non-empty countable sections is invariant, meaning that $P_x = P_{x'}$ for all $x, x' \in N$.*

Corollary 3.2.6. *There is a Δ^1_1 equivalence relation F on $\mathbb{N}^{\mathbb{N}}$ which is not smooth, and such that every Δ^1_1 F-invariant set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with non-empty countable sections admits a Δ^1_1 F-invariant uniformization.*

Proof. Let N, E be as in Theorem 3.2.5 and define

$$xFx' \iff (x = x') \vee (x, x' \in N \ \& \ xEx').$$

Suppose now that $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ were Δ^1_1 and F-invariant. Let $y \in A \iff \exists x \in N(P(x,y)) \iff \forall x \in N(P(x,y))$. Then A is countable and Δ^1_1, and $P_x = A$ for all $x \in N$. In particular, A contains a Δ^1_1 point, say y_0.

By the effective Lusin-Novikov theorem, there is a Δ^1_1 uniformization f of P. Letting $g(x) = f(x)$ for $x \notin N$, and $g(x) = y_0$ otherwise, gives a Δ^1_1 F-invariant uniformization of P. $\square$

Proof of Theorem 3.2.5. Consider the group $\mathbb{R}^{\mathbb{N}}$ and the translation action of $\ell^1 \subseteq \mathbb{R}^{\mathbb{N}}$ on $\mathbb{R}^{\mathbb{N}}$, which is turbulent by [Kec03, Section 10(ii)]. Let F be the induced equivalence relation, which is clearly Δ^1_1.

Let $\mathbb{F} \subseteq \mathbb{N}$ be the Π^1_1 set of codes for the Δ^1_1 functions from $\mathbb{R}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$, and for $n \in \mathbb{F}$ let f_n be the function that it codes. Let also $\mathbb{H} \subseteq \mathbb{F}$ be the Π^1_1 set of those n for which f_n is a homomorphism of F into $\mathbb{E}^{\mathbb{N}^\mathbb{N}}_{ctble}$.

Finally, let $\mathbb{D} \subseteq \mathbb{N}$ denote the usual Π^1_1 set of codes for the Δ^1_1 subsets of $\mathbb{R}^\mathbb{N}$, and for $n \in \mathbb{D}$ let $\mathbb{D}_n$ be the set that it codes.

By the proof of [Kec03, Theorem 12.5(i) $\implies$ (ii)], for each $n \in \mathbb{H}$ there is Δ^1_1 comeager F-invariant set $C_n \subseteq \mathbb{R}^\mathbb{N}$ which f_n maps to a single $\mathbb{E}^{\mathbb{N}^\mathbb{N}}_{ctble}$-class. Moreover, there is a computable map $n \mapsto n^*$ such that if $n \in \mathbb{H}$ then $n^* \in \mathbb{D}$ and $C_n = \mathbb{D}_{n^*}$.

Put $C = \bigcap_{n\in\mathbb{H}} C_n \subseteq \mathbb{R}^\mathbb{N}$. Then C is comeager, F-invariant and Σ^1_1, since

$$a \in C \iff \forall n(n \in \mathbb{H} \implies a \in \mathbb{D}_{n^*}).$$

Moreover, for each Δ^1_1 homomorphism f of F to $\mathbb{E}^{\mathbb{N}^\mathbb{N}}_{ctble}$, $f \restriction C$ maps into a single $\mathbb{E}^{\mathbb{N}^\mathbb{N}}_{ctble}$-class.

Let now $N \subseteq \mathbb{N}^\mathbb{N}$ be Π^0_1 and $c : N \to \mathbb{R}^\mathbb{N}$ be a Δ^1_1 map such that $c(N) = C$. Define the Δ^1_1 equivalence relation E on N by

$$xEx' \iff c(x)Fc(x').$$

We will show that this E works.

Let $P \subseteq N \times \mathbb{N}^\mathbb{N}$ be E-invariant with non-empty countable sections. Define $Q \subseteq C \times \mathbb{N}^\mathbb{N}$ by

$$\begin{aligned}(a, y) \in Q &\iff a \in C \ \& \ \exists x \in N(c(x) = a \ \& \ P(x, y)) \\ &\iff a \in C \ \& \ \forall x \in N(c(x) = a \implies P(x, y)).\end{aligned}$$

Note that Q is F-invariant. Moreover, Q is Δ^1_1 on the Σ^1_1 set $C \times \mathbb{N}^\mathbb{N}$, i.e., it is the intersection of $C \times \mathbb{N}^\mathbb{N}$ with a Σ^1_1 set in $\mathbb{R}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ as well as with a Π^1_1 set in $\mathbb{R}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$. By Σ^1_1 separation, there is a Δ^1_1 set $R \subseteq \mathbb{R}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ such that $R \cap (C \times \mathbb{N}^\mathbb{N}) = Q$.

Let now $C^* \subseteq \mathbb{R}^\mathbb{N}$ be defined by

$$a \in C^* \iff \forall a'[aFa' \implies R_a = R_{a'} \ \& \ R_a \text{ is countable and non-empty}].$$

Then C^* is Π^1_1, F-invariant and contains C, so there is a Δ^1_1 set B which is F-invariant and such that $C \subseteq B \subseteq C^*$. Finally, define $S \subseteq \mathbb{R}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ by

$$(a, y) \in S \iff [a \in B \ \& \ R(a, y)] \vee [a \notin B \ \& \ y = y_0]$$

for some fixed Δ^1_1 point y_0 in $\mathbb{N}^\mathbb{N}$. Then S is Δ^1_1, F-invariant, and has non-empty countable sections.

Let $s : \mathbb{R}^\mathbb{N} \to (\mathbb{N}^\mathbb{N})^\mathbb{N}$ be a Δ^1_1 homomorphism of F to $\mathbb{E}^{\mathbb{N}^\mathbb{N}}_{ctble}$ for which $S_a = \{s(a)_n\}$ for all $a \in \mathbb{R}^\mathbb{N}$, which exists by the effective Lusin-Novikov theorem. By the definition of C, we have that $s\restriction C$ maps into a single $\mathbb{E}^{\mathbb{N}^\mathbb{N}}_{ctble}$-class. Let A be the corresponding countable set. Then for $a \in C$ and any $x \in N$ with $c(x) = a$,

$$A = S_a = R_a = Q_a = P_x,$$

so $P_x = A$ for all $x \in N$.

It remains to check that E is not smooth. To see this, note that $F\restriction C$ has at least two classes (as every F-class is meager), and hence so does E. If E were smooth, then there would be a Δ^1_1 map $f : N \to \mathbb{N}^\mathbb{N}$ for which

$$xEx' \iff f(x) = f(x').$$

But then graph(f) would be Δ^1_1, E-invariant, have non-empty countable sections, and satisfy $P_x \neq P_{x'}$ for some $x, x' \in N$, a contradiction. $\square$

Problem 3.2.7. Is there a Δ^1_1 equivalence relation E on $\mathbb{N}^\mathbb{N}$ which is not smooth, and such that all Δ^1_1 E-invariant sets $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ satisfying one of (i)-(iii) in Theorem 3.2.4 admit a Δ^1_1 E-invariant uniformization?

Finally, we remark that if E is a Δ^1_1 equivalence relation which is not smooth, then there is a continuous embedding of $\mathbb{E}_0$ into E which is $\Delta^1_1(\mathcal{O})$. In particular, the converse of Theorem 3.2.4 holds if we consider all such $P \in \Delta^1_1(\mathcal{O})$.

3.3 Proofs of Theorems 3.1.6 and 3.1.7

(A) We first prove Theorem 3.1.7.

Let $F(Y)$ denote the Effros Borel space of closed subsets of Y (cf. [Kec95, 12.C]). Suppose $P_x \in F_\sigma$, for all $x \in X$, and that there is an E-invariant Borel map $x \mapsto F_x \in F(Y)$ such that P_x is non-meager in F_x for all $x \in X$. By [Kec95, 12.13], there is a sequence of E-invariant Borel functions $y_n : X \to Y$ such that $\{y_n(x)\}$ is dense in F_x for all $x \in X$. Since P_x is non-meager and F_σ in F_x, P_x contains an open set in F_x, and in particular contains some $y_n(x)$. Thus the map taking x to the least $y_n(x)$ such that $P(x, y_n(x))$ is an E-invariant Borel uniformization of P.

It remains only to show that in each of the cases (i), (ii), (iii), such an assignment $x \mapsto F_x$ exists. In (ii), we can take $F_x = Y$.

Consider case (i), that there is a Borel assignment $x \mapsto \mu_x$ of probability Borel measures on Y such that $P_x \in \mathbf{\Delta}^0_2$ and $\mu_x(P_x) > 0$, for all $x \in X$. Let ν_x denote the probability Borel measure μ_x restricted to P_x, i.e., $\nu_x(A) = \mu_x(A \cap P_x)/\mu_x(P_x)$, and define F_x to be the support of ν_x, i.e., the smallest ν_x-conull closed set in Y.

Since F_x is the support of ν_x, any open set in F_x is ν_x-positive, and therefore any ν_x-null F_σ set in F_x is meager. Now P_x is G_δ and ν_x-conull in F_x, so P_x is comeager in F_x, for all $x \in X$. Thus it remains only to show that the map $x \mapsto F_x$ is Borel. To see this, we observe that

$$F_x \cap U \neq \emptyset \iff \nu_x(U) > 0 \iff \mu_x(U \cap P_x) > 0$$

is Borel, by [Kec95, 17.25].

Finally, consider case (iii), that $P_x \in G_\delta$ and P_x is non-empty and K_σ for all $x \in X$. Let F_x be the closure of P_x in Y. Then P_x is dense G_δ in F_x, so it remains to check that $x \mapsto F_x$ is Borel. Indeed,

$$F_x \cap U \neq \emptyset \iff P_x \cap U \neq \emptyset,$$

and this is Borel by the Arsenin-Kunugui theorem [Kec95, 18.18], as $P_x \cap U$ is K_σ for all $x \in X$.

(B) We now prove Theorem 3.1.6.

Let $X = [\mathbb{N}]^{\aleph_0}$ denote the space of infinite subsets of $\mathbb{N}$. By identifying subsets of $\mathbb{N}$ with their characteristic functions, we can view X as an $\mathbb{E}_0$-invariant G_δ subspace of $2^{\mathbb{N}}$. Note that this is a dense G_δ in $2^{\mathbb{N}}$, and it is μ-conull, where μ is the uniform product measure on $2^{\mathbb{N}}$. We let E denote the equivalence relation $\mathbb{E}_0$ restricted to X.

Let $Y = 2^{\mathbb{N}}$, and define $P \subseteq X \times Y$ by

$$P(A, B) \iff |A \setminus B| = |A \cap B| = \aleph_0.$$

Then P is G_δ and E-invariant, and P_x is comeager for all $x \in X$. By the Borel-Cantelli lemma, one easily sees that $\mu(P_x) = 1$ for all $x \in X$.

We claim that P does not admit an E-invariant Borel uniformization. Indeed, suppose such a uniformization $f : X \to Y$ existed. By [Kec95, 19.19], there is some $A \in X$ such that $f\restriction[A]^{\aleph_0}$ is continuous, where $[A]^{\aleph_0}$ denotes the space of infinite subsets of A. Since E-classes are dense in $[A]^{\aleph_0}$, $f\restriction[A]^{\aleph_0}$ is constant, say with value B. Then $f(A) = B$, so $P(A, B)$ and $A \cap B$ is infinite. But then $A \cap B \in [A]^{\aleph_0}$, so $f(A \cap B) = B$. But $(A \cap B) \setminus B$ is not infinite, so $\neg P(A \cap B, B)$, a contradiction.

Remark 3.3.1. Using the same Ramsey-theoretic arguments, one can show that the following examples also do not admit E-invariant uniformizations:

1. Let Y be the space of graphs on $\mathbb{N}$ and set $Q(A, G)$ iff for all finite disjoint sets $x, y \subseteq \mathbb{N}$ there is some $a \in A$ which is adjacent (in G) to every element of x and no element of y, i.e., A contains witnesses that G is the random graph.
2. Let $Y = [\mathbb{N}]^{\aleph_0}$, and for $B \in Y$ let $f_B : \mathbb{N} \to \mathbb{N}$ denote its increasing enumeration. Then take $R(A, B)$ iff $f_B(A)$ contains infinitely many even and infinitely many odd elements.

As with P above, Q, R both have μ-conull dense G_δ sections.

3.4 Dichotomies and anti-dichotomies

3.4.1 Proof of Theorem 3.1.8

Here we derive Miller's dichotomy Theorem 3.1.8 for sets with countable sections, from Miller's $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy [Mil12] and Lecomte's $\aleph_0$-dimensional hypergraph dichotomy [Lec09].

We begin by noting the following equivalent formulations of the second alternative in Theorem 3.1.8.

Proposition 3.4.1. *Let X, Y be Polish spaces, E a Borel equivalence relation on X and $P \subseteq X \times Y$ an E-invariant Borel relation with countable non-empty sections. Then the following are equivalent:*

(2) There is a continuous embedding $\pi_X : 2^{\mathbb{N}} \times \mathbb{N} \to X$ of $\mathbb{E}_0 \times I_{\mathbb{N}}$ into E and a continuous injection $\pi_Y : 2^{\mathbb{N}} \times \mathbb{N} \to Y$ such that for all $x, x' \in 2^{\mathbb{N}} \times \mathbb{N}$,

$$\neg(x \ \mathbb{E}_0 \times I_{\mathbb{N}} \ x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset$$

and

$$P_{\pi_X(x)} = \pi_Y([x]_{\mathbb{E}_0 \times I_{\mathbb{N}}}).$$

(3) There is a continuous embedding $\pi_X : 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E and a continuous injection $\pi_Y : 2^{\mathbb{N}} \to Y$ such that for all $x, x' \in 2^{\mathbb{N}}$,

$$\neg(x \ \mathbb{E}_0 \ x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset$$

and

$$\pi_Y(x) \in P_{\pi_X(x)}.$$

(4) There is a continuous embedding $\pi_X : 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E such that for all $x, x' \in 2^{\mathbb{N}}$,

$$\neg(x \mathbin{\mathbb{E}_0} x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset.$$

Proof. Clearly (2) $\implies$ (3) $\implies$ (4). Assume now that (4) holds, and is witnessed by π_X. Let g be a uniformization of P and $\pi_Y = g \circ \pi_X$. Since π_Y is countable-to-one, by the Lusin-Novikov theorem there is a Borel non-meager set $B \subseteq 2^{\mathbb{N}}$ on which π_Y is injective. We then recursively construct a continuous embedding of $\mathbb{E}_0$ into $\mathbb{E}_0 \restriction B$, and compose this with π_X, π_Y to get maps witnessing (3).

Now suppose (3) holds, and is witnessed by π_X, π_Y. Let h be a continuous embedding of $\mathbb{E}_0 \times I_{\mathbb{N}}$ into $\mathbb{E}_0$, and let $\tilde{\pi}_X = \pi_X \circ h$. Let F be the equivalence relation on Y defined by yFy' iff $y = y'$ or there is some $x \in 2^{\mathbb{N}} \times \mathbb{N}$ such that $P(\tilde{\pi}_X(x), y)$ and $P(\tilde{\pi}_X(x), y')$. If $y \neq y'$, then the set of x witnessing that yFy' is a single $\mathbb{E}_0 \times I_{\mathbb{N}}$-class, so by Lusin-Novikov F is Borel. Thus, $\pi_Y \circ h$ is an embedding of $\mathbb{E}_0 \times I_{\mathbb{N}}$ into the countable Borel equivalence relation F, and by compressibility we can turn this into an invariant Borel embedding $\tilde{\pi}_Y$.

Now $\tilde{\pi}_X, \tilde{\pi}_Y$ would be witnesses to (2), except that $\tilde{\pi}_Y$ is not necessarily continuous. However, $\tilde{\pi}_Y$ is continuous when restricted to an $\mathbb{E}_0 \times I_{\mathbb{N}}$-invariant comeager Borel set C, so it suffices to find a continuous invariant embedding of $\mathbb{E}_0 \times I_{\mathbb{N}}$ into $(\mathbb{E}_0 \times I_{\mathbb{N}}) \restriction C$. One gets such an embedding by applying [Mild, Proposition 1.4] to the relation xRx' iff $x(\mathbb{E}_0 \times I_{\mathbb{N}})x'$ or $x \notin C$ or $x' \notin C$. $\square$

Remark 3.4.2. From the proof of (3) $\implies$ (2), one sees that if E is a countable Borel equivalence relation then actually one can strengthen (2) so that π_X is a continuous invariant embedding of $\mathbb{E}_0 \times I_{\mathbb{N}}$ into E, i.e., a continuous embedding such that additionally $\pi_X([x]_{\mathbb{E}_0 \times I_{\mathbb{N}}}) = [\pi_X(x)]_E$, for all $x \in 2^{\mathbb{N}} \times I_{\mathbb{N}}$.

The next two results will be used in the proof of Theorem 3.1.8.

Theorem 3.4.3 (Theorem 3.1.9)**.** *Let F be a smooth Borel equivalence relation on a Polish space X, Y be a Polish space, and $P \subseteq X \times Y$ be a Borel set with countable sections. Suppose that*

$$\bigcap_{x \in C} P_x \neq \emptyset$$

for every F-class C. Then P admits a Borel F-invariant uniformization.

Proof. Let Z be a Polish space and $S : X \to Z$ be a Borel map such that $xFx' \iff S(x) = S(x')$. Define $P^* \subseteq Z \times Y$ by

$$P^*(z, y) \iff \forall x(S(x) = z \implies P(x, y)).$$

Note that P^* is $\mathbf{\Pi^1_1}$, and that if $S(x) = z$ then

$$P^*_z = \bigcap_{xFx'} P_{x'}$$

is non-empty and countable.

By Lusin-Novikov, fix a sequence g_n of Borel maps $g_n : X \to Y$ such that $P = \bigcup_n \operatorname{graph}(g_n)$. Define $Q(x, n) \iff P^*(S(x), g_n(x))$. Then Q is $\mathbf{\Pi^1_1}$, so by the number uniformization property [Kec95, 35.1] we can fix a Borel map h uniformizing Q.

Let now $A(z, y) \iff \exists x(S(x) = z \ \& \ y = g_{h(x)}(x))$. Then $A \subseteq P^*$ is $\mathbf{\Sigma^1_1}$, so by the Lusin separation theorem there is a Borel set $A \subseteq P^{**} \subseteq P^*$. By [Kec95, 18.9], the set

$$C = \{z \mid P^{**}_z \text{ is countable}\}$$

is $\mathbf{\Pi^1_1}$, and it contains $S(X)$, so by the Lusin separation theorem again there is some Borel set $S(X) \subseteq B \subseteq C$.

By Lusin-Novikov, there is a Borel uniformization f of $R(z, y) \iff B(z) \ \& \ P^{**}(z, y)$. Then $f \circ S$ is an F-invariant Borel uniformization of P. $\square$

Proposition 3.4.4. *Let E be an analytic equivalence relation on a Polish space X, $F \supseteq E$ be a smooth Borel equivalence relation on X, Y be a Polish space, and $P \subseteq X \times Y$ be a Borel E-invariant set with countable sections. Suppose that*

$$xFx' \implies P_x \cap P_{x'} \neq \emptyset$$

for all $x, x' \in X$. Then there is a smooth equivalence relation $E \subseteq F' \subseteq F$ such that

$$\bigcap_{x \in C} P_x \neq \emptyset$$

for every F'-class C.

Proof. Let $G \subseteq X^{\mathbb{N}}$ be the $\aleph_0$-dimensional hypergraph of F-equivalent sequences x_n such that $\bigcap_n P_{x_n} = \emptyset$. By Lusin-Novikov, G is Borel.

We claim that G has a countable Borel colouring. By [Lec09, Lemma 2.1 and Theorem 1.6], it suffices to show that G has a countable $\sigma(\mathbf{\Sigma^1_1})$-colouring. Let S

be a $\sigma(\mathbf{\Sigma^1_1})$-measurable selector for F and g_n be a sequence of Borel functions such that $P = \bigcup_n \operatorname{graph}(g_n)$. Then the function $f(x)$ assigning to x the least n such that $P(x, g_n(S(x)))$ is such a colouring. (In fact, $x \mapsto g_{f(x)}(S(x))$ is a $\sigma(\mathbf{\Sigma^1_1})$-measurable F-invariant uniformization of P.)

If A is G-independent, then so is $[A]_E$. Thus, by repeated application of the first reflection theorem, any G-independent analytic set is contained in an E-invariant G-independent Borel set. We may therefore fix a countable cover B_n of X by E-invariant G-independent Borel sets.

Define $xF'x' \iff xFx' \ \& \ \forall n(x \in B_n \iff x' \in B_n)$. Then F' is a smooth Borel equivalence relation and $E \subseteq F' \subseteq F$. Fix $x = x_0 \in X$, in order to show that

$$\bigcap_{xF'x'} P_{x'} \neq \emptyset.$$

Fix an enumeration $y_n, n \geq 1$ of P_x, and suppose for the sake of contradiction that this intersection is empty. Then for each n, there is some $x_n F' x$ with $y_n \notin P_{x_n}$. Also, $x \in B_k$ for some k. But then $x_n \in B_k$ for all k, so B_k is not G-independent, a contradiction. $\square$

Proof of Theorem 3.1.8. Clearly the two cases are mutually exclusive. To see that at least one of them holds, define the graph G on X by $xGx' \iff P_x \cap P_{x'} = \emptyset$. By Lusin-Novikov, this is a Borel graph. We now apply the $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy [Mil12, Theorem 25] to (G, E), and consider the two cases.

Case 1: There is a countable Borel colouring of $G \cap F$, where $F \supseteq E$ is smooth. Let A be Borel and $(G \cap F)$-independent. By repeated applications of the first reflection theorem, we may assume that A is E-invariant. We can therefore refine F to a smooth equivalence relation $F' \supseteq E$ such that $xF'x' \implies P_x \cap P_{x'} \neq \emptyset$. The result now follows from Theorem 3.4.3 and Proposition 3.4.4.

Case 2: Let f be a continuous homorphism from $(\mathbb{G}_0, \mathbb{H}_0)$ to (G, E). It suffices to show that (4) holds in Proposition 3.4.1. To see this, consider $F = (f \times f)^{-1}(E), R = (f \times f)^{-1}(G)$. Then $\mathbb{H}_0 \subseteq F$ and each F-section is $\mathbb{G}_0$-independent, hence meager, so F is meager. We claim R is comeager. To see this, fix $x \in 2^{\mathbb{N}}$ and consider $R^c_x = \{x' : P_{f(x)} \cap P_{f(x')} \neq \emptyset\}$. Fix an enumeration y_n of $P_{f(x)}$, and let $A_n = \{x' : y_n \in P_{f(x')}\}$. Then each A_n is $\mathbb{G}_0$-independent, hence meager, and $R^c_x = \bigcup_n A_n$. Thus R has comeager sections, and by Kuratowski-Ulam R is comeager. One can now recursively construct a continuous homomorphism g from $((\Delta_{2^{\mathbb{N}}})^c, \mathbb{E}^c_0, \mathbb{E}_0)$ to $((f \times f)^{-1}(\Delta_X)^c, R, \mathbb{E}_0)$, see e.g. [Milb, Proposition 11]. Then $f \circ g$ satisfies (4). $\square$

3.4.2 An $\aleph_0$-dimensional $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy

In this section we state and prove an $\aleph_0$-dimensional analogue of Miller's $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy [Mil12, Theorem 25]. This dichotomy generalizes Lecomte's $\aleph_0$-dimensional $\mathbb{G}_0$ dichotomy [Lec09] (see also [Mil11]) in the same way that Miller's $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy generalizes the $\mathbb{G}_0$ dichotomy [KST99]. We then state an effective analogue of this theorem, and indicate the changes that must be made to prove it.

(A) Fix a strictly increasing sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ and dense sets $S \subseteq \bigcup_n \mathbb{N}^{2n}$, $T \subseteq \bigcup_n \mathbb{N}^{2n+1} \times \mathbb{N}^{2n+1}$, i.e., sets such that for all $u \in \mathbb{N}^{<\mathbb{N}}$ there is some $s \in S$ with $s \subseteq u$, and for all $(u, v) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}}$ there is some $t = (t_0, t_1) \in T$ such that $t_0 \subseteq u, t_1 \subseteq v$.

Let $X_\alpha = \{x \in \mathbb{N}^{\mathbb{N}} : \forall n \exists m \geq n(x{\restriction}m \in \alpha(m)^m)\}$. Note that X_α is dense G_δ in $\mathbb{N}^{\mathbb{N}}$.

Define the Borel $\aleph_0$-dimensional directed hypergraph $\mathbb{G}_0^\omega$ on X_α by

$$\mathbb{G}_0^\omega((x_n)) \iff \exists s \in S \exists z \in \mathbb{N}^{\mathbb{N}} \forall n(x_n = s^\frown n^\frown z),$$

and the Borel directed graph $\mathbb{H}_0^\omega$ on X_α by

$$x \mathbb{H}_0^\omega y \iff \exists (t_0, t_1) \in T \exists z \in \mathbb{N}^{\mathbb{N}}(x = t_0^\frown 0^\frown z \ \ \& \ \ y = t_1^\frown 1^\frown z).$$

We say $A \subseteq X_\alpha$ is $\mathbb{G}_0^\omega$-independent if $x \in A^{\mathbb{N}} \implies \neg \mathbb{G}_0^\omega(x)$.

Proposition 3.4.5 ([Lec09, Lemma 2.1])**.** *Let $A \subseteq X_\alpha$ be Baire measurable and $\mathbb{G}_0^\omega$-independent. Then A is meager.*

Proof. Suppose A is non-meager, and fix an open set $N_s = \{x \in \mathbb{N}^{\mathbb{N}} : s \subseteq x\}$ in which A is comeager. By density of S, we may assume wlog that $s \in S$. For each n, the set $A_n = \{x \in \mathbb{N}^{\mathbb{N}} : s^\frown n^\frown x \in A\}$ is comeager, so there is some $x \in \bigcap_n A_n$. But then $x_n = s^\frown n^\frown x \in A$, and $\mathbb{G}_0^\omega((x_n))$, so A is not $\mathbb{G}_0^\omega$-independent. □

Let R be a quasi-order on a Polish space X. We let $\equiv_R$ denote the equivalence relation $x \equiv_R y \iff xRy \ \ \& \ \ yRx$. We say R is **lexicographically reducible** if there is a Borel reduction of R to the lexicographic order $\leq_{\text{lex}}$ on 2^α, for some $\alpha < \omega_1$. If $A \subseteq X$, we let $[A]^R = \{y : \exists x \in A(xRy)\}, [A]_R = \{y : \exists x \in A(yRx)\}$, and say A is closed upwards (resp. downward) for R if $A = [A]^R$ (resp. $A = [A]_R$). If $A, B \subseteq X$, we say (A, B) is R-independent if $A \times B \cap R = \emptyset$.

Proposition 3.4.6 (Ess. [Milc, Proposition 5])**.** *Let $A \subseteq X_\alpha$ be Baire measurable and $\equiv_{\mathbb{H}_0^\omega}$-invariant. Then A is either meager or comeager.*

Proof. Suppose A is non-meager, and fix an open set N_u in which A is comeager. We show that A is non-meager in N_v for all $v \in \mathbb{N}^{<\mathbb{N}}$. By density of T, it suffices to show this assuming that $(u, v) \in T$. The set $A_0 = \{x \in \mathbb{N}^{\mathbb{N}} : u^\frown 0^\frown x \in A\}$ is comeager, and $x \in A_0 \implies v^\frown 1^\frown x \in A$, so A is comeager in $N_{v^\frown 1}$. □

Proposition 3.4.7 ([Milc, Proposition 1])**.** *Let R be an analytic quasi-order on a Polish space X and $A_0, A_1 \subseteq X$ be analytic such that (A_0, A_1) is R-independent. Then there are Borel sets $A_i \subseteq B_i$ such that (B_0, B_1) is R-independent, B_0 is closed upwards for R, and B_1 is closed downwards for R.*

Proof. Note that $([A_0]^R, [A_1]_R)$ is R-independent, and these sets are analytic. By the first reflection theorem, we can recursively construct a sequence of Borel sets B^i_n such that $A_i \subseteq B^i_0$, $[B^0_n]^R \subseteq B^0_{n+1}$, $[B^1_n]_R \subseteq B^1_{n+1}$, and (B^0_n, B^1_n) are R-independent. Take $B_i = \bigcup_n B^i_n$. □

Let F be an equivalence relation on X and G be an $\aleph_0$-dimensional directed hypergraph on X. We call $A \subseteq X$ F-locally G-independent if there is no sequence $x_n \in A$ of pairwise F-equivalent points with $G((x_n))$, and we call $c : X \to Y$ an F-local colouring of G if $c^{-1}(y)$ is F-locally G-independent for all $y \in Y$.

Theorem 3.4.8. *Let G be an analytic $\aleph_0$-dimensional directed hypergraph on a Polish space X, and R an analytic quasi-order on X. Then exactly one of the following holds:*

(1) There is a lexicographically reducible quasi-order R' on X such that $R \subseteq R'$ and there is a countable Borel $\equiv_{R'}$-local colouring of G.

(2) There is a continuous homomorphism from $(\mathbb{G}^\omega_0, \mathbb{H}^\omega_0)$ to (G, R).

Proof. To see these are mutually exclusive, it suffices to show that there is no smooth equivalence relation $F \supseteq \equiv_{\mathcal{H}^\omega_0}$ such that there is a countable Borel F-local colouring $c : X_\alpha \to \mathbb{N}$ of $\mathbb{G}^\omega_0$. Arguing by contradiction, suppose such F, c existed. By Proposition 3.4.6, we can fix $n \in \mathbb{N}$ and a single F-class C such that $A = c^{-1}(n) \cap C$ is non-meager. But then by Proposition 3.4.5, A is not $\mathbb{G}^\omega_0$-independent, a contradiction.

We now show that at least one of these alternatives hold. Fix continuous maps $\pi_G, \pi_R : \mathbb{N}^{\mathbb{N}} \to X$ such that $G = \pi_G(\mathbb{N}^{\mathbb{N}}), R = \pi_R(\mathbb{N}^{\mathbb{N}})$. Let d denote the usual metric on $\mathbb{N}^{\mathbb{N}}$, and d_X be a complete metric compatible with the Polish topology on X.

Let V be a set, H_0 be an $\aleph_0$-dimensional directed hypergraph on V with edge set E_0, and H_1 be a directed graph on V with edge set E_1. A **copy** of (H_0, H_1) in (G, R) is a triple $\varphi = (\varphi_X, \varphi_G, \varphi_R)$ where $\varphi_X : V \to X, \varphi_G : E_0 \to \mathbb{N}^\mathbb{N}, \varphi_R : E_1 \to \mathbb{N}^\mathbb{N}$, such that

$$e = (v_n) \in E_0 \implies \varphi_G(e) = (\varphi_X(v_n))_{n\in\mathbb{N}},$$

and

$$e = (v, u) \in E_1 \implies \varphi_R(e) = (\varphi_X(v), \varphi_X(u)).$$

Let $\mathrm{Hom}(H_0, H_1; G, R)$ denote the set of all copies of (H_0, H_1) in (G, R). Note that if V, E_0, E_1 are countable, then $\mathrm{Hom}(H_0, H_1; G, R) \subseteq X^V \times (\mathbb{N}^\mathbb{N})^{E_0} \times (\mathbb{N}^\mathbb{N})^{E_1}$ is closed, and hence Polish.

Suppose now we have H_0, H_1 as above, with V, E_0, E_1 countable, and consider $\mathcal{H} \subseteq \mathrm{Hom}(H_0, H_1; G, R)$. Let $\mathcal{H}(v) = \{\varphi_X(v) : \varphi \in \mathcal{H}\}$ for $v \in V$, and note that $\mathcal{H}(v)$ is analytic whenever $\mathcal{H}$ is analytic. Define $\mathcal{H}(e) \subseteq \mathbb{N}^\mathbb{N}$ similarly for $e \in E_0 \cup E_1$. Now call a set $\mathcal{H}$ **tiny** if it is Borel and there is a lexicographically reducible quasi-order R' on X such that $R \subseteq R'$ and one of the following holds:

(1) $\mathcal{H}(v)$ is $\equiv_{R'}$-locally G-independent for some $v \in V$.

(2) $\forall \varphi \in \mathcal{H} \exists u, v \in V(\varphi_X(u) \not\equiv_{R'} \varphi_X(v))$.

In this case, we call R' a **witness** that $\mathcal{H}$ is tiny, and say $\mathcal{H}$ is tiny of type 1 (resp. 2) if $\mathcal{H}$ satisfies (1) (resp. (2)). Finally, we say $\mathcal{H}$ is **small** if it is in the σ-ideal generated by the tiny sets, and otherwise we call $\mathcal{H}$ **large**.

Finally, fix H_0, H_1 as above with V, E_0, E_1 countable. For $v \in V$, we define the $\aleph_0$-dimensional directed hypergraph $\oplus_v H_0$ and the directed graph $\oplus_v H_1$ by taking a countable disjoint union of H_0 (resp. H_1), on vertex set $V \times \mathbb{N}$, and adding the edge $(v^\frown n)_{n\in\mathbb{N}}$ to $\oplus_v H_0$. Similarly, for $u, v \in V$, we define the $\aleph_0$-dimensional directed hypergraph $H_0\ {}_u{+}_v H_0$ and the directed graph $H_1\ {}_u{+}_v H_1$ by taking a countable disjoint union of H_0 (resp. H_1), on vertex set $V \times \mathbb{N}$, and adding the edge $(u^\frown 0, v^\frown 1)$ to $H_1\ {}_u{+}_v H_1$. Note that there are natural continuous projection maps

$$\mathrm{Hom}(\oplus_v H_0, \oplus_v H_1; G, R) \to \mathrm{Hom}(H_0, H_1; G, R)$$

and

$$\mathrm{Hom}(H_0\ {}_u{+}_v H_0, H_1\ {}_u{+}_v H_1; G, R) \to \mathrm{Hom}(H_0, H_1; G, R),$$

for all $n \in \mathbb{N}$, taking φ to its restriction φ^n to $V \times \{n\}$. If $\mathcal{H} \subseteq \mathrm{Hom}(H_0, H_1; G, R)$, we let

$$\oplus_v \mathcal{H} = \{\varphi \in \mathrm{Hom}(\oplus_v H_0, \oplus_v H_1; G, R) : \forall n(\varphi^n \in \mathcal{H})\},$$
$$\mathcal{H}\,{}_u{+}_v \mathcal{H} = \{\varphi \in \mathrm{Hom}(H_0\,{}_u{+}_v H_0, H_1\,{}_u{+}_v H_1; G, R) : \forall n(\varphi^n \in \mathcal{H})\}.$$

Claim 3.4.9. If $\mathrm{Hom}(\cdot, \cdot; G, R)$ is small, then there is a lexicographically reducible quasi-order R' on X such that $R \subseteq R'$ and there is a countable Borel $\equiv_{R'}$-local colouring of G.

Proof. Note that $\mathrm{Hom}(\cdot, \cdot; G, R)$ can be naturally identified with X, so that our assumption implies that X can be covered by countably-many Borel sets A_n such that for each n, there is a lexicographically reducible quasi-order R_n such that $R \subseteq R_n$ and A_n is $\equiv_{R_n}$-locally G-independent.

Let $f_n : X \to 2^{\alpha_n}$ be a Borel reduction of R_n to the lexicographic ordering on 2^{α_n}, $\alpha_n < \omega_1$. Let $\alpha = \sum_n \alpha_n$, and consider the map $f : X \to 2^{\alpha}$, $f(x) = f_0(x)^\frown f_1(x)^\frown f_2(n)^\frown \cdots$. Then f is Borel, so $xR'y \iff f(x) \leq_{\mathrm{lex}} f(y)$ is a lexicographically reducible quasi-order containing R. Note also that $\equiv_{R'} = \bigcap_n \equiv_{R_n}$. It follows that the map taking x to the least n for which $x \in A_n$ is a countable Borel $\equiv_{R'}$-local colouring of G. □

Claim 3.4.10. Let H_0, H_1 be as above with V, E_0, E_1 countable, $F \subseteq V \cup E_0 \cup E_1$ be finite, $\varepsilon > 0$, and $\mathcal{H} \subseteq \mathrm{Hom}(H_0, H_1; G, R)$ be large and Borel. Then there is a large Borel set $\mathcal{H}' \subseteq \mathcal{H}$ for which $\mathrm{diam}_{d_X}(\mathcal{H}'(v)) < \varepsilon$ for all $v \in F \cap V$ and $\mathrm{diam}_d(\mathcal{H}'(e)) < \varepsilon$ for all $e \in F \cap (E_0 \cup E_1)$.

Proof. This follows from the fact that we can cover $X, \mathbb{N}^{\mathbb{N}}$ with countably many sets of small diameter, and the small sets form a σ-ideal. □

Claim 3.4.11. Let H_0, H_1 be as above with V, E_0, E_1 countable, and suppose $\mathcal{H} \subseteq \mathrm{Hom}(H_0, H_1; G, R)$ is Borel and large. Then $\oplus_v \mathcal{H}, \mathcal{H}\,{}_u{+}_v \mathcal{H}$ are Borel and large.

Proof. That these sets are Borel is clear. Now suppose $\oplus_v \mathcal{H}$ is small and write $\oplus_v \mathcal{H} = \bigcup_{i \in 2, n \in \mathbb{N}} \mathcal{F}_n^i$, with F_n^i tiny of type i and witness R_n^i. Arguing as in the proof of Claim 3.4.9, we may assume that $R_n^i = R'$ for a single quasi-order R'. Let $v_n \in V$ be such that $\mathcal{F}_n^0(v_n)$ is $\equiv_{R'}$-locally G-independent. By the first reflection theorem, we may fix Borel sets $\mathcal{F}_n^0(v_n) \subseteq A_n$ which are $\equiv_{R'}$-locally G-independent. Define $\mathcal{H}_n = \{\varphi \in \mathcal{H} : \varphi_X(v_n) \in A_n\}$, and let

$$\mathcal{H}' = \mathcal{H} \setminus \left(\{\varphi \in \mathcal{H} : \exists u, v \in V(\varphi_X(u) \not\equiv_{R'} \varphi_X(v))\} \cup \bigcup_n \mathcal{H}_n \right).$$

We claim $\mathcal{H}'$ is tiny, which implies that $\mathcal{H}$ is small. Clearly $\mathcal{H}'$ is Borel, and we claim $\mathcal{H}'(v)$ is $\equiv_{R'}$-locally G-independent. Indeed, if $\varphi_n \in \mathcal{H}'$ and $G(((\varphi_n)_X(v))_{n\in\mathbb{N}})$, then there is some $\varphi \in \oplus_v \mathcal{H}$ with $\varphi^n = \varphi_n$ for all n. But then $\varphi \in \mathcal{F}_n^1$ for some n, so there are $u, w \in V \times \mathbb{N}$ such that $\varphi_X(u) \not\equiv_{R'} \varphi_X(w)$. Since $\varphi^n \in \mathcal{H}'$ for all n, we may assume that $u = v^\frown i, w = v^\frown j$ for some $i \neq j$. But then $\varphi_X^i(v) = (\varphi_i)_X(v) \not\equiv_{R'} (\varphi_j)_X(v) = \varphi_X^j(v)$.

Next suppose $\mathcal{H}\,{}_u{+}_v\mathcal{H}$ is small and write $\mathcal{H}\,{}_u{+}_v\mathcal{H} = \bigcup_{i\in 2, n\in\mathbb{N}} \mathcal{F}_n^i$, with $\mathcal{F}_n^i$ tiny of type i and witness R_n^i. As before, we may assume $R_n^i = R'$, and we define $\mathcal{H}_n, \mathcal{H}'$ in the same way, so that it suffices to show that $\mathcal{H}'$ is tiny of type 2.

Let $\varphi_i \in \mathcal{H}', i \in 2$, and suppose $(\varphi_0)_X(u) R (\varphi_1)_X(v)$. Then there is some $\varphi \in \mathcal{H}\,{}_u{+}_v\mathcal{H}$ with $\varphi^0 = \varphi_0$ and $\varphi^i = \varphi_1$ for $i > 0$. As before, we find that we must have $\varphi_X(u^\frown 0) \not\equiv_{R'} \varphi_X(v^\frown 1)$, so that $(\varphi_0)_X(u) \not\equiv_{R'} (\varphi_1)_X(v)$. Thus, $(\mathcal{H}'(u), \mathcal{H}'(v))$ is $(R\cap \equiv_{R'})$-independent, and by Proposition 3.4.7 we can find Borel sets $\mathcal{H}'(u) \subseteq A, \mathcal{H}'(v) \subseteq B$ such that A is closed upwards for $R\cap \equiv_{R'}$, B is closed downwards for $R\cap \equiv_{R'}$, and (A, B) is $(R\cap \equiv_{R'})$-independent. Then

$$xQy \iff xR'y \ \& \ (x \equiv_{R'} y \ \& \ x \in A \implies y \in A)$$

is a lexicographically reducible quasi-order containing R, and $\mathcal{H}'$ is tiny of type 2 with witness Q. $\square$

If $\mathrm{Hom}(\cdot,\cdot; G, R)$ is small, then by Claim 3.4.9 we are done. Suppose now that $\mathrm{Hom}(\cdot,\cdot; G, R)$ is large. We define a sequence G_n of $\aleph_0$-dimensional directed graphs on $\mathbb{N}^n$ and a sequence H_n of directed graphs on $\mathbb{N}^n$ as follows:

$$\begin{aligned} G_n(x_i) &\iff \exists k < n\, \exists s \in (S \cap \mathbb{N}^k)\, \exists u \in \mathbb{N}^{n-k-1}\, \forall i (x_i = s^\frown i^\frown u), \\ xH_n y &\iff \exists k < n\, \exists (t_0, t_1) \in (T \cap \mathbb{N}^k \times \mathbb{N}^k) \\ &\qquad\qquad \exists u \in \mathbb{N}^{n-k-1} (x = t_0^\frown 0^\frown u \ \& \ y = t_1^\frown 1^\frown y). \end{aligned}$$

Note that if $s \in S \cap \mathbb{N}^n$ then $G_{n+1} = \oplus_s G_n$ and $H_{n+1} = \oplus_s H_n$, and if $(t_0, t_1) \in T \cap \mathbb{N}^n \times \mathbb{N}^n$ then $G_{n+1} = G_n\,{}_{t_0}{+}_{t_1} G_n$ and $H_{n+1} = H_n\,{}_{t_0}{+}_{t_1} H_n$. Also,

$$\mathbb{G}_0^\omega((x_i)_{i\in\mathbb{N}}) \iff \exists N \forall n \geq N (G_n((x_i \restriction n)_{i\in\mathbb{N}}))$$

and

$$x\mathbb{H}_0^\omega y \iff \exists N\forall n \geq N(x{\restriction}n\, H_n\, y{\restriction}n),$$

and G_n, H_n have countably many vertices and edges.

By Claims 3.4.10 and 3.4.11, we can recursively construct a sequence of large Borel sets $\mathcal{H}_n \subseteq \mathrm{Hom}(G_n, H_n; G, R)$ such that $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n \oplus_s \mathcal{H}_n$ for $s \in S \cap \mathbb{N}^n$, $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n\, {}_{t_0}{+}_{t_1} \mathcal{H}_n$ for $(t_0, t_1) \in T \cap \mathbb{N}^n \times \mathbb{N}^n$, $\mathrm{diam}_{d_X}(\mathcal{H}_n(x)) < 2^{-n}$ for all $x \in \alpha(n)^n$, and $\mathrm{diam}_d(\mathcal{H}(e)) < 2^{-n}$ for all $e \in G_n \cup H_n$ with $e_0 \in \alpha(n)^n$, where e_0 denotes the first vertex in e. It follows that $\{f(x)\} = \bigcap_n \overline{\mathcal{H}_n(x{\restriction}n)}$ exists and is well defined for $x \in X_\alpha$, and that this map $f : X_\alpha \to X$ is continuous. To see that it is a homomorphism of $\mathbb{G}_0^\omega$ to G, suppose $\mathbb{G}_0^\omega((x_i)_{i\in\mathbb{N}})$ and let N be sufficiently large that $G_N((x_i{\restriction}N)_{i\in\mathbb{N}})$. Then $\{y\} = \bigcap_{n\geq N} \overline{\mathcal{H}_n((x_i{\restriction}n)_{i\in\mathbb{N}})}$ exists and is well defined, and by continuity we have $(f(x_i))_{i\in\mathbb{N}} = \pi_G(y) \in G$. A similar argument shows that f is a homomorphism from $\mathbb{H}_0^\omega$ to R. □

(B) This dichotomy admits the following effective refinement:

Theorem 3.4.12. *Let G be a Σ^1_1 $\aleph_0$-dimensional directed hypergraph on a Polish space X, and R a Σ^1_1 partial order on X. Then exactly one of the following holds:*

1. *There is a quasi-order R' on X such that $R \subseteq R'$, there is a countable Δ^1_1 $\equiv_{R'}$-local colouring of G, and there is a Δ^1_1 reduction of R' to the lexicographic order $\leq_{lex}$ on 2^α, for some $\alpha < \omega_1^{CK}$.*
2. *There is a continuous homomorphism from $(\mathbb{G}_0^\omega, \mathbb{H}_0^\omega)$ to (G, R).*

To prove this, we make the following modifications to the proof of Theorem 3.4.8. First, we choose π_G, π_H to be computable (restricting their domains appropriately to Π^0_1 sets). We then replace "Borel" with "Δ^1_1" and "lexicographically reducible" with "admitting a Δ^1_1 reduction to $\leq_{\mathrm{lex}}$ on 2^α, for some $\alpha < \omega_1^{CK}$" in the definition of tiny sets.

We now have the following:

Lemma 3.4.13. *Let V be a set, H_0 be an $\aleph_0$-dimensional directed hypergraph on V with edge set E_0 and H_1 be a directed graph on V with edge set E_1, with V, E_0, E_1 countable. Suppose $\mathcal{H} \subseteq \mathrm{Hom}(H_0, H_1; G, R)$ is small and Δ^1_1. Then one can find:*

(1) a uniformly Δ^1_1 sequence of tiny sets $(\mathcal{F}^i_n)_{i\in 2, n\in\mathbb{N}}$ covering $\mathcal{H}$,

(2) a uniformly Δ^1_1 sequence $(R^i_n)_{i\in 2,n\in\mathbb{N}}$ of quasi-orders on $\mathbb{N}$,

(3) a uniformly Δ^1_1 sequence of ordinals $\alpha^i_n < \omega_1^{CK}$,

(4) a uniformly Δ^1_1 sequence $(f^i_n)_{i\in 2,n\in\mathbb{N}}$ of maps $f^i_n : \mathbb{N}^\mathbb{N} \to 2^{\alpha^i_n}$, and

(5) a uniformly Δ^1_1 sequence $v_n \in V$,

such that the sets $\mathcal{F}^i_n$ are pairwise disjoint, $R \subseteq R^i_n$ for all i, n, each f^i_n is a reduction of R^i_n to $\leq_{lex}$ on $2^{\alpha^i_n}$, $\mathcal{F}^0_n(v_n)$ are $\equiv_{R^0_n}$-locally G-independent, and $\forall\varphi \in \mathcal{H}\exists u, v \in V(\varphi_X(u) \not\equiv_{R^1_n} \varphi_X(v))$.

Proof sketch. Fix a nice coding $D \ni n \mapsto D_n$ of the Δ^1_1 sets. The assertion that $(\mathcal{F}, R', \alpha, f, v, i)$ is a witness that $\mathcal{F}$ is tiny of type i is Π^1_1-on-Δ^1_1. It follows that the relation "$\varphi \notin \mathcal{H}$ or $n \in D$ codes such a tuple with $\varphi \in \mathcal{F}$" is Π^1_1, and hence by the number uniformization theorem for Π^1_1 there is a Δ^1_1 map $g : \mathcal{H} \to D$ taking each $\varphi \in \mathcal{H}$ to such a tuple. The image of $\mathcal{H}$ under g is Σ^1_1, so by the Lusin separation theorem there is a Δ^1_1 set $A \subseteq D$ containing $g(\mathcal{H})$ and such that every element in A codes a tuple as above. One can then fix a Δ^1_1 enumeration of A, which satisfies all of the above conditions except maybe pairwise disjointness of the family $\mathcal{F}^i_n$, and this can be fixed by a straightforward recursive construction. □

The effective analogue of Claim 3.4.9 follows immediately. We note that the first reflection theorem is effective enough that the proof of Proposition 3.4.7 is effective as well. Claim 3.4.11 now follows using Lemma 3.4.13. The rest of the proof is identical to that of Theorem 3.4.8.

3.4.3 Proof of Theorem 3.1.8 from the $\aleph_0$-dimensional $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy

(A) Clearly the two cases are mutually exclusive. To see that at least one of them holds, define the $\aleph_0$-dimensional hypergraph G on X by $G(x_n) \iff \bigcap_n P_{x_n} = \emptyset$. By Lusin-Novikov, G is Borel. We now apply Theorem 3.4.8 to (G, E), and consider the two cases.

Case 1: There is a lexicographically reducible quasi-order R containing E and a countable Borel $\equiv_R$-local colouring of G. Let $F = \equiv_R$, so that $E \subseteq F$ and F is smooth. Since P is E-invariant, if A is F-locally G-independent then so is $[A]_E$. It follows that there is a countable Borel E-invariant F-local colouring of G, so that after refining F with this colouring we may assume that X is F-locally G-independent, i.e., $\bigcap_{x\in C} P_x \neq \emptyset$ for every F-class C. Then P admits a Borel F-invariant uniformization by Theorem 3.4.3.

Case 2: There is a continuous homomorphism $\pi : X_\alpha \to X$ of $(\mathbb{G}_0^\omega, \mathbb{H}_0^\omega)$ to (G, E). We will show that (4) holds in Proposition 3.4.1. To see this, consider $F = (\pi \times \pi)^{-1}(E)$ and $R = (\pi \times \pi)^{-1}(R')$, where $xR'x' \iff P_x \cap P_{x'} = \emptyset$. Note that R' is Borel by Lusin-Novikov, and hence so is R. Also, $\mathbb{H}_0^\omega \subseteq F$ and $F \cap R = \emptyset$.

We claim that R is comeager. To see this, fix $x \in X_\alpha$ and consider

$$R_x^c = \{x' \in X_\alpha : P_{\pi(x)} \cap P_{\pi(x')} \neq \emptyset\}.$$

Fix an enumeration y_n of $P_{\pi(x)}$, and let $A_n = \{x' \in X_\alpha : y_n \in P_{\pi(x')}\}$. Then each A_n is $\mathbb{G}_0^\omega$-independent and hence meager; thus so is $R_x^c = \bigcup_n A_n$. Thus R_x is comeager for all $x \in X_\alpha$, and by Kuratowski-Ulam R is comeager.

One can now recursively construct a continuous homomorphism $f : 2^\omega \to X_\alpha$ from $(\Delta(2^\omega)^c, \mathbb{E}_0^c, \mathbb{E}_0)$ to $((\pi \times \pi)^{-1}(\Delta(X))^c, R, F)$, see e.g. [Milb, Proposition 11]. Then $\pi \circ f$ satisfies (4).

(B) We note the following effective version of Theorem 3.1.8:

Theorem 3.4.14. *Let E be a Δ_1^1 equivalence relation on $\mathbb{N}^\mathbb{N}$ and $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ an E-invariant Δ_1^1 relation with countable non-empty sections. Then exactly one of the following holds:*

1. *There is a Δ_1^1 E-invariant uniformization,*

2. *There is a continuous embedding $\pi_X \colon 2^\mathbb{N} \times \mathbb{N} \to X$ of $\mathbb{E}_0 \times I_\mathbb{N}$ into E and a continuous injection $\pi_Y \colon 2^\mathbb{N} \times \mathbb{N} \to Y$ such that for all $x, x' \in 2^\mathbb{N} \times \mathbb{N}$,*

$$\neg(x \; \mathbb{E}_0 \times I_\mathbb{N} \; x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset$$

and

$$P_{\pi_X(x)} = \pi_Y([x]_{\mathbb{E}_0 \times I_\mathbb{N}}).$$

This follows from the above proof, Theorem 3.4.12, and the fact that the proof of Theorem 3.4.3 is effective.

3.4.4 Proof of Theorem 3.1.10

(A) Note first that (1) is equivalent to the existence of a smooth Borel equivalence $F \supseteq E$ for which P is F-invariant, by Theorem 3.2.3.

To see that these are mutually exclusive, let $F \supseteq E$ be smooth so that P is F-invariant, and suppose that π_X, π_Y witness (2). Then there is a comeagre $\mathbb{E}_0$-invariant set C

that π_X maps into a single F-class, so $\pi_Y(C)$ is contained in a single P-section, a contradiction.

Now define the graph $xGx' \iff P_x \neq P_{x'}$. This graph is Borel by Lusin-Novikov. Apply the $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy to (G, E).

Case 1: There is a smooth $F \supseteq E$ such that G admits a countable Borel F-local colouring. If A is analytic and F-locally G-independent, then so is $[A]_E$, so by repeated applications of the first reflection theorem it is contained in a Borel E-invariant F-locally G-independent set. Thus we may assume that G admits a countable Borel E-invariant F-local colouring, and hence by refining F with this colouring we may assume that actually $G \cap F = \emptyset$, i.e., P is F-invariant. Thus (1) holds.

Case 2: There is a continuous homomorphism $\varphi : 2^{\mathbb{N}} \to X$ of $(\mathbb{G}_0, \mathbb{H}_0)$ to (G, E). Define $R(x, y) \iff P(\varphi(x), y)$, and let

$$Q(x, y) \iff R(x, y) \;\&\; \forall^* x' \neg R(x', y),$$

where $\forall^* x A(x)$ means A is comeager for $A \subseteq 2^{\mathbb{N}}$. Let $A = \mathrm{proj}(Q)$ and $xSx' \iff Q_x \cap Q_{x'} \neq \emptyset$. Then R is Borel with countable sections, and it follows that Q, A, S are Borel as well. Additionally, R, Q are $\mathbb{E}_0$-invariant.

We claim that A is comeager and S is meager. Granted this, we can find a continuous homomorphism $\psi : 2^{\mathbb{N}} \to A$ of $(\mathbb{E}_0, \mathbb{E}_0^c)$ to $(\mathbb{E}_0', S^c)$ such that $\varphi \circ \psi$ is injective, where $\mathbb{E}_0'$ is the smallest equivalence relation containing $\mathbb{H}_0$ (cf. [Milb, Proposition 11]). Now the set $Q'(x, y) \iff Q(\psi(x), y)$ has countable sections, so it admits a Borel uniformization g. Since ψ is a homomorphism from $\mathbb{E}_0^c$ to S^c, g is countable-to-one, so by Lusin-Novikov it is injective and continuous on a non-meager set B. Let τ be a continuous embedding of $\mathbb{E}_0$ into $\mathbb{E}_0 \restriction B$. Then $\pi_X = \varphi \circ \psi \circ \tau, \pi_Y = g \circ \tau$ satisfy (2).

Now suppose that A is comeager, in order to show that S is meager. By Kuratowski-Ulam, it suffices to show that S_x is meager for all $x \in A$. So consider $x \in A$, and let $y \in Q_x$ be arbitrary. Then $y \notin R_{x'}$ for comeagerly-many x', and so $y \notin Q_{x'}$ for comeagerly-many x'. Since Q_x is countable, it follows that $Q_x \cap Q_{x'} = \emptyset$ for comeagerly-many x', and so S_x is meager.

It remains to show that A is comeager. To see this, define $xBx' \iff R_x \subseteq R_{x'}$. For any x, $B_x = \bigcap_{y \in R_x} \{x' : R(x', y)\}$, so if B_x is meager then there is some $y \in R_x$ for which $\{x' : R(x', y)\}$ is not comeager. But this set is Borel and $\mathbb{E}_0$-invariant, so it is meager, and hence $y \in Q_x$ and $x \in A$. Thus by Kuratowski-Ulam it suffices to show that B is meager.

Suppose for the sake of contradiction that B is non-meager. Let C be the set of all x so that B_x is non-meager. Since B_x is $\mathbb{E}_0$-invariant, it must be comeager for all $x \in C$. Moreover C is non-meager and $\mathbb{E}_0$-invariant, and therefore it is comeager. It follows that B is comeager, and hence so is $B'(x, x') \iff B(x, x')\ \&\ B(x', x)$. In particular, $B'_x = C$ is comeager for some x. But then $x, x' \in C \implies R_x = R_{x'}$; thus C is $\mathbb{G}_0$-independent, a contradiction.

Remark 3.4.15. This proof actually shows that in case (2), we can take π_X, π_Y so that additionally $\pi_Y(x) \in P_{\pi_X(x')} \iff x\mathbb{E}_0 x'$.

(B) This proof can also be made effective, by Theorem 3.4.12:

Theorem 3.4.16. *Let E be a Δ^1_1 equivalence relation on X and $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ an E-invariant Δ^1_1 relation with countable non-empty sections. Then exactly one of the following holds:*

(1) There is a uniformly Δ^1_1 sequence $g_n : X \to Y$ of E-invariant uniformizations with $P = \bigcup_n \operatorname{graph}(g_n)$,

(2) There is a continuous embedding $\pi_X : 2^\mathbb{N} \to X$ of $\mathbb{E}_0$ into E and a continuous injection $\pi_Y : 2^\mathbb{N} \to Y$ such that for all $x \in 2^\mathbb{N}$, $P(\pi_X(x), \pi_Y(x))$.

3.4.5 Proofs of Proposition 3.1.11 and Theorem 3.1.12

Let us fix a parametrization of the Borel relations on $\mathbb{N}^\mathbb{N}$, as in [AK00, Section 5] (see also [Mos09, Section 3H]). This consists of a set $D \subseteq 2^\mathbb{N}$ and two sets $S, P \subseteq (\mathbb{N}^\mathbb{N})^3$ such that

(i) D is Π^1_1, S is Σ^1_1 and P is Π^1_1;

(ii) for $d \in D$, $S_d = P_d$, and we denote this set by D_d;

(iii) every Borel set in $(\mathbb{N}^\mathbb{N})^2$ appears as D_d for some $d \in D$; and

(iv) if $B \subseteq X \times (\mathbb{N}^\mathbb{N})^2$ is Borel, X a Polish space, there is a Borel function $p : X \to 2^\mathbb{N}$ so that $B_x = D_{p(x)}$ for all $x \in X$.

Define

$$\mathcal{P} = \{(d, e) : D_d \text{ is an equivalence relation on } \mathbb{N}^\mathbb{N} \text{ and } D_e \text{ is } D_d\text{-invariant}\},$$

and let $\mathcal{P}^{unif}$ denote the set of pairs $(d, e) \in \mathcal{P}$ for which D_e admits a D_d-invariant uniformization. More generally, for any set A of properties of sets $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, let $\mathcal{P}_A$ (resp. $\mathcal{P}^{unif}_A$) denote the set of pairs (d, e) in $\mathcal{P}$ (resp. $\mathcal{P}^{unif}$) such that D_e satisfies all of the properties in A. Let $\mathcal{P}_{ctble}$ (resp. $\mathcal{P}^{unif}_{ctble}$) denote $\mathcal{P}_A$ (resp. $\mathcal{P}^{unif}_A$) for A consisting of the property that P has countable sections.

We are interested in properties asserting that D_e, or its sections, are G_δ, F_σ, comeager, non-meager, μ-positive, μ-conull, countable, or K_σ, where μ varies over probability Borel measures on $\mathbb{N}^{\mathbb{N}}$. It is straightforward to check, using [Kec95, 16.1, 17.25, 18.9, 35.47], that for all such sets of properties A, $\mathcal{P}_A$ is $\mathbf{\Pi^1_1}$ and $\mathcal{P}^{unif}_A$ is $\mathbf{\Sigma^1_2}$.

By Theorem 3.4.14, we can bound the complexity of $\mathcal{P}^{unif}_{ctble}$:

Proposition 3.4.17 (Proposition 3.1.11)**.** *The set $\mathcal{P}^{unif}_{ctble}$ is $\mathbf{\Pi^1_1}$.*

Proof. By Theorem 3.4.14, $(d, e) \in \mathcal{P}^{unif}_{ctble}$ iff $(d, e) \in \mathcal{P}_{ctble}$ and there exists a $\Delta^1_1(d, e)$ function f which is a D_d-invariant uniformization of D_e. The assertion that a $\Delta^1_1(d, e)$ function f is a D_d-invariant uniformization of D_e is $\Pi^1_1(d, e)$, so by bounded quantification for Δ^1_1 [Mos09, 4D.3], $\mathcal{P}^{unif}_{ctble}$ is Π^1_1. $\square$

Recall that a set B in a Polish space X is called $\mathbf{\Sigma^1_2}$-complete if it is $\mathbf{\Sigma^1_2}$, and for all zero-dimensional Polish spaces Y and $\mathbf{\Sigma^1_2}$ sets $C \subseteq Y$ there is a continuous function $f : Y \to X$ such that $C = f^{-1}(B)$. Note that by [Sab12, Theorem 2], one could equivalently take f to be Borel in this definition (see also [Paw15]).

The following computes the exact complexity of the sets $\mathcal{P}^{unif}_A$, when A asserts that D_e has "large" sections.

Theorem 3.4.18 (Theorem 3.1.12)**.** *The set $\mathcal{P}^{unif}_A$ is $\mathbf{\Sigma^1_2}$-complete, where A is one of the following sets of properties of $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$:*

1. *P has non-meager sections;*
2. *P has non-meager G_δ sections;*
3. *P has non-meager sections and is G_δ;*
4. *P has μ-positive sections for some probability Borel measure μ on $\mathbb{N}^{\mathbb{N}}$;*
5. *P has μ-positive F_σ sections for some probability Borel measure μ on $\mathbb{N}^{\mathbb{N}}$;*
6. *P has μ-positive sections for some probability Borel measure μ on $\mathbb{N}^{\mathbb{N}}$ and is F_σ.*

The same holds for comeager instead of non-meager, and μ-conull instead of μ-positive.

In fact, there is a hyperfinite Borel equivalence relation E with code $d \in D$ such that for all such A above, the set of $e \in D$ such that $(d, e) \in \mathcal{P}^{unif}_A$ is $\mathbf{\Sigma^1_2}$-complete.

Proof. We will show this first when A asserts that P is G_δ and has comeager sections. Since $\mathbb{N}^\mathbb{N}$ is Borel isomorphic to $\mathbb{N}^\mathbb{N} \times 2^\mathbb{N}$, we may assume that D_d is instead an equivalence relation on $\mathbb{N}^\mathbb{N} \times 2^\mathbb{N}$, and that $D_e \subseteq (\mathbb{N}^\mathbb{N} \times 2^\mathbb{N}) \times \mathbb{N}^\mathbb{N}$.

Let E be the hyperfinite Borel equivalence relation on $\mathbb{N}^\mathbb{N} \times 2^\mathbb{N}$ given by

$$(x, y)E(x', y') \iff x = x' \ \& \ y\mathbb{E}_0 y',$$

fix a code $d \in D$ for E, and let $\mathcal{P}^{unif}_A(E)$ denote the set of all $e \in D$ so that $(d, e) \in \mathcal{P}^{unif}_A$. We will show that $\mathcal{P}^{unif}_A(E)$ is $\mathbf{\Sigma^1_2}$-complete.

Let now T be a tree on $\mathbb{N} \times \mathbb{N}$ (cf. [Kec95, 2.C]). Each such tree T defines a closed subset $[T] \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ given by

$$[T] = \{(x, y) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} : \forall n\, ((x{\restriction}n, y{\restriction}n) \in T)\}.$$

We say $[T]$ admits a **full Borel uniformization** if there is a Borel map $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ so that $(x, f(x)) \in [T]$ for all $x \in \mathbb{N}^\mathbb{N}$, and we denote by FBU the set of trees on $\mathbb{N} \times \mathbb{N}$ which admit full Borel uniformizations.

By the proof of Theorem 3.1.5, and considering $\mathbb{N}^\mathbb{N}$ as a co-countable set in $2^\mathbb{N}$, there is a G_δ set $P \subseteq 2^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ with comeager sections which is $\mathbb{E}_0$ invariant, and so that

$$\bigcap_{x \in C} P_x = \emptyset$$

whenever $C \subseteq 2^\mathbb{N}$ is μ-positive, where μ is the uniform product measure on $2^\mathbb{N}$. Given a tree T on $\mathbb{N} \times \mathbb{N}$, define $P_T \subseteq (\mathbb{N}^\mathbb{N} \times 2^\mathbb{N}) \times \mathbb{N}^\mathbb{N}$ by

$$P_T(x, y, z) \iff P(y, z) \vee (x, z) \in [T].$$

Note that P_T is G_δ, E-invariant, and has comeager sections.

Claim 3.4.19. $[T]$ admits a full Borel uniformization iff P_T admits a Borel E-invariant uniformization.

Proof. If f is a full Borel uniformization of $[T]$, then $g(x, y) = f(x)$ is an E-invariant Borel uniformization of P_T. Conversely, suppose g were an E-invariant Borel uniformization of P_T. For $x \in \mathbb{N}^\mathbb{N}$, let $g_x(y) = g(x, y)$. Then $g_x : 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is $\mathbb{E}_0$-invariant, hence constant on a μ-conull set $C \subseteq 2^{\mathbb{N}}$. Since

$$\bigcap_{y \in C} P_y = \emptyset,$$

we cannot have $P(y, g_x(y))$ for all $y \in C$, and so $(x, g_x(y)) \in [T]$ for all $y \in C$. Thus

$$f(x) = z \iff \forall^*_\mu y (g(x, y) = z)$$

is a full Borel uniformization of $[T]$ (cf. [Kec95, 17.26] and the paragraphs following it). $\square$

By identifying trees on $\mathbb{N} \times \mathbb{N}$ with their characteristic functions, we can view the space of trees as a closed subset of $2^{\mathbb{N}}$. The set B given by

$$B(T, x, y, z) \iff T \text{ is a tree and } P_T(x, y, z)$$

is clearly Borel, so there is a Borel map p such that for each tree T, $p(T) \in D$ and $D_{p(T)} = P_T$. It follows by Claim 3.4.19 that FBU $= p^{-1}(\mathcal{P}^{unif}_A(E))$. By [AK00, Lemma 5.3], the set FBU is $\mathbf{\Sigma}^1_2$-complete, and hence so is $\mathcal{P}^{unif}_A(E)$.

The cases 1–3 follow from this as well. For 4–6, simply replace P in the above proof with an F_σ set $Q \subseteq 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with μ-conull sections which is $\mathbb{E}_0$-invariant, and so that

$$\bigcap_{x \in C} Q_x = \emptyset$$

whenever $C \subseteq 2^{\mathbb{N}}$ is non-meager, which exists by the proof of Theorem 3.1.5. $\square$

Remark 3.4.20. We do not know the complexity of $\mathcal{P}^{unif}_A$ when A asserts that P is G_δ and has comeager μ-conull sections for a probability Borel measure μ. By the proof of Theorem 3.1.6, there is an $\mathbb{E}_0$-invariant G_δ set $R \subseteq [\mathbb{N}]^{\aleph_0} \times \mathbb{N}^{\mathbb{N}}$ with comeager μ-conull sections, such that

$$\bigcap_{x \in C} P_x = \emptyset$$

for all Ramsey-positive sets $C \subseteq [\mathbb{N}]^{\aleph_0}$. One can define P_T for a tree T on $\mathbb{N} \times \mathbb{N}$ as in the proof of Theorem 3.1.12, however the "if" direction of our proof of Claim 3.4.19 no longer works (cf. [Sab12]).

3.4.6 Proof of Proposition 3.1.14

By [Kec95, 18.17], there is a G_δ set $R \subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $\mathrm{proj}_{\mathbb{N}^{\mathbb{N}}}(R) = \mathbb{N}^{\mathbb{N}}$ which does not admit a Borel uniformization. Write $R = \bigcap_n Q_n$, $Q_n \subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ open, and define P by

$$P(n, x, y) \iff Q_n(x, y).$$

Let $(n, x)F(m, x') \iff x = x'$. Then F is a smooth countable Borel equivalence relation, P is open, and if $C = [(n, x)]_F$ is an F-class then

$$\bigcap_{u \in C} P_u = \bigcap_n P_{(n,x)} = \bigcap_n (Q_n)_x = R_x \neq \emptyset.$$

Suppose now towards a contradiction that $g : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ is an F-invariant uniformization of P. Define $f : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $f(x) = g(0, x)$. Then $f(x) = g(0, x) = g(n, x) \in P_{(n,x)}$ for all n, so $f(x) \in \bigcap_n P_{(n,x)} = R_x$, a contradiction.

3.5 On Conjecture 3.1.15

Concerning Conjecture 3.1.15, we first note the following analog of Lemma 3.2.1.

Lemma 3.5.1. *Let E, F be Borel equivalence relations on Polish spaces X, X', resp., such that $E \leq_B E'$. If E fails (b) (resp., (c), (d)), so does E'.*

The proof is identical to that of Lemma 3.2.1. Note now that any countable Borel equivalence relation E trivially satisfies (b), (c), and (d), so by Lemma 3.5.1, in Conjecture 3.1.15, (a) implies (b), (c), and (d).

To verify then Conjecture 3.1.15, one needs to show that if E is not reducible to countable, then (b), (c), and (d) fail. It is an open problem (see [HK01, end of Section 6]) whether the following holds:

Problem 3.5.2. Let E be a Borel equivalence relation which is not reducible to countable. Then one of the following holds:

(1) $\mathbb{E}_1 \leq_B E$, where $\mathbb{E}_1$ is the following equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$:

$$x \mathbb{E}_1 y \iff \exists m \forall n \geq m (x_n = y_n);$$

(2) There is a Borel equivalence relation F induced by a turbulent continuous action of a Polish group on a Polish space such that $F \leq_B E$;

(3) $\mathbb{E}_0^{\mathbb{N}} \leq_B E$, where $\mathbb{E}_0^{\mathbb{N}}$ is the following equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$:

$$x \mathbb{E}_0^{\mathbb{N}} y \iff \forall n (x_n \mathbb{E}_0 y_n).$$

It is therefore interesting to show that (b), (c), and (d) fail for $\mathbb{E}_1$, F as in (2) above, and $\mathbb{E}_0^{\mathbb{N}}$. Here are some partial results.

Proposition 3.5.3. *Let E be a Borel equivalence relation which is not reducible to countable but is Borel reducible to a Borel equivalence relation F with K_σ classes. Then E fails (d). In particular, $\mathbb{E}_1$ and $\mathbb{E}_2$ fail (d), where $\mathbb{E}_2$ is the following equivalence relation on $2^{\mathbb{N}}$:*

$$x \mathbb{E}_2 y \iff \sum_{n : x_n \neq y_n} \frac{1}{n+1} < \infty.$$

Proof. Suppose E, F live on the Polish spaces X, Y, resp., and let $g \colon X \to Y$ be a Borel reduction of E to F. Define $P \subseteq X \times X$ as follows:

$$(x, y) \in P \iff g(x) F y.$$

Clearly P is E-invariant and has K_σ sections. Suppose then that P admitted a Borel E-invariant countable uniformization $f \colon X \to Y^{\mathbb{N}}$. Then define $h \colon X \to X$ by $g(x) = f(x)_0$. Then by [Kec25, Proposition 3.7], h shows that E is reducible to countable, a contradiction. □

Concerning (b) and (c) for $\mathbb{E}_1$, the following is a possible example for their failure.

Problem 3.5.4. Let $X = (2^{\mathbb{N}})^{\mathbb{N}}, Y = 2^{\mathbb{N}}$ and define $P \subseteq X \times Y$ as follows:

$$(x, y) \in P \iff \exists m \forall n \geq m (x_n \neq y),$$

so that P is $\mathbb{E}_1$-invariant and each section P_x is co-countable, so has μ-measure 1 (for μ the product measure on Y) and is comeager. Is there a Borel $\mathbb{E}_1$-invariant countable uniformization of P?

One can show the following weaker result, which provides a Borel anti-diagonalization theorem for $\mathbb{E}_1$.

Proposition 3.5.5. *Let $f \colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel function such that $x \mathbb{E}_1 y \implies f(x) = f(y)$. Then there is $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that for infinitely many n, $f(x) = x_n$.*

Thus if X, Y, P are as in Problem 3.5.4, P does not admit a Borel $\mathbb{E}_1$-invariant uniformization.

Proof. For any nonempty countable set $S \subseteq 2^{\mathbb{N}}$ consider the product space $S^{\mathbb{N}}$ with the product topology, where S is taken to be discrete. Denote by $\mathbb{E}_0(S)$ the equivalence relation on $S^{\mathbb{N}}$ given by $x \mathbb{E}_0(S) y \iff \exists m \forall n \geq m (x_n = y_n)$. This is generically ergodic and for $x, y \in S^{\mathbb{N}}$ we have that $x \mathbb{E}_0(S) y \implies f(x) = f(y)$, so there is (unique)

$x_S \in 2^{\mathbb{N}}$ such that $f(x) = x_S$, for comeager many $x \in S^{\mathbb{N}}$. Clearly x_S can be computed in a Borel way given any $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ with $S = \{x_n : n \in \mathbb{N}\}$, i.e., we have a Borel function $F\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that

$$\{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\} = S \implies F((x_n)) = F((y_n)) = x_S.$$

We now use the following Borel anti-diagonalization theorem of H. Friedman, see [Sta85, Theorem 2, page 23]:

Theorem 3.5.6 (H. Friedman)**.** *Let E be a Borel (even analytic) equivalence relation on a Polish space X. Let $F\colon X^{\mathbb{N}} \to X$ be a Borel function such that*

$$\{[x_n]_E : n \in \mathbb{N}\} = \{[y_n]_E : n \in \mathbb{N}\} \implies F((x_n))\ E\ F((y_n)).$$

Then there is $x \in X^{\mathbb{N}}$ and $i \in \mathbb{N}$ such that $F(x) E x_i$.

Applying this to E being the equality relation on $2^{\mathbb{N}}$ and F as above, we conclude that for some S, we have that $x_S \in S$. Then for comeager many $x \in S^{\mathbb{N}}$ we have that $x_n = x_S$, for infinitely many n, and also $(x, x_S) \in P$, a contradiction. □

In response to a question by Andrew Marks, we note the following version of Proposition 3.5.5 for $\mathbb{E}_1$ restricted to injective sequences. Below $[2^{\mathbb{N}}]^{\mathbb{N}}$ is the Borel subset of $(2^{\mathbb{N}})^{\mathbb{N}}$ consisting of injective sequences and $x \leq_T y$ means that x is recursive in y.

Proposition 3.5.7. *Let $g\colon [2^{\mathbb{N}}]^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel function such that $x \mathbb{E}_1 y \implies g(x) = g(y)$. Then there is $y \in [2^{\mathbb{N}}]^{\mathbb{N}}$ such that for all n, $g(y) \leq_T y_n$.*

Proof. Fix a recursive bijection $x \mapsto \langle x \rangle$ from $(2^{\mathbb{N}})^{\mathbb{N}}$ to $2^{\mathbb{N}}$ and for each $i \in \mathbb{N}$ let $\bar{i} \in 2^{\mathbb{N}}$ be the characteristic function of $\{i\}$. Then for each $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $i \in \mathbb{N}$, put

$$\bar{x}^i = \langle \bar{i}, x_i, x_{i+1}, \dots \rangle \in 2^{\mathbb{N}}$$

and

$$x' = \langle \bar{x}^0, \bar{x}^1, \dots \rangle \in [2^{\mathbb{N}}]^{\mathbb{N}}.$$

Note that $x \mathbb{E}_1 y \implies x' \mathbb{E}_1 y'$. Finally define $f\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $f(x) = g(x')$. Then by Proposition 3.5.5, there is $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that for infinitely many n we have that $f(x) = x_n$. Let $y = x'$.

If n is such that $f(x) = g(y) = x_n$, then as $x_n \leq_T \bar{x}^k = y_k, \forall k \leq n$, we have that $g(y) \leq_T y_k, \forall k \leq n$. Since this happens for infinitely many n, we have that $g(y) \leq_T y_n$, for all n. □

We do not know anything about $\mathbb{E}_0^{\mathbb{N}}$ but if we let $\mathbb{E}_{ctble}$ be the equivalence relation $\mathbb{E}_{ctble}^{2^{\mathbb{N}}}$ (so that $\mathbb{E}_0^{\mathbb{N}} <_B \mathbb{E}_{ctble}$), we have:

Proposition 3.5.8. *$\mathbb{E}_{ctble}$ fails (b) and (c).*

Proof. We will prove that $\mathbb{E}_{ctble}$ fails (b), the proof that it also fails (c) being similar. Let $X = (2^{\mathbb{N}})^{\mathbb{N}}, Y = 2^{\mathbb{N}}$, let μ be the usual product measure on Y and put $E = \mathbb{E}_{ctble}$. Define $P \subseteq X \times Y$ by

$$(x, y) \in P \iff y \notin \{x_n : n \in \mathbb{N}\}.$$

Clearly $\mu(P_x) = 1$ and P is E-invariant. Assume now, towards a contradiction, that there is a Borel function $f : X \to Y^{\mathbb{N}}$ such that $\forall x \in X \forall n \in \mathbb{N}((x, f(x)_n) \in P)$ and $x_1 E x_2 \implies \{f(x_1)_n : n \in \mathbb{N}\} = \{f(x_2)_n : n \in \mathbb{N}\}$. Then

$$\forall x \in X\Big(\{f(x)_n : n \in \mathbb{N}\} \cap \{x_n : n \in \mathbb{N}\} = \emptyset\Big).$$

Define $F : X^{\mathbb{N}} \to Y^{\mathbb{N}}$ as follows: Fix a bijection $(i, j) \mapsto \langle i, j\rangle$ from $\mathbb{N}^2$ to $\mathbb{N}$ and for $n \in \mathbb{N}$ put $n = \langle n_0, n_1\rangle$. Given $x \in X^{\mathbb{N}}$, define $x' \in X$ by $x'_n = (x_{n_0})_{n_1}$. Then let $F(x) = f(x')$. First notice that for $x = (x_n), y = (y_n) \in X^{\mathbb{N}}$,

$$\{[x_n]_E : n \in \mathbb{N}\} = \{[y_n]_E : n \in \mathbb{N}\} \implies x' E y' \implies F(x) E F(y).$$

Thus by Theorem 3.5.6, there is some $x \in X^{\mathbb{N}}$ and $i \in \mathbb{N}$ such that $F(x) E x_i$, i.e., $f(x') E x_i$ or $\{f(x')_n : n \in \mathbb{N}\} = \{(x_i)_n : n \in \mathbb{N}\} = \{x'_{\langle i,n\rangle} : n \in \mathbb{N}\}$. Thus $\{f(x')_n : n \in \mathbb{N}\} \cap \{x'_n : n \in \mathbb{N}\} \neq \emptyset$, a contradiction. □

We do not know if $\mathbb{E}_{ctble}$ fails (d). We also do not know anything about equivalence relations induced by turbulent continuous actions of Polish groups on Polish spaces.

Finally, we note that by the dichotomy theorem of Hjorth concerning reducibility to countable (see [Hjo05] or [Kec25, Theorem 3.8]), in order to prove Conjecture 3.1.15 for Borel equivalence relations induced by Borel actions of Polish groups, it would be sufficient to prove it for Borel equivalence relations induced by stormy such actions.

Chapter 4

INVARIANT UNIFORMIZATION OVER QUOTIENTS

Michael S. Wolman

4.1 Introduction

Let X, Y be sets and $P \subseteq X \times Y$. For $x \in X$, we let $P_x = \{y \in Y : (x, y) \in P\}$ denote the **section of** P above x. When P has non-empty sections, we say a function $f : X \to Y$ is a **uniformization** of P if $(x, f(x)) \in P$ for $x \in X$. A **(proper) quasi-uniformization** of P is a set $U \subseteq X \times Y$ such that for all $x \in X$, U_x is a (proper) non-empty finite subset of P_x.

Let now E, F be **Borel equivalence relations** on Polish spaces X, Y, i.e., equivalence relations so that $E \subseteq X^2, F \subseteq Y^2$ are Borel. We say $P \subseteq X/E \times Y/F$ is **(weakly) Borel** if its *lift* $\tilde{P} = \{(x, y) \in X \times Y : ([x]_E, [y]_F) \in P\}$ is Borel, and a function $f : X/E \to Y/F$ is **(strongly) Borel** if its graph is Borel. (We note that in [dRM] these notions are referred to as weakly and strongly Borel. In this paper we simply refer to them as Borel, as there is no ambiguity in what we mean in various contexts.)

We are interested in characterizing exactly when a Borel set $P \subseteq X/E \times Y/F$ with countable sections admits a Borel (proper quasi-)uniformization. For notational convenience, we will always assume that the sections of P are non-empty, though in most cases this is equivalent to the general case (see e.g. [dRM, Theorem 2.12]).

When E, F are the equality relations $\Delta(X), \Delta(Y)$, Borel uniformizations always exist:

Theorem 4.1.1 (Lusin–Novikov)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ be a Borel set with countable non-empty sections. Then P admits a Borel uniformization. Moreover, there is a countable sequence of Borel uniformizations of P whose graphs cover P.*

More generally, this holds whenever E, F are **smooth**, meaning that the quotient spaces $X/E, Y/F$ are standard Borel (c.f. Theorem 3.2.3). When E, F are not smooth, this is no longer the case. Consider for example the *eventual equality* relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$,

$$x \mathbb{E}_0 y \iff \exists n \forall k \geq n (x_k = y_k).$$

This is a non-smooth *countable* Borel equivalence relation, meaning its equivalence classes are all countable. The image of $\mathbb{E}_0 \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ in $2^{\mathbb{N}}/\mathbb{E}_0 \times 2^{\mathbb{N}}$ is Borel, has countable sections, and does not admit a Borel proper quasi-uniformization.

There are a few ways one may consider generalizing the Lusin–Novikov Theorem to quotients, which are all equivalent when E, F are smooth but are not equivalent in general. Along one axis, we may consider either the existence of a single Borel uniformization of P, or a cover of P by graphs of countably many Borel uniformizations. In another direction, one may consider either Borel uniformizations or Borel proper quasi-uniformizations (though we note that these are equivalent when F is smooth).

These questions were studied by de Rancourt and Miller in [dRM], and by Miller in [Mild]. We summarize their results below, in the context of uniformizations over quotients. Before we do so, we need a few definitions.

Let E, F be Borel equivalence relations on Polish spaces X, Y. We let $E \times F$ denote the equivalence $(x, y)(E \times F)(x', y') \iff xEx' \ \& \ yFy'$ on $X \times Y$. For $B \subseteq X$, let $[B]_E = \{x \in X : \exists x' \in B(xEx')\}$ denote the **(E-)saturation of** B, and say B is **E-invariant** if $B = [B]_E$. A **homomorphism** from E to F is a map $f : X \to Y$ such that $xEx' \implies f(x)Ff(x')$, a **reduction** from E to F is a map $f : X \to Y$ such that $xEx' \iff f(x)Ff(x')$, and an **embedding** is an injective reduction. For $n \in \mathbb{N}$, we let $\mathbb{F}_n$ denote the equivalence relation on $2^{\mathbb{N}}$ given by

$$x\mathbb{F}_n y \iff \exists n \forall k \geq n \left(\sum_{i<k} x_i \equiv \sum_{i<k} y_i \pmod{n} \right).$$

We let $I(X) = X \times X$ be the trivial equivalence relation on any set X and $\Delta(X)$ denote the identity on X. Finally, we say a Borel equivalence relation E on a Polish space X is **strongly idealistic** if there is an E-invariant assignment $X \ni x \mapsto \mathcal{I}_x$ of σ-ideals on X to points in X that is **strongly Borel-on-Borel**, meaning that $\{(z, x) : R_{(z,x)} \in \mathcal{I}_x\}$ is Borel for all Borel sets $R \subseteq 2^{\mathbb{N}} \times X^2$.

Theorem 4.1.2 ([dRM, Theorem 4.11])**.** *Let E, F be Borel equivalence relations on Polish spaces X, Y with F strongly idealistic. Suppose that $P \subseteq X/E \times Y/F$ is Borel and has countable non-empty sections. Then exactly one of the following holds:*

1. *There is a countable sequence of Borel quasi-uniformizations of P that cover P.*

2. *There are continuous embeddings $\pi_X : 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E and $\pi_Y : 2^{\mathbb{N}} \to Y$ of $\Delta(2^{\mathbb{N}})$ into F such that $(\pi_X \times \pi_Y)(\mathbb{E}_0) \subseteq \tilde{P}$.*

Theorem 4.1.3 ([dRM, Theorem 2]). *Let E, F be Borel equivalence relations on Polish spaces X, Y with F strongly idealistic. Suppose that $P \subseteq X/E \times Y/F$ is Borel and has countable non-empty sections. Then exactly one of the following holds:*

1. *There is a countable sequence of Borel uniformizations of P whose graphs cover P.*

2. *There are continuous embeddings $\pi_X : 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E and $\pi_Y : 2^{\mathbb{N}} \to Y$ of $\mathbb{F}$ into F such that $(\pi_X \times \pi_Y)(\mathbb{E}_0) \subseteq \tilde{P}$, for some $\mathbb{F} \in \{\Delta(2^{\mathbb{N}})\} \cup \{\mathbb{F}_p : p \text{ prime}\}$.*

This fully characterizes generalization of Lusin–Novikov for quotients when considering covers by (quasi)-uniformizations, in the case where F is strongly idealistic; see also [dRM, Theorem 3.8] for proper quasi-transversals when P has sections of bounded finite cardinality. For the existence of a single quasi-uniformization, Miller has shown the following:

Theorem 4.1.4 ([Mild, Theorem 2.1]). *Let E, F be Borel equivalence relations on Polish spaces X, Y with F countable. Suppose that $P \subseteq X/E \times Y/F$ is Borel and has countable non-empty sections. Then exactly one of the following holds:*

1. *There is a Borel quasi-uniformization of P.*

2. *There are continuous embeddings $\pi_X : 2^{\mathbb{N}} \times \mathbb{N} \to X$ of $\mathbb{E}_0 \times I(\mathbb{N})$ into E and $\pi_Y : 2^{\mathbb{N}} \times \mathbb{N} \to Y$ of $\Delta(2^{\mathbb{N}} \times \mathbb{N})$ into F such that*

$$\tilde{P}_{\pi_X(z)} = [\pi_Y([z]_{\mathbb{E}_0 \times I(\mathbb{N})})]_F$$

for $z \in 2^{\mathbb{N}} \times \mathbb{N}$.

We consider next dichotomies characterizing the canonical obstructions to the existence of a single Borel uniformization. In order to do so, we must first look at Borel cocycles on Borel equivalence relations, and their essential values.

Let E be an equivalence relation on X and Γ be a group. A **cocycle** from E to Γ is a map $\rho : E \to \Gamma$ satisfying

$$\rho(x, y)\rho(y, z) = \rho(x, z)$$

for $xEyEz$. If F is an equivalence relation on Y and $\sigma : F \to \Gamma$ is a cocycle, then a **homomorphism** from ρ to σ is a homomorphism f from E to F such that

$\rho(x, y) = \sigma(f(x), f(y))$. An **embedding** from ρ into σ is a homomorphism from ρ to σ that is also an embedding from E into F.

Given an analytic equivalence relation E on a Polish space X, a countable discrete group Γ, and a Borel cocycle $\rho : E \to \Gamma$, we say that a set $\Lambda \subseteq \Gamma$ is an **essential value** of ρ if $\Lambda \neq \emptyset$ and for every cover of X by Borel sets $(B_n)_{n \in \mathbb{N}}$, there is some n such that $\Lambda \subseteq \rho(E \restriction B_n \setminus \Delta(B_n))$.

We say a sequence $\boldsymbol{\lambda} \in \Lambda^{\mathbb{N}}$ is a **redundant enumeration** of Λ if every element of Λ appears in this sequence infinitely often. For $\boldsymbol{\lambda} \in \Lambda^{\mathbb{N}}$ and $s \in 2^{<\mathbb{N}}$, let $\boldsymbol{\lambda}^s = \prod_{i<|s|} \boldsymbol{\lambda}(i)^{s(i)}$ and define $\wp_{\boldsymbol{\lambda}} : \mathbb{E}_0 \to \Gamma$ by

$$\wp_{\boldsymbol{\lambda}}(s^\frown x, t^\frown x) = \boldsymbol{\lambda}^s (\boldsymbol{\lambda}^t)^{-1}$$

for $s, t \in 2^{<\mathbb{N}}, x \in 2^{\mathbb{N}}$, where $^\frown$ denotes concatenation of sequences. Then $\wp_{\boldsymbol{\lambda}}$ is a Borel cocycle with values in $\langle \Lambda \rangle$, and if $\boldsymbol{\lambda}$ is a redundant enumeration of Λ then it has Λ as an essential value by [Mila, Proposition 1.5].

Miller has characterized the essential values of Borel cocycles into countable discrete groups:

Theorem 4.1.5 ([Mila, Theorem 1])**.** *Let $\Lambda \leq \Gamma$ be countable discrete non-trivial groups, $\boldsymbol{\lambda}$ be a redundant enumeration of Λ, X be a Polish space, E be an analytic equivalence relation on X, and $\rho : E \to \Gamma$ be a Borel cocycle. Then the following are equivalent:*

1. *Λ is an essential value of ρ.*
2. *There is a continuous embedding of $\wp_{\boldsymbol{\lambda}}$ into ρ.*

We return now to the existence of Borel uniformizations. Given $n \in \mathbb{N}$, let S_n denote the group of permutations of n. For $\Lambda \leq S_n$ and $\boldsymbol{\lambda} \in \Lambda^{\mathbb{N}}$, let $\mathbb{E}_{0,\boldsymbol{\lambda}}$ denote the equivalence relation on $2^{\mathbb{N}} \times n$ given by

$$(x, i) \mathbb{E}_{0,\boldsymbol{\lambda}} (y, j) \iff x \mathbb{E}_0 y \ \ \& \ \ \wp_{\boldsymbol{\lambda}}(x, y)(j) = i.$$

We let $\mathbb{E}_{0,\Lambda}$ denote $\mathbb{E}_{0,\boldsymbol{\lambda}}$ for some redundant enumeration of Λ; this does not depend on the choice of $\boldsymbol{\lambda}$ up to isomorphism by [Mila, Theorem 5] (and in fact, it depends only on the conjugacy class of Λ).

The following were proved independently by the author and Miller (personal communication), who pointed out that they follow from Theorem 4.1.5.

Theorem 4.1.6. *Let E, F be Borel equivalence relations on Polish spaces X, Y and fix $n \geq 2$. For any Borel set $P \subseteq X/E \times Y/F$ whose sections have size n, exactly one of the following holds:*

1. *There is a Borel uniformization of P.*

2. *There is a continuous reduction $\pi_X : 2^{\mathbb{N}} \times n \to X$ of $\mathbb{E}_0 \times I(n)$ to E and a continuous embedding $\pi_Y : 2^{\mathbb{N}} \times n \to Y$ of $\mathbb{E}_{0,\Lambda}$ into F such that $(\pi_X \times \pi_Y)(\mathbb{E}_0 \times I(n)) \subseteq \tilde{P}$, for some minimal fixed-point-free $\Lambda \leq S_n$.*

Theorem 4.1.7. *Let E, F be Borel equivalence relations on Polish spaces X, Y with F strongly idealistic and fix $n \geq 2$. For any Borel set $P \subseteq X/E \times Y/F$ whose sections are non-empty and have size $\leq n$, exactly one of the following holds:*

1. *There is a Borel uniformization of P.*

2. *There are minimal fixed-point-free $\Lambda \leq S_k$ for $k \leq n$, a continuous reduction $\pi_X : 2^{\mathbb{N}} \times k \to X$ of $\mathbb{E}_0 \times I(k)$ to E and a continuous embedding $\pi_Y : 2^{\mathbb{N}} \times k \to Y$ of $\mathbb{E}_{0,\Lambda}$ into F such that*

$$\tilde{P}_{\pi_X(z)} = [\pi_Y([z]_{\mathbb{E}_0 \times I(k)})]_F$$

   for $z \in 2^{\mathbb{N}} \times k$.

Theorem 4.1.8. *Let E, F be Borel equivalence relations on Polish spaces X, Y and fix $n \geq 2$. For any Borel set $P \subseteq X/E \times Y/F$ whose sections have size n, exactly one of the following holds:*

1. *There is a Borel proper quasi-uniformization of P.*

2. *There is a continuous reduction $\pi_X : 2^{\mathbb{N}} \times n \to X$ of $\mathbb{E}_0 \times I(n)$ to E and a continuous embedding $\pi_Y : 2^{\mathbb{N}} \times n \to Y$ of $\mathbb{E}_{0,\Lambda}$ into F such that $(\pi_X \times \pi_Y)(\mathbb{E}_0 \times I(n)) \subseteq \tilde{P}$, for some minimal transitive $\Lambda \leq S_n$.*

Theorem 4.1.9. *Let E, F be Borel equivalence relations on Polish spaces X, Y with F strongly idealistic and fix $n \geq 2$. For any Borel set $P \subseteq X/E \times Y/F$ whose sections are non-empty and have size $\leq n$, exactly one of the following holds:*

1. *There is a Borel proper quasi-uniformization of P.*

2. *There are minimal transitive* $\Lambda \leq S_k$ *for* $k \leq n$*, a continuous reduction* $\pi_X : 2^{\mathbb{N}} \times k \to X$ *of* $\mathbb{E}_0 \times I(k)$ *to* E *and a continuous embedding* $\pi_Y : 2^{\mathbb{N}} \times k \to Y$ *of* $\mathbb{E}_{0,\Lambda}$ *into* F *such that*
$$\tilde{P}_{\pi_X(z)} = [\pi_Y([z]_{\mathbb{E}_0 \times I(k)})]_F$$
for $z \in 2^{\mathbb{N}} \times k$*.*

For Borel equivalence relations E, F on X, Y and $P \subseteq X/E \times Y/F$ Borel, we say that P has **bounded finite sections** if the cardinality of the sections of P have a finite upper bound, and P has **σ-bounded finite sections** if there are Borel sets $P_n \subseteq P$ with bounded finite sections such that $P = \bigcup_n P_n$. We note the following:

Proposition 4.1.10. *Let* E, F *be Borel equivalence relations on Polish spaces* X, Y*, and let* $P \subseteq X/E \times Y/F$ *be Borel. The following are equivalent:*

1. P *has* σ*-bounded finite sections.*
2. $\tilde{P}$ *can be covered by countably many Borel sets* $\tilde{P}_n \subseteq \tilde{P}$*, not necessarily* $E \times F$*-invariant, for which* $\tilde{P}_n/(E \times F)$ *has bounded finite sections.*

Moreover, if F *is strongly idealistic then this is also equivalent to*

3. P *can be covered by countably many Borel sets* $P_n \subseteq P$ *such that for all* n*, the non-empty sections of* P_n *have the same finite cardinality.*

Note that if P has σ-bounded finite sections and F is strongly idealistic, then by [dRM, Theorem 2.12] there is a Borel quasi-uniformization of P.

In order to characterize the existence of Borel uniformizations when P has σ-bounded finite sections, we extend Miller's characterization of essential values of Borel cocycles to countable products of finite groups.

Given a sequence $\mathbf{\Gamma} = (\mathbf{\Gamma}_i)_{i \in \mathbb{N}}$ of countable groups, let $\tilde{\mathbf{\Gamma}}_k = \prod_{i \leq k} \mathbf{\Gamma}_i$ for $k \leq \infty$. We abuse notation and let proj_i denote the projections $\tilde{\mathbf{\Gamma}}_k \to \tilde{\mathbf{\Gamma}}_i$ for $i \leq k \leq \infty$.

Call a sequence of subgroups $\mathbf{\Lambda}_i \leq \tilde{\mathbf{\Gamma}}_i$ **coherent** if $\mathrm{proj}_i(\mathbf{\Lambda}_k) = \mathbf{\Lambda}_i$ for $i \leq k < \infty$. Given a coherent sequence $\mathbf{\Lambda}$, we say a sequence $\boldsymbol{\lambda}_i \in \mathbf{\Lambda}_i$ is a **redundant enumeration** of $\mathbf{\Lambda}$ if $\{\mathrm{proj}_i(\boldsymbol{\lambda}_k) : k \geq i\}$ is a redundant enumeration of $\mathbf{\Lambda}_i$ for all $i \in \mathbb{N}$. Note that every coherent sequence admits a redundant enumeration.

Suppose that $\boldsymbol{\mathcal{F}}_k$ is a family of subsets of $\boldsymbol{\Gamma}_k$ for $k \in \mathbb{N}$. We let $\tilde{\boldsymbol{\mathcal{F}}}_k$ denote the family of subgroups $\Lambda \leq \tilde{\boldsymbol{\Gamma}}_k$ for which $\mathrm{proj}_{\boldsymbol{\Gamma}_i}(\Lambda)$ contains an element of $\boldsymbol{\mathcal{F}}_i$ for all $i \leq k$.

Suppose now that E is an analytic equivalence relation on a Polish space X, $\boldsymbol{\Gamma}$ is a sequence of countable groups, $\boldsymbol{\mathcal{F}}_i$ is a family of subsets of $\boldsymbol{\Gamma}_i$, and $\rho : E \to \tilde{\boldsymbol{\Gamma}}_\infty$ is a Borel cocycle. We say $\boldsymbol{\mathcal{F}}$ is an **essential value** of ρ if for every cover $(B_{i,k})_{i,k\in\mathbb{N}}$ of X by Borel sets, there are $i, k \in \mathbb{N}$ for which $\mathrm{proj}_{\boldsymbol{\Gamma}_i}(\rho(E \upharpoonright B_{i,k} \setminus \Delta(B_{i,k}))$ contains an element of $\boldsymbol{\mathcal{F}}_i$.

Finally, given a sequence of countable groups $\boldsymbol{\Gamma}$, $\boldsymbol{\lambda}_i \in \tilde{\boldsymbol{\Gamma}}_i$, and a Borel cocycle $\wp : \mathbb{E}_0 \to \tilde{\boldsymbol{\Gamma}}_\infty$, we say $\wp$ is **consistent with $\boldsymbol{\lambda}$** if

$$\mathrm{proj}_k(\wp(0^k{}^\frown 1^\frown x, 0^k{}^\frown 0^\frown x)) = \boldsymbol{\lambda}_k$$

for all $k \in \mathbb{N}, x \in 2^{\mathbb{N}}$.

The following is an analogue of Theorem 4.1.5 for cocycles into countable products of finite groups.

Theorem 4.1.11. *Let $\boldsymbol{\Gamma}$ be a sequence of finite groups and $\boldsymbol{\mathcal{F}}_i$ be a family of subsets of $\boldsymbol{\Gamma}_i$ that is closed under conjugation, for which every set in $\boldsymbol{\mathcal{F}}_i$ contains a non-identity element. Let E be an analytic equivalence relation on a Polish space X and $\rho : E \to \tilde{\boldsymbol{\Gamma}}_\infty$ be a Borel cocycle. Then the following are equivalent:*

1. *The family $\boldsymbol{\mathcal{F}}$ is an essential value for ρ.*
2. *There is a coherent sequence $\boldsymbol{\Lambda}_i \in \tilde{\boldsymbol{\mathcal{F}}}_i$, a redundant enumeration $\boldsymbol{\lambda}$ of $\boldsymbol{\Lambda}$, a Borel cocycle $\wp : \mathbb{E}_0 \to \tilde{\boldsymbol{\Gamma}}_\infty$ consistent with $\boldsymbol{\lambda}$, and a continuous embedding of $\wp$ into ρ.*

Given a sequence $\alpha(n) \geq 2, n \in \mathbb{N}$, let $\boldsymbol{\Gamma}_i^\alpha = S_{\alpha(i)}$. Let also $n(\alpha, i) = \sum_{l \leq i} \alpha(l)$, so that $\tilde{\boldsymbol{\Gamma}}_i^\alpha = \prod_{l \leq i} S_{\alpha(l)} \leq S_{n(\alpha,i)}$ for $i \leq \infty$, and write $n(\alpha, -1) = 0$.

For a cocycle $\wp : \mathbb{E}_0 \to S_\infty$, let $\mathbb{E}_{0,\wp}$ denote the equivalence relation on $2^{\mathbb{N}}$ given by

$$(x, i)\mathbb{E}_{0,\wp}(y, j) \iff x\mathbb{E}_0 y \ \ \& \ \ \rho(x, y)(j) = i.$$

Using Theorem 4.1.11, we prove the following:

Theorem 4.1.12. *Let E, F be Borel equivalence relations on Polish spaces X, Y. Let $P_n \subseteq X/E \times Y/F$ be pairwise-disjoint Borel sets with sections of cardinality exactly $\alpha(n) \geq 2$, and $P = \bigcup_n P_n$. Then exactly one of the following holds:*

1. *There is a Borel uniformization of P.*

2. *There is a coherent sequence of fixed-point-free subgroups $\mathbf{\Lambda}_i \leq \tilde{\mathbf{\Gamma}}^{\alpha}_i$, a redundant enumeration $\boldsymbol{\lambda}$ of $\mathbf{\Lambda}$, a Borel cocycle $\rho : \mathbb{E}_0 \to \tilde{\mathbf{\Gamma}}^{\alpha}_{\infty}$ consistent with $\boldsymbol{\lambda}$, and continuous reductions $\pi_X : 2^{\mathbb{N}} \times \mathbb{N} \to X$ of $\mathbb{E}_0 \times I(\mathbb{N})$ to E and $\pi_Y : 2^{\mathbb{N}} \times \mathbb{N} \to Y$ of $\mathbb{E}_{0,\rho}$ to F, such that:*

   a) *$\tilde{P}_{\pi_X(z)} = [\pi_Y([z]_{\mathbb{E}_0 \times I(\mathbb{N})})]_F$ for $z \in 2^{\mathbb{N}} \times \mathbb{N}$; and*

   b) *π_Y is an embedding when restricted to $\bigcup_{i\in\mathbb{N}} N_{0^i} \times (n(\alpha, i) \setminus n(\alpha, i-1))$.*

We also prove a parametrized version of Theorem 4.1.11, and use this to give the following characterization of the existence of Borel (quasi-)uniformizations in the case of σ-bounded finite index.

Theorem 4.1.13. *Let E, F be Borel equivalence relations on Polish spaces X, Y with F strongly idealistic. For any Borel set $P \subseteq X/E \times Y/F$ with non-empty σ-bounded finite sections, exactly one of the following holds:*

1. *There is a Borel uniformization of P.*

2. *One of the following holds:*

   a) *There are minimal fixed-point-free $\Lambda \leq S_k$ for $k \in \mathbb{N}$, and continuous embeddings $\pi_X : 2^{\mathbb{N}} \times k \to X$ of $\mathbb{E}_0 \times I(k)$ into E and $\pi_Y : 2^{\mathbb{N}} \times k \to Y$ of $\mathbb{E}_{0,\Lambda}$ into F such that*

   $$\tilde{P}_{\pi_X(z)} = [\pi_Y([z]_{\mathbb{E}_0 \times I(k)})]_F$$

   for $z \in 2^{\mathbb{N}} \times k$.

   b) *There is some sequence $\alpha(n) \geq 2$, a coherent sequence of fixed-point-free subgroups $\mathbf{\Lambda}_i \leq \tilde{\mathbf{\Gamma}}^{\alpha}_i$, a redundant enumeration $\boldsymbol{\lambda}$ of $\mathbf{\Lambda}$, a Borel cocycle $\rho : \mathbb{E}_0 \to \tilde{\mathbf{\Gamma}}^{\alpha}_{\infty}$ consistent with $\boldsymbol{\lambda}$, and continuous reductions $\pi_X : 2^{\mathbb{N}} \times \mathbb{N} \to X$ of $\mathbb{E}_0 \times I(\mathbb{N})$ to E and $\pi_Y : 2^{\mathbb{N}} \times \mathbb{N} \to Y$ of $\mathbb{E}_{0,\rho}$ to F, such that:*

      i. *$\tilde{P}_{\pi_X(z)} = [\pi_Y([z]_{\mathbb{E}_0 \times I(\mathbb{N})})]_F$ for $z \in 2^{\mathbb{N}} \times \mathbb{N}$; and*

      ii. *π_Y is an embedding when restricted to $\bigcup_{i\in\mathbb{N}} N_{0^i} \times (n(\alpha, i) \setminus n(\alpha, i-1))$.*

It is unclear whether we can ensure that π_X, π_Y are embeddings in Theorem 4.1.12 (2) or Theorem 4.1.13 (2b).

	Uniformization	Proper quasi-uniformization
One	σ-bounded finite sections	1. All, for F countable [Mild] 2. σ-bounded finite sections
Cover	All [dRM]	All [dRM]

Table 4.1: A summary of Lusin–Novikov dichotomies over quotients.

We summarize what is known about Lusin–Novikov Theorems over quotients in Table 4.1. In this table, we assume F is strongly idealistic, and we show for which sets P there is a dichotomy theorem.

We prove Theorem 4.1.11 in Section 4.2, as well as a parametrized version. We then give proofs of Theorems 4.1.6 to 4.1.9, 4.1.12, and 4.1.13 and Proposition 4.1.10 in Section 4.3.

Remark 4.1.14. We have chosen to state and prove these results only for Borel equivalence relations on Polish spaces. However, we note that our proofs actually show that Theorems 4.1.6, 4.1.8, and 4.1.12 hold more generally in the case where X, Y are Hausdorff spaces, E is analytic, and $\tilde{P} \subseteq X \times Y$ is analytic and $E \times \Delta(Y)$-invariant, assuming that alternative 1 is adequately modified; see e.g. [dRM, Theorem 3.8, Theorem 4.11] and [Mild, Theorem 1].

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4.2 Dichotomies for essential values into pro-finite groups

4.2.1 Technical preliminaries

Let $\mathbf{\Gamma}$ be a sequence of countable groups, $\mathbf{\Lambda}_i \leq \tilde{\mathbf{\Gamma}}_i$ be a coherent sequence of subgroups, and $\boldsymbol{\lambda}$ be a redundant enumeration of $\mathbf{\Lambda}$. For $\gamma \in \tilde{\mathbf{\Gamma}}_\infty$, we let $\gamma\Lambda\gamma^{-1}, \gamma\boldsymbol{\lambda}\gamma^{-1}$ denote the sequences $(\mathrm{proj}_i(\gamma)\mathbf{\Lambda}_i \,\mathrm{proj}_i(\gamma)^{-1})_i$, $(\mathrm{proj}_i(\gamma)\boldsymbol{\lambda}_i \,\mathrm{proj}_i(\gamma)^{-1})_i$. Note that $\gamma\Lambda\gamma^{-1}$ (resp.$\gamma\boldsymbol{\lambda}\gamma^{-1}$) are coherent (resp. a redundant enumeration of $\gamma\Lambda\gamma^{-1}$).

Proposition 4.2.1. *Let $\mathbf{\Gamma}$ be a sequence of countable groups, $\mathbf{\Lambda}_i \leq \tilde{\mathbf{\Gamma}}_i$ be a coherent sequence of subgroups, $\boldsymbol{\lambda}$ be a redundant enumeration of $\mathbf{\Lambda}$, and $\wp : \mathbb{E}_0 \to \tilde{\mathbf{\Gamma}}_\infty$ be a Borel cocycle consistent with $\boldsymbol{\lambda}$.*

Suppose that $D_n \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ are closed and nowhere dense and $R \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is meagre. Then there is some $\gamma \in \tilde{\mathbf{\Gamma}}_\infty$, a Borel cocycle $\tilde{\wp} : \mathbb{E}_0 \to \tilde{\mathbf{\Gamma}}_\infty$ consistent with $\gamma\boldsymbol{\lambda}\gamma^{-1}$, and a continuous embedding $\pi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\tilde{\wp}$ into $\wp$ such that for $x, y \in 2^{\mathbb{N}}$,

$$x(n) \neq y(n) \implies \pi(x) \not{D}_n \pi(y) \ \ \& \ \ x \not{\mathbb{E}}_0 y \implies \pi(x) \not{R} \pi(y).$$

Proof. Let U_n be a decreasing sequence of symmetric dense open sets such that $D_n \cap U_n = R \cap \bigcap_n U_n = \emptyset$.

For all $s \in 2^{\leq\mathbb{N}}$ and $n \leq |s|$, let $z_n(s)$ denote the sequence where we replace the first n elements of s by 0, i.e., $z_n(s)(k) = s(k)$ for $n \leq k < |s|$ and $z_n(s)(k) = 0$ for $k < n$.

For $n, k \in \mathbb{N}$ and $s \in 2^k$, let $\boldsymbol{\lambda}^{0^n \frown s} = \prod_{i<k} \mathrm{proj}_n(\boldsymbol{\lambda}_{n+i})^{s(i)} \in \mathbf{\Lambda}_n \subseteq \tilde{\mathbf{\Gamma}}_n$. Note that there is some ambiguity as to which n is chosen in this definition, as s may begin with 0, though all such choices will be consistent with taking projections. In what follows, which n is chosen will be clear from context.

For all $n, k \in \mathbb{N}, x \in 2^{\mathbb{N}}$ and $u, v \in 2^k$ we have

$$\mathrm{proj}_n(\wp(0^n \frown u \frown x, 0^n \frown v \frown x)) = \boldsymbol{\lambda}^{0^n \frown u}(\boldsymbol{\lambda}^{0^n \frown v})^{-1}. \tag{†}$$

To see this, note that it suffices to show this for $v = 0^k$. Arguing inductively, suppose we have shown this for all $u \in 2^k$ and note that

$$\begin{aligned}\mathrm{proj}_n(\wp(0^n \frown u \frown i \frown x, 0^n \frown 0^k \frown 0 \frown x)) =& \,\mathrm{proj}_n(\wp(0^n \frown u \frown i \frown x, 0^n \frown 0^k \frown i \frown x)) \\ & \cdot \mathrm{proj}_n(\wp(0^n \frown 0^k \frown i \frown x, 0^n \frown 0^k \frown 0 \frown x)) \\ =& \boldsymbol{\lambda}^{0^n \frown u}(\mathrm{proj}_n(\boldsymbol{\lambda}_{n+k}))^i \\ =& \boldsymbol{\lambda}^{0^n \frown u \frown i}\end{aligned}$$

for all $u \in 2^k, i \in 2$, where the second equality follows from our inductive hypothesis and the fact that $\wp$ is consistent with $\boldsymbol{\lambda}$.

We will recursively construct $u_n, t_{i,n} \in 2^{<\mathbb{N}}$, $A_n, V_{n,l} \subseteq 2^{\mathbb{N}}$, $\gamma_n \in \tilde{\mathbf{\Gamma}}_n$, $\delta_n \in \mathbf{\Lambda}_n$, and $\phi_n : 2^n \to 2^{<\mathbb{N}}$ for $i \in 2, n, l \in \mathbb{N}$ so that for all $n \in \mathbb{N}$,

(a) $|t_{0,n}| = |t_{1,n}| > 0$, $t_{0,n} \neq t_{1,n}$, $\phi_{n+1}(s \frown i) = \phi_n(s) \frown u_n \frown t_{i,n}$ for all $i \in 2, s \in 2^n$, and $\phi_0(\emptyset) = \emptyset$ (so in particular $|\phi_n(s)| \geq n$ for $n \in \mathbb{N}$);

(b) $N_{\phi_n(s_0) \frown u_n \frown t_{0,n}} \times N_{\phi_n(s_1) \frown u_n \frown t_{1,n}} \subseteq U_n$ for all $(s_0, s_1) \in 2^n \times 2^n$;

(c) A_n is comeagre in $N_{\phi_n(0^n) \frown u_n}$ and $(V_{n,l})_{l \in \mathbb{N}}$ is a decreasing sequence of dense open sets in $N_{\phi_n(0^n) \frown u_n}$ satisfying $\bigcap_l V_{n,l} \subseteq A_n$;

(d) $N_{\phi_n(0^m \frown s)\frown u_n \frown t_{i,n}} \subseteq V_{m,n-m}$ for all $m \leq n$, $s \in 2^{n-m}$ and $i \in 2$;

(e) $\mathrm{proj}_n(\wp(x, z_n(x))) = \gamma_n$ for all $x \in A_n$;

(f) $\boldsymbol{\lambda}^{z_n(\phi_n(0^n))\frown u_n \frown t_{1,n}}(\boldsymbol{\lambda}^{z_n(\phi_n(0^n))\frown u_n \frown t_{0,n}})^{-1} = \delta_n \boldsymbol{\lambda}_n \delta_n^{-1}$; and

(g) $\mathrm{proj}_{n-1}(\gamma_n \delta_n) = \gamma_{n-1}\delta_{n-1}$ for $n > 0$.

To this end, suppose we have completed the construction below n and define ϕ_n as in (a). Let $\gamma_x = \mathrm{proj}_n(\wp(x, z_n(x)))$ for $x \in 2^{\mathbb{N}}$ and let $A_\gamma = \{x \in 2^{\mathbb{N}} : \gamma = \gamma_x\}$ for $\gamma \in \tilde{\boldsymbol{\Gamma}}_n$. The sets A_γ are Borel and cover $2^{\mathbb{N}}$, so there is some γ_n such that $A_n = A_{\gamma_n}$ is non-meagre in $N_{\phi_n(0^n)}$. Let $u_n \in 2^{<\mathbb{N}}$ be such that A_n is comeagre in $N_{\phi_n(0^n)\frown u_n}$, and fix a decreasing sequence of dense open sets $V_{n,l}$ in $N_{\phi_n(0^n)\frown u_n}$ satisfying (c).

Now recursively construct $t'_{i,n} \in 2^{<\mathbb{N}}$ such that (b) holds with $t'_{i,n}$ replacing $t_{i,n}$. This is possible as U_n is dense and open. We then recursively extend the $t'_{i,n}$ to some $t''_{i,n}$ such that (d) holds with $t''_{i,n}$ replacing $t_{i,n}$. This is possible as the $V_{m,n-m}$ are dense and open in $N_{\phi_m(0^m)\frown u_m}$ and $\phi_n(0^m \frown s)\frown u_n \supseteq \phi_m(0^m)\frown u_m$. We can make the $t''_{i,n}$ longer to ensure that $|t''_{0,n}| = |t''_{1,n}|$.

If $n = 0$, we let $\delta_0 = 1_{\boldsymbol{\Lambda}_0}$. Otherwise, we note that A_{n-1} is comeagre in $N_{\phi_n(0^n)}$, so there is some $x \in A_n \cap A_{n-1}$, which implies that

$$\begin{aligned}\mathrm{proj}_{n-1}(\gamma_n) &= \mathrm{proj}_{n-1}(\wp(x, z_n(x))) \\ &= \mathrm{proj}_{n-1}(\wp(x, z_{n-1}(x))\wp(z_{n-1}(x), z_n(x))) \\ &= \gamma_{n-1}\boldsymbol{\lambda}_{n-1}^{x(n-1)}\end{aligned}$$

by (†), so $\mathrm{proj}_{n-1}(\gamma_n) = \gamma_{n-1}\delta_{n-1}\delta$ for some $\delta \in \boldsymbol{\Lambda}_{n-1}$. Choose $\delta_n \in \boldsymbol{\Lambda}_n$ satisfying $\mathrm{proj}_{n-1}(\delta_n) = \delta^{-1}$, which is possible as $\boldsymbol{\Lambda}$ is a coherent sequence. This ensures that (g) is satisfied.

Finally, let $\lambda = (\boldsymbol{\lambda}^{z_n(\phi_n(0^n))\frown u_n \frown t''_{1,n}})^{-1}\delta_n \boldsymbol{\lambda}_n \delta_n^{-1}\boldsymbol{\lambda}^{z_n(\phi_n(0^n))\frown u_n \frown t''_{0,n}} \in \boldsymbol{\Lambda}_n \subseteq \tilde{\boldsymbol{\Gamma}}_n$. Find m such that $\mathrm{proj}_n(\boldsymbol{\lambda}_{|\phi_n(0^n)|+|u_n|+|t''_{i,n}|+m}) = \lambda$, and let $t_{i,n} = t''_{i,n}\frown 0^m \frown (i)$. Then

$$\begin{aligned}\boldsymbol{\lambda}^{z_n(\phi_n(0^n))\frown u_n \frown t_{1,n}}(\boldsymbol{\lambda}^{z_n(\phi_n(0^n))\frown u_n \frown t_{0,n}})^{-1} &= \boldsymbol{\lambda}^{z_n(\phi_n(0^n))\frown u_n \frown t''_{1,n}}\lambda(\boldsymbol{\lambda}^{z_n(\phi_n(0^n))\frown u_n \frown t''_{0,n}})^{-1} \\ &= \delta_n \boldsymbol{\lambda}_n \delta_n^{-1},\end{aligned}$$

so (f) holds. This completes the recursive construction.

Now define $\pi(x) = \bigcup_n \phi_n(x \restriction n)$ for $x \in 2^{\mathbb{N}}$. By (a), $\pi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous embedding of $\mathbb{E}_0$ into itself. By (b), if $x \not\mathbb{E}_0 y$ then $\pi(x) \not R \pi(y)$, and if $x(n) \neq y(n)$ then

$\pi(x)\not D_i \pi(y)$ for $i \leq n$. We claim that

$$\operatorname{proj}_n(\wp(\pi(0^n{}^\frown 1^\frown x), \pi(0^n{}^\frown 0^\frown x))) = \gamma_n \delta_n \boldsymbol{\lambda}_n \delta_n^{-1} \gamma_n^{-1} \tag{‡}$$

for all $n \in \mathbb{N}, x \in 2^{\mathbb{N}}$. To see this, note that by (d) we have $\pi(0^n{}^\frown i^\frown x) \in V_{n,l}$ for all l, hence by (c) $\pi(0^n{}^\frown i^\frown x) \in A_n$. By (e), $\operatorname{proj}_n(\wp(\pi(0^n{}^\frown i^\frown x), z_n(\pi(0^n{}^\frown i^\frown x)))) = \gamma_n$. Thus it suffices to check that

$$\operatorname{proj}_n(\wp(z_n(\pi(0^n{}^\frown 1^\frown x)), z_n(\pi(0^n{}^\frown 0^\frown x)))) = \delta_n \boldsymbol{\lambda}_n \delta_n^{-1}.$$

To see this, note that there is some $y \in 2^{\mathbb{N}}$ such that $\pi(0^n{}^\frown i^\frown x) = \phi_n(0^n)^\frown u_n^\frown t_{i,n}{}^\frown y$ for $i \in 2$, so

$$\begin{aligned}\operatorname{proj}_n(\wp(z_n(\pi(0^n{}^\frown 1^\frown x)), z_n(\pi(0^n{}^\frown 0^\frown x)))) &= \boldsymbol{\lambda}^{z_n(\phi_n(0^n))^\frown u_n^\frown t_{1,n}} (\boldsymbol{\lambda}^{z_n(\phi_n(0^n))^\frown u_n^\frown t_{0,n}})^{-1} \\ &= \delta_n \boldsymbol{\lambda}_n \delta_n^{-1}\end{aligned}$$

by (f) and (†).

By (g) the sequence $(\gamma_n \delta_n)_{n \in \mathbb{N}}$ is coherent, and therefore corresponds to some $\gamma \in \tilde{\boldsymbol{\Gamma}}_\infty$. If $\tilde{\wp}$ is the pullback of $\wp$ along π, then $\tilde{\wp}$ is consistent with $\gamma \boldsymbol{\lambda} \gamma^{-1}$ by (‡), so π is the desired embedding. □

Proposition 4.2.2. *Let $\boldsymbol{\Gamma}$ be a sequence of countable groups, $\boldsymbol{\Lambda}_i \leq \tilde{\boldsymbol{\Gamma}}_i$ be a coherent sequence of subgroups, $\boldsymbol{\lambda}$ be a redundant enumeration of $\boldsymbol{\Lambda}$, and $\wp : \mathbb{E}_0 \to \tilde{\boldsymbol{\Gamma}}_\infty$ be a Borel cocycle consistent with $\boldsymbol{\lambda}$.*

Suppose $B \subseteq 2^{\mathbb{N}}$ is Baire measurable and non-meagre. Then $\operatorname{proj}_i(\wp(\mathbb{E}_0 \restriction B \setminus \Delta(B)))$ *contains a conjugate of $\boldsymbol{\Lambda}_i$ for all $i \in \mathbb{N}$.*

Proof. Fix $i \in \mathbb{N}$. For all $x \in 2^{\mathbb{N}}$, let $\gamma_x = \operatorname{proj}_i(\wp(x, z_i(x)))$, where $z_i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is the function which replaces the first i elements of x with 0. Let $A_\gamma = \{x \in 2^{\mathbb{N}} : \gamma = \gamma_x\}$ for $\gamma \in \tilde{\boldsymbol{\Gamma}}_i$, and note that these sets are Baire-measurable and cover $2^{\mathbb{N}}$, so there is some γ for which $A_\gamma \cap B$ is non-meagre. Let $s \in 2^{<\mathbb{N}}$ be such that $A_\gamma \cap B$ is comeagre in N_s and $|s| \geq i$, and let $\lambda = \prod_{i \leq n < |s|} \operatorname{proj}_i(\boldsymbol{\lambda}_n)^{s(n)} \in \boldsymbol{\Lambda}_i$. As in the proof of Proposition 4.2.1, we see that for $n \in \mathbb{N}$ and $x \in 2^{\mathbb{N}}$ satisfying $s^\frown 0^n{}^\frown j^\frown x \in A_\gamma$ for $j \in 2$ we have

$$\operatorname{proj}_i(\wp(s^\frown 0^n{}^\frown 1^\frown x, s^\frown 0^n{}^\frown 0^\frown x)) = \gamma \lambda \operatorname{proj}_i(\boldsymbol{\lambda}_{|s|+n}) \lambda^{-1} \gamma^{-1}.$$

Since $\boldsymbol{\lambda}$ is a redundant enumeration of $\boldsymbol{\Lambda}$ and $\lambda \in \boldsymbol{\Lambda}_i$, $\{\gamma \lambda \operatorname{proj}_i(\boldsymbol{\lambda}_{|s|+n}) \lambda^{-1} \gamma^{-1}\}$ is a redundant enumeration of $\gamma \boldsymbol{\Lambda}_i \gamma^{-1}$, and since $A_\gamma \cap B$ is comeagre in N_s we can find for all $n \in \mathbb{N}$ some $x \in 2^{\mathbb{N}}$ so that $s^\frown 0^n{}^\frown j^\frown x \in A_\gamma \cap B$ for $j \in 2$. It follows that $\gamma \boldsymbol{\Lambda}_i \gamma^{-1} \subseteq \operatorname{proj}_i(\wp(\mathbb{E}_0 \restriction B \setminus \Delta(B)))$. □

4.2.2 The $\mathbb{G}_0$ dichotomy for sequences of graphs

We recall now Miller's $\mathbb{G}_0$ dichotomy for sequences of graphs.

We say $S \subseteq 2^{<\mathbb{N}}$ is **sparse** if it contains at most one sequence of every length, and **dense** if for all $t \in 2^{<\mathbb{N}}$ there is some $s \in S$ with $t \subseteq s$. We say a sequence $\boldsymbol{S}$ of subsets of $2^{<\mathbb{N}}$ is **sparse** if its union is sparse, and **dense** if every element of the sequence is dense.

For $s \in 2^{<\mathbb{N}}$, let $\mathbb{G}_s$ be the directed graph

$$\mathbb{G}_s = \{(s^\frown 0^\frown x, s^\frown 1^\frown x) : s \in S \ \ \& \ \ x \in 2^{\mathbb{N}}\}.$$

For $S \subseteq 2^{<\mathbb{N}}$, we write $\mathbb{G}_S = \bigcup_{s\in S} \mathbb{G}_s$, and for a sequence $\boldsymbol{S}$ of subsets of $2^{<\mathbb{N}}$ we let $\mathbb{G}_{\boldsymbol{S}} = (\mathbb{G}_{\boldsymbol{S}_i})_{i\in\mathbb{N}}$.

If $\boldsymbol{G}, \boldsymbol{H}$ are sequences of directed graphs on X, Y, a **homomorphism** from $\boldsymbol{G}$ to $\boldsymbol{H}$ is a map $f : X \to Y$ such that for all $i \in \mathbb{N}$, $x\boldsymbol{G}_i x' \implies f(x)\boldsymbol{H}_i f(x')$. We say a set $B \subseteq X$ is $\boldsymbol{G}$-independent if it is $\boldsymbol{G}_i$-independent for some $i \in \mathbb{N}$. If X is a Polish space, we write $\chi_B(\boldsymbol{G}) \leq \aleph_0$ if there is a cover of X by countably many Borel $\boldsymbol{G}$-independent Borel sets.

Theorem 4.2.3 (The $\mathbb{G}_0$ dichotomy for sequences, [Mil12, Theorem 21])**.** *Let $\boldsymbol{G}$ be a sequence of analytic directed graphs on a Polish space X and $\boldsymbol{S}$ be a sparse dense sequence of subsets of $2^{<\mathbb{N}}$. Then exactly one of the following holds:*

1. $\chi_B(\boldsymbol{G}) \leq \aleph_0$.
2. *There is a continuous homomorphism from $\mathbb{G}_{\boldsymbol{S}}$ to $\boldsymbol{G}$.*

4.2.3 Proof of Theorem 4.1.11

To see that 2 implies 1, fix $\boldsymbol{\Lambda}, \boldsymbol{\lambda}, \wp, \pi$ satisfying 2 and let $B_{i,k}$ be a cover of X by Borel sets. Then there are i, k so that $B = \pi^{-1}(B_{i,k})$ is non-meagre, so by Proposition 4.2.2 there is some $\gamma \in \tilde{\boldsymbol{\Gamma}}_i$ with $\gamma\boldsymbol{\Lambda}_i\gamma^{-1} \subseteq \mathrm{proj}_i(\wp(\mathbb{E}_0 \restriction B \setminus \Delta(B))) = \mathrm{proj}_i(\rho(E \restriction B_{i,k} \setminus \Delta(B_{i,k})))$. It follows that this set contains an element of $\boldsymbol{\mathcal{F}}_i$, as $\boldsymbol{\mathcal{F}}_i$ is closed under conjugation.

We now show that 1 implies 2. Let T be the tree of all sequences of the form $(\mathrm{proj}_0(\Lambda), \ldots, \mathrm{proj}_i(\Lambda))$ for $i \leq n < \mathbb{N}$ and $\Lambda \in \tilde{\boldsymbol{\mathcal{F}}}_n$. Since the groups $\tilde{\boldsymbol{\Gamma}}_i$ are finite, T is finitely branching.

For $n \in \mathbb{N}$ let $T_n = \{t(n) : t \in T \ \ \& \ \ |t| > n\}$ be the n-th level of T, and note that each T_n is finite. Let $(A^n_k)_{k<K_n}$ be an enumeration of all subsets of $\bigcup T_n$ which intersect every element of T_n and let $\boldsymbol{G}_{n,k} = (\mathrm{proj}_n \circ \rho)^{-1}(A^n_k) \setminus \Delta(X)$.

Suppose $B \subseteq X$ is $\boldsymbol{G}_{n,k}$-independent. Then $\mathrm{proj}_n(\rho(E{\upharpoonright}B \setminus \Delta(B)))$ does not contain any element of T_n. By [Mila, Proposition 1.4], we may partition B into countably many Borel sets B_l so that $\mathrm{proj}_n(\rho(E{\upharpoonright}B_l \setminus \Delta(B_l)))$ does not generate a group containing an element of T_n. In particular, for all l there is some i so that $\mathrm{proj}_{\boldsymbol{\Gamma}_i}(\rho(E{\upharpoonright}B_l \setminus \Delta(B_l)))$ does not contain an element of $\boldsymbol{\mathcal{F}}_i$. It follows that if $\chi_B(\boldsymbol{G}) \leq \aleph_0$, then there is a cover $B_{i,k}$ of X so that $\mathrm{proj}_{\boldsymbol{\Gamma}_i}(\rho(E{\upharpoonright}B_{i,k} \setminus \Delta(B_{i,k})))$ does not contain an element of $\boldsymbol{\mathcal{F}}_i$, and in particular $\boldsymbol{\mathcal{F}}$ is not an essential value of ρ.

Fix now a sparse dense sequence $(\boldsymbol{S}_{n,k})_{n,k\in\mathbb{N}}$ of subsets of $2^{<\mathbb{N}}$. By Theorem 4.2.3, we may assume that there is a continuous homomorphism $\phi : 2^{\mathbb{N}} \to X$ from $\mathbb{G}_{\boldsymbol{S}}$ to $\boldsymbol{G}$.

Claim 4.2.4. There is a coherent sequence $\boldsymbol{\Lambda}$ which is a path through T, a redundant enumeration $\boldsymbol{\lambda}$ of $\boldsymbol{\Lambda}$, a Borel cocycle $\wp : \mathbb{E}_0 \to \tilde{\boldsymbol{\Gamma}}_\infty$ consistent with $\boldsymbol{\lambda}$, and a continuous homomorphism $\psi : 2^{\mathbb{N}} \to X$ of $\wp$ to ρ.

We postpone the proof of the claim to the end.

Suppose now that we have $\boldsymbol{\Lambda}, \boldsymbol{\lambda}, \wp, \psi$ as in the claim, and let $D = (\psi \times \psi)^{-1}(\Delta(X))$, $E' = (\psi \times \psi)^{-1}(E)$, and $xF'x' \iff \mathrm{proj}_0(\rho(\psi(x), \psi(x'))) = 1_{\boldsymbol{\Gamma}_0}$. We claim that E' is meagre. Given this, $D \subseteq E'$ is closed and nowhere dense, so by Proposition 4.2.1 there is some $\gamma \in \tilde{\boldsymbol{\Gamma}}_\infty$, a cocycle $\tilde{\wp}$ consistent with $\gamma\boldsymbol{\lambda}\gamma^{-1}$ and a continuous embedding $\pi : \tilde{\wp} \to \wp$ which is moreover a homomorphism from $(\sim\Delta(2^{\mathbb{N}}), \sim\mathbb{E}_0)$ to $(\sim D, \sim E')$. It follows that $\psi \circ \pi$ witnesses alternative 2 of Theorem 4.1.11.

To see that E' is meagre, we first show that every F'-class is meagre. Indeed, let C be an F'-class and note that if $x, y \in C$ and $x\mathbb{E}_0 y$ then $\mathrm{proj}_0(\wp(x, y)) = 1$. By Proposition 4.2.2 and our assumption that $\boldsymbol{\Lambda}_0 \in \tilde{\boldsymbol{\mathcal{F}}}_0$ is non-trivial, C is meagre.

Next, note that every E'-class is a finite union of F'-classes. To see this, let C be an E'-class and let $x_0, \ldots, x_{n-1} \in C$ be a sequence of maximal length such that $\mathrm{proj}_0(\rho(\psi(x_0), \psi(x_i)))$ are distinct for all $i < n$ (such a sequence has length at most $|\boldsymbol{\Gamma}_0|$). If $y \in C$, let $i < n$ be such that $\mathrm{proj}_0(\rho(\psi(x_0), \psi(y))) = \mathrm{proj}_0(\rho(\psi(x_0), \psi(x_i)))$. Then $\mathrm{proj}_0(\rho(\psi(y), \psi(x_i))) = 1$, so $yF'x_i$, and hence $C = [x_0]_{F'} \cup \cdots \cup [x_{n-1}]_{F'}$.

Thus every E'-class is meagre, and hence so is E' by the Kuratowski-Ulam Theorem [Kec95, 8.41].

Finally, it remains to prove the claim.

Proof of claim. For $n \in \mathbb{N}, \lambda \in \tilde{\boldsymbol{\Gamma}}_n, s \in 2^{<\mathbb{N}}$, let

$$C_{\lambda,s} = \{x \in 2^{\mathbb{N}} : \mathrm{proj}_n(\rho(\phi(s^\frown 1^\frown x), \phi(s^\frown 0^\frown x))) = \lambda\}$$

and set

$$S_\lambda = \{s \in 2^{<\mathbb{N}} : C_{\lambda,s} \text{ is non-meagre}\}.$$

For $n \in \mathbb{N}, \Lambda \in T_n, u \in 2^{<\mathbb{N}}$, write

$$A(\Lambda, u) \iff \forall v \in 2^{<\mathbb{N}} \exists^\infty m \geq n \exists \Lambda' \in T_m(\mathrm{proj}_n(\Lambda') = \Lambda \ \& \\ \forall \lambda \in \Lambda' \exists w \in 2^{<\mathbb{N}}(S_\lambda \text{ is dense below } u^\frown v^\frown w)).$$

Here, $\exists^\infty$ means "there exists infinitely many". For all $n \in \mathbb{N}, k < K_n, s \in \boldsymbol{S}_{n,k}$ we have $2^{\mathbb{N}} = \bigcup_{\lambda \in A_k^n} C_{\lambda^{-1},s}$, since ϕ is a homomorphism from $\mathbb{G}_{\boldsymbol{S}_{n,k}}$ to $\boldsymbol{G}_{n,k}$. Note also that if $\lambda \in \tilde{\boldsymbol{\Gamma}}_m$ and $n \leq m$, then $S_\lambda \subseteq S_{\mathrm{proj}_n(\lambda)}$. In particular, if $A(\Lambda, u)$ then S_λ is dense below u for all $\lambda \in \Lambda$. Indeed, for $v \in 2^{<\mathbb{N}}$ we may fix $\Lambda' \in T_m, m \geq n$ so that $\mathrm{proj}_n(\Lambda') = \Lambda$ and $\forall \lambda' \in \Lambda'$ there is some element of $S_{\lambda'}$ below $u^\frown v$, so this holds also for S_λ for all $\lambda \in \Lambda$. Finally, note that $A(\Lambda, u) \implies A(\Lambda, u^\frown v)$ for all Λ and $u, v \in 2^{<\mathbb{N}}$.

Subclaim 4.2.5. Let $n \in \mathbb{N}, \Lambda \in T_n, u \in 2^{<\mathbb{N}}$. Then

$$A(\Lambda, u) \implies \forall v \in 2^{<\mathbb{N}} \exists w \in 2^{<\mathbb{N}} \exists \Lambda' \in T_{n+1}(\mathrm{proj}_n(\Lambda') = \Lambda \ \& \ A(\Lambda', u^\frown v^\frown w)).$$

Also, $A(\Lambda, u)$ holds for some $u \in 2^{<\mathbb{N}}, \Lambda \in T_0$.

Proof. We show the contrapositive. Suppose there is some u_0 extending u such that

$$\forall \Lambda' \in T_{n+1} \forall v \in 2^{<\mathbb{N}}(\mathrm{proj}_n(\Lambda') = \Lambda \implies \neg A(\Lambda', u_0{}^\frown v)),$$

in order to show that $\neg A(\Lambda, u)$. Let $\Lambda_0, \ldots, \Lambda_{l-1}$ be an enumeration of T_{n+1}, and recursively construct $u_0 \subseteq u_1 \subseteq \cdots \subseteq u_l$ and $N_1, \ldots, N_l \in \mathbb{N}$ as follows: Given $u_i, i < l$, if $\mathrm{proj}_n(\Lambda_i) \neq \Lambda'$ we let $N_{i+1} = n$ and $u_{i+1} = u_i$. Otherwise, we have $\neg A(\Lambda_i, u_i)$ by our assumption, so we may fix u_{i+1} extending u_i and $N_{i+1} > n$ so that for all $m \geq N_{i+1}$ and $\Lambda' \in T_m$,

$$\mathrm{proj}_{n+1}(\Lambda') = \Lambda_i \implies \exists \lambda \in \Lambda' \forall w \in 2^{<\mathbb{N}}(S_\lambda \text{ is not dense below } u_{i+1}{}^\frown w).$$

Let $N = \max_{i \leq l} N_i$. For all $m \geq N, \Lambda' \in T_m$, if $\mathrm{proj}_n(\Lambda') = \Lambda$ then $\mathrm{proj}_{n+1}(\Lambda') = \Lambda_i$ for some i, so by our construction we have

$$\exists \lambda \in \Lambda' \forall w \in 2^{<\mathbb{N}}(S_\lambda \text{ is not dense below } u_{i+1}{}^\frown w)$$

and in particular this holds for u_l replacing u_{i+1}. Thus u_l, N witness that $\neg A(\Lambda, u)$.

Suppose now for the sake of contradiction that $\neg A(\Lambda, u)$ for all $u \in 2^{<\mathbb{N}}, \Lambda \in T_0$. As above, by enumerating T_0 we can recursively construct some $u \in 2^{<\mathbb{N}}$ and some N so that

$$\forall \Lambda \in T_N \exists \lambda \in \Lambda \forall v \in 2^{<\mathbb{N}} (S_\lambda \text{ is not dense below } u^\frown v).$$

Let $A \subseteq \bigcup T_N$ be the set of all $\lambda \in \bigcup T_N$ such that $S_{\lambda^{-1}}$ is not dense below $u^\frown v$ for any $v \in 2^{<\mathbb{N}}$. Since A is finite, we can recursively build some $v \in 2^{<\mathbb{N}}$ so that $S_{\lambda^{-1}}$ contains no elements extending $u^\frown v$ for all $\lambda \in A$. By construction A intersects every element of T_N, so $A = A^N_k$ for some $k < K_N$, and by the density of $\boldsymbol{S}_{N,k}$ there is some $s \in \boldsymbol{S}_{N,k}$ extending $u^\frown v$. Now $2^{\mathbb{N}} = \bigcup_{\lambda \in A} C_{\lambda^{-1},s}$, so $C_{\lambda^{-1},s}$ is non-meagre for some $\lambda \in A$. But then $s \in S_{\lambda^{-1}}$, a contradiction. $\lhd\lhd$

Fix a bijection $p = (p_0, p_1) : \mathbb{N} \to \mathbb{N}^2$ satisfying $p_0(n) \leq n$ for all n. We will recursively construct $u_n \in 2^{<\mathbb{N}}, \boldsymbol{\Lambda}_n \in T_n, \boldsymbol{\lambda}_n, \lambda_{n,l} \in \boldsymbol{\Lambda}_n, V_{n,l} \subseteq 2^{<\mathbb{N}}, \psi_n : 2^n \to 2^{<\mathbb{N}}$ for $n, l \in \mathbb{N}$ satisfying:

(a) $\psi_0(\emptyset) = u_0$ and $\psi_{n+1}(s^\frown i) = \psi_n(s)^\frown i^\frown u_{n+1}$ for $s \in 2^n, i \in 2$;

(b) $\psi_n(0^n) \in S_{\boldsymbol{\lambda}_n}$;

(c) $C_{\boldsymbol{\lambda}_n, \psi_n(0^n)}$ is comeagre in $N_{u_{n+1}}$, $V_{n,l}$ is a decreasing sequence of sets which are dense and open in $N_{u_{n+1}}$ and satisfy $\bigcap_l V_{n,l} \subseteq C_{\boldsymbol{\lambda}_n, \psi_n(0^n)}$;

(d) $N_{u_{n+1}} \subseteq V_{n,0}$, and $N_{t^\frown i^\frown u_{n+1}} \subseteq V_{m,n-m}$ for all $m < n, s \in 2^{n-m}, i \in 2$, where $t \in 2^{<\mathbb{N}}$ is such that $\psi_n(0^{m\frown} s) = \psi_m(0^m)^\frown j^\frown t$ for some $j \in 2$;

(e) $A(\boldsymbol{\Lambda}_n, \psi_n(0^n))$;

(f) $\mathrm{proj}_n(\boldsymbol{\Lambda}_{n+1}) = \boldsymbol{\Lambda}_n$ and $(\lambda_{n,l})_{l \in \omega}$ is a redundant enumeration of $\boldsymbol{\Lambda}_n$;

(g) $\mathrm{proj}_{p_0(n)}(\boldsymbol{\lambda}_n) = \lambda_{p(n)}$.

We will construct these sequences in stages, so that at stage n of the construction we will have defined $u_k, \boldsymbol{\Lambda}_k, \boldsymbol{\lambda}_k, \lambda_{k,l}, \psi_k$ for $k \leq n$, and $V_{k,l}$ for $k < n$.

We begin with stage $n = 0$. By the sublemma, there are $\boldsymbol{\Lambda}_0 \in T_0$ and $u \in 2^{<\mathbb{N}}$ satisfying $A(\boldsymbol{\Lambda}_0, u)$. Let $(\lambda_{0,l})_l$ be a redundant enumeration of $\boldsymbol{\Lambda}_0$ and let $\boldsymbol{\lambda}_0 = \lambda_{p(0)}$. By the remarks preceding the sublemma, $S_{\boldsymbol{\lambda}_0}$ is dense below u, so we may choose $u_0 \in S_{\boldsymbol{\lambda}_0}$ extending u and set $\psi_0(\emptyset) = u_0$. Note that $A(\boldsymbol{\Lambda}_0, u_0)$ by the remarks preceding the sublemma.

Now suppose we have completed stage n of the construction. By (b), there is some u such that $C_{\boldsymbol{\lambda}_n,\psi_n(0^n)}$ is comeagre in N_u. Fix $V_{n,l}$ as in (c) (with u taking the place of u_{n+1}).

By (e) and the sublemma, we can (after possibly extending u) find some $\boldsymbol{\Lambda}_{n+1} \in T_{n+1}$ so that $\mathrm{proj}_n(\boldsymbol{\Lambda}_{n+1}) = \boldsymbol{\Lambda}_n$ and $A(\boldsymbol{\Lambda}_{n+1}, \psi_n(0^n)^\frown 0^\frown u)$. Let $(\lambda_{n+1,l})_l$ be a redundant enumeration of $\boldsymbol{\Lambda}_{n+1}$, and let $\boldsymbol{\lambda}_{n+1} \in \boldsymbol{\Lambda}_{n+1}$ satisfy (g) (which is possible as $\mathrm{proj}_{p_0(n+1)}(\boldsymbol{\Lambda}_{n+1}) = \boldsymbol{\Lambda}_{p_0(n+1)}$).

By extending u, we may assume that $N_u \subseteq V_{n,0}$. We may then recursively extend u so that the rest of (d) holds, as there are only finitely many such $m < n, s \in 2^{n-m}, i \in 2$ to consider, and given these and t as in (d) we have that $u_{m+1} \subseteq t$ and hence by (c) that $V_{m,n-m}$ is dense and open in N_t.

Finally, we may extend u to some u_{n+1} satisfying $\psi_n(0^n)^\frown 0^\frown u_{n+1} \in S_{\boldsymbol{\lambda}_{n+1}}$, as this set is dense below $\psi_n(0^n)^\frown 0^\frown u$ by the remarks preceding the sublemma. We define $\psi_{n+1}(s^\frown i) = \psi_n(s)^\frown i^\frown u_{n+1}$, and note that this completes stage $n+1$ of the construction as (c), (d), (e) continue to hold when we extend u.

This completes the recursive construction.

Let $\psi(x) = \bigcup_n \psi_n(x{\restriction}n)$. By (a), (b) this gives a continuous map $2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for all $n \in \mathbb{N}, x \in 2^{\mathbb{N}}$ there is some $y \in 2^{\mathbb{N}}$ with $\psi(0^{n\frown} i^\frown x) = \psi_n(0^n)^\frown i^\frown u_{n+1}{}^\frown y$ for $i \in 2$. By (c), (d) we have $u_{n+1}{}^\frown y \in \bigcap_l V_{n,l} \subseteq C_{\boldsymbol{\lambda}_n,\psi_n(0^n)}$, so

$$\mathrm{proj}_n(\rho((\phi \circ \psi)(0^{n\frown} 1^\frown x), (\phi \circ \psi)(0^{n\frown} 0^\frown x))) = \boldsymbol{\lambda}_n.$$

It follows that if $\wp$ is the restriction to $\mathbb{E}_0$ of the pullback of ρ along $\phi \circ \psi$, then $\wp$ is consistent with $\boldsymbol{\lambda}$ and $\phi \circ \psi$ is a continuous homomorphism of $\wp$ to ρ. By (f), (g), $\boldsymbol{\Lambda}$ is a coherent path through T and $\boldsymbol{\lambda}$ is a redundant enumeration of $\boldsymbol{\Lambda}$. $\triangleleft$

Remark 4.2.6. In the proof of the claim, we may replace $A(\Lambda, u)$ with $A'(\Lambda, u) \iff \forall \lambda \in \Lambda(S_\lambda$ is dense below $u)$, and then prove that for all $u \in 2^{<\mathbb{N}}$ and $n \in \mathbb{N}$ there are $v \in 2^{<\mathbb{N}}$ and $\Lambda \in T_n$ such that $A'(\Lambda, u^\frown v)$ (this is essentially the second part of the subclaim). This is enough to ensure that we can recursively construct ψ in the claim, except without the guarantee that the resulting sequence $\boldsymbol{\Lambda}$ is coherent. We can then build a tree from sequences of projections of elements of $\boldsymbol{\Lambda}$ (just as we did in the definition of T), and use König's lemma to find a branch through this tree, which would give a coherent sequence. One could then pre-compose ψ with an appropriate function π to complete the proof of the claim (the construction of this π being very explicit and straightforward). The definition of A and the first part of the sublemma essentially "unravel" the proof of König's lemma in this remark (this is why the "$\exists^\infty$" quantifier appears in the definition of A).

4.2.4 A parametrized version of Theorem 4.1.11

The proof of Theorem 4.1.11 is effective, meaning that if E is a Δ^1_1 equivalence relation on $\mathbb{N}^\mathbb{N}$, $\mathbf{\Gamma}$ is a Δ^1_1 sequence of finite groups (coded in some appropriate space), $\rho : E \to \tilde{\mathbf{\Gamma}}_\infty$ is a Δ^1_1 cocycle, the family $\boldsymbol{\mathcal{F}}$ is Δ^1_1, and $\boldsymbol{\mathcal{F}}$ is not an essential value of ρ, then there is a uniformly Δ^1_1 sequence of sets $(B_{i,k})_{i,k}$ covering $\mathbb{N}^\mathbb{N}$ so that for all $i, k \in \mathbb{N}$, $\mathrm{proj}_{\mathbf{\Gamma}_i}(\rho(E{\restriction}B_{i,k} \setminus \Delta(B_{i,k})))$ does not contain an element of $\boldsymbol{\mathcal{F}}_i$.

To see this, note that Theorem 4.2.3 is effective, i.e., if $\boldsymbol{G}$ is a uniformly Σ^1_1 sequence of graphs on $\mathbb{N}^\mathbb{N}$ and $\chi_B(\boldsymbol{G}) \leq \aleph_0$, then $\boldsymbol{G}$ admits a uniformly Δ^1_1 sequence of $\boldsymbol{G}$-independent sets covering $\mathbb{N}^\mathbb{N}$.

Now consider the proof of Theorem 4.1.11. It is easy to see that T, A^n_k are (uniformly) Δ^1_1, and hence so is $\boldsymbol{G}$. Thus, if $\boldsymbol{\mathcal{F}}$ is not an essential value of ρ, there is a uniformly Δ^1_1 sequence $(B_l)_l$ of sets covering $\mathbb{N}^\mathbb{N}$ that are $\boldsymbol{G}$-independent. Note that being $\boldsymbol{G}_{i,k}$-independent is a Π^1_1-onΣ^1_1 property, so that by the Number Uniformization Theorem [Mos09, 4B.5] there is a Δ^1_1 map $l \mapsto (i_l, k_l)$ so that B_l is $\boldsymbol{G}_{i_l,k_l}$-independent.

The proof of [Mila, Proposition 1.4] is effective enough that we may partition the sets B_l to get a uniformly Δ^1_1 sequence $(B_{l,j})_{l,j}$ so that $\mathrm{proj}_{i_l}(\rho(E{\restriction}B_{l,j} \setminus \Delta(B_{l,j})))$ does not generate an element of T_{i_l}. Then for all l, j there is some $m \leq i_l$ so that $\mathrm{proj}_{\mathbf{\Gamma}_m}(\rho(E{\restriction}B_{l,j} \setminus \Delta(B_{l,j})))$ does not contain an element of $\boldsymbol{\mathcal{F}}_m$. This is again a Π^1_1-on-Σ^1_1 property, so by the Number Uniformization Theorem there is a Δ^1_1 map $(l, j) \mapsto m_{l,j}$ witnessing this. Using these maps, we can relabel the sets $B_{l,j}$ so that we have a uniformly Δ^1_1 sequence $B_{i,k}$ witnessing that $\boldsymbol{\mathcal{F}}$ is not an essential value of ρ.

Thus, using [Mos09, 4D.4] we get the following parametrized version of Theorem 4.1.11.

Theorem 4.2.7. *Let X, Z be Polish spaces and $D \subseteq X \times Z$, $E \subseteq X^2 \times Z$ be Borel such that E^z is an equivalence relation on D^z for all $z \in Z$.*

Let FinGrp *be the countable discrete space of finite groups,* $E(\mathrm{FinGrp})$ *be the space of pairs* (Γ, γ) *where* $\gamma \in \Gamma \in \mathrm{FinGrp}$*, and* $F(\mathrm{FinGrp})$ *be the space of pairs* $(\Gamma, \mathcal{F})$ *where* $\Gamma \in \mathrm{FinGrp}$ *and* $\mathcal{F}$ *is a family of subsets of* Γ*.*

Let $z \mapsto \mathbf{\Gamma}^z$ be a Borel map from z to $\mathrm{FinGrp}^\mathbb{N}$*, and define $\tilde{\mathbf{\Gamma}}^z$ as in the usual setting. Let $\rho : E \to E(\mathrm{FinGrp})^\mathbb{N}$ be a Borel map such that ρ^z is a cocycle from E^z to $\tilde{\mathbf{\Gamma}}^z_\infty$ for all $z \in Z$.*

Let $z \mapsto \boldsymbol{\mathcal{F}}^z \in F(\mathrm{FinGrp})$ be a Borel map taking $z \in Z$ to a family of sets satisfying the hypotheses of Theorem 4.1.11 for the equivalence relation E^z, the family of finite groups $\boldsymbol{\Gamma}^z$, and the cocycle ρ^z.

The following are equivalent:

1. *The family $\boldsymbol{\mathcal{F}}$ is an essential value of ρ, meaning that for every cover of D by Borel sets $B_{i,k}$, there are $i, k \in \mathbb{N}$ and $z \in Z$ for which*

$$\mathrm{proj}_{\boldsymbol{\Gamma}_i^z}(\rho^z(E^z \restriction B_{i,k}^z \setminus \Delta(B_{i,k}^z)))$$

   contains an element of $\boldsymbol{\mathcal{F}}_i^z$.

2. *There are $z \in Z$, a coherent sequence $\boldsymbol{\Lambda}_i \in \tilde{\boldsymbol{\mathcal{F}}}_i^z$, a redundant enumeration $\boldsymbol{\lambda}$ of $\boldsymbol{\Lambda}$, a Borel cocycle $\wp : \mathbb{E}_0 \to \tilde{\boldsymbol{\Gamma}}_\infty^z$ consistent with $\boldsymbol{\lambda}$, and a continuous embedding of $\wp$ into ρ^z.*

Proof. By the usual transfer theorems, we may assume that $X = Z = \mathbb{N}^{\mathbb{N}}$. By relativizing, we may also assume that $D, E, \boldsymbol{\Gamma}, \rho, \boldsymbol{\mathcal{F}}$ are (uniformly) Δ_1^1.

To see that 2 implies 1, we argue as in the proof of Theorem 4.1.11. To see that 1 implies 2, suppose that 2 fails. Then $\boldsymbol{\mathcal{F}}^z$ is not an essential value of ρ^z for all $z \in Z$, so by the effectiveness of Theorem 4.1.11 discussed above, for all $z \in Z$ there is a $\Delta_1^1(z)$ witness that this is the case. By [Mos09, 4D.4], there are Borel sets $B_{i,k} \subseteq D$ so that for all $z \in Z$, $(B_{i,k}^z)_{i,k}$ witnesses that ρ^z is not an essential value of $\boldsymbol{\mathcal{F}}^z$. But then the sets $B_{i,k}$ witness that 1 fails. □

We include below a sketch of a proof that does not use effective descriptive set theory, for the convenience of the reader.

We first show that the proof of [Mila, Proposition 1.4] is uniform.

Proposition 4.2.8 ([Mila, Proposition 1.4])**.** *Let X, Z, D, E be as in Theorem 4.2.7. Let $\Gamma : Z \to \mathrm{FinGrp}$, $\rho : E \to E(\mathrm{FinGrp})$ be Borel maps so that $\rho^z : E^z \to \Gamma^z$ is a cocycle for all $z \in Z$.*

Suppose that $\Lambda : Z \to \mathrm{FinGrp}$ is a Borel map taking $z \in Z$ to a subgroup of Γ^z such that $\Lambda^z \not\subseteq \rho^z(E^z \setminus \Delta(D^z))$. Then there is a cover of D by countably many Borel sets $(B_l)_{l \in \mathbb{N}}$ such that the group generated by $\rho^z(E^z \restriction B_l^z \setminus \Delta(B_l^z))$ does not contain Λ^z for all $z \in Z, l \in \mathbb{N}$.

Proof. Let $(A^z_k)_{k\in\mathbb{N}}$ be a sequence of subsets of Λ^z which generate Λ^z, and such that every generating subset appears in this sequence at least once. Let K^z be such that every generating subset appears in $\{A^z_k : k < K^z\}$. This can be done in a uniformly Borel way.

We will show that for any k, there is a partition of D into countably many Borel sets $(B_l)_{l\in\mathbb{N}}$ so that $\rho^z(E^z\upharpoonright B^z_l \setminus \Delta(B^z_l))$ does not contain A^z_k. Suppose this has been done. Then we can let $(C^z_l)_{l\in\mathbb{N}}$ be the partition of B^z generated by each of these partitions for $k < K^z$, and again this can easily be done in a uniformly Borel way, completing the proof.

So let $z \mapsto A^z$ be a Borel map so that A^z generates Λ^z for all z. Let λ^z be an element of Λ^z which is not contained in $\rho^z(E^z\upharpoonright B^z \setminus \Delta(B^z))$, and note that this can be done in a uniformly Borel by the Number Uniformization Theorem (see [Kec95, 28.5] for a classical proof).

We may split Z into two Borel subsets Z_0, Z_1, in which λ^z is (resp. is not) the identity, and consider these cases separately.

Consider first the case where $z \in Z_1$, i.e., λ^z is not the identity. Let $w^z \in (A^z)^{<\mathbb{N}}$ be a word in A^z satisfying $\prod_{i<|w^z|} w^z(i) = \lambda^z$. This can again be done in a uniformly Borel way by the Number Uniformization Theorem. By splitting Z_1 according to the length of w^z, we may assume that we have Borel sets $Z_n, n \geq 1$ so that for $z \in Z_n$, $|w^z| = n$. We may therefore consider each Z_n separately.

Consider now $z \in Z_n, n \geq 1$. Define recursively $C^z_0 = B^z$ and

$$C^z_{i+1} = \{x \in C^z_i : \exists y \in C^z_i(\rho^z(x, y) = w^z(n-1-i))\}$$

for all $i < n$. The sets C_i are clearly analytic.

For $z \in Z_n$, $C^z_n = \emptyset$. Indeed, if $x_n \in C^z_n$ then by reverse recursion one could find $x_i \in C^z_i$ such that $\rho^z(x_{i+1}, x_i) = w^z(n-1-i)$ for $i < n$. But then $\rho^z(x_n, x_0) = w^z(0) \dots w^z(n-1) = \lambda^z$, and since this is not the identity we must have $x_n \neq x_0$, contradicting our choice of λ^z.

Thus, for $z \in Z_n, n \geq 1$ we have that $\forall x, y \in C^z_{n-1}(\rho^z(x, y) \neq w^z(0))$. By the First Reflection Theorem [Kec95, 35.10], we may enlarge C_{n-1} into a Borel set B_{n-1} for which this property still holds. But then $\forall x, y \in C^z_{n-2} \setminus B^z_{n-1}(\rho^z(x, y) \neq w^z(1))$, so we may enlarge C_{n-2} into a Borel set B_{n-2} for which this property still holds. Continuing recursively in this way, we obtain Borel sets $B_0, B_1, \dots, B_{n-1}$ so that for all $i < n$ and

$x, y \in B_i^z$, $\rho^z(x,y) \neq w^z(n-1-i)$. By construction, $C_0 \subseteq B_0 \cup \cdots \cup B_{n-1}$, so the sets $B_0, \ldots, B_{n-1}$ form the desired partition of D restricted to Z_n.

Consider now $z \in Z_0$, so that the identity is not contained in $\rho^z(E^z \restriction D^z \setminus \Delta(D^z))$. If $A^z = \{1\}$, then there is nothing to do, so we may assume wlog that $A^z \neq \{1\}$ for all $z \in Z_0$. Let γ^z be an arbitrary Borel assignment of a non-identity element of A^z to $z \in Z_0$. We will show that there is a partition of $D \cap X \times Z_0$ into Borel sets $(B_l)_{l \in \mathbb{N}}$ so that $\rho^z(E^z \restriction B_l^z \setminus \Delta(B_l^z))$ does not contain γ^z for all $z \in Z_0$.

As $D \cap X \times Z_0$ is Borel, there is a sequence $(W_l)_{l \in \mathbb{N}}$ of Borel sets in $D \cap X \times Z_0$ which separates points. Let $C_l^z = \{x \in W_l^z : \exists y \in {\sim}W_l^z(\gamma^z = \rho^z(x,y))\}$. Then the sets C_l are analytic, and we claim that $\gamma^z \notin \rho^z(E^z \restriction C_l^z \setminus \Delta(C_l^z))$ for all $z \in Z_0, l \in \mathbb{N}$. Indeed, if there were $x, y \in C_l^z$ with $\rho^z(x,y) = \gamma^z$, then we may fix $w \in {\sim}W_l^z$ satisfying $\gamma^z = \rho^z(x,w)$. But then $\rho^z(y,w) = 1$, so by our assumption that $\rho^z(E^z \restriction D^z \setminus \Delta(D^z))$ does not contain the identity we must have $y = w$, contradicting the fact that $y \in C_l^z \subseteq W_l^z$ and $w \notin W_l^z$.

We may therefore expand each C_l into Borel sets B_l for which $\rho^z(E^z \restriction B_l^z \setminus \Delta(B_l^z))$ does not contain γ^z for all $z \in Z_0$, so it suffices to show that for all $z \in Z_0$ and $x, y \in {\sim}\bigcup_{l \in \mathbb{N}} B_l^z$, $\rho^z(x,y) \neq \gamma^z$. To see this, note that if $x \in D^z$, $y \in {\sim}\bigcup_{l \in \mathbb{N}} B_l^z$ and $\rho^z(x,y) = \gamma^z$, then there is some l for which $x \in W_l^z, y \notin W_l^z$, so that $x \in C_l^z \subseteq B_l^z$. $\square$

Proof of Theorem 4.2.7. The proof that 2 implies 1 is the same as before.

Suppose now that 1 holds. For $z \in Z$, let T^z be the tree of sequences of the form $(\mathrm{proj}_0(\Lambda), \ldots, \mathrm{proj}_i(\Lambda))$ for $i \leq n < \mathbb{N}$ and $\Lambda \in \tilde{\boldsymbol{\mathcal{F}}}_n^z$. The map $z \mapsto T^z$ is easily seen to be Borel. Define T_n^z to be the n-th level of T^z, as in the proof of Theorem 4.1.11, and let $(A_k^{z,n})_{k \in \mathbb{N}}$ be a sequence of subsets of $\bigcup T_n^z$ which intersect every element of T_n^z, and for which every such subset appears at least once. Note that this can be done so that the map $(z, n, k) \mapsto A_k^{z,n}$ is Borel.

Define now a family of Borel directed graphs $\boldsymbol{G}_{n,k}$ on D by $(x,z)\boldsymbol{G}_{n,k}(y,w)$ iff $z = w, x \neq y, xE^z y$, and $\mathrm{proj}_n(\rho^z(x,y)) \in A_k^{z,n}$.

Suppose now that $B \subseteq D$ is $\boldsymbol{G}_{n,k}$-independent. Then $\mathrm{proj}_n(\rho^z(E^z \restriction B^z \setminus \Delta(B^z)))$ does not contain any element of T_n^z for all $z \in Z$. That is, for all z and all $\Lambda \in T_n^z$, $\Lambda \not\subseteq \mathrm{proj}_n(\rho^z(E^z \restriction B^z \setminus \Delta(B^z)))$.

Let $(\Lambda_k^z)_{k \in \mathbb{N}}$ be a sequence of elements of T_n^z for which every element of T_n^z appears at least once, and let K^z be such that all elements of T_n^z appear in $\{\Lambda_k^z : k < K^z\}$. As with $A_k^{z,n}$, this can be done in a uniformly Borel way. Then for all $k \in \mathbb{N}$ and all $z \in Z$

we have $\Lambda_k^z \not\subseteq \operatorname{proj}_n(\rho^z(E^z \restriction B^z \setminus \Delta(B^z)))$, so by Proposition 4.2.8 we have a Borel map $f : B \times \mathbb{N} \to \mathbb{N}$ so that for all $z \in Z, k, l \in \mathbb{N}$, $\operatorname{proj}_n(\rho^z(E^z \restriction (f^{z,k})^{-1}(l) \setminus \Delta((f^{z,k})^{-1}(l))))$ does not generate a group containing Λ_k^z.

Let $g : \mathbb{N} \times Z \to \mathbb{N}^{<\mathbb{N}}$ be a Borel map such that g^z maps $\mathbb{N}$ bijectively onto $\mathbb{N}^{K^z}$ for all $z \in Z$ and define $h : B \to \mathbb{N}$ so that $g(h(x,z),z) = (f(x,z,0), \ldots, f(x,z,K^z-1))$. Note that h is Borel. Let $B_l = h^{-1}(l)$ for $l \in \mathbb{N}$. Then $(B_l)_{l \in \mathbb{N}}$ is a cover of B by Borel sets, and we claim that for all $z \in Z, l \in \mathbb{N}$, $\operatorname{proj}_n(\rho^z(E^z \restriction B_l^z \setminus \Delta(B_l^z)))$ does not generate a group containing an element of T_n^z. Indeed, every such group appears as Λ_k^z for some $k < K^z$, and $x, y \in B_l^z \implies f(x,z,k) = f(y,z,k)$. In particular, $f(x,z,k) = m \in \mathbb{N}$ is constant for $x \in B_l^z$, and thus $B_l^z \subseteq (f^{z,k})^{-1}(m)$.

This implies that for all $z \in Z, l \in \mathbb{N}$ there is some i so that $\operatorname{proj}_{\boldsymbol{\Gamma}_i^z}(\rho^z(E^z \restriction B_l^z \setminus \Delta(B_l^z)))$ does not contain an element of $\boldsymbol{\mathcal{F}}_i^z$. Since the family $\boldsymbol{\mathcal{F}}_i^z$ is a finite collection of finite sets, the property that $\operatorname{proj}_{\boldsymbol{\Gamma}_i^z}(\rho^z(E^z \restriction B_l^z \setminus \Delta(B_l^z)))$ does not contain an element of $\boldsymbol{\mathcal{F}}_i^z$ is co-analytic (as a subset of $Z \times \mathbb{N}^2$). By the Number Uniformization Theorem, one can therefore refine the cover $(B_l)_{l \in \mathbb{N}}$ of B to a cover $(B_{i,l})_{i,l \in \mathbb{N}}$ of B by Borel sets so that for all $z \in Z, i, l \in \mathbb{N}$, $\operatorname{proj}_{\boldsymbol{\Gamma}_i^z}(\rho^z(E^z \restriction B_{i,l}^z \setminus \Delta(B_{i,l}^z)))$ does not contain an element of $\boldsymbol{\mathcal{F}}_i^z$.

It follows that if $\chi_B(\boldsymbol{G}) \leq \aleph_0$ then $\boldsymbol{\mathcal{F}}$ is not an essential value of ρ.

By Theorem 4.2.3, there is a sparse dense sequence of sets $\boldsymbol{S}_{n,k} \subseteq 2^{<\mathbb{N}}$, and a continuous homomorphism $\phi : 2^\omega \to D$ from $\mathbb{G}_{\boldsymbol{S}}$ to $\boldsymbol{G}$. We may choose $\boldsymbol{S}$ so that $\bigcup_{n,k} \boldsymbol{S}_{n,k}$ contains a sequence of every length. Then $\operatorname{proj}_Z \circ \phi : 2^\omega \to Z$ is a continuous homomorphism from $\bigcup_{n,k} \mathbb{G}_{\boldsymbol{S}_{n,k}}$ to $\Delta(Z)$, and it follows that $\operatorname{proj}_Z \circ \phi$ is a constant. Let $z \in Z$ be the value taken by $\operatorname{proj}_Z \circ \phi$.

We therefore have a continuous homomorphism $\operatorname{proj}_X \circ \phi : 2^\omega \to D^z$ from $\mathbb{G}_{\boldsymbol{S}}$ to $\boldsymbol{G}^z$. The rest follows as in the proof of Theorem 4.1.11. □

4.3 Proofs of Lusin–Novikov dichotomies over quotients

Let E, F be Borel equivalence relations on Polish spaces X, Y and $P \subseteq X \times Y$ be $E \times F$-invariant. A **uniformization** of P is a set $U \subseteq P$ whose sections U_x intersect P_x in exactly one F-class.

Let $F \subseteq E$ be Borel equivalence relations on a Polish space X. We say F has **index** n in E if every E-class contains n F-classes, **bounded finite index** in E if there is a finite bound on the number of F-classes inside each E-class, and **σ-bounded finite index** in E if there is a cover of X by countably many Borel sets on which E has bounded finite index over F.

A **transversal** of E over F is a set $B \subseteq X$ that intersects every E-class in exactly one F-class, and a **(proper) quasi-transversal** of E over F is a set $B \subseteq X$ whose intersection with every E-class has non-empty intersection with a non-empty (proper) finite subset of the F-classes it contains.

Proof of Theorem 4.1.6. To see that at most one of these alternatives hold, it suffices to show that there is no Borel $\mathbb{E}_{0,\Lambda}$-invariant transversal of $\mathbb{E}_0 \times I(n)$ over $\mathbb{E}_{0,\Lambda}$. Indeed, if π_X, π_Y are as in alternative 2 and U is a Borel uniformization of P, then $\{y \in 2^{\mathbb{N}} \times n : \exists x(\mathbb{E}_0 \times I(n))y((\pi_X(x), \pi_Y(y)) \in \tilde{U})\}$ is a Borel $\mathbb{E}_{0,\Lambda}$-invariant transversal of $\mathbb{E}_0 \times I(n)$ over $\mathbb{E}_{0,\Lambda}$. Suppose now that B were such a transversal. Then B would be non-meagre, as countably many homeomorphic copies of B cover $2^{\mathbb{N}} \times n$. It follows that for some $i < n$, $A = \{x \in 2^{\mathbb{N}} : (x, i) \in B\}$ is non-meagre. By [Mila, Proposition 1.5], $\Lambda \subseteq \wp_{\boldsymbol{\lambda}}(\mathbb{E}_0 \restriction A \setminus \Delta(A))$, so in particular there are $x, y \in A$ with $x \mathbb{E}_0 y$ and $\wp_{\boldsymbol{\lambda}}(x, y)(i) \neq i$ (as Λ is fixed-point-free). But then $(x, i) \cancel{\mathbb{E}_{0,\Lambda}} (y, i)$ and $(x, i), (y, i) \in B$, a contradiction.

To see that at least one of these alternatives hold, define $E' = (E \times I(Y)) \restriction \tilde{P}$, $F' = (I(X) \times F) \restriction \tilde{P}$, and

$$Z = \{z \in \tilde{P}^n : \forall i < j < n(z(i)(E' \setminus F')z(j))\}.$$

For $z, z' \in Z$ let $z \tilde{E} z' \iff z(0) E' z'(0)$, so that $\tilde{E}$ is a Borel equivalence relation on Z. There is a natural action of S_n on Z, namely the action $(\sigma z)(i) = z(\sigma^{-1}(i))$ permuting the coordinates, and for all $z \tilde{E} z'$ there is a unique $\sigma \in S_n$ such that $(\sigma z')(i) F' z(i)$ for $i < n$ (we are using here the fact that the sections of P have size exactly n). Let $\rho(z, z') \in S_n$ be this σ, and note that ρ is a Borel cocycle on $\tilde{E}$.

Case 1. Suppose that there is some fixed-point-free group $\Lambda \leq S_n$, a redundant enumeration $\boldsymbol{\lambda}$ of Λ, and a continuous embedding $\pi : 2^{\mathbb{N}} \to Z$ of $\wp_{\boldsymbol{\lambda}}$ to ρ. By [Mila, Proposition 1.6] and a straightforward recursive construction, we may assume that Λ is minimal. Let $x D y \iff \exists i, j < n(\pi(x)(i)(1) = \pi(x')(j)(1))$, and $x F'' y \iff \exists i, j < n(\pi(x)(i) F' \pi(y)(j))$. Note that F'' is an equivalence relation on $2^{\mathbb{N}}$, and that $D \subseteq F''$ is closed.

Claim 4.3.1. F'' is meagre.

Proof. By the Kuratowski–Ulam Theorem [Kec95, 8.41], it suffices to show that every F''-class C is meagre. Fix $x \in C$ and for $i, j < n$ let $C_{i,j} = \{y \in C : \pi(x)(i) F' \pi(y)(j)\}$. Suppose towards a contradiction that C is non-meagre, and fix i, j such that $C_{i,j}$ is non-meagre. By the proof of [dRM, Proposition 1.3] and the fact that $\boldsymbol{\lambda}$ is a redundant enumeration of Λ, for all $\lambda \in \Lambda$ there are $y, z \in C_{i,j}$ with $y \mathbb{E}_0 z$ and $\wp_{\boldsymbol{\lambda}}(y, z) = \lambda$. Since Λ is fixed-point-free, we may choose $y, z \in C_{i,j}$ such that $y \mathbb{E}_0 z$ and $\wp_{\boldsymbol{\lambda}}(y, z)(j) \neq j$. But then $\pi(y)(j) \not{F'} \pi(z)(j)$ and $\pi(y)(j) F' \pi(x)(i) F' \pi(z)(j)$, a contradiction. $\triangleleft$

By [Mila, Proposition 1.7], we may assume that $D = \Delta(2^{\mathbb{N}})$ and that $x \not{\mathbb{E}_0} x' \implies \pi(x) \not{F''} \pi(x')$. Let $\pi_X(x, i) = \mathrm{proj}_X(\pi(x)(i))$, $\pi_Y(x, i) = \mathrm{proj}_Y(\pi(x, i))$.

Note that

$$x \mathbb{E}_0 y \iff \pi(x) \tilde{E} \pi(y) \iff \pi(x)(i) E' \pi(y)(j) \iff \pi_X(x, i) E \pi_X(y, j)$$

for $x, y \in 2^{\mathbb{N}}$, $i, j < n$, and that

$$(x, i) \mathbb{E}_{0,\Lambda} (y, j) \iff x \mathbb{E}_0 y \ \& \ \wp_{\boldsymbol{\lambda}}(x, y)(j) = i \iff \pi(x)(i)(E' \cap F')\pi(y)(j),$$

so π_X is a reduction of $\mathbb{E}_0 \times I(n)$ to E and π_Y is a homomorphism from $\mathbb{E}_{0,\Lambda}$ to F. On the other hand, suppose that $(x, i) \not{\mathbb{E}_{0,\Lambda}} (y, j)$. If $x \mathbb{E}_0 y$, then $\rho(\pi(x), \pi(y))(j) \neq i$ so $\pi(x)(i) \not{F'} \pi(y)(j)$, and if $x \not{\mathbb{E}_0} y$ then $x \not{F''} y$ so $\pi(x)(i) \not{F'} \pi(y)(j)$. Thus π_Y is a reduction of $\mathbb{E}_{0,\Lambda}$ to F. Since $D = \Delta(2^{\mathbb{N}})$, it is easy to see that π_Y is injective.

Suppose now that $z(\mathbb{E}_0 \times I(n))z'$. Then $(\pi_X(z'), \pi_Y(z')) \in \tilde{P}$, and $\pi_X(z) E \pi_X(z')$, so by E-invariance $(\pi_X(z), \pi_Y(z')) \in \tilde{P}$. Therefore $(\pi_X \times \pi_Y)(\mathbb{E}_0 \times I(n)) \subseteq \tilde{P}$.

Thus π_X, π_Y witness alternative 2 in Theorem 4.1.6.

Case 2. Suppose now that we are not in case 1. By Theorem 4.1.5 and [Mila, Proposition 1.4], for every subset $\Lambda \subseteq S_n$ that generates a fixed-point-free subgroup there is a cover of Z by countably many Borel sets B_n for which Λ is not contained in $\rho(\tilde{E} \restriction B_n \setminus \Delta(B_n))$. Since there are only finitely many such sets Λ, we may find a cover B_n of Z by Borel sets satisfying that $\rho(\tilde{E} \restriction B_n)$ does not generate a fixed-point-free group, i.e., such that $\rho(\tilde{E} \restriction B_n)$ has a fixed point when acting naturally on n. For all n, let i_n be such a fixed point and let $A_n = \mathrm{proj}_{i_n}(B_n)$. Then $A_n \subseteq \tilde{P}$ is analytic, its sections intersect at most one F-class, and this F-class is invariant under E. By [dRM, Proposition 2.1], we may extend A_n to a Borel $E \times F$-invariant set $A'_n \subseteq \tilde{P}$ with this property. By [dRM, Proposition 3.4] the set $[A'_n]_{E'}$ is Borel, so $A = \bigcup_n (A'_n \setminus \bigcup_{k<n} [A'_k]_{E'})$ is Borel, $E \times F$-invariant, and its sections contain at most one F-class. To see that A has non-empty sections, note that the sets B_n cover Z and that for all $x \in X$ there is some $z \in Z$ with $x E z(0)(0)$. $\square$

The proof of Theorem 4.1.8 is identical, except we show that when Λ is transitive then there is no Borel $\mathbb{E}_{0,\Lambda}$-invariant proper quasi-transversal of $\mathbb{E}_0$ over $\mathbb{E}_{0,\Lambda}$, and in case 2 we simply consider the projection $\mathrm{proj}_0(B_n)$.

Proof of Theorem 4.1.7. That these are mutually exclusive follows from [Mila, Proposition 1.5], as in the proof of Theorem 4.1.6.

To see that at most one of them hold, we argue by induction on n. By [dRM, Theorem 2.12], the set A of $x \in X/E$ for which P_x contains exactly n points is Borel. Indeed, fix Borel maps $\phi_n : X \to Y$ so that $P_x = \bigcup_n [\phi_n(x)]_F$ for $x \in X$. Then

$$x \in \tilde{A} \iff \exists k_0, \dots, k_{n-1} \forall i < j < n(\phi_{k_i}(x) \not{F} \phi_{k_j}(x)).$$

Now we apply Theorem 4.1.6 to the restriction of E, P to $\tilde{A}$, and the inductive hypothesis to their restrictions to $X \setminus \tilde{A}$. □

Again, the proof of Theorem 4.1.9 is the same.

We now consider the case of σ-bounded finite index.

Proof of Proposition 4.1.10. Clearly $3 \implies 1 \implies 2$. To see that $2 \implies 1$, we apply [dRM, Proposition 2.1] to extend each $\tilde{P}_n$ to $E \times F$-invariant sets with the same property. To see that $1 \implies 3$ when F is strongly idealistic, we apply [dRM, Theorem 2.12] as in the proof of Theorem 4.1.7. □

Proof of Theorem 4.1.12. To see that at most one of these alternatives hold, it suffices to show (as in the proof of Theorem 4.1.6) that there is no Borel $\mathbb{E}_{0,\wp}$-invariant transversal of $\mathbb{E}_0 \times I(\mathbb{N})$ over $\mathbb{E}_{0,\wp}$, where $\mathbf{\Lambda}_i \leq \tilde{\mathbf{\Gamma}}_i^\alpha$ is a coherent sequence of fixed-point-free subgroups, $\boldsymbol{\lambda}$ is a redundant enumeration of $\mathbf{\Lambda}$, and $\wp : \mathbb{E}_0 \to \tilde{\mathbf{\Gamma}}_\infty^\alpha$ is a Borel cocycle consistent with $\boldsymbol{\lambda}$. So suppose that B were such a transversal. Then B would be non-meagre, as countably many homeomorphic copies of B would cover $2^\mathbb{N} \times \mathbb{N}$, so there is some $i \in \mathbb{N}$ such that $A = \{x \in 2^\mathbb{N} : (x, i) \in B\}$ is non-meagre. By Proposition 4.2.2 there is some $\gamma \in \tilde{\mathbf{\Gamma}}_{i+1}^\alpha$ so that $\gamma \mathbf{\Lambda}_{i+1} \gamma^{-1} \subseteq \mathrm{proj}_{i+1}(\wp(\mathbb{E}_0 \restriction A \setminus \Delta(A)))$. Note that $i < n(\alpha, i+1)$, so there is some $j < n(\alpha, i+1)$ with $\gamma(j) = i$. Since $\mathbf{\Lambda}_{i+1}$ is fixed-point-free, there is some $\lambda \in \mathbf{\Lambda}_{i+1}$ for which $\lambda(j) \neq j$. Taking $x, y \in A$ such that $\wp(x, y) = \gamma \lambda \gamma^{-1}$, we have $\wp(x, y)(i) \neq i$, so $(x, i) \not{\mathbb{E}}_{0,\wp} (y, i)$ and $(x, i), (y, i) \in B$, a contradiction.

To see that at least one of these alternatives hold, define $E' = (E \times I(Y))\upharpoonright\tilde{P}$, $F' = (I(X) \times F)\upharpoonright\tilde{P}$, and let

$$Z = \{z \in \prod_n (\tilde{P}_n)^{\alpha(n)} : \forall i < j(z(i)(E' \setminus F')z(j))\}.$$

As in the proof of Theorem 4.1.6, we let $z\tilde{E}z' \iff z(0)E'z'(0)$, and we note that for all $z\tilde{E}z'$ there is a unique element of S_∞, which we denote by $\rho(z, z')$, such that $(\rho(z, z') \cdot z')(i)F'z(i)$ for all $i \in \mathbb{N}$. Here, S_∞ acts on Z by permuting the coordinates, the choice of $\rho(z, z')$ is unique because the sets P_n are pairwise disjoint, and for the same reason it is easy to see that $\rho(z, z') \in \tilde{\mathbf{\Gamma}}^\alpha_\infty$. This makes $\rho : \tilde{E} \to \tilde{\mathbf{\Gamma}}^\alpha_\infty$ a Borel cocycle. For $i \in \mathbb{N}$, let $\boldsymbol{\mathcal{F}}_i$ denote the collection of subsets of $S_{\alpha(i)}$ that generate fixed-point-free subgroups of $S_{\alpha(i)}$.

Case 1. Suppose that $\boldsymbol{\mathcal{F}}$ is not an essential value of ρ, and fix Borel sets $B_{i,k}$ covering Z so that no element of $\boldsymbol{\mathcal{F}}_i$ is contained in $\mathrm{proj}_{S_{\alpha(i)}}(\rho(\tilde{E}\upharpoonright B_{i,k} \setminus \Delta(B_{i,k})))$, i.e., $\mathrm{proj}_{S_{\alpha(i)}}(\rho(\tilde{E}\upharpoonright B_{i,k} \setminus \Delta(B_{i,k})))$ has a fixed point $j(i, k) \in \alpha(i)$ for all $i, k \in \mathbb{N}$. Thus, $A_{i,k} = \mathrm{proj}_{n(\alpha,i-1)+j(i,k)}(B_{i,k})$ is an analytic set in $\tilde{P}_i$ whose sections intersect at most one F-class, and for which this F-class is invariant under E. By [dRM, Proposition 2.1] we can extend $A_{i,k}$ to a Borel $E \times F$-invariant set $A'_{i,k} \subseteq \tilde{P}_i$ with the same property, and by [dRM, Proposition 3.4] the set $[A'_{i,k}]_{E'\upharpoonright\tilde{P}_i}$ is Borel. It follows that $[A'_{i,k}]_{E'}$ is Borel: It is clearly analytic, and it is co-analytic because

$$x \in [A'_{i,k}]_{E'} \iff \forall y(yE'x \;\&\; y \in \tilde{P}_i \implies y \in [A'_{i,k}]_{E'\upharpoonright\tilde{P}_i}).$$

It follows as in the proof of Theorem 4.1.6 that $\bigcup_n (A'_n \setminus \bigcup_{k<n}[A'_k]_{E'})$ is a Borel $E \times F$-invariant uniformization of $\tilde{P}$, and hence its quotient gives a Borel uniformization of P.

Case 2. Suppose now that $\boldsymbol{\mathcal{F}}$ is an essential value of ρ. By Theorem 4.1.11, there is a coherent sequence $\mathbf{\Lambda}_i \in \tilde{\boldsymbol{\mathcal{F}}}_i$, a redundant enumeration $\boldsymbol{\lambda}$ of Λ, a Borel cocycle $\wp : \mathbb{E}_0 \to \tilde{\mathbf{\Gamma}}^\alpha_\infty$ consistent with $\boldsymbol{\lambda}$, and a continuous embedding $\pi : 2^\mathbb{N} \to Z$ of $\wp$ into ρ. Let $xD_ky \iff \exists i, j < n(\alpha, k)(\pi(x)(i)(1) = \pi(y)(j)(1))$, and $xF''y \iff \exists i, j(\pi(x)(i)F'\pi(y)(j))$. Note that F'' is an equivalence relation on $2^\mathbb{N}$, and that each $D_k \subseteq F''$ is closed.

Claim 4.3.2. *F'' is meagre.*

Proof. By the Kuratowski–Ulam Theorem, it suffices to show that every F''-class C is meagre. Fix $x \in C$, and for i, j let $C_{i,j} = \{y \in C : \pi(x)(i)F'\pi(y)(j)\}$. Suppose towards a contradiction that C is non-meagre, and fix i, j such that $C_{i,j}$ is non-meagre. Let k be such that $i, j < n(\alpha, k-1)$, and fix $s \in 2^{<\mathbb{N}}$ with $|s| \geq n(\alpha, k)$ so that $C_{i,j}$ is comeagre in N_s. Since $\tilde{\mathbf{\Gamma}}^{\alpha}_{k}$ is finite, there is some $\gamma \in \tilde{\mathbf{\Gamma}}^{\alpha}_{k}$ such that $\mathrm{proj}_k(\wp(s^\frown x, 0^{|s|\frown} x)) = \gamma$ for a non-meagre set A of $x \in 2^{\mathbb{N}}$. Let $B = \{0^{|s|\frown} x : x \in A \ \ \& \ \ s^\frown x \in C_{i,j}\}$ and note that B is non-meagre. By the proof of [dRM, Proposition 1.3], the fact that $\boldsymbol{\lambda}$ is a redundant enumeration of $\mathbf{\Lambda}$, and the fact that $\wp$ is consistent with $\boldsymbol{\lambda}$, for all $\lambda \in \mathbf{\Lambda}_k$ there are $y, z \in B$ with $y\mathbb{E}_0 z$ and $\mathrm{proj}_k(\wp(y, z)) = \lambda$. Since $\mathbf{\Lambda}_k$ has no fixed points, it follows that there are $y, z \in C_{i,j}$ so that $y\mathbb{E}_0 z$ and $\mathrm{proj}_k(\wp(y, z))(j) \neq j$. But then $\pi(y)(j) \cancel{F'} \pi(z)(j)$ and $\pi(y)(j) F' \pi(x)(i) F' \pi(z)(j)$, a contradiction. $\triangleleft$

By Proposition 4.2.1, we may assume that $x(k) \neq y(k) \implies x \cancel{D_k} y$ and that $x \cancel{\mathbb{E}_0} y \implies \pi(x) \cancel{F''} \pi(y)$. Let $\pi_X(x, i) = \mathrm{proj}_X(\pi(x)(i))$, $\pi_Y(x, i) = \mathrm{proj}_Y(\pi(x)(i))$.

Note that

$$x\mathbb{E}_0 y \iff \pi(x)\tilde{E}\pi(y) \iff \pi(x)(i)E'\pi(y)(j) \iff \pi_X(x, i)E\pi_X(y, j)$$

for $x, y \in 2^{\mathbb{N}}$, $i, j \in \mathbb{N}$, and that

$$(x, i)\mathbb{E}_{0,\wp}(y, j) \iff x\mathbb{E}_0 y \ \ \& \ \ \wp(x, y)(j) = i \iff \pi(x)(i)(E' \cap F')\pi(y)(j),$$

so π_X is a reduction of $\mathbb{E}_0 \times I(\mathbb{N})$ to E and π_Y is a homomorphism from $\mathbb{E}_{0,\wp}$ to F. On the other hand, suppose that $(x, i)\cancel{\mathbb{E}_{0,\wp}}(y, j)$. If $x\mathbb{E}_0 y$, then $\rho(\pi(x), \pi(y))(j) \neq i$ so $\pi(x)(i)\cancel{F'}\pi(y)(j)$, and if $x\cancel{\mathbb{E}_0}y$ then $x\cancel{F''}y$ so $\pi(x)(i)\cancel{F'}\pi(y)(j)$. Thus π_Y is a reduction from $\mathbb{E}_{0,\wp}$ to F. Since $x(k) \neq y(k) \implies x\cancel{D_k}y$, it is easy to see that π_Y is injective when restricted to $\bigcup_{i \in \mathbb{N}} N_{0^i} \times (n(\alpha, i) \setminus n(\alpha, i-1))$.

Suppose now that $z(\mathbb{E}_0 \times I(\mathbb{N}))z'$. Then $(\pi_X(z'), \pi_Y(z')) \in \tilde{P}$, and $\pi_X(z)E\pi_X(z')$, so by E-invariance $(\pi_X(z), \pi_Y(z')) \in \tilde{P}$. Therefore $(\pi_X \times \pi_Y)(\mathbb{E}_0 \times I(\mathbb{N})) \subseteq \tilde{P}$.

Thus π_X, π_Y witness alternative 2 in Theorem 4.1.12. $\square$

Proof of Theorem 4.1.13. That 1 and 2(a) are incompatible follows from (the proof of) Theorem 4.1.6. The same holds for 1 and 2(b) by Theorem 4.1.12.

We may assume that for every $x \in X$, the sections $(P_n)_x$ are pairwise disjoint. By (the proof of) Proposition 4.1.10, we may also assume that the sections of each P_n are either empty or have size exactly $n \geq 2$.

Note that by [dRM, Theorem 2.12], the map $x \mapsto A_x \in 2^{\mathbb{N}}$ given by $n \in A_x \iff (P_n)_x \neq \emptyset$ is Borel. For $A \in 2^{\mathbb{N}}$, let $X^A = \{x : A = A_x\}$. If $A \in 2^{\mathbb{N}}$ is finite, we restrict to X^A and apply Theorem 4.1.6. As there are only countably many such A, we may assume that 2(a) does not hold for any of them and hence that $X^A = \emptyset$ for finite A.

For infinite $A \in 2^{\mathbb{N}}$, let $\beta^A \in \mathbb{N}^{\mathbb{N}}$ be its increasing enumeration and define $\alpha^A(n) \geq 2$ to be the size of the non-empty sections of $P_{\beta^A(n)}$. Define $Z^A, \tilde{E}^A, \rho^A, \boldsymbol{\Gamma}^{\alpha^A} = \boldsymbol{\Gamma}^A, \boldsymbol{\mathcal{F}}^A$ as in the proof of Theorem 4.1.12 restricted to X^A. It is not hard to verify that these definitions are uniformly Borel, and hence we may apply Theorem 4.2.7 to $Z, \tilde{E}, \rho, \boldsymbol{\Gamma}, \boldsymbol{\mathcal{F}}$.

Case 1. If $\boldsymbol{\mathcal{F}}$ is not an essential value of ρ, let $B_{i,k}$ be Borel sets witnessing this. Then for all $i, k \in \mathbb{N}$ and infinite $A \in 2^{\mathbb{N}}$, the elements of $\mathrm{proj}_{\boldsymbol{\Gamma}_i^A}(\rho^A(\tilde{E}^A \restriction B_{i,k}^A \setminus \Delta(B_{i,k}^A)))$ fix some point in $\alpha^A(i)$. By the Number Uniformization Theorem, there is a Borel assignment $j^A(i,k)$ of such a point.

Define $n(\alpha^A, i)$ as in the non-parametrized case, and note that $(A, i) \mapsto n(\alpha^A, i)$ is Borel. For $i, k \in \mathbb{N}$, let

$$C_{i,k}(w, A) \iff \exists z \in B_{i,k}^A (z(n(\alpha^A, i-1) + j(i,k)) = w).$$

Then $C_{i,k} \subseteq \tilde{P}$ is an analytic set satisfying: for all $A \in 2^{\mathbb{N}}$ $C_{i,k}^A \subseteq \tilde{P}_{\beta^A(i)}$, its sections intersect at most one F-class, and this F-class is invariant under E. By [dRM, Proposition 2.1] we may extend $C_{i,k}$ to $E \times F \times \Delta(2^{\mathbb{N}})$-invariant Borel sets $C'_{i,k}$ with the same property. Note that the sets $D_{i,j} = \mathrm{proj}_{X \times Y}(C'_{i,k})$ are Borel, as the vertical sections of $C'_{i,k}$ are either empty or singletons, and these are $E \times F$-invariant sets whose sections contain at most one F-class. Note that $[D_{i,j}]_{E'}$ is Borel: It is analytic, and $x \notin [D_{i,j}]_{E'}$ iff there are $A \in 2^{\mathbb{N}}, y_0, \ldots, y_{\alpha^A(i)-1}$ such that $A = A_x$ and $y_l \in \tilde{P}_{\beta^A(i)} \setminus D_{i,j}$ for $l < \alpha^A(i)$. As in the proof of Theorem 4.1.6, we get a Borel uniformization of P.

Case 2. If $\boldsymbol{\mathcal{F}}$ is an essential value of ρ, then by Theorem 4.2.7 and the proof of Theorem 4.1.6 we see that 2(b) holds. $\square$

Chapter 5

DESCRIPTIVE DICHOTOMY THEOREMS

Michael S. Wolman

5.1 Introduction

There are many *dichotomy theorems* in descriptive set theory, that is, theorems showing that if a given structure is not "simple" then it contains a copy of a canonical "complicated" object. Perhaps the first example of such a dichotomy is Cantor's Perfect Set Theorem, which states that a closed set in a Polish space is either countable, or contains a homeomorphic copy of the Cantor space. This was later generalized to analytic sets, and then to various other types of structures, such as equivalence relations and graphs.

We recall now some of the most important dichotomy theorems for equivalence relations, namely Silver's Theorem and the Harrington–Kechris–Louveau Theorem, which we also call the $\mathbb{E}_0$ *dichotomy*. If E is an equivalence relation on X and F is an equivalence relation on Y, a *reduction* from E to F is a map $f : X \to Y$ such that $xEx' \iff f(x)Ff(x')$. If X, Y are Polish, we write $E \leq_B F$ if there is a Borel reduction from E to F, and $E \sqsubseteq_c F$ if there is a continuous *embedding* (i.e. injective reduction) from E to F. We let $\Delta(X)$ denote the equality relation on a set X, and say E is *smooth* if $E \leq_B \Delta(X)$ for a Polish space X. Finally, we write $\mathbb{E}_0$ for the "eventual equality" relation on $2^{\mathbb{N}}$:

$$x\mathbb{E}_0 y \iff \exists n \forall k \geq n(x_k = y_k).$$

Theorem 5.1.1 (Silver). *Let E be a co-analytic equivalence relation on a Polish space. Exactly one of the following holds:*

1. *E has countably many equivalence classes.*
2. *$\Delta(2^{\mathbb{N}}) \sqsubseteq_c E$.*

Theorem 5.1.2 (The $\mathbb{E}_0$ dichotomy, Harrington–Kechris–Louveau). *Let E be a Borel equivalence relation on a Polish space. Exactly one of the following holds:*

1. *E is smooth.*

2. $\mathbb{E}_0 \sqsubseteq_c E$.

We note that the $\mathbb{E}_0$ dichotomy extends prior results of Glimm and Effros, who proved this in the case that E is induced by a continuous action of a locally compact Polish group (and in particular when E is countable).

For graphs, we have the Kechris–Solecki–Todorčević Theorem, also referred to as the $\mathbb{G}_0$ *dichotomy*. Given directed graphs G, H on vertex sets X, Y, a *homomorphism* from G to H is a map $f : X \to Y$ such that $xGx' \implies f(x)Hf(x')$. If G is a graph on a Polish space X, we say G has *countable Borel chromatic number*, written $\chi_B(G) \leq \aleph_0$, if there is a Borel homomorphism from G to the complete graph on $\mathbb{N}$ (where $\mathbb{N}$ has the discrete topology). Put another way, $\chi_B(G) \leq \aleph_0$ if there is a cover of X by countably many Borel G-independent sets.

To define the canonical "complicated" graph $\mathbb{G}_0$, we need a few definitions. We say a set $S \subseteq 2^{<\mathbb{N}}$ is *sparse* if it contains at most one sequence of every length, and *dense* if for all $t \in 2^{<\mathbb{N}}$ there is some $s \in S$ for which $t \subseteq s$. For $s \in 2^{<\mathbb{N}}$, let $\mathbb{G}_s$ denote the directed graph on $2^{\mathbb{N}}$ given by

$$\mathbb{G}_s = \{(s^\frown 0^\frown x, s^\frown 1^\frown x) : x \in 2^{\mathbb{N}}\},$$

where $^\frown$ denotes concatenation of sequences and we identify $i \in 2$ with the sequence $(i) \in 2^1$. Let $\mathbb{G}_S = \bigcup_{s \in S} \mathbb{G}_s$ for $S \subseteq 2^{<\mathbb{N}}$, and write $\mathbb{G}_0$ to denote the graph $\mathbb{G}_S$ for some fixed sparse dense set S.

Theorem 5.1.3 (The $\mathbb{G}_0$ dichotomy, Kechris–Solecki–Todorčević)**.** *Let G be a directed analytic graph on a Polish space. Exactly one of the following holds:*

1. $\chi_B(G) \leq \aleph_0$.
2. *There is a continuous homomorphism of $\mathbb{G}_0$ into G.*

The graph $\mathbb{G}_0$ depends on the choice of S, however all such graphs are acyclic Borel graphs of uncountable Borel chromatic number, and $\mathbb{G}_S$ embeds continuously as a subgraph of $\mathbb{G}_{S'}$ for all such choices of S, S'. This was subsequently generalized to (directed) hypergraphs by Lecomte.

The original proofs of all of these dichotomies used techniques of effective descriptive set theory. Miller later found a classical proof of the $\mathbb{G}_0$ dichotomy, and used this to give a classical proof of Silver's Theorem and the $\mathbb{E}_0$ dichotomy for countable Borel equivalence relations. Miller also proved various generalizations of the $\mathbb{G}_0$ dichotomy, and used these generalizations to give a classical proof of the $\mathbb{E}_0$ dichotomy in full. Notably, Miller showed that many descriptive set theoretic dichotomies follow from appropriate graph-theoretic dichotomies and simple Mycielski-style Baire category arguments. We refer the reader to [Mil12] for a survey of these results, as well as many other examples.

As noted above, Miller's proof of the $\mathbb{E}_0$ dichotomy used a new generalization of the $\mathbb{G}_0$ dichotomy, and it has remained an open problem since then whether there is a direct proof of the $\mathbb{E}_0$ dichotomy from the $\mathbb{G}_0$ dichotomy. We show that this is indeed the case.

Theorem 5.1.4. *There is a proof of the $\mathbb{E}_0$ dichotomy directly from the $\mathbb{G}_0$ dichotomy.*

We give a proof of this in Section 5.2.

The remainder of this chapter explores other constructions related to these graph-theoretic dichotomies, which we summarize below. In what follows, we assume all graphs are directed, though all of our results and proofs apply identically to undirected graphs.

We first consider dichotomies of Miller for sequences of graphs. If $(G_n)_{n\in\mathbb{N}}$ is a sequence of graphs on a Polish space X, we write $\chi_B((G_n)) \le \aleph_0$ if there is a cover of X by countably many Borel sets so that each of these sets is G_n-independent for some $n \in \mathbb{N}$. If $(G_n)_{n\in\mathbb{N}}, (H_n)_{n\in\mathbb{N}}$ are sequences of graphs on X, Y respectively, we say a map $f : X \to Y$ is a *homomorphism* from (G_n) to (H_n) if f is a homomorphism from G_n to H_n for all $n \in \mathbb{N}$.

Let $\mathcal{S}$ be a collection of subsets of $2^{<\mathbb{N}}$. We say $\mathcal{S}$ is *sparse* if $\bigcup \mathcal{S}$ is sparse, and *dense* if every element of $\mathcal{S}$ is dense.

Theorem 5.1.5 ($\mathbb{G}_0$ for sequences, [Mil12, Theorem 21])**.** *Let $(S_n)_{n\in\mathbb{N}}$ be a sparse and dense sequence of subsets of $2^{<\mathbb{N}}$. For any sequence $(G_n)_{n\in\mathbb{N}}$ of analytic graphs on a Polish space, exactly one of the following holds:*

1. $\chi_B((G_n)) \le \aleph_0$.
2. *There is a continuous homomorphism from $(\mathbb{G}_{S_n})_{n\in\mathbb{N}}$ to $(G_n)_{n\in\mathbb{N}}$.*

We remark that this follows from the $\aleph_0$-dimensional $\mathbb{G}_0$ dichotomy (the generalization of the $\mathbb{G}_0$ dichotomy to $\aleph_0$-dimensional hypergraphs), as noted in [Mil12].

Theorem 5.1.6 ($\mathbb{G}_0$ for sequences-of-sequences, [Mil22, Theorem 1])**.** *Let $(S_n)_{n\in\mathbb{N}}$ be a sparse and dense sequence of subsets of $2^{<\mathbb{N}}$. For any doubly-indexed sequence $(G_{i,j})_{i,j\in\mathbb{N}}$ of analytic graphs on a Polish space X that is* increasing in the second coordinate*, i.e., $G_{i,j} \subseteq G_{i,j+1}$, exactly one of the following holds:*

1. *There is a cover of X by a sequence of Borel sets $(B_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $\chi_B((G_{i,j}\restriction B_i)_{j\in\mathbb{N}}) \leq \aleph_0$.*
2. *There is a map $f : \mathbb{N} \to \mathbb{N}$ and a continuous homomorphism from $(\mathbb{G}_{S_n})_{n\in\mathbb{N}}$ to $(G_{n,f(n)})_{n\in\mathbb{N}}$.*

We show in Section 5.3 that Theorem 5.1.6 follows from Theorem 5.1.5, and in particular that

Theorem 5.1.7. *The $\mathbb{G}_0$ dichotomy for doubly-indexed sequences of graphs follows from the $\aleph_0$-dimensional $\mathbb{G}_0$ dichotomy.*

Our proof uses in a crucial way the compactness of $2^{\mathbb{N}}$, and hence generalizes immediately to d-dimensional hypergraphs for d finite. We show that compactness is in some sense necessary in this result, by showing that it fails in the $\aleph_0$-dimensional case.

Theorem 5.1.8. *The d-dimensional analogue of the $\mathbb{G}_0$ dichotomy for doubly-indexed sequences of graphs is true for $d \in \mathbb{N}$, and false for $d = \aleph_0$.*

We also prove a dichotomy for triply-indexed sequences of graphs.

Suppose now that $G_0, \ldots, G_{n-1}$ is a finite sequence of graphs on a Polish space X, and let $G = \bigcup_{i<n} G_i$. It is easy to see that $\chi_B(G) \leq \aleph_0$ iff $\chi_B(G_i) \leq \aleph_0$ for all $i < n$. Thus, by the $\mathbb{G}_0$ dichotomy, if all of these graphs are analytic and there is a continuous homomorphism of $\mathbb{G}_0$ into G, then there is a continuous homomorphism of $\mathbb{G}_0$ into G_i for some $i < n$.

In Section 5.4, we consider the following question: Given a continuous homomorphism of $\mathbb{G}_0$ into G, can we show that there must be a continuous homomorphism of $\mathbb{G}_0$ into some G_i *without* applying the $\mathbb{G}_0$ dichotomy? We show that this is indeed the case:

Theorem 5.1.9. *Let $G_0, \ldots, G_{n-1}$ be a sequence of Baire measurable graphs on $2^{\mathbb{N}}$ for which $\mathbb{G}_0 = \bigcup_{i<n} G_i$. Then one can construct a continuous embedding of $\mathbb{G}_0$ into G_i for some $i < n$,* without *applying the $\mathbb{G}_0$ dichotomy.*

Our proof uses a Mycielski-style Baire category construction, and gives a concrete combinatorial condition that specifies which of the graphs G_i we will embed $\mathbb{G}_0$ into. This has potential applications to more complicated situations, where one cannot simply fix in advance some graphs to which we apply the $\mathbb{G}_0$ dichotomy. For example, this idea is used in the proofs of Proposition 4.2.1 and Theorem 4.1.11 where we recursively construct a graph-theoretic embedding, and at every step of the construction we have to choose one of finitely many subgraphs to continue to embed into in a way that depends on our previous choices.

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5.2 A proof of the $\mathbb{E}_0$ dichotomy from the $\mathbb{G}_0$ dichotomy

Let E be a Borel equivalence relation on a Polish space X. Let $Y = X \times X \setminus E$, and put a Polish topology on Y making the projections continuous. Define a *directed* Borel graph G on Y by setting $(a_0, b_0)G(a_1, b_1) \iff a_0 E b_1$.

Claim 5.2.1. E is smooth iff $\chi_B(G) \leq \aleph_0$.

Proof. Note that E is smooth iff it has a countable Borel separating family, i.e., a sequence of Borel sets $B_i \subseteq X$ so that $aEb \iff \forall i(a \in B_i \iff b \in B_i)$.

Suppose E is smooth, and let B_i be a countable Borel separating family for E. Then $Y = \bigcup_i (B_i \times \sim B_i) \cup (\sim B_i \times B_i)$, and these sets are G-independent.

Conversely, suppose that A_i is a cover of Y by countably many G-independent Borel sets. Then the pairs $(\mathrm{proj}_0(A_i), \mathrm{proj}_1(A_i))$ are E-independent, meaning that if $a \in \mathrm{proj}_0(A_i), b \in \mathrm{proj}_1(A_i)$ then $a \not{E} b$. By a standard reflection argument (see e.g. [dRM, Proposition 2.1]), there is a Borel E-invariant set $B_i \supseteq \mathrm{proj}_0(A_i)$ so that $(B_i, \mathrm{proj}_1(A_i))$ is E-independent, and it is easy to verify that the sets B_i form a countable Borel separating family for E. $\triangleleft$

Remark 5.2.2. Ben Miller has pointed out that the graph G we define here was first considered (at least in the context of descriptive set theory) by Louveau, and Claim 5.2.1 was essentially proven by Lecomte (see [LM08], bottom of page 2).

Suppose now that E is not smooth. Let $S \subseteq 2^{<\omega}$ be a dense set which contains exactly one sequence of every finite length $n \in \omega$. By the directed $\mathbb{G}_0$ dichotomy and the previous claim, there is a continuous homomorphism $\psi : 2^\omega \to Y$ from $\mathbb{G}_S$ to G.

Let now H be the (undirected) graph given by

$$xHz \iff \exists y(x\mathbb{G}_S y \ \& \ z\mathbb{G}_S y),$$

and define xFy iff x, y are in the same connected component of H. Let E' be the pullback of E along $\mathrm{proj}_0 \circ\psi$, and let D be the pullback of $\Delta(X)$ along $\mathrm{proj}_0 \circ\psi$.

Claim 5.2.3. E' is meagre and D is closed and nowhere dense.

Proof. As $\mathrm{proj}_0 \circ\psi$ is continuous and $D \subseteq E'$, it suffices to show that E' is meagre. By the Kuratowski–Ulam Theorem, it suffices to show that every E'-class is meagre. Let C be an E'-class, and suppose for the sake of contradiction that C is non-meagre. Then by [KST99, Proposition 6.2], there are $x, y \in C$ with $x\mathbb{G}_0 y$. If we write $\psi(x) = (a_0, b_0), \psi(y) = (a_1, b_1)$, then we have $a_0 E a_1$ (as x, y are in the same E'-class) and $a_0 E b_1$ (as ψ is a directed homomorphism from $\mathbb{G}_0$ to G). It follows that $a_1 E b_1$ and $\psi(y) = (a_1, b_1) \in Y = X \times X \setminus E$, a contradiction. $\triangleleft$

Claim 5.2.4. Let $R \subseteq 2^\omega \times 2^\omega$ be meagre and $D \subseteq 2^\omega \times 2^\omega$ be closed and nowhere dense. Then there is a continuous homomorphism $\phi : (\mathbb{G}_0, \sim\Delta(2^\omega), \sim\mathbb{E}_0) \to (F, \sim D, \sim R)$.

Apply these two claims to E', D, in order to get a continuous homomorphism $\phi : (\mathbb{G}_0, \sim\Delta(2^\omega), \sim\mathbb{E}_0) \to (F, \sim D, \sim E')$. Then $\mathrm{proj}_0 \circ\psi \circ \phi : 2^\omega \to X$ is a continuous embedding of $\mathbb{E}_0$ into E. Indeed, by definition of D this composition is injective. Moreover, $\mathrm{proj}_0 \circ\psi$ is a reduction of E' into E, and ϕ is a reduction of $\mathbb{E}_0$ into E'. (To see this last part, note that $H \subseteq E'$, and thus $F \subseteq E'$: if xHz, then there is some y so that $x\mathbb{G}_0 y$ and $z\mathbb{G}_0 y$, in which case $\mathrm{proj}_0(\psi(x)) E \,\mathrm{proj}_1(\psi(y)) E \,\mathrm{proj}_0(\psi(z))$.)

Thus it remains only to prove Claim 5.2.4.

Proof of Claim 5.2.4. We define directed graphs $\mathbb{G}_S$ on 2^n by setting, for $u, v \in 2^n$, $u\mathbb{G}_S v$ iff there are $s \in S \cap 2^{<n}, t \in 2^{n-1-|s|}$ so that $u = s^\frown 0^\frown t, v = s^\frown 1^\frown t$. We define (undirected) graphs H on 2^n by $uHv \iff \exists w(u\mathbb{G}_S w \ \& \ v\mathbb{G}_S w)$, and let F be the equivalence relation on 2^n given by the connected components of H. (We abuse notation here and let $\mathbb{G}_S, H, F$ denote graphs on 2^n for all $n \leq \omega$.) Note that

$$\forall \star \in \{\mathbb{G}_0, H, F\}, u, v \in 2^n, x \in 2^{\leq\omega}(u \star v \iff u^\frown x \star v^\frown x). \tag{$*$}$$

Subclaim 5.2.5. For any $u, v \in 2^n$, there are some $u', v' \in 2^k, k < \omega$ so that $u^\frown u' F v^\frown v'$.

Proof. It is a standard fact that for all $u, v \in 2^n, n < \omega$ there is an undirected path from u to v in $\mathbb{G}_S$. (One can prove this easily by induction on n, using $(*)$ and the fact that S contains a sequence of every finite length.) We prove the subclaim for all $u, v \in 2^n, n < \omega$, by induction on the distance from u to v in $\mathbb{G}_S$.

This is clearly true if the distance is 0, i.e., $u = v$. Suppose now that u, v are neighbours, i.e., $u \mathbb{G}_S v$ or $v \mathbb{G}_S u$. By the symmetry of F, we may assume wlog that $u \mathbb{G}_S v$. As S is dense, there is some $t \in 2^{<\omega}$ with $v^\frown t \in S$. But then $u^\frown t^\frown 1 \mathbb{G}_S v^\frown t^\frown 1$ by $(*)$, and $v^\frown t^\frown 0 \mathbb{G}_S v^\frown t^\frown 1$, so that $u' = t^\frown 1, v' = t^\frown 0$ works.

Finally, suppose that we have shown that this holds for all u, v of distance at most m in $\mathbb{G}_S$, for $m \geq 1$, and let u, v be of distance $m+1$ apart in $\mathbb{G}_S$. Let w be the last vertex before v on an undirected path from u to v of length $m+1$. By assumption, there are $u', v' \in 2^k$ with $u^\frown u' F w^\frown v'$. Note that $w^\frown v', v^\frown v'$ are adjacent in $\mathbb{G}_S$ by $(*)$, so by assumption there are $u'', v'' \in 2^l$ with $w^\frown v'^\frown u'' F v^\frown v'^\frown v''$. By $(*)$ and the transitivity of F,

$$u^\frown u'^\frown u'' F w^\frown v'^\frown u'' F v^\frown v'^\frown v'',$$

so the pair $u'^\frown u'', v'^\frown v'' \in 2^{k+l}$ satisfies the subclaim for u, v. ◁◁

Let now U_n be a decreasing sequence of dense open sets in $2^\omega \times 2^\omega$ so that $D \cap U_0 = \emptyset$ and $R \cap \bigcap_n U_n = \emptyset$. We will define recursively pairs of finite sequences $(t_0^n, t_1^n) \in \bigcup_{m>0} 2^m \times 2^m$ and maps $\phi_n : 2^n \to 2^{<\omega}$ satisfying for all $n < \omega$:

1. $\phi_0(\emptyset) = \emptyset$;
2. $\phi_{n+1}(u^\frown(i)) = \phi_n(u)^\frown t_i^n$ for $u \in 2^n, i \in 2$;
3. $N_{\phi_n(u)^\frown t_0^n} \times N_{\phi_n(v)^\frown t_1^n} \subseteq U_n$ for all $u, v \in 2^n$; and
4. if $s \in S \cap 2^n$, then $\phi_n(s)^\frown t_0^n F \phi_n(s)^\frown t_1^n$.

(Here, $N_t = \{x \in 2^\omega : t \subseteq x\}$ for $t \in 2^{<\omega}$.)

Assuming this has been done, define $\phi(x) = \bigcup_n \phi_n(x \restriction n)$ for $x \in 2^\omega$. Then $\phi : 2^\omega \to 2^\omega$ is continuous, and by 2, 3 it is a homomorphism $\sim\Delta(2^\omega) \to \sim D$ and $\sim\mathbb{E}_0 \to \sim R$. It is also easy to see by 2 that for any $u, v \in 2^n, x \in 2^\omega$, there is some $y \in 2^\omega$ so that $\phi(u^\frown x) = \phi_n(u)^\frown y, \phi(v^\frown x) = \phi_n(v)^\frown y$. By 2, 4, and (∗), it follows that ϕ is a homomorphism from $\mathbb{G}_0$ to F.

It remains to define the sequences (t_0^n, t_1^n) and maps ϕ_n. Suppose we have constructed these sequences and maps for all $m < n$. We define ϕ_n to be the unique map satisfying 1, 2. Because U_n is dense and open, there are sequences w_0, w_1 so that $N_{\phi_n(u)^\frown w_0} \times N_{\phi_n(v)^\frown w_1} \subseteq U_n$ for all $u, v \in 2^n$. (That this is true for a single pair u, v is immediate; we recursively apply this fact to all such pairs, extending w_0, w_1 at each step, in order to find sequences that work for all u, v.) We may ensure that $|w_0| = |w_1| > 0$.

Let now $s \in S \cap 2^n$. By the subclaim, there are $w_0', w_1' \in 2^k, k < \omega$ so that $\phi_n(s)^\frown w_0^\frown w_0' F \phi_n(s)^\frown w_1^\frown w_1'$. We let $t_i^n = w_i^\frown w_i'$, and note that 3, 4 are satisfied by this choice. $\triangleleft$

5.3 Dichotomies for doubly-indexed sequences of graphs

We begin by giving a proof of Theorem 5.1.6 from Theorem 5.1.5.

Proof of Theorem 5.1.6 from Theorem 5.1.5. Define graphs H_i on $\mathbb{N}^\mathbb{N} \times X$ by

$$(f, x) H_i (g, y) \iff f(i) = g(i) \ \ \& \ \ x G_{i,f(i)} y.$$

This is a sequence of analytic digraphs to which we apply Theorem 5.1.5.

Case 1: $\chi_B((H_i)) \leq \aleph_0$. For $B \subseteq \mathbb{N}^\mathbb{N} \times X$, let $B^{i,j} = \{(f, x) \in B : f(i) = j\}$. Note that if B is H_i-independent, then $\mathrm{proj}_X(B^{i,j})$ is $G_{i,j}$-independent for all j. Now fix a cover of $\mathbb{N}^\mathbb{N} \times X$ by Borel sets $B_{i,k}$ such that $B_{i,k}$ is H_i-independent for all i, k. Let $A_{i,j,k} = \mathrm{proj}_X(B_{i,k}^{i,j})$, and note that each $A_{i,j,k}$ is analytic and $G_{i,j}$-independent. Fix $G_{i,j}$-independent Borel sets $A'_{i,j,k} \supseteq A_{i,j,k}$, and let $A_i = \bigcap_j \bigcup_k A'_{i,j,k}$. Then alternative 1 of Theorem 5.1.6 holds for the sets A_i, so it remains to show that these sets cover X.

Suppose for the sake of contradiction that $x \notin \bigcup_i A_i$. Then $x \notin \bigcup_i \bigcap_j \bigcup_k A_{i,j,k}$, so there is some $f : \mathbb{N} \to \mathbb{N}$ such that $x \notin A_{i,f(i),k}$ for all i, k. By assumption there are i, k with $(f, x) \in B_{i,k}$, in which case $x \in \mathrm{proj}_X(B_{i,k}^{i,f(i)}) = A_{i,f(i),k}$, a contradiction.

Case 2: There is a continuous homomorphism $\phi : 2^\omega \to X$ from $(\mathbb{G}_{S_i})_i$ to $(H_i)_i$. Then $\mathrm{proj}_{\mathbb{N}^\mathbb{N}} \circ \phi : 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is continuous, so its image is compact and hence bounded (pointwise), say by $f \in \mathbb{N}^\mathbb{N}$. We verify that $f, \phi_X = \mathrm{proj}_X \circ \phi$ satisfy the second alternative of Theorem 5.1.6: Let $x \mathbb{G}_{S_i} y$ and $\phi(x) = (g, \phi_X(x)), \phi(y) = (h, \phi_X(y))$ for some $g, h \in \mathbb{N}^\mathbb{N}$. Then $g(i) = h(i) \leq f(i)$ and $\phi_X(x) G_{i,g(i)} \phi_X(y)$. Since the graphs G are increasing in the second coordinate, $\phi_X(x) G_{i,f(i)} \phi_X(y)$. $\square$

One can state and prove a d-dimensional analogue of Theorem 5.1.5 for $d \leq \aleph_0$, using the $\aleph_0$-dimensional $\mathbb{G}_0$ dichotomy. When d is finite, the graphs live on the compact space $d^{\mathbb{N}}$, so an identical proof to the one above for the $d = 2$ case gives a d-dimensional analogue of Theorem 5.1.6 for d finite.

We now give two examples to show that for Theorem 5.1.6 (a) the assumption that $(G_{i,j})_{i,j}$ is increasing in the second coordinate cannot be dropped, and (b) the $\aleph_0$-dimensional analogue of this theorem is false.

Example 5.3.1 (Sequences that are not increasing in the second coordinate)**.** Let $(S_i)_i$ be a dense sequence subsets of $2^{<\mathbb{N}}$ such that S_i contains only sequences of length at least i. For $i \in \mathbb{N}$ fix an enumeration t^i_j of 2^i and let $S_{i,j} = \{s \in S_i : t^i_j \subseteq s\}$. Let $G_{i,j} = \mathbb{G}_{S_{i,j}}$. This is a collection of Borel graphs such that for all $i \in \mathbb{N}$, there are only finitely-many $G_{i,j}$. Note moreover that this collection of graphs is not increasing in the second coordinate (in fact the graphs $(G_{i,j})_j$ for i fixed are pairwise disjoint).

For any i, $\bigcup_{j<2^i} G_{i,j} = \mathbb{G}_{S_i}$, so if $\chi_B(G_{i,j} \restriction B) \leq \aleph_0$ for all $j < 2^i$ then $\chi_B(\mathbb{G}_{S_i} \restriction B) \leq \aleph_0$. Thus by [KST99, Proposition 6.2], if B is Baire measurable then it is meagre. It follows that alternative 1 of Theorem 5.1.6 fails for $(G_{i,j})_{i,j}$.

Suppose for the sake of contradiction that there is some $f : \mathbb{N} \to \mathbb{N}$ and a continuous homomorphism $\phi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $(\mathbb{G}_{S_i})_i$ to $(G_{i,f(i)})_i$. Let $u_i = t^i_{f(i)}$ for $i \in \mathbb{N}$, which is possible as $f(i) < 2^i$. We claim that the image of ϕ is contained in N_{u_i} for all i, where $N_t = \{x \in 2^{\mathbb{N}} : t \subseteq x\}$. Otherwise, there is some open set $U \subseteq 2^\omega$ and some $u_i \neq v \in 2^i$ such that ϕ maps U into N_v. There are $x, y \in U$ with $x \mathbb{G}_{S_i} y$, so by assumption $\phi(x) G_{i,f(i)} \phi(y)$, and hence $\phi(x), \phi(y) \in N_{t^i_{f(i)}} = N_{u_i}$, a contradiction.

It follows that $u_0 \subseteq u_1 \subseteq u_2 \dots$, and if $u = \bigcup_i u_i \in 2^{\mathbb{N}}$ then ϕ is the constant function with value u, a contradiction. Thus alternative 2 of Theorem 5.1.6 fails for $(G_{i,j})_{i,j}$ as well.

Example 5.3.2 ($\aleph_0$-dimensional graphs)**.** We define the notion of density for subsets of $\mathbb{N}^{<\mathbb{N}}$ exactly as we did for subsets of $2^{<\mathbb{N}}$. We define $\mathbb{G}^{\aleph_0}_s, \mathbb{G}^{\aleph_0}_S$ analogously as well, e.g., by setting

$$\mathbb{G}^{\aleph_0}_s = \{(s^\frown n^\frown x)_{n \in \mathbb{N}} : x \in \mathbb{N}^{\mathbb{N}}\}$$

for $s \in \mathbb{N}^{<\mathbb{N}}$. Let $(S_i)_i$ be a dense sequence of subsets of $\mathbb{N}^{<\mathbb{N}}$ such that S_i contains only sequences of length at least i. For $i \in \mathbb{N}$, fix an enumeration t^i_j of $\mathbb{N}^i$, and let $S_{i,j} = \{s \in S_i : t^i_j \subseteq s\}$. Let $G_{i,j} = \bigcup_{l \leq j} \mathbb{G}^{\aleph_0}_{\boldsymbol{S}_{i,l}}$. This is a doubly-indexed sequence of Borel $\aleph_0$-dimensional digraphs which is increasing in the second coordinate.

Suppose B is Baire measurable, $i \in \mathbb{N}$, and $\chi_B(G_{i,j} \restriction B) \leq \aleph_0$ for all j. The usual argument shows that B is meagre in $N_{t^i_j}$ for all j, and hence B is meagre. It follows that alternative 1 of Theorem 5.1.6 fails for $(G_{i,j})_{i,j}$.

Suppose now that there was some $f : \mathbb{N} \to \mathbb{N}$, a comeagre set $C \subseteq \mathbb{N}^{\mathbb{N}}$, and a continuous homomorphism $\phi : C \to \mathbb{N}^{\mathbb{N}}$ of $(\mathbb{G}^{\aleph_0}_{S_i})_i$ to $(G_{i,f(i)})_i$. Let $A_i = \{t^i_j : j \leq f(i)\}$. We claim that the image of ϕ is contained in $\bigcup\{N_t : t \in A_i\}$ for all i, which can be seen just as in the previous example.

Let T be the tree of sequences whose initial segments are all contained in $\bigcup_i A_i$. Then the image of ϕ is contained in $[T]$, a compact set, which is bounded (pointwise). But every bounded set is $G_{i,j}$-independent for all i, j, a contradiction.

The dichotomy theorems we have discussed thus far all admit an effective analogue (see for example [KST99, Theorem 6.4]). In these effective dichotomies, the “simple” case has an effective witness, whereas the “complex” case does not.

We show that this is also the case for the function f in alternative 2 of Theorem 5.1.6.

Example 5.3.3 ((In)effective bounds on f)**.** Fix a coding $F \ni n \mapsto F_n$ of the Δ^1_1 points in $\mathbb{N}^{\mathbb{N}}$, i.e., sets $F \in \Pi^1_1(\mathbb{N}), P \in \Pi^1_1(\mathbb{N} \times \mathbb{N}^{\mathbb{N}}), S \in \Sigma^1_1(\mathbb{N} \times \mathbb{N}^{\mathbb{N}})$ so that if $n \in F$ then $P_n = S_n$ are singletons, which we denote $\{F_n\}$, and so that every Δ^1_1 point in $\mathbb{N}^{\mathbb{N}}$ appears in this way (see e.g. [Mos09, Section 4D]). Let $A(i,j) \iff \forall n \leq i(n \in F \implies F_n(i) < j)$. Then $A \in \Sigma^1_1$ and for all i, A_i is an interval $[j, \infty)$, but there is no $f \in \Delta^1_1$ with $A(i, f(i))$ for all $i \in \omega$. Indeed, if f were such a function, then there would be $i \in F$ with $f = F_i$, in which case we would have $f(i) = F_i(i)$ and thus $\neg A(i, f(i))$, a contradiction.

Let $S \subseteq 2^{<\mathbb{N}}$ be dense and computable, and let $x G_{i,j} y \iff x \mathbb{G}_S y \;\&\; A(i,j)$. Then $(G_{i,j})_{i,j}$ is a (uniformly) Σ^1_1 family of digraphs for which alternative 2 holds in Theorem 5.1.6. However, if f, ϕ witness this, then $A(i, f(i))$ for all $i \in \omega$, so $f \notin \Delta^1_1$.

Problem 5.3.4. Can we find such an example where $(G_{i,j})_{i,j}$ is uniformly Δ^1_1?

We give also a possible generalization of Theorem 5.1.6 to triply-indexed sequences of graphs.

Theorem 5.3.5. *Let $(S_{i,k})_{i,k}$ be a dense, sparse family of subsets of $2^{<\omega}$. Let $(G_{i,j,k})_{i,j,k}$ be a triply-indexed sequence of analytic digraphs on a Polish space X which is increasing in the second coordinate, i.e., $G_{i,j,k} \subseteq G_{i,j+1,k}$. For $i, j \in \mathbb{N}$, let $\boldsymbol{G}_{i,j} = (G_{i,j,k})_{k\in\omega}$. Then exactly one of the following holds:*

1. *There is a cover of X by countably-many Borel sets B_i such that $\chi_B(\boldsymbol{G}_{i,j} \restriction B_i) \leq \aleph_0$ for all $i, j \in \mathbb{N}$.*

2. *There is a map $f : \mathbb{N} \to \mathbb{N}$ and a continuous homomorphism $\phi : 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_{S_{i,k}})_{i,k \in \mathbb{N}}$ to $(G_{i,f(i),k})_{i,k \in \mathbb{N}}$.*

Proof sketch. We argue as in our proof of Theorem 5.1.6, but we apply Theorem 5.1.5 to the family of graphs

$$(f, x) \boldsymbol{H}_{i,k} (g, y) \iff f(i) = g(i) \ \& \ x \boldsymbol{G}_{i,f(i),k} y.$$

□

Problem 5.3.6. Is there a generalization of this to e.g. families of analytic graphs indexed by well-founded trees that satisfy some sort of monotonicity condition?

5.4 A notion of largeness for subgraphs of $\mathbb{G}_0$

Fix a sparse dense set $S \subseteq 2^{<\mathbb{N}}$ and write $\mathbb{G}_0 = \mathbb{G}_S$. Assume for convenience that S contains exactly one sequence of every finite length, and let $s_n \in S \cap 2^n$.

Let $(H_i)_{i \in l}$ be a finite sequence of Baire measurable graphs on $2^{\mathbb{N}}$ such that $\mathbb{G}_0 \subseteq \bigcup_{i \in l} H_i$. We will show that there is a continuous embedding of $\mathbb{G}_0$ into H_i for some $i \in l$, without applying the $\mathbb{G}_0$ dichotomy.

To begin, define $A_{n,i} = \{x \in 2^{\mathbb{N}} : (s_n^\frown 0^\frown x, s_n^\frown 1^\frown x) \in H_i\}$, and note that each $A_{n,i}$ is Baire measurable. Let $S_i = \{s_n : A_{n,i} \text{ is non-meagre}\}$, and let $T_i \subseteq 2^{<\mathbb{N}}$ be the smallest tree containing S_i.

Note that $\bigcup_{i \in l} S_i = S$ is dense, so $\bigcup_{i \in l} T_i = 2^{<\mathbb{N}}$. Moreover, we have

$$\bigcup_{i \in l} [T_i] = [\bigcup_{i \in l} T_i] = [2^{<\mathbb{N}}] = 2^{\mathbb{N}}$$

because this collection of trees is finite. It follows that some $[T_i]$ is non-meagre in $2^{\mathbb{N}}$. We may assume wlog that $[T_0]$ is non-meagre, and since it is closed, that there is some $t_0 \in 2^{<\mathbb{N}}$ such that $t_0 \subseteq t \implies t \in T_0$. In other words, S_0 is *dense below* t_0, meaning that if $t_0 \subseteq t$ then there is some $s \in S_0$ with $t \subseteq s$.

Finally, for each $s_n \in S_0$ we fix $\overline{s}_n \in 2^{<\mathbb{N}}$ such that $A_{n,0}$ is comeagre in $N_{\overline{s}_n}$, and we fix a decreasing sequence of open sets $(U_k^n)_{k \in \mathbb{N}}$ that are dense in $N_{\overline{s}_n}$ and which satisfy $U_0^n = N_{\overline{s}_n}$ and $A_{n,0} \supseteq \bigcap_k U_k^n$. For $B \subseteq 2^{\mathbb{N}}$ and $t \in 2^{<\mathbb{N}}$, write $t^\frown B = \{t^\frown x : x \in B\}$.

We will now recursively construct sequences $u_n \in 2^{<\mathbb{N}}$ and a map $f : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ satisfying:

(1) $t_0 \subseteq f(\emptyset)$ and $f(s^\frown i) = f(s)^\frown i^\frown u_n$ for all $s \in 2^n, i \in 2$;

(2) for all $n \in \mathbb{N}$ there is some $m = m(n)$ such that $f(s_n) = s_m \in S_0$;

(3) $N_{f(s_n{}^\frown i^\frown t)} \subseteq s_{m(n)}{}^\frown i^\frown U_k^{m(n)}$ for all $n \in \mathbb{N}, i \in 2, t \in 2^k$.

Suppose this has been done, and set $\pi(x) = \bigcup_n f(x{\restriction}n)$ for $x \in 2^{\mathbb{N}}$. By (1), π is a continuous embedding of $2^{\mathbb{N}}$ into itself. By (1) and (2), for all $n \in \mathbb{N}$ and $x \in 2^{\mathbb{N}}$ there is some m and $y \in 2^{\mathbb{N}}$ such that $s_m \in S_0$ and $\pi(s_n{}^\frown i^\frown x) = s_m{}^\frown i^\frown y$ for $i \in 2$, and by (3) $y \in \bigcap_k U_k^m \subseteq A_{m,0}$, so that $(\pi(s_n{}^\frown 0^\frown x), \pi(s_n{}^\frown 1^\frown x)) \in H_0$ and π is a continuous embedding of $\mathbb{G}_0$ into H_0.

We now construct the sequences u_n and the map f. First, we let $f(\emptyset) \in S_0$ be such that $t_0 \subseteq f(\emptyset)$, which is possible as S_0 is dense below t_0.

Suppose now that we have constructed $u_i, i < n$ and $f : 2^{<n} \to 2^{<\mathbb{N}}$ satisfying (1)-(3) on $2^{<n}$. We begin by finding a sequence $v \in 2^{<\mathbb{N}}$ such that $\overline{s}_{m(n-1)} \subseteq v$ and for all $k < n-1, t \in 2^{n-k-2}, i, j \in 2$ we have $N_{f(s_k{}^\frown i^\frown t)^\frown j^\frown v} \subseteq s_{m(k)}{}^\frown i^\frown U_{n-k}^{m(k)}$. To do so, it suffices to show that for any particular such k, t, i, j and any $v' \in 2^{<\mathbb{N}}$ there is some $v \supseteq v'$ which satisfies this condition, as we may then recursively extend $\overline{s}_{m(n-1)}$ to define a v which works for all such quadruples. Given k, t, i, j, v', this is always possible as $U_{n-k}^{m(k)}$ is open and dense in $N_{\overline{s}_{m(k)}}$, and $s_{m(k)}{}^\frown i^\frown \overline{s}_{m(k)} \subseteq f(s_k{}^\frown i^\frown t)$ by (3).

Next, let $t \in 2^{n-1}, j \in 2$ be such that $s_n = t^\frown j$. Since $t_0 \subseteq f(t)$ and S_0 is dense below t_0, there is some $u \supseteq v$ and $m \in \mathbb{N}$ such that $f(t)^\frown j^\frown u = s_m \in S_0$. We then set $u_n = u$ and $m(n) = m$, and define $f(s^\frown i) = f(s)^\frown i^\frown u_n$ for all $s \in 2^{n-1}, i \in 2$.

It is clear that $f : 2^{\leq n} \to 2^{<\mathbb{N}}$ satisfies (1) and (2) on $2^{\leq n}$. To show that (3) is satisfied, note that $f(s_{n-1}{}^\frown i) = f(s_{n-1})^\frown i^\frown u_n \supseteq s_{m(n-1)}{}^\frown i^\frown \overline{s}_{m(n-1)}$ and $U_0^{m(n-1)} = N_{\overline{s}_{m(n-1)}}$, so (3) holds for s_{n-1}. For $k < n-1, t \in 2^{n-k-2}, i, j \in 2$, we have $f(s_k{}^\frown i^\frown t^\frown j) \supseteq f(s_k{}^\frown i^\frown t)^\frown j^\frown v$, so (3) holds for this quadruple by our choice of v.

Appendix A

DITZEN'S EFFECTIVE VERSION OF NADKARNI'S THEOREM

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A.1 Introduction

In effective descriptive set theory one often considers the following type of question: Suppose we are given a (lightface) Δ^1_1 structure R on the Baire space $\mathcal{N}$ (like, e.g., an equivalence relation, graph, etc.) and a problem about R that admits a (classical) $\mathbf{\Delta^1_1}$ (i.e., Borel) solution. Is there an effective, i.e., Δ^1_1, solution?

For example, consider the case where $R = E$ is a Δ^1_1 equivalence relations which is **smooth**, i.e., admits a Borel function $f\colon \mathcal{N} \to \mathcal{N}$ such that $xEy \iff f(x) = f(y)$. Then it turns out that one can find such a function which is actually Δ^1_1.

One often derives such results via an effective version of a dichotomy theorem, For instance, for the example of smoothness above we have the following classical version of the so-called General Glimm-Effros Dichotomy proved in [HKL90]. Below $\mathbb{E}_0$ is the equivalence relation on the Cantor space $\mathcal{C}$ given by $x\mathbb{E}_0 y \iff \exists m \forall n \geq m(x(n) = y(n))$.

Theorem A.1.1 (General Glimm-Effros Dichotomy, see [HKL90]). *Let E be a Borel equivalence relation on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

(i) E is smooth, i.e., admits a Borel function $f\colon \mathcal{N} \to \mathcal{N}$ such that $xEy \iff f(x) = f(y)$,

(ii) There is a Borel injective function $g\colon \mathcal{C} \to \mathcal{N}$ such that $x\mathbb{E}_0 y \iff g(x)Eg(y)$.

Now it turns out that the proof of this result in [HKL90] actually gives the following effective version:

Theorem A.1.2 (Effective General Glimm-Effros Dichotomy, see [HKL90]). *Let E be a Δ^1_1 equivalence relation on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

(i) E admits a Δ^1_1 function $f\colon \mathcal{N} \to \mathcal{N}$ such that $xEy \iff f(x) = f(y)$.

(ii) There is a Borel injective function $g\colon \mathcal{C} \to \mathcal{N}$ such that $x\mathbb{E}_0 y \iff g(x)Eg(y)$.

From this it is immediate that the smoothness of E is witnessed effectively as mentioned earlier. For more examples of such effectivity results see also the recent paper [Tho24].

In Ditzen's unpublished PhD thesis [Dit92], it is shown that the notion of compressibility of a countable Borel equivalence relation (CBER) is effective, i.e., if a Δ^1_1 CBER on the Baire space $\mathcal{N}$ is compressible, then it admits a Δ^1_1 compression. This follows from an effective version of Nadkarni's Theorem that we state below.

First recall the following standard concepts. A **CBER** E on a standard Borel space X is a Borel equivalence relation all of whose classes are countable. A **compression** of E is an injective map $f : X \to X$ such that for each E-class C we have $f(C) \subsetneq C$. We say that E is **compressible** if it admits a Borel compression. Finally a Borel probability measure μ on X is **invariant** for E if for any Borel bijection $f\colon X \to X$ with $f(x)Ex, \forall x$, we have that $f_*\mu = \mu$.

We now have:

Theorem A.1.3 (Nadkarni's Theorem, see [Nad90] and [BK96])**.** *Let E be a CBER on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

(i) E is compressible, i.e., admits a Borel compression;

(ii) E admits an invariant probability Borel measure.

We include below Ditzen's proof of the following effective version of Nadkarni's Theorem:

Theorem A.1.4 (Effective Nadkarni's Theorem [Dit92])**.** *Let E be a (lightface) Δ^1_1 CBER on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

(i) E admits a Δ^1_1 compression;

(ii) E admits an invariant probability Borel measure.

As a consequence of the proof of the Effective Nadkarni Theorem we also obtain a proof of an effective version of the classical Ergodic Decomposition Theorem (see [Far62] and [Var63]). This provides a different proof, for the restricted case of invariant measures, of Ditzen's Effective Ergodic Decomposition Theorem for quasi-invariant measures [Dit92].

First we recall the classical Ergodic Decomposition Theorem. For a CBER E on a standard Borel space X, we let INV_E denote the space of E-invariant probability Borel measures on X. We say $\mu \in \mathrm{INV}_E$ is **ergodic** for E if $\mu(A) \in \{0,1\}$ for all E-invariant Borel sets $A \subseteq X$, and we let $\mathrm{EINV}_E \subseteq \mathrm{INV}_E$ denote the space of E-ergodic invariant probability Borel measures on X.

Theorem A.1.5 (Ergodic Decomposition Theorem, see [Far62] and [Var63])**.** *Let E be a CBER on the Baire space $\mathcal{N}$ and suppose that $\mathrm{INV}_E \neq \emptyset$. Then $\mathrm{EINV}_E \neq \emptyset$ and there is a Borel surjection $\pi : \mathcal{N} \to \mathrm{EINV}_E$ such that:*

(i) π is E-invariant;

(ii) if $S_e = \{x \colon \pi(x) = e\}$, for $e \in \mathrm{EINV}_E$, then $e(S_e) = 1$ and e is the unique E-ergodic invariant probability Borel measure on $E|S_e$;

(iii) for any $\mu \in \mathrm{INV}_E$, $\mu = \int \pi(x) d\mu(x)$.

Nadkarni in [Nad90] noted that his proof of Theorem A.1.3 can be also used to give a proof of Theorem A.1.5. We will show below that this argument can also be effectivized.

Let $P(\mathcal{N})$ denote the space of probability Borel measures on $\mathcal{N}$. One can identify a probability Borel measure μ on $\mathcal{N}$ with the map $\varphi_\mu \colon \mathbb{N}^{<\mathbb{N}} \to [0,1], \varphi_\mu(s) = \mu(N_s)$, where $N_s = \{x \in \mathcal{N} \colon s \subseteq x\}$ (cf. [Kec95, 17.7]). In this way, one may view $P(\mathcal{N})$ as the Π^0_2 subset of $[0,1]^{\mathbb{N}^{<\mathbb{N}}}$ consisting of all φ satisfying $\varphi(\emptyset) = 1$ and $\varphi(s) = \sum_n \varphi(s^\frown n)$ for all $s \in \mathbb{N}^{<\mathbb{N}}$. Via this identification, we will prove the following effective version of the Ergodic Decomposition Theorem:

Theorem A.1.6 (Effective Ergodic Decomposition Theorem, see [Dit92])**.** *Let E be a (lightface) Δ^1_1 CBER on the Baire space $\mathcal{N}$ and suppose that $\mathrm{INV}_E \neq \emptyset$. Then $\mathrm{EINV}_E \neq \emptyset$, and there is a Δ^1_1 E-invariant set $Z \subseteq \mathcal{N}$ and a Δ^1_1 map $\pi : Z \to [0,1]^{\mathbb{N}^{<\mathbb{N}}}$ such that:*

(i) $E|(\mathcal{N} \setminus Z)$ admits a Δ^1_1 compression, i.e. there is a Δ^1_1 injective map $f : \mathcal{N} \setminus Z \to \mathcal{N} \setminus Z$ such that $f(C) \subsetneq C$ for every E-class $C \subseteq \mathcal{N} \setminus Z$;

(ii) π maps Z onto EINV_E;

(iii) π is E-invariant;

(iv) if $S_e = \{x \in Z \colon \pi(x) = e\}$, for $e \in \mathrm{EINV}_E$, then $e(S_e) = 1$ and e is the unique E-ergodic invariant probability Borel measure on $E|S_e$;

(v) for any $\mu \in \mathrm{INV}_E$, $\mu = \int \pi(x) d\mu(x)$.

In Section A.4, we will show that there is a Δ^1_1 CBER E on $\mathcal{N}$ that admits an invariant probability Borel measure but does not admit a Δ^1_1 invariant probability measure. It follows that we cannot, in general, make the map π from Theorem A.1.6 total, because if we could then E would admit a Δ^1_1 invariant probability measure.

A.2 A representation of Δ^1_1 equivalence relations

In this section we will prove a representation of Δ^1_1 CBER that is needed for the proof of Theorem A.1.4. It can be viewed as a strengthening and effective refinement of the Feldman-Moore Theorem, which asserts that every CBER is obtained from a Borel action of a countable group. Below we use the following terminology:

Definition A.2.1. A sequence (A_n) of Δ^1_1 subsets of $\mathcal{N}$ is **uniformly** Δ^1_1 if the relation $A \subseteq \mathbb{N} \times \mathcal{N}$ given by

$$A(n.x) \iff x \in A_n,$$

is Δ^1_1. Similarly a sequence (f_n) of partial Δ^1_1 functions $f_n \colon \mathcal{N} \to \mathcal{N}$ (i.e., functions with Δ^1_1 graph) is uniformly Δ^1_1 if the partial function $f \colon \mathbb{N} \times \mathcal{N} \to \mathcal{N}$ given by

$$f(n,x) = f_n(x),$$

is Δ^1_1.

We also say that a countable collection of subsets of $\mathcal{N}$ is uniformly Δ^1_1 if it admits a uniformly Δ^1_1 enumeration. Similarly for a countable set of partial functions.

Theorem A.2.2 ([Dit92, Section 2.2.1])**.** *Let E be a Δ^1_1 CBER on the Baire space $\mathcal{N}$. Then*

(1) E is induced by a uniformly Δ^1_1 sequence of (total) involutions, i.e., there is a such a sequence (f_n) with $xEy \iff \exists n(f_n(x) = y)$.

(2) There is a Polish 0-dimensional topology τ on $\mathcal{N}$, extending the standard topology, and a uniformly Δ^1_1 countable Boolean algebra $\mathcal{U}$ of clopen sets in τ, which is a basis for τ and is closed under the group Γ generated by (f_n).

(3) There is a complete compatible metric d for τ such that for every $U \in \mathcal{U}$ and $k > 0$, there is a uniformly Δ^1_1, pairwise disjoint, sequence (U^k_n) with $U^k_n \in \mathcal{U}$, $U = \bigcup_n U^k_n$ and $diam_d(U^k_n) < \frac{1}{k}$, and such that moreover the sequence (U^k_n) is uniformly Δ^1_1 in U, k, n.

Proof. For (1): This follows immediately from the usual proof of the Feldman-Moore Theorem (see [FM77] or [Slu, Section 1.2]). So fix below such a sequence (f_n) and consider the corresponding Δ^1_1 action of the group Γ.

For (2), (3): We will first find a topology τ as in (2), which has a uniformly Δ^1_1 countable basis $\mathcal{B}$ of clopen sets closed under the Γ-action, because we can then take $\mathcal{U}$ to be the Boolean algebra generated by $\mathcal{B}$.

For (3) we will find a complete compatible Δ^1_1 metric d for τ (i.e., $d\colon \mathcal{N}^2 \to \mathbb{R}$ is Δ^1_1). Then if $(\mathcal{U}_n)$ is a uniformly Δ^1_1 enumeration of $\mathcal{U}$, we have that

$$A(k,n) \iff diam_d(\mathcal{U}_n) < \frac{1}{k+1}$$

is Π^1_1 and

$$\forall x \in \mathcal{N} \forall k \exists n (n \in A_k \ \&\ x \in \mathcal{U}_n),$$

where $A_k = \{n\colon (k,n) \in A\}$.

So, by the Number Uniformization Theorem for Π^1_1, there is a Δ^1_1 function $f\colon \mathcal{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\forall x \in \mathcal{N} \forall k (f(x,k) \in A_k \ \&\ x \in \mathcal{U}_{f(x,k)}).$$

Since $A'(k,n) \iff \exists x \in \mathcal{N}(n = f(x,k))$ is a Σ^1_1 subset of A, let A'' be Δ^1_1 such that $A' \subseteq A'' \subseteq A$. Since

$$\mathcal{N} \times \mathbb{N} = \bigcup_{(k,n) \in A''} \mathcal{U}_n \times \{k\},$$

we can find a uniformly Δ^1_1 sequence (X^k_n) of sets in $\mathcal{U}$, such that for all $k > 0$ the sequence $(X^k_n)_n$ is a partition of $\mathcal{N}$ of sets with d-diameter less than $\frac{1}{k}$. Finally given any $U \in \mathcal{U}$, let $U^k_n = X^k_n \cap U$.

It thus remains to find τ, d with these properties. We will need first a couple of lemmas.

Lemma A.2.3. *Let $A \subseteq \mathcal{N}$ be Δ^1_1. Then there is a Polish 0-dimensional topology τ_A on $\mathcal{N}$, which extends the standard topology, has a uniformly Δ^1_1 countable basis consisting of clopen sets containing A, and has a complete compatible Δ^1_1 metric d_A.*

Proof. Let $f\colon \mathcal{N} \to \mathcal{N}$ be computable and let $B \subseteq \mathcal{N}$ be Π^0_1 and such that $f|B$ is injective and $f(B) = A$. Use f to move the (relative) topology of B to A and the standard metric of B to A. Do the same for $\mathcal{N} \setminus A$ and then take the direct sum of these topologies and metrics on $A, \mathcal{N} \setminus A$ to find τ_A, d_A. $\square$

Lemma A.2.4. *Let $\mathcal{A} = (A_n)$ be a uniformly Δ^1_1 sequence of subsets of $\mathcal{N}$. Then there is a Polish 0-dimensional topology $\tau_{\mathcal{A}}$ on $\mathcal{N}$, which extends the standard topology, has a uniformly Δ^1_1 countable basis $\mathcal{B}_{\mathcal{A}}$ containing all the sets in $\mathcal{A}$, and has a complete compatible Δ^1_1 metric $d_{\mathcal{A}}$.*

Proof. Consider $\tau_{\mathcal{A}_n}, d_{\mathcal{A}_n}$ as in Lemma A.2.3. Then put

$$\tau_{\mathcal{A}} = \text{the topology generated by } \bigcup_n \tau_{\mathcal{A}_n}.$$

Then by [Kec95, Lemma 13.3], $\tau_{\mathcal{A}}$ is Polish (and contains the standard topology). A basis for $\tau_{\mathcal{A}}$ consists of all sets of the form

$$U_1 \cap U_2 \cap \cdots \cap U_n,$$

where $U_i \in \mathcal{B}_{A_{j_i}}, 1 \leq I \leq n$, and so it is 0-dimensional with a uniformly Δ^1_1 basis $\mathcal{B}_{\mathcal{A}}$ containing all the sets in $\mathcal{A}$.

Finally, as in the proof of [Kec95, Lemma 13.3] again, a complete compatible metric for $\tau_{\mathcal{A}}$ is

$$d_{\mathcal{A}}(x, y) = \sum_n 2^{-n-1} \cdot \frac{d_{A_n}(x, y)}{1 + d_{A_n}(x, y)}.$$

Because of the uniformity in A of the proof of Lemma A.2.3 this metric is also Δ^1_1. □

We finally find τ, d. To do this we recursively define a sequence of Polish 0-dimensional topologies $\tau_0, \tau_1, \ldots$ on $\mathcal{N}$, extending the standard topology, and uniformly Δ^1_1 countable bases $\mathcal{B}_n$ for τ_n and complete compatible Δ^1_1 metrics d_n for τ_n, *all uniformly in n as well*, and such that $\Gamma \cdot \mathcal{B}_n \subseteq \mathcal{B}_{n+1}$.

For $n = 0$, let $\tau_0, d_0, \mathcal{B}_0$ be the standard topology, metric and basis for $\mathcal{N}$.

Given $\tau_n, d_n, \mathcal{B}_n$, consider $\Gamma \cdot \mathcal{B}_n$ and use Lemma A.2.4 to define $\tau_{n+1}, \mathcal{B}_{n+1} \supseteq \Gamma \cdot \mathcal{B}_n, d_{n+1}$. The uniformity in n is clear from the construction.

Finally let τ be the topology generated by $\bigcup_n \tau_n$. It is 0-dimensional, Polish, with basis the sets of the form

$$U_1 \cap U_2 \cap \cdots \cap U_n,$$

with $U_i \in \mathcal{B}_{j_i}, 1 \leq i \leq n$, so this is a uniformly Δ^1_1 countable basis $\mathcal{B}$ consisting of clopen sets. Also clearly for any $\gamma \in \Gamma$,

$$\gamma \cdot (U_1 \cap U_2 \cap \cdots \cap U_n) = \gamma \cdot U_1 \cap \gamma \cdot U_2 \cap \cdots \cap \gamma \cdot U_n,$$

where $\gamma \cdot U_i \in \mathcal{B}_{j_i+1}$, thus $\gamma \cdot (U_1 \cap U_2 \cap \cdots \cap U_n) \in \mathcal{B}$ as well. Finally, as before, a complete compatible Δ^1_1 metric for τ is

$$d(x, y) = \sum_n 2^{-n-1} \cdot \frac{d_n(x, y)}{1 + d_n(x, y)}$$

and the proof is complete. □

A.3 Proof of Effective Nadkarni

In this section we show, using the representation of Δ^1_1 CBER constructed in Section A.2, that we can effectivize the proof of Nadkarni's Theorem. Our proof follows the exposition in [Dit92, Section 2.2.3]; see also the presentations of the classical proof in [BK96] or [Slu].

The classical proof of Nadkarni's Theorem proceeds as follows. Fix a CBER E on $\mathcal{N}$. We first define a way to compare the "size" of sets. For Borel sets $A, B \subseteq \mathcal{N}$ we write $A \sim_B B$ if there is a Borel bijection $g\colon A \to B$ with $xEg(x), \forall x \in A$. We write $A \prec_B B$ if there is some $B' \subseteq B$ with $A \sim_B B'$ and $[B]_E = [B \setminus B']_E$, and $A \approx_B nB$ if we can partition A into pieces $A_0, \ldots, A_n$ so that $A_i \sim_B B$ for $i < n$ and $A_n \prec_B B$. One thinks of $A \approx_B nB$ to mean that A is about n times the size of B. In particular, if $A \approx_B nB$ and μ is an E-invariant probability Borel measure, then $n\mu(B) \leq \mu(A) \leq (n+1)\mu(B)$.

Note that E is compressible iff $\mathcal{N} \prec_B \mathcal{N}$. More generally, we say that $A \subseteq \mathcal{N}$ is *compressible* if $A \prec_B A$, i.e., if the equivalence relation $E|A$ is compressible.

Next we construct a *fundamental sequence* for E, i.e., a decreasing sequence (F_n) of Borel sets such that $F_0 = \mathcal{N}$ and $F_{n+1} \sim_B F_n \setminus F_{n+1}$. Each F_n is a complete section for E, and is a piece of $\mathcal{N}$ of "size" 2^{-n}, in the sense that $\mathcal{N} \approx_B 2^n F_n$ and $\mu(F_n) = 2^{-n}$ for all E-invariant probability Borel measures μ. It follows that if $A \approx_B kF_n$ then $k2^{-n} \leq \mu(A) \leq (k+1)2^{-n}$ for any E-invariant probability Borel measure μ.

We then use the relative size of A with respect to the F_n to approximate what the measure of A would be with respect to some E-invariant probability Borel measure. To do this, we construct, for all m, a partition $[A]_E = \bigsqcup_{n \leq \infty} Q^{A,m}_n$ of $[A]_E$ into E-invariant Borel pieces such that $Q^{A,m}_\infty$ admits a Borel compression and $A \cap Q^{A,m}_n \approx_B n(F_m \cap Q^{A,m}_n)$ for $n < \infty$. We define the *fraction function* $[A/F_m]$ by setting $[A/F_m](x) = n$ if $x \in Q^{A,m}_n$ or if $n = 0 \quad \& \quad x \notin [A]_E$, and let the *local measure function* $m(A, x) = \lim_{m \to \infty} \frac{[A/F_m](x)}{[\mathcal{N}/F_m](x)}$. We show that $m(A, x)$ is well-defined modulo an E-invariant compressible set, meaning there is an E-invariant set $C \subseteq \mathcal{N}$ admitting a Borel compression and such that $m(A, x)$ is well-defined when $x \notin C$. We also show that for any partition $A = \bigsqcup_n A_n$ into Borel pieces we have $m(A, x) = \sum_n m(A_n, x)$ modulo an E-invariant compressible set, and if $A \sim B$ then $m(A, x) = m(B, x)$ modulo an E-invariant compressible set.

Finally, we show that the local measure function $m(\cdot, x)$ defines an E-invariant probability Borel measure, for all $x \in \mathcal{N} \setminus C$, where $C \subseteq \mathcal{N}$ is some E-invariant compressible set. To see this, we fix a Borel action $\Gamma \curvearrowright \mathcal{N}$ of a countable group Γ inducing E, a zero-dimensional Polish topology τ on $\mathcal{N}$ extending the usual one in which the action $\Gamma \curvearrowright \mathcal{N}$ is continuous, a complete compatible metric d for τ and a countable Boolean algebra of clopen-in-τ sets closed under the Γ action forming a basis for τ, and satisfying additionally that for every $U \in \mathcal{U}$ and $k > 0$ there is a pairwise disjoint sequence (U_n^k) of sets in $\mathcal{U}$ with $U = \bigcup_n U_n^k$ and $diam_d(U_n^k) < \frac{1}{k}$. For each $U \in \mathcal{U}, k > 0$ we fix such a sequence. Since the countable union of Borel E-invariant compressible sets is itself a Borel E-invariant compressible set, it follows that there is an E-invariant compressible set $C \subseteq \mathcal{N}$ such that for $x \notin C$ we have $m(U,x) = \sum_n m(U_n^k, x)$ for $U \in \mathcal{U}, k > 0$, $m(U \cup V, x) = m(U,x) + m(V,x)$ for $U, V \in \mathcal{U}$ disjoint, and $m(U,x) = m(\gamma U, x)$ for $U \in \mathcal{U}, \gamma \in \Gamma$. Using this, we show that for $x \notin C$ there is an E-invariant probability Borel measure μ with $\mu(U) = m(U,x)$ for $U \in \mathcal{U}$. It follows that either $C = \mathcal{N}$, in which case E is compressible, or E admits an invariant probability Borel measure.

In order to prove the effective version of Nadkarni's Theorem, we will show that the classical proof outlined above can be effectivized using the representation in Section A.2.

For the remainder of this section, we fix a Δ^1_1 CBER E on $\mathcal{N}$ and a uniformly Δ^1_1 sequence of (total) involutions (γ_n) inducing E, as in Theorem A.2.2(1). Moreover, we assume, without loss of generality, that E is **aperiodic**, meaning that every E-class is infinite, because if $C \subseteq \mathcal{N}$ were a finite E-class then the uniform measure on C would be an E-invariant probability Borel measure.

(A) Comparing the "size" of sets.

We begin by defining a way to compare the "size" of Δ^1_1 sets. The notation we use is the same as the notation typically used for the equivalent classical notions (cf. [Slu, Definition 2.2.4, Section 2.3]), which we denoted with the subscript B above. In this paper, these notions will *always* refer to the effective definitions below.

Definition A.3.1. Let $A, B \subseteq \mathcal{N}$ be Δ^1_1.

(1) We write $A \sim B$ if there is a Δ^1_1 bijection $f\colon A \to B$ and such that $xEf(x), \forall x \in A$. If f is such a function we write $f\colon A \sim B$.

(2) We write $A \preceq B$ if $A \sim B'$ for some Δ^1_1 subset $B' \subseteq B$. If f is such a function we write $f\colon A \preceq B$.

(3) We write $A \prec B$ if there is some $f\colon A \preceq B$ such that $[B \setminus f(A)]_E = [B]_E$. If f is such a function we write $f\colon A \prec B$.

(4) We say A **admits a Δ^1_1 compression** if $A \prec A$, and if $f\colon A \prec A$ then we call f a Δ^1_1 compression of A.

(5) We write $A \preceq nB$ if there are Δ^1_1 sets $A_i, i < n$ such that $A = \bigcup_{i<n} A_i$ and $A_i \preceq B$ for $i < n$. Note that $A \preceq 1B \iff A \preceq B$.

(6) We write $A \prec nB$ if in the previous definition there is some $i < n$ for which $A_i \prec B$. Note that $A \prec 1B \iff A \prec B$.

(7) We write $A \succeq nB$ if there are pairwise disjoint Δ^1_1 sets $B_i \subseteq A, i < n$ such that $B_i \sim B$.

(8) We write $A \approx nB$ if there is a partition $A = \bigsqcup_{i<n} B_i \sqcup R$ into Δ^1_1 pieces such that $B_i \sim B$ and $R \prec B$. In particular, $A \approx 0B \iff A \prec B$. Note that $A \approx nB$ implies that $A \succeq nB$ and $A \prec (n+1)B$.

We also let $\mathscr{H}$ denote the set of all E-invariant Δ^1_1 subsets $C \subseteq \mathcal{N}$ that admit a Δ^1_1 compression.

Lemma A.3.2. *(1) Let $A \subseteq \mathcal{N}$ be Δ^1_1. If $A \prec A$ then $[A]_E \prec [A]_E$.*

(2) Let $(A_n), (B_n)$ be uniformly Δ^1_1 families of E-invariant sets and let (f_n) be a uniformly Δ^1_1 sequence of maps satisfying $f_n\colon A_n \prec B_n$. Then $\bigcup_n A_n \prec \bigcup_n B_n$. The same holds when $\prec$ is replaced by $\preceq$ or $\sim$, or if these are sequences of pairwise disjoint but not necessarily E-invariant sets.

(3) Let $A, B, C \subseteq \mathcal{N}$ be Δ^1_1. If $A \succeq nB$ and $C \preceq mB$ for some $m \leq n$, then $C \preceq A$. If additionally $C \prec mB$ then $C \prec A$.

Proof. (1) Let $f\colon A \prec A$ and let $g(x) = f(x)$ for $x \in A$, $g(x) = x$ for $x \in [A]_E \setminus A$. Then $g\colon [A]_E \prec [A]_E$.

(2) For $x \in \bigcup_n A_n$ set $f(x) = f_n(x)$ where n is least with $x \in A_n$. Then $f\colon \bigcup_n A_n \prec \bigcup_n B_n$.

(3) Let $A_i, i < n$ be pairwise disjoint Δ^1_1 subsets of A, $f_i\colon A_i \sim B$ for $i < n$, $C_j, j < m$ be Δ^1_1 sets covering C and $g_j\colon C_j \preceq B$ for $j < m$. Define

$$h(x) = f_j^{-1} \circ g_j(x) \text{ for } j \text{ least with } x \in C_j.$$

Then $h\colon C \preceq A$, and if $g_j\colon C_j \prec B$ then, letting $C' = C_j \setminus \bigcup_{k<j} C_k$, we have

$$[A \setminus h(C)]_E \supseteq ([A]_E \setminus [B]_E) \cup [B \setminus g_j(C')]_E = ([A]_E \setminus [B]_E) \cup [B]_E = [A]_E,$$

so $f\colon C \prec A$. □

(B) Fundamental sequences.

Definition A.3.3. A **uniformly Δ^1_1 fundamental sequence** for E is a uniformly Δ^1_1 decreasing sequence (F_n) of sets and a uniformly Δ^1_1 sequence (f_n) of maps such that $F_0 = \mathcal{N}$ and $f_n\colon F_{n+1} \sim F_n \setminus F_{n+1}$ for all n.

Lemma A.3.4. *Let $X \subseteq \mathcal{N}$ be a Δ^1_1 set on which $E|X$ is aperiodic. Then there is a partition $X = A \sqcup B$ of X into Δ^1_1 pieces such that $A \sim B$. In particular, $E|A, E|B$ are also aperiodic.*

Proof. Let $<$ be a Δ^1_1 linear order on $\mathcal{N}$ (for example the lexicographic order) and let $x \in A_n \iff x < \gamma_n x$. Define recursively the sets

$$\tilde{A}_n = \{x \in X \cap A_n\colon x, \gamma_n x \in X \setminus \bigcup_{i<n} (\tilde{A}_i \cup \gamma_i \tilde{A}_i)\}.$$

Let $A = \bigsqcup_n \tilde{A}_n$ and define $f = \bigcup_n \gamma_n|\tilde{A}_n\colon A \to X$. Because of the uniformity of this construction, A, f are Δ^1_1. It is easy to see that f is injective and that $f(A) \cap A = \emptyset$, so in particular that $f\colon A \sim f(A)$.

We claim that $A \cup f(A)$ omits at most one point from each $E|X$-class. To see this, let $x < y \in X$ and suppose that xEy. Let $\gamma_n x = y$. If $x, y \notin \bigcup_{i<n} (\tilde{A}_i \cup \gamma_i \tilde{A}_i)$, then by definition we have $x, y \in \tilde{A}_n \cup \gamma_n \tilde{A}_n \subseteq A \cup f(A)$.

Now let $T = X \setminus (A \cup f(A)), Y = X \cap [T]_E, Z = X \setminus [T]_E$. Then T is a traversal of $E|Y$ and $f|(A \cap Z)\colon A \cap Z \sim f(A) \cap Z$. Thus it remains to prove the lemma for $E|Y$. In this case, using T and the sequence (γ_n), one can enumerate each $E|Y$-class, and since these are infinite we can take A (resp. B) to be the even (resp. odd) elements of this enumeration. □

Proposition A.3.5. *There exists a uniformly Δ^1_1 fundamental sequence for E.*

Proof. We construct the sequences recursively. Let $F_0 = \mathcal{N}$ and recursively apply Lemma A.3.4 to get F_{n+1} and $f_n\colon F_{n+1} \sim F_n \setminus F_{n+1}$. Uniformity of these sequences follows from the uniformity in the proof of Lemma A.3.4. □

For the remainder of this section, we fix a uniformly Δ^1_1 fundamental sequence (F_n) for E.

(C) Decompositions of Δ^1_1 sets.

Lemma A.3.6. *Let $A, B \subseteq \mathcal{N}$ be Δ^1_1 and let $Z = [A]_E \cap [B]_E$. There is a partition $Z = P \sqcup Q$ of Z into E-invariant uniformly Δ^1_1 sets such that $A \cap P \prec B \cap P$ and $B \cap Q \preceq A \cap Q$.*

Proof. Define recursively the sets

$$A_n = \{x \in A \setminus \bigcup_{i<n} A_i \colon \gamma_n x \in B \setminus \bigcup_{i<n} B_i\}, B_n = \gamma_n A_n.$$

Let $\tilde{A} = \bigcup_n A_n, \tilde{B} = \bigcup_n B_n$ and $f = \bigcup_n \gamma_n | A_n$. By the uniformity of this construction, $\tilde{A}, \tilde{B}, f$ are all Δ^1_1, so that $f \colon \tilde{A} \sim \tilde{B}$. If we set $P = Z \cap [B \setminus \tilde{B}]_E, Q = Z \setminus P$ then it is easy to see that $A \cap P \subseteq \tilde{A}, B \cap Q \subseteq \tilde{B}$ and hence that $f|(A \cap P) \colon A \cap P \prec B \cap P$ and $f^{-1}|(B \cap Q) \colon B \cap Q \preceq A \cap Q$. $\square$

Proposition A.3.7. *Let $A, B \subseteq \mathcal{N}$ be Δ^1_1 and let $Z = [A]_E \cap [B]_E$. There exists a partition $Z = \bigsqcup_{n \le \infty} Q_n$ of Z into E-invariant Δ^1_1 pieces such that $A \cap Q_n \approx n(B \cap Q_n)$ for $n < \infty$ and $Q_\infty \in \mathscr{H}$.*

Proof. We recursively construct sequences of sets

$$A_n, B_n, \tilde{P}_n, \tilde{Q}_n, f_n, g_n, \tilde{B}_n, Q_n, R_n, B^i_n, f^i_n$$

for $i < n$ such that $A \cap Q_n = \bigsqcup_{i<n} B^i_n \sqcup R$ for $n < \infty$, $f^i_n \colon B^i_n \sim B \cap Q_n$ for $i < n < \infty$, and $f_n \colon R_n \prec B \cap Q_n$ for $n < \infty$.

First we let $A_0 = A, B_0 = B$. We apply Lemma A.3.6 to these sets to get $\tilde{P}_0, \tilde{Q}_0, f_0, g_0$ and $\tilde{B}_0$ satisfying

$$f_0 \colon A_0 \cap \tilde{P}_0 \prec B_0 \cap \tilde{P}_0, \quad g_0 \colon B_0 \cap \tilde{Q}_0 \preceq A_0 \cap \tilde{Q}_0, \quad \tilde{B}_0 = \mathrm{Im}(g_0).$$

Define $Q_0 = \tilde{P}_0, R_0 = A_0 \cap Q_0$.

Now let $n > 0$ and suppose we have already constructed

$$A_k, B_k, \tilde{P}_k, \tilde{Q}_k, f_k, g_k, \tilde{B}_k, Q_k, R_k, B^i_k, f^i_k$$

for all $i < k < n$. Let $A_n = (A_{n-1} \cap \tilde{Q}_{n-1}) \setminus \tilde{B}_{n-1}, B_n = B \cap \tilde{Q}_{n-1}$. Apply Lemma A.3.6 to A_n, B_n to get $\tilde{P}_n, \tilde{Q}_n, f_n, g_n, \tilde{B}_n$ such that

$$f_n \colon A_n \cap \tilde{P}_n \prec B_n \cap \tilde{P}_n, \quad g_n \colon B_n \cap \tilde{Q}_n \preceq A_n \cap \tilde{Q}_n, \quad \tilde{B}_n = \mathrm{Im}(g_n).$$

Define $Q_n = \tilde{Q}_{n-1} \setminus \tilde{Q}_n, R_n = A_n \cap Q_n, B^i_n = \tilde{B}_i \cap Q_n, f^i_n = (g_i)^{-1}|B^i_n$.

By uniformity of this construction it is clear that these sequences are uniformly Δ^1_1. Additionally, $A \cap Q_n \approx n(B \cap Q_n)$ for $n < \infty$.

Now let $Q_\infty = Z \setminus \bigcup_n Q_n = \bigcap_n \tilde{Q}_n$. The sets $\tilde{B}_n$ are pairwise disjoint and $g_n \colon B \cap \tilde{Q}_n \sim \tilde{B}_n$ for all n. Therefore, if we define $B^n_\infty = \tilde{B}_n \cap Q_\infty$, $g^n_\infty = g_n|(B \cap Q_\infty)$ and $g^{n,m}_\infty = g^m_\infty \circ (g^n_\infty)^{-1}$, we have that the B^n_∞ are pairwise disjoint and $g^{n,m}_\infty \colon B^n_\infty \sim B^m_\infty$. Let $B_\infty = \bigcup_n B^n_\infty$ and $g_\infty = \bigcup_n g^{n,n+1}_\infty$. Then B_∞, g_∞ are Δ^1_1 and $g_\infty \colon B_\infty \prec B_\infty$. Since $[B_\infty]_E = [B^0_\infty]_E = [B \cap Q_\infty]_E = Q_\infty$, Q_∞ admits a Δ^1_1 compression by Lemma A.3.2(1). □

Notation A.3.8. For Δ^1_1 sets $A, B \subseteq \mathcal{N}$, we let $Q^{A,B}_n, n \leq \infty$ be the decomposition of $[A]_E \cap [B]_E$ constructed in Proposition A.3.7.

(D) The fraction functions.

Definition A.3.9. We associate to all Δ^1_1 sets $A, B \subseteq \mathcal{N}$ a **fraction function** $[A/B] : \mathcal{N} \to \mathbb{N}$ defined by

$$\left[\frac{A}{B}\right](x) = \begin{cases} n & \text{if } x \in Q^{A,B}_n \text{ for some } n \leq \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma A.3.10. *Let $A, A_0, A_1, A_2, B, S \subseteq \mathcal{N}$ be Δ^1_1.*

(1) If xEy then $[A/B](x) = [A/B](y)$.

(2) If $A_0 \preceq A_1$ then there is some $C \in \mathscr{H}$ such that $[A_0/B](x) \leq [A_1/B](x)$ for $x \notin C$.

(3) If $A_0 \sim A_1$ then there is some $C \in \mathscr{H}$ such that $[A_0/B](x) = [A_1/B](x)$ for $x \notin C$.

(4) If S is E-invariant then there is some $C \in \mathscr{H}$ such that for $x \in S \setminus C$ we have $[A/B](x) = [(A \cap S)/B](x)$.

(5) If A_0, A_1 are disjoint then there is some $C \in \mathscr{H}$ such that for $x \notin C$,

$$[A_0/B] + [A_1/B] \leq [(A_0 \cup A_1)/B] \leq [A_0/B] + 1 + [A_1/B] + 1.$$

(6) If A_1 is an E-complete section then there is some $C \in \mathscr{H}$ such that for $x \notin C$,

$$[A_0/A_1][A_1/A_2] \leq [A_0/A_2] < ([A_0/A_1] + 1)([A_1/A_2] + 1).$$

(7) There is some $C \in \mathscr{H}$ such that $[F_n/F_{n+m}] = 2^m$ holds for all $m, n \in \mathbb{N}, x \notin C$.

(8) There is some $C \in \mathscr{H}$ such that for all $x \in [A]_E \setminus C$ we have $[A/F_n](x) \to \infty$.

(9) The set $Y = \{x : [A_0/B](x) < [A_1/B](x)\}$ is Δ^1_1 and E-invariant and $A_0 \cap Y \preceq A_1 \cap Y$.

Proof. (1) This is clear, as the sets $Q^{A,B}_n$ are E-invariant.

(2) Let $C_{n,m} = Q^{A_0,B}_n \cap Q^{A_1,B}_m$ for $m < n$. Then $A_0 \cap C_{n,m} \approx n(B \cap C_{n,m})$ and $A_1 \cap C_{n,m} \approx m(B \cap C_{n,m})$ so by Lemma A.3.2(3) and our assumption we have $A_0 \cap C_{n,m} \preceq A_1 \cap C_{n,m} \prec A_0 \cap C_{n,m}$. By Lemma A.3.2(2) and the uniformity of the proofs of Proposition A.3.7 and Lemma A.3.2(3), $C = \bigcup_{m<n} C_{n,m} \in \mathscr{H}$, and $[A_0/B](x) \leq [A_1/B](x)$ for $x \notin C$.

(3) This follows from (2).

(4) As in the proof of (2), it suffices to show that $C = S \cap Q^{A,B}_k \cap Q^{A\cap S,B}_l$ admits a Δ^1_1 compression (in a uniform way) for $k \neq l$. But

$$A \cap C \approx k(B \cap C) \text{ and } A \cap C \approx l(B \cap C)$$

by E-invariance of C, so by Lemma A.3.2(1),(3) C admits a Δ^1_1 compression.

(5) Let $C = C_{i,j,k} = Q^{A_0,B}_i \cap Q^{A_1,B}_j \cap Q^{A_2,B}_k$. Then (5) fails to hold exactly when $x \in C_{i,j,k}$ for $k < i+j$ or $k > i+1+j+1$. Therefore, as in the proof of (2), it suffices to show that $C_{i,j,k}$ admits a Δ^1_1 compression (in a uniform way) for such i, j, k.

Now we know that $A_0 \cap C \approx i(B \cap C), A_1 \cap C \approx j(B \cap C), A_2 \cap C \approx k(B \cap C)$ by E-invariance of C. If $k < i+j$ then $(A_0 \cup A_1) \cap C \prec (i+j)(B \cap C)$ and (since A_0, A_1 are disjoint) we have $(A_0 \cap C) \cup (A_1 \cap C) \succeq (i+j)(B \cap C)$. Thus by Lemma A.3.2(1),(3) $C = [(A_0 \cup A_1) \cap C]_E$ admits a Δ^1_1 compression. On the other hand, if $k > i+1+j+1$ then $(A_0 \cap C) \cup (A_1 \cap C) \prec (i+1+j+1)(B \cap C)$ and $(A_0 \cup A_1) \cap C \succeq k(B \cap C)$, so again C admits a Δ^1_1 compression.

(6) If $x \notin [A_0]_E \cup [A_2]_E$ then this clearly holds. Thus if $C = C_{k,l,m} = Q^{A_0,A_1}_k \cap Q^{A_1,A_2}_l \cap Q^{A_0,A_2}_m$ then (6) fails to hold exactly when $x \in C_{k,l,m}$ for $m < kl$ or $m \geq (k+1)(l+1)$. Therefore, as in the proof of (2), it suffices to show that these sets admit a Δ^1_1 compression (in a uniform way).

Since $A_0 \cap C \approx k(A_1 \cap C)$ and $A_1 \cap C \approx l(A_2 \cap C)$ we have that $A_0 \cap C \succeq kl(A_2 \cap C)$. Also, $A_0 \cap C \approx m(A_2 \cap C)$, so if $kl > m$ then by Lemma A.3.2(1),(3) we are done. On the other hand, if $m \geq (k+1)(l+1)$ then $A_0 \cap C \succeq (k+1)(l+1)(A_2 \cap C)$, and since $A_1 \cap C \approx l(A_2 \cap C)$ one easily sees that $A_0 \cap C \succeq (k+1)(A_1 \cap C)$. Thus by Lemma A.3.2(1),(3) we are done.

(7) Again it suffices to show that $Q_k^{F_n,F_{n+m}}$ admits a Δ^1_1 compression in a uniform way for $k \neq 2^m$. When $k = \infty$ this is clear. Otherwise, one easily sees by definition of the fundamental sequence that $F_n \approx 2^m F_{n+m}$, and moreover there is a uniformly Δ^1_1 sequence of witnesses to this. It follows that $F_n \cap Q_k^{F_n,F_{n+m}} \approx 2^m(F_{n+m} \cap Q_k^{F_n,F_{n+m}})$ and $F_n \cap Q_k^{F_n,F_{n+m}} \approx k(F_{n+m} \cap Q_k^{F_n,F_{n+m}})$, so when $k \neq 2^m$ this follows from Lemma A.3.2(1),(3).

(8) Let C_0 be the set constructed in (7), $C(A_0, A_1, A_2)$ be the set constructed in (6), $C_1 = \bigcap_n Q_0^{A,F_n}$ and

$$C_2 = \bigcup_{n,m} C(A, F_n, F_{n+m}) \cup \bigcup_n C(F_0, F_n, A).$$

Let $C = C_0 \cup C_1 \cup C_2$. If $x \in [A]_E \setminus C$ then there is some n for which $[A/F_n](x) \neq 0$, so for all m we have

$$[A/F_{n+m}](x) \geq [A/F_n](x)[F_n/F_{n+m}](x) \geq 2^m,$$

and therefore $[A/F_n](x) \to \infty$. By the uniformity of the proofs of (6), (7) and Proposition A.3.7, C is Δ^1_1, so it remains to show that it admits a Δ^1_1 compression. By the uniformity of the proofs of (6), (7) and Lemma A.3.2(2), it suffices to show that $C_1 \setminus (C_0 \cup C_2)$ admits a Δ^1_1 compression.

First we show that $C_1 \cap \bigcup_n Q_0^{F_n,A}$ admits a Δ^1_1 compression. For this it suffices to show that $C_1 \cap Q_0^{F_n,A}$ admits a Δ^1_1 compression for all n (in a uniform way), by Lemma A.3.2(2). But by definition and E-invariance we have

$$F_n \cap C_1 \cap Q_0^{F_n,A} \prec A \cap C_1 \cap Q_0^{F_n,A} \prec F_n \cap C_1 \cap Q_0^{F_n,A},$$

so $F_n \cap C_1 \cap Q_0^{F_n,A}$ admits a Δ^1_1 compression, and since F_n is a complete section we are done by Lemma A.3.2(1).

Next we consider $C' = C_1 \setminus (C_0 \cup C_2 \cup \bigcup_n Q_0^{F_n,A})$. For any $x \in C', n \in \mathbb{N}$, we have

$$[F_0/A](x) \geq [F_0/F_n](x)[F_n/A](x) \geq 2^n,$$

so $[F_0/A](x) = \infty$ and $x \in Q_\infty^{F_0,A}$. Thus $C' \subseteq Q_\infty^{F_0,A}$ admits a Δ^1_1 compression.

(9) This set is clearly Δ^1_1 and it is E-invariant by (1). Next note that $Y \subseteq [B]_E \setminus Q_\infty^{A_0,B}$ so we can decompose Y into $Y_0 = Y \setminus [A_0]_E$ and $Y_1 = Y \cap [A_0]_E = \bigcup_n (Y \cap Q_n^{A_0,B})$. Since $Y_0 \cap A_0 = \emptyset$ we clearly have $Y_0 \cap A_0 \preceq Y_0 \cap A_1$, so it remains to show that $Y_1 \cap A_0 \preceq Y_1 \cap A_1$. But by Lemma A.3.2(3) we have that $A_0 \cap Q_m^{A_0,B} \cap Q_n^{A_1,B} \preceq A_1 \cap Q_m^{A_0,B} \cap Q_n^{A_1,B}$ for $m < n$, so by Lemma A.3.2(2) we are done. $\square$

(E) Local measures.

Proposition A.3.11. *Let $A \subseteq \mathcal{N}$ be Δ^1_1. Then there is some $C \in \mathscr{H}$ such that*

$$\lim_n \frac{[A/F_n](x)}{[\mathcal{N}/F_n](x)}$$

exists for $x \notin C$, and the limit is zero for $x \notin [A]_E \cup C$ and is non-zero and finite for $x \in [A]_E \setminus C$.

Proof. Let $C_0(A_0, A_1, A_2), C_1, C_2(A)$ be the sets we have constructed in the proofs of Lemma A.3.10(6)(7)(8), respectively, and take $C = \bigcup_{n,m} C_0(A, F_n, F_{n+m}) \cup C_1 \cup C_2(A) \cup \bigcup_n Q^{A,F_n}_\infty$. By Lemma A.3.2(2) and the uniformity of Lemma A.3.10, $C \in \mathscr{H}$. If $x \notin [A]_E \cup C$ then $[A/F_n](x) = 0$ and $[\mathcal{N}/F_n](x) = 2^n$ for all n, so the limit exists and is zero.

Now suppose that $x \in [A]_E \setminus C$. Then $[F_n/F_{n+m}](x) = 2^m$ for all $m, n \in \mathbb{N}$, and

$$[A/F_{n+m}](x) \leq ([A/F_n](x) + 1)([F_n/F_{n+m}](x) + 1),$$

so

$$\limsup_{m\to\infty} \frac{[A/F_{n+m}](x)}{[\mathcal{N}/F_{n+m}](x)} \leq \frac{[A/F_n](x) + 1}{[\mathcal{N}/F_n](x)}.$$

Thus the limit exists and is finite at x. To see that the limit is non-zero at x, note that $[A/F_{n+m}](x) \geq [A/F_n](x)[F_n/F_{n+m}](x)$ for all $m, n \in \mathbb{N}$, so

$$\liminf_{m\to\infty} \frac{[A/F_{n+m}](x)}{[\mathcal{N}/F_{n+m}](x)} \geq \frac{[A/F_n](x)}{[\mathcal{N}/F_n](x)}$$

for all n, and since $[A/F_n](x) \to \infty$ this lower bound must be non-zero for some n. □

Definition A.3.12. Let $A \subseteq \mathcal{N}$ be Δ^1_1 and let $C_A \in \mathscr{H}$ be the set constructed in the proof of Proposition A.3.11. We associate to A the **local measure function** $m(A, \cdot)\colon \mathcal{N} \setminus C_A \to \mathbb{R}$ defined by

$$m(A, x) = \lim_n \frac{[A/F_n](x)}{[\mathcal{N}/F_n](x)}.$$

Note that the local measure function is Δ^1_1, uniformly in A.

Lemma A.3.13. *Let $A, B, S \subseteq \mathcal{N}$ be Δ^1_1.*

(1) If xEy then $m(A, x) = m(A, y)$ for $x, y \notin C_A$.

(2) Let $Y = \{x \in \mathcal{N} \setminus (C_A \cup C_B) \colon m(A,x) < m(B,x)\}$. Then Y is Δ^1_1, E-invariant and $A \cap Y \preceq B \cap Y$.

(3) Suppose S is E-invariant. Then there is some $C \in \mathscr{H}$ such that for $x \notin C$,

$$m(S,x) = \begin{cases} 1 & x \in S, \\ 0 & x \notin S. \end{cases}$$

(4) If S is E-invariant, then there is some $C \in \mathscr{H}$ such that for $x \in S \setminus C$ we have $m(A,x) = m(A \cap S, x)$.

Proof. (1) This follows from Lemma A.3.10(1).

(2) This set is E-invariant by (1) and is Δ^1_1 because the local measure functions are Δ^1_1. Now let

$$Y_n = \{x \in Y \colon [A/F_n](x) < [B/F_n](x)\}.$$

The sets Y_n are E-invariant, Δ^1_1 and cover Y, so we have $A \cap Y \preceq B \cap Y$ by Lemma A.3.10(9) and Lemma A.3.2(2).

(3) If $x \notin S$ then $[S/F_n](x) = 0$ for all n, so $m(S,x) = 0$. On the other hand, if $x \in S$ then $[S/F_n](x) = k \iff x \in Q^{S,F_n}_k$, so it suffices to show that $\bigcup_{k \neq 2^n} Q^{S,F_n}_k \in \mathscr{H}$. This is done exactly as in the proof of Lemma A.3.10(7).

(4) Let $C_0(A,B,S)$ be the set constructed in the proof of Lemma A.3.10(4) and take $C = \bigcup_n C_0(A, F_n, S) \cup C_A \cup C_{A \cap S}$. Then $C \in \mathscr{H}$ by Lemma A.3.2(2) and clearly C works. $\square$

Proposition A.3.14. *Let $A, B, S \subseteq \mathcal{N}$ be Δ^1_1 and let (A_n) be a uniformly Δ^1_1 sequence of subsets of $\mathcal{N}$.*

(1) If $A \preceq B$ then there is some $C \in \mathscr{H}$ such that $m(A,x) \leq m(B,x)$ for $x \notin C$.

(2) If $A \sim B$ then there is some $C \in \mathscr{H}$ such that $m(A,x) = m(B,x)$ for $x \notin C$.

(3) If A, B are disjoint then there is some $C \in \mathscr{H}$ such that $m(A,x) + m(B,x) = m(A \sqcup B, x)$ for $x \notin C$.

(4) Suppose the (A_n) are pairwise disjoint, S is E-invariant and the partial maps $m(A,\cdot), m(A_n,\cdot)$ are defined on S. Suppose additionally that $m(A,x) > \sum_n m(A_n,x)$ for $x \in S$. Then there is some $C \in \mathscr{H}$ satisfying $(\bigsqcup_n A_n) \cap (S \setminus C) \preceq A \cap (S \setminus C)$.

(5) If $A = \bigsqcup_n A_n$ then there is some $C \in \mathscr{H}$ such that $m(A,x) = \sum_n m(A_n,x)$ for $x \notin C$.

Proof. (1) Let $C = \bigcup_n C_0(A, B, F_n) \cup C_A \cup C_B$, where $C_0(A_0, A_1, B)$ denotes the set constructed in the proof of Lemma A.3.10(2).

(2) This follows from (1).

(3) Let $C_0(A_0, A_1, B)$ and C_1 be the sets we have constructed in the proofs of Lemma A.3.10(5) and (7), respectively, and take $C = \bigcup_n C_0(A, B, F_n) \cup C_A \cup C_B \cup C_1$.

(4) We construct recursively a sequence of Δ^1_1 sets and functions $\tilde{A}_n$, B_n, C_n, S_n, f_n, g_n such that $\tilde{A}_{n+1} = \tilde{A}_n \setminus B_n$, $S_{n+1} = S_n \setminus C_n$, $f_n \colon A_n \cap S_n \sim B_n \cap S_n$, $g_n \colon C_n \prec C_n$, and $m(\tilde{A}_n, x) > \sum_{k \geq n} m(A_k, x)$ for $x \in S_n$. To do this, we first set $\tilde{A}_0 = A, S_0 = S$. Now suppose we have $\tilde{A}_n, S_n$ satisfying $m(\tilde{A}_n, x) > \sum_{k \geq n} m(A_k, x)$ for $x \in S_n$. Then $m(\tilde{A}_n, x) > m(A_n, x)$ for $x \in S_n$, so by Lemma A.3.13(2) we can find $B_n \subseteq \tilde{A}_n$ and $f_n \colon A_n \cap S_n \sim B_n \cap S_n$. By (2), (3) and Lemma A.3.13(4) there are $g_n \colon C_n \prec C_n$ such that for $x \in S_n \setminus C_n$ we have $m(A_n, x) = m(B_n, x)$ and $m(\tilde{A}_n, x) = m(B_n, x) + m(\tilde{A}_n \setminus B_n, x)$. We then define $\tilde{A}_{n+1} = \tilde{A}_n \setminus B_n, S_{n+1} = S_n \setminus C_n$.

By the uniformity of the proofs of (2), (3) and Lemma A.3.13, these sequences are uniformly Δ^1_1. Let $C = \bigcup_n C_n$, and note that $S \setminus C = \bigcap_n S_n$, so $A_n \cap (S \setminus C) \sim B_n \cap (S \setminus C)$ for all n. Thus by Lemma A.3.2(2) we have $C \in \mathscr{H}$ and

$$(\bigsqcup_n A_n) \cap (S \setminus C) \sim (\bigsqcup_n B_n) \cap (S \setminus C) \subseteq A \cap (S \setminus C).$$

(5) Let $C_0(A, B), C_1(A, B)$ be the sets constructed in the proofs of (1) and (3), respectively, and let

$$\tilde{C} = C_A \cup \bigcup_n [C_{A_n} \cup C_0(A_0 \cup \cdots \cup A_n, A) \cup C_1(A_0 \cup \cdots \cup A_n, A_{n+1})].$$

Then for $x \notin \tilde{C}$ and $n \in \mathbb{N}$ we have

$$\sum_{k<n} m(A_k, x) = m(\bigcup_{k<n} A_k, x) \leq m(A, x),$$

and therefore $\sum_n m(A_n, x) \leq m(A, x)$ for $x \notin \tilde{C}$.

Now let C_2 be the set constructed in the proof of Lemma A.3.10(7) and define

$$C = \tilde{C} \cup C_2 \cup C_{\mathcal{N} \setminus A} \cup C_1(A, \mathcal{N} \setminus A) \cup \bigcup_n [C_{F_n} \cup C_{\mathcal{N} \setminus F_n} \cup C_1(F_n, \mathcal{N} \setminus F_n)].$$

Then for $x \notin C$ we have

- $\sum_n m(A_n, x) \leq m(A, x)$,

- $m(A, x) + m(\mathcal{N} \setminus A, x) = m(\mathcal{N}, x)$,
- $\forall n(m(F_n, x) = 2^{-n})$, and
- $\forall n(m(F_n, x) + m(\mathcal{N} \setminus F_n, x) = m(\mathcal{N}, x))$.

Let $S_k = \{x \notin C : m(A, x) > \sum_n m(A_n, x) + 2^{-k}\}$. These sets are Δ_1^1 and E-invariant, and if $x \notin C \cup \bigcup_k S_k$ then $m(A, x) = \sum_n m(A_n, x)$. By the uniformity of the construction of C, S_k and Lemma A.3.2(2), it remains to show that each $S_k \in \mathscr{H}$.

For $x \in S_k$ we have

$$m(\mathcal{N} \setminus F_k, x) = m(A, x) + m(\mathcal{N} \setminus A, x) - m(F_k, x) > m(\mathcal{N} \setminus A, x) + \sum_n m(A_n, x).$$

By (4) there is some $C_k \in \mathscr{H}$ for which

$$S_k \setminus C_k = \left(\bigcup_n A_n \cup (\mathcal{N} \setminus A)\right) \cap (S_k \setminus C_k) \preceq (\mathcal{N} \setminus F_k) \cap (S_k \setminus C_k).$$

Since F_k is an E-complete section, this means that $S_k \setminus C_k \in \mathscr{H}$, and hence that $S_k \in \mathscr{H}$, as desired. $\square$

(F) Proof of the Effective Nadkarni's Theorem.

Recall that we have fixed some sequence of maps (γ_n) satisfying (1) of Theorem A.2.2. Fix now some $\tau, \mathcal{U}, d, (U_n^k)$ satisfying (2), (3) of Theorem A.2.2. Let C_A be the set defined in Definition A.3.12, and let $C_0(A, B)$, $C_1(A, B)$, and $C_2(A, (A_n))$ be the sets constructed in the proofs of Proposition A.3.14 (2), (3), and (5), respectively. Now define

$$\begin{aligned} C = & \bigcup \{C_U : U \in \mathcal{U}\} \\ & \cup \bigcup \{C_0(U, \gamma_n U) : U \in \mathcal{U}, n \in \mathbb{N}\} \\ & \cup \bigcup \{C_1(U, V \setminus U) : U, V \in \mathcal{U}\} \\ & \cup \bigcup \{C_2(U, (U_n^k)_n) : U \in \mathcal{U}, k > 0\}. \end{aligned}$$

By the uniformity of the constructions of the C_A, C_0, C_1, C_2, along with the fact that $\mathcal{U}, (U_n^k)$ are uniformly Δ_1^1, there is a uniformly Δ_1^1 enumeration of the sets in this union, so C is Δ_1^1. By this uniformity and Lemma A.3.2(2), C admits a Δ_1^1 compression.

If $\mathcal{N} = C$, then E admits a Δ_1^1 compression. So suppose $\mathcal{N} \neq C$ and fix some $x \in \mathcal{N} \setminus C$. By construction, the following hold for x:

- $m(\emptyset, x) = 0$ and $m(\mathcal{N}, x) = 1$;
- for all $U \in \mathcal{U}$, $m(U, x)$ exists, is zero for $x \notin [U]_E$, and is non-zero and finite for $x \in [U]_E$;
- $m(U, x) = m(\gamma_n U, x)$ for all $U \in \mathcal{U}, n \in \mathbb{N}$;
- $m(U \sqcup V, x) = m(U, x) + m(V, x)$ for all disjoint $U, V \in \mathcal{U}$; and
- for all $U \in \mathcal{U}$ and $k > 0$, $m(U, x) = \sum_n m(U_n^k, x)$.

Now define

$$\mu_x^*(A) = \inf\{\sum_n m(U_n, x) : U_n \in \mathcal{U} \ \ \& \ \ A \subseteq \bigcup_n U_n\}.$$

As in the classical proof of Nadkarni's Theorem (cf. [BK96, p. 51-52] or [Slu, Theorem 2.8.1]), μ_x^* is a metric outer measure whose restriction μ_x to the Borel sets is an E-invariant probability Borel measure satisfying $\mu_x(U) = m(U, x)$, for $U \in \mathcal{U}$. Thus, E admits an invariant probability Borel measure.

A.4 A counterexample

Let E be a Δ_1^1 CBER on $\mathcal{N}$. Nadkarni's Theorem says that either E is compressible or E admits an invariant probability Borel measure. We have seen in Theorem A.1.4 that if E is compressible, then actually there is a Δ_1^1 witness of this. On the other hand, if E is non-compressible, one may ask if there is an effective witness of this, i.e., if E admits a Δ_1^1 invariant probability measure. It turns out that this is true if, for example, E is induced by a continuous, Δ_1^1 action of a countable group on the Cantor space, but it is not true in general.

Let $P(\mathcal{C})$ denote the space of probability Borel measures on $\mathcal{C}$. As with $P(\mathcal{N})$, we identify $P(\mathcal{C})$ with the Π_1^0 set of all $\varphi \in [0,1]^{2^{<\mathbb{N}}}$ satisfying $\varphi(\emptyset) = 1$ and $\varphi(s) = \varphi(s^\frown 0) + \varphi(s^\frown 1)$ for $s \in 2^{<\mathbb{N}}$. We then have the following:

Proposition A.4.1. *Let E be a CBER on the Cantor space $\mathcal{C}$. Suppose there is a uniformly Δ_1^1 sequence (f_n) of homeomorphisms of $\mathcal{C}$ inducing E, i.e., such that $xEy \iff \exists n(f_n(x) = y)$. Then if E is non-compressible, E admits a Δ_1^1 invariant probability measure.*

Proof. Let $\mathrm{INV}_E \subseteq P(\mathcal{C})$ be the set of all E-invariant probability Borel measures on $\mathcal{C}$. If E is non-compressible, then INV_E is compact, Δ_1^1 and non-empty. By the basis theorem [Mos09, 4F.11], INV_E contains a Δ_1^1 point, which is a Δ_1^1 E-invariant probability measure on $\mathcal{C}$. $\square$

Let E, F be CBERs on the standard Borel spaces X, Y respectively. We say that E is **Borel invariantly embeddable** to F, denoted $E \sqsubseteq_B^i F$, if there is an injective Borel map $f : X \to Y$ such that $xEy \iff f(x)Ff(y)$, and such that additionally $f(X) \subseteq Y$ is F-invariant. We say F is **invariantly universal** if $E \sqsubseteq_B^i F$ for any CBER E. Clearly, all invariantly universal CBERs admit invariant probability Borel measures.

Proposition A.4.2. *There exists an invariantly universal Δ^1_1 CBER on $\mathcal{N}$ that does not admit a Δ^1_1 invariant probability measure.*

Proof. Let $\mathbb{F}_\infty$ be the free group on a countably infinite set of generators, and take F_0 to be the shift equivalence relation on $\mathcal{N}^{\mathbb{F}_\infty} \cong \mathcal{N}$. Note that F_0 is an invariantly universal Δ^1_1 CBER. Let F_1 be a compressible Δ^1_1 CBER on $\mathcal{N}$. Let T be an ill-founded computable tree on $\mathbb{N}$ with no Δ^1_1 branches (cf. [Mos09, 4D.10]), and define the equivalence relation E on $\mathcal{N} \times \mathcal{N}$ by

$$(w, x)E(y, z) \iff w = y \ \ \& \ \ [(w \in [T] \ \ \& \ \ xF_0z) \text{ or } (w \notin [T] \ \ \& \ \ xF_1z)].$$

Then E is a non-compressible invariantly universal Δ^1_1 CBER on $\mathcal{N} \times \mathcal{N} \cong \mathcal{N}$, because T is ill-founded and F_0 is non-compressible and invariantly universal.

Now suppose for the sake of contradiction that E admits a Δ^1_1 invariant probability measure μ. For $s \in \mathbb{N}^{<\mathbb{N}}$, let $N_s = \{x \in \mathcal{N} \colon s \subseteq x\}$, and define $S = \{s \in \mathbb{N}^{<\mathbb{N}} \colon \mu(N_s \times \mathcal{N}) > 0\}$. Then S is a non-empty pruned Δ^1_1 subtree of T, because if $s \notin T$ then $E|(N_s \times \mathcal{N})$ is compressible, so $\mu(N_s \times \mathcal{N}) = 0$. But then S, and hence T, has a Δ^1_1 branch, a contradiction. $\square$

Remark A.4.3. Let E be the equivalence relation induced by the shift action of $\mathbb{F}_\infty$ on $\mathcal{C}^{\mathbb{F}_\infty}$, and let $Fr(\mathcal{C}^{\mathbb{F}_\infty}) \subseteq \mathcal{C}^{\mathbb{F}_\infty}$ be the free part of $\mathcal{C}^{\mathbb{F}_\infty}$, i.e., the set of points x such that $\gamma x \neq x, \forall \gamma \in \mathbb{F}_\infty, \gamma \neq 1$. Then $E|Fr(\mathcal{C}^{\mathbb{F}_\infty})$ is invariantly universal for CBERs that can be induced by a free Borel action of $\mathbb{F}_\infty$.

Using the representation of Δ^1_1 CBERs constructed in Section A.2, and [Mos09, 4F.14], one sees that the proof of [FKSV23, Theorem 3.3.1] is effective. In particular, there is a Δ^1_1, compact, E-invariant set $K \subseteq \mathcal{C}^{\mathbb{F}_\infty}$ admitting a Δ^1_1 isomorphism $E|K \cong E|Fr(\mathcal{C}^{\mathbb{F}_\infty})$.

Now consider the equivalence relation F on $\mathcal{N} \times \mathcal{C}^{\mathbb{F}_\infty}$ given by

$$(w, x)F(y, z) \iff w = y \ \ \& \ \ xEz.$$

Let T be the tree from the proof of Proposition A.4.2 and let $X = [T] \times Fr(\mathcal{C}^{\mathbb{F}_\infty})$. Then $F|X$ is invariantly universal for CBERs that can be induced by a free action of $\mathbb{F}_\infty$, so there is a Borel isomorphism $F|X \cong E|Fr(\mathcal{C}^{\mathbb{F}_\infty})$, and $F|X$ does not admit a Δ^1_1 invariant probability Borel measure.

It follows that $F|X$ is Borel isomorphic to a Δ^1_1 compact subshift of $\mathcal{C}^{\mathbb{F}_\infty}$. However, by the proof of Proposition A.4.1, every such subshift admits a Δ^1_1 invariant probability Borel measure, so there is no Δ^1_1 isomorphism between $F|X$ and a Δ^1_1 compact subshift of $\mathcal{C}^{\mathbb{F}_\infty}$. In particular, $F|X$ is a concrete witness to [FKSV23, Proposition 3.8.15].

A.5 Proof of Effective Ergodic Decomposition

As noted in [Nad90], the proof of Nadkarni's Theorem can be used to provide a proof of the Ergodic Decomposition Theorem (see also [Slu, Section 2.9]). We will now show that this argument can also be effectivized, providing a proof of the Effective Ergodic Decomposition Theorem for invariant measures from the proof of the Effective Nadkarni's Theorem. This provides a different proof of a special case of Ditzen's Effective Ergodic Decomposition Theorem [Dit92], which is proved more generally for quasi-invariant measures.

Let E be a non-compressible CBER on the Baire space $\mathcal{N}$, in order to prove the Ergodic Decomposition Theorem for E. We may partition $\mathcal{N} = X \sqcup Y$ into Δ^1_1 E-invariant pieces so that $E|X$ is aperiodic and every $E|Y$-class $C \subseteq Y$ is finite. It is easy to see that the Ergodic Decomposition Theorem holds for $E|Y$, so we may assume that E is aperiodic.

Fix $(f_n), \tau, \mathcal{U}, d, (U^k_n)$ satisfying Theorem A.2.2 for E. By the proof of the Effective Nadkarni's Theorem, there is a Δ^1_1 E-invariant set $C \subseteq \mathcal{N}$ and a local measure function m, such that that C admits a Δ^1_1 compression and for each $x \in \mathcal{N} \setminus C$ there is a (unique) E-invariant probability Borel measure μ_x on X satisfying $\mu_x(U) = m(U, x)$ for all $U \in \mathcal{U}$.

For Δ^1_1 sets $A, B \subseteq \mathcal{N}$, let $Q^{A,B}_n$ be the associated decomposition (cf. Notation A.3.8). Let F_n be the uniformly Δ^1_1 fundamental sequence for E used in the proof of the Effective Nadkarni's Theorem, and for $s \in \mathbb{N}^{<\mathbb{N}}$, let $N_s = \{x \in \mathcal{N} : s \subseteq x\}$. For $s \in \mathbb{N}^{<\mathbb{N}}, n, k \in \mathbb{N}$ define

$$S_{s,n,k} = \begin{cases} (\mathcal{N} \setminus [N_s]_E) \cup Q^{N_s,F_n}_0 & k = 0, \\ Q^{N_s,F_n}_k & \text{otherwise.} \end{cases}$$

By the proof of Theorem A.2.2, we may assume that $S_{s,n,k} \in \mathcal{U}$ for all s, n, k.

Now let $Z = \mathcal{N} \setminus (C \cup \bigcup_{s,n,k} C_0(S_{s,n,k}))$, where $C_0(S)$ is the set constructed in the proof of Lemma A.3.13(3). By the uniformity of this construction and Lemma A.3.2(2), $\mathcal{N} \setminus Z$ is Δ^1_1 and admits a Δ^1_1 compression. By invariance of the local measure function, the assignment $x \mapsto \mu_x$ is E-invariant. Additionally, as noted in the introduction, we may identify $P(\mathcal{N})$ with the subspace of $\varphi \in [0,1]^{\mathbb{N}^{<\mathbb{N}}}$ satisfying $\varphi(\emptyset) = 1$ and $\varphi(s) = \sum_n \varphi(s^\frown n)$, for $s \in \mathbb{N}^{<\mathbb{N}}$. Then, by uniformity in A of the local measure function $m(A,x)$, the assignment $x \mapsto \mu_x$ defines a Δ^1_1 map $Z \to \mathrm{INV}_E \subseteq [0,1]^{\mathbb{N}^{<\mathbb{N}}}$.

For $x \in Z$, let $S_x = \{y \in Z \colon \mu_y = \mu_x\}$.

Lemma A.5.1. *For any $x \in Z$, $\mu_x(S_x) = 1$.*

Proof. If $x \in S_{s,n,k}$, then by definition of Z, E-invariance of $S_{s,n,k}$ and the fact that $S_{s,n,k} \in \mathcal{U}$, we have $\mu_x(S_{s,n,k}) = m(S_{s,n,k}, x) = 1$.

Now define $\tilde{S}_x = Z \cap \bigcap \{S_{s,n,k} \colon x \in S_{s,n,k}\}$. Since $\mathcal{N} \setminus Z$ is compressible, $\mu_x(Z) = 1$, and so $\mu_x(\tilde{S}_x) = 1$. If $y \in \tilde{S}_x$, then $[N_s/F_n](x) = [N_s/F_n](y)$ for all s, n, so $\mu_y(N_s) = m(N_s, y) = m(N_s, x) = \mu_x$ for all $s \in \mathbb{N}^{<\mathbb{N}}$, and hence $\mu_y = \mu_x$. Therefore $\tilde{S}_x \subseteq S_x$, and $\mu_x(S_x) = 1$. □

Lemma A.5.2. *Let $S \subseteq \mathcal{N}$ be E-invariant and Borel. Then there is an E-invariant compressible Borel set $C \subseteq \mathcal{N}$ such that for $x \notin C$ we have*

$$\mu_x(S) = m(S,x) = \begin{cases} 1 & x \in S, \\ 0 & x \notin S. \end{cases}$$

Proof. By relativizing, we may assume S is Δ^1_1. Repeat the proofs of this section, assuming this time that $S \in \mathcal{U}$, to get a Δ^1_1 set $Z' \subseteq \mathcal{N}$ and a Δ^1_1 assignment $Z' \ni x \mapsto \mu'_x \in \mathrm{INV}_E$ induced by a local measure function m'. Note that $m = m'$ by uniformity of the construction of the local measure function, and hence $\mu_x = \mu'_x$ for $x \in Z \cap Z'$.

Let $C = (\mathcal{N} \setminus Z \cap Z') \cup C_0(S)$, where $C_0(S)$ is the set constructed in the proof of Lemma A.3.13(3). Then C admits a Δ^1_1 compression, and if $x \notin C$ then

$$\mu_x(S) = \mu'_x(S) = m'(S,x) = \begin{cases} 1 & x \in S, \\ 0 & x \notin S. \end{cases}$$

□

Proposition A.5.3. *For any $x \in Z$, μ_x is the unique E-ergodic invariant probability Borel measure on $E|S_x$. Moreover, every E-ergodic invariant probability Borel measure is equal to μ_x, for some $x \in Z$.*

Proof. Fix $x \in Z$. Note that S_x is E-invariant, Borel and non-compressible (as it supports the E-invariant measure μ_x). Now let $Y \subseteq \mathcal{N}$ be E-invariant and Borel. By Lemma A.5.2 there is an E-invariant compressible Borel set $C \subseteq \mathcal{N}$ such that for $y \notin C$, $\mu_y(Y) \in \{0,1\}$. Since S_x is E-invariant and non-compressible, there must be some $y \in S_x \setminus C$. Then $\mu_x(Y) = \mu_y(Y) \in \{0,1\}$. Since Y was arbitrary, μ_x is E-ergodic.

Now let ν be any E-ergodic invariant probability Borel measure. For every $s \in \mathbb{N}^{<\mathbb{N}}, n \in \mathbb{N}$, there is a unique $k(s,n) \in \mathbb{N}$ such that $\nu(S_{s,n,k(s,n)}) = 1$. Define $S = \bigcap_{s,n} S_{s,n,k(s,n)}$. Then $\nu(S) = 1$, so in particular S is non-compressible, and hence $S \cap Z \neq \emptyset$. Let $x \in S \cap Z$.

We claim that $\mu_x = \nu$. To see this, fix some $s \in \mathbb{N}^{<\mathbb{N}}$, in order to show that $\mu_x(N_s) = \nu(N_s)$. Note that $[N_s/F_n](x) = k(s,n)$, for all s, n, so that $\mu_x(N_s) = \lim_n \frac{k(s,n)}{2^n}$ (cf. Definition A.3.9 and Definition A.3.12). We now consider two cases. If $\nu([N_s]_E) = 0$, then $k(s,n) = 0$ for all n, so $\mu_x(N_s) = 0 = \nu(N_s)$. Now suppose $\nu([N_s]_E) = 1$. For all n, we have $N_s \cap Q^{N_s,F_n}_{k(s,n)} \approx k(s,n)(F_n \cap Q^{N_s,F_n}_{k(s,n)})$, so, as noted at the start of Section A.3, $\nu(N_s) \in [k(s,n)2^{-n}, (k(s,n)+1)2^{-n}]$ for all n. Thus

$$\nu(N_s) = \lim_n \frac{k(s,n)}{2^n} = \mu_x(N_s).$$

Finally, it remains to show that μ_x is the unique E-ergodic invariant probability Borel measure on $E|S_x$. To see this, let ν be any other such measure and write $\nu = \mu_y$ for some $y \in Z$. Then $\nu(S_y) = \mu_y(S_y) = 1$, so $\nu(S_x \cap S_y) = 1$. Thus $S_x \cap S_y \neq \emptyset$, and so $\mu_x = \mu_y = \nu$. □

Proposition A.5.4. *Let $\mu, \nu \in \mathrm{INV}_E$. If $\mu(S) = \nu(S)$ for all E-invariant Borel sets $S \subseteq \mathcal{N}$, then $\mu = \nu$.*

Proof. Let $A \subseteq \mathcal{N}$ be Δ^1_1. As in the proof of Proposition A.5.3, we have

$$\mu(A \cap Q^{A,F_n}_k) \in [k2^{-n}\mu(Q^{A,F_n}_k), (k+1)2^{-n}\mu(Q^{A,F_n}_k)].$$

Similarly,

$$\nu(A \cap Q^{A,F_n}_k) \in [k2^{-n}\nu(Q^{A,F_n}_k), (k+1)2^{-n}\nu(Q^{A,F_n}_k)].$$

Since the sets Q_k^{A,F_n} are E-invariant, we have $\mu(Q_k^{A,F_n}) = \nu(Q_k^{A,F_n})$, and therefore

$$|\mu(A \cap Q_k^{A,F_n}) - \nu(A \cap Q_k^{A,F_n})| \leq 2^{-n}\mu(Q_k^{A,F_n}).$$

It follows that

$$|\mu(A) - \nu(A)| \leq \sum_k |\mu(A \cap Q_k^{A,F_n}) - \nu(A \cap Q_k^{A,F_n})| \leq 2^{-n} \sum_k \mu(Q_k^{A,F_n}) \leq 2^{-n}.$$

Since n was arbitrary, $\mu(A) = \nu(A)$. □

Proposition A.5.5. *For any $\nu \in \mathrm{INV}_E$, $\nu = \int \mu_x d\nu(x)$.*

Proof. Let $A \subseteq \mathcal{N}$ be E-invariant. Then $\int \mu_x(A) d\nu(x) = \nu(A \cap Z) = \nu(A)$. Thus, by Proposition A.5.4, $\nu = \int \mu_x d\nu(x)$. □

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