Soft Theorems from Spontaneous Symmetry Breaking

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"Oto wszystko, co mogę wam dać,
Kromkę księżyca.
Tym się nikt nie nakarmi,
Ale jak to zachwyca."

Kazimierz Wierzyński

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ABSTRACT

Spontaneous symmetry breaking occurs when the vacuum state is not preserved under (a subset of) symmetries in the theory. Instead, the symmetry is non-linearly realized by the associated massless degrees of freedom, the Nambu-Goldstone bosons. At the level of on-shell observables, the non-linearly realized symmetry is manifested as a universal structure of scattering amplitudes in the so-called soft limit, which means sending the momenta of a Nambu-Goldstone modes to zero.

In this dissertation, we further explore the link between spontaneous symmetry breaking and infrared dynamics of massless scalars. First, we derive soft theorems for theories with spontaneously broken Poincaré symmetries, corresponding to effective field theories for condensed matter systems such as solids, fluids, superfluids, and framids. We also implement a bootstrap in which the enhanced vanishing of amplitudes in the soft limit is taken as an input, thus sculpting out a subclass of exceptional solid, fluid, and framid theories.

Next, we consider spontaneous breaking of higher symmetries. We derive a new sub-leading double soft pion theorem in theories with a spontaneously-broken continuous 2-group global symmetry, which intertwines amplitudes with different numbers of pions and photons. We also provide a novel derivation of the leading soft photon theorem from the Ward identity of an emergent 1-form global symmetry in effective field theories where antiparticles are integrated out.

Finally, we turn to universal features in low-energy dynamics of generic effective field theories. We extend the scalar geometric soft theorem by allowing the massless scalar to couple to other scalars, fermions, and gauge bosons. The soft theorem keeps its geometric form, but where the field-space geometry now involves the full field content of the theory. As a bonus, we also present novel double soft theorems with fermions, which mimic the geometric structure of the double soft theorem for scalars.

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M. D. contributed to the calculations in the paper and preparation of the manuscript.

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Chapter 1

INTRODUCTION

The discovery of the Standard Model—our current theoretical description of elementary particles, tested to a high precision—was largely guided by the search for symmetries of the underlying fundamental interactions. Historically, one of the key insights to the Standard Model puzzle was provided by Nambu and Goldstone, who showed that spontaneous breaking of continuous symmetries mandates a presence of massless scalars in the spectrum, the Nambu-Goldstone (NG) bosons [1]–[3].

Spontaneous symmetry breaking refers to a scenario in which the vacuum state is not invariant under the full set of symmetries of the underlying Lagrangian. For the case of spontenously broken internal symmetries, there is a NG mode corresponding to each broken generator, which transforms non-linearly under the full symmetry. In turn, the low-energy dynamics of NG bosons is constrained by the algebra of spontaneously broken currents associated with the symmetries of the theory.

At the level of on-shell observables, the non-linearly realized symmetry dictates the form of a soft theorem—a universal behavior of NG amplitudes in the so-called soft limit, as we send a momentum of a gapless mode to zero and perform an expansion of the amplitude in the soft momenta. For a flagship example of such a relation, consider the non-linear sigma model (NLSM) which describes the interactions of massless pions, the NG bosons of spontaneously broken $SU(N)$ axial symmetry. As shown by Adler [4], the conservation of the axial current implies that an amplitude with n pions vanishes as we send the momentum of one of the pions to zero (single soft limit).

In fact, the algebra of non-abelian currents can be accessed from amplitudes with multiple soft pions; see [5]–[7] for early work on multiple-pion emission. In NLSM, in the simultaneous double soft limit (two of the pion momenta are sent to zero at the same rate) the amplitudes do not vanish; instead they have a universal structure, which can be expressed as a kinematic soft operator multiplying a lower point amplitude [8]. The form of a soft operator in the double soft theorem is controlled by the commutator of broken axial currents.

However, such underlying simplicity of the low-energy dynamics is often hard to diagnose just by looking at a particular form of the effective field theory (EFT) Lagrangian with non-linearly realized symmetry. First of all, the form of the Lagrangian is not unique, as we can perform field redefinitions which do not change the physical content of the theory.[1] Secondly, going back to the NLSM example, the pion Lagrangian contains a tower of two-derivative interactions, and so it is not clear that the amplitudes should vanish in the single soft limit. In fact, the Adler zero condition requires cancellations between different n-point interaction terms. Thus, for a systematic study of EFTs with "special" soft behavior, it is much more convenient to work at the level of scattering amplitudes, which is known as the *soft bootstrap* program [9]–[12].

The example of pion amplitudes revealed that the structures appearing in soft theorems are dictated by the spontaneous breaking of internal symmetry. However, such relation between non-linearly realized symmetries and low-energy on-shell observables should, of course, hold more generally. In particular, in this dissertation we explore three different directions in the study of how the spontaneous symmetry breaking pattern is encoded in the soft amplitudes. To this end, we derive soft theorems for massless scalars in

(*i*) a broad class of nonrelativistic EFTs with spontaneous breaking of *spacetime symmetries* (Chapter 2);

(*ii*) an example of an EFT with spontaneous breaking of *higher-group symmetry* (Chapter 3), where the set of Goldstone modes includes both spin-zero and spin-one particles;

(*iii*) general relativistic EFTs coupled to particles with higher spin, *without* assuming an internal symmetry structure (Chapter 4). This paper builds upon the work of [13], which developed a framework of geometric soft theorems for scalars.

We will now discuss each of these points in turn.

[1] The geometric approach to EFTs—which we will discuss in Chapter 4—exploits this fact to organize the EFT data into structures according to their transformations under field redefinitions.

1.1 Spontaneous breaking of spacetime symmetries

Famously, models with spontaneous spacetime symmetry breaking are relevant in cosmology, such as the proposed EFTs for inflation [14]–[16]. Specifically, in the limit as gravity decouples from inflation [17], the scalar metric mode interactions are captured by flat-space amplitudes of NG bosons for spontaneously broken time diffeomorphisms. Indeed, the first on-shell soft theorem for non-linearly realized boosts was derived in this context [18]. In fact, the same symmetry breaking pattern EFT describes superfluids [14], which brings us to another important class of models with spontaneous spacetime symmetry breaking, namely EFTs corresponding to condensed matter systems in the continuum limit [16], [19], [20].

Depending on the way spacetime symmetry is spontaneously broken, the number of NG bosons can be less than the number of broken symmetry generators [21]–[25]. An example of such scenario is the spontaneous breaking of translational and rotational symmetries in solids, yielding only phonon degrees of freedom associated with broken translations.

The coset construction, a standard procedure for deriving Lagrangians with spontaneous symmetry breaking [26]–[28], can be augmented to take that into account via the so-called inverse Higgs constraints [21], [29]. Using this "top-down" method, the authors of [19] obtained general Lagrangians for phases of matter classified by their possible symmetry breaking pattern. Such general considerations yielded—in addition to phases of matter found in nature—a toy model characterizing a framid, which only breaks boost invariance.

In Chapter 2, we present the soft theorems for a class of non-relativistic EFTs appearing in [19], which correspond to various condensed matter systems, such as solids and perfect fluids, as well as framids. As we will see, the form of the soft theorem encodes the symmetry braking pattern. In particular, in some cases only a subset of terms in the soft amplitude obeys a soft theorem, meaning that it can be expressed as a soft operator acting on lower point amplitudes in a given theory.[2]

Next, we apply soft bootstrap techniques to non-relativistic condensed matter EFTs to find theories with the enhanced soft limits—that is, vanishing in the

[2]The remaining terms in the soft amplitude also possess a universal structure, as dictated by the geometric soft theorem (see Chapter 4). However, that universal form involves a derivative acting on the space of couplings in the EFT, and so we will not discuss it here.

soft limit faster than naively expected from derivative counting—signaling a presence of additional "hidden" non-linearly realized symmetry.

In some special cases, knowing the soft limit of scattering amplitudes provides enough information to obtain higher-point amplitudes in the theory directly using on-shell methods [9], [30], [31]. For instance, NLSM amplitudes can be constructed in this "bottom-up" way, imposing Adler zero condition. Similarly, a non-vanishing soft theorem can be used as additional input. For the case of framids, requiring that the soft theorem is satisfied allows us to bootstrap higher-point interactions, bypassing the redundancies in the Lagrangian construction.

1.2 Spontaneous breaking of higher symmetries

Recent years have witnessed numerous developments in the field of *generalized symmetries*, motivated by extending—in several different ways—the standard notion of group-like symmetry acting on local objects; for a review, see [32], [33] and references therein. In particular, one can consider *higher-form* symmetries which act on non-local operators (ie., p-form symmetries act on p-dimensional objects). Crucially, associated charge operators always commute; therefore, higher-form symmetries are necessarily abelian. Furthermore, different p-form symmetries can mix in a non-trivial way, giving rise to a *higher-group* structure. In Chapter 3, we will focus on 1-form symmetries, which act on Wilson lines, and can mix with an ordinary (0-form) symmetry to form a 2-group.

Just like in the case of ordinary symmetry, higher-form symmetries can also be spontaneously broken. The higher-form analogue of the NG theorem [34]–[36] states that, for instance, the NG mode of spontaneously broken continuous 1-form symmetry is a massless spin-one boson. Consequently, one can identify the photon in pure Maxwell theory as a NG boson associated with spontaneously broken 1-form symmetry [34]. Given the link between soft theorems and spontaneous symmetry breaking, some immediate questions arise: is it possible to recast the Weinberg soft photon theorem [37], [38] as following from spontaneously broken 1-form symmetry? Also, are there any new soft structures associated to higher symmetries that we can discover?

At first glance, reinterpreting soft photon theorems with higher-form symmetry seems to be obstructed by the fact that charged matter breaks 1-form symmetry explicitly. However, in Chapter 3, we will argue that in the soft limit there is an *emergent* 1-form symmetry. The associated 1-form Ward identities provide constraints which allow us to derive the Weinberg soft photon theorem at leading order. Therefore this framework provides a unified picture for soft theorems for pions and photons, and moreover suggests that there are new soft structures as dictated by the spontaneously broken higher-group symmetries, which can intertwine NG bosons of different p-form symmetries.

For a non-trivial example of a new soft theorem, we consider a theory with a 2-group structure, which encodes a non-trivial mixing between non-abelian 0-form and $U(1)$ 1-form symmetry. In particular, the 2-group structure constant appears in a modification of the 0-form current algebra, which could affect the form of the double soft limit of 0-form NG bosons.

Concretely, we consider a theory with spontaneously broken 2-group symmetry discussed in [39], namely the NLSM with gauged vector $U(1)$ symmetry corresponding to baryon number. We show that the NLSM double soft theorem at leading order in soft momenta is not modified, but the sub-leading order has an additional contribution sensitive to the 2-group structure. Finally, we illustrate the new double soft theorem in explicit examples of soft amplitudes.

1.3 Soft theorems and field-space geometry

The intuition built from theories with non-linearly realized symmetries implies that the soft limit of amplitudes probes the neighborhood of the vacuum expectations value (VEV) of the spontaneously broken theory. For pions, a vanishing single-soft limit reveals there is an underlying moduli space of vacua in the unbroken phase, whereas double soft limit corresponds to different paths in moduli space [8]. In turn, this suggests that in a general EFT without internal symmetry, a soft limit of a massless scalar should reflect the EFT structure in the neighborhood of the VEV [13].

Parametrizing scalar EFTs using field-space geometry [28], [40]–[44] provides a suitable framework to address this question. In this setup, a multiplet of scalar fields can be viewed as functions from spacetime to a target scalar manifold, which encodes the EFT data.

In general, as we discussed above, a theory can be described by a family of Lagrangians that map into each other under field redefinitions. Choosing a field basis corresponds to picking a particular coordinate system for the underlying target manifold, yielding a set of couplings dependent on that basis.

In turn, a field redefinition with no derivatives[3] corresponds to a coordinate transformation, with associated mapping of couplings to a new basis.

With this insight, we can group higher-dimensional operators with a fixed number of derivatives into compact geometrical structures, which map onto themselves under field redefinitions. Clearly, physical observables such as scattering amplitudes depend only on geometrical data of the target manifold, such as Riemann curvature and its derivatives, and not on a choice of a particular basis. Organizing EFT data in the Lagrangian in such a way proved to be a practical tool for streamlining calculations [40], [46], [55]–[64] and also allows us to study infrared dynamics of scalars in theories with no internal symmetry.

Generally speaking, the scalar geometric soft theorem [13] relates a soft limit of a $(n+1)$-point amplitude to a field-space covariant derivative of n-point amplitude, which acts in the space of couplings. In other words, to apply the soft theorem, one needs to know the couplings in the amplitude as functions of the VEV. However, in the presence of non-linearly realized symmetry, the covariant derivative in the soft theorem can be expressed in terms of soft operators acting on lower-point amplitudes, recovering the familiar form of soft theorems for spontaneous symmetry breaking. Moreover, recently the geometric soft theorem has been extended to one-loop order [65].

Given the computational advantages of the geometric scalar construction, one would like to extend this framework to incorporate particles with higher spin. There are different approaches to this question [47], [66]–[69]; perhaps a minimal extension is dictated by considering field redefinitions depending on scalar fields only. Similarly, in Chapter 4, we extend the geometric soft theorem of [13] to include scalars coupling to fermions and gauge bosons. We employ the existing formulations of field-space geometry for fermions [66] and gauge bosons [69] and derive a general soft theorem for massless scalars coupled to particles with higher spin.

We find that the soft theorem still acts as a covariant derivative, but now in the full field-space accommodating scalars, fermions, and gauge bosons. Including massive fermions and gauge bosons gives rise to additional terms in soft theorem, which are in closely related to analogous terms for massive scalars. In addition, we derive a double soft theorem for both soft scalars and

[3]For extensions of field-space geometry to accommodate field redefinitions with derivatives, see [45]–[54].

soft fermions. Last but not least, we discuss several examples of geometric EFTs coupled to fermions and gauge bosons (with and without masses) and verify the validity of the new soft theorems explicitly.

Chapter 2

SOFT PHONON THEOREMS

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2.1 Introduction

The seminal work of Nambu and Goldstone [1]–[3] revealed a deep connection between spontaneously broken internal symmetries and a corresponding set of gapless degrees of freedom. These Nambu-Goldstone bosons (NGBs) parameterize a continuous degeneracy of vacua and transform nonlinearly under the broken symmetries. Notably, spontaneous symmetry breaking often mandates universal features in scattering, as perhaps best illustrated by the Adler zero [4], which refers to the vanishing of certain NGB amplitudes in the soft limit.

As is well-known, similar statements apply to the spontaneous breaking of *spacetime symmetries* [70]–[74], albeit with a fewer number of NGBs than naively expected [21]–[24]. More recently, it has also been suggested that spontaneous breaking of Poincaré invariance is not merely a calling card of certain condensed matter systems, but can actually be elevated to an *organizing principle* for these theories [19]. In this approach, nonrelativistic effective field theories (EFTs) are classified by their spacetime symmetry breaking pattern, yielding a rich array of physical systems corresponding to the phonon excitations at zero temperature of perfect solids and fluids, as well as modes of a superfluid. The authors of [19] also discovered some exotic, yet-to-be-experimentally-realized systems which include the framid, whose corresponding framon degree of freedom exhibits the minimal nonlinear realization of spontaneous Lorentz symmetry breaking.

In this paper we study the soft behavior of scattering amplitudes of NGBs arising from the spontaneous breaking of spacetime symmetries. Our analysis focuses on the nonrelativistic EFTs classified in [19] by the symmetry breaking pattern corresponding to solids, fluids, superfluids, and framids. In all of these systems, the group of spatial rotations is preserved at low energies, so the

NGBs reside in a linear representation of $SO(3)$. For example, the superfluid phonon is described by a scalar field π, while solid and fluid phonons and framons are described by a three-vector field π^i.[1] In general, the latter NGBs nonlinearly realize the underlying broken spacetime symmetries via

$$\pi^i \to \pi^i + \alpha^i + \beta^i{}_j \pi^j + \cdots \,, \tag{2.1.1}$$

for parameters α and β, which are in general *position-dependent*, and where the ellipses denote terms higher order in the field. In our analysis, we focus primarily on the case in which the broken spacetime symmetry generators are translations or boosts, but also consider the case of broken spatial diffeomorphisms in a fluid.

Following the logic of [18], we use current conservation to derive a broad class of soft theorems applicable to NGBs arising from the spontaneous breaking of *any* symmetry. Technically, our results apply to any symmetry breaking pattern involving spacetime or internal symmetries or both.[2] The schematic form of our soft theorem is

$$\lim_{q\to 0} \alpha[A_{n+1}] \sim -\lim_{q\to 0} \alpha\left[\sum V_3 \Delta A_n\right] - \sum \beta[A_n]\,, \tag{2.1.2}$$

where the soft limit $q \to 0$ corresponds to sending the energy and momentum components of the soft leg to zero. Here both sides of eq. (2.1.2) are $\mathcal{O}(q^0)$ in the soft momentum, which is to say that terms $\mathcal{O}(q^1)$ or higher have been dropped. The summations above run over all external legs in A_n, which are assumed to be hard.

Since α and β are in general functions of spacetime, they act as *differential operators* on the external momenta. In particular, in eq. (2.1.2), α acts on the soft energy or momentum, while β acts on the energy or momenta of each hard leg in A_n. Depending on the differential degree of α and β, they will extract different powers in the soft expansion of the amplitudes. For example, if α is

[1] Throughout, we use Greek letters $\mu, \nu, \rho, \cdots$, to denote four-vector indices, late Latin letters $i, j, k, \cdots$, to denote three-vector indices, and early Latin letters $a, b, c, \cdots$, to denote external particle labels. For products of three-momenta we will sometimes employ the shorthand, $p \cdot q = p^i q_i$.

[2] While the present work focuses solely on theories which preserve $SO(3)$ rotation symmetry, this is actually not required for our soft theorem. In particular, our results apply to any symmetry breaking pattern that preserves some version of spacetime translations in the broken phase such that energy and momentum are well-defined. We leave an analysis of more drastic symmetry breaking patterns for later work.

a constant, then eq. (2.1.2) extracts the $\mathcal{O}(q^0)$ piece of A_{n+1}, while if α is a single derivative with respect to the soft energy or momentum, then it probes the $\mathcal{O}(q^1)$ piece of A_{n+1}.

eq. (2.1.2) is an on-shell soft theorem because both the left- and right-hand sides are operations acting on the on-shell amplitudes A_{n+1} and A_n. Furthermore, as required of any physical on-shell scattering, the soft theorem in eq. (2.1.2) is satisfied irrespective of the choice of field basis. This feature is actually rather miraculous when one considers that eq. (2.1.2) depends explicitly on the *off-shell* three-point vertex V_3, which is field basis dependent along with the symmetry parameter β. However, as we will later argue, any change of field basis that sends V_3 and β to an alternative choice of V_3' and β' *necessarily* cancels in the soft theorem. The fact that the soft theorem is on-shell makes our results distinct from the soft theorems for correlation functions derived in [75]–[79].

The structure of this paper is as follows. In Sec. 2.2 we state the general soft theorem and present a proof as well as a discussion of its invariance under changes of field basis. We then turn to concrete examples of theories that satisfy the soft theorem: superfluids in Sec. 2.3, solids in Sec. 2.4, fluids in Sec. 2.5, and framids in Sec. 2.6. In Sec. 2.7 we discuss a soft bootstrap for nonrelativistic theories, and we conclude in Sec. 2.8.

2.2 Soft Theorem

2.2.1 Degrees of Freedom

In this work we focus on nonrelativistic theories describing a NGB arising from a spontaneously broken spacetime symmetry.[3] For concreteness, let us consider here the case of an $SO(3)$ vector field π^i, which transforms linearly under

$$\text{spatial rotations:} \quad \pi^i \to R^i{}_j \pi^j \,, \tag{2.2.1}$$

for a constant orthogonal matrix R. Here we emphasize that π^i does *not* describe a gauge theory in the conventional sense, since all three of its components are physical: they correspond to one longitudinal and two transverse modes of the NGB, which we describe in terms of one-particle states, $|\omega, p, L\rangle$

[3] For simplicity, we focus on the case of NGBs of type I with a linear dispersion relation. While we do not explicitly analyze NGBs of type II with quadratic dispersion relations (see [23], [25], [80]–[86]), all of our results, including the general soft theorem, should apply more generally.

and $|\omega, p, T\rangle$, respectively. These states overlap with the π^i field according to

$$\begin{aligned} \langle 0|\pi^i(t,x)|\omega,p,L\rangle &= e^i_L(p)e^{-i\omega t+ip\cdot x}\,, \\ \langle 0|\pi^i(t,x)|\omega,p,T\rangle &= e^i_T(p)e^{-i\omega t+ip\cdot x}\,. \end{aligned} \tag{2.2.2}$$

Here ω and p are the energy and three-momentum of the particle. Depending on the particle type, these quantities obey the dispersion relations,

$$\omega^2 - c_L^2 p^2 = 0 \qquad \text{or} \qquad \omega^2 - c_T^2 p^2 = 0\,, \tag{2.2.3}$$

where c_L and c_T are the speeds of sound for the longitudinal and transverse modes, respectively. The polarization vectors in eq. (2.2.2) satisfy

$$p_i e^i_T = 0 \qquad \text{and} \qquad \epsilon_{ijk} p^j e^k_L = 0\,. \tag{2.2.4}$$

This implies that e_L encodes the single longitudinal mode while e_T encodes the two transverse modes. For explicit calculations, we will use the unit normalized longitudinal polarization,

$$e^i_L = \frac{c_L p^i}{\omega}\,. \tag{2.2.5}$$

We can think of eq. (2.2.3) and eq. (2.2.4) as the on-shell conditions for the kinematic variables that characterize the phonon modes.

Here it will be convenient to define the projection operators,

$$\begin{aligned} \Pi^{ij}_L(p) &= \frac{p^i p^j}{p^2}\,, \\ \Pi^{ij}_T(p) &= \delta^{ij} - \frac{p^i p^j}{p^2}\,, \end{aligned} \tag{2.2.6}$$

which leave the polarizations invariant, so

$$\Pi^{ij}_L(p) e_{Lj} = e^i_L \qquad \text{and} \qquad \Pi^{ij}_T(p) e_{Tj} = e^i_T\,. \tag{2.2.7}$$

In terms of these projectors, the phonon propagator is

$$\Delta^{ij}(\omega,p) = \frac{\Pi^{ij}_L(p)}{\omega^2 - c_L^2 p^2} + \frac{\Pi^{ij}_T(p)}{\omega^2 - c_T^2 p^2}\,, \tag{2.2.8}$$

whose inverse, $\Delta^{-1}_{ij}(\omega,p)$, is the two-point Lagrangian term in momentum space.

To define the n-point scattering amplitude of phonons we define a set of external particles labelled by $a = 1, \cdots, n$, whose corresponding momenta are

$p_a^\mu = (\omega_a, p_a^i)$, with speed of sound c_a and polarization vector e_a^i which is chosen to be either longitudinal or transverse. The n-point scattering amplitude is

$$A_n^{i_1\cdots i_n}(p_1, \cdots, p_n) = \langle 0| \left(|\omega_1, p_1\rangle^{i_1} \cdots |\omega_n, p_n\rangle^{i_n} \right) , \tag{2.2.9}$$

where we have defined a shorthand for a state $|\omega, p\rangle^i$ that carries an arbitrary polarization and is related to the physical longitudinal and transverse states defined previously by $|\omega, p, L\rangle = e_L^i |\omega, p\rangle_i$ and $|\omega, p, T\rangle = e_T^i |\omega, p\rangle_i$, respectively.

The quantity $A_n^{i_1\cdots i_n}$ is simply the amputated correlation function of phonon fields, which can be computed straightforwardly using Feynman diagrams. To compute the physical amplitude we simply dot this object into external polarization vectors. Note that we have used a schematic notation in which $A_n^{i_1\cdots i_n}$ is written as a function of just the three-momenta $p_1, \cdots, p_n$. However, since we are interested in on-shell kinematics, momenta and energies can be interchanged freely. So in explicit calculations, our actual amplitudes may be functions of energies as well as the three-momenta.

2.2.2 Proof of Theorem

To begin, recall that the NGB of spontaneous spacetime symmetry breaking transforms nonlinearly under the broken symmetry transformations,

$$\pi^i \to \pi^i + \delta\pi^i . \tag{2.2.10}$$

In general, the nonlinearly realized symmetry transformation may also involve changes of coordinates, but this will not be important for our analysis. In addition, we will assume that the Lagrangian does not depend on the second and higher derivatives of fields. The statement that the Lagrangian is invariant implies that

$$L \to L + \delta L , \tag{2.2.11}$$

where the Lagrangian variation is

$$\delta L = \partial_\mu \left(\delta\pi^i \frac{\delta L}{\delta \partial_\mu \pi^i} \right) - \delta\pi^i E_i = \partial_\mu K^\mu . \tag{2.2.12}$$

Here K describes any shift of the Lagrangian by a total derivative, as would often appear in a spacetime symmetry transformation. Meanwhile, E denotes the equation of motion,

$$E_i = \partial_\mu \left(\frac{\delta L}{\delta \partial_\mu \pi^i} \right) - \frac{\delta L}{\delta \pi^i} \overset{\text{on-shell}}{=} 0 , \tag{2.2.13}$$

which vanishes on the support of on-shell, physical field configurations. Recalling the definition of the conserved current,

$$J^\mu = \delta\pi^i \frac{\delta L}{\delta \partial_\mu \pi^i} - K^\mu \,, \tag{2.2.14}$$

which satisfies the conservation equation,

$$\partial_\mu J^\mu = \delta\pi^i E_i \overset{\text{on-shell}}{=} 0 \,, \tag{2.2.15}$$

for on-shell configurations of fields. In order to derive our soft theorem we evaluate matrix elements of the above equation, keeping contributions up to $\mathcal{O}(q^1)$ in the soft limit.

For later convenience, let us define a bracket that acts on a local field operator $\mathcal{O}(t,x)$ via

$$\langle \mathcal{O} \rangle \delta^4(p_1 + \cdots + p_n) = \lim_{q\to 0} \int dt d^3x \, e^{-i\omega t} e^{iq\cdot x} \langle 0 | \mathcal{O}(t,x) \left(|\omega_1, p_1\rangle^{i_1} \cdots |\omega_n, p_n\rangle^{i_n} \right) \,, \tag{2.2.16}$$

which is the matrix element obtained by sandwiching the operator between a set of on-shell physical states with arbitrary polarizations. By construction, the field operator itself is imparted with energy ω and three-momentum q which are taken to zero, yielding a soft limit. Throughout, we assume that an on-shell momentum flows through the operator, so ω also scales as q and is implicitly sent to zero in the soft limit.

To derive a soft phonon theorem we evaluate eq. (2.2.15) as an operator equation sandwiched between on-shell physical states. To this end, let us define a general parameterization of the infinitesimal shift of the NGB field,

$$\delta\pi^i = \alpha^i + \beta^i{}_j \pi^j + \cdots \,, \tag{2.2.17}$$

where α and β are spacetime-dependent in general and the ellipses denote terms that are higher order in the field. Meanwhile, the equation of motion takes the general form,

$$E_i = V_{2ij}\pi^j + \tfrac{1}{2} V_{3ijk} \pi^j \pi^k + \cdots \,, \tag{2.2.18}$$

where V_{2ij} and V_{3ijk} correspond to the two- and three-point Lagrangian terms. Going to momentum space, the Feynman propagator and three-point Feynman vertex are equal to $\Delta_{ij} = V_{2ij}^{-1}$ and V_{3ijk}, all multiplied by i.

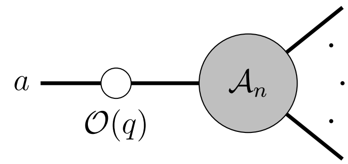

Figure 2.1: Diagrams computing the contribution from operator insertions on the external legs.

The classical conservation of the current in eq. (2.2.15) uplifts to the operator statement,

$$0 = \langle \partial_\mu J^\mu \rangle = \langle \delta\pi^i E_i \rangle = \langle (\alpha^i + \beta^i{}_j \pi^j + \cdots)((\Delta^{-1})_{ik}\pi^k + \tfrac{1}{2}V_{3ikl}\pi^k\pi^l + \cdots)\rangle \,. \tag{2.2.19}$$

To derive the soft theorem we must calculate the matrix elements in each term in eq. (2.2.19). In principle one should evaluate all possible insertions of each operator, both on internal and external lines. However, many terms can be neglected since we are only interested in terms at $\mathcal{O}(q^0)$ but not higher. In particular, since the first equality in eq. (2.2.19) implies that the matrix element is automatically equipped with an overall factor of q, any contributions to $\langle J^\mu \rangle$ which are analytic in q will only generate $\mathcal{O}(q^1)$ contributions to the matrix element. Conversely, $\mathcal{O}(q^0)$ contributions only arise from terms in $\langle J^\mu \rangle$ that go as $\mathcal{O}(q^{-1})$. Such terms appear due to soft pole contributions from q-dependent propagators, which in turn only arise from insertions of the operator on external legs (see Fig. 2.1). On the other hand, operator insertions on internal legs and terms of $\mathcal{O}(\pi^3)$ or higher yield terms that are analytic in q and thus vanish in the $q \to 0$ soft limit. Thus we can drop all such terms in the evaluation of the right-hand side of eq. (2.2.19).

Shuffling around terms in eq. (2.2.19), we arrive at

$$\langle \alpha_i (\Delta^{-1})^{ij}\pi_j \rangle = -\langle \tfrac{1}{2}\alpha_i V_3^{ijk}\pi_j\pi_k + \beta_i{}^j \pi_j (\Delta^{-1})^{ik}\pi_k \rangle_{\text{ext}} + \cdots \,, \tag{2.2.20}$$

where the ellipses denote irrelevant $\mathcal{O}(\pi^3)$ contributions and the "ext" subscript instructs that the matrix element should be evaluated with the operator inserted only on external legs. As described above, all insertions of the operator on internal legs are subleading in the soft limit and can be neglected. Importantly, each of the terms in eq. (2.2.20) can be recast in terms of on-shell scattering amplitudes.

As a warmup, let us consider the matrix element

$$\langle (\Delta^{-1})^{ij}\pi_j \rangle = -\lim_{q\to 0} A_{n+1}^{i_1\cdots i_n i}(p_1, \cdots, p_n, q) \,, \tag{2.2.21}$$

where as before, we use an abbreviated notation where we write out explicitly the three-momentum dependence of functions, but implicitly there is also energy dependence everywhere. eq. (2.2.21) simply says that the matrix element of the one-point function of an amputated field is precisely the amputated $(n+1)$-point amplitude. In the case where the operator includes the spacetime-dependent factor α, we obtain

$$\langle \alpha_i (\Delta^{-1})^{ij} \pi_j \rangle = -\lim_{q\to 0} \alpha_i (i\tfrac{\partial}{\partial \omega}, -i\tfrac{\partial}{\partial q}) \left[A_{n+1}^{i_1\cdots i_n i}(p_1, \cdots, p_n, q) \right] . \tag{2.2.22}$$

Note that in transforming to momentum space, the dependence of α on time t and position x becomes dependence on $i\frac{\partial}{\partial \omega}$ and $-i\frac{\partial}{\partial q}$, respectively.

Meanwhile, the terms involving β are written in terms of amplitudes as

$$\langle {\beta_i}^j \pi_j (\Delta^{-1})^{ik} \pi_k \rangle_{\text{ ext}} = -\sum_{a=1}^{n} {\beta^{i_a}}_{j_a} (i\tfrac{\partial}{\partial \omega_a}, -i\tfrac{\partial}{\partial p_a}) \left[A_n^{i_1\cdots j_a \cdots i_n}(\cdots, p_a, \cdots) \right] , \tag{2.2.23}$$

which corresponds to the sum over β acting on each external leg in the n-point amplitude. Last but not least, the term $\langle \frac{1}{2} \alpha_i V_3^{ijk} \pi_j \pi_k \rangle_{\text{ext}}$ is

$$-\lim_{q\to 0} \sum_{a=1}^{n} \alpha_i (i\tfrac{\partial}{\partial \omega}, -i\tfrac{\partial}{\partial q}) \left[{V_3^{i\, i_a}}_{j_a}(q, p_a) {\Delta^{j_a}}_{k_a}(p_a+q) A_n^{i_1\cdots k_a \cdots i_n}(\cdots, p_a+q, \cdots) \right] . \tag{2.2.24}$$

Here the propagator and n-point amplitude on the right-hand side are evaluated at *shifted* external energy and three-momentum, $\omega_a + \omega$ and $p_a + q$, where we have suppressed the dependence on the former in the various expressions for ease of notation. At low orders in the soft expansion we can express this alternatively as $(1 + \omega \frac{\partial}{\partial \omega_a} + q^i \frac{\partial}{\partial p_a^i})$ acting on these objects.

As noted earlier, the n-point amplitude is in general a function of three-momenta as well as energies—which is expected since these are generally interchangeable due to the on-shell conditions. Thus, the energies and three-momenta in the n-point amplitude should both be shifted. We will discuss later on how this shift is explicitly implemented in order to maintain the on-shell conditions.

In conclusion, each term in eq. (2.2.20) can be expressed in terms of differential operators acting on the $(n+1)$-point and n-point scattering amplitudes. Hence, eq. (2.2.20) implies that

$$\lim_{q\to 0} \alpha_i (i\tfrac{\partial}{\partial \omega}, -i\tfrac{\partial}{\partial q}) \left[A_{n+1}^{i_1\cdots i_n i}(p_1, \cdots, p_n, q) \right] =$$

$$-\lim_{q\to 0}\sum_{a=1}^{n}\alpha_i(i\tfrac{\partial}{\partial\omega},-i\tfrac{\partial}{\partial q})\left[V_3^{i\,i_a}{}_{j_a}(q,p_a)\Delta^{j_a}{}_{k_a}(p_a+q)A_n^{i_1\cdots k_a\cdots i_n}(\cdots,p_a+q,\cdots)\right]$$

$$-\sum_{a=1}^{n}\beta^{i_a}{}_{j_a}(i\tfrac{\partial}{\partial\omega_a},-i\tfrac{\partial}{\partial p_a})\left[A_n^{i_1\cdots j_a\cdots i_n}(\cdots,p_a,\cdots)\right]\,, \tag{2.2.25}$$

which is our final expression for the soft theorem after dropping terms $\mathcal{O}(q^1)$ or higher.

Let us comment on several subtle aspects of eq. (2.2.25). First of all, the limit $q\to 0$ with all other momenta unchanged will not, in general, preserve the on-shell conditions. Hence, a strict soft limit of this kind is not actually well-defined. The same is true for the shift of energy and momenta on the right-hand side of eq. (2.2.25). For these reasons, all of the amplitudes in eq. (2.2.25) should be evaluated in a minimal basis of kinematic invariants, which we will describe in great detail in Appendix A.[4] With this prescription, the amplitudes will be on-shell for any value of q and any value of p_a, so the soft limit and the shift of momentum are both well-defined on-shell operations.

2.2.3 Field Basis Independence

Next, we show how the soft theorem in eq. (2.2.25) is invariant under changes of field basis. To begin, we clarify that there are actually two physically distinct senses in which a soft theorem can be considered field basis invariant.

The first sense is simply the statement that the soft theorem is valid irrespective of which field basis the quantities β and V_3 are defined in, which enter explicitly into eq. (2.2.25). If we transform to a different field basis, these quantities will change to β' and V_3'. But crucially, all of the manipulations in the previous section still hold. Hence the soft theorem will still apply, so its validity is field basis independent.

The second sense is more nontrivial, and it is the statement that the on-shell amplitudes A_{n+1} and A_n can be computed in different field bases and the soft theorems will still be satisfied. As is well-known, changes of field basis induce new terms which always *vanish on-shell* in the amplitudes. However, since our soft theorems involve differential operators in energy and momentum, one can worry whether these vanishing terms end up contributing to the soft theorems

[4] Throughout, we assume complex kinematics, as commonly used in the study of gauge theory amplitudes.

anyway. We consider this possibility now, and show that such terms have no effect.

Concretely, we will now show how terms that vanish due to on-shell conditions in $A_n^{i_1\cdots i_n}$ will always cancel automatically in the soft theorem in eq. (2.2.25). Thus, the soft theorem in eq. (2.2.25) commutes with the on-shell condition, ensuring the field basis independence of the soft theorem. We assume $c_L \neq c_T$. The case where the transverse and longitudinal speeds of sound are equal is simpler, since then we don't need the projection operators.

A key relation we will need to show this cancellation comes from the symmetry transformation in eq. (2.2.17). Since this is an invariance of the Lagrangian, this symmetry transformation relates the inverse propagator and three-point vertex,

$$\lim_{q\to 0} \alpha_i [V_3^{i\,i_a}{}_{j_a}(q,p_a)] + \beta^{i_a}{}_{k_a}[\Delta^{-1}(p_a)^{k_a}{}_{j_a}] = 0\,, \tag{2.2.26}$$

which is even satisfied off-shell. In the following, we will assume that α and β are linear operators, which is indeed the case for all the examples we will discuss in this paper.

There are two types of off-shell contributions that will vanish on-shell. The first is the dispersion relation for a specific external particle a. Consider corrections to the amplitude of the form

$$\delta A_n^{i_1\cdots i_n} = (\omega_a^2 - c_a^2 p_a^2)\Pi^a(p_a)^{i_a}{}_{j_a}\mathcal{O}_n^{i_1\cdots j_a\cdots i_n}\,, \tag{2.2.27}$$

where we have inserted a projector Π^a that enforces that leg a has the correct corresponding longitudinal or transverse polarization, thus making the on-shell condition manifest. By construction, $\delta A_n^{i_1\cdots i_n}$ vanishes on-shell. Next, we apply the right-hand side of the soft theorem in eq. (2.2.25) to the above expression and then apply the on-shell conditions, yielding

$$-\Pi^a(p_a)^{i_a}{}_{j_a}\left[\lim_{q\to 0}\alpha_i\left(V_3^{i\,j_a\,k_a}\right) + \beta^{j_a\,k_a}\left(\omega_a^2 - c_a^2 p_a^2\right)\right]\Pi^a(p_a)_{k_a\,l_a}\mathcal{O}_n^{i_1\cdots l_a\cdots i_n}\,, \tag{2.2.28}$$

at the relevant order in the soft expansion. To show that eq. (2.2.28) vanishes, we sandwich eq. (2.2.26) between two projectors for particle a. Suppressing the indices, we obtain the relation

$$\begin{aligned}
\Pi^a\left[\lim_{q\to 0}\alpha(V_3)\right]\Pi^a &= -\Pi^a[\beta(\Delta^{-1})]\Pi^a \\
&= -\Pi^a\left[\beta\left((\omega^2 - c_a^2p^2)\Pi^a + (\omega^2 - c_{\bar a}^2 p^2)\Pi^{\bar a}\right)\right]\Pi^a \\
&= -\Pi^a[\beta(\omega^2 - c_a^2 p^2)]\Pi^a\,,
\end{aligned} \tag{2.2.29}$$

where $\bar{a}$ denotes the mode orthogonal to a. To obtain the last line in eq. (2.2.29), we have used that $\Pi^a\Pi^{\bar{a}} = 0$ and $\Pi^a[\beta(\Pi^a)]\Pi^a = \Pi^a[\beta(\Pi^{\bar{a}})]\Pi^a = 0$. This can be seen from

$$\Pi^a[\beta(\Pi^b)]\Pi^a = \Pi^a[\beta(\Pi^{b2})]\Pi^a = \Pi^a\left[\Pi^b\beta(\Pi^b) + \beta(\Pi^b)\Pi^b\right]\Pi^a\,, \tag{2.2.30}$$

which vanishes both when $b = a$ and $b = \bar{a}$. Hence the sum of field basis dependent contributions in eq. (2.2.28) vanishes.

The second off-shell contribution we consider, when the speeds of the two types of modes are different, $c_L \neq c_T$, is

$$\delta A_n^{i_1\cdots i_n} = \Pi^{\bar{a}}(p_a)^{i_a}{}_{j_a}\mathcal{O}_n^{i_1\cdots j_a\cdots i_n}\,, \tag{2.2.31}$$

corresponding to contributions that vanish due to the projectors coming from the choice of external polarizations. As before, $\bar{a}$ denotes the mode orthogonal to a, so $\delta A_n^{i_1\cdots i_n}$ again vanishes on-shell. First applying the soft theorem followed by the on-shell condition, we obtain

$$-\,\Pi^a(p_a)^{i_a}{}_{j_a}\left[\lim_{q\to 0}\alpha_i\left(V_3^{i\,j_a\,k_a}\right)\Delta^{\bar{a}}(p_a)_{k_a\,l_a} + \beta^{j_a\,k_a}\left(\Pi^{\bar{a}}(p_a)_{k_a\,l_a}\right)\right]\mathcal{O}_n^{i_1\cdots l_a\cdots i_n}\,. \tag{2.2.32}$$

To show that these terms vanish, we sandwich eq. (2.2.26) between Π^a and $\Delta^{\bar{a}}$, which yields

$$\begin{aligned}\Pi^a\left[\lim_{q\to 0}\alpha(V_3)\right]\Delta^{\bar{a}} &= -\Pi^a[\beta(\Delta^{-1})]\Delta^{\bar{a}}\\ &= -\Pi^a\Big[\beta\left((\omega^2 - c_a^2p^2)\Pi^a + (\omega^2 - c_{\bar{a}}^2p^2)\Pi^{\bar{a}}\right)\Big]\Delta^{\bar{a}}\\ &= -\Pi^a\Big[(\omega^2 - c_a^2p^2)\beta(\Pi^a) + (\omega^2 - c_{\bar{a}}^2p^2)\beta(\Pi^{\bar{a}})\Big]\Delta^{\bar{a}}\,,\end{aligned} \tag{2.2.33}$$

where again we have used $\Pi^a\Pi^{\bar{a}} = 0$ to get to the third line in eq. (2.2.33). The first term in the square brackets in the third line in eq. (2.2.33) vanishes on-shell. Hence we obtain

$$\Pi^a\left[\lim_{q\to 0}\alpha(V_3)\right]\Delta^{\bar{a}} = -\Pi^a\beta[\Pi^{\bar{a}}]\Pi^{\bar{a}}\,, \tag{2.2.34}$$

so that the off-shell terms in eq. (2.2.32) cancel out. This shows that the soft theorem in eq. (2.2.25) commutes with the on-shell conditions, thereby guaranteeing the field basis independence of the soft theorem.

2.3 Superfluids

To start, we will consider the soft theorems for superfluids corresponding to nonlinearly realized time translations and Lorentz boosts. These constrain the $\mathcal{O}(q^0)$ and $\mathcal{O}(q^1)$ terms in the amplitude in the soft limit, respectively. Note that the latter was exhaustively studied in the interesting recent work of [18] in the context of single field inflation, which in the flat space limit is described by the superfluid EFT [14]. For completeness, we recapitulate results for superfluids here even though the important insights on this theory were discussed already in [18].

2.3.1 Setup

The superfluid EFT arises from spontaneous symmetry breaking an internal $U(1)$ symmetry where the phase degree of freedom ϕ has a time-dependent vacuum expectation value (VEV),[5]

$$\langle\phi\rangle = t\,. \tag{2.3.1}$$

The VEV spontaneously breaks $U(1)$ symmetry and time translations down to a diagonal subgroup. Lorentz symmetry is also spontaneously broken. The fluctuations around the VEV are described by a field π, so

$$\phi = t + \pi\,. \tag{2.3.2}$$

Under time translations and Lorentz boosts, the field π transforms nonlinearly,

$$\text{time translations:} \quad \pi(x) \to \pi'(x') = \pi(x) + T\,, \tag{2.3.3}$$

$$\text{Lorentz boosts:} \quad \pi(x) \to \pi'(x') = \pi(x) + v_i x^i + v_i\left(x^i\partial_t + t\nabla^i\right)\pi(x)\,, \tag{2.3.4}$$

where T and v are constant parameters.

As is well-known, the superfluid Lagrangian can be written in terms of the spacetime translation and Lorentz invariant combination,

$$X = -\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + 1) = \dot{\pi} - \frac{1}{2}\partial_\mu\pi\partial^\mu\pi = \dot{\pi} + \frac{1}{2}\dot{\pi}^2 - \frac{1}{2}(\partial_i\pi)^2\,. \tag{2.3.5}$$

[5] Equivalently, the VEV of the field transforming linearly under internal $U(1)$ is $\langle e^{i\mu\phi}\rangle = e^{i\mu t}$, where μ is the chemical potential.

Considering terms with the fewest possible derivatives per field, we write down the leading terms in the superfluid EFT Lagrangian,

$$
\begin{aligned}
\mathcal{L}_{\text{superfluid}} &= M_1 X + \frac{M_2}{2}X^2 + \frac{M_3}{3!}X^3 + \cdots \\
&= M_1\dot{\pi} + \frac{M_1 + M_2}{2}\dot{\pi}^2 - \frac{M_1}{2}(\partial_i\pi)^2 + \frac{M_3}{3!}\dot{\pi}^3 + \frac{M_2}{2}\dot{\pi}(\dot{\pi}^2 - (\partial_i\pi)^2) + \cdots \\
&= c^2\dot{\pi} + \frac{1}{2}(\dot{\pi}^2 - c^2(\partial_i\pi)^2) + \frac{g_3}{3!}\dot{\pi}^3 + \frac{c^{-2}-1}{2}\dot{\pi}(\dot{\pi}^2 - c^2(\partial_i\pi)^2) + \ldots\,,
\end{aligned}
\tag{2.3.6}
$$

where c is the speed of sound, g is the coupling in the three-point on-shell amplitude, and the canonically normalized Lagrangian parameters are

$$
M_1 = c^2\,, \quad M_2 \;= 1 - c^2\,, \quad M_3 = g_3 + 3\frac{(1-c^2)^2}{c^2}\,. \tag{2.3.7}
$$

As is common for spontaneously broken spacetime symmetries, the interactions are related to the speed of sound by symmetry.

The Feynman rules for the superfluid are trivial to derive from the Lagrangian in eq. (2.3.6). The propagator for the superfluid scalar is

$$
\Delta(p) = \frac{1}{\omega^2 - c^2 p^2}\,. \tag{2.3.8}
$$

Given the convention defined in eq. (2.2.18), the cubic interaction vertex is

$$
V_3 = i g_3 \omega_1\omega_2\omega_3 + i(c^{-2}-1)\left(\omega_1(\omega_2\omega_3 - c^2 p_2\cdot p_3) + \text{cyclic}\right)\,. \tag{2.3.9}
$$

2.3.2 Amplitudes

For completeness, let us summarize here some amplitudes describing the scattering of superfluid modes. For example, the three-point scattering amplitude is

$$
A_3 = i g_3 \omega_1\omega_2\omega_3\,. \tag{2.3.10}
$$

For later convenience, let us define the kinematic variables

$$
\begin{aligned}
\omega_{ab}^2 &= (\omega_a + \omega_b)^2\,, \\
s_{ab} &= (\omega_a + \omega_b)^2 - c^2(p_a + p_b)^2\,,
\end{aligned}
\tag{2.3.11}
$$

where s_{ab} reduces to the familiar Mandelstam variables for $c = 1$. The four-point amplitude is

$$
A_4 = -\,g_3^2\left(\frac{\omega_{12}^2}{s_{12}} + \frac{\omega_{13}^2}{s_{13}} + \frac{\omega_{14}^2}{s_{14}}\right)\omega_1\omega_2\omega_3\omega_4 + \frac{1-c^2}{4c^4}(s_{12}^2 + s_{13}^2 + s_{14}^2) \tag{2.3.12}
$$

$$+\frac{g_3}{2c^2}(\omega_{12}^2 s_{12}+\omega_{13}^2 s_{13}+\omega_{14}^2 s_{14})+g_4\omega_1\omega_2\omega_3\omega_4\,,$$

where $g_4 = M_4 - 14g_3(1-c^2)/c^2 - 15(1-c^2)^3/c^4$. Note that the coefficient of the first term in A_4 is fixed by factorization. Naively, the coefficients of $(s_{12}^2+s_{13}^2+s_{14}^2)$ and $(\omega_{12}^2 s_{12}+\omega_{13}^2 s_{13}+\omega_{14}^2 s_{14})$ could have been independent contact terms. Nevertheless, the nonlinearly realized boost symmetry relates them to the speed of sound c and the tree-point coupling g_3.

2.3.3 Soft Theorem

As noted earlier, the superfluid exhibits the spontaneous breaking of time translations as well as Lorentz boosts. Let us now derive the soft theorems correspond to each of these broken symmetries. To achieve this, we take the general form of the soft theorem in eq. (2.2.25), plug in the cubic interaction vertex V_3 for the superfluid, and then insert the α and β parameters corresponding to either time translations or Lorentz boosts. These will constrain the $\mathcal{O}(q^0)$ and $\mathcal{O}(q^1)$ terms in the amplitudes, respectively.

2.3.3.1 Time Translations

For the case of time translations, we see by inspection from eq. (2.3.3) that the symmetry transformation parameters α and β defined in eq. (2.2.17) are simply

$$\alpha = T \qquad \text{and} \qquad \beta = 0\,. \tag{2.3.13}$$

Hence, the corresponding soft theorem is

$$\begin{aligned}\lim_{q\to 0} A_{n+1}(p_1,\cdots,p_n,q) &= -\lim_{q\to 0}\sum_{a=1}^{n} V_3(q,p_a)\Delta(p_a+q)A_n(p_1,\cdots,p_a+q,\cdots,p_n)\\ &= \frac{1}{2}\sum_{a=1}^{n}\frac{ig\omega\omega_a^2}{\omega\omega_a - c^2\,q\cdot p_a}A_n(p_1,\cdots,p_n)\,.\end{aligned} \tag{2.3.14}$$

To derive the second line in eq. (2.3.14) we have used that

$$\Delta(p_a+q) = \frac{1}{(\omega_a+\omega)^2 - c^2(p_a+q)^2} = \frac{1}{2(\omega\omega_a - c^2 q\cdot p_a)}\,, \tag{2.3.15}$$

and that the three-point interaction vertex with two legs on-shell,

$$\begin{aligned}V_3(q,p_a) &= -ig\omega\omega_a(\omega_a+\omega) + 2i(c^{-2}-1)(\omega_a+\omega)(\omega\omega_a - c^2 q\cdot p_a)\\ &\overset{q\to 0}{=} -ig\omega\omega_a^2 + 2i(c^{-2}-1)\omega_a(\omega\omega_a - c^2 q\cdot p_a)\,,\end{aligned} \tag{2.3.16}$$

where in the second line we have taken the soft limit. Plugging the above expression to eq. (2.3.14), we see that the first term persists while the total contribution from the second term, after cancellations with corresponding propagators, vanishes due to energy conservation $\sum_a \omega_a = 0$.

2.3.3.2 Lorentz Boosts

Next, we consider the soft theorem arising from the spontaneously broken Lorentz boosts. The parameters in eq. (2.2.17) are identified by comparing with eq. (2.3.4), giving

$$\alpha = v_i x^i \qquad \text{and} \qquad \beta = v_i \left(x^i \partial_t + t \nabla^i \right) . \tag{2.3.17}$$

By specifying to the values in eq. (2.3.17) for the general soft theorem in eq. (2.2.25), we get

$$\begin{aligned}
&\lim_{q \to 0} \tfrac{\partial}{\partial q^i} \left[A_{n+1}(p_1, \cdots, p_n, q) \right] = \\
&\qquad - \lim_{q \to 0} \tfrac{\partial}{\partial q^i} \sum_{a=1}^{n} \left[V_3(q, p_a) \Delta(p_a + q) A_n(\cdots, p_a + q, \cdots) \right] \\
&\qquad - \sum_{a=1}^{n} i \left(\omega_a \tfrac{\partial}{\partial p_{ai}} + p_a^i \tfrac{\partial}{\partial \omega_a} \right) \left[A_n(p_1, \cdots, p_n) \right] ,
\end{aligned} \tag{2.3.18}$$

where the propagator and three-point vertex are those in eqs (2.3.8) and (2.3.9) respectively, and we have stripped the constant vector v.

Our derivation of this soft theorem only differs from that in [18] in the field basis chosen, which changes the form of V_3 and β. The basis chosen in [18] is related to ours by

$$\pi \to \pi - (c^{-2} - 1)\pi \dot{\pi} \,. \tag{2.3.19}$$

In such basis the three-point vertex is

$$V_3' = i g \omega_1 \omega_2 \omega_3 \,, \tag{2.3.20}$$

and the symmetry transformation is

$$\alpha' = v_i x^i \qquad \text{and} \qquad \beta' = v_i \left(c^{-2} x^i \partial_t + t \nabla^i \right) , \tag{2.3.21}$$

so the soft theorem is as derived in [18]

$$\lim_{q \to 0} \tfrac{\partial}{\partial q^i} \left[A_{n+1}(p_1, \cdots, p_n, q) \right] =$$

$$-\lim_{q\to 0}\frac{\partial}{\partial q^i}\sum_{a=1}^{n}\left[V_3'(q,p_a)\Delta(p_a+q)A_n(\cdots,p_a+q,\cdots)\right]$$

$$-\sum_{a=1}^{n} i\left(c^{-2}\omega_a\frac{\partial}{\partial p_{ai}}+p_a^i\frac{\partial}{\partial\omega_a}\right)\left[A_n(p_1,\cdots,p_n)\right]\,. \tag{2.3.22}$$

The above version of the soft theorem features a boost operator that is non-relativistic, whereas the one in eq. (2.3.18) is relativistic. The former has the advantage that it annihilates the on-shell condition $\omega_a^2-c^2p_a^2=0$, thus making invariance under field redefinitions more manifest. Nevertheless, as explained in Sec. 2.2.3, the soft theorem can be written in any basis.

2.4 Solids

2.4.1 Setup

The Lagrangian description for solids utilizes a three-vector field which acquires a vacuum expectation value,

$$\langle\phi^i\rangle = x^i\,, \tag{2.4.1}$$

which spontaneously breaks part of the Poincaré symmetry. Fluctuations of this field are the NGBs for symmetry breaking, defined via

$$\phi^i = x^i+\pi^i\,, \tag{2.4.2}$$

where π is the phonon field. Under spatial translations, the phonon transforms as

$$\text{spatial translation:}\quad \pi^i(x)\to\pi'^i(x')=\pi^i(x)+w^i\,, \tag{2.4.3}$$

for a constant vector w. At the same time, the phonon transforms nonlinearly under boosts as

$$\text{Lorentz boost:}\quad \pi^i(x)\to\pi'^i(x')=\pi^i(x)+v^it+v_j\left(x^j\partial_t+t\nabla^j\right)\pi^i(x)\,, \tag{2.4.4}$$

for a constant vector v. Here the first term on the right-hand side arises because π transforms under boosts exactly like spatial position, as implied by eq. (2.4.2). The second term on the right-hand side arises because the spacetime argument of the phonon field actively transforms under boosts also.

To construct a Lagrangian that is invariant under nonlinearly realized boosts, we follow the procedure of [16], [19] and define

$$B^{ij}=\partial_\mu\phi^i\partial^\mu\phi^j-\delta^{ij}=\nabla^i\pi^j+\nabla^j\pi^i+\partial_\mu\pi^i\partial^\mu\pi^j\,. \tag{2.4.5}$$

In three spatial dimensions, the only independent scalar components of this matrix are $[B]$, $[B^2]$, and $[B^3]$, where the square brackets denote a trace over spatial indices. Hence, the general Lagrangian for the phonon is

$$L_{\text{solid}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{ijk}[B]^i[B^2]^j[B^3]^k\,, \tag{2.4.6}$$

corresponding to the modes of a solid.

The three components of the phonon can be further decomposed into a single longitudinal mode and two transverse modes via

$$\pi^i = \pi^i_L + \pi^i_T \qquad \text{where} \qquad \nabla_i \pi^i_T = 0 \qquad \text{and} \qquad \epsilon_{ijk}\nabla^j \pi^k_L = 0\,. \tag{2.4.7}$$

By expanding the Lagrangian to quadratic order, we learn that the longitudinal and transverse speeds of sound, c_L and c_T, are related to the coupling constants via

$$\lambda_{010} = \tfrac{1}{4}(1-c_T^2) \qquad \text{and} \qquad \lambda_{200} = -\tfrac{1}{8}(1+c_L^2-2c_T^2)\,. \tag{2.4.8}$$

Otherwise, the couplings are completely unfixed by the spontaneous symmetry breaking pattern.[6]

The soft theorem in eq. (2.2.25) depends on the propagator and cubic vertex, Δ and V_3. Let us briefly present expressions for these quantities in the case of a solid. The propagator for the phonons of the solid is given in eq. (2.2.8). From the solid Lagrangian in eq. (2.4.6), we also compute the cubic interaction vertex,

$$\begin{aligned} V_3^{i_1i_2i_3}(p_1,p_2,p_3) = &-\tfrac{i}{2}(1+c_L^2-2c_T^2)\big((\omega_2\omega_3 - p_2\cdot p_3)p_1^{i_1}\delta^{i_2i_3}\big)\\ &+ i(1-c_T^2)\big((\omega_2\omega_3 - p_2\cdot p_3)(p_1^{i_3}\delta^{i_1i_2}+p_1^{i_2}\delta^{i_1i_3})\big)\\ &-2i\lambda_{001}\big(p_1^{i_2}p_2^{i_3}p_3^{i_1}+3(p_2\cdot p_3)p_1^{i_2}\delta^{i_1i_3}\big)\big)\\ &-4i\lambda_{110}\big(p_1^{i_1}p_2^{i_3}p_3^{i_2}+(p_2\cdot p_3)p_1^{i_1}\delta^{i_2i_3}\big)\big)\\ &-8i\lambda_{300}p_1^{i_1}p_2^{i_2}p_3^{i_3}+\text{permutations}\,,\end{aligned} \tag{2.4.9}$$

which we can freely rewrite on the support of total momentum conservation.

2.4.2 Amplitudes

Next, let us briefly describe some explicit phonon amplitudes. The three-point scattering amplitudes for various combinations of longitudinal and transverse

[6] In general, there will be further thermodynamic constraints on the couplings, depending on the physical system of interest.

polarizations are

$$
\begin{aligned}
A_{LLL} &= -i\frac{3}{c_L^3}\big((1-c_L^2)^2 + 16(\lambda_{001}+\lambda_{110}+\lambda_{300})\big)\omega_1\omega_2\omega_3\,,\\
A_{TTT} &= -i\frac{1}{c_T^2}\big((1-c_T^2)^2 + 6\lambda_{001}\big)\omega_3(\omega_1-\omega_2)(e_1\cdot e_2)(p_2\cdot e_3) + \text{cyclic}\,,\\
A_{LLT} &= i\frac{\omega_1^2-\omega_2^2}{2c_L^2c_T^2\omega_1\omega_2}(p_1\cdot e_3)\Bigg(2c_L^2c_T^2\big((c_T^2-c_L^2)(\omega_1^2+\omega_2^2)+\omega_1\omega_2(c_T^2-3c_L^2)\big)\\
&\qquad + \big(2-c_T^2-3c_L^2+8(2\lambda_{110}+3\lambda_{001})\big)\big((c_T^2-c_L^2)(\omega_1^2+\omega_2^2)-2c_L^2\omega_1\omega_2)\big)\Bigg)\,,\\
A_{TTL} &= -i\frac{(p_2\cdot e_1)(p_1\cdot e_2)}{c_Lc_T^2\omega_3}\Bigg(2c_T^2(1-c_T^2+4\lambda_{110}+9\lambda_{001})\omega_3^2\\
&\qquad - c_L^2\big((1-c_T^4+6\lambda_{001})(\omega_1^2+\omega_2^2)+4c_T^2(1-c_T^2)\omega_1\omega_2\big)\Bigg)\\
&\quad + i\frac{e_1\cdot e_2}{2c_L^3c_T^4}\omega_3\Bigg(c_L^2c_T^2\big((c_T^2-c_L^2)(\omega_1^2+\omega_2^2)+2c_T^2(1-c_L^2+c_T^2)\omega_1\omega_2\big)\\
&\qquad + c_L^2\big((1-c_T^2)^2+6\lambda_{001}\big)\big((c_L^2-c_T^2)(\omega_1^2+\omega_2^2)-2c_L^2\omega_1\omega_2\big)\\
&\qquad + c_T^2\big(c_T^2+8\lambda_{110}+6\lambda_{001}\big)\big(c_L^2(\omega_1^2+\omega_2^2)-c_T^2\omega_3^2\big)\Bigg)\,,
\end{aligned}
\tag{2.4.10}
$$

where we have eliminated $p_i\cdot p_j$ using the minimal on-shell kinematic basis defined in eq. (A.0.7). As noted in [87], for real on-shell kinematics the external three-momenta are necessarily collinear, so the three-point amplitudes with an odd number of transverse polarizations are zero. However, collinearity is avoided in the case of complex kinematics.

2.4.3 Soft Theorem

Next, to evaluate eq. (2.2.25) we must specify α and β which depend on which symmetry is spontaneously broken. In what follows, we consider the case of spatial translations and Lorentz boosts, respectively.

2.4.3.1 Spatial Translations

By inspection, we see that the nonlinearly realized spatial translation in eq. (2.4.4) corresponds to eq. (2.2.17) by identifying

$$
\alpha^i = w^i \qquad \text{and} \qquad \beta^i{}_j = 0\,. \tag{2.4.11}
$$

Plugging this into eq. (2.2.25), we obtain the soft theorem corresponding to spatial translations of a phonon in a solid,

$$\lim_{q\to 0} A_{n+1}^{i_1\cdots i_n i}(p_1,\cdots,p_n,q) = \\ -\lim_{q\to 0}\sum_{a=1}^{n} V_3^{i\,i_a}{}_{j_a}(q,p_a)\Delta^{j_a}{}_{k_a}(p_a+q)A_n^{i_1\cdots j_a\cdots i_n}(\cdots,p_a+q,\cdots)\,,$$

where we have stripped off the constant translation vector w_i, leaving a free i index.

We have explicitly evaluated eq. (2.4.12) for the case of four- and three-point amplitudes and verified its validity. Here A_3 and A_4 should be evaluated in the minimal kinematic bases defined in eq. (A.0.7) and eq. (A.0.10). Also, to evaluate the above expression one must, in the end, contract the polarization indices of the hard particles, i_1, i_2, i_3, with explicit polarizations which are either longitudinal or transverse. The on-shell conditions for those legs should also correlate with the choice of polarizations, since the longitudinal and transverse speeds of sound are in general different. By computing all possible combinations of longitudinal and transverse combinations for the external legs, we have verified that the above formula holds.

Note that in general for the soft limit of A_n with $n > 4$ we do not encounter the fractional soft limits of [87]. In that setup, the authors assume real kinematics, for which taking the soft limit from four- to three-point yields collinear momenta for the latter. For five- and higher-point amplitudes the soft limit does not yield collinear kinematics in the amplitudes on the right-hand side of the soft theorem. Moreover, as noted earlier, we assume complex kinematics throughout, as is common in the study of gauge theory amplitudes.

2.4.3.2 Lorentz Boosts

Next, we consider the soft theorem corresponding to spontaneously broken Lorentz transformations. Comparing the nonlinearly realized Lorentz boost of the phonon in eq. (2.4.4) to eq. (2.2.17), we see that the transformation parameters are

$$\alpha^i = v^i t \qquad \text{and} \qquad \beta^i{}_j = v_k\left(x^k\partial_t + t\nabla^k\right)\delta^i{}_j\,. \tag{2.4.12}$$

Inserting eq. (2.4.12) into eq. (2.2.25), we obtain the soft theorem corresponding to Lorentz boosts of a phonon in a solid,

$$\begin{aligned}\lim_{q\to 0}\tfrac{\partial}{\partial\omega}\left[A_{n+1}^{i_1\cdots i_n i}(p_1,\cdots,p_n,q)\right] = \\ -\lim_{q\to 0}\sum_{a=1}^{n}\tfrac{\partial}{\partial\omega}\left[V_3^{i\,i_a}{}_{j_a}(q,p_a)\Delta^{j_a}{}_{k_a}(p_a+q)A_n^{i_1\cdots j_a\cdots i_n}(\cdots,p_a+q,\cdots)\right] \\ -\sum_{a=1}^{n} i\left(\omega_a\tfrac{\partial}{\partial p_{ai}}+p_a^i\tfrac{\partial}{\partial\omega_a}\right)\left[A_n^{i_1\cdots i_a\cdots i_n}(p_1,\cdots,p_a,\cdots,p_n)\right]\,,\end{aligned} \tag{2.4.13}$$

where the propagator Δ and cubic vertex V_3 are defined in eq. (2.2.8) and eq. (2.4.9), respectively. As before, we have stripped off the constant Lorentz boost vector v_i, leaving a free i index.

We have verified by explicit calculation that the soft theorem relating the four-point and three-point amplitudes is satisfied. As before, in order to verify this soft theorem it is important to go to the minimal kinematic bases for A_3 and A_4 in eq. (A.0.7) and eq. (A.0.10). Furthermore, we must contract with explicit longitudinal or transverse polarizations for the hard external states. Performing this for all combinations, we find that the above soft theorem is indeed satisfied.

2.5 Fluids

2.5.1 Setup

As emphasized in [19], fluids are nothing more than solids with an enhanced symmetry. In particular, the Lagrangian for fluids is the same as the one for solids except with couplings constrained to exhibit an additional invariance under infinitesimal volume-preserving diffeomorphisms,

$$\text{diffeomorphism:}\quad \phi^i \to \phi'^i = \phi^i + \xi^i(\phi)\,. \tag{2.5.1}$$

The volume-preserving condition implies that $\partial_i\xi^i = 0$. In terms of the physical phonon field this corresponds to

$$\text{diffeomorphism:}\quad \pi^i \to \pi'^i = \pi^i + \xi^i(x+\pi) = \pi^i + \xi^i(x) + \partial_j\xi^i(x)\pi^j + \cdots\,, \tag{2.5.2}$$

expanded to linear order in the phonon field.

Invariance under volume-preserving diffeomorphisms implies that the fluid Lagrangian can only depend on the combination

$$\det B' = \det(1+B) = 1 + [B] + \frac{1}{2}\left([B]^2 - [B^2]\right) + \frac{1}{3!}\left([B]^3 - 3[B][B^2] + 2[B^3]\right) \,,$$

and thus it takes the form

$$\mathcal{L}_{\text{fluid}} = -\frac{1}{2}\det B' + \tau_0 \det B'^2 + \tau_1 \det B'^3 + \tau_2 \det B'^4 + \cdots \,, \tag{2.5.3}$$

where $\tau_0 = (1 - c_L^2)/8$. This implies that fluid dynamics are obtained by imposing the following constraints on the solid Lagrangian,

$$\begin{aligned}
\lambda_{010} &= \tfrac{1}{4}\,, \\
\lambda_{001} &= -\tfrac{1}{6}\,, \\
\lambda_{200} &= -\tfrac{1}{8}(1 + c_L^2)\,, \\
\lambda_{020} &= \tfrac{1}{32}(1 - c_L^2)\,, \\
\lambda_{101} &= \tfrac{1}{12}(1 - c_L^2)\,, \\
\lambda_{110} &= \tfrac{1}{8}(1 + c_L^2)\,, \\
\lambda_{210} &= -\tfrac{3}{16}(1 - c_L^2) - \tfrac{3}{2}\tau_1\,, \\
\lambda_{300} &= \tfrac{1}{24}(1 - 3c_L^2) + \tau_1\,, \\
\lambda_{400} &= \tfrac{7}{96}(1 - c_L^2) + \tfrac{3}{2}\tau_1 + \tau_2\,,
\end{aligned} \tag{2.5.4}$$

where τ_1 and τ_2 are residual free parameters of the fluid Lagrangian.

As emphasized in [19], [20], the fluid EFT is peculiar because the transverse speed of sound is vanishing, so $c_T = 0$. Consequently, the transverse modes lack a gradient kinetic term and the corresponding degrees of freedom are not localized particles in any conventional sense.[7] Instead, we will consider the fluid case as a mathematically well-defined limit of small $c_T \to 0$. While strict vanishing may be ill-defined, the limit is a straightforward way to regulate the corresponding solid amplitudes on the approach to fluid dynamics.

[7] We will not shed any new insight on this particular problem, though there has been recent progress making sense of perfect fluids in two dimensions by recasting volume-preserving diffeomorphisms as $SU(N)$ transformations as $N \to \infty$ [88]. Curiously, a similar construction yields a nonperturbative formulation double copy relating scalar EFTs in two dimensions [89]. This double copy structure also arises in certain non-Abelian generalizations of the Navier-Stokes equations [90].

2.5.2 Amplitudes

Next, we record explicit fluid phonon amplitudes. The three-point scattering amplitude of longitudinal modes is

$$A_{LLL} = -i\frac{\tilde{\tau}_1}{c_L^3}\omega_1\omega_2\omega_3\,, \tag{2.5.5}$$

where $\tilde{\tau}_1 = 3((1-c_L^2)^2 + 16\tau_1)$, whereas the four-point amplitude is

$$\begin{aligned} A_{LLLL} = &-\frac{\tilde{\tau}_1^2}{c_L^6}\left(\frac{\omega_{12}^2}{s_{12}} + \frac{\omega_{13}^2}{s_{13}} + \frac{\omega_{14}^2}{s_{14}}\right)\omega_1\omega_2\omega_3\omega_4 + \frac{1-c_L^2}{4c_L^2}(s_{12}^2 + s_{13}^2 + s_{14}^2) \\ &+ \frac{\tilde{\tau}_1}{2c_L^4}(\omega_{12}^2 s_{12} + \omega_{13}^2 s_{13} + \omega_{14}^2 s_{14}) + \frac{\tilde{\tau}_2}{c_L^6}\omega_1\omega_2\omega_3\omega_4\,, \end{aligned} \tag{2.5.6}$$

where $\tilde{\tau}_2 = 3c_L^2\big(128\tau_2 - 5(1-c_L^2)^3\big) + \tilde{\tau}_1\big(3\tilde{\tau}_1 + 10c_L^2(1-c_L^2)\big)$.

2.5.3 Soft Theorem

2.5.3.1 Diffeomorphisms

In order to verify the soft theorem in eq. (2.2.25), we must compute the transformation parameters α and β for volume-preserving diffeomorphisms, and the propagator and cubic vertex Δ and V_3 for the fluid.

To begin, let us series expand a general volume-preserving diffeomorphism in powers of the space coordinate,

$$\xi^i(x) = \sum_{a=1}^{\infty} \xi^i{}_{j_1\cdots j_a} x^{j_1}\cdots x^{j_a} \qquad \text{where} \qquad \xi^i{}_{j_1\cdots i\cdots j_a} = 0\,. \tag{2.5.7}$$

Here we will be interested in verifying the leading nontrivial component of the diffeomorphism, which is linear in the space coordinate

$$\xi^i(x) = \xi^i{}_j x^j \qquad \text{where} \qquad \xi^i_i = 0\,. \tag{2.5.8}$$

Recasting this leading diffeomorphism in terms of the parameters in eq. (2.2.17), we find that

$$\alpha^i = \xi^i{}_j x^j \qquad \text{and} \qquad \beta^i{}_j = \xi^i{}_j\,. \tag{2.5.9}$$

Next, to obtain Δ and V_3 we simply take the expressions in eq. (2.2.8) and eq. (2.4.9) for the solid and insert eq. (2.5.4).

Putting this all together, we learn that the soft theorem in eq. (2.2.25) applied to the leading volume-preserving diffeomorphisms of a fluid is

$$\begin{aligned}
\lim_{q\to 0}\tfrac{\partial}{\partial q_j}\left[A_{n+1}^{i_1\cdots i_n i}(p_1,\cdots,p_n,q)\right] = \\
&-\lim_{q\to 0}\sum_{a=1}^{n}\tfrac{\partial}{\partial q_j}\left[V_3^{i\,i_a}{}_{j_a}(q,p_a)\Delta^{j_a}{}_{k_a}(p_a+q)A_n^{i_1\cdots j_a\cdots i_n}(\cdots,p_a+q,\cdots)\right] \\
&-\sum_{a=1}^{n} i\delta^{ii_a}\delta^{j}_{j_a}\left[A_n^{i_1\cdots j_a\cdots i_n}(p_1,\cdots,p_a,\cdots,p_n)\right] \\
&+\text{terms proportional to } \delta^{ij}\,,
\end{aligned} \tag{2.5.10}$$

where we have stripped off the constant diffeomorphism parameter ξ_{ij}, leaving free i and j indices. Thus we see that the terms proportional to δ^{ij} are projected out when the left- and right-hand sides are contracted into ξ_{ij}, which is by construction traceless.

To verify the above soft theorem we compute the three- and four-point amplitudes for fluid phonons by imposing the conditions on coupling constants in eq. (2.5.4) on our amplitudes for solid phonons. By explicit computation we have verified the validity of the above soft theorem relating the four- and three-point amplitudes. As before, this check requires going to minimal kinematic basis for A_3 and A_4. Furthermore, to avoid pathologies involving transverse polarizations of external states, we restrict to the case where all external polarizations are longitudinal.

Note that in principle, one can also derive soft theorems for high-order diffeomorphisms. In particular, we could consider the next-to-leading diffeomorphism defined by

$$\alpha^i = \tfrac{1}{2}\xi^i{}_{jk}x^j x^k \qquad \text{and} \qquad \beta^i{}_j = \xi^i{}_{jk}x^k\,, \tag{2.5.11}$$

where $\xi^i{}_{ik} = \xi^i{}_{ki} = 0$. In this case, the corresponding soft theorem will involve the action of the differential operator, $\alpha^i = -\frac{1}{2}\xi^i{}_{jk}\frac{\partial}{\partial q_j}\frac{\partial}{\partial q_k}$. This effectively extracts $\mathcal{O}(q^2)$ terms from the amplitude. The general soft theorem in eq. (2.2.25) applies for *any* spontaneously broken symmetry, including next-to-leading diffeomorphism. We do not explicitly construct and evaluate the soft theorem for next-to-leading diffeomorphism in this paper, but it is straightforward to do so by inserting eq. (2.5.11) into eq. (2.2.25).

2.6 Framids

2.6.1 Setup

The framid theory exhibits a minimal field content needed to represent the spontaneous breaking of Lorentz symmetry. The setup centers on a four-vector field whose vacuum expectation value,

$$\langle A_\mu(x) \rangle = \delta^0_\mu \,, \tag{2.6.1}$$

spontaneously breaks Lorentz symmetry. Fluctuations about this value are parameterized by framon fields which are the NGBs of boosts,

$$A_\mu = \exp(i\pi^j K_j)_\mu{}^\nu \delta^0_\nu \,, \tag{2.6.2}$$

where K^i is a three-vector parameterizing Lorentz boosts. Expanding in powers of the fields, we obtain explicit formulas for the four-vector field,

$$\begin{aligned} A_0 &= 1 + \tfrac{1}{2}\pi^2 + \cdots \,, \\ A_i &= \pi_i(1 + \tfrac{1}{6}\pi^2 + \cdots) \,. \end{aligned} \tag{2.6.3}$$

By construction, boosts are nonlinearly realized as constant shifts of the framon field,

$$\text{Lorentz boost:} \quad \pi^i(x) \to \pi'(x') = \pi^i(x) + v^i + v_j \left(x^j \partial_t + t\nabla^j\right) \pi^i(x) \,, \tag{2.6.4}$$

where, as before, the last term on the right-hand side appears because the spacetime position of the framon is boosted in the transformation. Note that the framon does not nonlinearly realize translation symmetries.

The leading order boost invariant Lagrangian for the framon is

$$\mathcal{L}_{\text{framid}} = -\tfrac{1}{2}M_3^2(\partial_\mu A^\mu)^2 - \tfrac{1}{2}M_2^2(\partial_\mu A_\nu)^2 - \tfrac{1}{2}(M_2^2 - M_1^2)(A^\rho \partial_\rho A_\mu)^2 \,. \tag{2.6.5}$$

As before, we can expand to quadratic order in the framons in order to express some of the couplings in terms of the speeds of sound of the longitudinal and transverse modes,

$$c_T^2 = \frac{M_2^2}{M_1^2} \qquad \text{and} \qquad c_L^2 = \frac{M_2^2 + M_3^2}{M_1^2} \,. \tag{2.6.6}$$

In order to evaluate the soft theorem in eq. (2.2.25) we must compute the propagator Δ and cubic vertex V_3 of the framid theory. Conveniently, the framon has an identical dispersion relation to the phonon, so Δ is defined as in eq. (2.2.8).

Meanwhile, the cubic interaction vertex is straightforwardly extracted from $\mathcal{L}_{\text{framid}}$, yielding

$$\begin{aligned} V_3^{i_1 i_2 i_3}(p_1, p_2, p_3) = &-(1-c_T^2)\left(\omega_1(\delta^{i_1 i_2} p_2^{i_3} + \delta^{i_1 i_3} p_3^{i_2})\right) \\ &-(c_T^2 - c_L^2)\left(\omega_1(\delta^{i_1 i_2} p_3^{i_3} + \delta^{i_1 i_3} p_2^{i_2})\right) + \text{cyclic}\,. \end{aligned} \tag{2.6.7}$$

As noted in [19], in the relativistic limit of $c_L = c_T = 1$, the framid coincides with the nonlinear sigma model (NLSM), which is why V_3 vanishes in this case.

2.6.2 Amplitudes

The three-point on-shell scattering amplitudes for framons are

$$\begin{aligned} A_{LLL} &= \frac{(1-c_L^2)}{c_L}(\omega_1^2 + \omega_2^2 + \omega_3^2)\,, \\ A_{TTT} &= (1-c_T^2)(\omega_1 - \omega_2)(e_1 \cdot e_2)(p_1 \cdot e_3) + \text{cyclic}\,, \\ A_{LLT} &= \omega_1 \left(\tfrac{(c_L^2 - c_T^2)}{c_T^2}\tfrac{\omega_3^2}{\omega_1 \omega_2} + (1-c_L^2)\right)(p_1 \cdot e_3) + (1 \leftrightarrow 2)\,, \\ A_{TTL} &= 3c_L(1-c_T^2)(p_2 \cdot e_1)(p_1 \cdot e_2) + \left(\tfrac{(c_L^2 - c_T^2)(1-3c_T^2)}{2c_L c_T^2}\omega_3^2 - \tfrac{2c_L(1-c_T^2)}{c_T^2}\omega_1\omega_2\right)(e_1 \cdot e_2)\,. \end{aligned} \tag{2.6.8}$$

2.6.3 Soft Theorem

2.6.3.1 Lorentz Boosts

Comparing eq. (2.2.17) to eq. (2.6.4), we see that the transformation parameters corresponding to the nonlinearly realized Lorentz boosts of the framon are

$$\alpha^i = v^i \qquad \text{and} \qquad \beta^i{}_j = v_k\left(x^k \partial_t + t\nabla^k\right)\delta^i{}_j\,. \tag{2.6.9}$$

Combining this with eq. (2.2.25), we obtain

$$\begin{aligned} \lim_{q\to 0} A_{n+1}^{i_1 \cdots i_n i}(p_1, \cdots, p_n, q) = & \\ &- \sum_{a=1}^{n} V_3^{i\, i_a}{}_{j_a}(q, p_a)\Delta^{j_a}{}_{p_a}(p_a + q) A_n^{i_1 \cdots j_a \cdots i_n}(\cdots, p_a + q, \cdots) \\ &- \sum_{a=1}^{n}\left(\omega_a \tfrac{\partial}{\partial p_{ai}} + p_a^i \tfrac{\partial}{\partial \omega_a}\right)\left[A_n^{i_1 \cdots i_a \cdots i_n}(p_1, \cdots, p_a, \cdots, p_n)\right]\,, \end{aligned} \tag{2.6.10}$$

which is the soft theorem corresponding to boosts in the framid.

By explicit calculation, we have verified the framon soft theorem at five-, four-, and three-point, by using a minimal kinematic basis and plugging in all possible combinations of longitudinal or transverse polarizations for the hard external legs.

2.7 Soft Bootstrap

Our analysis thus far has focused on incarnations of the soft theorem in eq. (2.2.25), which relate *nonzero* expressions involving the $(n+1)$-point and n-point amplitudes. However, in the special circumstance where soft limits vanish—also known as Adler zeros—the corresponding theories typically exhibit enhanced symmetry structures. Concretely, the Adler zero stipulates that

$$\lim_{q\to 0} A_{n+1}(p_1, \cdots, p_n, q) = \mathcal{O}(q^1)\,, \tag{2.7.1}$$

which is the case for, e.g., amplitudes of pions in the NLSM [4]. In special circumstances, scalar EFTs can exhibit an *enhanced* Adler zero,

$$\lim_{q\to 0} A_{n+1}(p_1, \cdots, p_n, q) = \mathcal{O}(q^2)\,. \tag{2.7.2}$$

This is the case for Dirac-Born-Infeld (DBI) theory and the Galileon. Remarkably, the NLSM, DBI, and the Galileon exhibit a soft behavior of amplitudes that is enhanced beyond what is naively expected simply from counting the number of derivatives per interaction vertex. Hence, by writing an ansatz and imposing eq. (2.7.1) or eq. (2.7.2) as constraints, one can bootstrap these theories from first principles [9], [91]–[93]. These resulting theories have highly constrained interactions, and were dubbed *exceptional* scalar EFTs. The soft bootstrap has also been extended to broader classes of theories, including theories with vectors or fermions [10], [30], [31], [94]–[105] as well as nonrelativistic theories [11], [12], [106].

In the context of spacetime symmetry breaking, it is natural to ask: Are there exceptional nonrelativistic EFTs? Are there exceptional theories of phonons and framons?

2.7.1 Exceptional Phonons

We start by considering effective theories of phonons, i.e., corresponding to spacetime symmetry breaking pattern of solids, fluids, and superfluids, which all have interaction vertices with one derivative per field. First, we want to establish the Adler zero for these theories, i.e., the requirement that the amplitudes vanish as $\mathcal{O}(q^1)$ in the soft limit.

For a theory with one derivative per field, all interaction vertices involving the soft particle will scale as $\mathcal{O}(q^1)$. This suggests that the on-shell amplitudes should also scale as $\mathcal{O}(q^1)$, thus exhibiting an Adler zero. However, as is well-known, this reasoning fails in the presence of three-point interaction vertices, which generically induce $\mathcal{O}(q^{-1})$ soft poles. These contributions can, in principle, conspire with $\mathcal{O}(q^1)$ contributions from the interaction vertex to give an amplitude that scales as $\mathcal{O}(q^0)$.

In relativistic theories of derivatively coupled scalars, there are no such soft poles because there are no on-shell three-point amplitudes. The only nonzero three-point amplitude for relativistic scalars is a constant arising from a cubic potential term, which is absent by definition for derivatively coupled scalars. The absence of relativistic three-point scalar amplitudes implies the existence of a field basis where the corresponding three-point vertex is zero and thus there are no singular terms in the soft limit.

In contrast, nonrelativistic theories can have nontrivial three-point amplitudes, even in theories with interaction vertices with one derivative per field. Thus, the Adler zero is not automatic. A sufficient condition for having an Adler zero for such nonrelativistic theories is to demand that all three-point amplitudes vanish.

2.7.1.1 Solids

Let us begin by imposing an Adler zero for phonons in a solid. The three-point solid amplitudes are given in eq. (2.4.10). Demanding that all of these vanish for any choice of external modes, transverse or longitudinal, implies a universal speed of sound, [8]

$$c_T = c_L = c\,, \tag{2.7.3}$$

in addition to the constraints

$$6\lambda_{001} = -8\lambda_{110} = 48\lambda_{300} = -(1-c^2)^2\,. \tag{2.7.4}$$

[8] Note that this constraint is not compatible with thermodynamic constraints imposed by bulk stability in ordinary solids [107].

Given the restrictions in Eqs. (2.7.3) and (2.7.4), the following field redefinition,

$$\pi^i \to \pi^i + (1-c^2)\pi^j \partial_j \pi^i \,, \tag{2.7.5}$$

sets the three-point vertex to zero. In this basis the soft theorem for spontaneously broken spatial translations in Eq. (2.4.12) with $V_3 = 0$ shows the existence an Adler zero for n-point amplitudes.

One might ask whether such restrictions on the solid couplings are technically natural. In other words, do the choices of couplings in Eqs. (2.7.3) and (2.7.4) enhance the symmetries of the solid EFT? Unfortunately, the answer is no. While the vanishing of the three-point amplitude and a relativistic dispersion relation suggest a possible emergent Lorentz symmetry, this is trivially broken by higher-point amplitudes.

Next, it is natural to further impose the condition that the $\mathcal{O}(q^1)$ term in the amplitude vanishes, yielding an enhanced $\mathcal{O}(q^2)$ soft limit. A necessary condition for this is given by the soft theorem for spontaneously broken boosts. In the basis where the three-point vertex is zero it takes the form

$$\begin{aligned}
&\lim_{q\to 0} \tfrac{\partial}{\partial \omega} \left[A_{n+1}^{i_1\cdots i_n i}(p_1, \cdots, p_n, q) \right] = \\
&\qquad - \sum_{a=1}^{n} i \left(\omega_a \tfrac{\partial}{\partial p_{ai}} + c^2 p_a^i \tfrac{\partial}{\partial \omega_a} \right) \left[A_n^{i_1 \cdots i_a \cdots i_n}(p_1, \cdots, p_n) \right] .
\end{aligned} \tag{2.7.6}$$

Interestingly, the enhanced Adler zero requires choosing couplings in the solid EFT such that the theory has an emergent boost symmetry with respect to the speed of sound c.

The boost soft theorem in eq. (2.4.13) does not capture all terms of $\mathcal{O}(q)$ in the soft expansion of the solid amplitude. Hence, the emergence of a relativistic symmetry is not sufficient to guarantee an enhanced Adler zero. By explicitly imposing the enhanced Adler zero on the four- and five-point scattering amplitudes of longitudinal phonons in a solid, we constrain the couplings in the solid Lagrangian in eq. (2.4.6) according to

$$\begin{aligned}
&252\lambda_{101} = 672\lambda_{020} = 1152\lambda_{400} = -224\lambda_{210} = 21(1-c^2)^3 \,, \\
&192\lambda_{310} = 320\lambda_{120} = -240\lambda_{201} = -120\lambda_{011} = -768\lambda_{500} = 5(1-c^2)^4 \,.
\end{aligned} \tag{2.7.7}$$

These conditions must be imposed in addition to the constraints in Eqs. (2.7.3) and (2.7.4) which are needed to ensure the ordinary Adler zero. We find that the five-point scattering amplitude vanishes identically for this choice of couplings.

This bootstrap suggests that there should exist a solid with an enhanced Adler zero. To investigate this possibility, we generalize to a solid in arbitrary spacetime dimension while including all $[B^n]$ operators in the Lagrangian. Remarkably, we find that the three-, four-, and five-point scattering amplitudes for a solid with the constraints in Eqs. (2.7.3), (2.7.4), and (2.7.7) agree with the scattering amplitudes derived from a physically equivalent Lagrangian,

$$\mathcal{L}_{\text{exc. solid}} = \frac{1}{\kappa}\sqrt{-\det\left(\eta_{\mu\nu} + \kappa\left(\partial'_\mu \pi_i \partial'_\nu \pi^i\right)\right)}\,, \tag{2.7.8}$$

with a coupling constant $\kappa = -(1-c^2)$, where we introduced ∂'_μ such that $\partial'_\mu \partial'^\mu = -\partial_t^2 + c^2\partial_i^2$. In addition, we have verified that the theory in eq. (2.7.8) yields a six-point amplitude that vanishes as $\mathcal{O}(q^2)$ in the soft limit.[9]

Since the phonon indices in eq. (2.7.8) are only contracted with each other, they effectively label an internal symmetry. Moreover, eq. (2.7.8) clearly describes a theory that linearly realizes the Lorentz symmetry with respect to the speed of sound c in eq. (2.7.6). Note that the coupling κ vanishes when $c = 1$, yielding a free theory. Thus, we have arrived at the Lagrangian for multiple relativistic DBI fields. The enhanced soft limit we have encountered is not surprising in light of the results of [9], [92], [93], which show that the only relativistic theory of single derivatively coupled scalar with an enhanced soft limit is DBI.

2.7.1.2 Fluids

Let us now attempt to construct an exceptional fluid theory. As noted earlier, we only consider the longitudinal external states. That makes the fluid case different from the solid case. The only condition for the fluid comes from requiring A_{LLL} in eq. (2.4.10) to vanish, subject to the fluid constraints in eq. (2.5.4). That fixes the free coupling in the fluid amplitude in terms of the speed of sound for the longitudinal modes,

$$\tau_1 = -\tfrac{1}{16}(1-c_L^2)^2\,. \tag{2.7.9}$$

[9]Since eq. (2.7.8) is expressed in terms of a determinant over spacetime indices, one must impose kinematics in a specific dimension in order to verify the enhanced soft theorem. In this case, the on-shell identities in Appendix A must be supplemented with dimensionally specific Gram determinant constraints.

As for the solid, this is sufficient to ensure an Adler zero thanks to the soft theorem for spontaneously broken spatial translations in eq. (2.4.12). Similarly, the soft theorem for spontaneously broken boosts in eq. (2.4.13) shows that an enhanced Adler zero requires an emergent relativistic symmetry.

Next we demand that the four-point amplitude for all longitudinal polarizations has an enhanced Adler zero. Imposing the constraints in Eqs. (2.5.4) and (2.7.9), we find that the four-point amplitude is

$$A_{LLLL} = -\frac{3}{c_L^4}\left(5(1-c_L^2)^3 - 128\tau_2\right)\omega_1\omega_2\omega_3\omega_4 + \frac{1-c_L^2}{4c_L^2}(s_{12}^2 + s_{13}^2 + s_{14}^2)\,, \tag{2.7.10}$$

where s_{ab} was defined in eq. (2.3.11) and here $c = c_L$. In the soft limit, the first term scales as $\mathcal{O}(q)$ and the second as $\mathcal{O}(q^2)$. Imposing the enhanced Adler zero constraints the coupling to be

$$\tau_2 = \tfrac{5}{128}(1-c_L^2)^3\,. \tag{2.7.11}$$

From the choice of coefficients in Eqs. (2.7.9) and (2.7.11) it is easy to guess the pattern for the exceptional fluid Lagrangian

$$\mathcal{L}_{\text{exc. fluid}} = \frac{1}{1-c_L^2}\sqrt{1+(1-c_L^2)\det B'} - \frac{1}{1-c_L^2} \tag{2.7.12}$$

$$= -\frac{1}{2}\det B' + \frac{1}{8}(1-c_L^2)\det B'^2 - \frac{1}{16}(1-c_L^2)^2\det B'^3 + \frac{5}{128}(1-c_L^2)^3\det B'^4 + \cdots\,. \tag{2.7.13}$$

Although not obvious, can we identify a different Lagrangian which reproduces exactly these fluid amplitudes which have an enhanced soft limit. That Lagrangian takes the form

$$\mathcal{L}'_{\text{exc. fluid}} = \frac{1}{\kappa'}\sqrt{1+\kappa'\left(\dot{\pi}^2 - c_L^2(\partial_i\pi^i)^2\right)}\,. \tag{2.7.14}$$

The coupling constant is $\kappa' = -(1-c_L^2)/c_L^2$. As it turns out, this Lagrangian is tree-level equivalent to relativistic DBI with the longitudinal phonon playing the role of the DBI scalar. To understand why this is so, we simply expand the phonon field in terms of its longitudinal and transverse components as in eq. (2.4.7), yielding

$$\mathcal{L}'_{\text{exc. fluid}} = \frac{1}{\kappa'}\sqrt{1+\kappa'\left(\dot{\pi}_T^2 + \dot{\pi}_L^2 - c_L^2(\partial_i\pi_L^i)^2\right)}\,. \tag{2.7.15}$$

Crucially, since π_T enters quadratically, it can only be pair-produced. Thus, for tree-level amplitudes with all external legs with longitudinal polarizations, the transverse modes decouple completely. Hence, the resulting Lagrangian is equivalent to that of DBI for the longitudinal phonon mode at tree level.

2.7.1.3 Superfluids

A similar analysis was carried out for the superfluid in [18], [108]. In order to have an Adler zero, one demands that the tree-point amplitude in eq. (2.3.10) vanishes, which imposes $g_3 = 0$. Then the soft theorem in eq. (2.3.22) shows that an enhanced Adler zero requires an emergent boost symmetry. From directly requiring the $\mathcal{O}(q^2)$ vanishing of the soft limits of higher-point amplitudes it follows that the only such theory is

$$\mathcal{L}_{\text{exc. superfluid}} = \frac{1}{c}\sqrt{1 + (1-c^2)\left(\partial_\mu\phi\partial^\mu\phi\right)}\,, \tag{2.7.16}$$

written in terms of the field ϕ defined in eq. (2.3.2). This superfluid DBI Lagrangian corresponds to one of the symmetric superfluid theories in [108], found by considering all possible new symmetries that form a consistent algebra with Poincaré and $U(1)$ shift symmetries.

As before, we can identify a classically equivalent Lagrangian

$$\mathcal{L}'_{\text{exc. superfluid}} = \frac{1}{\kappa''}\sqrt{1 + \kappa''\left(\dot{\varphi}^2 - c^2(\partial_i\varphi)^2\right)}\,, \tag{2.7.17}$$

with $\kappa'' = -(1-c^2)/c^4$, which also follows from the emergent boost symmetry and the results in Refs. [9], [92], [93]. This Lagrangian describes a brane moving with constant velocity in an extra dimension [109].

A more general soft bootstrap for nonrelativistic theories with a single scalar was also performed in [106], including the case of NGB with quadratic dispersion relations. By imposing the enhanced Adler zero for a theory with a single scalar with one derivative per field—such as the superfluid—the resulting theory was also found to be effectively relativistic.

2.7.2 Exceptional Framons

The analysis of enhanced soft limits for framids is slightly different. Framon interactions, unlike phonons, do not involve one derivative per field. This is analogous to what happens in the relativistic NLSM, where at leading order in the derivative expansion the interactions have the structure $\pi^n(\partial_\mu\pi)^2$.

The framon soft theorem in eq. (2.6.10) shows that an Adler zero requires the vanishing of the three-point amplitude. This is achieved for

$$c_T = c_L = 1\,, \tag{2.7.18}$$

corresponding to a genuine relativistic dispersion relation for all modes. With this choice the soft theorem still yields

$$\lim_{q\to 0} A_{n+1}^{i_1\cdots i_n i}(p_1,\cdots,p_n,q) = -\sum_{a=1}^{n}\left(\omega_a \frac{\partial}{\partial p_{ai}} + p_a^i \frac{\partial}{\partial \omega_a}\right)\left[A_n^{i_1\cdots i_a\cdots i_n}(\cdots,p_a,\cdots)\right]\,, \tag{2.7.19}$$

so an Adler zero requires full boost symmetry of the framon amplitudes. This is indeed a consequence of the choice in eq. (2.7.18) which corresponds to the Lagrangian

$$\mathcal{L}_{\text{exc. framid}} = -\tfrac{1}{2}M_2^2(\partial_\mu A_\nu)^2\,, \tag{2.7.20}$$

at the leading order in the EFT derivative expansion. Note that the derivative indices are independent from the Lorentz indices of A_μ. Thus, this theory simply describes a relativistic NLSM realizing the spontaneous breaking of an internal $SU(2)$ symmetry corresponding to the boosts of A_μ. That such choice of couplings at this order corresponds to the relativistic NLSM was already pointed out in [19]. Here we have derived this condition from the bottom up as a necessary condition for an Adler zero. Furthermore, since our soft theorem is a consequence of symmetry it is valid to all orders in the EFT derivative expansion.

2.7.3 Alternative Bootstraps

Given the soft theorem in eq. (2.2.25), can we extend the approach of on-shell soft bootstrap to theories with nonzero soft limits? Immediately, we see an obstruction: as we have noted, the soft theorem in eq. (2.2.25) contains field basis dependent terms: the off-shell three-point vertex V_3 and the field transformation under the symmetry β. If we do not know these beforehand, how can we initiate a bootstrap procedure?

Despite this apparent lack of data, such a bootstrap is actually possible in some circumstances, e.g., for the framid. Suppose we want to find theories with two derivatives per interaction vertex which spontaneously break Lorentz boosts. Let us start with the general soft theorem in eq. (2.2.25), together with eq. (2.6.4) for the framons. For this particular case, the soft theorem for the on-shell three-point amplitude encodes enough information to constrain all couplings in V_3. Then, imposing the soft theorem on higher point amplitudes recursively, we obtain a set of constraints on the couplings in a Lagrangian ansatz with only rotational symmetry. We find that the only solution at the two-derivative order coincides with the expansion of the framid Lagrangian in [19], which was constructed using the top-down approach.

Similarly, higher-derivative deformations of the framid Lagrangian can be obtained either from the top-down or the bottom-up construction (by adding higher dimensional operators to the Lagrangian ansatz). The two methods are complementary, however the bottom-up on-shell approach allows us to bypass the field-dependent redundancies of the Lagrangian and directly obtain the on-shell framid amplitudes.

2.8 Conclusions

In this paper we have initiated a systematic analysis of the soft behavior of scattering amplitudes in a broad class of condensed matter systems. The common thread linking these theories is that their gapless modes are the NGBs of spontaneously broken spacetime symmetries. As per the classification of [19], the dynamics of superfluids, solids, fluids, and framids can all be derived from universal principles governing nonlinearly realized symmetries.

Using current conservation equation, we have derived the general soft theorem in eq. (2.2.25), which encodes the action of broken symmetry generators on the NGBs. The ingredients entering the soft theorem are the parameters of the broken symmetry transformation, α and β, together with the propagator and cubic vertex of the theory, Δ and V_3. While β and V_3 are generally dependent on the field basis, and thus not individually invariant, they enter into eq. (2.2.25) in a way that is field basis independent. Furthermore, eq. (2.2.25) should be viewed as a soft theorem because it is an operation relating on-shell scattering amplitudes.

Applying this construction to various EFTs, we present and check a broad array of soft theorems, including those corresponding to temporal translations and Lorentz boosts of the superfluid, spatial translations and Lorentz boosts of the solid, volume-preserving diffeomorphisms of the fluid, and Lorentz boosts of the framid.

Last but not least, we have applied a soft bootstrap approach to these condensed matter systems. In this analysis we take as input the assumption of an enhanced Adler zero condition for soft NGBs. While we have identified exceptional theories of the solid and fluid with these enhanced infrared properties, they are all closely related in structure to relativistic DBI.

Our analysis leaves a number of directions for future work. For example, as noted earlier, our soft theorem does not actually require $SO(3)$ rotation invariance in the broken phase. Indeed, the only requirement is that symmetry breaking preserves some notion of a conserved energy and momentum. For this reason one can in principle study condensed matter systems with even less rotational symmetry. It would be interesting to classify these theories and their corresponding soft theorems.

Throughout this paper we have focused on scattering induced by the self-interactions of phonons. However, the interactions of phonons with other degrees of freedom, e.g., crystal defects or vortices are also of interest for many condensed matter systems. As long as these other modes can be incorporated consistently into the EFT of spontaneous spacetime symmetry breaking, it should be possible to mechanically derive new soft theorems for scattering processes involving these other degrees of freedom.

Another avenue for exploration is more elaborate variations of the soft bootstrap. In particular, it should be possible to *assume* a general ansatz for the broken symmetry parameters α and β, as well as a general ansatz for the propagator and cubic vertex, Δ and V_3. Sculpting out the space of amplitudes satisfying our soft theorem might offer insight into new condensed matter systems of interest.

Finally, it would be interesting to understand whether the geometric perspective on soft theorems presented in [13], [47] extends to a nonrelativistic setting. A geometric description of the corresponding EFTs has already been described in [85], so generalizing to this case should be relatively straightforward.

Chapter 3

SOFT THEOREMS FROM HIGHER SYMMETRIES

J. Berean-Dutcher, M. Derda, and J. Parra-Martinez, "Soft Theorems from Higher Symmetries," 2025. arXiv: 2505.03566 [hep-th].

3.1 Introduction

Symmetry is one of the cornerstones of our understanding of nature at all scales, from the world of fundamental particles to the physics of phase transitions and the early universe. Perhaps the most striking consequence of global symmetry in quantum field theory (QFT) is the existence of massless scalar particles, predicted by Nambu [1] and Goldstone [2], [3], when such symmetry is continuous and spontaneously broken. Beyond their existence, the dynamics of such Nambu-Goldstone bosons (NGB) is weakly coupled and heavily constrained by the broken symmetry. This allows a detailed effective field theory (EFT) description of their long-distance interactions, which lets us understand a variety of systems: from ferromagnetic materials to the dynamics of pions in low-energy quantum chromodynamics (QCD) [110]–[112].

A direct implication of spontaneously broken symmetry in the dynamics of NGB are *soft theorems*, which govern the universal behavior of the scattering amplitudes of NGB in the limit in which some of their momenta are taken to zero. An early example is the so-called Adler zero [4] in the low-energy dynamics of pions, and its generalizations [5], [6] including to multi-soft pion limits, [7], [8], [94], [97], [113], subleading [97], [114], [115], higher-derivative corrections [116], and other theories of NGB [18], [40], [117]–[119]. Soft limits are not limited to scalar NGB, and are found to be universal for other massless particles, including scalar moduli [13], [47], [65], [120], photons [37], [38], [121], [122], gluons [123]–[125] and gravitons [37], [38], [118], [126], [127]. However, our traditional description of such soft theorems often requires going beyond the language of spontaneous symmetry breaking or ordinary symmetries, and utilizes notions of asymptotic symmetries [128]–[139], or other considerations [127], [140].

Recently, there has been a leap in our understanding of what constitutes a symmetry in QFT. Beyond the familiar textbook symmetries that act on local operators and whose charges are carried by particles, there exists a zoo of so-called *generalized symmetries* or *higher symmetries* (for references, see [33]). These include higher-form symmetries, which act on lower-codimension operators (lines, surfaces, etc.), and are carried by extended excitations (strings, membranes, etc.), higher-group symmetries [141]–[148], which mix symmetries of various ranks, and even non-invertible symmetries [149]–[154]. A simple example of these is the (electric) 1-form symmetry of Maxwell theory, which acts on Wilson lines and has as current the electromagnetic field-strength, conserved by the vacuum Maxwell's equations. From this modern perspective, non-scalar massless particles, such as the photon, can often be interpreted as the NBG (or pseudo-NGB) associated with the higher symmetry [34], [155], [156]. These novel symmetries have found wide application in high-energy and condensed-matter physics [33], [157], but their consequences on the on-shell S-matrix describing scattering processes remain largely unexplored (with a few notable exceptions [158]–[160]).

In this paper we argue that these new notions of symmetry enable the unification of our understanding of soft theorems for the various kinds of NGB. In particular, we will explain that the familiar soft photon theorem can be derived as a consequence of 1-form symmetries which act on line operators and as a shift symmetry on the electromagnetic field. Such symmetry is emergent in the limit where particle production is suppressed, which can be made precise using the language of heavy-particle effective theories such as Heavy Quark EFT (HQET) [161], [162]. Furthermore we will derive a new double soft pion theorem, in theories with a spontaneously broken continuous 2-group symmetry, wherein the current algebra of ordinary (or 0-form, in modern parlance) symmetry is intertwined with that of a 1-form symmetry. This double soft theorem includes contributions which change particle species—between NGB of the 0-form symmetry (pions) and of the 1-form symmetry (photons)—thereby reflecting the intertwined current algebra. As an example, we focus on the low-energy effective theory of QCD with massless quarks and with gauged $U(1)_V$ vector symmetry corresponding to Baryon number, which shows that such structure is far from exotic.

The rest of this paper is organized as follows: in section 3.2 we review the familiar understanding of scalar NGB, which henceforth we call pions, and the derivation of their single and double soft theorems from the current algebra of 0-form symmetry. In section 3.3 we review the modern perspective on the photon as a NGB and show that leading soft photon theorems follow from the Ward identities of emergent 1-form symmetries. Finally in section 3.4 we derive the new double soft pion theorem for theories with spontaneously broken continuous 2-group symmetry using its current algebra, and illustrate it with some examples at tree level. Finally, in an appendix we provide some technical details for the current algebra proof of the known sub-leading double soft pion theorem (which is presented here for the first time).

Conventions: This paper deals with scattering amplitudes, so we work in Lorentzian signature with a mostly-minus metric $(+\,,\,-\,,\,-\,,\,-)$. This is in contrast to most of the literature on generalized symmetries which uses Euclidean signature. Our scattering amplitudes are defined with all external momenta taken to be outgoing.

3.2 Review: Soft pions from ordinary symmetry

In this section we will review the spontaneous breaking of continuous 0-form symmetry and the associated soft theorems for the scattering amplitudes of pions. This is all textbook material, which we choose to collect here with the purpose of highlighting the similarities with our discussion of 1-form symmetry in Section 3.3, and to provide the background for our new double soft theorem in Section 3.4. The only novelty here is the current algebra derivation of the known sub-leading double soft pion theorem, whose technical details are collected in Appendix B.

Continuous 0-form symmetries have conserved currents

$$\partial^\mu j^a_\mu(x) = 0\,, \tag{3.2.1}$$

which we can integrate over codimension-1 cycles, Σ_{D-1} to define an associated conserved charges:

$$Q^a(\Sigma_{D-1}) = \int_{\Sigma_{D-1}} d^{d-1}x\,\hat{n}^\mu j^a_\mu\,. \tag{3.2.2}$$

Here $\hat{n}$ is a unit normal vector to Σ_{D-1}. These conserved charges are in one-to-one correspondence with the Lie algebra generators of G, t^a, satisfying

$$[t^a, t^b] = if^{abc}t^c\,. \tag{3.2.3}$$

Local operators may be charged under 0-form symmetries. Given a 0-form symmetry with group G, an operator transforming under G with representation $\mathcal{R}$ will satisfy the Ward identity,

$$\partial^\mu j^a_\mu(x)\mathcal{O}_\mathcal{R}(y) = \delta^{(4)}(x-y)t^a_\mathcal{R}\mathcal{O}_\mathcal{R}(y)\,, \tag{3.2.4}$$

where t^a are the generators of G. If such a local operator develops a vacuum expectation value

$$\langle\mathcal{O}_\mathcal{R}(y)\rangle \sim v^a\,. \tag{3.2.5}$$

the symmetry is spontaneously broken down to a subgroup H which is generated by the charges $\{T^a\} \subset \{t_a\}$ that leave the vacuum expectation value invariant $[v^a, T^a] = 0$. The rest of the generators, X^a, form a coset G/H. This implies the following decomposition of the commutation relations for the generators of G,

$$[T^a, T^b] = if_T^{abc}T^c\,, \tag{3.2.6}$$

$$[X^a, X^b] = iF^{abc}T^c\,, \tag{3.2.7}$$

$$[T^a, X^b] = if_X^{abc}X^c\,. \tag{3.2.8}$$

The f_T^{abc}, f_X^{abc}, and F^{abc} are structure constants which satisfy the appropriate Jacobi identities, and coincide in the case of the chiral symmetry breaking in QCD. Here we have assumed that the coset G/H is a symmetric space, i.e., that it has the additional symmetry $X^a \to -X^a$, sometimes called G-parity.

Goldstone's theorem [3] implies that the spectrum contains a massless NGB – a pion π^a – for every broken generator X^a. More concretely, there exists conserved currents $\mathcal{J}^a_\mu(x)$ corresponding to the broken generators, which interpolate the pion,

$$\langle\pi^b(p)|\mathcal{J}^a_\mu(x)\rangle = if_\pi p_\mu\delta^{ab}e^{ip\cdot x}. \tag{3.2.9}$$

Here f_π is the dimensionful pion decay constant. The interpolation property implies that this current contains a contribution that is linear in the pion field,

$$\mathcal{J}^a_\mu(x) = f_\pi\partial_\mu\pi^a(x) + \mathcal{O}(\pi^3)\,. \tag{3.2.10}$$

Thus, the symmetry is non-linearly realized and at leading order acts as a constant shift of the pion field

$$\pi^a(x) \to \pi^a(x) + c^a + \cdots\,, \qquad \text{with} \qquad \partial_\mu c^a = 0\,. \tag{3.2.11}$$

In contrast, the π^a transform as the broken generators, that is, in a linear representation of the unbroken subgroup H, which infinitesimally takes

$$\pi^a(x) \to \pi^a(x) - f_X^{abc}\pi^b(x)\lambda^c\,. \tag{3.2.12}$$

The associated symmetry current then has the form

$$V_\mu^a(x) = f_X^{abc}\pi^b(x)\partial_\mu\pi^c(x) + \mathcal{O}(\pi^4). \tag{3.2.13}$$

The symmetry structure furnished by eq. (3.2.8) implies the following set of operator equations for the currents,

$$\partial^\mu V_\mu^a(x)V_\nu^b(y) = if_T^{abc}\delta^{(4)}(x-y)V_\nu^c(y) \tag{3.2.14}$$

$$\partial^\mu \mathcal{J}_\mu^a(x)\mathcal{J}_\nu^b(y) = iF^{abc}\delta^{(4)}(x-y)V_\nu^c(y) \tag{3.2.15}$$

$$\partial^\mu \mathcal{J}_\mu^a(x)V_\nu^b(y) = if_X^{abc}\delta^{(4)}(x-y)\mathcal{J}_\nu^c(y)\,. \tag{3.2.16}$$

From this point on we will focus on theories possessing a chiral 0-form global symmetry $G = G_L \times G_R$ which is spontaneously broken down to a vector subgroup $H = G_V$, so we will refer to $\mathcal{J}_\mu^a(x)$ as the axial current, and to $V_\mu^a(x)$ as the vector current. In the example of the chiral effective theory of pions describing (massless) QCD at low energies, $G = SU(N_f)_L \times SU(N_f)_R$ and is broken down to the vector subgroup, $SU(N_f)_V$.

An effective theory for these pion NGB can be written in terms of the charged operator

$$U(x) = \exp\left(\frac{2i}{f_\pi}\pi(x)^a t^a\right), \tag{3.2.17}$$

parameterizing fluctuations of the vacuum expectation value in Eq. (3.2.5), where we choose the normalization $\mathrm{Tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab}$. The effective Lagrangian for a general 0-form spontaneous symmetry breaking pattern $G_L \times G_R \to G_V$ is given by [26], [27], [41], [110], [163]–[165]

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \cdots\,. \tag{3.2.18}$$

where the dots include higher derivative terms. This is the famous non-linear sigma model (NLSM).

3.2.1 Pion Adler zero

It follows from eq. (3.2.9) that momentum-space form factors describing the overlap of the axial current with an on-shell scattering state of n-many outgoing pions, $|\alpha\rangle$, admit a decomposition

$$\langle\alpha|\mathcal{J}^a_\mu(q)\rangle = -\frac{f_\pi q_\mu}{q^2}\langle\alpha+\pi^a(q)|0\rangle + \langle\alpha|\mathcal{J}_H{}^a_\mu(q)\rangle\,. \tag{3.2.19}$$

The first term carries the single-pion pole, whose coefficient is the matrix element corresponding to the amplitude with a additional pion with momentum q

$$\langle\alpha+\pi^a(q)|0\rangle = i\mathcal{A}_{n+\pi(q)}\,. \tag{3.2.20}$$

where we leave a momentum-conserving delta function implicit. The second term contains the 'hard part of the current' $\mathcal{J}_H{}^a_\mu$, which includes the infinite number of higher-order contributions that make up the axial current. The axial current is conserved in the correlation function on the LHS of the above equation since there are no other insertions of charged operators in the correlator. Conservation then implies the relation

$$\mathcal{A}_{n+\pi(q)} = -i\frac{1}{f_\pi}q^\mu\langle\alpha|\mathcal{J}_H{}^a_\mu(q)\rangle\,. \tag{3.2.21}$$

Taking the limit $q\to 0$ of this equation yields the soft theorem known as the Adler zero [4],

$$\lim_{q\to 0}\mathcal{A}_{n+\pi(q)} = 0+\mathcal{O}(q)\,, \tag{3.2.22}$$

where we have assumed that $\langle\alpha|\mathcal{J}_H{}^a_\mu(q)\rangle$ is regular in this limit. This regularity follows from spontaneous symmetry breaking, and is realized in the theory by the lack of any cubic coupling for the pions.

In summary, we have seen that the Adler zero follows from the spontaneously broken symmetry as a consequence of Goldstone's theorem and the conservation of the axial current.

3.2.2 Double soft pion theorem

Now we will discuss the double soft pion theorem [5], [7] which encodes the non-abelian current algebra described by the operator equations in eqs. (3.2.14) to (3.2.16). Several different derivations of the double soft theorem for pions exist in the literature, for example in [8], [113]. The sub-leading contribution for flavor-dressed amplitudes was first derived in [97], [114], [115] at tree-level or using diagrammatic arguments. Here we follow Ref. [94] and extend the analysis to emphasize how the leading and sub-leading theorem are a direct consequence of current algebra.

The starting point is the momentum-space correlator with two axial current operator insertions, which satisfies the Ward identity eq. (3.2.15) in either of the currents. For the current with momentum q_1 we have

$$q_1^\mu \langle\alpha|\mathcal{J}_\mu^a(q_1)\mathcal{J}_\nu^b(q_2)\rangle = F^{abc}\langle\alpha|V_\nu^c(q_1+q_2)\rangle\,, \tag{3.2.23}$$

and there is an analogous Ward identity for current with momentum q_2. Hence we can write the RHS of eq. (3.2.23) in the manifestly symmetric form

$$q_1^\mu q_2^\nu \langle\alpha|\mathcal{J}_\mu^a(q_1)\mathcal{J}_\nu^b(q_2)\rangle = -\frac{1}{2}F^{abc}(q_1-q_2)^\mu\langle\alpha|V_\mu^c(q_1+q_2)\rangle\,. \tag{3.2.24}$$

On the other hand, decomposing axial currents according to eq. (3.2.19) yields

$$\begin{aligned} q_1^\mu q_2^\nu \langle\alpha|\mathcal{J}_\mu^a(q_1)\mathcal{J}_\nu^b(q_2)\rangle = &- f_\pi^2\langle\alpha+\pi^a(q_1)\pi^b(q_2)|0\rangle \\ &+ i f_\pi q_1^\mu\langle\alpha+\pi^b(q_2)|\mathcal{J}_{H}{}_\mu^a(q_1)\rangle \\ &+ i f_\pi q_2^\nu\langle\alpha+\pi^a(q_1)|\mathcal{J}_{H}{}_\nu^b(q_2)\rangle \\ &+ q_1^\mu q_2^\nu\langle\alpha|\mathcal{J}_{H}{}_\mu^a(q_1)\mathcal{J}_{H}{}_\nu^b(q_2)\rangle\,, \end{aligned} \tag{3.2.25}$$

which with the help of eq. (3.2.21) we can rewrite as

$$q_1^\mu q_2^\nu \langle\alpha|\mathcal{J}_\mu^a(q_1)\mathcal{J}_\mu^b(q_2)\rangle = f_\pi^2\langle\alpha+\pi^a(q_1)\pi^b(q_2)|0\rangle + q_1^\mu q_2^\nu\langle\alpha|\mathcal{J}_{H}{}_\mu^a(q_1)\mathcal{J}_{H}{}_\nu^b(q_2)\rangle. \tag{3.2.26}$$

The first term on the RHS of eq. (3.2.26) contains a scattering amplitude of $(n+2)$-many pions. Combining eq. (3.2.24) and eq. (3.2.26) we obtain

$$\begin{aligned} f_\pi^2\langle\alpha+\pi^a(q_1)\pi^b(q_2)|0\rangle = &-\frac{1}{2}F^{abc}(q_1-q_2)^\mu\langle\alpha|V_\mu^c(q_1+q_2)\rangle \\ &- q_1^\mu q_2^\nu\langle\alpha|\mathcal{J}_{H}{}_\mu^a(q_1)\mathcal{J}_{H}{}_\nu^b(q_2)\rangle\,. \end{aligned} \tag{3.2.27}$$

To arrive at the double soft theorem we take the simultaneous limit $q_1, q_2 \to 0$ of this expression. In Appendix B we derive the applicable soft limits of the two form factors on the RHS which are shown to be related to n-point scattering amplitudes of the remaining n hard pions $\pi^{a_i}(p_i)$ in the state $\langle\alpha|$.

More concretely we find that at sub-leading order the soft vector current form factor is given by

$$\lim_{q\to 0}\langle\alpha|V^c_\mu(q)\rangle = \sum_{i=1}^{n} f^{aa_i d}\left(\frac{(2p_i+q)_\mu}{(p_i+q)^2} - \frac{iq^\nu L_{i\mu\nu}}{(p_i\cdot q)}\right)\mathcal{A}_n^{a_1\ldots d\ldots a_n}\,, \tag{3.2.28}$$

where the angular momentum operator $L_i^{\mu\nu}$ is defined as

$$L_i^{\mu\nu} = i\left(p_i^\mu\frac{\partial}{\partial p_i^\nu} - p_i^\nu\frac{\partial}{\partial p_i^\mu}\right). \tag{3.2.29}$$

Meanwhile, the hard axial currents form factor at leading order in soft momenta is

$$\lim_{q_1 q_2\to 0}\langle\alpha|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)\rangle = \sum_{i=1}^{n}(F^{aa_i e}f_X^{ebd} + F^{ba_i e}f_X^{ead})\frac{\eta_{\mu\nu}}{2p_i\cdot(q_1+q_2)}\mathcal{A}_n^{a_1\cdots d\cdots a_n}\,. \tag{3.2.30}$$

Combining these results we obtain for an $(n+2)$-particle amplitude with soft particles $\pi^a(q_1)$ and $\pi^b(q_2)$ and n hard particles $\pi^{a_i}(p_i)$,

$$\lim_{q_1,q_2\to 0}\mathcal{A}_{n+\pi^a(q_1)\pi^b(q_2)} = \left(S^{(0)} + S^{(1)}\right)\mathcal{A}_n\,, \tag{3.2.31}$$

where we scale both soft momenta to zero simultaneously. The leading soft factor is

$$S^{(0)}\mathcal{A}_n = -\frac{1}{f_\pi^2}\sum_{i=1}^{n} F^{abc}f_X^{ca_i d}\frac{p_i\cdot(q_1-q_2)}{2p_i\cdot(q_1+q_2)}\mathcal{A}_n^{a_1\ldots d\ldots a_n}, \tag{3.2.32}$$

and the sub-leading soft factor is[1]

$$\begin{aligned} S^{(1)}\mathcal{A}_n = &-\frac{1}{f_\pi^2}\sum_{i=1}^{n}(F^{aa_i c}f_X^{cbd} + F^{ba_i c}f_X^{cad})\frac{(q_1\cdot q_2)}{2p_i\cdot(q_1+q_2)}\mathcal{A}_n^{a_1\ldots d\ldots a_n} \\ &+\frac{1}{f_\pi^2}\sum_{i=1}^{n} F^{abc}f_X^{ca_i d}\frac{(q_1\cdot q_2)(p_i\cdot(q_1-q_2))}{2(p_i\cdot(q_1+q_2))^2}\mathcal{A}_n^{a_1\ldots d\ldots a_n} \\ &+\frac{1}{f_\pi^2}\sum_{i=1}^{n} F^{abc}f_X^{ca_i d}\frac{iq_1^\mu q_2^\nu L_{i\mu\nu}}{p_i\cdot(q_1+q_2)}\mathcal{A}_n^{a_1\ldots d\ldots a_n}\,, \end{aligned} \tag{3.2.33}$$

with the angular momentum operator $L_i^{\mu\nu}$ given by eq. (3.2.29).

[1] The sub-leading order soft factor is modified in the presence of four-derivative operators in the chiral Lagrangian; see [116] for a full expression. It also receives loop corrections that will be explored elsewhere.

3.3 Soft photons from higher-form symmetry

In this section we will discuss photons as NGB for 1-form global symmetries. We will show how the development of the previous section for 0-form NGB is suitably generalized and how the familiar soft photon theorems derive from corresponding Ward identities.

3.3.1 Photons as Nambu-Goldstone bosons

Let us begin by quickly reviewing some basic facts about continuous 1-form symmetries, and the modern perspective of the photon as a NGB. We aim to be pedagogical and to stress the similarities with our discussion of 0-form symmetry breaking in the previous section. This discussion can (and probably should) be skipped by the experts.

Continuous 1-form symmetries have conserved currents which are antisymmetric rank-2 tensors

$$J_{\mu\nu}(x) = J_{[\mu\nu]}(x)\,, \qquad \text{with} \qquad \partial^\mu J_{\mu\nu}(x) = 0\,. \tag{3.3.1}$$

The corresponding conserved charges are integrals over codimension-2 cycles, Σ_{D-2}

$$Q(\Sigma_{D-2}) = \int_{\Sigma_{D-2}} dS^{\mu\nu} J_{\mu\nu} \tag{3.3.2}$$

and, as such, are always commuting. Hence, 1-form symmetries are always abelian, and are in correspondence with group elements of $U(1)$ when the symmetry is continuous.

Local operators are not charged under 1-form symmetries, since they trivially commute with charges of the form eq. (3.3.2). Line operators, however, can be charged under a 1-form symmetry $U(1)^{(1)}$[2]. Consider a line operator, $W_Q(C)$, with charge Q supported on a curve C, then the corresponding Ward identity is

$$\partial^\mu J_{\mu\nu}(x) W_Q(C) = Q\,\delta^{(3)}_\nu(x-C) W_Q(C). \tag{3.3.3}$$

where we have defined a higher-codimension delta function,

$$\delta^{(3)}_\nu(x-C) = \int ds\,\frac{dy_\nu}{ds}\delta^{(4)}(x-y(s))\,, \tag{3.3.4}$$

[2]Henceforth, we use a superscript $^{(1)}$ to denote 1-form symmetry.

with $y(s)$ a given parameterization of the curve C, such that its integral over any cycle intersecting the curve equals one. This is analogous to the contact term in the Ward identity of eq. (3.2.4) for local operators charged under a 0-form symmetry. The Ward identity of eq. (3.3.3) constitutes a non-trivial constraint on correlation functions involving the line operators $W_Q(C)$. We will show momentarily that it also implies the familiar soft photon theorem for scattering amplitudes in quantum electrodynamics.

1-form symmetries can be spontaneously broken if a charged operator develops a vacuum expectation value [34]

$$\langle W_Q(C)\rangle \sim 1\,. \tag{3.3.5}$$

This is equivalent to the perhaps more familiar "perimeter law" for a Wilson loop $\langle W_Q(C)\rangle \sim e^{-aL(C)}$, where $L(C)$ is the length of the loop, by a scheme choice in which a local counterterm on the line is added and tuned to cancel the perimeter scaling. Eq. (3.3.5) indeed indicates a deconfined phase, and a version of Golstone's theorem [34], [36] implies that the current must overlap with a one-photon state, which for helicity h and momentum p yields[3]

$$\langle \gamma_h(p)|J^{\mu\nu}(x)\rangle = \frac{i}{g}\,(p^\mu \varepsilon_h^{\nu *} - p^\nu \varepsilon_h^{\mu *})\, e^{ip\cdot x}\,, \tag{3.3.6}$$

where g is the gauge coupling and ε_h^μ is the corresponding polarization vector, which satisfies the on-shell and normalization conditions

$$\varepsilon\cdot q = 0\,, \qquad \varepsilon\cdot\varepsilon = 0\,, \qquad \varepsilon\cdot\varepsilon^* = 1\,. \tag{3.3.7}$$

This is in precise correspondence with eq. (3.2.9).

Eq. (3.3.6) implies that the current must contain a contribution linear in the NGB field,

$$J^{\mu\nu}(x) = \frac{1}{g^2}(\partial^\mu A^\nu - \partial^\nu A^\mu) + \cdots\,, \tag{3.3.8}$$

where the dots denote possible higher-order contributions. Thus under $U(1)^{(1)}$ the photon transforms by a shift analogous to that of π^a in eq. (3.2.11)

$$A_\mu(x) \mapsto A_\mu(x) + \lambda_\mu(x)\,, \qquad \text{with} \qquad \partial_{[\mu}\lambda_{\nu]}(x) = 0\,. \tag{3.3.9}$$

[3] The polarization vector appears complex conjugate because the current creates an incoming photon.

Note that $\lambda_\mu(x)$ are the components of a closed, but not exact 1-form, so this is not equivalent to a small gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$.

As in eq. (3.2.17) the charge operators can be constructed by exponentiating the NGB field

$$W_Q(C) = \exp\left(iQ\int_C A_\mu dx^\mu\right)\,, \tag{3.3.10}$$

so the NGB parameterizes fluctuations of the vacuum expectation value. In the present case, these are of course the familiar expressions for the Wilson line operators, and by combining Eqs. (3.3.2) and (3.3.8) their 1-form charges simply correspond to the electric flux.

Finally, we must discuss the fate of the 1-form symmetry in the presence of electrically charged matter. Indeed this induces the explicit breaking of the 1-form symmetry. The current for the 1-form symmetry is no longer conserved:

$$\partial^\mu J_{\mu\nu}(x) = j_\nu(x)\,, \tag{3.3.11}$$

where j_ν is the electromagnetic current.

Physically, the presence of electric charges allows the vacuum to polarize by pair production with the result that Wilson lines are screened and do not induce a non-trivial electric flux. Thus they cannot carry a 1-form charge. This is closely related to the fact that the Wilson lines can now have endpoints on local charged operators, so the putative 1-form symmetry charges would act trivially on them.

3.3.2 Soft photon theorems

The famous soft photon theorem [37], [38], [121], [122] concerns scattering amplitudes with charged particles. It states that the leading term in the soft expansion of an amplitude in the photon's momentum, q, is given by

$$\lim_{q\to 0}\mathcal{A}_{n+\gamma(q)} = g\sum_i Q_i \frac{\varepsilon\cdot p_i}{q\cdot p_i}\mathcal{A}_n \tag{3.3.12}$$

where Q_i and p_i are the charges and momenta of the hard particles.

In light of our discussion of breaking of 1-form symmetry by charged matter, it might naively seem that it is not possible to interpret eq. (3.3.12) using this language. We are then forced to ask: is there any limit in which the 1-form symmetry re-emerges and explains the soft theorem? We will explain below that the answer to this question is positive, but let us begin by making some general comments that might give us hope that this is the case.

From the modern viewpoint, the masslessness of the photon is a consequence of the spontaneously broken 1-form symmetry and not of gauge invariance. From this perspective, one might also wonder why the photon remains massless even in the presence of charged particles. The answer is that 1-form symmetries are *robust* in the sense that they cannot be broken by adding any local operators to the effective action, as these are not charged under the symmetry. Incidentally, this also implies that when 1-form symmetries emerge in some limit they are exact. This robustness protects the masslessness of the photon (though not completely, as it could be Higgsed by the charged matter). More generally, the breaking of the 1-form symmetry is universal: it can only happen by adding charged degrees of freedom and in the form of eq. (3.3.11). One might expect that this universality is the moral reason for the existence of the soft photon theorems. In the rest of this section we will try to sharpen this perspective.

3.3.2.1 A photon Adler zero

Let us first consider a trivial limit in which the 1-form symmetry is emergent. If we consider charged particles with mass m, then the scattering of photons with momenta $k \ll m$ is described by an effective theory *à la* Euler-Heisenberg [166], in which the charged matter is integrated out. This results in the non-linear Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} + \frac{c_1}{m^4} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{c_2}{m^4} (F_{\mu\nu} F^{\nu\sigma} F_{\sigma\rho} F^{\rho\mu}) + \mathcal{O}\left(\frac{1}{m^6}\right) . \tag{3.3.13}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength, and c_i are dimensionless Wilson coefficients which depend on the details of the matter that is integrated out. This theory has an exact electric 1-form symmetry $U(1)_e^{(1)}$ with current

$$J_e^{\mu\nu} = \frac{1}{g^2} F^{\mu\nu} + \mathcal{O}(F^3), \tag{3.3.14}$$

conserved by the equations of motion derived from (3.3.13). It also has a magnetic $U(1)_m^{(1)}$ 1-form global symmetry with the current

$$J_m^{\mu\nu} = \frac{1}{4\pi} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \, , \tag{3.3.15}$$

conserved by the Bianchi identity and which measures magnetic flux. The NGB associated to $U(1)^{(1)}_m$ is a dual photon which does not appear explicitly in eq. (3.3.13) but likewise transforms non-linearly by a shift under $U(1)^{(1)}_m$. This symmetry will play an important role in section 3.4, but not in our current discussion.

Let us now follow the steps analogous to the derivation of the pion Adler zero. We separate the single-photon and hard contributions in the form factor for the 1-form symmetry current within a multi-photon state $|\alpha\rangle$,

$$\langle\alpha|J_e^{\mu\nu}(q)\rangle = \frac{1}{g}\frac{(q^\mu\varepsilon^{\nu*} - q^\nu\varepsilon^{\mu*})}{q^2}\langle\alpha+\gamma(q)|0\rangle + \langle\alpha|J_{eH}^{\mu\nu}(q)\rangle\,. \tag{3.3.16}$$

On the RHS we have left a sum over helicities implicit. The coefficient of the pole is simply related to the scattering amplitude with an additional photon with momentum q

$$\langle\alpha+\gamma(q)|0\rangle = i\mathcal{A}_{n+\gamma(q)}\,. \tag{3.3.17}$$

Then the conservation of the 1-form current implies the relation

$$\mathcal{A}_{n+\gamma(q)} = -ig\,q_\mu\varepsilon_\nu\langle\alpha|J_{eH}^{\mu\nu}(q)\rangle\,, \tag{3.3.18}$$

where we made use of eq. (3.3.7). Taking the soft limit we find that the RHS vanishes, since as before, the hard current contributions are regular in the soft limit due to the shift symmetry of the NGB implying that there is no cubic coupling. Thus we find an Adler zero for photon amplitudes in this theory

$$\lim_{q\to 0}\mathcal{A}_{n+\gamma(q)} = 0 \tag{3.3.19}$$

This is in exact analogy with Eqs. (3.2.21) and (3.2.22) in the previous section.

Such a zero is perhaps an unsurprising statement, since the effective theory in eq. (3.3.13) is derivatively coupled with at least one derivative by field, that is, it is a theory of abelian NGB. This soft photon theorem is, nevertheless, a direct consequence of the electric 1-form symmetry of the theory.

3.3.2.2 Low's soft photon theorem

By integrating out the charged matter we have seemingly thrown away the baby with the bathwater, and have only been able to reproduce eq. (3.3.12) in the case where the scattering amplitude involves no external charged particles.

However there is a middle way in which we can keep both external charged states and see the 1-form symmetry emerge.

Consider a scattering amplitude involving n massive charged particles with 'hard' momenta, p_i. In the limit that all hard momenta are neighboring the mass shell of either external particle or anti-particle states, we can write $\ell_i^\mu = p_i^\mu + k^\mu$ with p_i on-shell and $k_i \ll p_i$. Expanding in this limit, the scattering amplitude will be given by a correlation function of Wilson lines plus a hard (local) operator insertion, O_H,

$$i\mathcal{A}_n \sim \langle W_{Q_1}(C_1)\cdots W_{Q_n}(C_n)O_H\rangle \tag{3.3.20}$$

where $\sim$ denotes that the equality holds up to corrections in inverse powers of the masses. The Wilson lines are extended from the origin to infinity along trajectories with four-velocities $v_i^\mu = p_i^\mu/m_i$ (that is, $C_i : y_i^\mu(s) = sv_i^\mu$) with $s \in [0,\infty]$, and carry the corresponding charges, Q_i,

$$W_{Q_i}(C_i) = \exp\left(iQ_i \int_0^\infty ds\, v_i^\mu A_\mu(sv)\right) \tag{3.3.21}$$

The hard operator O_H is inserted at the origin. Its form will not be important in what follows.

We claim that the electric 1-form symmetry emerges in this limit. We will show that this is the case momentarily, but first let us consider the consequence of the Ward identity eq. (3.3.3) upon the correlation function appearing in the expanded amplitude eq. (3.3.20)

$$\begin{aligned}&\partial_\mu\langle J_e^{\mu\nu}(x)\prod_i W_{Q_i}(C_i)O_H\rangle \\ &\qquad = \left(\sum_i Q_i \int ds\, \frac{dy_i^\nu}{ds}\delta^{(4)}(x-y_i(s))\right)\langle\prod_i W_{Q_i}(C_i)O_H\rangle\,. \end{aligned} \tag{3.3.22}$$

We will contract this equation with a polarization $\varepsilon^\nu(q)$, and Fourier transform in x. On the one hand, the LHS yields,

$$\varepsilon_\nu \int d^4x\, e^{-iq\cdot x}\partial_\mu\langle J_e^{\mu\nu}\prod_i W_{Q_i}(C_i)O_H\rangle \tag{3.3.23}$$

$$= -\frac{i}{g}\langle\gamma_h(q)|\prod_i W_{Q_i}(C_i)O_H\rangle + iq_\mu\varepsilon_\nu\langle J_{eH}^{\mu\nu}(q)\prod_i W_{Q_i}(C_i)O_H\rangle\,, \tag{3.3.24}$$

where the the derivative operator has effected an amputation of an external photon $\gamma_h(q)$. The first term is then identified with the amplitude with an additional external photon

$$\langle\gamma(q)|\prod_i W_{Q_i}(C_i)O_H\rangle \sim i\mathcal{A}_{n+\gamma(q)} \tag{3.3.25}$$

On the RHS of eq. (3.3.22) we just need to compute the Fourier transform of the contact terms

$$\begin{aligned}\int d^4x\, e^{-iq\cdot x}\left(Q_i\int_0^\infty ds\,\varepsilon\cdot v_i\delta^{(4)}(x-vs)\right) &= Q_i\varepsilon\cdot v_i\int_0^\infty ds\, e^{isq\cdot v}\\ &= iQ_i\frac{\varepsilon\cdot v_i}{q\cdot v_i}\end{aligned} \tag{3.3.26}$$

which shows that the the RHS of the Ward identity gives

$$\sum_i Q_i\frac{\varepsilon\cdot v_i}{q\cdot v_i}\langle\prod_i W_{Q_i}(C_i)O_H\rangle \sim \sum_i Q_i\frac{\varepsilon\cdot p_i}{q\cdot p_i}\mathcal{A}_n \tag{3.3.27}$$

Putting all together yield the relation

$$\mathcal{A}_{n+\gamma(q)} = g\sum_i Q_i\frac{\varepsilon\cdot p_i}{q\cdot p_i}\mathcal{A}_n + igq_\mu\varepsilon_\nu\langle J^{\mu\nu}_{eH}(q)\prod_i W_{Q_i}(C_i)O_H\rangle\,. \tag{3.3.28}$$

In the soft limit the matrix element of the hard current can only give terms which are at most singular as $\mathcal{O}(q^{-1})$, so taking the small q limit of this relation gives the soft theorem in eq. (3.3.12).

3.3.3 Emergent higher-form symmetry in heavy-particle EFT

Above we have claimed that in the limit where the soft photon leaves particles (or antiparticles) close to the mass shell there is an emergent 1-form symmetry, which we then used to derive the soft photon theorem. Let us now make this precise using the language of effective field theory. The manipulations in this section are not new, and very familiar in the context of HQET [161], but we will describe them emphasizing the physics of the emergence of the 1-form symmetry.

We begin by considering a charged massive particle associated to a field $\phi(x)$ interacting with the photon. In order to manifest the excitations close to the mass shell we perform a field redefinition which pulls out the large part of the momentum

$$\phi(x) = \frac{e^{imv\cdot x}}{\sqrt{2m}}\left(\varphi_v^+ + \varphi_v^-\right) \tag{3.3.29}$$

where $v^\mu = p^\mu/m$ with p on-shell, and we have also separated the field into positive and negative frequency modes corresponding to the particle and antiparticle.[4] This field redefinition turns large derivatives of the field $\partial\phi \sim p\phi \sim m\phi$ into manifest couplings, and leaves only small derivatives $\partial\varphi \sim k\varphi \ll m\varphi$ in the effective Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2} F^{\mu\nu}F_{\mu\nu} + i\varphi_v^{+*}(v\cdot D)\varphi_v^+ + i\varphi_v^{-*}(v\cdot D + 2m)\varphi_v^- + \mathcal{O}\Big(\frac{1}{m}\Big)\,, \quad (3.3.30)$$

where $D_\mu = \partial_\mu + iA_\mu$ is the usual covariant derivative. Thus we see that the positive frequency φ_v^+ fields describe excitations close to the particle mass shell, and φ_v^- describe antiparticle excitations which have a large gap $k \sim 2m$. This is illustrated in Fig. 3.1. We can integrate out the antiparticle excitations to get an effective theory describing only particles

$$\mathcal{L} = i\varphi_v^*(v\cdot D)\varphi_v + \varphi_v^* \frac{D_\perp^2}{2m}\varphi_v + d\,\varphi_v^* \frac{F_{\mu\nu}M^{\mu\nu}}{4m}\varphi_v + \cdots\,. \quad (3.3.31)$$

Here we have kept terms only up to subleading power in $1/m$, including the kinetic energy written in terms of $D_\perp^\mu = D^\mu - v^\mu(v\cdot D)$; and the magnetic dipole moment operator with Wilson coefficient d, written using the Lorentz generator, $M^{\mu\nu}$, in the appropriate representation.

At this point an astute reader might guess that this theory has an emergent 1-form symmetry. As explained above, the explicit breaking of 1-form symmetries by charged matter is closely related to the phenomenon of pair creation or vacuum polarization. However in constructing this effective theory we have integrated out the antiparticles, so such processes cannot occur. A further consequence of this is that the coupling constant g does not run in the EFT and is frozen at the matching scale, m, which makes the current $J^{\mu\nu} = \frac{1}{g^2}F^{\mu\nu} + \cdots$ well-defined.

The symmetry can be made manifest by performing one further field redefinition (due to BPS [167]) which strips a Wilson line from the particle fields

$$\varphi_v = W_Q(C)\tilde{\varphi}_v = \exp\left(iQ\int_0^\infty v^\mu A_\mu(sv)\right)\tilde{\varphi}_v\,. \quad (3.3.32)$$

This justifies our claim in eq. (3.3.20) that the amplitude of charged particles can be related to a correlation function of Wilson lines. Furthermore, by the

[4]These can be extracted using projection operators $\varphi_v^\pm = P_\pm\phi$ which depend on the spin of the particle. For instance, for scalars $P_\pm \propto (iv\cdot D \pm m)$ and for Dirac fermions $P_\pm \propto 1 \pm v_\mu\gamma^\mu$.

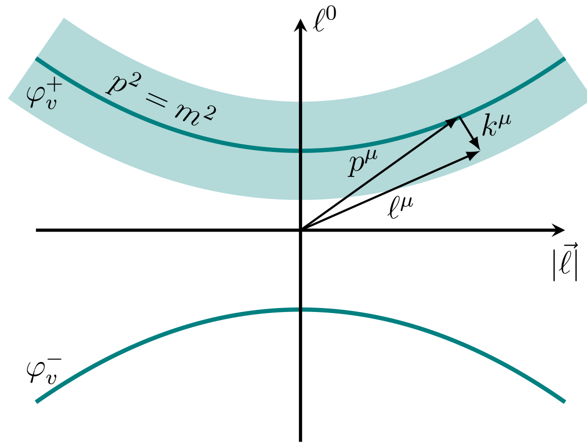

Figure 3.1: Illustration of the decomposition of a momentum $\ell^\mu = p^\mu + k^\mu$, as a large component on the mass shell, p^μ, and a small component $k^\mu \ll p^\mu$. The field φ_v^+ describes small fluctuations around the mass shell in the shaded region, whereas all modes outside this region are integrated out in the effective theory.

identity[5]

$$D_\mu \varphi_v = W_Q(C) \left(\partial_\mu + \frac{1}{v \cdot \partial} F_{\mu\nu} v^\nu \right) \tilde{\varphi}_v \, , \tag{3.3.33}$$

it removes the dependence on bare gauge fields without derivatives in the effective Lagrangian

$$\mathcal{L} = \tilde{\varphi}_v^* (v \cdot \partial) \tilde{\varphi}_v + \tilde{\varphi}_v^* \frac{1}{2m} \left(\partial_\mu^\perp + \frac{1}{v \cdot \partial} F_{\mu\nu} v^\nu \right)^2 \tilde{\varphi}_v + d \, \varphi_v^* \frac{F_{\mu\nu} M^{\mu\nu}}{4m} \varphi_v + \cdots \, , \tag{3.3.34}$$

which now manifestly exhibits the shift symmetry in the gauge field eq. (3.3.9), as it only depends on field strengths. In fact, this observation is not restricted to the leading powers in the $k/m \ll 1$ limit, but holds to all orders. We have shown that, as expected, the one-form symmetry emerges as an exact symmetry in the heavy-particle limit, which is dual to the soft photon limit, justifying our analysis of soft photon theorems.

While we have focused here on the case of emergent $U(1)$ 1-form symmetries in theories with an abelian gauge field, we expect this to be a generic feature of theories in which anti-particle production is suppressed and Wilson lines play

[5]The notation with $1/(v \cdot \partial)$ is shorthand for operator insertions on the Wilson line, e.g.,

$$\frac{1}{v \cdot \partial} F_{\mu\nu}(x) = \int_0^\infty ds \, F_{\mu\nu}(x + sv) \, .$$

a significant role. For instance, we think it is likely that the discrete 1-form symmetries of pure Yang-Mills theory emerge as symmetries of HQET or of the ultrasoft sector of Soft Collinear Effective Theory (SCET) [168].

3.4 New soft theorem from higher-group symmetry

The fact that higher-form symmetries are always abelian means that the operator equations satisfied by their currents do not contain non-trivial contributions at coincident points,[6] and hence there are no interesting double soft theorems for the associated NGB. However there exists a generalized symmetry structure known as higher-group global symmetry which mixes global symmetries of different rank in a non-trivial way [39], [141], [144], [169]–[171]. We will focus on the case of 2-group symmetry and show that, when spontaneously broken, such structure leads to interesting double soft theorems.

The hallmark of a continuous 2-group symmetry is an additional contact term in the Ward identity of 0-form currents which depends on the 1-form symmetry current

$$\partial^\mu j^a_\mu(x) j^b_\nu(y) \supset i\frac{\kappa}{2\pi}\delta^{ab}\partial^\lambda\delta^{(4)}(x-y)J_{\nu\lambda}(y), \tag{3.4.1}$$

where κ is a quantized constant. The 2-group symmetry is denoted by $G \times_\kappa U(1)^{(1)}$, where G is generated by the j^a and $U(1)^{(1)}$ is a 1-form symmetry generated by J. Due to the form of the Ward identity the 0-form currents do not form a closed subalgebra, so the group G does not label a 0-form symmetry. One might think of the 2-group as a non-trivial extension of the 0-form G by the 1-form $U(1)^{(1)}$.[7]

Intuitively, one might think of κ as an f^{abc} structure constant in which one of the indices points in the 1-form "direction". Hence, when the symmetry is spontaneously broken one can expect that the double soft theorem will describe mixing of the 0-form NGB (pions) and the 1-form NGB (photons). Furthermore, the fact that the contact term in eq. (3.4.1) has an additional derivative suggests that the 2-group structure will be visible at the sub-leading order. We will see that this is precisely correct.

[6] 't Hooft anomalies are, of course, the exception to this rule.

[7] The 0-form analog of this is the fact that a coset G/H need not be a subgroup of G, but one can think of G as an extension of G/H by H.

3.4.1 2-group double soft pion theorem

In what follows we focus on a theory possessing a 2-group global symmetry,

$$(G_L \times G_R) \times_\kappa U(1)^{(1)}_m \,. \tag{3.4.2}$$

where $G_L \times G_R$ is a non-abelian chiral group, and $U(1)^{(1)}_m$ a magnetic 1-form symmetry. When this 2-group symmetry structure is spontaneously broken to G_V, the low-energy dynamics are universally described at low energies by a theory of NGB featuring pions π^a associated to $G_L \times G_R$, as well as a photon associated to the $U(1)^{(1)}_m$, and currents

$$\mathcal{J}^a_\mu(x) = f_\pi \partial_\mu \pi^a(x) + \cdots \qquad \text{and} \qquad J_{\mu\nu}(x) = \frac{1}{4\pi}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}(x)\,, \tag{3.4.3}$$

as well as a vector current $V^a_\mu(x)$ quadratic in the fields. The 2-group is encoded in the structure of the Ward identities

$$\partial^\mu V^a_\mu(x) V^b_\nu(y) = i f^{abc}_T \delta^{(4)}(x-y) V^c_\nu(y) \tag{3.4.4}$$

$$\partial^\mu \mathcal{J}^a_\mu(x) \mathcal{J}^b_\nu(y) = i F^{abc} \delta^{(4)}(x-y) V^c_\nu(y) \tag{3.4.5}$$

$$\partial^\mu \mathcal{J}^a_\mu(x) V^b_\nu(y) = i f^{abc}_X \delta^{(4)}(x-y) \mathcal{J}^c_\nu(y) + i\frac{\kappa}{2\pi}\delta^{ab}\partial^\lambda \delta^{(4)}(x-y) J_{\nu\lambda}(y)\,. \tag{3.4.6}$$

The new term appearing in eq. (3.4.6) contains the symmetry current associated to magnetic 1-form symmetry $U(1)^{(1)}_m$ and the quantized constant κ.

The effective Lagrangian realizing such symmetry breaking pattern[39] is,

$$\begin{aligned}\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} \\ - \frac{i\kappa}{24\pi^2}\,\epsilon^{\mu\nu\alpha\beta}A_\mu \mathrm{Tr}\left[(iU^\dagger\partial_\nu U)(iU^\dagger\partial_\alpha U)(iU^\dagger\partial_\beta U)\right] + \cdots\,,\end{aligned} \tag{3.4.7}$$

Comparing to the action for 0-form spontaneous breaking eq. (3.2.18), we have added a dynamical $U(1)$ gauge field – a photon – and a term which couples it to the pion. One consequence of this coupling is to explicitly break the putative 1-form $U(1)^{(1)}_e$ symmetry associated to the photon. However $U(1)^{(1)}_m$ is the 1-form symmetry participating in the 2-group eq. (3.4.2). This means that strictly speaking the photon is a pseudo-NGB and the dual photon is a NGB.

The pions are uncharged under the $U(1)$ gauge group, however the pion-photon interaction term in eq. (3.4.7) is a coupling of the photon to a topological symmetry current,

$$B^\mu = i\frac{\kappa}{24\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[(iU^\dagger\partial_\nu U)(iU^\dagger\partial_\alpha U)(iU^\dagger\partial_\beta U)\right]. \tag{3.4.8}$$

In the absence of a dynamical photon, and hence in the absence of the 2-group global symmetry, the current B^μ would generate a topological 0-form $U(1)$ global symmetry.

This theory is far from exotic, as for $G_L \times G_R = SU(N_f)_L \times SU(N_f)_R$ it describes the low-energy limit of massless QCD with gauged $U(1)_V$ vector symmetry corresponding to Baryon number, which acts diagonally on quarks.[8] Such symmetry, with current (3.4.8), is associated to the non-trivial homotopy group $\pi_3(SU(N_f)) = \mathbb{Z}$, and the field configurations where B^μ integrates to a non-trivial winding number are identified with baryons [172]. In fact, the photon-pion coupling term was first suggested in [173], where it was introduced as an anomalous contribution to the baryon current in the chiral effective theory.[9] As explained in [144], 2-group global symmetries may arise from gauging a 0-form global symmetry with a mixed anomaly. This is possible, for example, when the original theory possesses a mixed 't Hooft anomaly which is quadratic in a non-abelian G and linear in a $U(1)$ symmetry, which is then gauged. Indeed this is the case in massless QCD where there is a mixed anomaly between the $U(1)_V$ and $SU(N_f)_L \times SU(N_f)_R$ which gives rise to the 2-group structure.

Let us comment on the discrete symmetries of this theory. In the absence of the pion-photon interaction term the theory has four distinct $\mathbb{Z}_2$ symmetries: parity, $P_0 : x_i \mapsto -x_i$ for $i = 1, 2$, and 3, charge conjugation of the photon $C_1 : A_\mu \mapsto -A_\mu$, charge conjugation of the chiral field $C_2 : U \mapsto U^T$, and pion number mod-2 $(-1)^{N_\pi} : U \mapsto U^{-1}$ (or $\pi^a \to -\pi^a$). The pion-photon interaction breaks one of these, leaving the $(\mathbb{Z}_2)^3$ discrete symmetry associated to the combinations [39]

$$P = P_0(-1)^{n_\pi}, \qquad C = C_1C_2, \qquad \widetilde{C} = C_1(-1)^{n_\pi}\,, \tag{3.4.9}$$

which include a new parity, P, and charge conjugation C, as well as pion + photon number mod-2, $\widetilde{C}$. We see that the 2-group theory allows, for example, scattering amplitudes involving an odd-number of pions so long as there is also an odd-number of photons such that $\widetilde{C}$ is conserved.

[8] Technically, one must also add a Wess-Zumino-Witten term to match the various 't Hooft anomalies in the chiral symmetry, but we drop this here for simplicity.

[9] Note that our photon is not the usual photon of electromagnetism, which arises from gauging a linear combination of $U(1)_V$ and a $U(1)$ subgroup of $SU(N_f)_V$. An easy way to see this is that the pions in our effective theory are not charged under $U(1)_V$.

The main result of this paper is a new double soft pion theorem for amplitudes with $n_\pi + 2$ pions and n_γ photons stemming from the continuous 2-group symmetry, which takes the form

$$\lim_{q_1,q_2\to 0} \mathcal{A}_{(n_\pi+\pi^a(q_1)\pi^b(q_2),n_\gamma)} = \left(S^{(0)} + S^{(1)} + S^{(1)}_\kappa\right) \mathcal{A}_{(n_\pi,n_\gamma)} \tag{3.4.10}$$

where the $S^{(i)}$ are those given in eq. (3.2.33) and

$$\begin{aligned} S^{(1)}_\kappa \mathcal{A}_{(n_\pi,n_\gamma)} =& \frac{i\kappa}{2f_\pi^3\pi^2}\sum_{i=1}^{n_\pi}\sum_{\mathrm{h}} f^{aba_i}\frac{\epsilon(q_1q_2p_i\varepsilon^*_{ih})}{2p_i\cdot(q_1+q_2)}\mathcal{A}^{a_1\ldots a_{i-1}a_{i+1}\ldots a_n}_{(n_\pi-1,n_\gamma+1)} \\ &- \frac{i\kappa}{2f_\pi^3\pi^2}\sum_{j=1}^{n_\gamma} f^{abd}\frac{\epsilon(q_1q_2k_j\varepsilon_j)}{2k_j\cdot(q_1+q_2)}\mathcal{A}^{da_1\ldots a_n}_{(n_\pi+1,n_\gamma-1)}\,, \end{aligned} \tag{3.4.11}$$

where in the first line we sum over helicities of internal photons and we introduced the notation $\epsilon(abcd) = \epsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma$. This new soft factor $S^{(1)}_\kappa$ is the consequence of the spontaneously broken 2-group global symmetry. Its salient feature is that it is odd under parity and thus allow for external pions 'rotating' into photons and vice versa. As we will see, the presence of those terms is required precisely by the appearance of the new term with the 1-form symmetry current in the 2-group Ward identities in eq. (3.4.6). We also note that the non-trivial 2-group soft factor depends on the antisymmetric non-abelian structure constant of the 0-form group, and thus at this order we do not expect an analogous non-trivial 2-group soft factor for abelian 2-groups [32].

3.4.2 Proof

We will now explain how the derivation of the double soft theorem is augmented by the 2-group. We emphasize that since the proof uses current algebra it is valid to all loop orders. As in section 3.2, our starting point is the following consequence of the Ward identity

$$q_1^\mu q_2^\nu \langle\alpha|\mathcal{J}^a_\mu(q_1)\mathcal{J}^b_\mu(q_2)\rangle = -\frac{1}{2}F^{abc}(q_1-q_2)^\mu\langle\alpha|V^c_\mu(q_1+q_2)\rangle. \tag{3.4.12}$$

Notice that this relation is unmodified by the 2-group current algebra, since the Ward identity involving two axial currents is unmodified. Thus, the intermediate relation eq. (3.2.27) follows in the new theory,

$$\begin{aligned} f_\pi^2\langle\alpha+\pi^a(q_1)\pi^b(q_2)|0\rangle =& -\frac{1}{2}F^{abc}(q_1-q_2)^\mu\langle\alpha|V^c_\mu(q_1+q_2)\rangle \\ &- q_1^\mu q_2^\nu\langle\alpha|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)\rangle\,. \end{aligned}$$

Next, we need to analyze additional contributions to form factors $\langle\alpha|V^c_\mu(q)\rangle$ and $\langle\alpha|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)\rangle$ in the new theory.

3.4.2.1 Soft vector form factor

We follow the same strategy as in the case of spontaneous 0-form symmetry breaking (see Appendix B.1) and insert a complete set of states into the form factor, where $\langle\alpha|$ now denotes a state with n pions and m photons. The dominant contribution in the soft limit comes from single-particle states. We have

$$\lim_{q\to 0}\langle\alpha|V^a_\mu(q)\rangle = \sum_{i=1}^{n+m}\sum\!\!\!\!\!\!\!\int_X \langle X_i|V^a_\mu(q)|X\rangle\Delta_X\langle X+\hat{\alpha}_i|0\rangle\,, \tag{3.4.13}$$

where by $\hat{\alpha}_i$ we denote the state α with the i-th particle X_i removed, $|X\rangle$ stands for a single-particle state of either pions or photons, and Δ_X is the corresponding propagator.

The $\langle\pi^b(p)|V^a_\mu(q)|\pi^c(k)\rangle$ form factor is the same as in the 0-form case derived in Appendix B.1. Since there is no flavor-structure corresponding to $\langle\gamma_h(p)|V^a_\mu(q)|\gamma_{h'}(k)\rangle$, the form factor vanishes. Thus the only additional form factor we need is $\langle\pi^b(p)|V^a_\mu(q)|\gamma_h(k)\rangle$, which at leading order in soft momentum can be parametrized as

$$\langle\pi^b(p)|V^a_\mu(q)|\gamma_h(k)\rangle = A(p,q)B^{ab}\epsilon_{\mu\nu\rho\sigma}q^\nu p^\rho\varepsilon^{*\sigma}_h + \mathcal{O}(q^2)\,, \tag{3.4.14}$$

where $k=q+p$, $A(p,q)$ is a Lorentz-invariant structure function and B^{ab} is an arbitrary flavor-structure. Here we have specialized to a parity-odd ansatz for the form factor. As discussed above, the theory is invariant under the product of any two of the four $\mathbb{Z}_2$ transformations: C_1, C_2, P_0, and $(-1)^{N_\pi}$. It follows that the form factor eq. (3.4.14) must be parity-odd. We can constrain the coefficient in the ansatz by considering a related object $\langle\mathcal{J}^b_\nu(p)V^a_\mu(q)|\gamma_h(k)\rangle$ and its axial current decomposition.

We write an ansatz for $\langle\mathcal{J}^b_\nu(p)V^a_\mu(q)|\gamma_h(k)\rangle$ and impose the Ward identity for the axial current

$$p^\nu\langle\mathcal{J}^b_\nu(p)V^a_\mu(q)|\gamma_h(k)\rangle = i\frac{\kappa}{2\pi}\delta^{ab}p^\nu\langle J_{\mu\nu}(p+q)|\gamma_h(k)\rangle\,, \tag{3.4.15}$$

where we used the fact that $\langle\mathcal{J}_\mu(p+q)|\gamma_h(p+q)\rangle = 0$. Similarly, we require that the Ward identity for the vector current is satisfied. Those two conditions allow us to constrain the correlator at the leading order in soft momentum

$$\langle V^a_\mu(q)\mathcal{J}^b_\nu(p)|\gamma_h(k)\rangle =\delta^{ab}\frac{i\kappa}{8\pi^2 f_\pi}\left(\epsilon_{\mu\nu\sigma\lambda}(p+q)^\lambda - \frac{2p_\nu}{p^2}\epsilon_{\mu\lambda\rho\sigma}q^\lambda p^\rho\right)\varepsilon^{*\sigma}_h + \mathcal{O}(q^2,p^2)\,, \tag{3.4.16}$$

which in turn implies, using eq. (3.2.19), that

$$\langle\pi^b(p)|V^a_\mu(q)|\gamma_h(k)\rangle = -\delta^{ab}\frac{\kappa}{4\pi^2 f_\pi}\epsilon_{\mu\nu\rho\sigma}q^\nu p^\rho\varepsilon^{*\sigma}_h + \mathcal{O}(q^2)\,. \tag{3.4.17}$$

Finally, putting everything together we arrive at the soft limit of the vector current in the theory with a 2-group global symmetry,

$$\begin{aligned}\lim_{q\to 0}\langle\alpha|V^a_\mu(q)\rangle =& \sum_{i=1}^{n_\pi} f^{aa_i d}\left(\frac{(2p_i+q)_\mu}{(p_i+q)^2} - \frac{iq^\nu L_{i\mu\nu}}{(p_i\cdot q)}\right)\mathcal{A}^{a_1\ldots d\ldots a_{n_\pi}}_{(n_\pi,n_\gamma)}\\ &- \frac{i\kappa}{4\pi^2 f_\pi}\sum_{i=1}^{n_\pi}\sum_h \delta^{aa_i}\frac{\epsilon_{\mu\nu\rho\sigma}q^\nu p_i^\rho\varepsilon^{*\sigma}_i}{2p_j\cdot q}\mathcal{A}^{a_1\ldots a_{i-1}a_{i+1}\ldots a_{n_\pi}}_{(n_\pi-1,n_\gamma+1)}\\ &+ \frac{i\kappa}{4\pi^2 f_\pi}\sum_{j=1}^{n_\gamma}\delta^{aa_i}\frac{\epsilon_{\mu\nu\rho\sigma}q^\nu k_j^\rho\varepsilon^{\sigma}_j}{2k_j\cdot q}\mathcal{A}^{da_1\ldots a_{n_\pi}}_{(n_\pi+1,n_\gamma-1)}\,,\end{aligned} \tag{3.4.18}$$

where $\langle\alpha|$ again denotes a state with n_π pions and n_γ photons. The first line of eq. (3.4.18) is the contribution from the $G_L\times G_R$ symmetry as derived in Appendix B.1.

3.4.2.2 Soft axial-axial hard current form factor

Repeating the same analysis for the soft axial-axial hard current form factor $\langle\alpha|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)\rangle$ we see that the new relevant form factor we need is $\langle\pi^c(p)|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)|\gamma_h(k)\rangle$,[10] which can be obtained from the 2-group current algebra. As is consistent with the discrete $\mathbb{Z}_2$ symmetry structure of the theory, we start with the parity-odd ansatz

$$\langle\pi^c(p)|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)|\gamma_h(k)\rangle = A(p,q_1,q_2)B^{abc}\epsilon_{\mu\nu\rho\sigma}p^\rho\varepsilon^{*\sigma}_h + \mathcal{O}(q_1,q_2)\,, \tag{3.4.19}$$

where $A(p,q_1,q_2)$ is a Lorentz-invariant structure function and B^{abc} is an arbitrary flavor-structure. We consider a related object,

$$\langle\mathcal{J}^c_\lambda(p)\mathcal{J}^a_\mu(q_1)\mathcal{J}^b_\nu(q_2)|\gamma_h(k)\rangle\,,$$

[10] The form factor $\langle\gamma_h|\mathcal{J}_{H}{}^a_\mu\mathcal{J}_{H}{}^b_\nu|\gamma_{h'}\rangle$ is suppressed in the soft limit, as can be deduced from an analogous bootstrap argument imposing Ward identities. One can also explicitly perform a perturbative check and see that the loop diagrams contributing to $\langle\gamma_h|\mathcal{J}_{H}{}^a_\mu\mathcal{J}_{H}{}^b_\nu|\gamma_{h'}\rangle$ are suppressed in the soft limit.

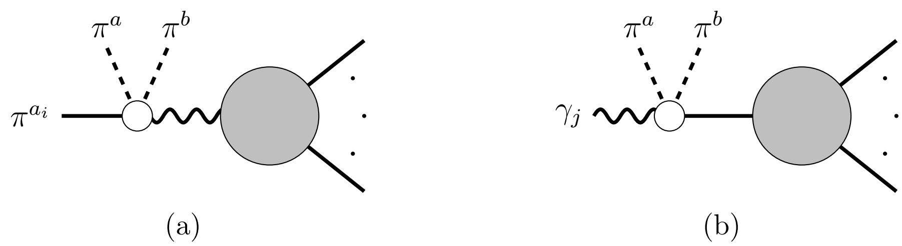

Figure 3.2: Graphical representation of the singular contributions comprising the 2-group soft factor $S_\kappa^{(1)}$ in eq. (3.4.10) and eq. (3.4.11).

which we constrain using the Ward identity

$$p^\lambda \langle \mathcal{J}_\lambda^c(p) \mathcal{J}_\mu^a(q_1) \mathcal{J}_\nu^b(q_2) | \gamma_h(k) \rangle = F^{cae} \langle V_\mu^e(p+q_1) \mathcal{J}_\nu^b(q_2) | \gamma_h(k) \rangle + F^{cbe} \langle \mathcal{J}_\mu^a(q_1) V_\nu^e(p+q_2) | \gamma_h(k) \rangle \,, \quad (3.4.20)$$

where the RHS can be evaluated using the result from the previous section eq. (3.4.16). As before, from axial current decomposition we can deduce at leading order in soft momenta

$$\langle \pi^c(p) | \mathcal{J}_{H\mu}^{\,a}(q_1) \mathcal{J}_{H\nu}^{\,b}(q_2) | \gamma_h(k) \rangle = -F^{abc} \frac{\kappa}{4\pi^2 f_\pi} \epsilon_{\mu\nu\rho\sigma} p^\rho \varepsilon_h^{*\sigma} + \mathcal{O}(q_1, q_2) \,. \quad (3.4.21)$$

Therefore we have the 2-group soft axial-axial remnant theorem

$$\begin{aligned} \lim_{q_1 q_2 \to 0} \langle \alpha | \mathcal{J}_{H\mu}^{\,a}(q_1) \mathcal{J}_{H\nu}^{\,b}(q_2) \rangle = & \sum_{i=1}^{n_\pi} (F^{aa_ie} f_X^{ebd} + F^{ba_ie} f_X^{ead}) \frac{\eta_{\mu\nu}}{2p_i \cdot (q_1+q_2)} \mathcal{A}_{(n_\pi, n_\gamma)}^{a_1 \ldots d \ldots a_n} \\ & - \frac{i\kappa}{4\pi^2 f_\pi} \sum_{i=1}^{n_\pi} F^{aba_i} \frac{\epsilon_{\mu\nu\rho\sigma} p_i^\rho \varepsilon_i^{*\sigma}}{2p_i \cdot (q_1+q_2)} \mathcal{A}_{(n_\pi-1, n_\gamma+1)}^{a_1 \ldots a_{i-1} a_{i+1} \ldots a_n} \\ & + \frac{i\kappa}{4\pi^2 f_\pi} \sum_{j=1}^{n_\gamma} F^{abd} \frac{\epsilon_{\mu\nu\rho\sigma} k_j^\rho \varepsilon_j^\sigma}{2k_j \cdot (q_1+q_2)} \mathcal{A}_{(n_\pi+1, n_\gamma-1)}^{da_1 \ldots a_n} \,, \end{aligned} \quad (3.4.22)$$

where again $\langle \alpha |$ denotes a state with n_π pions and n_γ photons. The first line of eq. (3.4.22) is the contribution from the $G_L \times G_R$ symmetry as derived in Appendix B.2.

Finally, plugging in eq. (3.4.18) and eq. (3.4.22) into eq. (3.4.13), we obtain the full double soft theorem eq. (3.4.10). The 2-group contributions in eq. (3.4.11) are represented graphically in Fig. 3.2.

3.4.3 Examples

Here we provide some examples of tree-level amplitudes in the theory in eq. (3.4.7). We have verified the double soft theorem in all amplitudes with up to six external particles, with any allowed combinations of pions and photons. We only present explicit checks in some simple cases.

The first nontrivial amplitudes appear at 4-point, $\mathcal{A}_{(4,0)}$ and $\mathcal{A}_{(3,1)}$, and all other 4-point tree-level amplitudes vanish. The four pion amplitude is the same as in the NLSM eq. (3.2.18) for spontaneous 0-form symmetry breaking. The amplitude with three pions (with momenta p_1, p_2, p_3) and a photon (momentum p_4) is

$$\mathcal{A}_{(3,1)}^{a_1a_2a_3}(p_1,p_2,p_3,p_4) = i\mathcal{K}f^{a_1a_2a_3}\epsilon(p_1p_2p_4\varepsilon_4)\,, \qquad \mathcal{K} = \frac{\kappa}{2\pi^2 f_\pi^3}\,, \tag{3.4.23}$$

where we used $\mathcal{K}$ to denote a common combination of constants.

To illustrate the double soft theorem in a simple example, we consider an amplitude with four pions (momenta $p_1, \ldots, p_4$) and two photons (momenta p_5 and p_6) given by

$$\begin{aligned}\mathcal{A}_{(4,2)}^{a_1a_2a_3a_4} =& f^{a_1a_2c}f^{ca_3a_4}\mathcal{K}^2\left[\frac{\epsilon(p_1p_2p_5\varepsilon_5)\epsilon(P_{125}p_3p_6\varepsilon_6)}{s_{12}+s_{15}+s_{25}} + \frac{\epsilon(p_1p_2p_6\varepsilon_6)\epsilon(P_{126}p_3p_5\varepsilon_5)}{s_{12}+s_{16}+s_{26}}\right] \\ &+ (1\leftrightarrow 3) + (2\leftrightarrow 3)\,, \end{aligned} \tag{3.4.24}$$

where $P_{ijk}^\mu = p_i^\mu + p_j^\mu + p_k^\mu$ and we introduced Mandelstam variables $s_{ij} = 2p_i \cdot p_j$. Taking the double soft limit of two pions, $p_1 \to 0$ and $p_2 \to 0$, we obtain

$$\begin{aligned}&\lim_{p_1,p_2\to 0}\mathcal{A}_{(4,2)}^{a_1a_2a_3a_4} \\ &\quad = f^{a_1a_2c}f^{ca_3a_4}\mathcal{K}^2\left[\frac{\epsilon(p_1p_2p_5\varepsilon_5)\epsilon(p_5p_3p_6\varepsilon_6)}{s_{15}+s_{25}} + \frac{\epsilon(p_1p_2p_6\varepsilon_6)\epsilon(p_6p_3p_5\varepsilon_5)}{s_{16}+s_{26}}\right],\end{aligned} \tag{3.4.25}$$

which is at next-to-leading order $\mathcal{O}(p_i)$ in soft momenta.

Now we evaluate the double soft limit of $\mathcal{A}_{(4,2)}$ using the soft theorem in eq. (3.4.10). Note that since $\mathcal{A}_{(2,2)}$ and $\mathcal{A}_{(1,3)}$ vanish, we only get non-trivial contributions from the soft operator in eq. (3.4.11) acting on external photons

$$\begin{aligned}S_\kappa^{(1)}\mathcal{A}_{(2,2)} =& -i\mathcal{K}f^{a_1a_2c}\left[\frac{\epsilon(q_1q_2p_5\varepsilon_5)}{s_{15}+s_{25}}\mathcal{A}_{(3,1)}^{ca_3a_4}(p_5,p_3,p_4,p_6)\right. \\ &\left. + \frac{\epsilon(q_1q_2p_6\varepsilon_6)}{s_{16}+s_{26}}\mathcal{A}_{(3,1)}^{ca_3a_4}(p_6,p_3,p_4,p_5)\right]\end{aligned}$$

$$=\mathcal{K}^2 f^{a_1a_2c}f^{ca_3a_4}\left[\frac{\epsilon(q_1q_2p_5\varepsilon_5)}{s_{15}+s_{25}}\epsilon(p_5p_3p_6\varepsilon_6)+\frac{\epsilon(q_1q_2p_6\varepsilon_6)}{s_{16}+s_{26}}\epsilon(p_6p_3p_5\varepsilon_5)\right], \tag{3.4.26}$$

where in the second line we plugged in for the amplitudes using eq. (3.4.23). This recovers the soft limit obtained in eq. (3.4.25), showing that the double soft theorem is satisfied.

Next, consider the 6-point pion amplitude, which decomposes into a NLSM part and $\mathcal{A}^{\kappa}_{(6,0)}$ denoting terms with two separate color-structures

$$\mathcal{A}_{(6,0)} = \mathcal{A}^{\mathrm{NLSM}}_{(6,0)} + \mathcal{A}^{\kappa}_{(6,0)}\,. \tag{3.4.27}$$

The contribution $\mathcal{A}^{\kappa}_{(6,0)}$ is a sum over ten factorization channels with different color-structures. Writing the (123)(456) channel explicitly, we have

$$\begin{aligned}\mathcal{A}^{\kappa}_{(6,0)} = \frac{f^{a_1a_2a_3}f^{a_4a_5a_6}\mathcal{K}^2}{8(s_{12}+s_{13}+s_{23})}\Big[&s_{34}(-s_{15}s_{23}+s_{13}s_{25})+s_{35}(-s_{13}s_{24}+s_{14}s_{23})\\ &+s_{12}s_{34}(s_{25}-s_{15})+s_{12}s_{35}(s_{14}-s_{24})\\ &+(s_{13}+s_{23})(s_{15}s_{24}-s_{14}s_{25})\Big]+\cdots\,,\end{aligned} \tag{3.4.28}$$

where the dots denote the other nine channels, which can be obtained by permutation.

Now we consider the double soft limit of the six pion amplitude in eq. (3.4.27) and keep the terms to sub-leading order. The $\mathcal{A}^{\mathrm{NLSM}}_{(6,0)}$ satisfies the NLSM double soft theorem; here we focus on $\mathcal{A}^{\kappa}_{(6,0)}$. Clearly, the double soft limit of eq. (3.4.28) vanishes at leading order in soft momenta $\mathcal{O}(q^0)$. At next-to-leading order, only the terms with a soft pole survive, and so four channels contribute. Again, focusing on the (123)(456) channel, the double soft limit yields

$$\begin{aligned}\lim_{p_1,p_2\to 0}\mathcal{A}^{\kappa}_{(6,0)} =&\frac{1}{8}\frac{f^{a_1a_2a_3}f^{a_4a_5a_6}\mathcal{K}^2}{s_{13}+s_{23}}\Big[s_{34}(-s_{15}s_{23}+s_{13}s_{25})+s_{35}(-s_{13}s_{24}+s_{14}s_{23})\Big]\\ &+(3\leftrightarrow 4)+(3\leftrightarrow 5)+(3\leftrightarrow 6)\,,\end{aligned} \tag{3.4.29}$$

where we used that the second and third lines in eq. (3.4.28) are higher order in soft momenta.

On the other hand, we can evaluate the new contributions to the double soft limit using the soft theorem in eq. (3.4.11), which is a sum of soft operator acting on all external particles. For instance, when the soft operator acts on the pion with momentum p_3, we have

$$S_\kappa^{(1)}(p_3)\mathcal{A}_{(4,0)} = i\mathcal{K}\sum_{\text{spins}}\frac{f^{a_1a_2a_3}\epsilon(p_1p_2p_3\varepsilon_3^*)}{s_{13}+s_{23}}\mathcal{A}_{(3,1)}^{a_4a_5a_6}(p_4,p_5,p_6,p_3)\,, \tag{3.4.30}$$

which we can evaluate using eq. (3.4.23). Hence we obtain

$$\begin{aligned} S_\kappa^{(1)}(p_3)\mathcal{A}_{(4,0)} &= -\mathcal{K}^2\sum_{\text{spins}}\frac{f^{a_1a_2a_3}\epsilon(p_1p_2p_3\varepsilon_3^*)}{s_{13}+s_{23}}f^{a_4a_5a_6}\epsilon(p_4p_5p_3e_3) \\ &= -\mathcal{K}^2\frac{f^{a_1a_2a_3}f^{a_4a_5a_6}}{s_{13}+s_{23}}\frac{1}{8}\Big[s_{34}(s_{15}s_{23}-s_{13}s_{25})+s_{35}(s_{13}s_{24}-s_{14}s_{23})\Big] \end{aligned} \tag{3.4.31}$$

which agrees with eq. (3.4.29). Repeating the same steps for the other three channels, we see explicitly that the double soft theorem holds for the 6-point pion amplitude.

3.5 Conclusions

In this paper, we explored the implications of spontaneously broken higher symmetries for the soft behavior of scattering amplitudes. Our principal result was the derivation of a new double soft theorem for NGB in theories possessing a spontaneously broken continuous 2-group global symmetry. This structure is characterized by a current algebra which mixes 0-form and 1-form symmetry currents. We showed that the corresponding Ward identities imply a universal subleading soft factor acting on lower point amplitudes. This soft factor contains terms where particles in the lower point amplitude change species, from the 0-form NGB pion to the 1-form NGB photon, and vice versa. We have also illustrated the new soft theorem with explicit examples of amplitudes in theories with such symmetry.

Along the way, we presented a unified picture for soft theorems from spontaneously broken symmetries. This allowed us to recast well-known results, such as the leading-order soft photon theorem as a consequence of 1-form symmetry emergent in the soft limit. In this limit, the energy of a soft NGB is much smaller than the energies of massive particles. The leading soft behavior of the amplitude is then captured by an EFT with Wilson lines of charged particles treated as background insertions, similar to those familiar from particle physics such as HQET [161], SCET [168], and their relatives. Our analysis shows that higher-form symmetries can generically emerge as accidental symmetries of such EFTs. It would be interesting explore the connections between this observation to factorization phenomena, and derive the associated soft theorems or selection rules on matrix elements. We expect that such an EFT perspective will also be useful in extending our analysis to sub-leading soft theorems.

Throughout the paper, we carried out the derivations of all soft theorems without making reference to diagrammatics or perturbation theory, and instead using the Ward identities of currents which take the form of ordinary local operators in spacetime. We have not attempted to connect our analysis to the derivation of soft theorems from asymptotic symmetries [128], [131], [133]. This seems a worthwhile exercise, which is left for the interested reader.

While our analysis of soft theorems focused on soft pions and photons, we expect that it will extend to other interesting cases. For instance, it is natural to consider the free gluons of non-abelian gauge theories at weak coupling as the (pseudo-)NGB of emergent 1-form symmetries in the zero-coupling limit, around which point their scattering amplitudes are well-defined (see [174], [175] for related observations). Hence, one can likely derive their soft theorems using analogous symmetry arguments. One might also wonder if a similar picture holds for soft gravitons, perhaps in connection to the symmetries in Refs. [176]–[179].

Finally, we believe that further soft theorems might be discovered using higher symmetry as a guiding principle, perhaps in connection to higher-rank or more exotic symmetries in other dimensions, as well as non-invertible symmetries [150], [151], [154], [180]–[185] which might be related to the existence of massless (or light) particles. We leave such explorations for another time.

Chapter 4

SOFT SCALARS IN EFFECTIVE FIELD THEORY

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4.1 Introduction

Low-energy modes are often related to the symmetry properties of a theory. In scattering amplitudes, this connection takes the form of a soft limit, where the momentum of a particle is sent to zero. If this limit exhibits a universal pattern, we declare it a soft theorem. Salient examples of such relations are the pion soft theorem—the Adler zero [4]—which is a consequence of the spontaneously-broken chiral symmetry, the soft theorem for gauge theories [37], [121], [122], which follows from charge conservation, and the graviton soft theorem [37], due to energy-momentum conservation. In general, a theory with a nonlinearly realized symmetry manifests this fact in scattering amplitudes through soft theorems. Also in condensed matter systems, such as solids, fluids, and superfluids, phonon soft theorems are direct consequences of spontaneous symmetry breaking [18], [119]. Finally, there is a close connection between soft theorems and asymptotic symmetries [128]–[138] (see also ref. [186] and references therein).

However, symmetry is not the only possible origin of these universal relations between scattering amplitudes. A geometric soft theorem for scalar effective field theories was derived solely as a consequence of the geometry of field space [13], which did not rely on any symmetry of the theory. In the simplest case with no potential, the geometric soft theorem takes the form

$$\lim_{q\to 0} A_{n+1} = \nabla_i A_n, \tag{4.1.1}$$

where A_n is an n-particle scattering amplitude, ∇_i is the field-space covariant derivative with respect to the vacuum expectation value (VEV), and i is the flavor index of the soft scalar. Mathematically, as explained in ref. [13], scattering amplitudes of scalars take values in the tangent bundle of the field-space manifold and the soft theorem is described by the familiar Levi-Civita connection on the tangent space. This geometric picture is general for any effective field theory and manifests the invariance of scattering amplitudes under changes of field basis.[1]

In this paper, we extend the geometric soft theorem for a massless scalar by allowing the scalar to couple to fermions and gauge bosons. The geometry must be extended to include the full field content of the theory, since we can perform field redefinitions for any field in our theory. Remarkably, this is precisely what we need to complete the geometric soft theorem, which takes a form similar to eq. (4.1.1) but with the upgrade

$$\nabla \to \bar{\nabla} = \partial + \Gamma^s + \Gamma^f + \Gamma^g\,, \tag{4.1.2}$$

i.e., the covariant derivative for the full field-space geometry which includes a connection for scalars, Γ^s, fermions, Γ^f, and gauge bosons, Γ^g. More precisely, the additional fields take values in a vector bundle over the field space, with an associated connection which features in the soft theorem.

We can also reverse this logic and use the new geometric soft theorem as justification for the extension of the geometric picture to include particles with spin. For example, the scalar soft theorem for a theory of scalars and fermions involves the connection $\bar{\Gamma}^p_{ir}$, where i is a scalar flavor index and p, r are fermion flavor indices. This shows that the definition of a scalar-fermion geometry is not simply a formal exercise but that it has physical consequences manifested in the soft scalar limit.

The geometric soft theorems have wide applicability and are realized in many theories of interest. For instance, when the massless scalars are Nambu-Goldstone bosons (NGBs), they generalize the Adler zero and describe the coupling of NGBs to other species. They also describe the dependence of amplitudes in supersymmetric theories on the VEV of scalar moduli [187]–[191]. Furthermore, they provide a vast generalization of the well-known low-energy theorems for a light Higgs (see, e.g., ref. [192], [193]). This is, of course, not an exhaustive list.

The paper is organized as follows. First, we review the geometry of field space for scalars, fermions, and gauge bosons. Then we derive the geometric soft theorem, valid for any effective field theory with a massless scalar. We present

[1] The geometric soft theorem has also found an interpretation in the context of celestial holography [135], [136].

the geometric soft theorem in three parts: first with only scalars, then with fermions, and last with gauge bosons. In the following section, we present a novel double soft theorem, where the momenta of two scalars are sent to zero. In this case, the soft theorem involves the curvature of the full field-space geometry, including components for fermions and gauge bosons. Then we present new double-soft theorems for fermions. These soft theorems are almost identical to the double-scalar soft theorem, up to the simple replacement of a kinematic factor. Numerous examples are listed in section 4.5. We end with a discussion and outlook.

4.2 Geometry of Field Space

We consider an effective theory that includes scalars, fermions, and gauge bosons. To low orders in the derivative expansion, the Lagrangian is

$$\begin{aligned}\mathcal{L} =& \frac{1}{2} h_{IJ}(\phi)(D_\mu\phi)^I(D^\mu\phi)^J - V(\phi) + i\frac{1}{2}k_{\bar{p}r}(\phi)(\bar{\psi}^{\bar{p}}\gamma^\mu \overleftrightarrow{D}_\mu\psi^r) \\ &+ i\omega_{\bar{p}rI}(\phi)(\bar{\psi}^{\bar{p}}\gamma_\mu\psi^r)(D^\mu\phi)^I - \bar{\psi}^{\bar{p}}\mathcal{M}(\phi)_{\bar{p}r}\psi^r + c_{\bar{p}r\bar{s}t}(\phi)(\bar{\psi}^{\bar{p}}\gamma_\mu\psi^r)(\bar{\psi}^{\bar{s}}\gamma^\mu\psi^t) \\ &- \frac{1}{4}g_{AB}(\phi)F^A_{\mu\nu}F^{B\mu\nu} + d_{\bar{p}rA}(\phi)(\bar{\psi}^{\bar{p}}\sigma^{\mu\nu}\psi^r)F^A_{\mu\nu} + \dots,\end{aligned} \tag{4.2.1}$$

where we allow for higher-derivative operators and operators with more fermion fields, but do not list them explicitly. To keep the discussion simple, we omit the CP-odd scalar–gauge-boson couplings, $-\frac{1}{4}\tilde{g}_{AB}(\phi)F^A_{\mu\nu}\tilde{F}^{B\mu\nu}$, but all results generalize straightforwardly when they are present. We group all fields with the same spin into multiplets, with scalar indices $I, J, \dots$, fermion indices $p, \bar{p}, \dots$, and gauge indices $A, B, \dots$ The functions $h_{IJ}(\phi)$, $V(\phi)$, $k_{\bar{p}r}(\phi)$ etc., are functions of the scalar fields. By introducing these functions, we group infinite families of higher-dimensional operators into compact structures [58]. This grouping of operators underlies the geometric construction. The scalars ϕ^I and fermions ψ^r can be charged under the gauge symmetry through the covariant derivative, which we describe in more detail below.

As in any effective field theory, the number of independent operators is less than the number of possible composite operators consistent with the symmetries of the theory. This is because integration-by-parts relations and field redefinitions can be used to write the Lagrangian in a form with a minimal number of operators, i.e., a nonredundant operator basis. This freedom of redefining the fields (at least when the field redefinition does not involve derivatives) takes on a geometric meaning, paralleling coordinate changes in differential geometry. The field-space geometry for scalars is by now a standard quantum-field-theory technique. See refs. [28], [41]–[44] for some early works on the connection between differential geometry and field redefinitions and refs. [40], [46], [55]–[64] for modern applications of the scalar geometry in effective field theories. Recently, this geometric picture has been extended to include both fermions and gauge bosons [47], [66], [67], [69], and several proposals attempt to extend the geometric description to accommodate field redefinitions with derivatives [47]–[52].

4.2.1 Scalars

The geometry of the scalar field space is dictated by the metric h_{IJ}. From this metric, we can derive the Christoffel symbol

$$\Gamma^I_{JK} = \frac{1}{2} h^{IL} \left(h_{JL,K} + h_{LK,J} - h_{JK,L} \right), \tag{4.2.2}$$

where $h_{IJ,K} = \partial_K h_{IJ}$, and the Riemann curvature

$$R_{IJKL} = h_{IM} \left(\partial_K \Gamma^M_{LJ} + \Gamma^M_{KN} \Gamma^N_{LJ} - (K \leftrightarrow L) \right). \tag{4.2.3}$$

The covariant derivative ∇_I uses the connection in eq. (4.2.2). The field-space geometry for scalars captures field redefinitions of the form $\phi \to F(\phi)$, where $F'(v) \neq 0$ at the VEV v^I, and was used to describe the geometric soft theorem for scalar effective field theories [13].

The scalar field in the Lagrangian, ϕ^I, can be used as an interpolating field between the vacuum and a one-particle state,

$$\langle p_i | \phi^I(x) | 0 \rangle = e^I_i(v) e^{ip\cdot x}, \tag{4.2.4}$$

where the momentum is on the mass shell, $p^2 = m_i^2(v)$, and $e^I_i(v)$ is the tetrad, which is defined from the metric

$$h_{IJ}(v) = e_{Ii}(v) e^i_J(v). \tag{4.2.5}$$

The tetrad is the wavefunction factor in the LSZ reduction formula. Its role is to canonically normalize and rotate between the flavor-eigenstate fields in the Lagrangian and the mass eigenstates used in scattering amplitudes. Therefore, a scattering amplitude is a tensor with lowercase tetrad indices. Further details on the geometric construction for scalars can be found in ref. [13].

4.2.2 Fermions

We follow the setup in ref. [66] to describe fermions geometrically. A similar approach, but with certain differences in the technical steps, is described in refs. [67], [68]. The main novelty for the fermion geometry compared with the scalar geometry discussed above is that we now must accommodate anti-commuting fields into the geometric picture. This can be conveniently done by replacing the Riemannian manifold with a supermanifold, which involves Grassmann coordinates [194]. Note that the notion of a supermanifold is distinct from supersymmetry, and we do not require our theories to possess supersymmetry.

The fermion geometry is defined by the metric[2]

$$\bar{g}_{\mathfrak{ij}} = \begin{pmatrix} h_{IJ} & (\bar{\psi}\omega^-)_{rI} & (\omega^+\psi)_{\bar{r}I} \\ -(\bar{\psi}\omega^-)_{pJ} & 0 & k_{\bar{r}p} + c_{\bar{r}p} \\ -(\omega^+\psi)_{\bar{p}J} & -k_{\bar{p}r} - c_{\bar{p}r} & 0 \end{pmatrix}, \tag{4.2.6}$$

where $\omega^{\pm}_{\bar{p}rI} = \omega_{\bar{p}rI} \pm \frac{1}{2}k_{\bar{p}r,I}$. The scalar indices $I, J, \ldots$ and the fermion indices $p, \bar{p}, \ldots$ are unified in the indices $\mathfrak{i}, \mathfrak{j}, \ldots$. The metric and descendant quantities are denoted with a bar to distinguish them from the corresponding quantities in the scalar geometry.

Four-fermion operators were not included in the geometric construction in ref. [66]. We include them in the metric in eq. (4.2.6) through the term $c_{\bar{p}r} = 4(c_{\bar{p}r\bar{s}t} + c_{\bar{p}t\bar{s}r})\psi^t\bar{\psi}^{\bar{s}}$. There are several reasons why this construction is sensible. First, the four-fermion operators transform as tensors under redefinitions of the fermion fields that depend on the scalar fields. Thus, they are fine objects to add to the metric, as they do not spoil any of the transformation properties used to bootstrap the metric for the two-fermion sector. Second, the other operators which make up the scalar-fermion metric are combinations of two scalar currents or one scalar current and one fermion current. Thus, it is natural to expect that operators with two fermion currents can also reside in the metric. Lastly, in the supersymmetric nonlinear sigma model, the coefficient of the four-fermion operator is the Riemann curvature. Therefore, these operators must be included in the metric even for a general theory without

[2] Following the conventions of ref. [194], the metric should be written as ${}_{\mathfrak{i}}\bar{g}_{\mathfrak{j}}$, and shifting the indices to the right will pick up additional signs, $\bar{g}_{\mathfrak{ij}} = (-1)^{\mathfrak{i}}\ {}_{\mathfrak{i}}\bar{g}_{\mathfrak{j}}$. Here, we exclusively deal with the metric and allow ourselves to abuse the notation by having the indices on the right from the start.

supersymmetry, since the supersymmetric theory should be obtainable from the general theory by picking the correct field content and tuning the coefficients. The virtue of this definition will be apparent when we consider single and double soft theorems of scalars and fermions.

From this metric we can also calculate the Christoffel symbol and the curvature, but with the definitions appropriate for a supermanifold. In particular, the relevant connection coefficients are [66]

$$\bar{\Gamma}^K_{IJ} = \Gamma^K_{IJ}, \tag{4.2.7}$$

$$\bar{\Gamma}^p_{Ir} = k^{p\bar{s}}\omega^+_{\bar{s}rI}, \tag{4.2.8}$$

$$\bar{\Gamma}^{\bar{p}}_{I\bar{r}} = -\omega^-_{\bar{r}sI}k^{s\bar{p}}, \tag{4.2.9}$$

and the corresponding curvatures are

$$\bar{R}_{\bar{p}rIJ} = \omega_{\bar{p}rJ,I} + \omega^-_{\bar{p}sI}k^{s\bar{t}}\omega^+_{\bar{t}rJ} - (I \leftrightarrow J), \tag{4.2.10}$$

$$\bar{R}_{\bar{p}r\bar{s}t} = 4(c_{\bar{p}r\bar{s}t} + c_{\bar{p}t\bar{s}r}), \tag{4.2.11}$$

all evaluated at the VEV. The covariant derivative $\bar{\nabla}$ uses the connections in eqs. (4.2.7) to (4.2.9). For our purposes, where we analyze the geometric structure of scattering amplitudes, we only need the geometric quantities evaluated at the VEV. Other applications, such as background-field calculations [61], [64], [66], also use the geometric information away from the VEV.

Similar to the scalars above, the flavor-basis field $\bar{\psi}^{\bar{R}}$ sandwiched between the one-particle fermion state and the vacuum is

$$\langle p_{\bar{r}}|\bar{\psi}^{\bar{R}}(x)|0\rangle = \bar{u}(p)e^{\bar{R}}_{\bar{r}}(v)e^{ip\cdot x}. \tag{4.2.12}$$

Note that we here used capital indices for the flavor-basis field $\bar{\psi}^{\bar{R}}$ to distinguish them from the lowercase indices mass-eigenstate basis. However, for esthetic reasons, we used lowercase indices in the Lagrangian in eq. (4.2.1). Hopefully, this slight abuse of notation will not cause confusion. The tetrad, which is derived from the metric, will implicitly be used to transform between the two bases,

$$\begin{pmatrix} 0 & k_{R\bar{P}} \\ -k_{\bar{P}R} & 0 \end{pmatrix} = e^{\bar{p}}_{\bar{P}} \begin{pmatrix} 0 & \delta_{r\bar{p}} \\ -\delta_{\bar{p}r} & 0 \end{pmatrix} e^r_R, \tag{4.2.13}$$

where $\delta_{\bar{p}r}$ is the Kronecker delta. The fermions are canonically normalized and rotated to the mass-eigenstate basis via the tetrad. The tetrad shows up in the LSZ reduction formula for the fermions as the wavefunction factor, exactly as for the scalars.

4.2.3 Gauge bosons

There is a larger freedom in how to construct a geometric field space which includes gauge bosons. One option is to use the geometry-kinematics map [47], where essentially the gauge bosons act like scalars, and all the geometric quantities in the scalar field space get upgraded to depend on both the scalars and the gauge fields. As an added bonus, the geometry-kinematics duality allows all higher-derivative operators to be placed on the same footing as the two-derivative operators, thus providing a geometric understanding of derivative field redefinitions. The advantage of using the geometry-kinematics map is that statements which hold for scalar effective field theories immediately get upgraded to statements which hold for general bosonic effective field theory. This includes the geometric soft theorem. Some drawbacks of this approach are that the notation is rather compact and that there are some ambiguities in the initial choice for the metric.

Another option is to treat the gauge fields separately from the scalar fields. One such formulation was introduced in ref. [69]. By using a geometric gauge fixing [57], the metric takes the form

$$\bar{g}_{\alpha\beta} = \begin{pmatrix} \bar{g}_{\mathsf{ij}} & 0 \\ 0 & g_{AB} \end{pmatrix}, \tag{4.2.14}$$

where we also include the fermions via the metric in eq. (4.2.6). For a theory without fermions, we simply replace $\bar{g}_{\mathsf{ij}} \to h_{IJ}$ in eq. (4.2.14). We have stripped off a factor $(-\eta_{\mu\nu})$ compared to the metric in ref. [69]. This factor can be trivially reinstated with the replacement $g_{AB} \to -g_{AB}\eta_{\mu_A\mu_B}$. The indices $\alpha, \beta, \ldots$ include all scalar and fermion indices, as well as the gauge-field indices $A, B, \ldots$ Here we slightly abuse the notation by denoting the full scalar–fermion–gauge-boson metric with a bar, as we did in the scalar-fermion metric in eq. (4.2.6). If we included the CP-odd scalar–gauge-boson couplings, $-\frac{1}{4}\tilde{g}_{AB}(\phi)F^A_{\mu\nu}\tilde{F}^{B\mu\nu}$, the metric in eq. (4.2.14) would change to $g_{AB} \to g^{\pm}_{AB} = g_{AB} \pm \tilde{g}_{AB}$ for positive/negative helicity gauge fields. This is analogous to how positive/negative helicity fermions couple through the vertex $\omega^{\pm}_{\bar{p}rI}$. For simplicity, we omit the CP-odd couplings in the gauge metric.

In this paper, we opt for using eq. (4.2.14) for concreteness. This cleanly separates the particles of different spin. However, we will in passing mention how our results change when using the geometry-kinematics map. Intriguingly, both definitions of the gauge-boson metric lead to a new geometric soft theorem. These soft theorems are equivalent but differ in form.

With this choice of metric, we can calculate the connection [69],

$$\bar{\Gamma}^a_{bi} = \frac{1}{2} g^{ac} (\nabla_i g_{cb}), \tag{4.2.15}$$

and the curvature,

$$\bar{R}_{aijb} = \frac{1}{2} \nabla_i \nabla_j g_{ab} - \frac{1}{4} (\nabla_j g_{ac}) g^{cd} (\nabla_i g_{db}). \tag{4.2.16}$$

Next, we need to relate the gauge field to the scattering state. The gauge field creates a one-particle state,

$$\langle p_b, \epsilon | A^B_\mu(x) | 0 \rangle = \epsilon^*_\mu(p) e^B_b(v) e^{ip\cdot x}. \tag{4.2.17}$$

The polarization vector ϵ_μ encodes the two degrees of freedom for a massless gauge field, or the three degrees of freedom for a massive gauge field. Sometimes, we combine the polarization vector and the tetrad, which is defined as $g_{AB}(v) = e_{Aa}(v) e^a_B(v)$, into a new polarization tensor, $\epsilon^B_{b\mu}(v) = \epsilon_\mu e^B_b(v)$, which carry the tetrad indices. The scattering amplitude is multilinear in these polarization vectors, and it will be a tensor with gauge-boson indices in the mass-eigenstate basis.

We also consider massive gauge bosons which get their mass through the Higgs mechanism. As is well known (and reviewed in ref. [13]), a global symmetry in the scalar sector is associated with a set of Killing vectors, $t^I{}_A(\phi)$, such that

$$\phi^I \to \phi^I + c^A \, t^I{}_A(\phi) \tag{4.2.18}$$

leaves the Lagrangian invariant for any c^A. The Killing vectors satisfy commutation relations

$$t^J{}_A(\phi) \partial_J t^I{}_B(\phi) - t^J{}_B(\phi) \partial_J t^I{}_A(\phi) = f_{AB}{}^C t^I{}_C(\phi) \tag{4.2.19}$$

corresponding to a Lie algebra[3]. When this symmetry is gauged, the covariant derivative which describes the coupling of the scalars to gauge bosons is

$$(D_\mu \phi)^I = \partial_\mu \phi^I + t^I_B(\phi) A^B_\mu, \tag{4.2.20}$$

[3]Note that we use a convention different from the usual one in the amplitudes literature ($[t^A, t^B] = \sqrt{2} f_{AB}{}^C t_C$, see, e.g., ref. [195]), so our gauge-boson amplitudes carry additional factors of $\sqrt{2}$ in comparison.

where $t^I_A(\phi)$ is a Killing vector of the scalar field-space manifold. The gauge bosons can acquire mass through the Higgs mechanism. Some of the scalar fields then take on a nonvanishing VEV, which spontaneously breaks the gauge symmetry. In this case, some of the Killing vectors are nonzero at the VEV, $t^I_A(v) \neq 0$. The mass of a gauge boson is generally given by the square of the Killing vectors evaluated at the VEV,

$$m^2_{ab}(v) = t^I{}_a(v) t_{Ib}(v)\,. \tag{4.2.21}$$

However, if the gauge group is not broken, then the Killing vectors vanish at the VEV, $t^I_A(v) = 0$, and the gauge bosons remain massless. We will not commit to either case, and allow for having both charged and neutral scalars as well as massless and massive gauge bosons in our effective field theory.

For later reference, it is useful to quote the Goldstone boson equivalence theorem in the geometric notation:

$$\lim_{p_1 \to \infty} A_{a_1 \cdots}(1_L \cdots) = \frac{t^{i_1}{}_{a_1}(v)}{m_a} A_{i_1 \cdots}(1_{scalar} \cdots)\,, \tag{4.2.22}$$

where the left-hand side is the amplitude of a longitudinal massive gauge boson, and the right-hand side is the amplitude of the "would-be" NGB scalar which is eaten in the Higgs mechanism.

4.3 Geometric Soft Theorem

Below, we present the geometric soft theorem for a massless scalar in a general effective field theory with other (possibly massive) scalars, fermions, and gauge bosons. The derivation of this result is analogous to the derivation for scalar effective field theories [13]. We first review the case for scalars before also including fermions and gauge bosons. The general soft theorem is the union of these results.

4.3.1 Scalars

The geometric soft theorem for scalars was derived in ref. [13]. We reproduce it here. It involves the covariant derivative in field space acting on either the lower-point amplitude or the mass matrix of the external particles. The index j on the covariant derivative corresponds to the index of the particle with momentum q, whose momentum is sent to zero. In full, the geometric soft theorem is

$$\lim_{q\to 0} A_{n+1,i_1\cdots i_n j} = \nabla_j A_{n,i_1\cdots i_n} + \sum_{a=1}^{n} \frac{\nabla_j V_{i_a}{}^{j_a}}{(p_a+q)^2 - m_{j_a}^2} \left(1 + q^\mu \frac{\partial}{\partial p_a^\mu}\right) A_{n,i_1\cdots j_a \cdots i_n}, \tag{4.3.1}$$

where $V_{ij} \equiv V_{;ij}$. The first term in the soft theorem acts on all coupling constants and masses in the amplitude, which are viewed as functions of the VEV. The second term is essential to be consistent with the on-shell conditions for all particles.

This geometric soft theorem unifies the Adler zero for Nambu-Goldstone bosons on a symmetric coset [4], soft theorems for more general Nambu-Goldstone bosons [117], and the dilaton soft theorem [118], [196]–[198]. For illustration, we have listed in section 4.5 examples of scattering amplitudes for four and five scalar particles and shown how they are connected through the geometric soft theorem.

4.3.2 Fermions

Next, we add fermions to the mix. The geometric soft theorem for a massless scalar in the presence of both scalars and fermions is new. It bears stark resemblance to the soft theorem above. The geometric soft theorem again depends on the covariant derivative in field space, but this time for the combined scalar-fermion geometry defined through the metric in eq. (4.2.6). This covariant derivative $\bar{\nabla}_i$ is denoted with a bar to indicate that it is also sensitive to fermionic flavor indices.

The full scalar-fermion soft theorem is

$$\begin{aligned} \lim_{q\to 0} A_{n+1,\mathrm{i}_1\cdots \mathrm{i}_n j} &= \bar{\nabla}_j A_{n,\mathrm{i}_1\cdots \mathrm{i}_n} \\ &+ \sum_{a\in\{\text{scalar}\}} \frac{\bar{\nabla}_j V_{i_a}{}^{j_a}}{(p_a+q)^2 - m_{j_a}^2} \left(1 + q^\mu \frac{\partial}{\partial p_a^\mu}\right) A_{n,\mathrm{i}_1\cdots j_a \cdots \mathrm{i}_n} \\ &+ \sum_{b\in\{\text{fermion}\}} \sum_{\text{spin}} \frac{\bar{\nabla}_j \mathcal{M}^{r_b}{}_{p_b} (\bar{u}(p+q) u(p))}{(p_b+q)^2 - m_{r_b}^2} \left(1 + q^\mu \frac{\partial}{\partial p_b^\mu}\right) A_{n,\mathrm{i}_1\cdots r_b \cdots \mathrm{i}_n} \\ &+ \sum_{b\in\{\text{anti-fermion}\}} \sum_{\text{spin}} \frac{\bar{\nabla}_j \mathcal{M}_{\bar{p}_b}{}^{\bar{r}_b} (-\bar{v}(p) v(p+q))}{(p_b+q)^2 - m_{r_b}^2} \left(1 + q^\mu \frac{\partial}{\partial p_b^\mu}\right) A_{n,\mathrm{i}_1\cdots \bar{r}_b \cdots \mathrm{i}_n}, \end{aligned} \tag{4.3.2}$$

where $\mathcal{M}$ is the fermion mass matrix. Let us unpack this soft theorem. We take all momenta to be incoming and write the amplitude with lowered flavor indices. Note that the tetrads are implicitly included for both scalars and fermions, although we use the same index for the fermion flavors in the amplitude as in the Lagrangian. The tetrads canonically normalize and rotate the states to the mass-eigenstate basis, where the mass matrix is diagonal. Another thing we have kept implicit is the label for the spin component of the fermion wave functions. The spin is summed over for the external fermion wavefunction in the n-point amplitude and the shifted spinor in the prefactors.

The first line in eq. (4.3.2) is similar to the scalar soft theorem in eq. (4.3.1), with the replacement $\nabla_i \to \bar{\nabla}_i$, while the second and third lines are the covariant derivative acting on the external fermion propagators. The last three terms can be unified to the covariant derivative of a single mass matrix, where the indices run over both scalar and fermion flavors, but we choose to write out all the terms explicitly for clarity.

The geometric soft theorem in eq. (4.3.2) holds at tree level. However, in the case where the potential and fermion mass matrix vanish, $V(\phi)=0$ and $\mathcal{M}(\phi) = 0$, we believe the soft theorem holds at all loop orders, for the same reasons as in the soft scalar theorem [13].

For this soft theorem to have a sensible on-shell interpretation, it must commute with the on-shell conditions. Consider the action of the soft theorem on the on-shell condition for an incoming fermion,

$$(\not{p}_s \delta^r{}_s - \mathcal{M}^r{}_s) u(p_s) = 0\,. \tag{4.3.3}$$

The covariant derivative shifts the mass matrix,

$$\bar{\nabla}_j (\not{p}_s \delta^r{}_s - \mathcal{M}^r{}_s) u(p_s) = -(\bar{\nabla}_j \mathcal{M}^r{}_s) u(p_s), \tag{4.3.4}$$

and the third term in eq. (4.3.2) acts on the spinor,

$$\begin{aligned}
&\sum_{\text{spin}} \frac{\bar{\nabla}_j \mathcal{M}^t{}_s (\bar{u}(p_s+q)u(p_s))}{(p_s+q)^2 - m_r^2} \left(1 + q^\mu \frac{\partial}{\partial p_s^\mu}\right) (\not{p}\delta^r{}_t - \mathcal{M}^r{}_t) u(p_s) \\
=&\sum_{\text{spin}} \frac{\bar{\nabla}_j \mathcal{M}^t{}_s (\bar{u}(p_s+q)u(p_s))}{(p_s+q)^2 - m_r^2} ((\not{p}_s + \not{q})\delta^r{}_t - \mathcal{M}^r{}_t) u(p_s+q) = (\bar{\nabla}_j \mathcal{M}^r{}_s) u(p_s),
\end{aligned} \tag{4.3.5}$$

where we have used that the sum over spins is

$$\sum_{\text{spin}} u(p)\bar{u}(p) = (\not{p} + m). \tag{4.3.6}$$

Clearly, eqs. (4.3.4) and (4.3.5) cancel, which means that the soft theorem does not spoil the on-shell conditions for incoming fermions and can therefore be applied unambiguously to scattering amplitudes.

The soft theorem also commutes with the on-shell condition for incoming anti-fermions. In this case, the cancellation happens between the covariant derivative and the fourth term in eq. (4.3.2), where we now have to use that the sum over spins is

$$\sum_{\text{spin}} v(p)\bar{v}(p) = (\not{p} - m). \tag{4.3.7}$$

Perhaps the most well-known case of low-energy dynamics for relativistic scalars is the theory of pions. The soft limit of pion–pion scattering vanishes, known as the Adler zero [4]. In contrast, the soft limit of a pion scattering off nucleons does not vanish. However, this limit is universal and can be derived using current algebra methods. The nonzero soft limit of pion-nucleon scattering is related to the coupling in the so-called gradient-coupling theory [199]. This is nothing but the couplings $\omega^{\pm}$, which enter the geometric soft theorem through the covariant derivative $\bar{\nabla}$. Thus, the pion-nucleon soft theorem is a special case of the geometric soft theorem [199]–[202]. Another special case of the geometric soft theorem is the low-energy limit of the η' particle in large-N QCD described long ago by Witten [203]. This is a pseudoscalar NGB for the axial $U(1)$ symmetry of QCD which remains unbroken in the planar limit. Its soft limit computes derivatives of scattering amplitudes with respect to the QCD θ angle, or equivalently the η' VEV. More generally, the leading term in the soft limit goes as $1/(p \cdot q)$ and comes from the covariant derivative acting on the scalar potential $V(\phi)$ or the fermion mass matrix $\mathcal{M}(\phi)$. This universal soft behavior of scalars is analogous to the leading soft limit of photons and gravitons, and it follows from similar polology considerations [204], [205].

The derivation of the geometric soft theorem in eq. (4.3.2) is analogous to the derivation for a scalar effective field theory in ref. [13]. We will here highlight the main novelties compared to the scalar case. The derivation begins by using the Euler-Lagrange equations,

$$\partial_\mu \mathcal{J}_I^\mu = \partial_I L, \tag{4.3.8}$$

where

$$\mathcal{J}_I^\mu = \frac{\delta L}{\delta(\partial_\mu \phi^I)} \qquad \text{and} \qquad \partial_I L = \frac{\delta L}{\delta \phi^I}. \tag{4.3.9}$$

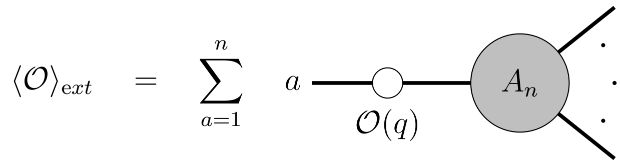

Figure 4.1: Diagrams computing $\langle\mathcal{O}\rangle_{ext}$, which sums over the insertion of an operator $\mathcal{O}$ on each external leg a of the n-particle amplitude. This figure is directly reproduced from ref. [13].

Since the scalar field ϕ^I is expanded around the VEV v^I, their appearance in the Lagrangian is identical, and we can equivalently calculate the variation of the Lagrangian with respect to the VEV to obtain $\partial_I L$.

The only terms that will affect the soft theorem are operators that are at most cubic in the field. We split the contributions coming from scalar and fermion operators. We find that

$$\mathcal{J}_I^\mu = (\mathcal{J}_I^\mu)_{\text{scalar}} + i\omega_{\bar{p}rI}(v)(\bar{\psi}^{\bar{p}}\gamma^\mu\psi^r) + \ldots, \tag{4.3.10}$$

$$\partial_I L = (\partial_I L)_{\text{scalar}} + \frac{1}{2}ik_{\bar{p}r,I}(v)(\bar{\psi}^{\bar{p}}\overleftrightarrow{\not{\partial}}\psi^r) - \mathcal{M}_{\bar{p}r,I}(v)(\bar{\psi}^{\bar{p}}\psi^r) + \ldots \tag{4.3.11}$$

We now collect the contributions from the fermion operators, and insert them on external fermion lines,

$$\begin{aligned}\langle\partial_I L - \partial_\mu\mathcal{J}_I^\mu\rangle_{\text{ext. fermion}} = &- \langle(\bar{\psi}\omega^-)_{tI}(i\delta^t{}_r\not{\partial} - \mathcal{M}^t{}_r)\psi^r\rangle_{\text{ext}} \\ &- \langle\bar{\psi}^{\bar{p}}(i\delta_{\bar{p}}{}^{\bar{t}}\overleftarrow{\not{\partial}} + \mathcal{M}_{\bar{p}}{}^{\bar{t}})(\omega^+\psi)_{\bar{t}I}\rangle_{\text{ext}} \\ &- \langle\bar{\nabla}_I\mathcal{M}_{\bar{p}r}\bar{\psi}^{\bar{p}}\psi^r\rangle_{\text{ext}}\,,\end{aligned} \tag{4.3.12}$$

where the notation is defined in fig. 4.1. By evaluating these operator insertions, we find that the first line in eq. (4.3.12) either vanishes due to the on-shell condition, or it cancels a propagator and becomes a local term multiplying the amplitude. These local terms are $-\bar{\Gamma}^p_{Ir}$ or $-\bar{\Gamma}^{\bar{p}}_{I\bar{r}}$, depending on whether the operator is inserted on an incoming fermion or anti-fermion line. This is precisely the fermion connections in eqs. (4.2.8) and (4.2.9), and when combined with the scalar contributions, they complete the covariant derivative $\bar{\nabla}_I$ in the soft theorem in eq. (4.3.2). What is left is the insertion of the operator in the second line in eq. (4.3.12), which becomes the second and third lines of the soft theorem in eq. (4.3.2).

In section 4.5 we will check some examples of the soft theorem for an effective field theory with scalars and fermions.

4.3.3 Gauge bosons

The last particle to make an appearance is the gauge boson. In this case, we use the geometric construction in eq. (4.2.14). Due to the block-diagonal structure of the metric, the fermions and gauge bosons do not couple directly through the geometry, and we can simply ignore the fermions for the moment.

The soft theorem for a massless scalar in a theory with scalars and gauge bosons is

$$
\begin{aligned}
\lim_{q\to 0} A_{n+1,\alpha_1\cdots\alpha_n j} =& \bar{\nabla}_j A_{n,\alpha_1\cdots\alpha_n} \\
&+ \sum_{a\in\{\text{scalar}\}} \frac{\nabla_j V_{i_a}{}^{j_a}}{(p_a+q)^2 - m_{j_a}^2} \left(1 + q^\mu \frac{\partial}{\partial p_a^\mu}\right) A_{n,\alpha_1\cdots j_a\cdots\alpha_n} \\
&+ \sum_{a\in\{\text{scalar}\}} \sum_{\text{spin}} i \frac{(2\nabla_j t_{i_a\,B})(\epsilon^{Bb*}\cdot q)}{(p_a+q)^2 - m_b^2} A_{n,\alpha_1\cdots b\cdots\alpha_n} \\
&+ \sum_{b\in\{\text{gauge}\}} i \frac{(2\nabla_j t_B^{j_a})(\epsilon_b^B\cdot q)}{(p_b+q)^2 - m_{j_a}^2} A_{n,\alpha_1\cdots j_a\cdots\alpha_n} \\
&- \sum_{b\in\{\text{gauge}\}} \sum_{\text{spin}} \frac{(\bar{\nabla}_j m_{BA}^2)(\epsilon_b^B\cdot\epsilon^{Aa*})}{(p_b+q)^2 - m_a^2} \left(1 + q^\mu \frac{\partial}{\partial p_a^\mu}\right) A_{n,\alpha_1\cdots a\cdots\alpha_n}\,.
\end{aligned}
\tag{4.3.13}
$$

Note that the covariant derivative $\bar{\nabla}_i$ now uses the connections derived from the metric in eq. (4.2.14) and sees the gauge group indices of the gauge bosons. Here, the spin is summed over for the external gauge-boson polarization vector in the n-point amplitude and the polarization vector in the prefactors $\epsilon^{Aa*}(p+q)$, evaluated at shifted momentum.

As a first check, we act the soft theorem on the on-shell condition for the gauge boson,

$$
(p_b^2 g_{AB} - m_{AB}^2)\epsilon_\mu^{Bb} = 0. \tag{4.3.14}
$$

The covariant derivative picks up the variation of the mass matrix,

$$
\bar{\nabla}_j (p_b^2 g_{AB} - m_{AB}^2)\epsilon_\mu^{Bb} = -(\bar{\nabla}_j m_{AB}^2)\epsilon_\mu^{Bb}. \tag{4.3.15}
$$

The first term vanishes due to metric compatibility, $\bar{\nabla} g = 0$.

Then we contract the on-shell condition with the last term of the soft theorem,

$$
-\sum_{\text{spin}} \frac{(\bar{\nabla}_j m_{BC}^2)(\epsilon_b^B\cdot\epsilon^{Cc*})}{(p_b+q)^2 - m_c^2} \left(1 + q^\mu \frac{\partial}{\partial p_b^\mu}\right) (p_b^2 g_{AC'} - m_{AC'}^2)\epsilon_{c\mu}^{C'}
$$

$$= -\sum_{\text{spin}} \frac{(\bar{\nabla}_j m^2_{BC})(\epsilon^B_b \cdot \epsilon^{Cc*})}{(p_b+q)^2 - m^2_c}((p_b+q)^2 g_{AC'} - m^2_{AC'})\epsilon^{C'}_{c\mu} = +(\bar{\nabla}_j m^2_{BA})\epsilon^B_{b\mu}. \tag{4.3.16}$$

We see that eqs. (4.3.15) and (4.3.16) cancel, which means that the soft theorem commutes with the on-shell conditions.

Note that there are no terms of the form ∇m^2 in the soft theorem for massless gauge bosons since

$$\nabla_K m^2_{ab}(v) = \nabla_K \left(t^I{}_a(v)\, t_{Ib}(v)\right) = \nabla_K t^I{}_a(v)\, t_{Ib}(v) + t^I{}_a(v) \nabla_K t_{Ib}(v), \tag{4.3.17}$$

and in the unbroken phase, one has $t^I_a(v) = 0$, so

$$\nabla_K m^2_{ab}(v) = 0. \tag{4.3.18}$$

This means that the gauge boson masses vanish even in an infinitesimal neighborhood of the unbroken VEV.

To fully understand the form of the soft theorem, we need to consider the interplay between longitudinal gauge bosons and Goldstone bosons. We will show how different representations of the soft theorem are linked via the Goldstone boson equivalence theorem. Consider the second and third lines in eq. (4.3.13). In unitary gauge, the Goldstone boson decouples, and we only exchange massive gauge bosons

$$+ i \frac{(2\nabla_j t^b_{i_a})(q^\mu)}{2p_a \cdot q}(-1)\left(\eta^{\mu\nu} - \frac{p^\mu_a p^\nu_a}{m^2_a}\right) A_{n,\alpha_1 \cdots b\nu \cdots \alpha_n}. \tag{4.3.19}$$

In R_ξ gauge, we instead get

$$\begin{aligned}
&+ i \frac{(2\nabla_j t^b_{i_a})(q^\mu)}{(p_a+q)^2 - m^2_a}(-1)\left(\eta^{\mu\nu} - (1-\xi)\frac{p^\mu_a p^\nu_a}{p^2_a - \xi m^2_a}\right) A_{n,\alpha_1 \cdots b\nu \cdots \alpha_n} \\
&+ i \frac{(2\nabla_j t^{j_a}_B)(\epsilon^B_b \cdot q)}{(p_b+q)^2 - \xi m^2_a} A_{n,\alpha_1 \cdots j_a \cdots \alpha_n} \\
= &+ i \frac{(2\nabla_j t^b_{i_a})(q^\mu)}{2p_a \cdot q}(-1)\left(\eta^{\mu\nu} - \frac{p^\mu_a p^\nu_a}{m^2_a}\right) A_{n,\alpha_1 \cdots b\nu \cdots \alpha_n} \\
&+ i \frac{(2\nabla_j t^b_{i_a})(q^\mu)}{(p_a+q)^2 - \xi m^2_a}(-1)\left(\frac{p^\mu_a p^\nu_a}{m^2_a}\right) A_{n,\alpha_1 \cdots b\nu \cdots \alpha_n} \\
&+ i \frac{(2\nabla_j t^{j_a}_B)(\epsilon^B_b \cdot q)}{(p_b+q)^2 - \xi m^2_a} A_{n,\alpha_1 \cdots j_a \cdots \alpha_n}.
\end{aligned} \tag{4.3.20}$$

If we identify the longitudinal polarization with the momentum $\epsilon_L^\mu \to p^\mu/m$, then the last two terms in eq. (4.3.20) cancel, and we end up with the same result as in eq. (4.3.19). This is due to the Goldstone boson equivalence theorem in eq. (4.2.22). Here, instead of taking the high-energy limit for the longitudinal gauge boson, we take the soft limit for a scalar. These limits yield the same result because the longitudinal gauge boson has a large energy relative to the soft scalar.

Incidentally, if we instead used the geometry-kinematics map, the soft theorem would take the form

$$\lim_{q\to 0} A_{n+1,\alpha_1\cdots\alpha_n j} = \nabla'_j A_{n,\alpha_1\cdots\alpha_n}. \tag{4.3.21}$$

This coincides with the first term in eq. (4.3.13), but the greater freedom in the mapping also puts the nonlocal terms into the covariant derivative as extensions of the connection. Ref. [47] showed that this soft theorem also captures the leading and subleading soft photon theorem.

The proof of the geometric soft theorem with gauge bosons is completely analogous to that for scalars and vectors, so we will not describe it here. Instead, we will directly check the soft theorem in various examples in section 4.5.

4.4 Double Soft Theorems

Another way to study scattering amplitudes is to send the momenta of multiple particles to zero. If we do so in a consecutive order, we simply need to apply the geometric soft theorem multiple times. However, if the momenta are sent to zero simultaneously, we will discover a genuinely new geometric structure in the scattering amplitudes: the curvature. This demonstrates the non-abelian nature of pion scattering [7].

To ease the presentation, we turn off all couplings which appear in the nonlocal terms in the geometric soft theorem, i.e., the scalar potential $V(\phi)$, the fermion mass matrix $\mathcal{M}(\phi)$, and we make the particles neutral, i.e., $t = 0$ and $\nabla t = 0$. This avoids multiple soft poles in the expressions.

First, we consider the double soft limit where the momenta of two scalars are taken to zero at the same rate. This will be an extension of the double soft theorem in a scalar effective field theory [13]. Then, we change the protagonists and consider the double soft limit where the momenta of two fermions of opposite helicity are sent to zero at the same rate. This new double soft fermion theorem has striking similarities to the double soft scalar theorem. Here, all momenta are outgoing.

4.4.1 Scalars

The double scalar soft theorem is identical to the form derived in ref. [13], when using the appropriate geometric extensions when fermions and gauge bosons are present. Here, the potential and other terms singular in the soft limit are neglected. The simultaneous double soft theorem is

$$\begin{aligned}\lim_{q_1,q_2\to 0} A_{n+2,\alpha_1\cdots\alpha_n j_1 j_2} =& \bar{\nabla}_{(j_1}\bar{\nabla}_{j_2)} A_{n,\alpha_1\cdots\alpha_n} \\ &+ \frac{1}{2}\sum_{a=1}^{n} \frac{p_a\cdot(q_1-q_2)}{p_a\cdot(q_1+q_2)} \bar{R}_{j_1 j_2}{}^{\beta_a}{}_{\alpha_a}\, A_{n,\alpha_1\cdots\beta_a\cdots\alpha_n}. \end{aligned} \tag{4.4.1}$$

The particles with flavor labels $\{\alpha_1,\cdots,\alpha_n\}$ can be any combination of massless scalars, fermions, or gauge bosons. Remarkably, the same double soft theorem holds regardless of whether the scalars couple to fermions, gauge bosons, or other scalars; the various interactions are captured by the combined curvature $\bar{R}_{j_1 j_2 \alpha_a \beta_a}$. We present several examples of soft limits of scattering amplitudes in section 4.5, where the double scalar soft theorem can be checked.

4.4.2 Fermions

We can also consider the soft limit of two fermions with opposite helicities. For convenience, let us use the spinor-helicity formalism (following the conventions in ref. [195]). The result is

$$\begin{aligned}\lim_{q_1,q_2\to 0} A_{n+2,\alpha_1\cdots\alpha_n \bar{r}_1 r_2} =& \frac{1}{2}\left\{\lim_{q_1\to 0}, \lim_{q_2\to 0}\right\} A_{n+2,\alpha_1\cdots\alpha_n \bar{r}_1 r_2} \\ &+ \frac{1}{2}\sum_{a=1}^{n} \frac{[q_1|p_a|q_2\rangle}{p_a\cdot(q_1+q_2)} \bar{R}_{\bar{r}_1 r_2}{}^{\beta_a}{}_{\alpha_a}\, A_{n,\alpha_1\cdots\beta_a\cdots\alpha_n}. \end{aligned} \tag{4.4.2}$$

The double fermion soft theorem is equal to the double scalar soft theorem under the replacement $p_a\cdot(q_1-q_2)\to[q_1|p_a|q_2\rangle$, as first noted in ref. [206] for supersymmetric theories. The proof is diagrammatic and the same as for two soft scalars (see ref. [13, sec. 6.2]). The first term in eq. (4.4.2) is written in terms of the anticommutator of two consecutive soft limits, rather than as covariant derivatives acting on the lower-point amplitude. This is because we do not have a geometric way of writing the soft limit of a single fermion in terms of lower-point amplitudes. However, the single soft fermion limit vanishes in many instances, and then we end up with the simpler form of eq. (4.4.2),

$$\lim_{q_1,q_2\to 0} A_{n+2,\alpha_1\cdots\alpha_n\bar{r}_1 r_2} = \frac{1}{2}\sum_{a=1}^{n} \frac{[q_1|p_a|q_2\rangle}{p_a\cdot(q_1+q_2)} \bar{R}_{\bar{r}_1 r_2}{}^{\beta_a}{}_{\alpha_a}\, A_{n,\alpha_1\cdots\beta_a\cdots\alpha_n}. \tag{4.4.3}$$

The curvature that enters the double soft theorem depends on the other particles in the theory. In the presence of scalar particles, the mixed scalar-fermion curvature $\bar{R}_{\bar{r}_1 r_2 ij}$ controls the nonlocal term, whose expression is given in eq. (4.2.10). For fermion-fermion interactions, the four-fermion curvature $\bar{R}_{\bar{r}_1 r_2 \bar{r}_3 r_4}$ in eq. (4.2.11) enters the double soft theorem. Even though the kinematic expressions that come with these different curvature components in scattering amplitudes are very different, they reduce to the exact same term in the double soft limit.

4.4.3 Other simultaneous soft limits

Now, the door is open to consider even more exotic simultaneous soft limits. Take, as an example, the simultaneous soft limit of one scalar and one fermion. Concretely, we take a positive-helicity fermion with a holomorphic soft scaling, in spinor-helicity variables $(\lambda,\tilde{\lambda})\to(z\lambda,\tilde{\lambda})$, with z small. Rather than deriving the double soft theorem as we did above, let us try to guess the answer from the intuition we have accrued. A natural guess for the double soft limit is

$$\begin{aligned}\lim_{q_1,q_2\to 0} A_{n+2,\alpha_1\cdots\alpha_n\bar{r}_1 j_2} = &\lim_{q_1\to 0} \bar{\nabla}_{j_2} A_{n+1,\alpha_1\cdots\alpha_n\bar{r}_1} \\ &+ \frac{1}{2}\sum_{a\in\{\text{fermion}\}}^{n} \frac{[q_1|q_2|p_a\rangle}{p_a\cdot(q_1+q_2)} \bar{R}_{\bar{r}_1 r_a}{}^{j_a}{}_{j_2}\, A_{n,\alpha_1\cdots j_a\cdots\alpha_n} \\ &+ \frac{1}{2}\sum_{a\in\{\text{scalar}\}} \frac{[q_1|q_2|p_a\rangle}{p_a\cdot(q_1+q_2)} \bar{R}_{\bar{r}_1}{}^{\bar{s}_a}{}_{i_a j_2}\, A_{n,\alpha_1\cdots\bar{s}_a\cdots\alpha_n}\,.\end{aligned} \tag{4.4.4}$$

This mixed double soft theorem can be derived via a diagrammatic approach analogous to the double scalar and fermion soft theorems. We have verified in the examples in section 4.5 that this mixed double soft theorem indeed holds. Again, the soft theorem is identical in form to the double scalar and double fermion soft theorem, up to a simple replacement of a kinematic factor. The first term in eq. (4.4.4) is written in terms of the single soft limit of a lower-point amplitude rather than as $\bar{\nabla}_{(\bar{r}_1}\bar{\nabla}_{i_2)} A_{n,\alpha_1\cdots\alpha_n}$. The single soft limit of a fermion is hard to interpret in terms of scattering amplitudes because it would involve the derivative of a would-be amplitude with an odd number of fermions[4].

Based on this, we expect that *any* double soft limit will be universal. It will involve the curvature in field space, accompanied by an appropriate kinematic factor to account for the helicity weight of the soft particles. Indeed, we know that this is true for double soft limits involving gauge bosons, through the geometry-kinematics map. In this case, the double soft limit will be identical to eq. (4.4.1), with replacements $\nabla \to \nabla'$ and $R \to R'$. The kinematic factors that carry the helicity weights are folded into the geometry, which now also depends on the kinematics.

One can also consider more soft particles. The case with three soft scalars was analyzed in ref. [13]. As one might have guessed, the triple soft theorem involves various terms with ∇^3, $R\nabla$, and ∇R acting on the lower-point amplitude. We expect that the generalization of multiple soft limits with a mixture of particles will be the natural generalizations of the scalar case, but where the kinematic factors are replaced and the geometry is extended. We will not explore this direction further here.

4.5 Examples

We now present tree-level scattering amplitudes for scalars, fermions, and gauge bosons. With these amplitudes, we can check the single and double soft theorems. All momenta are outgoing, and we use the spinor-helicity conventions for massless and massive particles in refs. [195], [209], [210].

4.5.1 Scalars

We start by listing some scattering amplitudes for scalars with two-derivative interactions. The corresponding Lagrangian is

$$\mathcal{L} = \frac{1}{2} h_{IJ}(\phi)(\partial_\mu \phi)^I (\partial^\mu \phi)^J. \tag{4.5.1}$$

The scattering amplitudes for four and five particles are

$$A_{4,i_1 i_2 i_3 i_4} = R_{i_1 i_3 i_2 i_4} s_{34} + R_{i_1 i_2 i_3 i_4} s_{24}, \tag{4.5.2}$$

$$\begin{aligned} A_{5,i_1 i_2 i_3 i_4 i_5} =& \nabla_{i_3} R_{i_1 i_4 i_2 i_5} s_{45} + \nabla_{i_4} R_{i_1 i_3 i_2 i_5} s_{35} + \nabla_{i_4} R_{i_1 i_2 i_3 i_5} s_{25} \\ &+ \nabla_{i_5} R_{i_1 i_3 i_2 i_4} s_{34} + \nabla_{i_5} R_{i_1 i_2 i_3 i_4} (s_{24} + s_{45}), \end{aligned} \tag{4.5.3}$$

[4]The exception is for supersymmetric theories, where the soft fermion theorem is related to a soft boson theorem (e.g., for Goldstinos [207] or photinos [208]).

where $s_{ij} = (p_i + p_j)^2$. We will use these amplitudes to illustrate the geometric soft theorem for scalar effective field theories in eq. (4.3.1). For more examples and the original derivation, see ref. [13].

Take the limit $p_4 \to 0$ of eq. (4.5.2),

$$\lim_{p_4 \to 0} A_{4,i_1 i_2 i_3 i_4} = 0. \tag{4.5.4}$$

This adheres to the geometric soft theorem, because the scalar three-particle amplitude is zero when the potential is absent.

Next, look at the limit $p_5 \to 0$ of eq. (4.5.3):

$$\lim_{p_5 \to 0} A_{5,i_1 i_2 i_3 i_4 i_5} = \nabla_{i_5} R_{i_1 i_3 i_2 i_4} s_{34} + \nabla_{i_5} R_{i_1 i_2 i_3 i_4} s_{24} = \nabla_{i_5} A_{4,i_1 i_2 i_3 i_4}. \tag{4.5.5}$$

This is precisely the statement of the geometric soft theorem with no potential: the soft limit of the amplitude is equal to the covariant derivative acting on the lower-point amplitude. This is the cleanest illustration of the geometric soft theorem. However, the soft theorem is valid for general scalar effective field theories, including potential and higher-derivative interactions. Additional examples can be found in Ref. [13].

4.5.2 Fermions

Next, we look at scattering amplitudes with fermions and scalars, coming from the one-derivative fermion bilinear operators and scalar operators with two derivatives as well as from the four-fermion operators. These are the operators which appear in the scalar-fermion metric in eq. (4.2.6). The Lagrangian is

$$\begin{aligned}\mathcal{L} =& \frac{1}{2} h_{IJ}(\phi)(\partial_\mu \phi^I)(\partial^\mu \phi^J) + i\frac{1}{2} k_{\bar{p}r}(\phi)(\bar{\psi}^{\bar{p}} \overleftrightarrow{\not{D}} \psi^r) + i\omega_{\bar{p}rI}(\phi)(\bar{\psi}^{\bar{p}} \gamma_\mu \psi^r)(\partial^\mu \phi^I) \\ &+ c_{\bar{p}r\bar{s}t}(\phi)(\bar{\psi}^{\bar{p}} \gamma_\mu \psi^r)(\bar{\psi}^{\bar{s}} \gamma^\mu \psi^t).\end{aligned} \tag{4.5.6}$$

The scattering amplitude with two fermions and one scalar vanishes. The scattering amplitudes with two, three, or four scalars are

$$A_{4,\bar{r}_1 r_2 i_3 i_4} = - [1|p_4|2\rangle \bar{R}_{\bar{r}_1 r_2 i_3 i_4}, \tag{4.5.7}$$

$$A_{5,\bar{r}_1 r_2 i_3 i_4 i_5} = - [1|p_4|2\rangle \bar{\nabla}_{i_5} \bar{R}_{\bar{r}_1 r_2 i_3 i_4} - [1|p_5|2\rangle \bar{\nabla}_{i_4} \bar{R}_{\bar{r}_1 r_2 i_3 i_5}, \tag{4.5.8}$$

$$A_{6,\bar{r}_1 r_2 i_3 i_4 i_5 i_6} = \Bigg\{ -[1|p_4|2\rangle \Bigg(\frac{1}{4} \bar{\nabla}_{i_5} \bar{\nabla}_{i_6} \bar{R}_{\bar{r}_1 r_2 i_3 i_4} + \frac{1}{6} \bar{R}_{\bar{r}_1 r_2 i_3}{}^{j} \bar{R}_{i_4 i_5 i_6 j}$$

$$
\begin{aligned}
&\qquad\qquad -\frac{1}{4}\bar{R}_{\bar{r}_1 s i_3 i_5}\bar{R}^s{}_{r_2 i_4 i_6} - (3\leftrightarrow 4) + (5\leftrightarrow 6)\\
&\qquad\qquad \left. - (3\leftrightarrow 4, 5\leftrightarrow 6)\Big) + \mathrm{cycl}(456)\right\}\\
&+\left\{\frac{[1|(p_5-p_6)P_{234}(p_3-p_4)|2\rangle}{16 s_{234}}\bar{R}_{\bar{r}_1 s i_5 i_6}\bar{R}^s{}_{r_2 i_3 i_4} + \mathrm{perm}(3456)\right\}\\
&+\left\{\frac{[1|p_6|2\rangle}{3 s_{345}}\bar{R}_{\bar{r}_1 r_2 i_6}{}^j\big[s_{34}(\bar{R}_{i_3 i_4 i_5 j} - 2\bar{R}_{i_3 i_5 i_4 j})\right.\\
&\qquad\qquad + s_{35}(\bar{R}_{i_3 i_5 i_4 j} - 2\bar{R}_{i_3 i_4 i_5 j})\\
&\qquad\qquad \left. + s_{45}(\bar{R}_{i_3 i_4 i_5 j} + \bar{R}_{i_3 i_5 i_4 j})\big] + \mathrm{cycl}(3456)\right\}.
\end{aligned}
\tag{4.5.9}
$$

Here, $s_{ijk} = (p_i + p_j + p_k)^2$ and $P^\mu_{ijk} = p^\mu_i + p^\mu_j + p^\mu_k$, and we sum over all or cyclic permutations denoted by perm() or cycl().

The scattering amplitudes with four or six fermions but no scalars are

$$
A_{4,\bar{r}_1 r_2 \bar{r}_3 r_4} = [13]\langle 42\rangle \bar{R}_{\bar{r}_1 r_2 \bar{r}_3 r_4}, \tag{4.5.10}
$$

$$
\begin{aligned}
A_{6,\bar{r}_1 r_2 \bar{r}_3 r_4 \bar{r}_5 r_6} &= \left(-\frac{[5|(p_1+p_3)|2\rangle[13]\langle 64\rangle}{s_{123}}\bar{R}_{\bar{r}_1 r_2 \bar{r}_3}{}^{\bar{s}}\bar{R}_{\bar{s} r_4 \bar{r}_5 r_6} + \mathrm{cycl}(135)\right)\\
&\quad + \mathrm{cycl}(246).
\end{aligned}
\tag{4.5.11}
$$

Let us now check the new geometric soft theorem in the presence of fermions. First, the limit $p_4 \to 0$ for the two scalar, two fermion amplitude is

$$
\lim_{p_4\to 0} A_{4,\bar{r}_1 r_2 i_3 i_4} = 0, \tag{4.5.12}
$$

which is consistent with the soft theorem in eq. (4.3.2).

A more nontrivial example is the $p_5 \to 0$ soft limit of the five-particle amplitude,

$$
\lim_{p_5\to 0} A_{5,\bar{r}_1 r_2 i_3 i_4 i_5} = -[1|p_4|2\rangle \bar{\nabla}_{i_5}\bar{R}_{\bar{r}_1 r_2 i_3 i_4} = \bar{\nabla}_{i_5} A_{4,\bar{r}_1 r_2 i_3 i_4}. \tag{4.5.13}
$$

This is the scalar-fermion soft theorem in eq. (4.3.2) with the potential and fermion mass matrix turned off. Structurally, it is identical to the geometric soft theorem for scalars, but with the crucial difference that $\nabla_i \to \bar{\nabla}_i$. The geometric soft theorem depends on the combined scalar-fermion geometry dictated by the metric in eq. (4.2.6).

Next, consider the limit $p_6 \to 0$ of the six-particle amplitude in eq. (4.5.9),

$$\begin{aligned}\lim_{p_6\to 0} A_{6,\bar{r}_1 r_2 i_3 i_4 i_5 i_6} &= -[1|p_4|2\rangle \bar{\nabla}_{i_6}\bar{\nabla}_{i_5}\bar{R}_{\bar{r}_1 r_2 i_3 i_4} - [1|p_5|2\rangle \bar{\nabla}_{i_6}\bar{\nabla}_{i_4}\bar{R}_{\bar{r}_1 r_2 i_3 i_5}\\ &= \bar{\nabla}_{i_6} A_{5,\bar{r}_1 r_2 i_3 i_4 i_5} \,. \end{aligned} \tag{4.5.14}$$

This example showcases an intricate cancellation between local R^2 terms and R^2 terms with factorization channels which become localized in the soft limit.

We can also study these amplitudes in the double soft limit. Take the simultaneous soft limit $p_5, p_6 \to 0$ of the six-particle amplitude in eq. (4.5.9),

$$\begin{aligned}\lim_{p_5,p_6\to 0} A_{6,\bar{r}_1 r_2 i_3 i_4 i_5 i_6} =& - [1|p_4|2\rangle \bar{\nabla}_{(i_5}\bar{\nabla}_{i_6)}\bar{R}_{\bar{r}_1 r_2 i_3 i_4}\\ &+ \frac{(s_{15}-s_{16})\bar{R}_{i_5 i_6 \bar{r}_1}{}^{\bar{s}}}{2(s_{15}+s_{16})}\left(-[1|p_4|2\rangle \bar{R}_{s r_2 i_3 i_4}\right)\\ &+ \frac{(s_{25}-s_{26})\bar{R}_{i_5 i_6 r_2}{}^{s}}{2(s_{25}+s_{26})}\left(-[1|p_4|2\rangle \bar{R}_{\bar{r}_1 s i_3 i_4}\right)\\ &- \frac{(s_{35}-s_{36})\bar{R}_{i_5 i_6 i_3}{}^{j}}{2(s_{35}+s_{36})}\left(-[1|p_4|2\rangle \bar{R}_{\bar{r}_1 r_2 j i_4}\right)\\ &- \frac{(s_{45}-s_{46})\bar{R}_{i_5 i_6 i_4}{}^{j}}{2(s_{45}+s_{46})}\left(-[1|p_4|2\rangle \bar{R}_{\bar{r}_1 r_2 i_3 j}\right)\\ =& \bar{\nabla}_{(i_5}\bar{\nabla}_{i_6)} A_{4,\bar{r}_1 r_2 i_3 i_4} + \frac{1}{2}\sum_{a=1}^{4}\frac{p_a\cdot(p_5-p_6)}{p_a\cdot(p_5+p_6)}\bar{R}_{i_5 i_6}{}^{\beta_a}{}_{\alpha_a} A_{4,\ldots\beta_a\ldots} \,. \end{aligned} \tag{4.5.15}$$

This novel double soft theorem is again structurally similar to the corresponding double soft theorem for scalar theories, but with the uplifts $\nabla_i \to \bar{\nabla}_i$ and $R \to \bar{R}$.

With these amplitudes in hand, we can ask a different question. What happens when the momenta of two fermions are sent to zero? Take two fermions with opposite helicity and democratically scale their spinors in the soft limit. The double fermion soft limit of the six-particle amplitude in eq. (4.5.9) is

$$\lim_{p_1,p_2\to 0} A_{6,\bar{r}_1 r_2 i_3 i_4 i_5 i_6} = \frac{1}{2}\frac{[1|p_6|2\rangle}{p_6\cdot(p_1+p_2)}\bar{R}_{\bar{r}_1 r_2}{}^{j}{}_{i_6}\, A_{4,i_3 i_4 i_5 j} + \mathrm{cycl}(3456). \tag{4.5.16}$$

This agrees with eq. (4.4.3), since the single soft fermion limit vanishes.

As a last example, we take the double fermion soft limit of the six-fermion amplitude in eq. (4.5.11), which gives

$$\lim_{p_1,p_2\to 0} A_{6,\bar{r}_1 r_2 \bar{r}_3 r_4 \bar{r}_5 r_6} = \frac{[1|p_3|2\rangle \bar{R}_{\bar{r}_1 r_2 \bar{r}_3}{}^{\bar{s}}}{s_{13}+s_{23}}\left([35]\langle 64\rangle \bar{R}_{\bar{s} r_4 \bar{r}_5 r_6}\right)$$

$$
\begin{aligned}
&+\frac{[1|p_4|2\rangle \bar{R}_{\bar{r}_1 r_2 r_4}{}^{s}}{s_{14}+s_{24}}\left([35]\langle 64\rangle \bar{R}_{\bar{r}_3 s \bar{r}_5 r_6}\right)\\
&+\frac{[1|p_5|2\rangle \bar{R}_{\bar{r}_1 r_2 \bar{r}_5}{}^{\bar{s}}}{s_{15}+s_{25}}\left([35]\langle 64\rangle \bar{R}_{\bar{r}_3 r_4 \bar{s} r_6}\right)\\
&+\frac{[1|p_6|2\rangle \bar{R}_{\bar{r}_1 r_2 r_6}{}^{s}}{s_{16}+s_{26}}\left([35]\langle 64\rangle \bar{R}_{\bar{r}_3 r_4 \bar{r}_5 s}\right). \qquad (4.5.17)
\end{aligned}
$$

Again, this agrees with eq. (4.4.3).

4.5.3 Gauge bosons

Third, we consider the scattering of scalars and gauge bosons. For the sake of illustration, we take the scalars to be neutral and massless. The relevant Lagrangian is

$$
\mathcal{L} = \frac{1}{2} h_{IJ}(\phi)(\partial_\mu \phi)^I (\partial^\mu \phi)^J - \frac{1}{4} g_{AB}(\phi) F^A_{\mu\nu} F^{B\mu\nu}. \qquad (4.5.18)
$$

The scattering amplitudes for two positive-helicity gauge bosons and one, two, or three scalars are

$$
A_{3,a_1a_2i_3} = [12]^2 \frac{1}{2} \nabla_{i_3} g_{a_1a_2}, \qquad (4.5.19)
$$

$$
A_{4,a_1a_2i_3i_4} = [12]^2 \frac{1}{2} \bar{\nabla}_{i_4} \nabla_{i_3} g_{a_1a_2}, \qquad (4.5.20)
$$

$$
\begin{aligned}
A_{5,a_1a_2i_3i_4i_5} =& [12]^2 \frac{1}{2} \bar{\nabla}_{(i_4} \bar{\nabla}_{i_5} \nabla_{i_3)} g_{a_1a_2}\\
&+\left\{ \frac{(\nabla_{i_5} g_{a_1b_1}) g^{b_1b_2} (\nabla_{i_4} g_{b_2b_3}) g^{b_3b_4} (\nabla_{i_3} g_{b_4a_2})}{s_{15}s_{23}} \right.\\
&\qquad \left. \times \left[\frac{1}{8} [1|p_5p_3|2]^2 - \frac{1}{24} s_{15}s_{23}[12]^2 \right] + \text{perm}(345) \right\}\\
&+\frac{[12]^2 \nabla^j g_{a_1a_2}}{6 s_{345}} \left[s_{34}(\bar{R}_{i_3i_4i_5j} - 2\bar{R}_{i_3i_5i_4j}) + s_{35}(\bar{R}_{i_3i_5i_4j} - 2\bar{R}_{i_3i_4i_5j}) \right.\\
&\qquad \left. + s_{45}(\bar{R}_{i_3i_4i_5j} + \bar{R}_{i_3i_5i_4j}) \right]. \qquad (4.5.21)
\end{aligned}
$$

Note that the amplitudes do not vanish due to metric compatibility, $\bar{\nabla} g = 0$, because the connection in the covariant derivative ∇_i is for the scalar bundle, i.e., $\nabla_i g_{ab} = g_{ab,i}$. However, for the four-particle amplitude the connection for the full scalar–gauge-boson geometry is in play:

$$
\bar{\nabla}_{i_4} \nabla_{i_3} g_{a_1a_2} = \nabla_{i_4} \nabla_{i_3} g_{a_1a_2} - \bar{\Gamma}^b_{a_1i_4} \nabla_{i_3} g_{ba_2} - \bar{\Gamma}^b_{a_2i_4} \nabla_{i_3} g_{a_1b}. \qquad (4.5.22)
$$

Now we can study the single soft limit, starting with $p_4 \to 0$ in the four-particle amplitude,

$$
\lim_{p_4 \to 0} A_{4,a_1a_2i_3i_4} = [12]^2 \frac{1}{2} \bar{\nabla}_{i_4} \nabla_{i_3} g_{a_1a_2} = \bar{\nabla}_{i_4} A_{3,a_1a_2i_3}. \qquad (4.5.23)
$$

A more involved example is the soft limit $p_5 \to 0$ for the five-particle amplitude,

$$
\begin{aligned}
\lim_{p_5\to 0} A_{5,a_1a_2i_3i_4i_5} =&[12]^2\frac{1}{2}\bar{\nabla}_{i_5}\bar{\nabla}_{i_4}\nabla_{i_3}g_{a_1a_2}\\
&+[12]^2\frac{1}{6}\left(\bar{R}_{i_4i_5i_3}{}^{j}\nabla_j g_{a_1a_2}+\bar{R}_{i_4i_5a_1}{}^{b}\nabla_{i_3}g_{ba_2}+\bar{R}_{i_4i_5a_2}{}^{b}\nabla_{i_3}g_{a_1b}\right)\\
&+[12]^2\frac{1}{6}\left(\bar{R}_{i_3i_5i_4}{}^{j}\nabla_j g_{a_1a_2}+\bar{R}_{i_3i_5a_1}{}^{b}\nabla_{i_4}g_{ba_2}+\bar{R}_{i_3i_5a_2}{}^{b}\nabla_{i_4}g_{a_1b}\right)\\
&+\Bigg\{(\nabla_{i_5}g_{a_1b_1})g^{b_1b_2}(\nabla_{i_4}g_{b_2b_3})g^{b_3b_4}(\nabla_{i_3}g_{b_4a_2})\left[-\frac{1}{24}[12]^2\right]\\
&\qquad+\text{perm}(345)\Bigg\}\\
&+\Bigg\{(\nabla_{i_4}g_{a_1b_1})g^{b_1b_2}(\nabla_{i_5}g_{b_2b_3})g^{b_3b_4}(\nabla_{i_3}g_{b_4a_2})\left[\frac{1}{8}[12]^2\right]\\
&\qquad+(3\leftrightarrow 4)\Bigg\}\\
&+\frac{[12]^2\nabla^j g_{a_1a_2}}{6}\left[(\bar{R}_{i_3i_4i_5j}-2\bar{R}_{i_3i_5i_4j})\right]\\
=&[12]^2\frac{1}{2}\bar{\nabla}_{i_5}\bar{\nabla}_{i_4}\nabla_{i_3}g_{a_1a_2}=\bar{\nabla}_{i_5}A_{4,a_1a_2i_3i_4}. \qquad (4.5.24)
\end{aligned}
$$

Here again there are intricate cancellations between local curvature terms and curvature terms coming from factorization channels which localize in the soft limit.

As a last example for the scalar–gauge-boson theory, let us send the momenta of two scalars to zero. For the five-particle amplitude, where $p_4, p_5 \to 0$, we get

$$
\begin{aligned}
\lim_{p_4,p_5\to 0} A_{5,a_1a_2i_3i_4i_5} =& [12]^2\frac{1}{2}\bar{\nabla}_{(i_4}\bar{\nabla}_{i_5)}\nabla_{i_3}g_{a_1a_2}\\
&+[12]^2\frac{1}{12}\left(\bar{R}_{i_3i_4i_5}{}^{j}\nabla_j g_{a_1a_2}+\bar{R}_{i_3i_4a_1}{}^{b}\nabla_{i_5}g_{ba_2}+\bar{R}_{i_3i_4a_2}{}^{b}\nabla_{i_5}g_{a_1b}\right)\\
&+[12]^2\frac{1}{12}\left(\bar{R}_{i_3i_5i_4}{}^{j}\nabla_j g_{a_1a_2}+\bar{R}_{i_3i_5a_1}{}^{b}\nabla_{i_4}g_{ba_2}+\bar{R}_{i_3i_5a_2}{}^{b}\nabla_{i_4}g_{a_1b}\right)\\
&+\Bigg\{(\nabla_{i_5}g_{a_1b_1})g^{b_1b_2}(\nabla_{i_4}g_{b_2b_3})g^{b_3b_4}(\nabla_{i_3}g_{b_4a_2})\left[-\frac{1}{24}[12]^2\right]\\
&\qquad+\frac{(\nabla_{i_5}g_{a_1b_1})g^{b_1b_2}(\nabla_{i_4}g_{b_2b_3})g^{b_3b_4}(\nabla_{i_3}g_{b_4a_2})}{s_{15}s_{23}}\left[\frac{1}{8}[1|p_5p_3|2]^2\right]\\
&\qquad+\text{perm}(345)\Bigg\}\\
&+\frac{[12]^2\nabla^j g_{a_1a_2}}{6s_{345}}\left[s_{34}(\bar{R}_{i_3i_4i_5j}-2\bar{R}_{i_3i_5i_4j})+s_{35}(\bar{R}_{i_3i_5i_4j}-2\bar{R}_{i_3i_4i_5j})\right]
\end{aligned}
$$

$$
\begin{aligned}
&= [12]^2 \frac{1}{2} \bar{\nabla}_{(i_4} \bar{\nabla}_{i_5)} \nabla_{i_3} g_{a_1 a_2} \\
&- \frac{(s_{34} - s_{35})}{2(s_{34} + s_{35})} \bar{R}_{i_4 i_5 i_3}{}^{j} \left[[12]^2 \frac{1}{2} \nabla_j g_{a_1 a_2} \right] \\
&- \frac{(s_{14} - s_{15})}{2(s_{14} + s_{15})} \bar{R}_{i_4 i_5 a_1}{}^{b} \left[[12]^2 \frac{1}{2} \nabla_{i_3} g_{b a_2} \right] \\
&- \frac{(s_{24} - s_{25})}{2(s_{24} + s_{25})} \bar{R}_{i_4 i_5 a_2}{}^{b} \left[[12]^2 \frac{1}{2} \nabla_{i_3} g_{a_1 b} \right] \\
&= \bar{\nabla}_{(i_4} \bar{\nabla}_{i_5)} A_{3, a_1 a_2 i_3} + \frac{1}{2} \sum_{a=1}^{3} \frac{p_a \cdot (p_4 - p_5)}{p_a \cdot (p_4 + p_5)} \bar{R}_{i_4 i_5}{}^{\beta_a}{}_{\alpha_a} A_{3, \cdots \beta_a \cdots}.
\end{aligned}
\tag{4.5.25}
$$

This is the double soft theorem in eq. (4.4.1) where the scalars interact with gauge bosons.

4.5.4 Scalars, fermions, and gauge bosons

In this example, we will combine scalars, fermions, and gauge bosons in the same scattering amplitude. The relevant Lagrangian is

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{2} h_{IJ}(\phi) (\partial_\mu \phi)^I (\partial^\mu \phi)^J + i \frac{1}{2} k_{\bar{p}r}(\phi) (\bar{\psi}^{\bar{p}} \overleftrightarrow{\not{\partial}} \psi^r) + i \omega_{\bar{p}rI}(\phi) (\bar{\psi}^{\bar{p}} \gamma_\mu \psi^r) (\partial^\mu \phi)^I \\
& - \frac{1}{4} g_{AB}(\phi) F^A_{\mu\nu} F^{B\mu\nu} + \frac{-1}{2\sqrt{2}} d_{\bar{p}rA}(\phi) (\bar{\psi}^{\bar{p}} \sigma^{\mu\nu} \psi^r) F^A_{\mu\nu} \,,
\end{aligned}
\tag{4.5.26}
$$

where the fermions and gauge bosons couple through the dipole term $d_{\bar{p}rA}$. The normalization for the dipole term is chosen for later convenience. The scattering amplitudes with two negative-helicity fermions, one negative-helicity gauge boson, and zero, one, or two scalars are

$$
A_{3, \bar{r}_1 r_2 a_3} = \langle 13 \rangle \langle 23 \rangle d_{\bar{r}_1 r_2 a_3} \,, \tag{4.5.27}
$$

$$
A_{4, \bar{r}_1 r_2 a_3 i_4} = \langle 13 \rangle \langle 23 \rangle \bar{\nabla}_{i_4} d_{\bar{r}_1 r_2 a_3} \,, \tag{4.5.28}
$$

$$
\begin{aligned}
A_{5, \bar{r}_1 r_2 a_3 i_4 i_5} =& \langle 13 \rangle \langle 23 \rangle \bar{\nabla}_{(i_5} \bar{\nabla}_{i_4)} d_{\bar{r}_1 r_2 a_3} \\
& + \left(\langle 13 \rangle \langle 23 \rangle \frac{p_1 \cdot (p_4 - p_5)}{2 p_2 \cdot p_3} + \frac{\langle 1 | p_4 p_5 - p_5 p_4 | 3 \rangle \langle 23 \rangle}{4 p_2 \cdot p_3} \right) \bar{R}_{i_4 i_5 \bar{r}_1}{}^{\bar{s}} d_{\bar{s} r_2 a_3} \\
& + \left(\langle 13 \rangle \langle 23 \rangle \frac{p_2 \cdot (p_4 - p_5)}{2 p_1 \cdot p_3} + \frac{\langle 2 | p_4 p_5 - p_5 p_4 | 3 \rangle \langle 13 \rangle}{4 p_1 \cdot p_3} \right) \bar{R}_{i_4 i_5 r_2}{}^{s} d_{\bar{r}_1 s a_3} \\
& + \Bigg\{ d_{\bar{r}_1 r_2 b} g^{bc} (\nabla_{i_5} g_{cd}) g^{de} (\nabla_{i_4} g_{e a_3}) \times \Bigg[\langle 23 \rangle \langle 13 \rangle \frac{s_{34} - s_{35} - s_{45}}{8 s_{345}} \\
& \qquad + \left(\langle 25 \rangle \langle 31 \rangle + \langle 23 \rangle \langle 51 \rangle \right) \frac{s_{345} + s_{34} - s_{35}}{8 s_{345} s_{34}} \langle 34 \rangle [54]
\end{aligned}
$$

$$-\left(\langle 24\rangle\langle 51\rangle+\langle 25\rangle\langle 41\rangle\right)\frac{\langle 34\rangle\langle 35\rangle[45]^2}{8s_{345}s_{34}}\Bigg]+(4\leftrightarrow 5)\Bigg\}. \tag{4.5.29}$$

Here we see the interplay between the various sectors in the full field-space geometry. Consider the four-particle amplitude. It depends on the covariant derivative of the dipole coupling, which is

$$\bar{\nabla}_i d_{\bar{p}ra}=\partial_i d_{\bar{p}ra}-\bar{\Gamma}^{\bar{s}}_{i\bar{p}}d_{\bar{s}ra}-\bar{\Gamma}^{s}_{ir}d_{\bar{p}sa}-\bar{\Gamma}^{b}_{ia}d_{\bar{p}rb}. \tag{4.5.30}$$

Recall from eqs. (4.2.8), (4.2.9) and (4.2.15) that

$$\bar{\Gamma}^{p}_{Ir}=k^{p\bar{s}}\omega^{+}_{\bar{s}rI}, \tag{4.5.31}$$

$$\bar{\Gamma}^{\bar{p}}_{I\bar{r}}=-\omega^{-}_{\bar{r}sI}k^{s\bar{p}}, \tag{4.5.32}$$

$$\bar{\Gamma}^{a}_{ib}=\frac{1}{2}g^{ac}(\nabla_i g_{cb})\,. \tag{4.5.33}$$

Now we can investigate the soft limits of these amplitudes. Taking the soft limit $p_4\to 0$ in the four-particle amplitude in eq. (4.5.28), we immediately land on the covariant derivative of the three-particle amplitude in eq. (4.5.27).

The soft limit $p_5\to 0$ of the five-particle amplitude is a bit more involved. By collecting all the terms, we find that

$$\begin{aligned}\lim_{p_5\to 0}A_{5,\bar{r}_1r_2a_3i_4i_5}=&\langle 13\rangle\langle 23\rangle\left(\bar{\nabla}_{(i_5}\bar{\nabla}_{i_4)}d_{\bar{r}_1r_2a_3}\right)\\&+\langle 13\rangle\langle 23\rangle\frac{1}{2}\left(\bar{R}_{i_4i_5\bar{r}_1}{}^{\bar{s}}d_{\bar{s}r_2a_3}+\bar{R}_{i_4i_5r_2}{}^{s}d_{\bar{r}_1sa_3}-\bar{R}_{i_4i_5a_3}{}^{b}d_{\bar{r}_1r_2b}\right)\\=&\langle 13\rangle\langle 23\rangle\bar{\nabla}_{i_5}\bar{\nabla}_{i_4}d_{\bar{r}_1r_2a_3}=\bar{\nabla}_{i_5}A_{4,\bar{r}_1r_2a_3i_4}\,.\end{aligned} \tag{4.5.34}$$

This is the geometric soft theorem.

Finally, we look at the double soft limit of the five-particle amplitude. The scalar double soft limit is

$$\begin{aligned}\lim_{p_4,p_5\to 0}A_{5,\bar{r}_1r_2a_3i_4i_5}=&\langle 13\rangle\langle 23\rangle\bar{\nabla}_{(i_5}\bar{\nabla}_{i_4)}d_{\bar{r}_1r_2a_3}\\&+\left(\langle 13\rangle\langle 23\rangle\frac{p_1\cdot(p_4-p_5)}{2p_1\cdot(p_4+p_5)}\right)\bar{R}_{i_4i_5\bar{r}_1}{}^{\bar{s}}d_{\bar{s}r_2a_3}\\&+\left(\langle 13\rangle\langle 23\rangle\frac{p_2\cdot(p_4-p_5)}{2p_2\cdot(p_4+p_5)}\right)\bar{R}_{i_4i_5r_2}{}^{s}d_{\bar{r}_1sa_3}\\&-\left(\langle 13\rangle\langle 23\rangle\frac{p_3\cdot(p_4-p_5)}{2p_3\cdot(p_4+p_5)}\right)\bar{R}_{i_4i_5a_3}{}^{b}d_{\bar{r}_1r_2b}\end{aligned}$$

$$=\bar{\nabla}_{(i_4}\bar{\nabla}_{i_5)}A_{3,a_1a_2i_3}+\frac{1}{2}\sum_{a=1}^{3}\frac{p_a\cdot(p_4-p_5)}{p_a\cdot(p_4+p_5)}\bar{R}_{i_4i_5}{}^{\beta_a}{}_{\alpha_a}\,A_{3,\cdots\beta_a\cdots}. \tag{4.5.35}$$

This agrees with eq. (4.4.1).

4.5.5 Massive gauge bosons

Lastly, we will consider an example with massive gauge bosons. To keep the expressions manageable, we restrict to a flat field-space geometry for the gauge fields with the Lagrangian

$$\mathcal{L}=\frac{1}{2}h_{IJ}(\phi)(D_\mu\phi)^I(D^\mu\phi)^J-V(\phi)-\frac{1}{4}\delta_{AB}F^A_{\mu\nu}F^{B\mu\nu}. \tag{4.5.36}$$

Furthermore, we assume the following spectrum: scalars with arbitrary masses m_j (could be either massive or massless) and massive gauge bosons with mass m. The three- and four-point amplitudes for massive gauge bosons and massless scalars are

$$A_{3,a_1a_2i_3}=\frac{\nabla_{i_3}m^2_{a_1a_2}}{m^2}\langle\mathbf{12}\rangle[\mathbf{21}]\,, \tag{4.5.37}$$

$$A_{3,a_1i_2i_3}=-i\sqrt{2}\nabla_{i_3}t_{a_1i_2}\frac{\langle\mathbf{1}|p_3|\mathbf{1}]}{m}\,, \tag{4.5.38}$$

$$\begin{aligned}A_{4,a_1a_2i_3i_4}=&\frac{1}{m^2}\nabla_{i_4}\nabla_{i_3}m^2_{a_1a_2}\langle\mathbf{12}\rangle[\mathbf{21}]+\sum_j\frac{\nabla_{i_4}V_{i_3}{}^j}{s_{34}-m_j^2}\frac{\nabla_j m^2_{a_1a_2}}{m^2}\langle\mathbf{12}\rangle[\mathbf{21}]\\&+\frac{1}{m^2}\left[\frac{1}{s_{14}-m^2}\nabla_{i_4}m^2_{a_1}{}^{b_1}\nabla_{i_3}m^2_{a_2b_1}\left(\langle\mathbf{12}\rangle[\mathbf{21}]+\frac{1}{2m^2}\langle\mathbf{1}|p_4|\mathbf{1}]\langle\mathbf{2}|p_3|\mathbf{2}]\right)\right.\\&\left.+\sum_j\frac{2\nabla_{i_4}t_{a_1}{}^j\langle\mathbf{1}|p_4|\mathbf{1}]}{s_{14}-m_j^2}\nabla_{i_3}t_{a_2j}\langle\mathbf{2}|p_3|\mathbf{2}]+(1\leftrightarrow 2)\right].\end{aligned} \tag{4.5.39}$$

Note that in eq. (4.5.39) we allow for the exchange of scalars of arbitrary masses m_j.

We will verify the soft theorem for the four-point amplitude in eq. (4.5.39). Sending the momentum of the massless scalar p_4 to zero, we obtain in the soft limit

$$\begin{aligned}\lim_{p_4\to 0}A_{4,a_1a_2i_3i_4}=&\frac{1}{m^2}\nabla_{i_4}\nabla_{i_3}m^2_{a_1a_2}\langle\mathbf{12}\rangle[\mathbf{21}]\\&+\frac{\nabla_{i_4}V_{i_3}{}^j}{2p_3\cdot p_4}\frac{\nabla_j m^2_{a_1a_2}}{m^2}\langle\mathbf{12}\rangle[\mathbf{21}]\end{aligned}$$

$$+\frac{1}{m^2}\Bigg[\frac{\nabla_{i_4}m^{2\ \ b_1}_{a_1}}{2p_1\cdot p_4}\nabla_{i_3}m^2_{a_2b_1}\Big(\langle\mathbf{12}\rangle[\mathbf{21}]+\frac{1}{2m^2}\langle\mathbf{1}|p_4|\mathbf{1}]\langle\mathbf{2}|p_3|\mathbf{2}]\Big)$$
$$+\frac{2(\nabla_{i_4}t_{a_1I})\epsilon^{Ij}\langle\mathbf{1}|p_4|\mathbf{1}]}{2p_1\cdot p_4}\nabla_{i_3}t_{a_2j}\langle\mathbf{2}|p_3|\mathbf{2}]+(1\leftrightarrow 2)\Bigg]. \tag{4.5.40}$$

On the other hand, the soft limit is given by the soft operator in eq. (4.3.13) acting on the lower-point amplitude. Note that this requires a choice for the off-shell continuation of $A_{3,a_1a_2i_3}$. As discussed in section 4.3.3, the soft theorem is independent of that particular choice, as we will see later on.

In our example, it is convenient to write the normalization of polarization vectors in terms of momenta $|p_i|=\sqrt{p_i^2}$ in eq. (4.5.37). Evaluating the soft theorem for this case, we find that

$$\lim_{p_4\to 0}A_{4,a_1a_2i_3i_4}= \tag{4.5.41}$$
$$\nabla_{i_4}\left(\frac{\nabla_{i_3}m^2_{a_1a_2}}{|p_1||p_2|}\langle\mathbf{12}\rangle[\mathbf{21}]\right)$$
$$+\frac{\nabla_{i_4}V_{i_3}^{\ \ j}}{2p_3\cdot p_4}\left(1+p_4^\mu\frac{\partial}{\partial p_3^\mu}\right)\left(\frac{\nabla_j m^2_{a_1a_2}}{|p_1||p_2|}\langle\mathbf{12}\rangle[\mathbf{21}]\right)$$
$$+\Bigg[-\sum_{\text{spin}}\frac{(\nabla_{i_4}m^2_{i_1A})}{2p_1\cdot p_4}\frac{\langle\mathbf{1}|\epsilon^{*Ab_1}|\mathbf{1}\rangle}{m}\left(1+p_4^\mu\frac{\partial}{\partial p_1^\mu}\right)\left(\nabla_{i_3}m^2_{a_2B}\frac{\langle\mathbf{2}|\epsilon^B_{b_1}|\mathbf{2}]}{|p_2|}\right)$$
$$+\frac{2i(\nabla_{i_4}t_{a_1I})\epsilon^{Ij}}{2p_1\cdot p_4}\frac{\langle\mathbf{1}|p_4|\mathbf{1}]}{\sqrt{2}m}\left(-i\sqrt{2}\nabla_{i_3}t_{a_2j}\frac{\langle\mathbf{2}|p_3|\mathbf{2}]}{|p_2|}\right)+(1\leftrightarrow 2)\Bigg]. \tag{4.5.42}$$

Comparing the two expressions, we see that first, second and fourth lines in eq. (4.5.40) match with eq. (4.5.41). Next, we need to implement the soft-momentum-shift operator acting on the lower-point amplitude in terms of the spinors (see ref. [211]). One such option of a shift by soft momentum q is given by

$$\langle\mathbf{p}|\to\langle\mathbf{p}|+\frac{\langle\mathbf{p}|pq}{2m^2}\,,$$
$$|\mathbf{p}]\to|\mathbf{p}]+\frac{qp|\mathbf{p}]}{2m^2}\,. \tag{4.5.43}$$

Applying the above shift with soft p_4 to hard momentum p_1, we obtain

$$\left(1+p_4^\mu\frac{\partial}{\partial p_1^\mu}\right)\frac{\langle\mathbf{21}\rangle[\mathbf{12}]}{|p_1||p_2|}\to\frac{\langle\mathbf{21}\rangle[\mathbf{12}]}{|p_1||p_2|}-\frac{\langle\mathbf{1}|p_4|\mathbf{1}]}{2p_1^2}\frac{\langle\mathbf{2}|p_1|\mathbf{2}]}{|p_1||p_2|}+\mathcal{O}(p_4^2)$$

$$=\frac{\langle\mathbf{21}\rangle[\mathbf{12}]}{m^2}+\frac{\langle\mathbf{1}|p_4|\mathbf{1}]}{2m^2}\frac{\langle\mathbf{2}|p_3|\mathbf{2}]}{m^2}+\mathcal{O}(p_4^2)\,, \qquad (4.5.44)$$

where we used that in the soft limit, $\langle\mathbf{2}|p_3|\mathbf{2}] = -\langle\mathbf{2}|p_1|\mathbf{2}] + \mathcal{O}(p_4)$. This matches with the third line in eq. (4.5.40). Hence, we have verified that the soft limit of $A_{4,a_1a_2i_3i_4}$ is given by the soft theorem in the presence of massive gauge bosons.

Let us briefly comment on a different choice of an off-shell continuation of the lower-point amplitude. Suppose we chose to evaluate the soft limit by directly applying the soft operator to eq. (4.5.37), instead of using eq. (4.5.41). We see that the covariant derivative $\bar{\nabla}A_3$ will now pick up additional terms. At the same time, the soft shift eq. (4.5.43) acting on A_3 will also have extra terms. Those two contributions precisely cancel, as required, and the soft theorem again agrees with eq. (4.5.40).

All these examples demonstrate that the universal behavior of the soft limits for massless scalars is captured by the geometric soft theorem.

4.6 Conclusion

Scattering amplitudes in any effective field theory have a universal feature; they are invariant under changes of field basis. This invariance is manifest when we express all couplings in the theory as geometric structures, such as the Riemann curvature in field space. This was initially appreciated for scalars, and now this geometric picture has been extended to both fermions and gauge fields.

The geometry also exposes new relations *b*etween scattering amplitudes. The geometric soft theorem for scalar effective field theories [13] relates scattering amplitudes with different number of particles via the covariant derivative. In this paper, we complete this story by extending the geometric soft theorem to generic effective field theories with scalars, fermions, and gauge bosons. The more general soft theorem is still linked to the covariant derivative but now for the full field space.

Soft theorems in effective field theories can be leveraged to recursively calculate higher-point scattering amplitudes from lower-point amplitudes. The bad high-energy behavior of effective-field-theory amplitudes can be ameliorated via an appropriate subtraction which uses the knowledge of the soft behavior [9], [10], [30], [31], [92]–[95], [101], [104], [105]. This also applies to general massless scalars using the geometric soft theorem in ref. [13]. Of course, in the latter case there is no free lunch. Information about higher-point contact terms is encoded in the Riemann curvature, which appears in the four-point amplitude when viewing the curvature as a function of the VEV. Using the more general geometric soft theorems presented here we can enroll many additional effective field theories (e.g., eq. (4.5.6)) in the list of on-shell constructible theories, whose amplitudes satisfy recursion relations. We look forward to studying such recursion relations in future work.

Even though the field-space geometry has proven valuable for understanding effective field theories, there is still a larger landscape of invariances which is not accounted for. Namely, field redefinitions with derivatives also leave the scattering amplitudes unchanged. A full geometric explanation of this invariance is a topic of current investigations [47]–[52]. However, any extension of the geometric picture to accommodate such field redefinitions will not affect the geometric soft theorem because the derivative deformations needed to accomplish this would vanish in the soft limit.

A natural question to ask is whether there is a version of the geometric soft theorem that holds beyond tree level. In the simpler case where singular terms in the soft limit are absent, we believe that the soft theorem remains valid at all loop orders, perhaps even non-perturbatively. In this case, the derivation is nearly identical to one derivation of the Adler zero for pions, or the geometric soft theorem for scalar effective field theories. It will be instructive to find a rigorous proof of this, and also to investigate the fate of the geometric soft theorem at loop-level when the singular terms are present.

BIBLIOGRAPHY

[1] Y. Nambu, "Quasiparticles and Gauge Invariance in the Theory of Superconductivity," *Phys. Rev.*, vol. 117, J. C. Taylor, Ed., pp. 648–663, 1960. DOI: 10.1103/PhysRev.117.648.

[2] J. Goldstone, "Field Theories with Superconductor Solutions," *Nuovo Cim.*, vol. 19, pp. 154–164, 1961. DOI: 10.1007/BF02812722.

[3] J. Goldstone, A. Salam, and S. Weinberg, "Broken Symmetries," *Phys. Rev.*, vol. 127, pp. 965–970, 1962. DOI: 10.1103/PhysRev.127.965.

[4] S. L. Adler, "Consistency conditions on the strong interactions implied by a partially conserved axial vector current," *Phys. Rev.*, vol. 137, B1022–B1033, 1965. DOI: 10.1103/PhysRev.137.B1022.

[5] S. Weinberg, "Current-Commutator Theory of Multiple Pion Production," *Phys. Rev. Lett.*, vol. 16, no. 19, pp. 879–883, 1966. DOI: 10.1103/PhysRevLett.16.879.

[6] R. F. Dashen, "Chiral $SU(3) \times SU(3)$ as a symmetry of the strong interactions," *Phys. Rev.*, vol. 183, pp. 1245–1260, 1969. DOI: 10.1103/PhysRev.183.1245.

[7] S. Weinberg, "Pion scattering lengths," *Phys. Rev. Lett.*, vol. 17, pp. 616–621, 1966. DOI: 10.1103/PhysRevLett.17.616.

[8] N. Arkani-Hamed, F. Cachazo, and J. Kaplan, "What is the Simplest Quantum Field Theory?" *JHEP*, vol. 09, p. 016, 2010. DOI: 10.1007/JHEP09(2010)016. arXiv: 0808.1446 [hep-th].

[9] C. Cheung, K. Kampf, J. Novotny, and J. Trnka, "Effective Field Theories from Soft Limits of Scattering Amplitudes," *Phys. Rev. Lett.*, vol. 114, no. 22, p. 221 602, 2015. DOI: 10.1103/PhysRevLett.114.221602. arXiv: 1412.4095 [hep-th].

[10] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen, J. Trnka, and C. Wen, "Vector Effective Field Theories from Soft Limits," *Phys. Rev. Lett.*, vol. 120, no. 26, p. 261 602, 2018. DOI: 10.1103/PhysRevLett.120.261602. arXiv: 1801.01496 [hep-th].

[11] E. Pajer, D. Stefanyszyn, and J. Supeł, "The Boostless Bootstrap: Amplitudes without Lorentz boosts," *JHEP*, vol. 12, p. 198, 2020, [Erratum: JHEP 04, 023 (2022)]. DOI: 10.1007/JHEP12(2020)198. arXiv: 2007.00027 [hep-th].

[12] D. Stefanyszyn and J. Supeł, "The Boostless Bootstrap and BCFW Momentum Shifts," *JHEP*, vol. 03, p. 091, 2021. DOI: 10.1007/JHEP03(2021)091. arXiv: 2009.14289 [hep-th].

[13] C. Cheung, A. Helset, and J. Parra-Martinez, "Geometric soft theorems," *JHEP*, vol. 04, p. 011, 2022. DOI: 10.1007/JHEP04(2022)011. arXiv: 2111.03045 [hep-th].

[14] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, "The Effective Field Theory of Inflation," *JHEP*, vol. 03, p. 014, 2008. DOI: 10.1088/1126-6708/2008/03/014. arXiv: 0709.0293 [hep-th].

[15] P. Creminelli, M. A. Luty, A. Nicolis, and L. Senatore, "Starting the Universe: Stable Violation of the Null Energy Condition and Non-standard Cosmologies," *JHEP*, vol. 12, p. 080, 2006. DOI: 10.1088/1126-6708/2006/12/080. arXiv: hep-th/0606090.

[16] S. Endlich, A. Nicolis, and J. Wang, "Solid Inflation," *JCAP*, vol. 10, p. 011, 2013. DOI: 10.1088/1475-7516/2013/10/011. arXiv: 1210.0569 [hep-th].

[17] D. Baumann and D. Green, "Equilateral Non-Gaussianity and New Physics on the Horizon," *JCAP*, vol. 09, p. 014, 2011. DOI: 10.1088/1475-7516/2011/09/014. arXiv: 1102.5343 [hep-th].

[18] D. Green, Y. Huang, and C.-H. Shen, "Inflationary Adler conditions," *Phys. Rev. D*, vol. 107, no. 4, p. 043 534, 2023. DOI: 10.1103/PhysRevD.107.043534. arXiv: 2208.14544 [hep-th].

[19] A. Nicolis, R. Penco, F. Piazza, and R. Rattazzi, "Zoology of condensed matter: Framids, ordinary stuff, extra-ordinary stuff," *JHEP*, vol. 06, p. 155, 2015. DOI: 10.1007/JHEP06(2015)155. arXiv: 1501.03845 [hep-th].

[20] S. Endlich, A. Nicolis, R. Rattazzi, and J. Wang, "The Quantum mechanics of perfect fluids," *JHEP*, vol. 04, p. 102, 2011. DOI: 10.1007/JHEP04(2011)102. arXiv: 1011.6396 [hep-th].

[21] E. A. Ivanov and V. I. Ogievetsky, "The Inverse Higgs Phenomenon in Nonlinear Realizations," *Teor. Mat. Fiz.*, vol. 25, pp. 164–177, 1975. DOI: 10.1007/BF01028947.

[22] I. Low and A. V. Manohar, "Spontaneously broken space-time symmetries and Goldstone's theorem," *Phys. Rev. Lett.*, vol. 88, p. 101 602, 2002. DOI: 10.1103/PhysRevLett.88.101602. arXiv: hep-th/0110285.

[23] H. Watanabe and H. Murayama, "Redundancies in Nambu-Goldstone Bosons," *Physical Review Letters*, vol. 110, no. 18, p. 181 601, May 2013, ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.110.181601. arXiv: 1302.4800 [cond-mat.other].

[24] T. Brauner and H. Watanabe, "Spontaneous breaking of spacetime symmetries and the inverse Higgs effect," *Phys. Rev. D*, vol. 89, no. 8, p. 085 004, 2014. DOI: 10.1103/PhysRevD.89.085004. arXiv: 1401.5596 [hep-ph].

[25] A. Kapustin, “Remarks on nonrelativistic Goldstone bosons,” Jul. 2012. arXiv: 1207.0457 [hep-ph].

[26] S. R. Coleman, J. Wess, and B. Zumino, “Structure of phenomenological Lagrangians. 1.,” *Phys. Rev.*, vol. 177, pp. 2239–2247, 1969. DOI: 10.1103/PhysRev.177.2239.

[27] C. G. Callan Jr., S. R. Coleman, J. Wess, and B. Zumino, “Structure of phenomenological Lagrangians. 2.,” *Phys. Rev.*, vol. 177, pp. 2247–2250, 1969. DOI: 10.1103/PhysRev.177.2247.

[28] D. V. Volkov, “Phenomenological lagrangians,” *Sov. J. Particles Nucl.*, vol. 4, pp. 1–17, 1973.

[29] A. Nicolis, R. Penco, F. Piazza, and R. A. Rosen, “More on gapped Goldstones at finite density: More gapped Goldstones,” *JHEP*, vol. 11, p. 055, 2013. DOI: 10.1007/JHEP11(2013)055. arXiv: 1306.1240 [hep-th].

[30] H. Luo and C. Wen, “Recursion relations from soft theorems,” *JHEP*, vol. 03, p. 088, 2016. DOI: 10.1007/JHEP03(2016)088. arXiv: 1512.06801 [hep-th].

[31] H. Elvang, M. Hadjiantonis, C. R. T. Jones, and S. Paranjape, “Soft Bootstrap and Supersymmetry,” *JHEP*, vol. 01, p. 195, 2019. DOI: 10.1007/JHEP01(2019)195. arXiv: 1806.06079 [hep-th].

[32] T. D. Brennan and S. Hong, “Introduction to Generalized Global Symmetries in QFT and Particle Physics,” Jun. 2023. arXiv: 2306.00912 [hep-ph].

[33] C. Córdova, T. T. Dumitrescu, K. Intriligator, and S.-H. Shao, “Snowmass White Paper: Generalized Symmetries in Quantum Field Theory and Beyond,” in *Snowmass 2021*, May 2022. arXiv: 2205.09545 [hep-th].

[34] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, “Generalized Global Symmetries,” *JHEP*, vol. 02, p. 172, 2015. DOI: 10.1007/JHEP02(2015)172. arXiv: 1412.5148 [hep-th].

[35] Y. Hidaka, Y. Hirono, and R. Yokokura, “Counting Nambu-Goldstone Modes of Higher-Form Global Symmetries,” *Phys. Rev. Lett.*, vol. 126, no. 7, p. 071601, 2021. DOI: 10.1103/PhysRevLett.126.071601. arXiv: 2007.15901 [hep-th].

[36] E. Lake, “Higher-form symmetries and spontaneous symmetry breaking,” Feb. 2018. arXiv: 1802.07747 [hep-th].

[37] S. Weinberg, “Infrared photons and gravitons,” *Phys. Rev.*, vol. 140, B516–B524, 1965. DOI: 10.1103/PhysRev.140.B516.

[38] S. Weinberg, “Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass,” *Phys. Rev.*, vol. 135, B1049–B1056, 1964. DOI: 10.1103/PhysRev.135.B1049.

[39] A. Antinucci, F. Benini, and G. Rizi, “Holographic Duals of Symmetry Broken Phases,” *Fortsch. Phys.*, vol. 72, no. 12, p. 2 400 172, 2024. DOI: 10.1002/prop.202400172. arXiv: 2408.01418 [hep-th].

[40] R. Alonso, E. E. Jenkins, and A. V. Manohar, “Geometry of the Scalar Sector,” *JHEP*, vol. 08, p. 101, 2016. DOI: 10.1007/JHEP08(2016)101. arXiv: 1605.03602 [hep-ph].

[41] K. Meetz, “Realization of chiral symmetry in a curved isospin space,” *J. Math. Phys.*, vol. 10, pp. 589–593, 1969. DOI: 10.1063/1.1664881.

[42] J. Honerkamp, “Chiral multiloops,” *Nucl. Phys. B*, vol. 36, pp. 130–140, 1972. DOI: 10.1016/0550-3213(72)90299-4.

[43] G. Ecker and J. Honerkamp, “Application of invariant renormalization to the nonlinear chiral invariant pion lagrangian in the one-loop approximation,” *Nucl. Phys. B*, vol. 35, pp. 481–492, 1971. DOI: 10.1016/0550-3213(71)90468-8.

[44] L. Tataru, “One Loop Divergences of the Nonlinear Chiral Theory,” *Phys. Rev. D*, vol. 12, pp. 3351–3352, 1975. DOI: 10.1103/PhysRevD.12.3351.

[45] Y.-T. Lee, “Field Space Geometry and Nonlinear Supersymmetry,” Oct. 2024. arXiv: 2410.21395 [hep-th].

[46] K. Finn, S. Karamitsos, and A. Pilaftsis, “Frame Covariance in Quantum Gravity,” *Phys. Rev. D*, vol. 102, no. 4, p. 045 014, 2020. DOI: 10.1103/PhysRevD.102.045014. arXiv: 1910.06661 [hep-th].

[47] C. Cheung, A. Helset, and J. Parra-Martinez, “Geometry-kinematics duality,” *Phys. Rev. D*, vol. 106, no. 4, p. 045 016, 2022. DOI: 10.1103/PhysRevD.106.045016. arXiv: 2202.06972 [hep-th].

[48] T. Cohen, N. Craig, X. Lu, and D. Sutherland, “On-Shell Covariance of Quantum Field Theory Amplitudes,” *Phys. Rev. Lett.*, vol. 130, no. 4, p. 041 603, 2023. DOI: 10.1103/PhysRevLett.130.041603. arXiv: 2202.06965 [hep-th].

[49] T. Cohen, X. Lu, and D. Sutherland, “On Amplitudes and Field Redefinitions,” Dec. 2023. arXiv: 2312.06748 [hep-th].

[50] N. Craig, Y.-T. Lee, X. Lu, and D. Sutherland, “Effective Field Theories as Lagrange Spaces,” May 2023. arXiv: 2305.09722 [hep-th].

[51] N. Craig, Yu-Tse, and Lee, “Effective Field Theories on the Jet Bundle,” Jul. 2023. arXiv: 2307.15742 [hep-th].

[52] M. Alminawi, I. Brivio, and J. Davighi, “Jet Bundle Geometry of Scalar Field Theories,” Jul. 2023. arXiv: 2308.00017 [hep-ph].

[53] T. Cohen, X. Lu, and Z. Zhang, “What is the geometry of effective field theories?” *Phys. Rev. D*, vol. 111, no. 8, p. 085 012, 2025. DOI: 10.1103/PhysRevD.111.085012. arXiv: 2410.21378 [hep-th].

[54] P. Aigner, L. Bellafronte, E. Gendy, D. Haslehner, and A. Weiler, “Renormalising the Field-Space Geometry,” Mar. 2025. arXiv: 2503.09785 [hep-th].

[55] R. Alonso, E. E. Jenkins, and A. V. Manohar, “A Geometric Formulation of Higgs Effective Field Theory: Measuring the Curvature of Scalar Field Space,” *Phys. Lett. B*, vol. 754, pp. 335–342, 2016. DOI: 10.1016/j.physletb.2016.01.041. arXiv: 1511.00724 [hep-ph].

[56] R. Alonso, K. Kanshin, and S. Saa, “Renormalization group evolution of Higgs effective field theory,” *Phys. Rev. D*, vol. 97, no. 3, p. 035 010, 2018. DOI: 10.1103/PhysRevD.97.035010. arXiv: 1710.06848 [hep-ph].

[57] A. Helset, M. Paraskevas, and M. Trott, “Gauge fixing the Standard Model Effective Field Theory,” *Phys. Rev. Lett.*, vol. 120, no. 25, p. 251 801, 2018. DOI: 10.1103/PhysRevLett.120.251801. arXiv: 1803.08001 [hep-ph].

[58] A. Helset, A. Martin, and M. Trott, “The Geometric Standard Model Effective Field Theory,” *JHEP*, vol. 03, p. 163, 2020. DOI: 10.1007/JHEP03(2020)163. arXiv: 2001.01453 [hep-ph].

[59] T. Cohen, N. Craig, X. Lu, and D. Sutherland, “Is SMEFT Enough?” *JHEP*, vol. 03, p. 237, 2021. DOI: 10.1007/JHEP03(2021)237. arXiv: 2008.08597 [hep-ph].

[60] T. Cohen, N. Craig, X. Lu, and D. Sutherland, “Unitarity violation and the geometry of Higgs EFTs,” *JHEP*, vol. 12, p. 003, 2021. DOI: 10.1007/JHEP12(2021)003. arXiv: 2108.03240 [hep-ph].

[61] A. Helset, E. E. Jenkins, and A. V. Manohar, “Renormalization of the Standard Model Effective Field Theory from geometry,” *JHEP*, vol. 02, p. 063, 2023. DOI: 10.1007/JHEP02(2023)063. arXiv: 2212.03253 [hep-ph].

[62] R. Alonso and M. West, “On the effective action for scalars in a general manifold to any loop order,” *Phys. Lett. B*, vol. 841, p. 137 937, 2023. DOI: 10.1016/j.physletb.2023.137937. arXiv: 2207.02050 [hep-th].

[63] R. Alonso, “A primer on Higgs Effective Field Theory with Geometry,” Jul. 2023. arXiv: 2307.14301 [hep-ph].

[64] E. E. Jenkins, A. V. Manohar, L. Naterop, and J. Pagès, "Two Loop Renormalization of Scalar Theories using a Geometric Approach," Oct. 2023. arXiv: 2310.19883 [hep-ph].

[65] T. Cohen, I. Fadakar, A. Helset, and F. Nardi, "Geometry of soft scalars at one loop," Apr. 2025. arXiv: 2504.12371 [hep-th].

[66] B. Assi, A. Helset, A. V. Manohar, J. Pagès, and C.-H. Shen, "Fermion geometry and the renormalization of the Standard Model Effective Field Theory," *JHEP*, vol. 11, p. 201, 2023. DOI: 10.1007/JHEP11(2023)201. arXiv: 2307.03187 [hep-ph].

[67] K. Finn, S. Karamitsos, and A. Pilaftsis, "Frame covariant formalism for fermionic theories," *Eur. Phys. J. C*, vol. 81, no. 7, p. 572, 2021. DOI: 10.1140/epjc/s10052-021-09360-w. arXiv: 2006.05831 [hep-th].

[68] V. Gattus and A. Pilaftsis, "Minimal supergeometric quantum field theories," *Phys. Lett. B*, vol. 846, p. 138 234, 2023. DOI: 10.1016/j.physletb.2023.138234. arXiv: 2307.01126 [hep-th].

[69] A. Helset, E. E. Jenkins, and A. V. Manohar, "Geometry in scattering amplitudes," *Phys. Rev. D*, vol. 106, no. 11, p. 116 018, 2022. DOI: 10.1103/PhysRevD.106.116018. arXiv: 2210.08000 [hep-ph].

[70] H. Leutwyler, "Phonons as goldstone bosons," *Helv. Phys. Acta*, vol. 70, pp. 275–286, 1997. arXiv: hep-ph/9609466.

[71] H. Leutwyler, "Nonrelativistic effective Lagrangians," *Phys. Rev. D*, vol. 49, pp. 3033–3043, 1994. DOI: 10.1103/PhysRevD.49.3033. arXiv: hep-ph/9311264.

[72] D. T. Son, "Low-energy quantum effective action for relativistic superfluids," Apr. 2002. arXiv: hep-ph/0204199.

[73] J. O. Andersen, "Effective field theory for Goldstone bosons in nonrelativistic superfluids," Sep. 2002. arXiv: cond-mat/0209243.

[74] T. Brauner, "Spontaneous Symmetry Breaking and Nambu-Goldstone Bosons in Quantum Many-Body Systems," *Symmetry*, vol. 2, pp. 609–657, 2010. DOI: 10.3390/sym2020609. arXiv: 1001.5212 [hep-th].

[75] V. Assassi, D. Baumann, and D. Green, "On Soft Limits of Inflationary Correlation Functions," *JCAP*, vol. 11, p. 047, 2012. DOI: 10.1088/1475-7516/2012/11/047. arXiv: 1204.4207 [hep-th].

[76] B. Finelli, G. Goon, E. Pajer, and L. Santoni, "Soft Theorems For Shift-Symmetric Cosmologies," *Phys. Rev. D*, vol. 97, no. 6, p. 063 531, 2018. DOI: 10.1103/PhysRevD.97.063531. arXiv: 1711.03737 [hep-th].

[77] L. Hui, A. Joyce, and S. S. C. Wong, "Inflationary soft theorems revisited: A generalized consistency relation," *JCAP*, vol. 02, p. 060, 2019. DOI: 10.1088/1475-7516/2019/02/060. arXiv: 1811.05951 [hep-th].

[78] S. Jazayeri, E. Pajer, and D. van der Woude, "Solid Soft Theorems," *JCAP*, vol. 06, p. 011, 2019. DOI: 10.1088/1475-7516/2019/06/011. arXiv: 1902.09020 [hep-th].

[79] L. Hui, A. Joyce, I. Komissarov, K. Parmentier, L. Santoni, and S. S. C. Wong, "Soft theorems for boosts and other time symmetries," Oct. 2022. arXiv: 2210.16276 [hep-th].

[80] R. V. Lange, "Goldstone Theorem in Nonrelativistic Theories," *Physical Review Letters*, vol. 14, no. 1, pp. 3–6, 1965, ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.14.3.

[81] R. V. Lange, "Nonrelativistic Theorem Analogous to the Goldstone Theorem," *Physical Review*, vol. 146, no. 1, pp. 301–303, 1966, ISSN: 0031-899X. DOI: 10.1103/PhysRev.146.301.

[82] H. Nielsen and S. Chadha, "On how to count Goldstone bosons," *Nuclear Physics B*, vol. 105, no. 3, pp. 445–453, 1976, ISSN: 05503213. DOI: 10.1016/0550-3213(76)90025-0.

[83] H. Watanabe and T. Brauner, "Spontaneous breaking of continuous translational invariance," *Physical Review D*, vol. 85, no. 8, p. 085 010, 2012, ISSN: 1550-7998, 1550-2368. DOI: 10.1103/PhysRevD.85.085010. arXiv: 1112.3890 [cond-mat.stat-mech].

[84] H. Watanabe and T. Brauner, "On the number of Nambu-Goldstone bosons and its relation to charge densities," *Physical Review D*, vol. 84, no. 12, p. 125 013, 2011, ISSN: 1550-7998, 1550-2368. DOI: 10.1103/PhysRevD.84.125013. arXiv: 1109.6327 [hep-ph].

[85] H. Watanabe and H. Murayama, "Unified Description of Nambu-Goldstone Bosons without Lorentz Invariance," *Physical Review Letters*, vol. 108, no. 25, p. 251 602, 2012, ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.108.251602. arXiv: 1203.0609 [hep-th].

[86] H. Watanabe and H. Murayama, "Effective Lagrangian for Nonrelativistic Systems," *Physical Review X*, vol. 4, no. 3, p. 031 057, Sep. 2014, ISSN: 2160-3308. DOI: 10.1103/PhysRevX.4.031057. arXiv: 1402.7066 [hep-th].

[87] T. Brauner, A. Esposito, and R. Penco, "Fractional Soft Limits of Scattering Amplitudes," *Phys. Rev. Lett.*, vol. 128, no. 23, p. 231 601, 2022. DOI: 10.1103/PhysRevLett.128.231601. arXiv: 2203.00022 [hep-th].

[88] A. Dersy, A. Khmelnitsky, and R. Rattazzi, "The Quantum Perfect Fluid in 2D," Nov. 2022. arXiv: 2211.09820 [hep-th].

[89] C. Cheung, J. Mangan, J. Parra-Martinez, and N. Shah, "Non-perturbative Double Copy in Flatland," *Phys. Rev. Lett.*, vol. 129, no. 22, p. 221 602, 2022. DOI: 10.1103/PhysRevLett.129.221602. arXiv: 2204.07130 [hep-th].

[90] C. Cheung and J. Mangan, "Scattering Amplitudes and the Navier-Stokes Equation," Oct. 2020. arXiv: 2010.15970 [hep-th].

[91] L. Susskind and G. Frye, "Algebraic aspects of pionic duality diagrams," *Phys. Rev. D*, vol. 1, pp. 1682–1686, 1970. DOI: 10.1103/PhysRevD.1.1682.

[92] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen, and J. Trnka, "A Periodic Table of Effective Field Theories," *JHEP*, vol. 02, p. 020, 2017. DOI: 10.1007/JHEP02(2017)020. arXiv: 1611.03137 [hep-th].

[93] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen, and J. Trnka, "On-Shell Recursion Relations for Effective Field Theories," *Phys. Rev. Lett.*, vol. 116, no. 4, p. 041 601, 2016. DOI: 10.1103/PhysRevLett.116.041601. arXiv: 1509.03309 [hep-th].

[94] K. Kampf, J. Novotny, and J. Trnka, "Tree-level Amplitudes in the Nonlinear Sigma Model," *JHEP*, vol. 05, p. 032, 2013. DOI: 10.1007/JHEP05(2013)032. arXiv: 1304.3048 [hep-th].

[95] H. Elvang, M. Hadjiantonis, C. R. T. Jones, and S. Paranjape, "On the Supersymmetrization of Galileon Theories in Four Dimensions," *Phys. Lett. B*, vol. 781, pp. 656–663, 2018. DOI: 10.1016/j.physletb.2018.04.032. arXiv: 1712.09937 [hep-th].

[96] I. Low, "Adler's zero and effective Lagrangians for nonlinearly realized symmetry," *Phys. Rev. D*, vol. 91, no. 10, p. 105 017, 2015. DOI: 10.1103/PhysRevD.91.105017. arXiv: 1412.2145 [hep-th].

[97] I. Low, "Double Soft Theorems and Shift Symmetry in Nonlinear Sigma Models," *Phys. Rev. D*, vol. 93, no. 4, p. 045 032, 2016. DOI: 10.1103/PhysRevD.93.045032. arXiv: 1512.01232 [hep-th].

[98] I. Low and Z. Yin, "Ward Identity and Scattering Amplitudes for Nonlinear Sigma Models," *Phys. Rev. Lett.*, vol. 120, no. 6, p. 061 601, 2018. DOI: 10.1103/PhysRevLett.120.061601. arXiv: 1709.08639 [hep-th].

[99] I. Low and Z. Yin, "The Infrared Structure of Nambu-Goldstone Bosons," *JHEP*, vol. 10, p. 078, 2018. DOI: 10.1007/JHEP10(2018)078. arXiv: 1804.08629 [hep-th].

[100] D. Liu, I. Low, and R. Vega-Morales, "A Golden Probe of Nonlinear Higgs Dynamics," *Eur. Phys. J. C*, vol. 80, no. 9, p. 829, 2020. DOI: 10.1140/epjc/s10052-020-8244-8. arXiv: 1904.00026 [hep-ph].

[101] I. Low and Z. Yin, "Soft Bootstrap and Effective Field Theories," *JHEP*, vol. 11, p. 078, 2019. DOI: 10.1007/JHEP11(2019)078. arXiv: 1904.12859 [hep-th].

[102] L. Dai, I. Low, T. Mehen, and A. Mohapatra, "Operator Counting and Soft Blocks in Chiral Perturbation Theory," *Phys. Rev. D*, vol. 102, p. 116 011, 2020. DOI: 10.1103/PhysRevD.102.116011. arXiv: 2009.01819 [hep-ph].

[103] D. Roest, D. Stefanyszyn, and P. Werkman, "An Algebraic Classification of Exceptional EFTs," *JHEP*, vol. 08, p. 081, 2019. DOI: 10.1007/JHEP08(2019)081. arXiv: 1903.08222 [hep-th].

[104] K. Kampf, J. Novotny, F. Preucil, and J. Trnka, "Multi-spin soft bootstrap and scalar-vector Galileon," *JHEP*, vol. 07, p. 153, 2021. DOI: 10.1007/JHEP07(2021)153. arXiv: 2104.10693 [hep-th].

[105] K. Kampf, J. Novotny, and P. Vasko, "Extended DBI and its generalizations from graded soft theorems," *JHEP*, vol. 10, p. 101, 2021. DOI: 10.1007/JHEP10(2021)101. arXiv: 2107.04587 [hep-th].

[106] M. A. Mojahed and T. Brauner, "Nonrelativistic effective field theories with enhanced symmetries and soft behavior," *JHEP*, vol. 03, p. 086, 2022. DOI: 10.1007/JHEP03(2022)086. arXiv: 2201.01393 [hep-th].

[107] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Course of Theoretical Physics). New York: Elsevier Butterworth-Heinemann, 1986, vol. 7, ISBN: 978-0-7506-2633-0. DOI: 10.1016/C2009-0-25521-8.

[108] E. Pajer and D. Stefanyszyn, "Symmetric Superfluids," *JHEP*, vol. 06, p. 008, 2019. DOI: 10.1007/JHEP06(2019)008. arXiv: 1812.05133 [hep-th].

[109] T. Grall, S. Jazayeri, and D. Stefanyszyn, "The cosmological phonon: symmetries and amplitudes on sub-horizon scales," *JHEP*, vol. 11, p. 097, 2020. DOI: 10.1007/JHEP11(2020)097. arXiv: 2005.12937 [hep-th].

[110] S. Weinberg, "Phenomenological Lagrangians," *Physica A*, vol. 96, no. 1-2, S. Deser, Ed., pp. 327–340, 1979. DOI: 10.1016/0378-4371(79)90223-1.

[111] A. V. Manohar, "Introduction to Effective Field Theories," S. Davidson, P. Gambino, M. Laine, M. Neubert, and C. Salomon, Eds., Apr. 2018. DOI: 10.1093/oso/9780198855743.003.0002. arXiv: 1804.05863 [hep-ph].

[112] C. P. Burgess, *Introduction to Effective Field Theory*. Cambridge University Press, Dec. 2020, ISBN: 978-1-139-04804-0, 978-0-521-19547-8. DOI: 10.1017/9781139048040.

[113] J. Distler, R. Flauger, and B. Horn, "Double-soft graviton amplitudes and the extended BMS charge algebra," *JHEP*, vol. 08, p. 021, 2019. DOI: 10.1007/JHEP08(2019)021. arXiv: 1808.09965 [hep-th].

[114] F. Cachazo, S. He, and E. Y. Yuan, "New Double Soft Emission Theorems," *Phys. Rev. D*, vol. 92, no. 6, p. 065 030, 2015. DOI: 10.1103/PhysRevD.92.065030. arXiv: 1503.04816 [hep-th].

[115] Y.-J. Du and H. Luo, "On single and double soft behaviors in NLSM," *JHEP*, vol. 08, p. 058, 2015. DOI: 10.1007/JHEP08(2015)058. arXiv: 1505.04411 [hep-th].

[116] L. Rodina and Z. Yin, "Exploring the landscape for soft theorems of nonlinear sigma models," *JHEP*, vol. 08, p. 096, 2021. DOI: 10.1007/JHEP08(2021)096. arXiv: 2102.08396 [hep-th].

[117] K. Kampf, J. Novotny, M. Shifman, and J. Trnka, "New Soft Theorems for Goldstone Boson Amplitudes," *Phys. Rev. Lett.*, vol. 124, no. 11, p. 111 601, 2020. DOI: 10.1103/PhysRevLett.124.111601. arXiv: 1910.04766 [hep-th].

[118] P. Di Vecchia, R. Marotta, M. Mojaza, and J. Nohle, "New soft theorems for the gravity dilaton and the Nambu-Goldstone dilaton at subsubleading order," *Phys. Rev. D*, vol. 93, no. 8, p. 085 015, 2016. DOI: 10.1103/PhysRevD.93.085015. arXiv: 1512.03316 [hep-th].

[119] C. Cheung, M. Derda, A. Helset, and J. Parra-Martinez, "Soft phonon theorems," *JHEP*, vol. 08, p. 103, 2023. DOI: 10.1007/JHEP08(2023)103. arXiv: 2301.11363 [hep-th].

[120] M. Derda, A. Helset, and J. Parra-Martinez, "Soft scalars in effective field theory," *JHEP*, vol. 06, p. 133, 2024. DOI: 10.1007/JHEP06(2024)133. arXiv: 2403.12142 [hep-th].

[121] F. E. Low, "Bremsstrahlung of very low-energy quanta in elementary particle collisions," *Phys. Rev.*, vol. 110, pp. 974–977, 1958. DOI: 10.1103/PhysRev.110.974.

[122] T. H. Burnett and N. M. Kroll, "Extension of the low soft photon theorem," *Phys. Rev. Lett.*, vol. 20, p. 86, 1968. DOI: 10.1103/PhysRevLett.20.86.

[123] A. Bassetto, M. Ciafaloni, and G. Marchesini, "Jet Structure and Infrared Sensitive Quantities in Perturbative QCD," *Phys. Rept.*, vol. 100, pp. 201–272, 1983. DOI: 10.1016/0370-1573(83)90083-2.

[124] F. A. Berends and W. T. Giele, "Multiple Soft Gluon Radiation in Parton Processes," *Nucl. Phys. B*, vol. 313, pp. 595–633, 1989. DOI: 10.1016/0550-3213(89)90398-2.

[125] E. Casali, "Soft sub-leading divergences in Yang-Mills amplitudes," *JHEP*, vol. 08, p. 077, 2014. DOI: 10.1007/JHEP08(2014)077. arXiv: 1404.5551 [hep-th].

[126] F. Cachazo and A. Strominger, "Evidence for a New Soft Graviton Theorem," Apr. 2014. arXiv: 1404.4091 [hep-th].

[127] Z. Bern, S. Davies, P. Di Vecchia, and J. Nohle, "Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance," *Phys. Rev. D*, vol. 90, no. 8, p. 084 035, 2014. DOI: 10.1103/PhysRevD.90.084035. arXiv: 1406.6987 [hep-th].

[128] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*. Mar. 2017, ISBN: 978-0-691-17973-5. arXiv: 1703.05448 [hep-th].

[129] A. Strominger, "On BMS Invariance of Gravitational Scattering," *JHEP*, vol. 07, p. 152, 2014. DOI: 10.1007/JHEP07(2014)152. arXiv: 1312.2229 [hep-th].

[130] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, "Higher-dimensional supertranslations and Weinberg's soft graviton theorem," *Ann. Math. Sci. Appl.*, vol. 02, pp. 69–94, 2017. DOI: 10.4310/AMSA.2017.v2.n1.a2. arXiv: 1502.07644 [gr-qc].

[131] M. Campiglia and A. Laddha, "Asymptotic symmetries of QED and Weinberg's soft photon theorem," *JHEP*, vol. 07, p. 115, 2015. DOI: 10.1007/JHEP07(2015)115. arXiv: 1505.05346 [hep-th].

[132] E. Himwich and A. Strominger, "Celestial current algebra from Low's subleading soft theorem," *Phys. Rev. D*, vol. 100, no. 6, p. 065 001, 2019. DOI: 10.1103/PhysRevD.100.065001. arXiv: 1901.01622 [hep-th].

[133] T. He and P. Mitra, "Asymptotic symmetries and Weinberg's soft photon theorem in Mink_{d+2}," *JHEP*, vol. 10, p. 213, 2019. DOI: 10.1007/JHEP10(2019)213. arXiv: 1903.02608 [hep-th].

[134] D. Kapec and P. Mitra, "Shadows and soft exchange in celestial CFT," *Phys. Rev. D*, vol. 105, no. 2, p. 026 009, 2022. DOI: 10.1103/PhysRevD.105.026009. arXiv: 2109.00073 [hep-th].

[135] D. Kapec, Y. T. A. Law, and S. A. Narayanan, "Soft scalars and the geometry of the space of celestial conformal field theories," *Phys. Rev. D*, vol. 107, no. 4, p. 046 024, 2023. DOI: 10.1103/PhysRevD.107.046024. arXiv: 2205.10935 [hep-th].

[136] D. Kapec, "Soft particles and infinite-dimensional geometry," *Class. Quant. Grav.*, vol. 41, no. 1, p. 015 001, 2024. DOI: 10.1088/1361-6382/ad0514. arXiv: 2210.00606 [hep-th].

[137] T. Adamo, W. Bu, and B. Zhu, "Infrared structures of scattering on self-dual radiative backgrounds," Sep. 2023. arXiv: 2309.01810 [hep-th].

[138] H. Z. Chen, R. C. Myers, and A.-M. Raclariu, "Entanglement, Soft Modes, and Celestial Holography," Aug. 2023. arXiv: 2308.12341 [hep-th].

[139] S. Agrawal and K. Nguyen, "Soft theorems and spontaneous symmetry breaking," Apr. 2025. arXiv: 2504.10577 [hep-th].

[140] A. J. Larkoski, D. Neill, and I. W. Stewart, "Soft Theorems from Effective Field Theory," *JHEP*, vol. 06, p. 077, 2015. DOI: 10.1007/JHEP06(2015)077. arXiv: 1412.3108 [hep-th].

[141] A. Kapustin and R. Thorngren, "Higher Symmetry and Gapped Phases of Gauge Theories," *Prog. Math.*, vol. 324, pp. 177–202, 2017. DOI: 10.1007/978-3-319-59939-7_5. arXiv: 1309.4721 [hep-th].

[142] E. Sharpe, "Notes on generalized global symmetries in QFT," *Fortsch. Phys.*, vol. 63, pp. 659–682, 2015. DOI: 10.1002/prop.201500048. arXiv: 1508.04770 [hep-th].

[143] Y. Tachikawa, "On gauging finite subgroups," *SciPost Phys.*, vol. 8, no. 1, p. 015, 2020. DOI: 10.21468/SciPostPhys.8.1.015. arXiv: 1712.09542 [hep-th].

[144] C. Córdova, T. T. Dumitrescu, and K. Intriligator, "Exploring 2-Group Global Symmetries," *JHEP*, vol. 02, p. 184, 2019. DOI: 10.1007/JHEP02(2019)184. arXiv: 1802.04790 [hep-th].

[145] F. Benini, C. Córdova, and P.-S. Hsin, "On 2-Group Global Symmetries and their Anomalies," *JHEP*, vol. 03, p. 118, 2019. DOI: 10.1007/JHEP03(2019)118. arXiv: 1803.09336 [hep-th].

[146] J. Baez and U. Schreiber, "Higher gauge theory: 2-connections on 2-bundles," Dec. 2004. arXiv: hep-th/0412325.

[147] T. D. Brennan and C. Cordova, "Axions, higher-groups, and emergent symmetry," *JHEP*, vol. 02, p. 145, 2022. DOI: 10.1007/JHEP02(2022)145. arXiv: 2011.09600 [hep-th].

[148] A. Arbalestrier, R. Argurio, and L. Tizzano, "U(1) Gauging, Continuous TQFTs, and Higher Symmetry Structures," Feb. 2025. arXiv: 2502.12997 [hep-th].

[149] Z. Komargodski, K. Ohmori, K. Roumpedakis, and S. Seifnashri, "Symmetries and strings of adjoint QCD_2," *JHEP*, vol. 03, p. 103, 2021. DOI: 10.1007/JHEP03(2021)103. arXiv: 2008.07567 [hep-th].

[150] J. Kaidi, K. Ohmori, and Y. Zheng, "Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories," *Phys. Rev. Lett.*, vol. 128, no. 11, p. 111601, 2022. DOI: 10.1103/PhysRevLett.128.111601. arXiv: 2111.01141 [hep-th].

[151] Y. Choi, C. Córdova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, "Non-invertible duality defects in 3+1 dimensions," *Phys. Rev. D*, vol. 105, no. 12, p. 125 016, 2022. DOI: 10.1103/PhysRevD.105.125016. arXiv: 2111.01139 [hep-th].

[152] C. Córdova and K. Ohmori, "Noninvertible Chiral Symmetry and Exponential Hierarchies," *Phys. Rev. X*, vol. 13, no. 1, p. 011 034, 2023. DOI: 10.1103/PhysRevX.13.011034. arXiv: 2205.06243 [hep-th].

[153] I. García Etxebarria and N. Iqbal, "A Goldstone theorem for continuous non-invertible symmetries," *JHEP*, vol. 09, p. 145, 2023. DOI: 10.1007/JHEP09(2023)145. arXiv: 2211.09570 [hep-th].

[154] Y. Choi, H. T. Lam, and S.-H. Shao, "Noninvertible Global Symmetries in the Standard Model," *Phys. Rev. Lett.*, vol. 129, no. 16, p. 161 601, 2022. DOI: 10.1103/PhysRevLett.129.161601. arXiv: 2205.05086 [hep-th].

[155] D. M. Hofman and N. Iqbal, "Goldstone modes and photonization for higher form symmetries," *SciPost Phys.*, vol. 6, no. 1, p. 006, 2019. DOI: 10.21468/SciPostPhys.6.1.006. arXiv: 1802.09512 [hep-th].

[156] A. Kovner and B. Rosenstein, "New look at QED in four-dimensions: The Photon as a Goldstone boson and the topological interpretation of electric charge," *Phys. Rev. D*, vol. 49, pp. 5571–5581, 1994. DOI: 10.1103/PhysRevD.49.5571. arXiv: hep-th/9210154.

[157] J. McGreevy, "Generalized Symmetries in Condensed Matter," *Ann. Rev. Condensed Matter Phys.*, vol. 14, pp. 57–82, 2023. DOI: 10.1146/annurev-conmatphys-040721-021029. arXiv: 2204.03045 [cond-mat.str-el].

[158] C. Copetti, L. Córdova, and S. Komatsu, "Noninvertible Symmetries, Anomalies, and Scattering Amplitudes," *Phys. Rev. Lett.*, vol. 133, no. 18, p. 181 601, 2024. DOI: 10.1103/PhysRevLett.133.181601. arXiv: 2403.04835 [hep-th].

[159] C. Copetti, L. Córdova, and S. Komatsu, "S-matrix bootstrap and non-invertible symmetries," *JHEP*, vol. 03, p. 204, 2025. DOI: 10.1007/JHEP03(2025)204. arXiv: 2408.13132 [hep-th].

[160] M. van Beest, P. Boyle Smith, D. Delmastro, Z. Komargodski, and D. Tong, "Monopoles, scattering, and generalized symmetries," *JHEP*, vol. 03, p. 014, 2025. DOI: 10.1007/JHEP03(2025)014. arXiv: 2306.07318 [hep-th].

[161] A. V. Manohar and M. B. Wise, *Heavy quark physics*. Cambridge University Press, 2000, vol. 10, ISBN: 978-0-521-03757-0, 978-1-009-40212-5. DOI: 10.1017/9781009402125.

[162] P. H. Damgaard, K. Haddad, and A. Helset, "Heavy Black Hole Effective Theory," *JHEP*, vol. 11, p. 070, 2019. DOI: 10.1007/JHEP11(2019)070. arXiv: 1908.10308 [hep-ph].

[163] D. V. Volkov, "Phenomenological Lagrangians," *Fiz. Elem. Chast. Atom. Yadra*, vol. 4, pp. 3–41, 1973.

[164] J. Honerkamp and K. Meetz, "Chiral-invariant perturbation theory," *Phys. Rev. D*, vol. 3, pp. 1996–1998, 1971. DOI: 10.1103/PhysRevD.3.1996.

[165] D. G. Boulware and L. S. Brown, "Symmetric Space Scalar Field Theory," *Annals Phys.*, vol. 138, p. 392, 1982. DOI: 10.1016/0003-4916(82)90192-0.

[166] W. Heisenberg and H. Euler, "Consequences of Dirac's theory of positrons," *Z. Phys.*, vol. 98, no. 11-12, pp. 714–732, 1936. DOI: 10.1007/BF01343663. arXiv: physics/0605038.

[167] C. W. Bauer, D. Pirjol, and I. W. Stewart, "Soft collinear factorization in effective field theory," *Phys. Rev. D*, vol. 65, p. 054 022, 2002. DOI: 10.1103/PhysRevD.65.054022. arXiv: hep-ph/0109045.

[168] C. W. Bauer, S. Fleming, D. Pirjol, and I. W. Stewart, "An Effective field theory for collinear and soft gluons: Heavy to light decays," *Phys. Rev. D*, vol. 63, p. 114 020, 2001. DOI: 10.1103/PhysRevD.63.114020. arXiv: hep-ph/0011336.

[169] J. C. Baez and A. D. Lauda, "Higher-dimensional algebra V: 2-Groups," *Theory and Applications of Categories*, vol. 12, pp. 423–491, 2004. arXiv: math/0307200 [math.QA]. [Online]. Available: http://eudml.org/doc/124217.

[170] T. Nakajima, T. Sakai, and R. Yokokura, "BCF anomaly and higher-group structure in the low energy effective theories of mesons," *JHEP*, vol. 01, p. 175, 2023. DOI: 10.1007/JHEP01(2023)175. arXiv: 2212.12987 [hep-th].

[171] J. Davighi and N. Lohitsiri, "WZW terms without anomalies: Generalised symmetries in chiral Lagrangians," *SciPost Phys.*, vol. 17, no. 6, p. 168, 2024. DOI: 10.21468/SciPostPhys.17.6.168. arXiv: 2407.20340 [hep-th].

[172] E. Witten, "Current Algebra, Baryons, and Quark Confinement," *Nucl. Phys. B*, vol. 223, pp. 433–444, 1983. DOI: 10.1016/0550-3213(83)90064-0.

[173] E. Witten, "Global Aspects of Current Algebra," *Nucl. Phys. B*, vol. 223, pp. 422–432, 1983. DOI: 10.1016/0550-3213(83)90063-9.

[174] C. Córdova, K. Ohmori, and T. Rudelius, "Generalized symmetry breaking scales and weak gravity conjectures," *JHEP*, vol. 11, p. 154, 2022. DOI: 10.1007/JHEP11(2022)154. arXiv: 2202.05866 [hep-th].

[175] A. Antinucci, G. Galati, and G. Rizi, "On continuous 2-category symmetries and Yang-Mills theory," *JHEP*, vol. 12, p. 061, 2022. DOI: 10.1007/JHEP12(2022)061. arXiv: 2206.05646 [hep-th].

[176] V. Benedetti, H. Casini, and J. M. Magan, "Generalized symmetries of the graviton," *JHEP*, vol. 05, p. 045, 2022. DOI: 10.1007/JHEP05(2022)045. arXiv: 2111.12089 [hep-th].

[177] K. Hinterbichler, D. M. Hofman, A. Joyce, and G. Mathys, "Gravity as a gapless phase and biform symmetries," *JHEP*, vol. 02, p. 151, 2023. DOI: 10.1007/JHEP02(2023)151. arXiv: 2205.12272 [hep-th].

[178] V. Benedetti, P. Bueno, and J. M. Magan, "Generalized Symmetries for Generalized Gravitons," *Phys. Rev. Lett.*, vol. 131, no. 11, p. 111 603, 2023. DOI: 10.1103/PhysRevLett.131.111603. arXiv: 2305.13361 [hep-th].

[179] C. Cheung, M. Derda, J.-H. Kim, V. Nevoa, I. Rothstein, and N. Shah, "Generalized symmetry in dynamical gravity," *JHEP*, vol. 10, p. 007, 2024. DOI: 10.1007/JHEP10(2024)007. arXiv: 2403.01837 [hep-th].

[180] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, "Topological Defect Lines and Renormalization Group Flows in Two Dimensions," *JHEP*, vol. 01, p. 026, 2019. DOI: 10.1007/JHEP01(2019)026. arXiv: 1802.04445 [hep-th].

[181] C. Córdova, S. Hong, S. Koren, and K. Ohmori, "Neutrino Masses from Generalized Symmetry Breaking," *Phys. Rev. X*, vol. 14, no. 3, p. 031 033, 2024. DOI: 10.1103/PhysRevX.14.031033. arXiv: 2211.07639 [hep-ph].

[182] J. A. Damia, R. Argurio, and L. Tizzano, "Continuous Generalized Symmetries in Three Dimensions," *JHEP*, vol. 05, p. 164, 2023. DOI: 10.1007/JHEP05(2023)164. arXiv: 2206.14093 [hep-th].

[183] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki, and A. Tiwari, "Non-invertible higher-categorical symmetries," *SciPost Phys.*, vol. 14, no. 1, p. 007, 2023. DOI: 10.21468/SciPostPhys.14.1.007. arXiv: 2204.06564 [hep-th].

[184] J. Kaidi, G. Zafrir, and Y. Zheng, "Non-invertible symmetries of $\mathcal{N} = 4$ SYM and twisted compactification," *JHEP*, vol. 08, p. 053, 2022. DOI: 10.1007/JHEP08(2022)053. arXiv: 2205.01104 [hep-th].

[185] Y. Choi, C. Córdova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, "Non-invertible Condensation, Duality, and Triality Defects in 3+1 Dimensions," *Commun. Math. Phys.*, vol. 402, no. 1, pp. 489–542, 2023. DOI: 10.1007/s00220-023-04727-4. arXiv: 2204.09025 [hep-th].

[186] S. Pasterski, M. Pate, and A.-M. Raclariu, "Celestial Holography," in *Snowmass 2021*, Nov. 2021. arXiv: 2111.11392 [hep-th].

[187] Y. Wang and X. Yin, "Constraining Higher Derivative Supergravity with Scattering Amplitudes," *Phys. Rev. D*, vol. 92, no. 4, p. 041 701, 2015. DOI: 10.1103/PhysRevD.92.041701. arXiv: 1502.03810 [hep-th].

[188] Y. Wang and X. Yin, "Supervertices and Non-renormalization Conditions in Maximal Supergravity Theories," May 2015. arXiv: 1505.05861 [hep-th].

[189] Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, "Supersymmetry Constraints and String Theory on K3," *JHEP*, vol. 12, p. 142, 2015. DOI: 10.1007/JHEP12(2015)142. arXiv: 1508.07305 [hep-th].

[190] M. B. Green and C. Wen, "Modular Forms and $SL(2,\mathbb{Z})$-covariance of type IIB superstring theory," *JHEP*, vol. 06, p. 087, 2019. DOI: 10.1007/JHEP06(2019)087. arXiv: 1904.13394 [hep-th].

[191] A. Tropper, "Supersymmetric soft theorems," *Class. Quant. Grav.*, vol. 42, no. 8, p. 085 004, 2025. DOI: 10.1088/1361-6382/adc536. arXiv: 2404.03717 [hep-th].

[192] B. A. Kniehl and M. Spira, "Low-energy theorems in Higgs physics," *Z. Phys. C*, vol. 69, pp. 77–88, 1995. DOI: 10.1007/s002880050007. arXiv: hep-ph/9505225.

[193] R. Balkin, G. Durieux, T. Kitahara, Y. Shadmi, and Y. Weiss, "On-shell Higgsing for EFTs," *JHEP*, vol. 03, p. 129, 2022. DOI: 10.1007/JHEP03(2022)129. arXiv: 2112.09688 [hep-ph].

[194] B. S. DeWitt, *Supermanifolds* (Cambridge Monographs on Mathematical Physics). Cambridge, UK: Cambridge Univ. Press, May 2012, ISBN: 978-0-521-42377-9. DOI: 10.1017/CBO9780511564000.

[195] L. J. Dixon, "Calculating scattering amplitudes efficiently," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 95): QCD and Beyond*, Jan. 1996, pp. 539–584. arXiv: hep-ph/9601359.

[196] C. G. Callan Jr., "Broken scale invariance in scalar field theory," *Phys. Rev. D*, vol. 2, pp. 1541–1547, 1970. DOI: 10.1103/PhysRevD.2.1541.

[197] R. H. Boels and W. Wormsbecher, "Spontaneously broken conformal invariance in observables," Jul. 2015. arXiv: 1507.08162 [hep-th].

[198] Y.-t. Huang and C. Wen, “Soft theorems from anomalous symmetries,” *JHEP*, vol. 12, p. 143, 2015. DOI: 10.1007/JHEP12(2015)143. arXiv: 1509.07840 [hep-th].

[199] S. L. Adler, “Consistency conditions on the strong interactions implied by a partially conserved axial-vector current. II,” *Phys. Rev.*, vol. 139, B1638–B1643, 1965. DOI: 10.1103/PhysRev.139.B1638.

[200] Y. Nambu and E. Shrauner, “Soft pion emission induced by electromagnetic and weak interactions,” *Phys. Rev.*, vol. 128, T. Eguchi, Ed., pp. 862–868, 1962. DOI: 10.1103/PhysRev.128.862.

[201] Y. Nambu and D. Lurie, “Chirality conservation and soft pion production,” *Phys. Rev.*, vol. 125, T. Eguchi, Ed., pp. 1429–1436, 1962. DOI: 10.1103/PhysRev.125.1429.

[202] S. Coleman, “Soft pions,” in *Aspects of Symmetry: Selected Erice Lectures*. Cambridge University Press, 1985, pp. 36–66. DOI: 10.1017/CBO9780511565045.003.

[203] E. Witten, “Current Algebra Theorems for the U(1) Goldstone Boson,” *Nucl. Phys. B*, vol. 156, pp. 269–283, 1979. DOI: 10.1016/0550-3213(79)90031-2.

[204] M. Campiglia, L. Coito, and S. Mizera, “Can scalars have asymptotic symmetries?” *Phys. Rev. D*, vol. 97, no. 4, p. 046 002, 2018. DOI: 10.1103/PhysRevD.97.046002. arXiv: 1703.07885 [hep-th].

[205] S. Biswas and G. W. Semenoff, “Soft scalars do not decouple,” *Phys. Rev. D*, vol. 106, no. 10, p. 105 023, 2022. DOI: 10.1103/PhysRevD.106.105023. arXiv: 2208.05023 [hep-th].

[206] W.-M. Chen, Y.-t. Huang, and C. Wen, “New Fermionic Soft Theorems for Supergravity Amplitudes,” *Phys. Rev. Lett.*, vol. 115, no. 2, p. 021 603, 2015. DOI: 10.1103/PhysRevLett.115.021603. arXiv: 1412.1809 [hep-th].

[207] T. Uematsu and C. K. Zachos, “Structure of Phenomenological Lagrangians for Broken Supersymmetry,” *Nucl. Phys. B*, vol. 201, pp. 250–268, 1982. DOI: 10.1016/0550-3213(82)90431-X.

[208] T. T. Dumitrescu, T. He, P. Mitra, and A. Strominger, “Infinite-dimensional fermionic symmetry in supersymmetric gauge theories,” *JHEP*, vol. 08, p. 051, 2021. DOI: 10.1007/JHEP08(2021)051. arXiv: 1511.07429 [hep-th].

[209] N. Arkani-Hamed, T.-C. Huang, and Y.-t. Huang, “Scattering amplitudes for all masses and spins,” *JHEP*, vol. 11, p. 070, 2021. DOI: 10.1007/JHEP11(2021)070. arXiv: 1709.04891 [hep-th].

[210] G. Durieux, T. Kitahara, Y. Shadmi, and Y. Weiss, "The electroweak effective field theory from on-shell amplitudes," *JHEP*, vol. 01, p. 119, 2020. DOI: 10.1007/JHEP01(2020)119. arXiv: 1909.10551 [hep-ph].

[211] A. Falkowski and C. S. Machado, "Soft Matters, or the Recursions with Massive Spinors," *JHEP*, vol. 05, p. 238, 2021. DOI: 10.1007/JHEP05(2021)238. arXiv: 2005.08981 [hep-th].

[212] F. Cachazo, P. Cha, and S. Mizera, "Extensions of Theories from Soft Limits," *JHEP*, vol. 06, p. 170, 2016. DOI: 10.1007/JHEP06(2016)170. arXiv: 1604.03893 [hep-th].

[213] M. A. Mojahed and T. Brauner, "On-shell recursion relations for non-relativistic effective field theories," *Phys. Lett. B*, vol. 822, p. 136 705, 2021. DOI: 10.1016/j.physletb.2021.136705. arXiv: 2108.03189 [hep-th].

Appendix A

NONRELATIVISTIC KINEMATICS

Our analysis will require a careful treatment of on-shell kinematics for scattering amplitudes with nonrelativistic dynamics. For the n-point amplitude, A_n, we define the four-momenta of the n hard legs to be

$$p_a^\mu = (\omega_a, p_a^i)\,, \tag{A.0.1}$$

where the external particle index is an integer in the range $1 \le a \le n$.

It will be crucial to define the notion of a minimal on-shell basis of kinematic invariants. A priori, the n-point amplitude is an $SO(3)$ invariant quantity that is a function of the energies ω_a and all inner products of three-momenta, $p_a \cdot p_b$ for $1 \le a \le b \le n$. A minimal on-shell basis defines a set of kinematic variables for which all on-shell constraints are automatically imposed.

To achieve this, we first eliminate the energy and three-momentum of some leg, chosen here to be leg n, so

$$\omega_n = -\sum_{a=1}^{n-1} \omega_a \qquad \text{and} \qquad p_n^i = -\sum_{a=1}^{n-1} p_a^i\,. \tag{A.0.2}$$

The elimination of the energy ω_n and three-momentum p_n^i of leg n via the above equations then automatically enforces total momentum conservation. Second, we impose the on-shell conditions for the external legs, allowing us to eliminate the kinematic invariants

$$p_a^2 = \frac{\omega_a^2}{c_a^2} \qquad \text{for} \qquad 1 \le a \le n-1\,, \tag{A.0.3}$$

where c_a is the speed of sound for the corresponding leg. For example, for phonons it would be the longitudinal or transverse speeds of sound, c_L or c_T. The above condition allows us to eliminate p_a^2 for $1 \le a \le n-1$. However since we have already eliminated p_n^i by momentum conservation, for the case of $a = n$ the on-shell condition imposes a more elaborate constraint,

$$p_n^2 = \left(\sum_{a=1}^{n-1} p_a\right)^2 = \frac{1}{c_n^2}\left(\sum_{a=1}^{n-1} \omega_a\right)^2\,, \tag{A.0.4}$$

which should can be used to eliminate one more kinematic invariant, which we can choose to be $p_{n-2} \cdot p_{n-1}$ without loss of generality. In summary, the minimal kinematic basis is comprised of the variables

$$\begin{aligned} &\omega_a && \text{for} && 1 \leq a \leq n-1\,, \\ &p_a \cdot p_b && \text{for} && 1 \leq a < b \leq n-2 \qquad \text{and} \qquad 1 \leq a \leq n-3,\;\; b = n-1\,, \end{aligned} \tag{A.0.5}$$

with all other kinematic variables fixed by on-shell conditions.

With the inclusion of external polarization vectors, e_a^i for $1 \leq a \leq n$, similar logic applies. Without assuming any special properties of the external polarizations, for example whether they are longitudinal or transverse, we simply eliminate all invariants involving p_n^i. Hence, we obtain

$$\begin{aligned} &e_a \cdot e_b && \text{for} && 1 \leq a < b \leq n\,, \\ &p_a \cdot e_b && \text{for} && 1 \leq a \leq n-1,\;\; 1 \leq b \leq n\,, \end{aligned} \tag{A.0.6}$$

for the elements of the minimal kinematic basis involving polarizations.

Let us also write down the explicit minimal kinematic basis for three-point scattering,

$$\begin{aligned} \text{basis for } A_3: \qquad & \omega_1,\;\; \omega_2\,, \\ & p_1 \cdot e_1,\;\; p_1 \cdot e_2,\;\; p_1 \cdot e_3,\;\; p_2 \cdot e_1,\;\; p_2 \cdot e_2,\;\; p_2 \cdot e_3\,, \\ & e_1 \cdot e_2,\;\; e_1 \cdot e_3,\;\; e_2 \cdot e_3\,. \end{aligned} \tag{A.0.7}$$

The utility of these variables is that we can freely change them while remaining on the kinematic surface that defines on-shell configurations. From here on, we will write all on-shell quantities in terms of these minimal bases.

Finally, to evaluate the amplitude for a specific configuration of external modes, we plug in explicit longitudinal or transverse polarizations. The transverse conditions put additional constraints on the minimal basis

$$\begin{aligned} p_a \cdot e_a &= 0 \qquad \text{for} \qquad 1 \leq a < n\,, \\ p_{n-1} \cdot e_n &= -\sum_{a=1}^{n-2} p_a \cdot e_n\,. \end{aligned} \tag{A.0.8}$$

For our analysis, we will be interested in how the soft limit of the $(n+1)$-point amplitude, A_{n+1}, can be expressed in terms of the n-point amplitude, A_n. For this reason, we define legs $1, \cdots, n$ to be hard, since they are present in both

A_{n+1} and A_n. On the other hand, leg $n+1$, with external polarization e^i, will be taken soft, so we parameterize it with a special four-momentum

$$q^\mu = (\omega, q^i)\,. \tag{A.0.9}$$

To be very explicit, A_{n+1} is a function of $p_1, \cdots, p_n, q$ while A_n is a function of $, \cdots, p_n$. Both should be evaluated in a minimal kinematic basis in which the energy ω_n, the three-momentum p_n^i, and the invariant $p_{n-2} \cdot p_{n-1}$ have been eliminated. Consequently, for *a*ny values of the soft momentum q, the amplitudes remain on-shell. This ensures that the soft limit is taken while maintaining all on-shell conditions. The minimal basis for four-point scattering is then

$$\begin{aligned}
\text{basis for } A_4: \qquad & \omega,\ \omega_1,\ \omega_2,\ q\cdot p_1,\ q\cdot p_2,\ q\cdot e,\ q\cdot e_1,\ q\cdot e_2,\ q\cdot e_3\,,\\
& p_1\cdot e,\ p_1\cdot e_1,\ p_1\cdot e_2,\ p_1\cdot e_3,\ p_2\cdot e,\ p_2\cdot e_1,\ p_2\cdot e_2,\ p_2\cdot e_3\,,\\
& e\cdot e_1,\ e\cdot e_2,\ e\cdot e_3,\ e_1\cdot e_2,\ e_1\cdot e_3,\ e_2\cdot e_3\,.
\end{aligned} \tag{A.0.10}$$

By construction, the minimal basis for A_4 in eq. (A.0.10) reduces to the minimal basis for A_3 in eq. (A.0.7) in the soft limit, $q \to 0$.

While the above approach is somewhat convoluted, we emphasize that any definition of the soft limit of an on-shell amplitude requires something analogous. In general, simply changing the momentum q of a leg to be soft will not maintain on-shell conditions. For the case of on-shell soft recursion [30], [93], [212], [213], the soft limit of a given leg must always be compensated by a slight shift of one of the hard legs. The minimal basis construction we have described above achieves this automatically.

Appendix B

SELECT FORM FACTOR SOFT LIMITS

In this appendix we provide the technical details for the derivation of the soft behavior of the correlators $\langle\alpha|V^c_\mu(q)\rangle$ and $\langle\alpha|\mathcal{J}_H{}^a_\mu(q_1)\mathcal{J}_H{}^b_\nu(q_2)\rangle$ which appeared in eq. (3.2.27) of the main text.

B.1 Soft vector current

Consider the behavior of the following matrix element where an off-shell vector current is inserted into a scattering amplitude of on-shell NGB,

$$\langle\pi^{a_1}(p_1)\dots\pi^{a_i}(p_i)\dots\pi^{a_n}(p_n)|V^a_\mu(q)\rangle. \tag{B.1.1}$$

In [94] the leading order soft behavior of eq. (B.1.1) was proven. Here we will extend that proof to the sub-leading order in the soft momentum q. This extension will closely follow the development appearing in [127], wherein sub-leading soft theorems are proven for photons, gluons and gravitons.

In the soft limit, the singular behavior of eq. (B.1.1) will derive from processes in which a propagating single particle state is created. Such a pole structure will be generated for each of the outgoing NGB states $\langle\pi^{a_i}(p_i)|$. We insert a complete set of states $\mathbb{1} = \sum\!\!\!\!\!\!\int_X |X\rangle\langle X|$ between the soft current operator $V^a_\mu(q)$ and the rest of the process,

$$\sum\!\!\!\!\!\!\!\int_X \sum_{i=1}^n \langle\pi^{a_i}(p_i)|V^a_\mu(q)|X\rangle\Delta_X\langle X+\pi^{a_1}(p_1)\dots\widehat{\pi^{a_i}(p_i)}\dots\pi^{a_n}(p_n)|0\rangle \tag{B.1.2}$$

$$= \sum_{i=1}^n\langle\pi^{a_i}(p_i)|V^a_\mu(q)|\pi^c(p_i+q)\rangle\Delta^{cd}(p_i+q)\langle\pi^{a_1}(p_1)\dots\pi^d(p_i+q)\dots\pi^{a_n}(p_n)|0\rangle$$

with $\widehat{\pi^{a_i}}$ denoting the omission of the corresponding particle. Where $|X\rangle$ is a multi-particle state, the kinematics in the soft limit will not produce singular behavior. The singular behavior will receive contributions from only those $|X\rangle$ which are single-particle NGB states, where $\Delta^{cd}(p_i+q)$ is the associated propagator.

To obtain the sub-leading contribution we begin from the decomposition,

$$\lim_{q\to 0}\langle\pi^{a_1}(p_1)\dots\pi^{a_i}(p_i)\dots\pi^{a_n}(p_n)|V^a_\mu(q)\rangle$$

$$= \sum_{i=1}^{n} f_X^{aa_id} \frac{(2p_i+q)_\mu}{(p_i+q)^2} \langle \pi^{a_1}(p_1)\dots\pi^d(p_i+q)\dots\pi^{a_n}(p_n)|0\rangle$$
$$+ R_\mu^{aa_1\dots a_n}(q;p_1,\dots,p_n). \tag{B.1.3}$$

Here the first term is the result from [94] without having expanded the propagator pole, and the second, remnant term, parameterizes insertions of the current which do not contain said pole. The entire matrix element satisfies the Ward identity,

$$0 = q^\mu \langle \pi^{a_1}\cdots\pi^{a_n}|V_\mu^a(q)\rangle$$
$$= \sum_{i=1}^{n} f_X^{aa_id} \langle \pi^{a_1}(p_1)\dots\pi^d(p_i+q)\dots\pi^{a_n}(p_n)|0\rangle + q^\mu R_\mu^{aa_1\dots a_n}(q;p_1,\dots,p_n). \tag{B.1.4}$$

Expanding around $q=0$, we have

$$0 = \sum_{i=1}^{n} f_X^{aa_id}\left(1+q^\mu\frac{\partial}{\partial p_i^\mu}\right)\langle \pi^{a_1}(p_1)\dots\pi^d(p_i)\dots\pi^{a_n}(p_n)|0\rangle$$
$$+ q^\mu R_\mu^{aa_1\dots a_n}(0;p_1,\dots,p_n) + \mathcal{O}(q^2). \tag{B.1.5}$$

This produces a set of relations for each order in q. At leading order we have,

$$0 = \sum_{i=1}^{n} f_X^{aa_id} \langle \pi^{a_1}(p_1)\dots\pi^d(p_i)\dots\pi^{a_n}(p_n)|0\rangle, \tag{B.1.6}$$

which is a consequence of invariance under the unbroken subgroup H, and was demonstrated in [94]. At the next order we have

$$\sum_{i=1}^{n} f_X^{aa_id} q^\mu \frac{\partial}{\partial p_i^\mu}\langle \pi^{a_1}(p_1)\dots\pi^d(p_i)\dots\pi^{a_n}(p_n)|0\rangle = -q^\mu R_\mu^{aa_1\dots a_n}(0;p_1,\dots,p_n). \tag{B.1.7}$$

This relation determines the remnant $R_\mu^{aa_1\dots a_n}(0;p_1,\dots,p_n)$ up to potential terms that are separately vanishing under the Ward identity. Furthermore, the requirement that such terms be local in q implies that they must also be at least linear in q. Therefore these terms would be suppressed in the soft expansion. We can remove the q^μ contracted with both sides of eq. (B.1.7), leaving

$$\sum_{i=1}^{n} f_X^{aa_id} \frac{\partial}{\partial p_i^\mu}\langle \pi^{a_1}(p_1)\dots\pi^d(p_i)\dots\pi^{a_n}(p_n)|0\rangle = -R_\mu^{aa_1\dots a_n}(0;p_1,\dots,p_n). \tag{B.1.8}$$

Inserting this into the decomposition eq. (B.1.3) completes the sub-leading theorem,

$$\begin{aligned}&\lim_{q\to 0}\langle \pi^{a_1}\cdots\pi^{a_n}|V^a_\mu(q)\rangle \\ &=\sum_{i=1}^{n} f^{aa_id}\left(\frac{(2p_i+q)_\mu}{(p_i+q)^2}-\frac{iq^\nu L_{i\mu\nu}}{(p_i\cdot q)}\right)\mathcal{A}_n^{a_1\ldots d\ldots a_n}(p_1,\ldots,p_n)+\mathcal{O}(q)\,, \quad \text{(B.1.9)}\end{aligned}$$

where as before $L_{i\mu\nu}$ is the angular momentum operator defined as

$$L_i^{\mu\nu}=i\left(p_i^\mu\frac{\partial}{\partial p_{i\nu}}-p_i^\nu\frac{\partial}{\partial p_{i\mu}}\right). \tag{B.1.10}$$

B.2 Soft axial-axial remnant

We will proceed with a similar analysis for the matrix element:

$$\langle \pi^{a_1}(p_1)\ldots\pi^{a_i}(p_i)\ldots\pi^{a_n}(p_n)|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)\rangle. \tag{B.2.1}$$

We will follow the same line of reasoning as for the soft vector current $V^a_\mu(q)$ in the matrix element eq. (B.2.1). The singular behavior of eq. (B.2.1) will derive from contributions of the form,

$$\begin{aligned}&\sum_{i=1}^{n}\langle \pi^{a_i}(p_i)|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)|\pi^c(p_i+q_1+q_2)\rangle \\ &\qquad\times\Delta^{cd}(p_i+q_1+q_2)\langle\pi^{a_1}(p_1)\ldots\pi^d(p_i+q_1+q_2)\ldots\pi^{a_n}(p_n)|0\rangle,\end{aligned} \tag{B.2.2}$$

which are generated by the propagation of single particle NGB states.

The isolated form factor $\langle \pi^{a_i}(p_i)|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)|\pi^c(p_i+q_1+q_2)\rangle$ can be determined to leading order. The effective theory is invariant under parity inversions, thus we consider a parity-even ansatz,

$$\begin{aligned}&\langle \pi^{a_i}(p_i)|\mathcal{J}_{H}{}^a_\mu(q_1)\mathcal{J}_{H}{}^b_\nu(q_2)|\pi^c(p_i+q_1+q_2)\rangle \\ &=A_1(p_i,q_1,q_2)B_1^{a_iabc}\eta_{\mu\nu}+A_2(p_i,q_1,q_2)B_2^{a_iabc}p_{i\mu}p_{i\nu}+\mathcal{O}(q_1,q_2),\end{aligned} \tag{B.2.3}$$

where $A_i(p_i,q_1,q_2)$ are Lorentz-invariant structure functions and $B_i^{a_iabc}$ are arbitrary flavor-structures. The $\mathcal{J}_{H}{}^a_\mu$ operators appearing in this form factor do not satisfy any Ward identities, but we can instead consider the Ward identity,

$$p_i^\alpha\langle \mathcal{J}_\alpha^{a_i}(p_i)\mathcal{J}^a_\mu(q_1)\mathcal{J}^b_\nu(q_2)|\pi^c(p_i+q_1+q_2)\rangle \tag{B.2.4}$$

$$= F^{a_i a d}\langle V^d_\mu(p_i+q_1)\mathcal{J}^b_\nu(q_2)|\pi^c(p_i+q_1+q_2)\rangle$$
$$+ F^{a_i b d}\langle \mathcal{J}^a_\mu(q_1)V^d_\nu(p_i+q_2)|\pi^c(p_i+q_1+q_2)\rangle\,.$$

Through a procedure of writing analogous ansätze for the correlators on the RHS of the Ward identity eq. (B.2.4), imposing the subsequent Ward identities these correlators in turn satisfy, and finally matching the contained pole structures with our initial ansatz eq. (B.2.3), we arrive at

$$\begin{aligned}&\langle \pi^{a_i}(p_i)|\mathcal{J}_H{}^a_\mu(q_1)\mathcal{J}_H{}^b_\nu(q_2)|\pi^c(p_i+q_1+q_2)\rangle\\ &= i(F^{a_i a e}f_X^{ebc}+F^{a_i b e}f_X^{eac})\eta_{\mu\nu}+A_2(p_i,q_1,q_2)B_2^{a_i abc}p_{i\mu}p_{i\nu}+\mathcal{O}(q_1,q_2)\,.\end{aligned} \tag{B.2.5}$$

The product $A_2(p_i,q_1,q_2)B_2^{a_i abc}$ cannot be fixed by current algebra. This reflects the fact that the sub-leading double soft limit is sensitive to higher-derivative operators in the effective field theory, which was studied in [116], and also one loop corrections. For simplicity, we will ignore these.

At tree-level, we can conclude the soft behavior,

$$\begin{aligned}&\lim_{q_1 q_2\to 0}\langle\alpha|\mathcal{J}_H{}^a_\mu(q_1)\mathcal{J}_H{}^b_\nu(q_2)\rangle\\ &=\sum_{i=1}^n(F^{a a_i e}f_X^{ebd}+F^{b a_i e}f_X^{ead})\frac{\eta_{\mu\nu}}{2p_i\cdot(q_1+q_2)}\mathcal{A}_n^{a_1\cdots d\cdots a_n}+\mathcal{O}(q^0)\,.\end{aligned} \tag{B.2.6}$$

Combining the results eq. (B.2.6) and eq. (B.1.9) in eq. (3.2.27), we arrive at the NLSM double soft theorem eq. (3.2.31) in the main text.